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A GELFAND-GRAEV FORMULA AND STABLE TRANSFER FACTORS FOR SL_N

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DANIEL LENTINE JOHNSTONE

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Abstract

A result of Gelfand and Graev shows that the supercuspidal representations of $\mathrm{SL}_2(F)$ for a nonarchimedean local field F are neatly parameterized by characters of elliptic tori, and that the stable character data for all such representations may be collected into a single function by means of a Fourier Transform. For the group $\mathrm{SL}_n(F)$, we state a conjectural formula for the character values of supercuspidal representations arising from unramified tori of $\mathrm{SL}_n(F)$ and use this formula to prove analogous to those of Gelfand and Graev in this case.

Contents

Introduction

In the 1960s, Gelfand and Graev gave a construction of the supercuspidal representations of $\mathrm{SL}_2(F)$ for F a nonarchimedean local field. For a character ψ of a maximal elliptic torus T of $\mathrm{SL}_2(F)$ their construction gave rise to a supercuspidal representation V_ψ of $\mathrm{SL}_2(F)$ corresponding to ψ which splits into two subrepresentations $V_\psi^\pm$; each $V_\psi^\pm$ is irreducible and supercuspidal except in the case $\psi = \psi_0$ where ψ_0 is the quadratic character of T. They established various properties of characters $R_\psi^\pm$ of $V_\psi^\pm$, notably that $R_\psi^+ - R_\psi^-$ is supported on the stable class of T and that $R_\psi = R_\psi^+ + R_\psi^-$ is a stable character. Moreover, they noticed a curious phenomenon: computing the Fourier transform in the second variable of the map $(\gamma, \psi) \mapsto R_\psi(\gamma)$ yields

$$L(\gamma, t) = \int_{\widehat{T}} R_\psi(\gamma)\psi(t^{-1})d\psi = 2\frac{\mathrm{sgn}_E(\mathrm{Tr}(\gamma) - \mathrm{Tr}(t))}{|\mathrm{Tr}(\gamma) - \mathrm{Tr}(t)|}$$

where sgn_E is the quadratic character of F for a field extension of E associated to T. The proof of this fact appearing in [22] could likely be made rigorous but, as written, leaves much to be desired. Nonetheless, the result can be established by computing the integral on the right hand side explicitly, a calculation appearing in [31] as well as in §5 of this thesis.

In addition to being a generalization of the result of Gelfand and Graev, the distribution $L(\gamma, t)$ also plays the role of the stable transfer factor associated to the pair (G, T) in the sense of [31]. Indeed, a distribution $\Theta(a_T, a_G)$ is considered therein where $a_T \in \mathfrak{A}_T$ and $a_G \in \mathfrak{A}_G$, the Steinberg-Hitchin bases of T and G respectively, such that for a stable character $\chi_{\pi_G^{\mathrm{st}}}$ arising from a stable character $\chi_{\pi_T^{\mathrm{st}}}$ of T, we have

$$\chi_{\pi_G^{\mathrm{st}}}(a_G) = \int_{\mathfrak{A}_T} \chi_{\pi_T^{\mathrm{st}}}(a_T)\Theta(a_T, a_G)da_T$$

Translating the notation of [31] to that of this thesis, and knowing the structure of the aforementioned stable characters, it can be shown that $\Theta(a_H, a_G) = L(\gamma, t)$ where a_G and a_T are the images of γ and t in $\mathfrak{A}_G$ and $\mathfrak{A}_T$, respectively.

It is known that the stable characters of $\mathrm{SL}_n(F)$ may be realized as the restriction to $\mathrm{SL}_n(F)$ of a character of $\mathrm{GL}_n(F)$. If $\widetilde{T}$ is a maximal elliptic torus of $\mathrm{GL}_n(F)$ and $T = \widetilde{T} \cap \mathrm{SL}_n(F)$, and $\widetilde{\psi}$ is a character of $\widetilde{T}$ with $\widetilde{\psi}|_T = \psi$, then the stable character Θ_ψ associated to ψ agrees with the restriction of $\mathrm{SL}_n(F)$ of the character of the representation of $\mathrm{GL}_n(F)$ associated to $\widetilde{T}$. In general, the discrete spectrum of $\mathrm{GL}_n(F)$ for $p \nmid n$ is parameterized by the dual groups of the various maximal elliptic tori of $\mathrm{GL}_n(F)$. A similar result holds for $\mathrm{SL}_n(F)$.

The above suggests that to compute the distributions $L(\gamma, t)$ we require a deep understanding of, as well as an ability to calculate, precise values of the characters of the supercuspidal representations of GL_n. Moreover, we will also need information concerning the so-called special representations, the discrete series representations associated to the non-admissible characters of elliptic tori of GL_n.

Computing explicit character values is substantially simpler in the case of GL_ℓ where ℓ is a prime, such computations appear in [8] and [38]. Moreover, if T is unramified, we have from [35] exhaustive data concerning the local character expansion and hence we have full knowledge of the character values on the regular set in $\mathrm{GL}_\ell(F)$. It is thus feasible to compute the distribution $L(\gamma, t)$.

For $\gamma \in \mathrm{GL}_\ell(F)$ not in the stable class of T we have

$$L(\gamma,t) = \sum_{\mathcal{O}\leq\mathcal{O}_\gamma} E_{\mathcal{O}}(\gamma) + C_{\mathcal{O}}(\gamma,t) q^{\min\{d(t),\lceil d(\gamma)-1\rceil\}\left(\frac{|\Phi_{\mathcal{O}}|}{2}+\ell-1\right)}$$

where $\mathcal{O}_\gamma$ is the nilpotent orbit in $\mathfrak{gl}_\ell(F)$ induced from the centralizer of γ and where

$$E(\gamma) = \ell(-1)^{\ell+r_{\mathcal{O}}}(r_{\mathcal{O}}-1)!\left(|T(\mathfrak{f})|\left(\frac{q^{\frac{|\Phi_{\mathcal{O}}|}{2}}-1}{q^{\frac{|\Phi_{\mathcal{O}}|}{2}+\ell-1}-1}-1\right)-1\right)\frac{|D_{C(\gamma)}(\gamma)|^{\frac{1}{2}}}{|D_{\mathrm{SL}_\ell}(\gamma)|^{\frac{1}{2}}}$$

and

$$C_{\mathcal{O}}(\gamma,t) = \begin{cases} \ell(-1)^{\ell+r_{\mathcal{O}}}(r_{\mathcal{O}}-1)!|T(\mathfrak{f})|\dfrac{q^{\frac{|\Phi_{\mathcal{O}}|}{2}}-1}{q^{\frac{|\Phi_{\mathcal{O}}|}{2}+\ell-1}-1}\dfrac{|D_{C(\gamma)}(\gamma)|^{\frac{1}{2}}}{|D_{\mathrm{SL}_\ell}(\gamma)|^{\frac{1}{2}}} & d(t)<d(\gamma) \\ \ell(-1)^{\ell+r_{\mathcal{O}}}(r_{\mathcal{O}}-1)!|T(\mathfrak{f})|\left(\dfrac{q^{\frac{|\Phi_{\mathcal{O}}|}{2}}-1}{q^{\frac{|\Phi_{\mathcal{O}}|}{2}+\ell-1}-1}-1\right)\dfrac{|D_{C(\gamma)}(\gamma)|^{\frac{1}{2}}}{|D_{\mathrm{SL}_\ell}(\gamma)|^{\frac{1}{2}}} & d(t)\geq d(\gamma) \end{cases}$$

where $r_{\mathcal{O}}$ is such that $\mathcal{O}$ is induced from a Levi subgroup of the form $\prod_{i=1}^{r_{\mathcal{O}}}\mathrm{GL}_{n_i}$ for some $n_1,\ldots,n_{r_{\mathcal{O}}}$ and $\Phi_{\mathcal{O}}$ is the root system of this Levi subgroup. Moreover, we have that

$$\frac{|D_{C(\gamma)}(\gamma)|^{\frac{1}{2}}}{|D_{\mathrm{SL}_\ell}(\gamma)|^{\frac{1}{2}}} = q^{X_{\mathcal{O}}}$$

where $q=|\mathfrak{f}|$, the size of the residue field of F, and where $X_{\mathcal{O}} \geq d(\gamma)\frac{\dim(\mathcal{O})}{2}$.

For $\gamma \in T$ our formula has the novel feature that it is genuinely a distribution; it cannot be represented by a smooth function. We have

$$L(\gamma,t) = E_T + C_T q^{\min\{d(\gamma),d(t)\}\dim(B)} + \frac{1}{|D_G(\gamma)|^{\frac{1}{2}}}\sum_{\beta\in\gamma^{\mathrm{Gal}(E/F)}}\delta_\beta$$

for constants $E_T, C_T \in \mathbb{C}$ and where E is such that $T(F)\simeq E^\times$. Notably, where $L(\gamma,t)$ in the case $\ell=2$ has poles when γ and t are stably conjugate, when ℓ is prime the component of our distribution represented by a function is bounded but we have multiples of delta mass measures at the points where γ and t are stably conjugate. In the case of SL_2, when computing $L(\gamma,t)$ using character data, when the roots of γ and t are not too close to one another we have that

$$L(\gamma,t) = 2\mathrm{sgn}_E(\mathrm{Tr}(\gamma)-\mathrm{Tr}(t))q^{2\min\{d(\gamma),d(t)\}}$$

This equation, while lacking some of the elegance of the original formula, is more closely analogous to the formula for ℓ odd.

The difficulty which arises when attempting to compute the distribution $L(\gamma, t)$ is that we must compute an integral over the character group $\widehat{T}$ which does not necessarily converge. To deal with issues of convergence we consider, for fixed $\gamma \in G$, the map $\psi \mapsto \Theta_\psi$ as a distribution on $\widehat{T}$. We may then compute the Fourier transform of this distribution to obtain $L(\gamma, t)$.

Extending an analogue of our result to the case of SL_n with n composite presents a considerable number of difficulties not present when n is prime. In this case we must deal with the so-called Howe factorizations of the characters to which our representations correspond; such a factorization is rather trivial when ℓ is prime but can become quite complicated in general. It is difficult to compute characters in this case; we may only use the [7] formula in the case where the character ψ is strongly primitive. The constant terms of all such representations are calculated in [17], though we must work harder to compute character values at shallow elements. Character formulas appear in [2] but these have the twofold deficiency of having been computed using the matching theorem and, graver still, that the proofs thereof do not appear.

In [42] Yu gave a very general construction of supercuspidal characters for reductive p-adic groups. This construction has been shown to be exhaustive in many cases by Kim in [28]. In a series of papers including [7] and [20] a character formula was given for supercuspidal representations arising from Yu's construction which satisfy a compactness condition. For such a representation π with $d(\pi) = r$ if we can write $\gamma \in G(F)^{\mathrm{rss}}$ as a product $\gamma = \gamma_{<r}\gamma_{\geq r}$ in the sense of [6] we have

$$\Theta_\pi(\gamma) = \phi_d(\gamma)\frac{1}{|D_G(\gamma)|^{\frac{1}{2}}}\sum_{g \in H\backslash G(F)/G'(F):\gamma^g \in G'(F)} \epsilon_d(\gamma^g_{<r})\Phi_{\pi_{d-1}}(\gamma^g_{<r})\widehat{\iota}^H_{gX^*_\psi}(\mathbf{e}_x^{-1}(\gamma_{\geq r}))$$

The notation of this formula is explained in §3.3 below. This formula simplifies somewhat for γ of the form $\gamma = \gamma_{<r}$ but in general the orbital integral term $\widehat{\iota}^H_{gX^*_\psi}(\mathbf{e}_x^{-1}(\gamma_{\geq r}))$ is very difficult to compute. Moreover, having to decompose γ in this way massively obstructs our computation of $L(\gamma, t)$ since such decompositions depend on the depth of the representation.

In order to compute $L(\gamma, t)$ at least for $\gamma \in T$ we require a character formula for the values of Θ_ψ on T. We make the following conjecture: if T is unramified and $\gamma \in T$ is regular we may compute

$$\Theta_\psi(\gamma) = |D_G(\gamma_{<r})|^{-\frac{1}{2}}|D_G(X^*_{\psi,<d(\gamma)})|^{\frac{1}{2}}\epsilon_\psi(\gamma)\sum_{\sigma \in \Gamma_{E/F}} \psi(\gamma^\sigma)$$

where $\epsilon_\psi(\gamma)$ is a root of unity depending on ψ and γ. With such a character formula it becomes possible to compute $L(\gamma, t)$ in the case of SL_n. We give for evidence this conjecture, establishing it in a number of cases, and show that it follows from a small number of natural assumptions.

In addition to requiring extremely precise character data, calculating the Fourier transform $L(\gamma, t)$ for $\mathrm{GL}_n(F)$ with n composite presents a number of additional challenges. The values of stable characters depend heavily on their Howe factorizations so that our computations become far more complicated than in the case where n is prime. We overcome this difficulty by carefully grouping characters in terms of their Howe factorizations and compute our Fourier transform in pieces. Indeed, to any admissible character ψ of an elliptic torus of $\mathrm{GL}_n(F)$, the Howe factorization of ψ assigns to ψ a decreasing tower of fields $E = E^0 \supset E^1 \supset \ldots \supset E^{d-1} \supsetneq F$ for some $d \in \mathbb{N}$. Setting $E^\psi = E^{d-1}$ we consider separately the distributions

$$L^M(\gamma, t) = \sum_{\psi : E^\psi = M} \Theta_\psi(\gamma)\psi(t^{-1})$$

as M ranges over the intermediate extensions of F contained in E, so that we have

$$L(\gamma, t) = \sum_{E \supset M \supsetneq F} L^M(\gamma, t)$$

We may then compute $L^M(\gamma, t)$ with relative ease assuming we know the formula in the case of $\mathrm{GL}_m(F)$ for $m = [M : F]$; our calculations thus proceed essentially via an inductive argument.

This thesis is organized as follows:

In section 1 we state and prove an analogue of the result of Gelfand and Graev in the case of finite groups of Lie type. To do so we use facts about the Deligne-Lusztig representations of these groups first considered in [21].

In section 2 we give an explicit construction of the building $\mathcal{B}(\mathrm{GL}_n, F)$ of $\mathrm{GL}_n(F)$ as the collection of additive norms on the space F^n and use this description to prove some properties of $\mathcal{B}(\mathrm{GL}_n, F)$ which will be crucial to our character computations. We will discuss how the building relates to the maximal compact subgroups of $\mathrm{GL}_n(F)$ and use it to define the parahoric subgroups of $\mathrm{GL}_n(F)$ and the Moy-Prasad filtrations thereof. We will also use the building to define various notions of the depth of an element of $\mathrm{GL}_n(F)$.

In section 3 we discuss properties of supercuspidal representations. We introduce Yu's construction of supercuspidal representations for reductive p-adic groups in general and discuss ideas appearing in [6] so that we can state the character formula of [7] for such representations.

In section 4 we state and discuss our conjectural character formula for representations arising from characters of unramified Tori. We also compute explicitly the roots of unity appearing in the character formula and perform a number of computations of the orbital integrals appearing in the character formula. We also use the character formula of [20] and results from [35] to compute explicit character formulas for GL_ℓ.

In section 5 we calculate $L(\gamma, t)$ for the case $\mathrm{SL}_\ell(F)$ where ℓ is prime in the unramified case. We are exhaustive here, computing $L(\gamma, t)$ explicitly for any regular semisimple $\gamma \in \mathrm{GL}_\ell(F)$.

In section 6, we restrict to the case where T is unramified and demonstrate how to generalize our computations to the case of $\mathrm{SL}_n(F)$ where n is composite. As mentioned above, we do so by breaking up our computation based on the Howe factorizations of the characters in question and are able to compute $L(\gamma, t)$ inductively based on the knowledge of the formula in the cases $\mathrm{GL}_m(F)$ for all $m|n$. In the simplest case, where n is the product of two primes, we give formulas for $L(\gamma, t)$ in certain cases.

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0 Notation and Conventions

In this section we set notations as well as state basic facts about nonarchimedean local fields and reductive groups in general. Basic references for this material include [11], [15] and [39].

Nonarchimedean Local Fields

Let F denote a nonarchimedean local field, $\mathcal{O}_F$ its ring of integers, $\varpi = \varpi_F \in \mathcal{O}_F$ a uniformizer and $\mathfrak{f} = \mathcal{O}_F/\varpi\mathcal{O}_F$ its residue field. Denote by $\mathrm{ord} : F \to \mathbb{Z} \cup \{\infty\}$ the additive norm on F, $q = |\mathfrak{f}|$ and $|\cdot| : F \to \mathbb{R}_{\geq 0}$ the multiplicative norm on F given by

$$|a| = q^{-\mathrm{ord}(a)}$$

Fix an algebraic closure $\overline{F}$ of F. For an algebraic extension $E \subset \overline{F}$ of F we denote the ring of integers and residue field by $\mathcal{O}_E$ and $\mathfrak{f}_E$, respectively. Also, denote by $\varpi_E \in \mathcal{O}_E$ a uniformizer of E. We denote by $\mathrm{ord}_E : E \to \mathbb{Q} \cup \{\infty\}$ the additive norm extending ord so that $\mathrm{ord}_E(E^\times) \subset \frac{1}{e}\mathbb{Z}$ where e is the ramification degree of E over F and $|\cdot|_E : E \to \mathbb{R}_{\geq 0}$ the multiplicative norm $|a| = q^{-\mathrm{ord}_E(a)}$. Moreover, when E/F is Galois, we write $\Gamma_{E/F} = \mathrm{Gal}(E/F)$ or simply Γ when the field E is understood.

For G an algebraic group defined over F we denote by $G(R)$ the R-points of G for any F-algebra R. Similarly, we denote by $G \otimes R$ the base change of G from F to R. For $\sigma \in \Gamma_{E/F}$ we also denote by σ its action on $G(E)$. Also, we denote by $\mathfrak{g}$ the Lie algebra of G and by $\mathfrak{g}(R)$ the R-points of $\mathfrak{g}$ for any F-algebra R.

For an algebraic group (a smooth algebraic group scheme) $\mathcal{G}$ defined over $\mathcal{O}_F$ we denote by π the quotient map

$$\pi : \mathcal{G}(\mathcal{O}_F) \to \mathcal{G}(\mathfrak{f})$$

Tori and Root Systems

Let G be a reductive group defined over F and let $T \subset G$ be a torus defined over F. We set

$$X^*(T) = \mathrm{Hom}(T, \mathbb{G}_m)$$

and

$$X_*(T) = \mathrm{Hom}(\mathbb{G}_m, T)$$

and denote by $\langle \cdot, \cdot \rangle : X^*(T) \times X_*(T) \to \mathbb{Z}$ the natural pairing

$$\chi \circ \lambda(a) = a^{\langle \chi, \lambda \rangle}$$

for $a \in F^\times$.

We define the (finite) algebraic group

$$\Omega(G, T) = N_G(T)/C_G(T)$$

Also, let $N_{T,F} = N_{G(F)}(T(F))$ and $W_{T,F} = \Omega(G, T)(F)$ or simply N_F and W_F if the torus T is understood.

There is a natural action of the Weyl group W_F on $X^*(T)$ and $X_*(T)$. Indeed, we set

$$^{w}\chi(t) = \chi(t^w)$$

and

$$\lambda^w(a) = \lambda(\alpha)^w$$

for $w \in W$, $\chi \in X^*(T)$, $\lambda \in X_*(T)$, $t \in T$ and $a \in \mathbb{G}_m$. Moreover, we see that

$$\langle \chi, \lambda^w \rangle = \langle {}^w\chi, \lambda \rangle$$

If E/F is a Galois extension there is a natural action of $\Gamma_{E/F}$ on $X^*(T)$ and $X_*(T)$ via

$$\sigma * \chi = \sigma_{\mathbb{G}_m} \circ \chi \circ \sigma_T^{-1}$$

and

$$\sigma * \lambda = \sigma_T \circ \lambda \circ \sigma_{\mathbb{G}_m}^{-1}$$

For an extension E of F we let $\Phi_E = \Phi(G \otimes E, T \otimes E)$ be the root system of $G \otimes E$ with respect to $T \otimes E$. We write $\Phi(G,T) = \Phi(G \otimes \overline{F}, T \otimes \overline{F})$ for the absolute root system of T.

For $\alpha \in \Phi_F$ denote by $F_\alpha \subset \overline{F}$ be the splitting field for α. We say that α is unramified if F_α is and that α is ramified if F_α is. Also, we let $F_{\pm\alpha}$ be the compositum of F_α and $F_{-\alpha}$.

For $\alpha \in \Phi_F$ we say that α is (E)-symmetric if $-\alpha \in \Gamma_{E/F} \cdot \alpha$ and (E)-asymmetric otherwise. If the extension E is understood, we simply refer to α as symmetric or asymmetric, respectively.

For $\alpha \in \Phi_F$ we denote by U_α denote the root subgroup of G associated to α. There is an action of N_F on the groups $U_\alpha(F)$ via

$$U_\alpha{}^n = U_{\alpha^w}$$

for any $n \in N_F$ with image $w \in W_F$. Also, for $\alpha, \beta \in \Phi_F$ we have the commutation relations

$$[U_\alpha, U_\beta] \subseteq \langle U_\gamma : \gamma = k\alpha + \ell\beta, k, \ell > 0 \rangle$$

and

$$[U_\alpha, U_{-\alpha}] \subseteq \langle T, U_\alpha, U_{-\alpha} \rangle$$

Moreover, the root system Φ may be ordered such that the product map

$$T \times \prod_{\alpha \in \Phi} U_\alpha \to G$$

is an isomorphism of varieties.

For $\alpha \in \Phi(G,T)$ denote by $\alpha^\vee \in \Phi(G,T)^\vee$ associated to α. Let $H_\alpha \in \mathfrak{g}(\overline{F})$ be the element defined via $H_\alpha = d\alpha^\vee(1)$.

Notation for $\mathbf{GL}_n(F)$

Let $T = D_n \subset \mathrm{GL}_n$ denote the diagonal torus; we note that T is maximal and split. In this case we may identify $X^*(T) \simeq \mathbb{Z}^n$ and $X_*(T) \simeq \mathbb{Z}^n$ via

$$\chi_{\vec{m}} \begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix} = t_1^{m_1} \cdots t_n^{m_n}$$

and

$$\lambda_{\vec{k}}(a) = \begin{pmatrix} a^{k_1} & & \\ & \ddots & \\ & & a^{k_n} \end{pmatrix}$$

Under this identification, the natural pairing is given by

$$\left\langle \chi_{\vec{m}}, \lambda_{\vec{k}} \right\rangle = \vec{m} \cdot \vec{k}$$

We have that $W \simeq \mathfrak{S}_n$, the symmetric group on n letter, and moreover that the subgroup $\mathrm{Perm}_n \subset \mathrm{GL}_n(\mathcal{O}_F)$ of permutation matrices is contained in N_F and maps isomorphically to W_F. For $\tau \in \mathfrak{S}_n$ we denote by $P_\tau \in \mathrm{Perm}_n$ the matrix with the property

$$P_\tau(v_1, \ldots, v_n) = (v_{\tau(1)}, \ldots, v_{\tau(n)})$$

for $\vec{v} = (v_1, \ldots, v_n) \in F^n$.

Identifying W with $\mathfrak{S}_n$, for $w \in W$ with $w \simeq \tau \in \mathfrak{S}_n$, the action of W on $X^*(T)$ and $X_*(T)$ may be computed explicitly as

$${}^w\chi_{\vec{m}} = \chi_{\tau^{-1} \cdot \vec{m}}$$

and

$$\lambda_{\vec{k}}{}^w = \lambda_{\tau \cdot \vec{k}}$$

For $1 \leq i, j \leq n$, $i \neq j$ we denote by α_{ij} the root of T defined via

$$\alpha_{ij} \begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix} = \frac{t_i}{t_j}$$

Fix an additive character $\Lambda : F \to \mathbb{C}$ of depth 0. We identify $\mathfrak{gl}_n(F)$ with its dual via the identification

$$X \mapsto (Y \mapsto \Lambda(\mathrm{Tr}(XY)))$$

We endow $\mathfrak{gl}_n(F)$ with the Haar measure $dY = d_\Lambda Y$ which is self-dual with respect to this identification.

1 Finite Groups of Lie Type

We prove a result in the spirit of that of Gelfand and Graev for finite groups of Lie type.

Let G be a connected reductive group defined over a finite field $\mathfrak{f}$ with Frobenius map σ. Let $T \subset G$ be a maximal torus defined over $\mathfrak{f}$ and let $B = TU$ be a Borel subgroup containing T with unipotent radical U. Also, let $L : G \to G$ be the Lang map

$$L(g) = g^{-1}F(g) \tag{1.1}$$

We define the variety $\widetilde{X} = \widetilde{X}_T^G = L^{-1}(U)$. There is an action of $G(\mathfrak{f}) \times T(\mathfrak{f})$ on $\widetilde{X}$ defined via

$$(k,t) \cdot g = kgt$$

Indeed, $L(kg) = L(g)$ for all $k \in G(\mathfrak{f})$ and we note that $T(\mathfrak{f})$ normalizes U. This gives rise to an action of $G(\mathfrak{f}) \times T(\mathfrak{f})$ on the $H_c^i(\widetilde{X}, \overline{\mathbb{Q}}_\ell)$, the ℓ-adic cohomology groups of $\widetilde{X}$ with compact support.

For $\vartheta \in \widehat{T^F}$ we define the virtual character

$$R_{T,\vartheta}(g) = \sum_{i \geq 0} (-1)^i \mathrm{Tr}(g, H_c^i(\widetilde{X}, \overline{\mathbb{Q}}_\ell)_\vartheta) \tag{1.2}$$

where $H_c^i(\widetilde{X}, \overline{\mathbb{Q}}_\ell)_\vartheta \subset H_c^i(\widetilde{X}, \overline{\mathbb{Q}}_\ell)$ is the subspace on which T^F acts via ϑ. It is shown in [21] that every irreducible character χ of $G(\mathfrak{f})$ is a constituent of one of the form $\pm R_{T,\vartheta}$ for some T and $\vartheta \in \widehat{T(\mathfrak{f})}$. Following [15] we may rearrange the defining formula for $R_{T,\vartheta}(g)$ to obtain

$$R_{T,\vartheta}(g) = \frac{1}{|T(\mathfrak{f})|} \sum_{t \in T(\mathfrak{f})} \vartheta(t^{-1}) \mathcal{L}((g,t), \widetilde{X}) \tag{1.3}$$

where $\mathcal{L}((g,t), \widetilde{X})$ is the Lefschetz number of the automorphism (g,t) of $\widetilde{X}$.

We define the function

$$L(g,t) = \frac{1}{|\widehat{T(\mathfrak{f})}|} \sum_{\vartheta \in \widehat{T(\mathfrak{f})}} R_{T,\vartheta}(g)\vartheta(t^{-1}) = \frac{1}{|T(\mathfrak{f})|} \sum_{\vartheta \in \widehat{T(\mathfrak{f})}} R_{T,\vartheta}(g)\vartheta(t^{-1}) \tag{1.4}$$

which is the Fourier transform of the map $\vartheta \mapsto R_{T,\vartheta}(g)$. By (1.3) we have that

$$L(g,t) = \mathcal{L}\left((g,t^{-1}), \widetilde{X}\right) \tag{1.5}$$

We wish to be able to give a more explicit formula for $L(g,t)$.

From [21] if $u \in G(\mathfrak{f})$ is unipotent we have that $R_{T,\vartheta}(u)$ is independent of ϑ and we set

$$Q_T^G(u) = Q_T(u) = R_{T,\vartheta}(u)$$

for any $\vartheta \in \widehat{T(\mathfrak{f})}$. We call Q_T a Green function.

If $g \in G(\mathfrak{f})$ has Jordan decomposition $g = su = us$ we have by [21] the following character formula for $R_{T,\vartheta}(g)$:

$$R_{T,\vartheta}(g) = \frac{1}{|C_G(s)^\circ(\mathfrak{f})|} \sum_{x \in G(\mathfrak{f}), s^x \in T(\mathfrak{f})} \vartheta(s^x) Q_{^xT}^{C_G(s)^\circ}(u) \tag{1.6}$$

Using this formula, we compute

$$\begin{aligned} L(g,t) &= \frac{1}{|T(\mathfrak{f})|} \sum_{\vartheta \in \widehat{T(\mathfrak{f})}} \frac{1}{|C_G(s)^\circ(\mathfrak{f})|} \sum_{x \in G(\mathfrak{f}), s^x \in T(\mathfrak{f})} \vartheta(s^x) Q_{^xT}^{C_G(s)^\circ}(u) \vartheta(t^{-1}) \\ &= \frac{1}{|T(\mathfrak{f})||C_G(s)^\circ(\mathfrak{f})|} \sum_{x \in G(\mathfrak{f}), s^x \in T(\mathfrak{f})} \sum_{\vartheta \in \widehat{T(\mathfrak{f})}} \vartheta(s^x t^{-1}) Q_{^xT}^{C_G(s)^\circ}(u) \\ &= \frac{1}{|C_G(s)^\circ(\mathfrak{f})|} \sum_{x \in G(\mathfrak{f})^{(s,t)}} Q_{^xT}^{C_G(s)^\circ}(u) \end{aligned} \tag{1.7}$$

where we define

$$G(\mathfrak{f})^{(s,t^{-1})} = \{x \in G(\mathfrak{f}) : xt = sx\}$$

We note that $G(\mathfrak{f})^{(s,t^{-1})} = \emptyset$ unless s is conjugate to t. Otherwise, we have that

$$G(\mathfrak{f})^{(s,t^{-1})} = x \cdot C_G(s)(\mathfrak{f})$$

for any $x \in G(\mathfrak{f})^{(s,t^{-1})}$. Moreover, we note that ${}^xT = {}^{x'}T$ for any $x, x' \in G(\mathfrak{f})^{(s,t^{-1})}$. Setting $S = {}^xT$ it follows from (1.7) that

$$L(g,t) = [C_G(s)(\mathfrak{f}) : C_G(s)^\circ(\mathfrak{f})] Q_{^xT}^{C_G(s)^\circ}(u) \tag{1.8}$$

Going further, applying (1.3) to $C_G(s)^\circ$ and S we obtain

$$Q_S^{C_G(s)^\circ}(u) = \frac{1}{|S(\mathfrak{f})|} \sum_{t \in S(\mathfrak{f})} \mathcal{L}\left((u,t), \widetilde{X}_S^{C_G(s)^\circ}\right) = \frac{1}{|T(\mathfrak{f})|} \mathcal{L}\left(u, \widetilde{X}_S^{C_G(s)^\circ}\right) \tag{1.9}$$

for $\widetilde{X}_S^{C_G(s)^\circ} = L^{-1}(U')$ for the unipotent radical of some Borel subgroup $B' = TU'$ of $C_G(s)^\circ$. Indeed, by [15] the terms $\mathcal{L}\left((u,t), \widetilde{X}_S^{C_G(s)^\circ}\right)$ for $t \neq 1$ each vanish since the action of t on $\widetilde{X}_S^{C_G(s)^\circ}$ has no fixed points.

Combining (1.8) and (1.9) yields

Theorem 1.1. *We have that*

$$L(g,t) = \begin{cases} \dfrac{[C_G(s)(\mathfrak{f}) : C_G(s)^\circ(\mathfrak{f})]}{|T(\mathfrak{f})|} \mathcal{L}\left(u, \widetilde{X}_S^{C_G(s)^\circ}\right) & \text{if } t \text{ is } G(\mathfrak{f})\text{-conjugate to } s \\ 0 & \text{otherwise} \end{cases} \tag{1.10}$$

As a special case, suppose $g \in T(\mathfrak{f})$ with $C_G(g) = T$. In this case $\widetilde{X}_T^T = T(\mathfrak{f})$ is finite so that $\mathcal{L}(1, T(\mathfrak{f})) = |T(\mathfrak{f})|$ by [15]. Moreover, $C_G(g) = C_G(g)^\circ$ it this case. As such, we have

Corollary 1.2. *If $g \in T(\mathfrak{f})$ and $C_G(g) = T$ we compute*

$$L(g, t) = 1 \tag{1.11}$$

for any t $G(\mathfrak{f})$-conjugate to g.

2 The Bruhat-Tits Building and Moy-Prasad Filtrations

An important tool in understanding the structure of p-adic groups is the Bruhat-Tits building $\mathcal{B}(G,F)$, a polysimplicial complex on which the group $G(F)$ acts. The building collects and nicely parametrizes a vast amount of data about the group $G(F)$, notably that of its compact open subgroups; for every $x \in \mathcal{B}(G,F)$ the group $G_x(F) = \operatorname{stab}_{G(F)}(x)$ is compact open and, conversely, every maximal compact open subgroup of $G(F)$ has this form. Moreover, we may use the building to define filtrations of compact open subgroups of $G(F)$ which are crucial in the study of its representation theory.

While the construction of the building for general reductive p-adic groups is somewhat abstract, in the case of $G = \mathrm{GL}_n$ we may realize $\mathcal{B}(G,F)$ as the collection of additive norms on the space F^n. This approach is also discussed in other sources such as [41]. With this concrete description we can study the structure of the building very closely in this case. We will use this realization to prove a number of properties of the building. Furthermore, we show that this definition agrees with the standard definition.

We may also use the building to parametrize a families of lattices $\mathfrak{g}_x(F)$ of $\mathfrak{g}(F) = \mathfrak{gl}_n(F)$ and filtrations thereof such that each $\mathfrak{g}_x(F)$ may be identified with the Lie algebra of the group $G_x(F)$. Moreover, we show that there are isomorphisms between quotients of compact subgroups of $G(F)$ with quotients of lattices in $\mathfrak{g}(F)$ which are necessary to employ for the construction of supercuspidal representations of $G(F)$.

We will also use the machinery of the building to define the notion of depth of elements of $G(F)$ which is crucial for our character computations.

2.1 The Building of $\mathbf{GL}_n(F)$

Additive Norms on Vector Spaces over F

Let V be a finite dimensional F-vector space.

Definition 2.1. *We say a map $\mathbf{x} : V \to \mathbb{R} \cup \{\infty\}$ is an additive norm provided that*

i) $\mathbf{x}(\vec{v}) = \infty$ *if and only if* $\vec{v} = 0$

ii) $\mathbf{x}(\vec{v} + \vec{w}) \geq \min\{\mathbf{x}(\vec{v}), \mathbf{x}(\vec{w})\}$ *for any* $\vec{v}, \vec{w} \in V$

iii) $\mathbf{x}(a\vec{v}) = \operatorname{ord}(a) + \mathbf{x}(\vec{v})$ *for any* $\vec{v} \in V$, $a \in F$

The prototypical additive norm on $V = F^n$ is the standard norm $\mathbf{x}_{\vec{0}}$ given by

$$\mathbf{x}_0(\vec{v}) = \min_{1 \leq i \leq n} \operatorname{ord}(v_i)$$

where $\vec{v} = \sum_{i=1}^n v_i e_i$ and $\{e_i\}$ is the standard basis for F^n. For any basis $\{b_i\}$ of F^n and any $\vec{c} \in \mathbb{R}^n$ we may define an additive norm $\mathbf{x} = \mathbf{x}_{\{b_i\},\vec{c}}$ on F^n via

$$\mathbf{x}_{\{b_i\},\vec{c}}\left(\sum_{i=1}^n a_i b_i\right) = \min_{1 \leq i \leq n} \operatorname{ord}(a_i) + c_i \tag{2.1}$$

We write

$$\mathbf{x}_{\vec{c}} = \mathbf{x}_{\{e_i\},\vec{c}}$$

The above construction of norms is exhaustive. We have

Proposition 2.2. *Let* $\mathbf{x}$ *be an additive norm on* F^n*. Then there is a basis* $\{b_i\}$ *of* F^n *and* $\vec{c} \in \mathbb{R}^n$ *such that* $\mathbf{x} = \mathbf{x}_{\{b_i\},\vec{c}}$*.*

Proof. See [41] p.61. □

The Building and Apartment

We begin with the following definition:

Definition 2.3. *Let* $\mathcal{B}$ *denote the collection of additive norms on the* F*-vector space* F^n*.*

The action of $G(F)$ on F^n gives rise to an action of $G(F)$ on $\mathcal{B}$; we set

$$g \cdot \mathbf{x}(\vec{v}) = \mathbf{x}(g^{-1} \cdot \vec{v})$$

for $g \in G(F)$, $\mathbf{x} \in \mathcal{B}$ and $\vec{v} \in F^n$.

Moreover, we define a subset $\mathcal{A} \subset \mathcal{B}$ via

$$\mathcal{A} = \{\mathbf{x}_{\vec{c}} : \vec{c} \in \mathbb{R}^n\}$$

We call $\mathcal{A}$ the standard apartment of $\mathcal{B}$.

Proposition 2.2 immediately implies the following result:

Corollary 2.4. *For* $\boldsymbol{x}_{\{g \cdot e_i\},\vec{c}}$ *is as defined in* (2.1) *we have*

a) *The map* $G(F) \times \mathbb{R}^n \to \mathcal{B}$ *given by* $(g, \vec{c}) \mapsto \mathbf{x}_{\{g \cdot e_i\},\vec{c}}$ *is surjective.*

b) *We have*

$$\mathcal{B} = G(F) \cdot \mathcal{A}$$

The map $G(F) \times \mathbb{R}^n \to \mathcal{B}$ defined in Corollary 2.4 is not injective. We have

Corollary 2.5. *Suppose* $(\{b_i\}, \vec{c})$ *and* $(\{b'_i\}, \vec{c}')$ *are such that* $\boldsymbol{x}_{\{b_i\},\vec{c}} = \boldsymbol{x}_{\{b'_i\},\vec{c}'}$*. Then there is some* $\sigma \in \mathfrak{S}_n$ *and* $m_1, \ldots, m_n \in \mathbb{Z}$ *such that*

$$d_i = c_{\sigma(i)} + m_i$$

for $1 \leq i \leq n$*.*

In particular, if $\vec{c}$ *and* $\vec{c}'$ *are such that* $1 > c_1 \geq \ldots \geq c_n \geq 0$ *and* $1 > c'_1 \geq \ldots \geq c'_n \geq 0$ *then* $\vec{c} = \vec{c}'$*.*

This is a consequence of the following technical lemma:

Lemma 2.6. *Suppose* $\mathbf{x} \in \mathcal{B}$ *is such that* $\mathbf{x} = \mathbf{x}_{(\{b_i\},\vec{c})} = \mathbf{x}_{(\{b'_i\},\vec{c}')}$. *Let* $S_i = c_i + \mathbb{Z}$ *and let* $i_1 < \ldots < i_k$ *be such that* $S_{i_1}, \ldots, S_{i_k}$ *are pairwise distinct and that for each* $1 \leq i \leq n$ *we have* $S_i = S_{i_j}$ *for some* $1 \leq j \leq k$. *Similarly, let* $S'_i = c'_i + \mathbb{Z}$ *and let* $i'_1 < \ldots < i'_{k'}$ *be such that* $S'_{i'_1}, \ldots, S'_{i'_{k'}}$ *are pairwise distinct and that for each* $1 \leq i \leq n$ *we have* $S'_i = S'_{i'_j}$ *for some* $1 \leq j \leq k'$. *Let*

$$V_j = \operatorname{span}\left\{b_i : c_i \in S_{i_j}\right\}$$

for $1 \leq j \leq k$ *and*

$$W_j = \operatorname{span}\left\{b'_i : c'_i \in S_{i'_j}\right\}$$

for $1 \leq j \leq k'$. *Then* $k = k'$ *and there is some* $\sigma \in \mathfrak{S}_k$ *such that* $\dim(W_j) = \dim(V_{\sigma(j)})$ *for each* $1 \leq j \leq k$.

Proof. Since $\operatorname{im}(\mathbf{x}) \subset \{S_{i_1}, \ldots, S_{i_k}\}$ and $\operatorname{im}(\mathbf{x}) \subset \left\{S'_{i'_1}, \ldots, S'_{i'_{k'}}\right\}$ it follows that for each S_i we have $S_{i_j} = S'_{i'_j}$ for some j and, conversely, that each $S'_i = S_{i_j}$ for some j. It follows that $k = k'$; for convenience, reorder the $S'_{i'_j}$ so that $S_{i_j} = S'_{i'_j}$ for each $1 \leq j \leq k$ and write $N_j = S_{i_j}$.

Let $v_{j,1}, \ldots, v_{j,m_j}$ be a basis for V_j and $X_j = \operatorname{span}\{V_i : i \neq j\}$. For each $1 \leq l \leq m_j$ we cannot have that $v_{j,l} \in X_j$ since none of the elements of X_j have norms lying in N_j. Moreover, writing

$$v_{j,l} = w_{j,l} + x_{j,l}$$

for $w_{j,l} \in W_j$ and $x_{j,l} \in X_j$ we have that the $w_{j,l}$ must be linearly independent; indeed, otherwise a non-zero linear combination of the elements $v_{j,1}, \ldots, v_{j,m_j}$ could have a norm lying in $\bigcup_{i \neq j} N_i \cup \{\infty\}$; this is impossible since this set does not intersect N_j. It follows that $\dim(V_j) \leq \dim(W_j)$ for each $1 \leq j \leq k$ and hence that $\dim(V_j) = \dim(W_j)$ for each $1 \leq j \leq k$. □

The Action of N on the Apartment

We will henceforth make the following identification: we identify $\mathcal{A}$ with (an affine space under) the vector space $X_*(T) \otimes \mathbb{R} \simeq \mathbb{R}^n$. This is to say that we identify

$$\mathcal{A} = \{\mathbf{x}_{\vec{c}} : \vec{c} \in X_*(T) \otimes \mathbb{R}\}$$

The natural action of W on $X_*(T)$ gives rise to an action of W on $X_*(T) \otimes \mathbb{R}$ and hence on $\mathcal{A}$ by reflections. We will show that $N \cdot \mathcal{A} = \mathcal{A}$ and that there is a natural action $\nu : N \to \operatorname{Aff}(X_*(T) \otimes \mathbb{R})$ such that for $\mathbf{x}_{\vec{c}} \in \mathcal{A}$ we have

$$n \cdot \mathbf{x}_{\vec{c}} = \mathbf{x}_{\nu(n)\vec{c}}$$

For $t \in T$ we define a map $\kappa : T \to X_*(T) \otimes \mathbb{R}$ via

$$\chi(\kappa(t)) = -\operatorname{ord}(\chi(t))$$

for $\chi \in X^*(T)$ and $t \in T$. Further, let $\nu(t) \in \mathrm{Aff}(X_*(T) \otimes \mathbb{R})$ be the translation by $\kappa(t)$. For $t = \mathrm{diag}(t_1, \ldots, t_n)$ we compute

$$\kappa(t) = (-\mathrm{ord}(t_1), \ldots, -\mathrm{ord}(t_n))$$

Letting

$$T_0(F) = T(\mathcal{O}_F)$$

we see that $\ker(\nu) = T_0(F)$.

We introduce the following group, an extension of the Weyl group W_F of $G(F)$:

Definition 2.7. *We define $\widetilde{W}_F$, the Iwahori-Weyl group of $G(F)$, to be*

$$\widetilde{W}_F = N_F / T_0(F)$$

Since $N_F = \mathrm{Perm}_n \cdot T(F)$ we may decompose

$$\widetilde{W} \simeq \mathrm{Perm}_n \cdot T_0(F)/T_0(F) \rtimes T(F)/T_0(F) \simeq W_F \rtimes T(F)/T_0(F)$$

Denote by $r(w) \in \mathrm{Aff}(X_*(T) \otimes \mathbb{R})$ the action of w^{-1} on $X_*(T) \otimes \mathbb{R}$. For $n \in N$ with image $\widetilde{w} \in \widetilde{W}_F$ with $\widetilde{w} \sim w \rtimes tT_0$ we define our action via

$$\nu(n) = r(w) \rtimes \nu(t)$$

We may now establish the required properties of ν:

Proposition 2.8. *We have $N \cdot \mathcal{A} = \mathcal{A}$ and that $n \cdot \mathbf{x}_{\vec{c}} = \mathbf{x}_{\nu(n)\vec{c}}$.*

Proof. It suffices to compute the actions of $T(F)$ and of Perm_n on $\mathcal{A}$. Let $\vec{v} = \sum_{i=1}^n v_i e_i$.

For $t \in T$ we have

$$t \cdot \mathbf{x}_{\vec{c}}(\vec{v}) = \mathbf{x}_{\vec{c}}(t^{-1} \cdot \vec{v}) = \min_{1 \le i \le n} \mathrm{ord}(t_i^{-1} v_i) + c_i = \min_{1 \le i \le n} \mathrm{ord}(v_i) + c_i - \mathrm{ord}(t_i) = \mathbf{x}_{\nu(t)\vec{c}}(\vec{v})$$

For w with representative $P_\tau \in \mathrm{Perm}_n$ we have

$$\nu(w)\vec{c} = r(w)\vec{c} = (c_{\sigma^{-1}(1)}, \ldots, c_{\sigma^{-1}(n)})$$

so that

$$P_\sigma \cdot \mathbf{x}_{\vec{c}}(\vec{v}) = \mathbf{x}_{\vec{c}}(P_\sigma^{-1}\vec{v}) = \min_{1 \le i \le n} \mathrm{ord}(v_{\sigma(i)}) + c_i = \min_{1 \le i \le n} \mathrm{ord}(v_i) + c_{\sigma^{-1}(i)} = \mathbf{x}_{\nu(P_\sigma)\vec{c}}(\vec{v})$$

□

Proposition 2.8 and Corollary 2.5 together imply

Proposition 2.9. *Let*

$$\Delta = \{\mathbf{x}_{\vec{c}} \in \mathcal{A} : 1 > c_1 \geq c_2 \geq \ldots \geq c_n \geq 0\}$$

Then Δ *is a fundamental domain for the action of* N *on* $\mathcal{A}$*. Moreover, for any* $\mathbf{x} \in \mathcal{B}$ *there is a* $g \in G(F)$ *with* $g \cdot \mathbf{x} \in \Delta$*.*

Proof. The first statement is immediate from Proposition 2.8 since the action ν of N on $\mathcal{A}$ acts via translation by integers and permutations of indices. The second follows from the first and the fact that there is some $g' \in G(F)$ with $g' \cdot \mathbf{x} \in \mathcal{A}$ by Corollary 2.4. □

We further have

Corollary 2.10. *Let* $\mathbf{x}_{\vec{c}}, \mathbf{x}_{\vec{d}} \in \mathcal{A}$ *be such that there is some* $g \in G(F)$ *with* $g \cdot \mathbf{x}_{\vec{c}} = \mathbf{x}_{\vec{d}}$*. Then there is some* $n \in N$ *such that* $\nu(n)\vec{d} = \vec{c}$*. It follows that* Δ *is a fundamental domain for the action of* $G(F)$ *on* $\mathcal{B}$*.*

Proof. Let $\vec{c}_0, \vec{d}_0 \in \Delta$ be such that there are $n_1, n_2 \in N$ with $\vec{c} = \nu(n_1)\vec{c}_0$ and $\vec{d} = \nu(n_2)\vec{d}_0$. Then we have

$$(n_2^{-1} g n_1) \cdot \mathbf{x}_{\vec{c}_0} = \mathbf{x}_{\vec{d}_0}$$

By Corollary 2.5 we have that $\vec{c}_0 = \vec{d}_0$. It follows that $\vec{d} = \nu(n_2 n_1^{-1})\vec{c}$. □

The Action of the Root Subgroups on $\mathcal{B}$

For each $\alpha \in \Phi$ let $f_\alpha : F \to U_\alpha$ be such that for $\alpha = \alpha_{ij}$ we have $f_\alpha(a)$ is a matrix with a in its ij^{th} entry, 1s down the diagonal and 0s elsewhere. Furthermore, for any $t \in T(F)$ this map has the property that

$${}^t f_\alpha(y) = f_\alpha(\alpha(t)y)$$

We use the map f_α to define a decreasing filtration $\{U_{\alpha,m}\}_{m\in\mathbb{Z}}$ on each U_α by setting

$$U_{\alpha,m}(F) = f_\alpha(\varpi^m \mathcal{O}_F)$$

By the proof of Proposition 2.8, for $n \in N$ with image $w \in W$ we have

$${}^n U_{\alpha,m}(F) = U_{{}^w\alpha, m+\mathrm{ord}(\alpha(t))}(F)$$

We will give a description of the groups $G_{\mathbf{x}}(F)$ via the root subgroups $U_\alpha(F)$ and their filtrations. Towards this goal, we define a set of affine functionals on $\mathcal{A}$ referred to as the affine roots. For any $\alpha \in \Phi$ and $m \in \mathbb{Z}$ we define a map $\alpha + m : \mathcal{A} \to \mathbb{R}$ via

$$(\alpha + m)(\vec{c}) = \langle \alpha, \vec{c} \rangle + m$$

We identify $\alpha = \alpha + 0$. Moreover, we set

$$\Psi = \{\alpha + m : \alpha \in \Phi, m \in \mathbb{Z}\}$$

For any $\psi \in \Psi$, $\psi = \alpha + m$ we write $\dot{\psi} = \alpha$ and define

$$U_\psi(F) = U_{\dot{\psi},m}(F)$$

The groups $U_\psi(F)$ have the following property:

Proposition 2.11. *For any $\psi \in \Psi$ the group $U_\psi(F)$ fixes the half-plane $\psi^{-1}([0,\infty)) \subset \mathcal{A}$.*

Proof. Suppose $\psi = \alpha_{ij} + m$ so that

$$\psi^{-1}([0,\infty)) = \{\mathbf{x}_{\vec{c}} : c_i - c_j \geq -m\}$$

For $u \in U_\psi(F) = U_{\alpha,m}(F)$ we have that the ij-entry u_{ij} is such that $\text{ord}(u_{ij}) \geq m$ so that we may compute

$$u \cdot \mathbf{x}_{\vec{c}}(\vec{v}) = \mathbf{x}_{\vec{c}}(u^{-1} \cdot \vec{v}) = \max\{\text{ord}(v_i - u_{ij}v_j) + c_i, \text{ord}(v_l) + c_l : l \neq i\} = \mathbf{x}_{\vec{c}}(\vec{v})$$

Indeed $\text{ord}(v_i + u_{ij}v_j) + c_i \geq \text{ord}(v_j) + c_j$ since $\text{ord}(u_{ij}) \geq m$. □

We use the groups of the form $U_\psi(F)$ to define the following subgroups:

Definition 2.12. *To $\mathbf{x} \in \mathcal{B}$ we associate the subgroup*

$$U_{\mathbf{x}}(F) = \langle U_\psi(F) : \psi(\mathbf{x}) \geq 0 \rangle$$

Using facts about how conjugation by $n \in N_F$ permutes the groups of the form $U_\psi(F)$ we obtain

Proposition 2.13. *For $\mathbf{x} \in \mathcal{A}$ and $n \in N_F$ we have*

$$^{n}U_{\mathbf{x}}(F) = U_{\nu(n)\cdot\mathbf{x}}(F)$$

Proof. It suffices to show that $^{t}U_{\mathbf{x}}(F) = U_{\nu(t)\mathbf{x}}(F)$ for any $t \in T$ and that $^{P_\tau}U_{\mathbf{x}}(F) = U_{\nu(P_\tau)\cdot\mathbf{x}}(F)$ for any $P_\tau \in \text{Perm}_n$.

For $t \in T(F)$ and $\psi \in \Psi$ we compute $\psi \circ t^{-1} = \psi + \text{ord}(\dot{\psi}(t))$ so that

$$\psi(\mathbf{x}) \geq 0 \iff (\psi + \text{ord}(\dot{\psi}(t)))(t \cdot \mathbf{x}) \geq 0$$

It follows that

$$U_{t\cdot\mathbf{x}}(F) = \langle U_\psi(F) : \psi(t \cdot \mathbf{x}) \geq 0 \rangle = \left\langle U_{\psi+\text{ord}(\dot{\psi}(t))}(F) : \psi(\mathbf{x}) \geq 0 \right\rangle = \left\langle {}^{t}U_\psi(F) : \psi(\mathbf{x}) \geq 0 \right\rangle = {}^{t}U_{\mathbf{x}}(F)$$

For $P_\tau \in \text{Perm}_n$ with image $w \in W$ and $\psi = \alpha + m$ we compute $\psi \circ P_\tau^{-1} = {}^{w}\alpha + m =: {}^{w}\psi$ so that

$$\psi(\mathbf{x}) \geq 0 \iff {}^{w}\psi(P_\tau \cdot \mathbf{x}) \geq 0$$

and hence

$$U_{P_\tau\cdot\mathbf{x}}(F) = \langle U_\psi(F) : \psi(P \cdot \mathbf{x}) \geq 0 \rangle = \langle U_{^{w}\psi}(F) : \psi(\mathbf{x}) \geq 0 \rangle = \left\langle {}^{P_\tau}U_\psi(F) : \psi(\mathbf{x}) \geq 0 \right\rangle = {}^{P_\tau}U_{\mathbf{x}}(F)$$

□

For $\mathbf{x} \in \mathcal{B}$ and $g \in G(F)$ we note that

$$U_{\mathbf{x}}(F) = {}^{g^{-1}}U_{g\cdot\mathbf{x}}(F)$$

The Groups $G_{\mathbf{x}}(F)$ and their Filtrations

Let V be a finite dimensional F-vector space. We refer to a finitely generated $\mathcal{O}_F$-submodule $\mathcal{L}$ of V of maximal rank as a lattice. Let $\mathcal{L}att(V)$ denote the collection of lattices in V. The action of $\mathrm{GL}(V)$ on V induces an action on $\mathcal{L}att(V)$.

The Theorem of Principle Divisors shows that $\mathrm{GL}(V)$ acts simply transitively on $\mathcal{L}att(V)$. Moreover, there is a one-to-one correspondence between $\mathcal{L}att(V)$ and the maximal compact subgroups of $\mathrm{GL}(V)$.

For $V = F^n$ write $\mathcal{L}att(n) = \mathcal{L}att(n)$. Let $\mathcal{L}_0 = {\mathcal{O}_F}^n$ and set $K_0 = \mathrm{stab}_{G(F)}(\mathcal{L}_0)$. We have the following:

Proposition 2.14. *We have $K_0 = GL_n(\mathcal{O}_F)$ and that K_0 is a maximal compact subgroup of $G(F)$ which is open in $G(F)$. Moreover, if K is a maximal compact subgroup of $G(F)$ then there is some $g \in G(F)$ such that $K = {}^gK_0$ so that $K = stab_{G(F)}(g \cdot \mathcal{L})$.*

Proof. See [41]. □

For $\mathcal{L} \in \mathcal{L}att(n)$ we may define a decreasing filtration $\{\mathcal{L}(m)\}_{m \geq 0}$ by setting $\mathcal{L}(0) = \mathcal{L}$ and $\mathcal{L}(n) = \varpi^m \mathcal{L}$. Also, to any $\mathbf{x} \in \mathcal{B}$ we may assign a lattice $\mathcal{L}_{\mathbf{x}} \in \mathcal{L}att(n)$ via

$$\mathcal{L}_{\mathbf{x}} = \{\vec{v} \in F^n : \mathbf{x}(\vec{v}) \geq 0\} \tag{2.2}$$

For example, we have $\mathcal{L}_0 = \mathcal{L}_{\mathbf{x}_{\vec{0}}}$. In the case of $\mathcal{L} = \mathcal{L}_{\mathbf{x}}$ we may reinterpret our filtrations via

$$\mathcal{L}_{\mathbf{x}}(m) = \{\vec{v} \in V : \mathbf{x}(\vec{v}) \geq m\} \tag{2.3}$$

We make the following crucial definition:

Definition 2.15. *For $\mathbf{x} \in \mathcal{B}$ we define $G_{\mathbf{x}}(F) = \mathrm{stab}_{G(F)}(\mathbf{x})$*

We call a subgroup of the form $G_{\mathbf{x}}(F)$ of $G(F)$ a parahoric subgroup.

We claim that each $G_{\mathbf{x}}(F)$ is a compact open subgroup of $G(F)$. To see that $G_x(F)$ is open, and hence closed, we note that it contains the subgroup

$$\left\{g \in \mathrm{stab}_{G(F)}(\mathcal{L}_{\mathbf{x}}) : g \text{ acts on } \mathcal{L}_{\mathbf{x}}/\varpi\mathcal{L}_{\mathbf{x}} \text{ via the identity}\right\}$$

The action of $G(F)$ allows us to replace $\mathbf{x}$ with $\mathbf{x}_0$ in which case we may compute the subgroup described in (2.1) explicitly as

$$I_n + \varpi M_n(\mathcal{O}_F)$$

which is open in $G(F)$.

To see that $G_{\mathbf{x}}(F)$ is compact, we note that

$$G_{\mathbf{x}}(F) \subset \mathrm{stab}_{G(F)}(\mathcal{L}_{\mathbf{x}})$$

The group $G_{\mathbf{x}}(F)$ preserves $\mathcal{L}_{\mathbf{x}}$ as well as $\varpi^m \mathcal{L}_{\mathbf{x}}$ for each $m > 0$, and therefore acts on $\mathcal{L}_{\mathbf{x}}/\varpi^n \mathcal{L}_{\mathbf{x}}$ for each $n \geq 0$. As such, we may define a filtration on $G_{\mathbf{x}}(F)$ as follows

$$G_{\mathbf{x},m}(F) = \{g \in G_{\mathbf{x}}(F) : g \text{ acts on } \mathcal{L}_{\mathbf{x}}/\varpi^m \mathcal{L}_{\mathbf{x}} \text{ via the identity}\} \tag{2.4}$$

The most basic example of such a filtration is for $G_{\mathbf{x}_0}(F) = K_0 = G(\mathcal{O}_F)$ where for $m > 0$ we have

$$G_{\mathbf{x}_0,m}(F) = I_n + \varpi^m M_n(\mathcal{O}_F)$$

It will be of use to us going forward to be able to concretely express the groups $G_{\mathbf{x}_{\vec{c}}}(F)$; by Corollary 2.10 it suffices to consider $\vec{c}$ for which $1 > c_1 \geq c_2 \ldots \geq c_n \geq 0$.

Proposition 2.16. *Let $\mathbf{x} = \mathbf{x}_{\vec{c}}$ with $1 > c_1 \geq \ldots c_n \geq 0$ and, moreover, that $n_1, n_2, \ldots, n_k \in \mathbf{N}$ are such that*

$$c_1 = \ldots = c_{n_1},\ c_{n_1+1} = \ldots = c_{n_1+n_2},\ \ldots,\ c_{n_1+\ldots+n_{k-1}+1} = \ldots = c_{n_1+\ldots+n_k}$$

Let

$$P(\mathbf{x}) = \begin{pmatrix} \mathrm{GL}_{n_1}(\mathfrak{f}) & * & & * \\ 0 & \mathrm{GL}_{n_2}(\mathfrak{f}) & & \\ & & \ddots & * \\ 0 & & 0 & \mathrm{GL}_{n_k}(\mathfrak{f}) \end{pmatrix}$$

and

$$P_{\mathbf{x}}(F) = \{g \in K_0 : \pi(g) \in P(\mathbf{x})\}$$

Then $G_{\mathbf{x}}(F) = P_{\mathbf{x}}(F)$.

Proof. A straightforward computation shows that the action of $P_{\mathbf{x}}(F)$ preserves $\mathbf{x}$ so that $P_{\mathbf{x}}(F) \subset G_{\mathbf{x}}(F)$. Conversely, suppose $k \in G_{\mathbf{x}}(F)$ and write $k = k_{ij}$. Computing $\mathbf{x}_{\vec{c}}(e_j)$ for each standard basis vector e_j we see that

$$c_j = \mathbf{x}_{\vec{c}}(e_j) = \mathbf{x}_{\vec{c}}(k \cdot e_j) = \mathbf{x}_{\vec{c}}\left(\sum_{i=1}^{n} k_{ij} e_i\right) = \min_{1 \leq i \leq n} c_i + \mathrm{ord}(k_{ij})$$

This forces the condition that $\mathrm{ord}(k_{ij}) \geq 1$ for each i such that $c_j > c_i$. It follows that $G_{\mathbf{x}}(F) \subset P_{\mathbf{x}}(F)$, completing the proof. □

We give an alternate description of $G_{\mathbf{x}}(F)$ using root subgroups as follows:

Proposition 2.17. *Let $\mathbf{x} \in \mathcal{A}$. Then $G_{\mathbf{x}}(F) = T_0(F)U_{\mathbf{x}}(F)$.*

Proof. By Proposition 2.13 it suffices to prove the result for $\mathbf{x} = \mathbf{x}_{\vec{c}} \in \Delta$ since

$$T_0(F)U_{n\cdot\mathbf{x}}(F) = T_0(F)^n U_{\mathbf{x}}(F) = {}^n(T_0(F)U_{\mathbf{x}}(F))$$

Since $G_{\mathbf{x}}(F) = P_{\mathbf{x}}(F)$ by Proposition 2.16 for $\alpha = \alpha_{ij}$ let

$$k_{\alpha_{ij}} = \begin{cases} 1 & \text{if } j < i \text{ and } c_i > c_j \\ 0 & \text{else} \end{cases}$$

We easily see that

$$P_{\mathbf{x}}(F) = T_0(F) \langle U_{\alpha,k_\alpha}(F) : \alpha \in \Phi \rangle$$

Conversely, if $\psi = \alpha_{ij} + m$ is such that $\psi(\mathbf{x}_{\vec{c}}) \geq 0$ we must have that $c_i - c_j \geq -m$. Letting

$$k'_{\alpha_{ij}} = \lfloor c_i - c_j \rfloor$$

we see that $k'_{\alpha_{ij}} = k_{\alpha_{ij}}$ and

$$U_{\mathbf{x}}(F) = \left\langle U_{\alpha,k'_\alpha}(F) : \alpha \in \Phi \right\rangle = \langle U_{\alpha,k_\alpha}(F) : \alpha \in \Phi \rangle$$

It follows that $T_0(F)U_{\mathbf{x}}(F) = P_{\mathbf{x}}(F)$. □

Summary of the Structure of $\mathcal{B}$

We are now in a position to prove that the definition we gave for the building $\mathcal{B}$ coincides with the standard. Indeed, by [40] any two $G(F)$-sets containing $\mathcal{A}$ with the following properties are isomorphic:

Proposition 2.18. *$\mathcal{B}$ satisfies the following properties:*

a) $\mathcal{B} = G(F) \cdot \mathcal{A}$.

b) *N stabilizes $\mathcal{A}$ and acts on $\mathcal{A}$ via ν.*

c) *For any $\psi \in \Psi$ the group $U_\psi(F)$ stabilizes the half-plane $\psi^{-1}([0,\infty)) \subset \mathcal{A}$.*

Proof. This follows from Corollary 2.4, Proposition 2.8 and Proposition 2.11. □

In a similar vein, we have

Proposition 2.19. *We may identify $\mathcal{B}$ with the quotient of $G(F) \times X_*(T) \otimes \mathbb{R}$ by the equivalence relation*

$$(g_1, \vec{c}_1) \sim (g_2, \vec{c}_2)$$

if and only if there is some $n \in N$ such that $\vec{c}_2 = \nu(n)\vec{c}_1$ and $g_1^{-1}g_2 n \in G_{\mathbf{x}}(F)$.

The Building for General Reductive Groups

For G a reductive group over F there exists a set $\mathcal{B}(G,F)$ which is, in general, the product of a finite dimensional affine space and a polysimplicial complex, on which $G(F)$ acts, see [12] for a general construction. This set has the following properties:

- For any maximally F-split maximal torus $S \subset G$ defined over F there is an affine space $A(G,S,F) \simeq X_*(S) \otimes \mathbb{R}$ with $A(G,S,F) \subset \mathcal{B}(G,F)$ such that

$$\mathcal{B}(G,F) = G(F) \cdot A(G,S,F)$$

  Moreover, if $S' = S^g$ for some $g \in G(F)$ we have $A(G,S',F) = g \cdot A(G,S,F)$.

- With S as above, for $N_S = N_{G(F)}(S(F))$ there is an action ν of N_S on $A(G,S,F)$. Moreover, there is a set Ψ of affine roots of $G(F)$ and a group $U_\psi(F)$ for each $\psi \in \Psi$ such that the group $U_\psi(F)$ fixes a corresponding half-plane in $A(G,S',F)$ (that is to say, $\mathcal{B}(G,F)$ satisfies properties analogous to those in Proposition 2.18).

- Let $\kappa_G : G(F) \to X^*(Z(\widehat{G}))^{\Gamma_{\overline{F}/F}}$ be the Kottwitz map defined in [29]. In general, for $\mathbf{x} \in \mathcal{B}(G,F)$ we define

$$G_{\mathbf{x}}(F) = \mathrm{stab}_{G(F)}(\mathbf{x}) \cap \ker(\kappa_G)$$

- We may canonically identify

$$\mathcal{B}(G,F) \simeq \mathcal{B}^{\mathrm{red}}(G,F) \times X_*(Z(G)) \otimes \mathbb{R}$$

  where $\mathcal{B}^{\mathrm{red}}(G,F) = \mathcal{B}(G_{\mathrm{der}},F)$; the action of $G_{\mathrm{der}}(F)$ on $\mathcal{B}^{\mathrm{red}}(G,F)$ extends to one of $G(F)$ and $G(F)$ acts trivially on $X_*(Z(G))\otimes\mathbb{R}$. Denoting by $[\cdot] : \mathcal{B}(G,F) \to \mathcal{B}^{\mathrm{red}}(G,F)$ the projection map we have for $\mathbf{x},\mathbf{y} \in \mathcal{B}(G,F)$ with $[\mathbf{x}] = [\mathbf{y}]$ that

$$G_{\mathbf{x}}(F) = G_{\mathbf{y}}(F)$$

- Let $H \subset G$ be an F-Levi subgroup defined over F, that is to say that $H(F)$ is a Levi subgroup of $G(F)$. Then there is an embedding $\mathcal{B}(H,F) \hookrightarrow \mathcal{B}(G,F)$ which induces an embedding $\mathcal{B}^{\mathrm{red}}(H,F) \hookrightarrow \mathcal{B}^{\mathrm{red}}(G,F)$. These embeddings are not necessarily unique but any two embeddings in either case have the same image. If $H/Z(G)$ is F-anisotropic then the embedding $\mathcal{B}^{\mathrm{red}}(H,F) \hookrightarrow \mathcal{B}^{\mathrm{red}}(G,F)$ is unique so that, for the purposes of defining filtrations of subgroups, we may identify

$$\mathcal{B}^{\mathrm{red}}(H,F) \subset \mathcal{B}^{\mathrm{red}}(G,F)$$

  In fact, all we require is for H to be defined over F and be such that there is a tame extension E of F such that $H \otimes E$ is a Levi subgroup of $G \otimes E$.

- The set $\mathcal{B}(G,F)$ may be constructed via an equivalence relation analogous to Proposition 2.19.

There are additional difficulties when constructing the building in the case where G is not split, as well as additional issues with constructing the filtrations of parahoric subgroups, especially in the case where G is of type triality D_4; see [40], [34].

The Building and Field Extensions

For an extension E/F we may define the building $\mathcal{B}(G,E)$ of G over E via

$$\mathcal{B}(G,E) = \mathcal{B}(G \otimes E, E)$$

When E/F is Galois and tamely ramified, the relationship between $\mathcal{B}(G,F)$ and $\mathcal{B}(G,E)$ is contained in the following theorem:

Theorem 2.20. *If E/F is Galois then there is an action of $\Gamma_{E/F}$ on $\mathcal{B}(G,E)$. Moreover, if E/F is tamely ramified then*

a) *We may identify $\mathcal{B}(G,F) = \mathcal{B}(G,E)^{\Gamma_{E/F}}$.*

b) *Let T be a maximal torus of G defined over F which splits over E. Then*

$$\mathcal{A}(G,T,F) := \mathcal{A}(G,T,E) \cap G(F)$$

may be identified with $X_*(T)^{\Gamma_E} \otimes \mathbb{R}$. Notably, $\mathcal{A}(G,T,F)$ is non-empty.

Proof. See [42]. □

Theorem 2.20 need not hold for E wildly ramified: for example, if E is wildly ramified we have

$$\mathcal{B}(\mathbf{Res}_{E/F}G, F) = \mathcal{B}(G,F)$$

by Lemma 2.1.1 of [3].

Suppose $G = \mathrm{GL}_n$ and let E/F be a finite extension. We may similarly define $\mathcal{B}(G,E)$ to be the set of all additive norms on E^n. In this way, we may define groups $G_{\mathbf{x}}(E)$ for $\mathbf{x} \in \mathcal{B}(E)$ exactly as in (2.15) as well as filtrations thereof analogously to (2.4). A minor difference is that we must index these filtrations by $\frac{1}{e}\mathbb{Z}$ instead of $\mathbb{Z}$ where e is the ramification degree of E/F.

Letting $\vec{v} \mapsto \vec{v}^{\sigma}$ denote the (coordinate-wise) action of $\sigma \in \Gamma_{E/F}$ on E^n we may realize the action of $\Gamma_{E/F}$ on $\mathcal{B}(G,E)$ as

$$\mathbf{x}^{\sigma}(\vec{v}) = \mathbf{x}(\vec{v}^{\sigma})$$

In what follows, it will be necessary to understand the sets $\mathcal{A}(G,T,F)$ in the case where $T \simeq \mathbf{Res}_{E/F}(\mathbb{G}_m)$ for a cyclic extension E with $[E:F] = n$. Suppose $n = ef$ where e is the ramification degree of E and f is the degree of the residual extension. In this case T is F-anisotropic modulo the centre of $G(F)$ so that the image of $\mathcal{A}(G,T,F)$ in $\mathcal{B}^{\mathrm{red}}(G,F)$ is a single point $\mathbf{x}_T$.

Suppose further that $\mathbf{x}_T$ lies in the image of Δ in $\mathcal{B}^{\mathrm{red}}(G,F)$. Then by [14] the group $G_{\mathbf{x}_T}(F)$ has the following form:

Proposition 2.21. *Let $K_e \subset G(\mathcal{O}_F)$ be the subset of matrices A which may be decomposed into an $f \times f$ matrix $A = (A)_{kl}$ of $e \times e$ block matrices of the form*

$$\begin{cases} A_{ij} \in \mathrm{M}_e(\mathcal{O}_F) & \text{for } j > i \\ A_{ij} \in \mathrm{GL}_e(\mathcal{O}_F) & \text{for } i = j \\ A_{ij} \in \varpi \mathrm{M}_e(\mathcal{O}_F) & \text{for } j > i \end{cases}$$

and let

$$z_e = \begin{pmatrix} 0_e & I_e & \cdots & 0_e \\ \vdots & & \ddots & \vdots \\ 0_e & & & I_e \\ \varpi I_e & 0_e & \cdots & 0_e \end{pmatrix}$$

Then $G_{x_T}(F)$ is the group generated by K_e and z_e.

Combining the above with Proposition 2.17 we see that $\mathbf{x}_T \in \mathcal{B}^{\mathrm{red}}(G, F)$ is a vertex if E is unramified and that $\mathbf{x}_T$ lies in the interior of the simplex if E is totally ramified.

2.2 Moy-Prasad Filtrations for $\mathbf{GL}_n(F)$

We define the filtration $\{G_{x,r}(F)\}$ for $r \in \mathbb{R}$ of the groups $G_x(F)$ described in [33] and [34]. For $r \in \mathbb{Z}_{\geq 0}$ these filtrations agree with those defined in (2.4). We will also define lattices $\mathfrak{g}_x(F)$ in $\mathfrak{gl}_n(F)$ and analogous filtrations $\{\mathfrak{g}_{x,r}(F)\}$ for $r \in \mathbb{R}$ thereof. Using this machinery, we will define isomorphisms between quotients of groups of the form $G_{x,r}(F)$ with quotients of lattices of the form $\{\mathfrak{g}_{x,r}(F)\}$, the Moy-Prasad isomorphisms, which will be necessary to use in our construction of supercuspidal characters.

We discuss the case of $G = \mathrm{GL}_n$ in detail but note by [33] and [34] that analogous results hold in full generality.

Filtrations of Parahoric Subgroups of $\mathbf{GL}_n$

Above we set $T_0(F) = T(\mathcal{O}_F)$. We define a decreasing filtration on $T_0(F)$ via

$$T_m(F) = \{t \in T_0(F) : \mathrm{ord}(\alpha(t) - 1) \geq m \ \forall \alpha \in \Phi\}$$

The proof of Proposition 2.17 gives us an easy recipe for a decomposition of the filtrations of $G_{\mathbf{x}}(F)$ defined in (2.4) in terms of the root subgroups.

Proposition 2.22. *For $\mathbf{x} \in \mathcal{A}$ and $m \geq 0$ let*

$$U_{\mathbf{x},m}(F) = \langle U_\psi(F) : \psi(\mathbf{x}) \geq m \rangle$$

Then we have

$$G_{\mathbf{x},m}(F) = T_m(F) U_{\mathbf{x},m}(F)$$

Proof. For details see [33]. $\square$

We wish to enlarge our indexing set of the filtrations of our subgroups. Let $\widetilde{\mathbb{R}}$ denote the set of subsets of $\mathbb{R}$ satisfying

$$I \in \widetilde{\mathbb{R}} \iff x \in I, x < y \implies y \in I$$

Then $\widetilde{\mathbb{R}}$ is an ordered monoid where the ordering is given by inclusion and where we set

$$I_1 + I_2 = \{x \in \mathbb{R} : x = x_1 + x_2, x_i \in I_i\}$$

Moreover, we observe that

$$\widetilde{\mathbb{R}} = \{[a, \infty) : a \in \mathbb{R}\} \cup \{(a, \infty) : a \in \mathbb{R}\}$$

Define maps $\iota : \mathbb{R} \to \widetilde{\mathbb{R}}$ and $\iota^+ : \mathbb{R} \to \widetilde{\mathbb{R}}$ via $\iota(r) = [r, \infty)$ and $\iota^+(r) = (r, \infty)$, respectively.

We define subgroups

$$G_{x,I}(F) = \bigcup_{m \in \mathbb{Z}_{\geq 0} \cap I} G_{x,m}(F) \tag{2.5}$$

It is clear that $G_{x,I}(F) \subset G_{x,J}(F)$ if and only if $I \leq J$. For brevity we shall write

$$G_{x,r}(F) = G_{x,\iota(r)}(F)$$

and

$$G_{x,r^+}(F) = G_{x,\iota^+(r)}(F)$$

As is frequently seen in the literature, we abuse notation and write r for $\iota(r) \in \widetilde{\mathbb{R}}$ and r^+ for $\iota^+(r) \in \widetilde{\mathbb{R}}$. In this way we identify $\mathbb{R}$ as a subset of $\widetilde{\mathbb{R}}$.

Lattices in $\mathfrak{gl}_n(F)$ and their Filtrations

We give filtrations of the Lie algebra $\mathfrak{g}(F)$ of $G(F)$ corresponding to $x \in \mathcal{B}(G, F)$. By [13] there is a smooth group scheme $\mathcal{G}_{\mathbf{x}}$ defined over $\mathcal{O}_F$ with $\mathcal{G}_{\mathbf{x}}(\mathcal{O}_F) = G_{\mathbf{x}}(F)$ and with generic fibre G. We set

$$\mathfrak{g}_{\mathbf{x}}(F) = \mathrm{Lie}(\mathcal{G}_{\mathbf{x}})(\mathcal{O}_F)$$

which is a lattice in $\mathfrak{g}(F)$. For $\mathbf{x} \in \mathcal{B}$ and $g \in G(F)$ we note that ${}^g\mathfrak{g}_{\mathbf{x}}(F) = \mathfrak{g}_{g \cdot \mathbf{x}}(F)$.

For $x \in \Delta$ recall that there is a parabolic subgroup $P(x) \subset G(\mathfrak{f})$ such that

$$G_{\mathbf{x}}(F) = \{g \in G(\mathcal{O}_F) : \pi(g) \in P(x)\}$$

We have in this case that

$$\mathfrak{g}_{\mathbf{x}}(F) = \{X \in \mathfrak{g}(\mathcal{O}_F) : \pi(X) \in \mathrm{Lie}(P(x))\}$$

For $m \in \mathbb{Z}$ we define a filtration on $\mathfrak{t}(F)$ via

$$\mathfrak{t}_m(F) = \{X \in \mathfrak{t}(F) : X \in \varpi^m \mathfrak{t}(\mathcal{O}_F)\}$$

For each $\alpha \in \Phi$ there is a map $g_\alpha : F \to \mathfrak{u}_\alpha(F)$ which, for $\alpha = \alpha_{ij}$, maps a to the matrix with a in its ij-coordinate and 0s elsewhere. For $\psi \in \Psi$ with $\psi = \alpha + m$ we set

$$\mathfrak{u}_\psi = g_\alpha(\varpi^m \mathcal{O}_F)$$

Similarly to the case of groups, the following is easily established:

Proposition 2.23. *For* $\mathbf{x} \in \Delta$ *we have that*

$$\mathfrak{g}_{\mathbf{x}}(F) = \mathfrak{t}_0(F) \oplus \langle \mathfrak{u}_\psi : \psi(\mathbf{x}) \geq 0 \rangle$$

We define filtrations of the lattices $\mathfrak{g}_{\mathbf{x}}(F)$ by, for $m \geq 0$, setting

$$\mathfrak{g}_{\mathbf{x},m}(F) = \mathfrak{t}_m(F) \oplus \langle \mathfrak{u}_\psi : \psi(\mathbf{x}) \geq m \rangle$$

Furthermore, we define lattices $\mathfrak{g}_{\mathbf{x},r}(F)$ for $r \in \widetilde{\mathbb{R}}$ identically to how they were defined for groups in (2.5).

The Moy-Prasad Isomorphism

We combine the isomorphisms $f_\alpha : F \to U_\alpha(F)$ and $g_\alpha : F \to \mathfrak{u}_\alpha(F)$ defined above to obtain an isomorphism

$$h_\alpha = g_\alpha \circ f_\alpha^{-1} : U_\alpha(F) \to \mathfrak{u}_\alpha(F)$$

We clearly have $h_\alpha(U_{\alpha,m}(F)) = \mathfrak{u}_{\alpha,m}(F)$. Also, for $t \in T_{m'}(F)$ we have that

$$h_\alpha(U_{\alpha,m}(F))^t = \mathfrak{u}_{\alpha,m+m'}(F)$$

Using facts about commutation for the root subgroups $U_\alpha(F)$ we have for $\alpha, \beta \in \Phi$ that

$$[U_{\alpha,m_1}(F), U_{\beta,m_2}(F)] \subset \langle U_{\gamma,m_1+m_2}(F) : \gamma = k\alpha + \ell\beta, k, \ell > 0 \rangle \tag{2.6}$$

and

$$[U_{\alpha,m_1}(F), U_{-\alpha,m_2}(F)] \subset \langle T_{m_1+m_2}, U_{\alpha,m_1+m_2}(F), U_{-\alpha,m_1+m_2}(F) \rangle \tag{2.7}$$

With regards to the torus, we observe that the map $h_T : T_{0^+}(F) \to \mathfrak{t}_{0^+}(F)$ defined via

$$h_T(t) = t - 1$$

has the property that $h_T(T_m(F)) = \mathfrak{t}_m(F)$.

Collecting the maps h_T and h_α for $\alpha \in \Phi$ we obtain the following:

Theorem 2.24 (Moy-Prasad Isomorphism)**.** *Let* $\mathbf{x} \in \mathcal{B}$ *and let* $r, s \in \widetilde{\mathbb{R}}$ *be such that* $0 \leq r \leq s \leq 2r$. *Write*

$$G_{\mathbf{x},r:s}(F) = G_{\mathbf{x},r}(F)/G_{\mathbf{x},s}(F)$$

and

$$\mathfrak{g}_{\mathbf{x},r:s}(F) = \mathfrak{g}_{\mathbf{x},r}(F)/\mathfrak{g}_{\mathbf{x},s}(F)$$

Then we have an isomorphism

$$\mathbf{e}_{\mathbf{x},r;s} : G_{\mathbf{x},r:s}(F) \simeq \mathfrak{g}_{\mathbf{x},r:s}(F) \tag{2.8}$$

Proof. Suppose $\mathbf{x} \in \mathcal{B}$ is such that

$$U_{\mathbf{x}}(F) = \langle U_{\alpha,m_\alpha}(F) : \alpha \in \Phi \rangle$$

Then for $m > 0$ the map

$$G_{\mathbf{x},m}(F) \to T_m(F) \times \prod_{\alpha\in\Phi} U_{\alpha,m+m_\alpha}(F) \to \mathfrak{t}_m(F) \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{u}_{\alpha,m+m_\alpha}(F) \Big/ \mathfrak{t}_{2m}(F) \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{u}_{\alpha,2(m+m_\alpha)}(F)$$

$$\simeq \mathfrak{g}_{\mathbf{x},m:2m}(F)$$

has kernel $G_{\mathbf{x},2m}(F)$. The first map is the standard isomorphism of varieties restricted to $G_{\mathbf{x},m}(F)$ and the second map is defined to be the product of h_T and the maps h_α for $\alpha \in \Phi$. This follows largely due to (2.6) and (2.7) above; see [4] for additional details. □

In cases of interest we may realize these isomorphisms explicitly; it is further shown in [4] that

Proposition 2.25. *Let* $\mathbf{x} = \mathbf{x}_0$. *Then for any* $r, s \in \widetilde{\mathbb{R}}$ *with* $0 \leq r \leq s \leq 2r$ *and* $X \in \mathfrak{g}_{x,r}(G)$ *we have that*

$$\mathbf{e}_{\mathbf{x},r;s}(X)G_{x,2r}(F) = (1+X)G_{x,2r}(F)$$

Similarly, for $\gamma \in G_{x,r}(F)$ *we have*

$$\mathbf{e}^{-1}_{\mathbf{x},r;s}(\gamma)\mathfrak{g}_{x,2r}(F) = (\gamma - 1)\mathfrak{g}_{x,2r}(F)$$

We interpret the above as saying that we may often replace the abstractly defined Moy-Prasad isomorphisms and their inverses with the mutually inverse maps $X \mapsto 1 + X$ and $\gamma \mapsto \gamma - 1$.

2.3 The Depth of an Element of $GL_n(F)$

We will use the building $\mathcal{B}(G,F)$ to define various notions of the depth of an element for elements of $G(F)$. We will also define analogous notions for elements of the Lie algebra $\mathfrak{g}(F)$. We also collect a number of related conjugacy results which will play a necessary role in our character computations.

Depth of Elements

Following [4] we set

$$G_r(F) = \bigcup_{x\in\mathcal{B}(G,F)} G_{x,r}(F) \tag{2.9}$$

We see that $G_r(F)$ is open and closed and that the action of $G(F)$ preserves $G_r(F)$ since $g \cdot G_{x,r}(F) = G_{g\cdot x,r}(F)$ for any $g \in G(F)$ and $x \in \mathcal{B}(G,F)$.

We say that $g \in G(F)$ is unipotent if there exists $\lambda \in X_*(G)^\Gamma$ such that $\lim_{t\to 0}^{\lambda(t)} g = 1$. Let $\mathcal{U}$ denote the set of unipotent elements of $G(F)$. Since $\mathfrak{f}$ is finite, by [4] $g \in \mathcal{U}$ if and only if the p-adic closure of $G(F) \cdot g$ contains the identity.

We define the notion of the depth of an element as follows.

Definition 2.26. *There exists a map $d : G_0(F) \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ such that, for any $g \in G(F)$, either $d(g) = \infty$ or $d(g)$ is the unique number such that $g \in G_{d(g)}(F) \setminus G_{d(g)^+}(F)$. We call $d(g)$ the depth of g. Moreover, we have that $d(g) = \infty$ if and only if $g \in \mathcal{U}$ and that d is locally constant on $G(F) \setminus \mathcal{U}$.*

Let d_T denote the depth function on the group T. We note that we may have $d(\gamma) > d(z\gamma)$ for some $z \in Z(G)$ whereas the definition of the filtration on $T(F)$ implies $d_T(z\gamma) = d_T(\gamma)$ for all $z \in Z(G)$. We make the following definition

Definition 2.27. *We define $d^+ : G(F) \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ via*

$$d^+(\gamma) = \sup \{d(z\gamma) : z \in Z(G)\}$$

These notions of depth for elements of $T(F)$ are related in the following manner

Proposition 2.28. *For $\gamma \in T(F)$ we have $d_T(\gamma) = d^+(\gamma)$.*

Proof. This follows from Lemma 8.3 of [6]. □

Good Elements and Conjugacy

We have the following definitions from [6]

- For a reductive group H over F we say $\gamma \in H(F)$ is absolutely semisimple if the eigenvalues of γ lie in $\mathcal{O}_{\overline{F}}^\times$. We let $\mathcal{G}_0^{G(F)} \subset G(F)$ denote the set of elements in $G(F)$ such that the image of γ in $G(F)/Z(G)$ is absolutely semisimple.
- For $d > 0$ we let $\mathcal{G}_d^{G(F)} \subset G(F)$ be the set of elements $\gamma \in G(F)$ such that there exists a tame-modulo-centre torus S with $\gamma \in S_d \setminus S_{d^+}$ and either $\alpha(\gamma) = 1$ or $\operatorname{ord}(\alpha(\gamma) - 1) = d$ for all $\alpha \in \Phi(G, S)$. If $\gamma \in \mathcal{G}_d^{G(F)}$ we say γ is good of depth d.

We can say much about the situation of how conjugacy of elements influences their depth. From [4] we have the following fact:

Proposition 2.29. *We have ${}^{G(F)}G_r(F) \subset G_{x,r}(F) \cdot \mathcal{U}$ for any $x \in \mathcal{B}(G, F)$.*

We may say much more if γ is a good element.

Proposition 2.30. *Let $\gamma \in G_x(F)$ be regular and absolutely semisimple. Then ${}^g\gamma \in G_x(F)$ if and only if $g \in G_x(F)$.*

Proof. This follows from §3.6 of [40]. □

We may now upgrade Proposition 2.30 in the case of $G = \mathrm{GL}_n$ to the following fact which will be crucial to our character computations

Proposition 2.31. *Let $G = \mathrm{GL}_n$ and let $T \subset G$ be an unramified maximal torus such that $G_{x_T}(F) = \mathrm{GL}_n(\mathcal{O}_F)$. Let $\gamma \in T(F)$ be regular and good of depth r. Then $\gamma^g \in G_{x_T,r}(F)$ if and only if $g \in N_{G(F)}(T(F))G_{x_T}(F)$.*

Proof. We have that $\gamma - 1$ is of the form $\varpi^r Y$ for some $Y \in M_n(\mathcal{O}_F)$. Moreover, since γ is good of depth r we have that each of the eigenvalues of Y has order 0. As such, we may consider Y as an element of $\mathrm{GL}_n(F)$ which is regular and absolutely semisimple.

Now we observe

$$d(\gamma^g) \geq r \iff d((1 + \varpi^r Y)^g) \geq r \iff d(1 + \varpi^r Y^g) \geq r \iff Y^g \in G_x(F)$$

The result follows from Proposition 2.30. □

Depth and the Characteristic Polynomial

Here we establish a result relating the depth of an element of $\mathrm{GL}_n(F)$ with its characteristic polynomial:

Proposition 2.32. *Suppose char($\mathfrak{f}$) $> n$. For $\gamma \in \mathrm{GL}_n(F)^{\mathrm{rss}}$ and $\gamma \in T_0(F)$ for a torus $T \subset \mathrm{GL}_n$ we have that $d_T(\gamma)$ is equal to the value of the minimal slope of the Newton polygon of the polynomial*

$$p_\gamma(x) = \det\left(\gamma - \left(x + \frac{\mathrm{Tr}(\gamma)}{n}\right) I_n\right) = \mathrm{ch}_\gamma\left(x - \left(-\frac{\mathrm{Tr}(\gamma)}{n}\right)\right)$$

Proof. Let $d = d_T(\gamma)$. We note that the matrix $X = \gamma - \frac{\mathrm{Tr}(\gamma)}{n} I_n$ is such that $\mathrm{Tr}(X) = 0$. By potentially extending our base field, we may suppose γ is diagonal with roots $r_1, \ldots, r_n$ so that X is also diagonal with roots $x_i = r_i - \frac{\mathrm{Tr}(\gamma)}{n}$ for $1 \leq i \leq n$. Moreover, by definition we have that $\mathrm{ord}(r_i - r_j) \geq d$ for all $i \neq j$ and hence we also have $\mathrm{ord}(x_i - x_j) \geq d$ for all $i \neq j$. It suffices to show that there is some k such that $\mathrm{ord}(x_k) = d$ and for $i \neq k$ we have $\mathrm{ord}(x_i) \geq d$.

Let

$$d' = \min_{1 \leq i \leq n} \mathrm{ord}(x_i)$$

By potentially reordering the x_i, choose $1 \leq i_0 \leq n$ such that

$$\mathrm{ord}(x_1) = \ldots = \mathrm{ord}(x_{i_0}) = d'$$

and $\mathrm{ord}(x_i) > d'$ for $i > i_0$. We note that $d' \leq d$ since $\mathrm{ord}(x_i - x_j) \geq d'$ for any $i \neq j$. We are done if we can show that $d' = d$. Also, we are clearly done if $i_0 = 1$ so henceforth assume $i_0 > 1$.

Suppose $d > d'$. Since $\mathrm{Tr}(X) = 0$ we have that

$$\sum_{i=1}^{n} x_i \equiv 0 \bmod \varpi^{d'}$$

so that

$$-\sum_{i=1}^{i_0} x_i \equiv \sum_{i=i_0+1}^{n} x_i \equiv 0 \bmod \varpi^{d'} \tag{2.10}$$

since $\mathrm{ord}(x_i) > d'$ for $i > i_0$. Also, we have

$$x_1 - x_i \equiv 0 \bmod \varpi^{d'} \tag{2.11}$$

for each $2 \leq i \leq i_0$. Combining (2.10) and (2.11) we obtain

$$\begin{aligned} 0 &\equiv \sum_{i=2}^{i_0} x_1 - x_i && \bmod \varpi^{d'} \\ &\equiv (i_0 - 1)x_1 - \sum_{i=2}^{i_0} x_i && \bmod \varpi^{d'} \\ &\equiv i_0 x_1 - \sum_{i=1}^{i_0} x_i && \bmod \varpi^{d'} \\ &\equiv i_0 x_1 + \sum_{i=i_0+1}^{n} x_i && \bmod \varpi^{d'} \\ &\equiv i_0 x_1 && \bmod \varpi^{d'} \end{aligned}$$

Since $i_0 \not\equiv 0 \bmod \varpi$ the above implies $\mathrm{ord}(x_1) > d'$, a contradiction. Therefore $d' = d$, completing the proof. □

Depth and Generic Elements of the Lie Algebra

Results analogous to those for the group $G(F)$ hold for the Lie algebra $\mathfrak{g}(F)$. We set

$$\mathfrak{g}_r(F) = \bigcup_{x \in \mathcal{B}(G,F)} \mathfrak{g}_{x,r}(F) \tag{2.12}$$

which is similarly a $G(F)$-set. Let $\mathcal{N}$ be the set of nilpotent elements of $\mathfrak{g}(F)$. Again by [4] we have

Proposition 2.33. *For any $x \in \mathcal{B}(G,F)$ we have ${}^{G(F)}\mathfrak{g}_r(F) \subset \mathfrak{g}_{x,r}(F) + \mathcal{N}$.*

We define the following depth functions on $\mathfrak{g}(F)$

Definition 2.34. *Let* $X \in \mathfrak{g}(X)$*. We define a map* $d : \mathfrak{g}(F) \to \mathbb{R} \cup \{\infty\}$ *via*

$$d(X) = \sup \{r : X \in \mathfrak{g}_r(F)\}$$

We have that d *is locally constant on* $\mathfrak{g}(F) \setminus \mathcal{N}$*. Further we define*

$$d^+(X) = \sup \{d(X + Z) : Z \in \mathfrak{z}(F)\}$$

Let $T \subset G$ be a maximal torus defined over F. We make the following technical definition:

Definition 2.35. *Suppose* $\mathrm{char}(\mathfrak{f})$ *isn't a bad prime for* $\Phi(G, T)^\vee$ *(see [42] for details). Let* $G' \subset G$ *be a subgroup defined over* F *with* $T \subset G'$ *and such that* $G' \otimes E$ *is a Levi subgroup of* $G \otimes E$ *for some tame extension* E *of* F*. We identify* $(\mathfrak{g}')^*(F) \subset \mathfrak{g}^*(F)$ *as the subspace fixed by the action of* $Z'(F) = Z(G')(F)$*. We say* $X^* \in \mathfrak{z}'(F)$ *is* G*-generic of depth* r *if for all* $\alpha \in \Phi(G, T) \setminus \Phi(G', T)$ *we have* $\mathrm{ord}(X^*(H_\alpha)) = -r$*.*

A related notion is the the following:

Definition 2.36. *Let* $T \subset G$ *be a maximal torus. We say that* $X \in \mathfrak{g}(F)$ *is good of depth* d *if for all* $\alpha \in \Phi(G, T)$ *we have* $\mathrm{ord}(d\alpha(X)) = d$ *or* $d\alpha(X) = 0$*.*

We can say more in the case $G = \mathrm{GL}_n$. The above notions are related in the sense that X is good of depth r if and only if $X^* \in \mathfrak{g}^*(F)$ defined via $X^*(Y) = \Lambda(\mathrm{Tr}(XY))$ is $C_G(X)$-generic of depth r. Also, if $\gamma \in G_{0+}(F)$ is good of depth r we have that $\gamma - 1 \in \mathfrak{g}_{0+}(F)$ is good of depth r.

3 Tame Supercuspidal Representations

An exhaustive construction of tame supercuspidal representations of $\mathrm{GL}_n(F)$ in the tame case was first given by Howe in [25]. Constructions of tame supercuspidal representations of general groups were later given by Adler in [1] and J. K. Yu in [42]. Yu's construction was shown to be exhaustive under certain assumptions by Ju-Lee Kim in [28], notably that the residual characteristic of F be sufficiently large.

How to compute the characters of supercuspidal representations was a long standing open problem. Character computations in the case of $\mathrm{SL}_2(F)$ were known by Sally and Shalika [37] and Gelfand and Graev [22] but additional progress in general was not made for decades. Much more recently, in [7], a character formula is established for representations given by Yu's construction. In its present form, the formula requires a compactness assumption to be used. Many terms appearing in the formula of [7] were greatly simplified in [20]. Furthermore, it was shown recently in [26] that compactness assumptions in the formula can be dropped in certain cases; also discussed is the relation between this formula and analogous results for real groups.

3.1 Admissible Representations and their Characters

Let (π, V) be an admissible representation of $G(F)$, see [24] for basic definitions. Then there is an action of the Hecke algebra $\mathcal{H}(G)$ on V which we will also denote by π such that for each $f \in \mathcal{H}(G)$ the operator $\pi(f)$ on V has finite dimensional image. We may thus define a distribution $\Theta_\pi : \mathcal{H}(G) \to \mathbb{C}$ via

$$\Theta_\pi(f) = \mathrm{Tr}_V \pi(f)$$

We call Θ_π the character of π. This distribution is very well behaved, especially when π is supercuspidal.

Representability of Characters and the Harish-Chandra Integral Formula

A basic theorem of Harish-Chandra concerning Θ_π of (π, V) is the following:

Theorem 3.1. *Suppose* $\mathrm{char}(F) = 0$. *For any* $f \in \mathcal{H}(G)$ *there is a locally constant function* Ω *on* $G(F)^{\mathrm{rss}}$ *such that*

$$\Theta_\pi(f) = \int_{G(F)} \Omega(\gamma) f(\gamma) d\gamma$$

By convention, we write $\Omega(\gamma) = \Theta_\pi(\gamma)$.

Moreover, the function $\gamma \mapsto |D_G(\gamma)|^{\frac{1}{2}} \Theta_\pi(\gamma)$ *may be extended to a locally bounded function on* $G(F)$.

Proof. See [24]. □

In fact, this result holds in arbitrary characteristic:

Corollary 3.2. *The result of* Theorem 3.1 *holds even for* $\mathrm{char}(F) > 0$.

Proof. See the Appendix B of [3] by Gopal Prasad. □

Theorem 3.1 and Corollary 3.2 allow us to henceforth consider the character Θ_ψ as a function. Moreover, since Θ_ψ is locally constant and therefore continuous, its values are unambiguously defined for all $\gamma \in G(F)^{\mathrm{rss}}$.

If π is supercuspidal, character values of Θ_π may be computed via an integral formula.

Theorem 3.3. *Let θ be a matrix coefficient of π, $K \subset G(F)$ be a compact open subgroup of $G(F)$ and dk the Haar measure on K normalized so that $\mathrm{meas}_{dk}(K) = 1$. Then for any $\gamma \in G(F)^{\mathrm{rss}}$ we have*

$$\Theta_\pi(\gamma) = \frac{\deg(\pi)}{\theta(1)} \int_{Z(F)\backslash G(F)} \int_K \theta(\gamma^{kg}) dk d\dot{g} \tag{3.1}$$

If γ is elliptic we also have

$$\Theta_\pi(\gamma) = \frac{\deg(\pi)}{\theta(1)} \int_{Z(F)\backslash G(F)} \theta(\gamma^{g}) d\dot{g} \tag{3.2}$$

In [36] Rader and Silberger proved that (3.2) holds for any discrete series representation.

The Local Character Expansion

The character Θ_π of a supercuspidal representation π has a geometric expansion for elements γ of sufficiently large depth. We have the following

Theorem 3.4 (Local Character Expansion)**.** *Suppose* $\mathrm{char}(F) = 0$*. Moreover, suppose that char*$(\mathfrak{f})$ *is sufficiently large with respect to G (see* [24]*). Then the following properties hold:*

- *The set of nilpotent orbits $\mathcal{N}$ of $\mathfrak{g}(F)$ is finite. Moreover, for each nilpotent orbit $\mathcal{O} \subset \mathfrak{g}(F)$ there is a $G(F)$-invariant radon measure $\mu_{\mathcal{O}}$ on $\mathcal{O}$ whose Fourier transform $\widehat{\mu}_{\mathcal{O}}$ is represented by a locally constant function on $\mathfrak{g}(F)$ which we will also denote by $\widehat{\mu}_{\mathcal{O}}$.*

- *Let π be a supercuspidal representation of $G(F)$. Then there are constants $c_{\mathcal{O}}(\pi) \in \mathbb{C}$ for each $\mathcal{O} \in \mathcal{N}$ such that*

$$\Theta_\pi(\exp(X)) = \sum_{\mathcal{O}\in\mathcal{N}} c_{\mathcal{O}}(\pi)\widehat{\mu}_{\mathcal{O}}(X) \tag{3.3}$$

for any $X \in \mathfrak{g}(F)$ with $\exp(X) \in G(F)^{rss}$ and $d(\exp(X)) > d(\pi)$.

Proof. See [24]. □

The result of Theorem 3.4 still holds in the case $\mathrm{char}(F) > 0$ under certain restrictions on $\mathrm{char}(\mathfrak{f})$ and when a suitable analog of the exponential map exists, see [5] for details.

Exponential Maps

Suppose $\mathrm{char}(F) = 0$. Then the map exp may be defined without issue by its well-known power series. Under certain restrictions on (?) the map exp has the following properties:

Proposition 3.5. *The map* exp *is defined on* $\mathfrak{g}_{0^+}(F)$ *and is such that*

$$\exp(\mathfrak{g}_{0^+}(F)) = G_{0^+}(F)$$

It is invertible with inverse log*, also defined by its well-known power series. Moreover, for any* $\mathbf{x} \in \mathcal{B}(G, F)$ *we have*

$$\exp(\mathfrak{g}_{\mathbf{x},0^+}(F)) = G_{\mathbf{x},0^+}(F)$$

We have that $d(\exp(X)) = d(X)$ *for any* $X \in \mathfrak{g}_{0^+}(F)$ *and* $d_{\mathbf{x}}(\exp(X)) = d_{\mathbf{x}}(X)$ *for any* $X \in \mathfrak{g}_{\mathbf{x},0^+}(F)$*. Also, for any* $g \in G(F)$ *we have* $\exp(X^g) = \exp(X)^g$*.*

For $r, s \in \widetilde{\mathbb{R}}$ *with* $0 < \frac{r}{2} \leq s \leq r$ *we have that Moy-Prasad isomorphism*

$$\mathbf{e}_{x,r;s} : G_{\mathbf{x},r;s}(F) \to \mathfrak{g}_{\mathbf{x},r;s}(F)$$

may be realized explicitly by the restriction of log *to* $G_{\mathbf{x},r}(F)$*.*

Proof. See [3]. □

The existence of a suitable analogue of the exponential map in the case $\mathrm{char}(F) > 0$ appears to still be open in general, see [3], but such a map is known to exist in a number of cases. Families of maps with not all but many of the properties of the exponential map, however, exist in abundance:

Proposition 3.6. *Let* $T \subset G$ *be a maximal torus defined over* F *which splits over a tame extension of* F*. Then there are isomorphisms* $\mathbf{e}_{T,x} : \mathfrak{g}_{x,0^+}(F) \to G_{x,0^+}(F)$ *for each* $x \in \mathcal{B}(T, F)$ *which, for* $X \in \mathfrak{g}_{x,r}(F)$ *with* $r > 0$*, satisfy*

(1) $\mathbf{e}_{T,x}(X) \in \mathbf{e}_{x,r;2r}(X)\mathfrak{g}_{x,2r}(F)$.

(2) *For* $g \in G(F)$ *we have* $X^g \in \mathfrak{g}_{x,r}(F)$ *if and only if* $\mathbf{e}_{T,x}(X)^g \in G_{x,r}(F)$*.*

(3) *If* H *is a reductive* F*-subgroup of* G *containing* T *then* $\mathbf{e}_{T,x}(\mathfrak{h}_{x,0^+}) = H_{x,0^+}(F)$*.*

Proof. See Appendix A of [7]. □

In the case $G = \mathrm{GL}_n$ we do in fact have an analogue of exp which works in arbitrary characteristic

Proposition 3.7. *Let* $G = \mathrm{GL}_n$ *and suppose* $\mathrm{char}(\mathfrak{f}) > n$*. Then in* Proposition 3.5 *the map* exp *may be replaced with the map*

$$X \mapsto \sum_{k=0}^{n-1} \frac{X^k}{k!} \tag{3.4}$$

Moreover, we have

$$\widehat{\mu}_{X^*}^{G(F)}(\log(\gamma)) = \widehat{\mu}_{X^*}^{G(F)}(\gamma - 1) \tag{3.5}$$

Proof. See [3] for the first fact and [18] for the second. □

The Depth of Admissible Representations

To an admissible representation (π, V_π) of $G(F)$ we may use the building $\mathcal{B}(G,F)$ to assign a positive number to π which we call the depth of π.

Theorem 3.8. *Let (π, V_π) be an admissible representation of $G(F)$. There exists a number $d(\pi) \in \mathbb{R}_{\geq 0}$ with the following property: there is some $x \in \mathcal{B}(G,T)$ such that $V_\pi^{G_{x,d(\pi)+}(F)} \neq \emptyset$ and $d(\pi)$ is unique with this property.*

Proof. See [34]. □

In the case where $G = T$ is a torus and $\psi \in \widehat{T(F)}$ we have that $d(\psi) \in \mathbb{R}_{\geq 0}$ is the smallest number for which $\psi|_{T_{d(\psi)+}(F)}$ is trivial but $\psi|_{T_{d(\psi)}(F)}$ is non-trivial.

A crucial observation concerning the depth of representations is that depth is preserved by compact induction:

Proposition 3.9. *Let $K \subset G(F)$ be a compact open subgroup and let σ be a representation of K. Then for $\pi = \text{c-Ind}_K^{G(F)}\sigma$ we have $d(\pi) = d(\sigma)$.*

3.2 Yu's Construction

The basic building blocks of the representations Yu constructs are a point $y \in \mathcal{B}(G,F)$, a tower of Levi subgroups of G, and characters thereof of increasing depths. It is noted in [26] that this process is analogous to the notion of a Howe factorization of a character of a maximal torus of GL_n in [25].

Compact Subgroups Defined by Concave Functions

In order to construct the characters we need, we must use the Moy-Prasad filtrations of the groups $G_{\mathbf{x}}(F)$ in order to build some special compact open subgroups of $G(F)$.

We begin with the following definition:

Definition 3.10. *We say $\vec{G} = (G^0, \ldots, G^d)$ is a tamely ramified twisted Levi sequence if $G^0 \subset G^1 \subset \ldots \subset G^d = G$ is a tower of subgroups of G defined over F such that there is a tamely ramified extension E of F such that each $G^i \otimes E$ is a Levi subgroup of $G \otimes E$.*

Let $\vec{G} = (G^0, \ldots, G^d)$ be a tamely ramified twisted Levi sequence and $T \subset G^0$ a maximal torus which splits over a tamely ramified extension of F. Also, choose a point $\mathbf{x}_T \in \mathcal{A}(G,T,F)$.

Let $\vec{r} = (r_0, \ldots, r_d)$ be a sequence of real numbers such that for some $0 \leq j \leq d$ we have

$$0 \leq r_0 = \ldots = r_j, \frac{1}{2}r_j \leq r_{j+1} \leq \ldots \leq r_d$$

We refer to such a $\vec{r}$ as an admissible sequence. Letting $\Phi^i = \Phi(G^i, T)$ we set $f_{\vec{r}} : \Phi \cup \{0\} \to \mathbb{R}$ via $f_{\vec{r}}(0) = r_0$ and $f_{\vec{r}}(\alpha) = r_i$ if $\alpha \in \Phi^i \setminus \Phi^{i-1}$. We have that $f_{\vec{r}}$ is a concave function in the sense of [12].

Theorem 3.11. *We define the subgroup $\vec{G}_{\mathbf{x}_T,\vec{r}}(E)$ via*

$$\vec{G}_{\mathbf{x}_T,\vec{r}}(E) = \left\langle T_{f_{\vec{r}}(0)}(E), U_{\alpha,f_{\vec{r}}(\alpha)}(E) : \alpha \in \Phi \right\rangle$$

and we set

$$\vec{G}_{\mathbf{x}_T,\vec{r}}(F) = \vec{G}_{\mathbf{x}_T,\vec{r}}(E) \cap G(F)$$

The following properties of the group $\vec{G}_{\mathbf{x}_T,\vec{r}}(F)$ hold:

a) *There is a group scheme $\vec{\mathcal{G}}_{\mathbf{x}_T,\vec{r}}$ defined over $\mathcal{O}_F$ with generic fibre G and $\vec{\mathcal{G}}_{\mathbf{x}_T,\vec{r}}(\mathcal{O}_F) = \vec{G}_{\mathbf{x}_T,\vec{r}}(F)$. The Lie algebra $\mathrm{Lie}(\vec{\mathcal{G}}_{\mathbf{x}_T,\vec{r}})$ is such that $\mathrm{Lie}(\vec{\mathcal{G}}_{\mathbf{x}_T,\vec{r}})(\mathcal{O}_E) = \vec{\mathfrak{g}}_{\mathbf{x}_T,\vec{r}}(E)$ for*

$$\vec{\mathfrak{g}}_{\mathbf{x}_T,\vec{r}}(E) = \mathfrak{t}_{f_{\vec{r}}(0)}(E) \bigoplus_{\alpha\in\Phi} \mathfrak{u}_{\alpha,f_{\vec{r}}(\alpha)}(E)$$

b) *If $\vec{r}$ is such that $r_0 \leq r_1 \leq \ldots r_d$ we may write*

$$\vec{G}_{\mathbf{x}_T,\vec{r}}(E) = T_{r_0}(E) G^1_{\mathbf{x}_T,r_1}(E) \cdots G^d_{\mathbf{x}_T,r_d}(E)$$

c) *Let $\vec{r}$ and $\vec{s}$ be admissible sequences such that*

$$0 < r_i \leq s_i \leq \min\{r_i, \ldots, r_d\} + \min\{r_1, \ldots, r_d\} \quad \textit{for } 0 \leq i \leq d$$

Setting

$$\vec{G}_{\mathbf{x}_T,\vec{r};\vec{s}}(F) = \vec{G}_{\mathbf{x}_T,\vec{r}}(F)/\vec{G}_{\mathbf{x}_T,\vec{s}}(F)$$

and

$$\vec{\mathfrak{g}}_{\mathbf{x}_T,\vec{r};\vec{s}}(F) = \vec{\mathfrak{g}}_{\mathbf{x}_T,\vec{r}}(F)/\vec{\mathfrak{g}}_{\mathbf{x}_T,\vec{s}}(F)$$

we have the following analogue of the Moy-Prasad isomorphism; there are isomorphisms

$$\mathbf{e}_{\mathbf{x}_T,\vec{r};\vec{s}} : \vec{G}_{\mathbf{x}_T,\vec{r};\vec{s}}(F) \to \vec{\mathfrak{g}}_{\mathbf{x}_T,\vec{r};\vec{s}}(F)$$

Proof. See [42], [43]. □

Generic Characters and Weil Representations

Fix a linear character $\Lambda : F \to \mathbb{C}$ of depth 0. We begin with the following definitions

Definition 3.12. *We say that a character ϕ of $G_x(F)$ of depth r is represented by $X^* \in \mathfrak{g}^*(F)$ if for all $\gamma \in G_{x,\frac{r}{2}+}(F)$ we have*

$$\psi(\gamma) = \Lambda(X^*(\mathbf{e}_x^{-1}(\gamma)))$$

Moreover, we say ψ is generic if X^* is.

Let (G', G) be a tame twisted Levi sequence and $x \in \mathcal{B}(G', F)$. Suppose that ϕ is a G-generic character of G' with $d(\psi) = r$. Let $J = (G', G)_{x,(r,\frac{r}{2})}$, $J_+ = (G', G)_{x,(r,\frac{r}{2}+)}$, $K' = G'(F) \cap \mathrm{stab}_{G(F)}([x])$ and $K'_+ = G'_{x,0^+}(F)$

Let $(V, \langle \cdot, \cdot \rangle)$ be a finite dimensional vector space over $\mathfrak{f}$ where the bracket $\langle \cdot, \cdot \rangle$ takes values in the finite dimensional vector space C over $\mathfrak{f}$. From V we may construct the Heisenberg p-group $V^\sharp$ with underlying space $V \times C$ and group action

$$(v, a) \cdot (w, b) = \left(v + w, a + b + \frac{1}{2} \langle v, w \rangle\right)$$

For any $\psi \in \widehat{C}$ it can be shown that there exists a unique irreducible representation ω_ψ of $V^\sharp$, the Heisenberg representation, with central character ψ. Moreover, letting $\mathrm{Sp}(V)$ be the subgroup of $\mathrm{GL}(V)$ preserving $\langle \cdot, \cdot \rangle$, it can be shown that there is an action of $\mathrm{Sp}(V)$ on $V^\sharp$ and that ω_ψ may be extended canonically to a representation of $\mathrm{Sp}(V) \ltimes V^\sharp$, the Weil representation, which we will also denote via ω_ψ.

In §11 of [42] it is shown that the group J/J_+ endowed with the bracket $\langle x, y \rangle \mapsto \widehat{\phi}(xyx^{-1}y^{-1})$ is a symplectic space and the associated Heisenberg p-group $(J/J_+)^\sharp$ may be identified with $J/\ker \widehat{\phi}$. Moreover, it is carefully shown in Theorem 11.5 of [42] that the Weil representation of $\mathrm{Sp}(J/J^+) \ltimes (J/\ker \widehat{\phi})$ may be pulled back to a representation $\widetilde{\phi}$ of $K' \ltimes J$ with the property that $\widetilde{\phi}|_{1 \ltimes J_+}$ is $\widehat{\phi}$-isotypic and $\widetilde{\phi}|_{K'_+ \ltimes 1}$ is $\mathbf{1}$-isotypic.

The Datum

Let G be a connected reductive group defined over F. A Yu-datum Σ is a quintuple $\Sigma = (\vec{G}, \mathbf{x}, \vec{r}, \vec{\phi}, \rho)$ where

- $\vec{G} = (G^0, \ldots, G^d)$ is a tower of subgroups defined over F with $G^d = G$ and such that there is a tamely ramified extension E of F such that $G \vec{\otimes} E = (G^0 \otimes E, \ldots, G^d \otimes E)$ is a tower of E-Levi subgroups of $G \otimes E$. Moreover, we assume that $Z(G^0)/Z(G)$ is F-anisotropic.

- $x \in \mathcal{A}(G, T, F)$ where T is a maximal torus of G^0 which splits over a tamely ramified extension of F.

- $\vec{r} = (r_0, \ldots, r_d)$ is a non-decreasing sequence of real numbers with $r_0 \geq 0$ if $d = 0$ and

$$0 < r_0 < \ldots < r_{d-1} \leq r_d$$

  otherwise.

- $\vec{\phi} = (\phi_0, \ldots, \phi_d)$ where ϕ_i is a generic character of $G^i_x(F)$ of depth r_i for $0 \leq i \leq d-1$. Furthermore, ϕ_i is a generic character of $G^d_x(F)$ of depth r_d if $r_d > r_{d-1}$ and is the trivial character of $G^d_x(F)$ otherwise.

- ρ is a depth zero representation of $G^0_{[x]}(F)$ such that $\rho|_{G^0_{0^+}(F)}$ is $\mathbf{1}$-isotypic and $\text{c-Ind}_{G^0_{[x]}(F)}^{G^0(F)} \rho$ is irreducible supercuspidal.

The Construction

In [42] representations π_i of $G^i(F)$ of depth r_i for each $0 \leq i \leq d$ are constructed inductively. For simplicity, we will assume $G^0 = T$, $\rho = \mathbf{1}_{T_0(F)}$ and that T is a maximal torus of G^i for each $1 \leq i \leq d$. First, we construct some subgroups: we set $K^0 = T_0(F)$ and $K^0_+ = T_{0^+}(F)$ and, for $1 \leq i \leq d$ we set

$$K^i = K^0 G^1_{x,\frac{r_0}{2}}(F) \cdots G^i_{x,\frac{r_{i-1}}{2}}(F)$$

and

$$K^i_+ = K^0_+ G^1_{x,\frac{r_0}{2}+}(F) \cdots G^i_{x,\frac{r_{i-1}}{2}+}(F)$$

Further, following [42] we set

$$J^i = (G^{i-1}, G^i)(F)_{x,r_{i-1},\frac{r_{i-1}}{2}}$$

and

$$J^i_+ = (G^{i-1}, G^i)(F)_{x,r_{i-1},\frac{r_{i-1}}{2}+}$$

By [42] we have $K^{i-1}J^i = K^i$ and $K^{i-1}_+ J^i_+ = K^i_+$. Also, set $Z^i = Z(G^i)^\circ$.

By the Moy-Prasad isomorphism we have $G^i_{x,\frac{r_i}{2}+}(F)/G^i_{x,r_i^+}(F) \simeq \mathfrak{g}_{x,\frac{r_i}{2}+}(F)/\mathfrak{g}_{x,r_i^+}(F)$. This allows us to consider $\phi_i|_{G^i_{x,\frac{r_i}{2}+}(F)}$ as a character of $\mathfrak{g}_{x,\frac{r_i}{2}+}(F)$. Denoting by $\mathfrak{n}^i$ the sum of the isotypic subspaces of $\mathfrak{g}^i$ on which Z^i acts non-trivially, we have that

$$\mathfrak{g}_{x,\frac{r_i}{2}+}(F) = \mathfrak{g}^i_{x,\frac{r_i}{2}+}(F) \oplus \mathfrak{n}^i(F)$$

As such, we may extend $\phi_i|_{\mathfrak{g}_{x,\frac{r_i}{2}+}(F)}$ to a character $\widehat{\phi}_i$ of $\mathfrak{g}_{x,\frac{r_i}{2}+}(F)$ by setting it to be 0 on $\mathfrak{n}^i(F)$. Subsequently, again using the Moy-Prasad isomorphism, we obtain a character of $G^i_{x,\frac{r_i}{2}+}(F)$ which we will also denote by $\widehat{\phi}_i$. Furthermore, there is a unique extension of $\widehat{\phi}_i$ to $T_0(F)G^i_x(F)G_{x,\frac{r_i}{2}+}(F)$ since we observe that $\widehat{\phi}_i|_{G^i_{x,\frac{r_i}{2}+}(F)} = \phi_i|_{G^i_{x,\frac{r_i}{2}+}(F)}$. We will also denote this extension by $\widehat{\phi}_i$.

We begin by inductively constructing representations ρ'_i and ρ_i of K^i. Suppose ρ'_{i-1} and ρ_{i-1} have already been constructed. Our discussion of Weil representations above shows that there exists a representation $\widetilde{\phi}_{i-1}$ of $K^{i-1} \ltimes J^i$ such that the restriction of $\widetilde{\phi}_{i-1}$ to $1 \ltimes J^i_+$ is $(\widehat{\phi}_{i-1}|_{J^{i+1}_+})$-isotypic and the restriction of $\widetilde{\phi}_{i-1}$ to $K^{i-1}_+ \ltimes 1$ is $\mathbf{1}$-isotypic. Now if we denote by $\inf(\phi_{i-1})$ the inflation of $\phi|_{K^{i-1}}$ to $K^{i-1} \ltimes J^i$, it is shown in §4 of [42] that $\inf(\phi_{i-1}) \otimes \widetilde{\phi}_{i-1}$ factors through the natural map $K^{i-1} \ltimes J^i \to K^{i-1}J^i = K^i$. Denote by ϕ'_{i-1} the representation of K^i whose inflation to $K^{i-1} \ltimes J^i$ is $\inf(\phi_{i-1}) \otimes \widetilde{\phi}_{i-1}$.

Inflating ρ'_{i-1} to a representation $\inf(\rho'_{i-1})$ of K^i (see [42] for details) we set $\rho'_i = \inf(\rho'_{i-1}) \otimes \phi'_{i-1}$ and $\rho_i = \rho'_i \otimes (\phi_i|_{K^i})$.

Now we have the following:

Theorem 3.13 (Yu, 2001)**.** *For $0 \le i \le d$ the representation*

$$\pi_i = \text{c-Ind}_{K^i}^{G^i(F)} \rho_i' \otimes \phi_i \tag{3.6}$$

is irreducible supercuspidal of depth r_i.

3.3 Characters of Tame Supercuspidal Representations

Here we discuss the character formula for supercuspidal representations arising via Yu's construction which first appeared in [7] and was later refined in [20]. We discuss a number of preliminaries before we present the formula.

Good Product Expansions

To use the character formula of [7] we require the notion of a good product expansion introduced in [6]. For $\gamma \in G(F)^{\text{rss}}$ and $r \ge 0$ we wish to write $\gamma = \gamma_{<r}\gamma_{\ge r}$ where $\gamma_{<r}$ and $\gamma_{\ge r}$ commute and where $d(\gamma_{\ge r}) \ge r$. Colloquially, we think of $\gamma_{<r}$ as the head of γ and $\gamma_{\ge r}$ as the tail.

We begin with the following definition

Definition 3.14. *For $r' \ge 0$ we call $\underline{\gamma} = (\gamma_i)_{0 \le i < r'}$ a good sequence in $G(F)$ if all but finitely $\gamma_i = 1$ and there is a tame-modulo-centre torus $S \subset G$ defined over F such that $\gamma_i \in S(F)$ and $\gamma_i \in \mathcal{G}_i^{G(F)}$ for each $0 \le i < r'$. Moreover, for $r \le r'$ we set*

$$C_{G(F)}^{(r)}(\underline{\gamma}) = C_{G(F)}\left(\prod_{0 \le i < r} \gamma_i\right) \tag{3.7}$$

We may use the notion of a good sequence to define the types of approximations we seek:

Definition 3.15. *Let $\gamma \in G(F)^{rss}$. We say that a good sequence $\underline{\gamma} = (\gamma_i)_{0 \le i < r'}$ is an r-approximation to $\gamma \in G(F)$ if there is some $x \in \mathcal{B}(C_{G(F)}^{(r)}, F)$ such that $\gamma \in \left(\prod_{0 \le i < r} \gamma_i\right) G_{\mathbf{x},r}(F)$. Sometimes we refer to such a $\underline{\gamma}$ as a (r, x)-approximation.*

We say that $\underline{\gamma}$ is a normal r-approximation (normal (r, x)-approximation) if, further, we have that $\gamma \in C_{G(F)}^{(r)}(\underline{\gamma})$.

Let us note, once and for all, that we must only consider normal r-approximations: by Theorem 9.2 of [6] we have that if γ has an r-approximation then it has a normal r-approximation.

If $\underline{\gamma}$ is a normal r-approximation to γ we write

$$\gamma_{<r} = \prod_{0 \le i < r} \gamma_i \tag{3.8}$$

and

$$\gamma_{\ge r} = \gamma_{<r}^{-1}\gamma \tag{3.9}$$

Though normal approximations are not unique, the decomposition $\gamma = \gamma_{<r}\gamma_{\geq r}$ is as unique as we require; we have

Proposition 3.16. *If $\underline{\gamma}$ and $\underline{\gamma}'$ are normal r-approximations to γ we have*

$$C^{(r)}_{G(F)}(\underline{\gamma}) = C^{(r)}_{G(F)}(\underline{\gamma}')$$

Henceforth we define

$$C^{(r)}_{G(F)}(\gamma) = C^{(r)}_{G(F)}(\underline{\gamma})$$

We observe that $C^{(r)}_{G(F)}(\gamma)$ is a reductive subgroup of $G(F)$ defined over F.

Proof. See [6]. □

We also introduce the following subset of $\mathcal{B}(G,F)$ discussed in [6] and [7]:

Definition 3.17. *For $\gamma \in G(F)^{rss}$ and $\underline{\gamma}$ a normal r-approximation to γ we define*

$$\mathcal{B}_r(\gamma) = \left\{ x \in \mathcal{B}(C^{(r)}_{G(F)}(\gamma), F) : d_x(\gamma_{\geq r}) \geq r \right\}$$

For a normal r-approximation $\underline{\gamma}$ for γ we have that $\mathcal{B}_r(\gamma)$ is the set of all x for which $\underline{\gamma}$ is a normal (r,x)-approximation.

Normalized Characters and Orbital Integrals

Instead of the standard character Θ_π of an admissible representation π, occasionally we prefer to consider the normalized character Φ_π defined as follows:

Definition 3.18. *For an admissible representation π of $G(F)$ we define*

$$\Phi_\pi(\gamma) = |D_G(\gamma)|^{\frac{1}{2}}\Theta_\pi(\gamma)$$

If π is supercuspidal, arises from Yu's construction and satisfies the required techincal conditions allowing us to apply the character formula of [20], the Fourier transform of an orbital integral appears in the formula for Φ_π. Recall that for $X^* \in \mathfrak{g}^*(F)$ and $f^* \in \mathcal{C}_c^\infty(\mathfrak{g}^*(F))$ we define the orbital integral $\mu^{G(F)}_{X^*}(f^*)$ via

$$\mu^{G(F)}_{X^*}(f^*) = \int_{G(F)/C_{G(F)}(X^*)} f^*(\mathrm{Ad}^*(g)X^*)d\dot{g}$$

and that the Fourier transform $\widehat{\mu}^{G(F)}_{X^*}$ of $\mu^{G(F)}_{X^*}$ is represented by a locally constant function $\widehat{\mu}^{G(F)}_{X^*}$ on $\mathfrak{g}(F)^{\mathrm{rss}}$ in the sense that

$$\widehat{\mu}^{G(F)}_{X^*}(f) = \int_{\mathfrak{g}(F)} \widehat{\mu}^{G(F)}_{X^*}(Y)f(Y)dY \qquad (3.10)$$

for $f \in \mathcal{C}_c^\infty(\mathfrak{g}(F))$.

We defined the normalized orbital $\widehat{\iota}^{G(F)}_{X^*}(Y)$, a term which will appear in our character formulas, via

$$\widehat{\iota}^{G(F)}_{X^*}(Y) = |D_G(Y)|^{\frac{1}{2}}|D_G(X^*)|^{\frac{1}{2}}\widehat{\mu}^{G(F)}_{X^*}(Y) \qquad (3.11)$$

Definition of the Roots of Unity

A number of roots of unity are introduced into the character formula which, by §3 of [7], arise from the character theory of Weil representations of finite symplectic groups. With a view to compute them below in the case $G = \mathrm{GL}_n$, here we include their descriptions as they appear in [26].

Let $\Sigma = (\vec{G}, \mathbf{x}, \vec{r}, \vec{\phi}, \rho)$ be a Yu-datum and let $r = r_{d-1}$. Suppose $\gamma \in G(F)$ has a normal r-approximation and set $\gamma = \gamma_{<r}\gamma_{\geq r}$. We define sets

$$R_{\gamma_{<r}} = \{\alpha \in \Phi(G,T) \setminus \Phi(G',T) : \alpha(\gamma_{<r}) \neq 1\}$$

as well as

$$R_{\frac{r}{2}} = \{\alpha \in R_{\gamma_{<r}} : r \in 2\mathrm{ord}_x(\alpha)\}$$

and

$$R_{\frac{r-\mathrm{ord}(\gamma_{<r})}{2}} = \{\alpha \in R_{\gamma_{<r}} : r - \mathrm{ord}(\alpha(\gamma_{<r}) - 1) \in 2\mathrm{ord}_x(\alpha)\}$$

For a subset $S \subset \Phi(G,T)$ we denote by S_{sym} the set of those roots in S which are symmetric and by S^{sym} the set of those roots in S which are asymmetric.

We define the following roots of unity. First, we let

$$\epsilon^r(\gamma_{<r}) = \prod_{\alpha \in \Gamma \times \pm 1 \backslash \left(R_{\frac{r}{2}}\right)^{\mathrm{sym}}} \mathrm{sgn}_{\mathfrak{f}_\alpha^\times}(\alpha(\gamma_{<r})) \prod_{\alpha \in \Gamma \backslash \left(R_{\frac{r}{2}}\right)_{\text{sym, unram}}} \mathrm{sgn}_{\mathfrak{f}_\alpha^1}(\alpha(\gamma_{<r})) \tag{3.12}$$

where $\mathrm{sgn}_{\mathfrak{f}_\alpha^1}$ is the quadratic character of the elements of norm 1 of the residue field of F_α. By Remark 4.3.4 of [20] we have that $\epsilon^r|_{T(F)}$ is a character of $T(F)$ which is $\Omega(G,T)(F)$-invariant.

We let

$$\widetilde{e}(\gamma_{<r}) = \prod_{\alpha \in \Gamma \backslash \left(R_{\frac{r-\mathrm{ord}(\gamma_{<r})}{2}}\right)_{\mathrm{sym}}} (-1) \tag{3.13}$$

The most complicated root of unity appearing is

$$\epsilon_{s,r}(\gamma_{<r}) = \prod_{\alpha \in \Gamma \backslash \left(R_{\frac{r-\mathrm{ord}(\gamma_{<r})}{2}}\right)_{\text{sym, ram}}} \mathrm{sgn}_{F_{\pm\alpha}}(G_{\pm\alpha})(-\mathfrak{G})^{f_\alpha}\mathrm{sgn}_{\mathfrak{f}_\alpha^\times}(t_\alpha) \tag{3.14}$$

Here, we have that $\mathfrak{f}_\alpha$ is the residue field of F_α, f_α is the degree of the extension $[\mathfrak{f}_\alpha : \mathfrak{f}]$, and $\mathrm{sgn}_{F_{\pm\alpha}}(G_{\pm\alpha})$ is the Kottwitz sign defined in [30] of the group $G_{\pm\alpha}$ generated by the root subgroups U_α and $U_{-\alpha}$. Moreover, for an additive character Λ of F we have that $\mathfrak{G}$ is a Gauss sum depending on Λ and $t_\alpha \in \mathcal{O}_{F_\alpha}^\times$ is an element of the form

$$t_\alpha = \frac{1}{2} e_\alpha N_{F_{\pm\alpha}/F_\alpha}(w_\alpha) X_{d-1}^*(H_\alpha)(\alpha(\gamma_{<r}) - 1) \tag{3.15}$$

where X^*_{d-1} is the element of $\mathfrak{g}^*(F)$ representing ϕ_{d-1} (with respect to Λ) and where $w_\alpha \in F_\alpha^\times$ is any element of valuation $\frac{\mathrm{ord}(\alpha(\gamma_{<r})-1)-r}{2}$. The product defining $\epsilon_{s,r}(\gamma_{<r})$ is independent of choice of Λ.

To streamline notation we set

$$\epsilon_d(\gamma_{<r}) = \epsilon^r(\gamma_{<r})\widetilde{e}(\gamma_{<r})\epsilon_{s,r}(\gamma_{<r}) \tag{3.16}$$

Formulas for Supercuspidal Characters

We now state various forms of the character formula appearing in [7] and [20]. Let $H = C_{G(F)}(\gamma_{<r})$.

Let $\pi = \pi_d$ be the supercuspidal representation associated to the Yu-datum $\Sigma = (\vec{G}, \mathbf{x}, \vec{r}, \vec{\phi}, \rho)$. The following is the character formula as it appears in [20].

Theorem 3.19. *Suppose G^{d-1} is F-anisotropic mod $Z(G)$. Then*

$$\Phi_\pi(\gamma) = \phi_d(\gamma) \sum_{g \in H\backslash G(F)/G'(F) : \gamma^g \in G'(F)} \epsilon_d(\gamma^g_{<r})\Phi_{\pi_{d-1}}(\gamma^g_{<r})\widehat{\iota}^H_{{}^gX^*_\psi}(\mathbf{e}_x^{-1}(\gamma_{\geq r})) \tag{3.17}$$

where ϵ_d is the root of unity defined in (3.16)*.*

We note that if G^{d-1} is not F-anisotropic mod $Z(G)$ the term $\Phi_{\pi_{d-1}}(\gamma^g_{<r})$ is undefined if $\gamma_{<r}$ is not regular in G^{d-1}. With a view to apply our character formula more generally, we state an alternate form of the formula. Following [7] we set $K_\sigma = \mathrm{stab}_{G^{d-1}(F)}(\overline{x})G_{x,0^+}(F)$ and

$$\sigma = \mathrm{Ind}_{K^d}^{K_\sigma}\rho'_d \tag{3.18}$$

Then we have

Theorem 3.20. *Suppose G^{d-1} is F-anisotropic mod $Z(G)$. Then*

$$\Theta_\pi(\gamma) = \phi_d(\gamma) \sum_{g \in H\backslash G(F)/ : \gamma^g_{<r} \in G'(F)} \theta_\sigma(\gamma^g_{<r})\widehat{\mu}^H_{{}^gX^*_\psi}(\mathbf{e}_x^{-1}(\gamma)) \tag{3.19}$$

If $\gamma = \gamma_{\geq r}$ this becomes

$$\Theta_\pi(\gamma) = \phi_d(\gamma)\deg(\sigma)\widehat{\mu}^{G(F)}_{X^*_\psi}(\mathbf{e}_x^{-1}(\gamma)) \tag{3.20}$$

Proof. This is the second statement of Theorem 6.4 of [7]. □

Furthermore we set

$$\tau = \mathrm{Ind}_{K^d}^{\mathrm{stab}_{G(F)}(\overline{x})}\rho_d \tag{3.21}$$

A formula for θ_τ which we will make use of below is given in [7]; it has the novel feature that it does not rely on the assumption that G^{d-1} is F-anisotropic mod $Z(G)$. The form of this formula we will use is as follows:

Theorem 3.21. *For $\gamma \in G_r(F)$ the character $\theta_\tau(\gamma)$ may be computed via*

$$\theta_\tau(\gamma) = \phi_d(\gamma)\deg(\sigma)\widehat{\mu}^{\mathrm{stab}_{G(F)}(\overline{x})}_{X^*_\psi}(\mathbf{e}_x^{-1}(\gamma)) \tag{3.22}$$

Proof. This follows from the first statement of Theorem 6.4 of [7]. □

4 Character Computations for $\mathbf{GL}_n(F)$

To compute $L(\gamma, t)$, especially when n is not prime, we require very precise information about values of stable characters. We will summarize properties of the characters of supercuspidal representations and discuss techniques relevant to the computation of $L(\gamma, t)$.

In §4.5 we state a conjectural character formula for supercuspidal characters of $\mathrm{GL}_n(F)$ in cases where (3.19) does not necessarily apply. We give evidence that this formula holds by establishing it in several cases. Moreover, we show that this formula follows from a small number of natural assumptions.

4.1 Howe Factorizations and Notations

Here we will fix notation for the remainder of the chapter. Let F be a nonarchimedean local field and let E be a finite extension of F with $n = [E : F]$. We will assume E/F is cyclic so that $\Gamma_{E/F} \simeq \mathbb{Z}/n\mathbb{Z}$. Write $G = \mathrm{GL}_n$ and let $T \subset G$ be a maximal torus defined over F with $T(F) \simeq E^\times$.

Following [25] we make the following definition:

Definition 4.1. *We say $\psi \in \widehat{E^\times}$ is admissible if*

- *There is no subextension $E \supsetneq M \supset F$ and $\phi \in \widehat{M^\times}$ such that $\psi = \phi \circ N_{E/M}$.*
- *If $\psi|_{1+\varpi_E \mathcal{O}_E} = (\phi \circ N_{E/M})|_{1+\varpi_M \mathcal{O}_M}$ for a subextension $E \supsetneq M \supset F$ and $\phi \in \widehat{M^\times}$ then E is unramified over M.*

We say $\psi \in \widehat{T(F)}$ is admissible if it is admissible when considered as a character of $E^\times$ via the identification $T(F) \simeq E^\times$.

Admissible characters have the following property:

Proposition 4.2. *Let $\psi \in \widehat{T(F)} \simeq \widehat{E^\times}$ be admissible. Then ψ has a factorization of the form*

$$\psi = \psi_0 \psi_1 \cdot \ldots \cdot \psi_{d-1} \psi_d$$

with $\psi_i = \varphi_i \circ N_{E/E^i}$ for fields

$$E = E^0 \supsetneq E^1 \supsetneq \ldots \supsetneq E^{d-1} \supsetneq E^d = F$$

with $d(\psi_0) < d(\psi_1) < \ldots < d(\psi_{d-1})$.

Proof. See [25]. □

We call a factorization of the form appearing in Proposition 4.2 a Howe factorization for ψ. We also make the following definition:

Definition 4.3. *We say that $\psi \in \widehat{T(F)} \simeq \widehat{E^\times}$ is strongly primitive if it is admissible and has a Howe factorization of the form $\psi = \psi_0 \psi_1$.*

Assume ψ is such that $\psi_d = 1$ so that $d(\psi) = d(\psi_{d-1})$. Let $T^i \subset T$ be such that $T^i(F) = (E^i)^\times$ and let $G^i = C_G(T^i)$. We have that $G^i \simeq \mathrm{Res}_{E^i/F}\mathrm{GL}_{[E:E^i]}$ and that we have a determinant map $\det_i : G^i \to T^i$. Define $\phi_i = \varphi_i \circ \det_i \in \widehat{G^i(F)}$. Now, letting $r_i = d(\psi_i) = d(\phi_i)$ for $0 \leq i \leq d-1$, we define $\widetilde{\pi}_\psi$ to be supercupsidal representation corresponding to the datum $\Sigma = (\vec{G}, x, \vec{r}, \vec{\phi}, \mathbf{1}_{T_0(F)})$ where

- $\vec{G} = (G^0, G^1, \ldots, G^{d-1}, G^d)$
- $x = x_T \in A(G, T, F)$
- $\vec{r} = (r_0, r_1, \ldots, r_{d-1}, r_{d-1})$
- $\vec{\phi} = (\phi_0, \phi_1, \ldots, \phi_{d-1}, \mathbf{1})$

It is shown in §3.5 of [23] that the representation $\widetilde{\pi}_\psi$ coincides with the supercuspidal representation associated to ψ in [25]. Moreover, though Howe factorizations are not unique, it is shown that the representation $\widetilde{\pi}_\psi$ is independent of the choice of Howe factorization.

Let $X^*_\psi \in \mathfrak{t}^*(F)$ be the element representing ψ in the sense of Definition 3.12. If $X^*_i \in (\mathfrak{g}^i)^*(F)$ represents ϕ_i for $0 \leq i \leq d$ we have that X^*_i may be chosen to lie in $\mathfrak{t}(F)$ and such that we have

$$X^*_\psi = \sum_{i=0}^{d} X^*_i \tag{4.1}$$

since $\psi_d = 1$ we have $X^*_d = 0$. Also, for $s \geq 0$ we set

$$X^*_{\psi,<s} = \sum_{i:r_i<s} X^*_i \tag{4.2}$$

We further note that the character ψ^g represented by the element ${}^gX_\psi$.

We set

$$\Theta_\psi(\gamma) = \Theta_{\widetilde{\pi}_\psi}(\gamma) \tag{4.3}$$

and

$$\Phi_\psi(\gamma) = \Phi_{\widetilde{\pi}_\psi}(\gamma) \tag{4.4}$$

In order to compute values of Θ_ψ we require additional notation. Let

$$\vec{G}^{(i)} = (G^0, G^1, \ldots, G^i)$$

be the truncation of $\vec{G}$ to its first $i+1$ components; define $\vec{\phi}^{(i)}$ and $\vec{r}^{(i)}$ analogously. Let $\Sigma^{(i)} = (\vec{G}^{(i)}, x_T, \vec{r}^{(i)}, \vec{\phi}^{(i)}, \mathbf{1}_{T_0(F)})$ and $\widetilde{\pi}^i_\psi$ be the corresponding representation of $G^i(F)$ with character $\Theta^i_\psi(\gamma)$ and normalized character $\Phi^i_\psi(\gamma)$. For simplicity we write $G^{d-1} = G'$, $\Theta^{d-1}_\psi = \Theta'_\psi$, $\Phi^{d-1}_\psi = \Phi'_\psi$ and $r = r_{d-1}$.

Denote by $\epsilon_d(\psi, \gamma_{<r})$ the root of unity for $\widetilde{\pi}_\psi$ defined in (3.16) of §3.3.

For $\gamma \in T(F)$ we observe that $x \in \mathcal{B}_r(\gamma)$. Fix a normal (r,x)-approximation $(\gamma_j)_{0\le j<r}$ for γ and let

$$\gamma_{<r_i} = \prod_{0\le j<r_i} \gamma_j$$

for $0 \le i \le d-1$ and $\gamma_{\ge r_i} = \gamma_{<r_i}^{-1}\gamma$. Further, we set

$$H^i = C_{G^{i+1}}(\gamma_{<r_i})$$

For brevity we simply write H^i for $H^i(F)$.

We also let σ and τ be as in §3.3.

Special Representations

Equation (4.3) assigns to each admissible character $\psi \in \widehat{T(F)}$ a character Θ_ψ of a supercuspidal representation of $G(F)$. We wish to also consider characters which are not admissible; we make the following definition:

Definition 4.4. *We say $\psi \in \widehat{E^\times}$ is subadmissible if there is a subextension $E \supsetneq E' \supset F$ such that $\psi = \varphi \circ N_{E/E'}$ where $\varphi \in \widehat{(E')^\times}$ is admissible.*

We say $\psi \in \widehat{T(F)}$ is subadmissible if it is subadmissible when considered as a character of $E^\times$ via the identification $T(F) \simeq E^\times$.

In [17] it is shown that to a subadmissible character $\psi \in \widehat{T(F)}$ we may assign a discrete series representation, a generalized special representation, constructed in [10]. We denote this representation by $\widetilde{\pi}_\psi$ and denote by Θ_ψ and Φ_ψ its character and normalized character.

We note in the case where T is unramified that every $\psi \in \widehat{T(F)}$ is admissible or subadmissible. We may thus define a Howe factorization for any $\psi \in \widehat{T(F)}$.

4.2 Computations of Roots of Unity

We first compute the action of $\Gamma_{E/F}$ on $\Phi = \Phi(G,T)$. By [26] we have in our case that

$$W_F = \Omega(G,T)(F) = N_{G(F)}(T(F))/T(F)$$

We will see that for any $\gamma \in T(F)$ the orbits γ^{W_F} and $\gamma^{\Gamma_{E/F}}$ coincide. We have

Proposition 4.5. *The group W_F is cyclic of order n and possesses a generator w_σ with $t^{w_\sigma} = t^\sigma$ for all $t \in T(F)$. Moreover, $\Phi_{\mathrm{sym}} = \emptyset$ unless n is even, in which case $|\Phi_{\mathrm{sym}}| = n$ and $\Phi_{\mathrm{sym}} = \Gamma_{E/F} \cdot \alpha$ for any $\alpha \in \Phi_{\mathrm{sym}}$.*

Proof. Let $\gamma \in T(F)$ be regular semisimple and let $\sigma^i(\theta)$ for $0 \le i \le n-1$ be the roots of γ. Let $\mathcal{B}$ be the F-basis $\mathcal{B} = \{1, \theta, \ldots, \theta^{n-1}\}$ of E and let $[\gamma]_{\mathcal{B}}$ be the matrix representation of

γ with respect to this basis. Let V_θ be the Vandermonde matrix

$$V_\theta = \begin{pmatrix} 1 & \theta & \cdots & \theta^{n-1} \\ 1 & \sigma(\theta) & \cdots & \sigma(\theta^{n-1}) \\ \vdots & \vdots & & \vdots \\ 1 & \sigma^{n-1}(\theta) & \cdots & \sigma^{n-1}(\theta^{n-1}) \end{pmatrix}$$

Also, let $D_\gamma = \operatorname{diag}(\theta, \sigma(\theta), \ldots, \sigma^{n-1}(\theta))$. By direct computation we see that

$$V_\theta \cdot [\gamma]_{\mathcal{B}} = D_\gamma \cdot V_\theta$$

since the ij-entry of either matrix can be seen to equal $\sigma^{i-1}(\sigma^j)$. Since

$$V_\theta [\gamma]_{\mathcal{B}} V_\theta^{-1} \in \mathcal{D}_n(E)$$

It follows that $V_\theta T(F) V_\theta^{-1} \subset C_{\mathrm{GL}_n(E)}(D_\gamma) = \mathcal{D}_n(E)$ and hence that conjugation by V_θ diagonalizes $T(F)$. As such, we may realize the root system Φ explicitly as

$$\Phi = \{\widetilde{\alpha}_{i,j} : 1 \leq i \neq j \leq n\} \tag{4.5}$$

for

$$\widetilde{\alpha}_{i,j}(\gamma') = \alpha_{i,j}(V_\theta \gamma' V_\theta^{-1})$$

We see that σ permutes the rows of V_θ; define $P_\sigma \in \mathrm{Perm}_n$ via

$$\sigma(V_\theta) = P_\sigma V_\theta \tag{4.6}$$

Notably, we have $\sigma^{-1}(V_\theta) = P_\sigma^{-1} V_\theta$. For $\gamma' \in T(F)$ with $V_\theta \gamma' V_\theta^{-1} = \operatorname{diag}(\theta', \ldots, \sigma^{n-1}(\theta'))$ and $\sigma(\gamma') = \gamma'$ we compute

$$\begin{aligned} \sigma * \widetilde{\alpha}_{i,j}(\gamma') &= \sigma(\widetilde{\alpha}_{i,j}(\sigma^{-1}(\gamma'))) \\ &= \sigma(\widetilde{\alpha}_{i,j}(\gamma')) \\ &= \sigma\left(\frac{\sigma^i(\theta)}{\sigma^j(\theta)}\right) \\ &= \widetilde{\alpha}_{i+1,j+1}(\gamma) \end{aligned}$$

so that $\sigma * \widetilde{\alpha}_{i,j} = \widetilde{\alpha}_{i+1,j+1}$. It follows that $-\alpha \in \Gamma_{E/F} \cdot \alpha$ if and only if $\alpha = \widetilde{\alpha}_{i,j}$ with $j \equiv i + \frac{n}{2} \bmod n$ which can only occur if n is even. Furthermore, we have $\Phi_{\mathrm{sym}} = \Gamma_{E/F} \cdot \widetilde{\alpha}_{1,1+\frac{n}{2}}$ in this case, establishing the second claim.

We now see that W_F has a set of generators lying in $V_\theta^{-1} \mathrm{Perm}_n V_\theta$. Moreover, for $P_\tau \in \mathrm{Perm}_n$ we have that $V_\theta^{-1} P_\tau V_\theta$ has image in W_F if and only if $\sigma(V_\theta^{-1} P_\tau V_\theta) = V_\theta^{-1} P_\tau V_\theta$ and we have

$$\sigma(V_\theta^{-1} P_\tau V_\theta) = V_\theta^{-1} P_\sigma^{-1} P_\tau P_\sigma V_\theta$$

so that P_σ must commute with P_τ. The centralizer of P_σ in Perm_n is $\langle P_\sigma \rangle$, a group of order n.

Letting w_σ be the image of $V_\theta^{-1}P_\sigma V_\theta$ in W_F and for γ' as above, we have that

$$(\gamma')^\sigma = V_\theta^{-1}\mathrm{diag}(\sigma(\theta'),\dots,\sigma^n(\theta'))V_\theta$$

so we may compute

$$(\gamma')^{w_\sigma} = V_\theta^{-1}P_\sigma\mathrm{diag}(\theta',\dots,\sigma^{n-1}(\theta'))P_\sigma^{-1}V_\theta = V_\theta^{-1}\mathrm{diag}(\sigma(\theta'),\dots,\sigma^n(\theta'))V_\theta = (\gamma')^\sigma$$

□

Understanding the action of $\Gamma_{E/F}$ on Φ allows us to compute various subroot systems of Φ corresponding to intermediate extensions $E \supset E' \supset F$:

Corollary 4.6. *Let $E \supset E' \supset F$ be a subextension with $E' = E^{\sigma^k}$ with $[E : E'] = m$. Let $T' \subset T$ be such that $T'(F) = (E')^\times$. Then $\widetilde{\alpha}_{i,j} \in \Phi(G, T')$ if and only if $i \not\equiv j \mod k$. Moreover, we have*

$$|\Phi(G, T')| = n^2 - nm$$

Proof. The second claim follows immediately from the first.

We have that $\widetilde{\alpha}_{i,j} \in \Phi(G, T')$ if and only if $\widetilde{\alpha}_{i,j}$ is non-trivial on T'. Moreover, $\gamma' \in T'(F)$ if and only if $\theta' \in E'$. If $\widetilde{\alpha}_{i,j}(\gamma') = 1$ we have

$$1 = \widetilde{\alpha}_{i,j}(\gamma') = \frac{\sigma^i(\theta')}{\sigma^j(\theta')}$$

so that $i \equiv j \mod m$. On the other hand, if $i \equiv j \mod m$ then $\widetilde{\alpha}_{i,j}(\gamma') = 1$ for all $\gamma' \in T'(F)$. □

We now compute the root of unity ϵ_d. We break into cases of increasing complexity:

- n odd

  In this case $\Phi_{\mathrm{sym}} = \emptyset$ so that $\epsilon_{s,r} = \widetilde{e} = 1$. Therefore $\epsilon_d(\psi, \gamma_{<r}) = \epsilon^r(\gamma_{<r})$.

- n even, E unramfied

  We again have $\epsilon_{s,r} = 1$ since E is unramified. Since Φ_{sym} is a single Γ orbit we compute

$$\widetilde{e}(\gamma_{<r}) = \begin{cases} -1 & d(\gamma) - d(\psi) \text{ is even} \\ 1 & d(\gamma) - d(\psi) \text{ is odd} \end{cases}$$

  Therefore $\epsilon_d(\psi, \gamma_{<r}) = (-1)^{d(\gamma)-d(\psi)}\epsilon^r(\gamma_{<r})$.

- n even, E ramified

  In contrast to the previous cases, $\epsilon_{s,r}$ depends on more information than just the depths of $\gamma_{<r}$ and ψ. Indeed, for $n = 2$ we compute

$$\mathrm{sgn}_{\mathfrak{f}_\alpha^\times}(t_\alpha) = \mathrm{sgn}_{\mathfrak{f}^\times}(\beta_\psi v_{\gamma_{<r}})$$

for t_α defined in (3.15) and where

$$\gamma_{<r} = \begin{pmatrix} a_\gamma & v_{\gamma_{<r}}\varpi^{d(\gamma_{<r})+\frac{1}{2}} \\ v_{\gamma_{<r}}\varpi^{d(\gamma_{<r})-\frac{1}{2}} & a_\gamma \end{pmatrix}$$

and X_ψ^* is given by $Y \mapsto \mathrm{Tr}(X_\psi Y)$ for

$$X_\psi = \begin{pmatrix} 0 & \beta_\psi\varpi^{d(\psi)+\frac{1}{2}} \\ \beta_\psi\varpi^{d(\psi)-\frac{1}{2}} & 0 \end{pmatrix}$$

The computation of this term for $n > 2$ is similar but more complicated. We omit these calculations.

For $0 \leq i \leq d$ define roots of unity ϵ_i and $\epsilon_i^{r_i}$ for each representation π_i of $\widetilde{G}^i(F)$ for $0 \leq i \leq d-1$. Define

$$\epsilon^j(\psi,\gamma) = \prod_{i=1}^{j} \epsilon_j(\psi,\gamma_{<r_{i-1}}) \tag{4.7}$$

and set

$$\epsilon_\psi(\gamma) = \epsilon^d(\psi,\gamma) \tag{4.8}$$

Also let

$$\epsilon_\psi^j(\gamma) = \prod_{i=1}^{d} \epsilon_j^{r_j}(\psi,\gamma_{<r_{i-1}}) \tag{4.9}$$

We may use the calculations above to give simple formulas for $\epsilon_\psi(\gamma)$ in various cases. We have

Proposition 4.7. *Let $\gamma \in T(F)$.*

a) *If E is unramified then*

$$\epsilon_\psi(\gamma) = \epsilon_\psi^r(\gamma)\prod_{i=1}^{d}(-1)^{(n-1)\max\{d(\gamma)-r_i,0\}}$$

As such, $\epsilon_\psi(\gamma)$ depends only on the depth of γ and the depths of the characters in the Howe factorization of ψ. Moreover, $\epsilon^d(\psi,\gamma^\sigma) = \epsilon^d(\psi,\gamma)$ for all $\sigma \in \Gamma$.

b) *If n is odd then*

$$\epsilon_\psi(\gamma) = \epsilon_\psi^r(\gamma)$$

4.3 Some Orbital Integral Computations

Suppose T is unramified. We collect some results which will enable us to compute a character table for $\mathrm{GL}_\ell(F)$ in the unramified case. Further, we establish a generalization of Theorem 3.20 for regular elements of the torus which requires no compactness assumptions on $\vec{G}$.

The following vanishing results which appear in [18] allow us to simplify our orbital integral calculations considerably

Lemma 4.8. *We have*

a) *For* $W \in \mathfrak{g}_{x,m}(F) \setminus (\mathfrak{z}_m(F) + \mathfrak{g}_{x,m^+}(F))$ *for* $m < r$. *If* $W^{G_x(F)} \cap (\mathfrak{t}_m(F) + \mathfrak{g}_{x,\frac{m+r}{2}}) = \emptyset$ *then*

$$\int_{G_x(F)} \widehat{\psi}(W^k)dk = 0$$

b) *If* n *is prime,* $Y \in \mathfrak{g}(F)$ *is such that* $d^+(Y) = r$, $Y^{G(F)} \cap \mathfrak{g}_{x,r}(F)$ *and* $Y^{G(F)} \cap \mathfrak{t}(F) = \emptyset$ *then*

$$\widehat{\mu}^{G(F)}_{X_\psi}(Y) = 0$$

c) *For* $Y \in \mathfrak{t}(F)$ *with* $d(Y) = d^+(Y) = r$ *and* $g \in G(F)$ *for which* $W = {}^gY$ *is such that* $W \notin \mathfrak{g}_{x,r}(F)$ *we have*

$$\int_{G_x(F)} \widehat{\psi}(W^k)dk = 0$$

Proof. Parts a) and b) are Lemmata 5.2.5 and 5.2.7 of [18], respectively. We note that a) does not require n to be prime. Part c) follows from a) precisely as in the proof of Lemma 5.2.13 of [18]. □

The following results, which are of a similar flavor, will help us compute character values at elements whose depth equals the depth of the representation.

Proposition 4.9. *Let* $Y \in \mathfrak{t}_{0^+}(F)^{\mathrm{reg}}$ *and* $\gamma = 1 + Y \in T_{0^+}(F)$.

a) *We have*

$$\widehat{\mu}^{G(F)}_{X_\psi}(Y) = \int_{Z\setminus G(F)} \mathbf{1}_{G_{x,r}(F)}(\gamma^g)\widehat{\psi}(\gamma^g)d\dot{g}$$

b) *If* $\gamma \in T_r(F)$ *is good of depth* $r > 0$ *we have*

$$\widehat{\mu}^{G(F)}_{X_\psi}(Y) = \int_{G_x(F)} \widehat{\psi}(\gamma^k)dk = \sum_{k \in G(\mathfrak{f})} \widehat{\psi}(\gamma^k)$$

Proof. a) follows immediately from Lemma B4 of [7] and the fact that γ is elliptic.

For b), we use a) and Fubini's theorem to compute

$$\begin{aligned}\widehat{\mu}^{G(F)}_{X_\psi}(Y) &= \int_{Z\backslash G(F)} \mathbf{1}_{G_{x,r}(F)}(\gamma^g)\widehat{\psi}(\gamma^g)d\dot{g} \\ &= \int_{Z\backslash G(F)}\int_{G_x(F)} \mathbf{1}_{G_{x,r}(F)}(\gamma^{gk})\widehat{\psi}(\gamma^{gk})dkd\dot{g}\end{aligned}$$

Since γ is good of depth r by Proposition 2.31 we have that $\gamma^{gk} \in G_{x,r}(F)$ implies $g \in G_x(F)$. Now

$$\begin{aligned}&\int_{Z\backslash G(F)}\int_{G_x(F)} \mathbf{1}_{G_{x,r}(F)}(\gamma^{gk})\widehat{\psi}(\gamma^{gk})dkd\dot{g} \\ =&\int_{Z\backslash ZG_x(F)}\int_{G_x(F)} \widehat{\psi}(\gamma^{gk})dkd\dot{g} \\ =&\int_{G_x(F)} \widehat{\psi}(\gamma^{k})dk\end{aligned}$$

□

We now use the facts we have collected above to prove a generalization of Theorem 3.20:

Proposition 4.10. *Let $\gamma \in T(F)$ be regular with $\gamma = \gamma_{\geq r}$. Then*

$$\Phi_\psi(\gamma) = \widehat{\iota}^{G(F)}_{X_\psi}(\gamma - 1)$$

Proof. We follow the argument of the proof of Theorem 6.4 in [7] where we substitute various convergence assumptions used there for the fact that γ is elliptic. We may realize $\pi = \text{c-Ind}^{G(F)}_{K_\sigma}\sigma$ for which the map $\dot{\theta}_\sigma(\gamma)$ defined via

$$\gamma \mapsto \begin{cases} \theta_\sigma(\gamma) & \gamma \in K_\sigma \\ 0 & \text{else}\end{cases}$$

is a matrix coefficient. By Theorem 3.3 we have

$$\Theta_\psi(\gamma) = \frac{\deg(\pi)}{\deg(\sigma)}\int_{Z\backslash G} \theta_\sigma(\gamma^g)d\dot{g}$$

Let $\mathcal{Y}$ be the set of $G_x(F)$-conjugacy classes in $G_x(F)/G_{x,r}(F)$ and let $\mathcal{Z}$ be a set of representatives of the classes in $\mathcal{Y}$. For $\eta \in \mathcal{Z}$ define

$$S_\eta = \{g \in G(F) : \gamma^g \in \eta G_{x,r}(F)\}$$

We note that each S_η is compact since $(\eta G_{x,r}(F))^{G_x(F)}$ is compact and $g \mapsto \gamma^g$ is proper. Moreover, each S_η is open since S_η is a union of left $G_x(F)$-cosets. It follows that each S_η is a finite union of left $G_x(F)$-cosets.

Let $\eta \in \mathcal{Z}$ be such that $S_\eta \neq \emptyset$ and $\eta G_{x,r} \neq G_{x,r}$. We claim that

$$\int_{Z\backslash S_\eta} \theta_\sigma(\gamma^g) d\dot{g} = 0$$

We write

$$S_\eta = \bigcup_{i=1}^{M} g_i G_x(F)$$

for some $g_1, \ldots, g_M \in G(F)$ and compute

$$\begin{aligned}
\int_{Z\backslash S_\eta} \dot{\theta}_\sigma(\gamma^g) d\dot{g} &= \text{meas}_{d\dot{g}}(ZG_x(F)) \int_{S_\eta} \dot{\theta}_\sigma(\gamma^g) dg \\
&= \text{meas}_{d\dot{g}}(ZG_x(F)) \sum_{i=1}^{M} \int_{g_i G_x(F)} \dot{\theta}_\sigma(\gamma^g) dg \\
&= \text{meas}_{d\dot{g}}(ZG_x(F)) \sum_{i=1}^{M} \int_{G_x(F)} \dot{\theta}_\sigma(\gamma^{g_i g}) dg \\
&= M \cdot \text{meas}_{d\dot{g}}(ZG_x(F)) \int_{G_x(F)} \dot{\theta}_\sigma(\eta^g) dg \\
&= M \cdot \frac{\text{meas}_{d\dot{g}}(ZG_x(F))}{\text{meas}_{dg}(K_\sigma)} \sum_{g \in K_\sigma \backslash G_x(F)} \dot{\theta}_\sigma(\eta^g) \\
&= M \cdot \frac{\text{meas}_{d\dot{g}}(ZG_x(F))}{\text{meas}_{dg}(K_\sigma)} \theta_\tau(\eta)
\end{aligned}$$

where the last line follows from the Frobenius formula for θ_τ. It suffices to show that $\theta_\tau(\eta) = 0$. By Theorem 3.21 we have

$$\theta_\tau(\eta) = \widehat{\mu}_{X_\psi^*}^{G_x(F)}(\eta - 1)$$

where we observe that

$$\widehat{\mu}_{X_\psi^*}^{G_x(F)}(\eta - 1) = \int_{G_x(F)} \widehat{\psi}((\eta - 1)^k) dk$$

Since $d_x(\eta - 1) < d_x(\gamma)$ we have by Lemma 4.8 c) that $\widehat{\mu}_{X_\psi}^{G_x(F)}(\eta - 1) = 0$.

We may suppose $1 \in \mathcal{Z}$ is the representative of the class of $G_{x,r}(F)$. By Theorem 4.4 of [7] we have that if $\gamma^g \in G_{x,r}(F)$ then

$$\theta_\sigma(\gamma^g) = \deg(\sigma) \widehat{\psi}(\gamma^g) \tag{4.10}$$

Now we compute

$$\begin{aligned}
&\frac{\deg(\pi)}{\deg(\sigma)}\int_{Z\backslash G(F)}\theta_\sigma(\gamma^g)d\dot{g}\\
=&\frac{\deg(\pi)}{\deg(\sigma)}\sum_{\eta\in\mathcal{Z}}\int_{Z\backslash S_\eta}\theta_\sigma(\gamma^g)d\dot{g}\\
=&\frac{\deg(\pi)}{\deg(\sigma)}\int_{Z\backslash S_1}\theta_\sigma(\gamma^g)d\dot{g}\\
=&\frac{\deg(\pi)}{\deg(\sigma)}\int_{Z\backslash G(F)}\mathbf{1}_{G_{x,r}(F)}(\gamma^g)\theta_\sigma(\gamma^g)d\dot{g}\\
=&\deg(\pi)\int_{Z\backslash G(F)}\mathbf{1}_{G_{x,r}(F)}(\gamma^g)\widehat{\psi}(\gamma^g)d\dot{g}\\
=&\deg(\pi)\widehat{\mu}^{G(F)}_{X^*_\psi}(\gamma-1)
\end{aligned}$$

where the final line follows from Proposition 4.9 a). □

4.4 Overview of Supercuspidal Character Computations for $\mathbf{GL}_\ell(F)$ in the Unramified Case

In order to perform our calculation of $L(\gamma,t)$ in §5 we require exhaustive character data for $\mathrm{GL}_\ell(F)$ in the unramified case. We have the following

Theorem 4.11. *Suppose $n=\ell$ is prime and T is unramified. Then for $\gamma\in G(F)^{\mathrm{rss}}$ we have*

$$\Theta_\psi(\gamma)=\begin{cases}(-1)^\ell(-1)^{(\ell-1)(d(\gamma)-d(\psi))}q^{\min\{d(\gamma),d(\psi)\}\frac{\ell^2-\ell}{2}}\sum_{\sigma\in\Gamma_{E/F}}\psi(\gamma^\sigma) & \gamma\in T(F)\setminus T_r(F)\\[2ex]
(-1)^\ell q^{\min\{d(\gamma),d(\psi)\}\frac{\ell^2-\ell}{2}}\sum_{\sigma\in\Gamma_{E/F}}\psi(\gamma^\sigma) & \gamma\in T_r(F)\\[2ex]
\sum_{\mathcal{O}\leq\mathcal{O}_\gamma}\ell(-1)^{\ell+r_{\mathcal{O}}}(r_{\mathcal{O}}-1)!q^{d(\gamma)\frac{dim(\mathcal{O}}{2}+e_{\mathcal{O}}} & \gamma\in G_{d(\psi)^+}\\[2ex]
0 & \gamma\notin G_{d(\psi)^+}(F),\gamma\notin{}^{\widetilde{G}(F)}T(F)\end{cases}$$

We break the proof of Theorem 4.11 into pieces

Shallow Elements not in the Stable Class of $T(F)$

Suppose $\gamma\in G(F)^{\mathrm{rss}}$ is such that $\gamma\notin T(F)^{G(F)}$. If $d(\gamma)<r$ we have that $\Phi_\psi(\gamma)=0$ by Theorem 3.19. If $d(\gamma)=r$ we have by Theorem 3.19 that

$$\Theta_\psi(\gamma)=\frac{1}{|D_G(\gamma)|^{\frac{1}{2}}}\widehat{\iota}^{G(F)}_{X^*_\psi}(\gamma-1)$$

which vanishes due to Lemma 4.8. This justifies the last row of the character table of Theorem 4.11.

The Local Character Expansion for $\mathbf{GL}_n(F)$

Here we compute the terms $c_{\mathcal{O}}(\widetilde{\pi}_\psi)$ and $\widehat{\mu}_{\mathcal{O}}(\gamma-1)$ appearing in the local character expansion. The facts we use concerning nilpotent orbits appear in [16]. Also, these results do not rely on n being prime.

Denote by $\mathcal{N}il$ the set of nilpotent orbits in $\mathfrak{g}(F)$. Any $\mathcal{O} \in \mathcal{N}il$ corresponds to some partition $\vec{m}$ of n in the sense that $\mathcal{O} = G \cdot X_{\vec{m}}$ where $X_{\vec{m}}$ is the nilpotent matrix in Jordan normal form corresponding to $\vec{m}$. In this case we write $\mathcal{O} = \mathcal{O}_{\vec{m}}$. If $\vec{m}' = (m_1', \ldots, m_{d'}')$ is the dual partition to $\vec{m}$ we have

$$\dim(\mathcal{O}) = n^2 - \sum_{i=1}^{d'} m_i' \tag{4.11}$$

For a Levi subgroup $M \subset G$ with Lie algebra there is a sense in which nilpotent orbits in $\mathfrak{g}$ may be induced from the 0 orbit of $\mathfrak{m}$, the Lie algebra of M; denote this orbit by $\{0\}_M^G$. In [16] it is shown that every $\mathcal{O} \in \mathcal{N}il$ is of the form $\mathcal{O} = \{0\}_M^G$ for some Levi subgroup M. If $\mathcal{O} = \{0\}_M^G$ write $M = M_{\mathcal{O}}$. If $\vec{m} = (m_1, \ldots, m_d)$ is a partition of n and $M_{\vec{m}}$ the Levi subgroup of G given by

$$M = \prod_{i=1}^{d} \mathrm{GL}_{m_i}(F)$$

it can be shown that $\{0\}_{M_{\vec{m}}}^G = \mathcal{O}_{\vec{m}'}$. Since $(\vec{m}')' = \vec{m}$ it follows from (4.11) that

$$\dim(\{0\}_{M_{\vec{m}}}^G) = n^2 - \sum_{i=1}^{d} m_i^2 \tag{4.12}$$

If $\Phi_{M_{\vec{m}}}$ denotes the root system of $M_{\vec{m}}$ we compute

$$|\Phi_{M_{\vec{m}}}| = \sum_{i=1}^{d} m_i^2 - m_i = -n + \sum_{i=1}^{d} m_i^2 \tag{4.13}$$

Letting $\Phi_{\mathcal{O}} = \Phi_{M_{\mathcal{O}}}$ we have by (4.11) and (4.13) that

$$\dim(\mathcal{O}) = |\Phi| - |\Phi_{\mathcal{O}}| \tag{4.14}$$

Let $\mathcal{O} \in \mathcal{N}il$. By [35] we have that

$$\widehat{\mu}_{\mathcal{O}}(\gamma - 1) = \sum_{s \in W_{\mathcal{O}}(\gamma)} \frac{|D_{M_{\mathcal{O}}}(\gamma^s)|^{\frac{1}{2}}}{|D_G(\gamma)|^{\frac{1}{2}}}$$

for

$$W_{\mathcal{O}}(\gamma) = \{s \in \mathrm{Hom}(A_{\mathcal{O}} \to A_\gamma) : s \text{ is injective }, s(a) = a^y \text{ for some } y \in G(F)\}$$

where A_M and A_γ are the split components of maximal tori of $M_{\mathcal{O}}$ and $C_G(\gamma)$, respectively. As such, we have $\mu_{\mathcal{O}}(\gamma - 1) = 0$ unless A_M injects into A_γ in the above sense. Letting $\mathcal{O}_\gamma = \{0\}^G_{C_G(\gamma)}$ this is equivalent to the condition $\mathcal{O} \leq \mathcal{O}_\gamma$. It follows that

$$\mathcal{O} \not\leq \mathcal{O}_\gamma \implies \widehat{\mu}_{\mathcal{O}}(\gamma - 1) = 0 \tag{4.15}$$

If $W_{\mathcal{O}}$ is non-empty we may assume $\gamma \in M_{\mathcal{O}}$. For $w_{\mathcal{O}} = |W_{\mathcal{O}}|$ we compute

$$\widehat{\mu}_{\mathcal{O}}(\gamma - 1) = w_{\mathcal{O}} \frac{|D_{M_{\mathcal{O}}}(\gamma)|^{\frac{1}{2}}}{|D_G(\gamma)|^{\frac{1}{2}}}$$

We may also compute

$$|D_G(\gamma)| = \exp_q\left(\sum_{\alpha \in \Phi} d_\alpha(\gamma)\right)$$

and

$$|D_{M_{\mathcal{O}}}(\gamma)| = \exp_q\left(\sum_{\alpha \in \Phi_{\mathcal{O}}} d_\alpha(\gamma)\right)$$

so that

$$\widehat{\mu}_{\mathcal{O}}(\gamma - 1) = w_{\mathcal{O}} \exp_q\left(\frac{1}{2} \sum_{\alpha \in \Phi \setminus \Phi_{\mathcal{O}}} d_\alpha(\gamma)\right)$$

Letting

$$e_{\mathcal{O}}(\gamma) = \frac{1}{2} \sum_{\alpha \in \Phi \setminus \Phi_{\mathcal{O}} : \alpha(\gamma_{>d(\gamma)}) \neq 1} d_\alpha(\gamma) - d(\gamma) = \frac{1}{2} \sum_{\alpha \in \Phi \setminus \Phi_{\mathcal{O}} : \alpha(\gamma_{>d(\gamma)}) \neq 1} d_\alpha(\gamma_{>d(\gamma)}) - d(\gamma)$$

we may write

$$\widehat{\mu}_{\mathcal{O}}(\gamma - 1) = w_{\mathcal{O}} q^{d(\gamma)\frac{\dim(\mathcal{O})}{2} + e_{\mathcal{O}}(\gamma)} \tag{4.16}$$

The constants $c_{\mathcal{O}}(\widetilde{\pi}_\psi)$ are computed in [35]; combining results therein with (4.14) we obtain

$$c_{\mathcal{O}}(\widetilde{\pi}_\psi) = n(-1)^{n + r_{\mathcal{O}}}(r_{\mathcal{O}} - 1)! \frac{1}{w_{\mathcal{O}}} q^{d(\psi)\frac{|\Phi| - \dim(\mathcal{O})}{2}} = n(-1)^{n + r_{\mathcal{O}}}(r_{\mathcal{O}} - 1)! \frac{1}{w_{\mathcal{O}}} q^{d(\psi)\frac{|\Phi_{\mathcal{O}}|}{2}} \tag{4.17}$$

Combining (4.15), (4.16) and (4.17) we obtain

$$\Theta_\psi(\gamma) = \sum_{\mathcal{O} \leq \mathcal{O}_\gamma} n(-1)^{n + r_{\mathcal{O}}}(r_{\mathcal{O}} - 1)! q^{d(\psi)\frac{|\Phi_{\mathcal{O}}|}{2} + d(\gamma)\frac{\dim(\mathcal{O})}{2} + e_{\mathcal{O}}(\gamma)}$$

This justifies the penultimate row of the character table of Theorem 4.11.

Character Values on the Torus

Let $\gamma \in T(F)$. If $d(\gamma) < r$ we have

$$(-1)^{(\ell-1)(d(\gamma)-d(\psi))} q^{d(\gamma)\frac{\ell^2-\ell}{2}} \sum_{\sigma \in \Gamma_{E/F}} \psi(\gamma^\sigma)$$

by Theorem 3.19 and Proposition 4.7.

If $d(\gamma) = r$ we again use Theorem 3.19 to obtain

$$\Theta_\psi(\gamma) = \frac{1}{|D_G(\gamma)|^{\frac{1}{2}}} \widehat{\iota}_{X_\psi^*}^{G(F)}(\gamma - 1) = q^{d(\gamma)\frac{\ell^2-\ell}{2}} \sum_{k \in G(\mathfrak{f})} \widehat{\psi}(\gamma^k)$$

where the second equality follows by Proposition 4.9 b). It remains to show that we have

$$\sum_{k \in G(\mathfrak{f})} \widehat{\psi}(\gamma^k) = \sum_{\sigma \in \Gamma_{E/F}} \psi(\gamma^\sigma) \tag{4.18}$$

Establishing (4.18) is surprisingly difficult. It can be shown to follow from results in §6 of [38] which rely on the global result of Kazhdan in [27]. We desire to find a direct and purely local proof of (4.18) in future.

If $d(\gamma) > r$ since γ is elliptic we have

$$\Theta_\psi(\gamma) = c_0(\widetilde{\pi}_\psi) = (-1)^\ell \ell q^{d(\psi)\frac{\ell^2-\ell}{2}} = (-1)^\ell q^{d(\psi)\frac{\ell^2-\ell}{2}} \sum_{\sigma \in \Gamma_{E/F}} \psi(\gamma^\sigma)$$

This justifies the first two rows of the character table of Theorem 4.11.

4.5 A Conjectural Character Formula

Suppose T is unramified. In §4.1 we defined a map $\psi \mapsto \Theta_\psi$ from $\widehat{T(F)}$ to a collection of characters of discrete series representations of $G(F)$. While current techniques are unable to fully compute the values of the characters Θ_ψ, even when restricted to $T(F)$, we conjecture the following:

Conjecture 4.12. *For $\psi \in \widehat{T(F)}$ and $\gamma \in T(F)$ the character value $\Theta_\psi(\gamma)$ is given by*

$$\Theta_\psi(\gamma) = |D_G(\gamma_{<r})|^{-\frac{1}{2}} |D_G(X^*_{\psi,<d(\gamma)})|^{\frac{1}{2}} \epsilon_\psi(\gamma) \sum_{\sigma \in \Gamma_{E/F}} \psi(\gamma^\sigma) \tag{4.19}$$

where ϵ_ψ is the root of unity defined in (4.8) *below.*

In this section we show that (4.19) follows from a small number of natural assumptions. We will also discuss cases where these assumptions are known to hold.

An Extension of the Supercuspidal Character Formula

The character formula (3.17) of Theorem 3.19 requires the restrictive assumption that G' be anisotropic modulo $Z(G)$ which is only satisfied when ψ is strongly primitive. To proceed, we make the following assumption

Assumption 4.13. *If* $\gamma \in T(F)^{\mathrm{rss}}$ *and* $\gamma_{<r} \in G'(F)^{\mathrm{rss}}$ *then* (3.17) *holds for* $\Phi_\psi(\gamma)$.

Assumption 4.13 is known to hold for absolutely semisimple elements:

Proposition 4.14. *If* G *is a connected reductive group defined over* F *and* $\gamma = \gamma_0 \in G(F)^{\mathrm{rss}}$ *is absolutely semisimple then* (3.17) *holds for* $\Phi_\psi(\gamma)$.

Proof. This is shown in §4.3 of [26]. □

Removing the compactness requirement from (3.17) allows one to leverage the inductive nature of the formula as demonstrated in [26] as well as in Theorem 4.19 below.

A Nice Orbital Integral Identity

Another major obstacle one encounters when trying to unwind (3.17) is the orbital integral term. Evidence from the case where n is prime suggests the following assumption:

Assumption 4.15. *Let* $T' \subset T$ *be a subtorus with* $T'(F) = M^\times$ *for an intermediate extension* $E \supset M \supset F$. *Then for* $\gamma \in T_r(F)$ *we have*

$$\widehat{\mu}_{X_\psi}^{C_{G(F)}(T'(F))}(\gamma - 1) = \sum_{\sigma \in \Gamma_{E/M}} \psi(\gamma^\sigma) \tag{4.20}$$

We note by Proposition 4.5 that we are free to replace

$$\sum_{\sigma \in \Gamma_{E/M}} \psi(\gamma^\sigma)$$

with

$$\sum_{n \in T(F) \backslash N_{G(F)}(T)} \psi(\gamma^n)$$

By Theorem 4.11 we indeed know that Assumption 4.15 holds when n is prime. We may also show rather easily that it holds in the case $T = T'$:

Proposition 4.16. *For* $\gamma = \gamma_{\geq r}$ *we have* $\widehat{\mu}_{X_\psi}^{T(F)}(\gamma - 1) = \psi(\gamma)$.

Proof. For $f \in C_c^\infty(\mathfrak{t})$ we compute

$$\begin{aligned}
\widehat{\mu}^{T(F)}_{X^*_\psi}(f) &= \mu^{T(F)}_{X^*_\psi}\left(\widehat{f}\right) \\
&= \widehat{f}(X^*_\psi) \\
&= \int_{\mathfrak{t}} f(Y)\Lambda\left\langle X^*_\psi, Y\right\rangle dY \\
&= \int_{\mathfrak{t}} f(Y)\widehat{\psi}(Y)dY
\end{aligned}$$

It follows that $\widehat{\mu}^{T(F)}_{X_\psi}(\gamma - 1) = \widehat{\psi}(\gamma - 1)$. □

In §4.6 we show that Assumption 4.15 also holds in the case $T' = \{1\}$ and $d(\gamma) > r$.

Derivation of the Formula

To deal with potential convergence issues at various stages of the formula, we make the following definition:

Definition 4.17. *We say that $\gamma \in T(F)$ is computable for ψ if γ is regular and for $0 \le i \le d-1$ at least one of the following conditions holds:*

- $\gamma_{<r_i} \in G^i(F)^{\mathrm{rss}}$
- $\gamma = \gamma_{\ge r_i}$
- *G^i is F-anisotropic modulo $Z(G)$*

We observe that if $\gamma \in T(F)$ is regular and good of depth $d(\gamma)$ then γ is computable for ψ for any $\psi \in \widehat{T(F)}$.

To perform our computations we also require the following lemma

Lemma 4.18. *Let A be the centralizer of a subtorus of T. If $\gamma \in T(F)$ and $g \in G(F)$ are such that $\gamma^g \in A(F)$ then $\gamma \in T(F)^{A(F)}$. Moreover, we have*

$$A(F)\backslash N_{G(F)}(A(F)) \simeq N_{A(F)}(T(F))\backslash N_{G(F)}(T(F))$$

Proof. This follows from the fact that $A \simeq \mathrm{Res}_{M/F}\mathrm{GL}_{[E:M]}$ and that T is a maximal torus of A. □

Now we may establish the following:

Theorem 4.19. *Assume that Assumption 4.13 and Assumption 4.15 hold. Suppose $\gamma \in T(F)$ is computable for ψ. Then*

$$\Phi_\psi(\gamma) = |D_G(\gamma_{\ge r})|^{\frac{1}{2}}|D_G(X^*_{\psi,<d(\gamma)})|^{\frac{1}{2}}\epsilon_\psi(\gamma)\sum_{\sigma\in\Gamma_{E/F}}\psi(\gamma^\sigma) \tag{4.21}$$

Proof. Let M be such that $C_G(\gamma_{<r}) = C_G(T')$ with $T'(F) \simeq M^\times$.

To begin, note that if $\gamma = \gamma_{\geq r}$ the result follows immediately from Assumption 4.15. Otherwise, we proceed by induction on d.

For $d = 1$ we have $\vec{G} = (T, G)$ so that we may apply (3.19) to compute Φ_ψ: using Lemma 4.18 we obtain

$$\begin{aligned}
\Phi_\psi(\gamma) &= \sum_{g \in H\backslash G(F)/T(F): \gamma^g_{<r} \in T(F)} \epsilon_d(\psi, \gamma^g)\Phi'_\psi(\gamma^g_{<r})\widehat{\iota_g}^H_{X^*_\psi}(\gamma_{\geq r} - 1) \\
&= \sum_{n \in N_H(T(F))\backslash N_{G(F)}(T(F))} \epsilon_d(\psi, \gamma^n)\psi(\gamma^n_{<r})\widehat{\iota_n}^H_{X^*_\psi}(\gamma_{\geq r} - 1) \\
&= |D_H(\gamma_{<r})|^{\frac{1}{2}}|D_H(X^*_\psi)|^{\frac{1}{2}} \sum_{\sigma \in \Gamma_{M/F}\backslash \Gamma_{E/F}} \epsilon_d(\psi, \gamma^\sigma)\psi(\gamma^\sigma_{<r}) \sum_{\sigma' \in \Gamma_{M/F}} \psi((\gamma^\sigma_{\geq r})^{\sigma'}) \\
&= |D_H(\gamma_{<r})|^{\frac{1}{2}}|D_H(X^*_\psi)|^{\frac{1}{2}}\epsilon_\psi(\gamma) \sum_{\sigma \in \Gamma_{E/F}} \psi(\gamma^\sigma)
\end{aligned}$$

Now suppose $d > 1$ and assume (4.21) holds for Φ'_ψ and that $\gamma_{<r} \in G'(F)^{\text{rss}}$ so that $C_{G'(F)}(\gamma_{<r}) = T(F)$. By Assumption 4.13 we may write

$$\Phi_\psi(\gamma) = \sum_{g \in H\backslash G(F)/G'(F): \gamma^g_{<r} \in G'(F)} \epsilon_d(\psi, \gamma^g)\Phi'_\psi(\gamma^g_{<r})\widehat{\iota_g}^H_{X^*_\psi}(\gamma_{\geq r} - 1) \tag{4.22}$$

If $\gamma^g_{<r} \in G'(F)$ we have by Lemma 4.18 that $\gamma^g_{<r} \in T(F)^{G'(F)}$ which allows us to rewrite (4.22) as

$$\Phi_\psi(\gamma) = \sum_{g \in H\backslash N_{G(F)}(H)/N_{G'(F)}(T(F))} \epsilon_d(\psi, \gamma^g_{<r})\Phi'_\psi(\gamma^g_{<r})\widehat{\iota_g}^H_{X^*_\psi}(\gamma_{\geq r} - 1) \tag{4.23}$$

Letting

$$C = |D_H(\gamma_{\geq r})|^{\frac{1}{2}}|D_H(X^*_\psi)|^{\frac{1}{2}}$$

we use Lemma 4.18 and Assumption 4.15 to compute

$$\begin{aligned}
\Phi_\psi(\gamma) &= \sum_{g \in H\backslash N_{G(F)}(H)/N_{G'(F)}(T(F))} \epsilon_d(\psi, \gamma^g_{<r})\Phi'_\psi(\gamma^g_{<r})\widehat{\iota_g}^H_{X^*_\psi}(\gamma_{\geq r} - 1) \\
&= C \cdot \sum_{g \in H\backslash N_{G(F)}(H)/N_{G'(F)}(T(F))} \epsilon_d(\psi, \gamma^g_{<r})\Phi'_\psi(\gamma^g_{<r}) \sum_{h \in T(F)\backslash N_H(T(F))} \psi(\gamma^{hg}_{\geq r}) \\
&= C \cdot \sum_{g \in H\backslash N_{G(F)}(H)/N_{G'(F)}(T(F))} \sum_{h \in T(F)\backslash N_H(T(F))} \epsilon_d(\psi, \gamma^{hg})\psi(\gamma^{hg}_{\geq r})\Phi'_\psi(\gamma^{hg}_{<r}) \\
&= C \cdot \sum_{g \in N_H(T)\backslash N_{G(F)}(T)/N_{G'(F)}(T(F))} \sum_{h \in T(F)\backslash N_H(T(F))} \epsilon_d(\psi, \gamma^{hg})\psi(\gamma^{hg}_{\geq r})\Phi'_\psi(\gamma^{hg}_{<r}) \\
&= C \cdot \sum_{g \in T(F)\backslash N_{G(F)}(T)/N_{G'(F)}(T(F))} \epsilon_d(\psi, \gamma^g)\psi(\gamma^g_{\geq r})\Phi'_\psi(\gamma^g_{<r})
\end{aligned}$$

Applying (4.21) to $\Phi'_\psi(\gamma^g)$ we obtain

$$\begin{aligned}
&C \cdot \sum_{g \in T(F)\backslash N_{G(F)}(T)/N_{G'(F)}(T(F))} \epsilon_d(\psi,\gamma^g)\psi(\gamma^g_{\geq r})\Phi'_\psi(\gamma^g_{<r}) \\
=&CC' \cdot \sum_{g \in T(F)\backslash N_{G(F)}(T)/N_{G'(F)}(T(F))} \epsilon_d(\psi,\gamma^g)\psi(\gamma^g_{\geq r}) \sum_{h \in N_{G'(F)}(T(F))/T(F)} \epsilon^{d-1}(\psi,\gamma^{gh}_{<r})\psi(\gamma^{gh}_{<r}) \\
=&CC' \cdot \sum_{g \in T(F)\backslash N_{G(F)}(T)/N_{G'(F)}(T(F))} \sum_{h \in N_{G'(F)}(T(F))/T(F)} \epsilon_d(\psi,\gamma^g)\epsilon^{d-1}(\psi,\gamma^{gh}_{<r})\psi(\gamma^g_{\geq r})\psi(\gamma^{gh}_{<r}) \\
=&CC' \cdot \sum_{g \in T(F)\backslash N_{G(F)}(T)/N_{G'(F)}(T(F))} \sum_{h \in N_{G'(F)}(T(F))/T(F)} \epsilon_d(\psi,\gamma^{gh})\epsilon^{d-1}(\psi,\gamma^{gh}_{<r})\psi(\gamma^{gh}_{\geq r})\psi(\gamma^{gh}_{<r}) \\
=&CC' \cdot \sum_{g \in T(F)\backslash N_{G(F)}(T)/N_{G'(F)}(T(F))} \sum_{h \in N_{G'(F)}(T(F))/T(F)} \epsilon^{d}(\psi,\gamma^{gh})\psi(\gamma^{gh}) \\
=&CC' \cdot \sum_{g \in T(F)\backslash N_{G(F)}(T)} \epsilon^{d}(\psi,\gamma^{g})\psi(\gamma^{g})
\end{aligned}$$

for

$$C' = |D_{H'}(\gamma_{\geq r_{d-2}})|^{\frac{1}{2}}|D_{H'}(X^*_{\psi,<d(\gamma_{<r})})|^{\frac{1}{2}}$$

Lastly, we observe that

$$CC' = |D_G(\gamma_{\geq r})|^{\frac{1}{2}}|D_G(X^*_{\psi,<d(\gamma)})|^{\frac{1}{2}}$$

completing the proof. □

We may now compute the un-normalized character as follows:

Corollary 4.20. *Maintain the notation and assumptions of Theorem* 4.19. *We have*

$$\Phi_\psi(\gamma) = |D_G(\gamma_{<r})|^{-\frac{1}{2}}|D_G(X^*_{\psi,<d(\gamma)})|^{\frac{1}{2}}\epsilon_\psi(\gamma)\sum_{\sigma \in \Gamma_{E/F}} \psi(\gamma^\sigma) \tag{4.24}$$

4.6 Computation of $c_0(\widetilde{\pi}_\psi)$

For our computations in §6 it will be beneficial to have an explicit formula for the constant terms $c_0(\widetilde{\pi}_\psi)$ of our representations. We will also show that (assumption) holds for...

$$\widehat{\mu}^{G(F)}_{X^*_\psi}(\gamma - 1) = n \tag{4.25}$$

We proceed by computing $c_0(\widetilde{\pi}_\psi)$ in two ways.

If $\gamma \in T(F)^{\text{reg}}$ and $d(\gamma) > r$ we have by Proposition 4.10 that

$$c_0(\widetilde{\pi}_\psi) = |D_G(\gamma)|^{\frac{1}{2}}\widehat{\iota}^{G(F)}_{X^*_\psi}(\gamma - 1) = |D_G(X^*_\psi)|^{\frac{1}{2}}\widehat{\mu}^{G(F)}_{X^*_\psi}(\gamma - 1)$$

where we compute

$$
\begin{aligned}
|D_G(X_\psi^*)| &= \prod_{\alpha\in\Phi(G,T^0)} |d\alpha^\wedge(X_\psi^*)| \\
&= \prod_{i=0}^{d-1} \prod_{\alpha\in\Phi(G,T^i)\setminus\Phi(G,T^{i+1})} |d\alpha^\wedge(X_\psi^*)| \\
&= \prod_{i=0}^{d-1} \prod_{\alpha\in\Phi(G,T^i)\setminus\Phi(G,T^{i+1})} |d\alpha^\wedge(X_{\psi_i}^*)| \\
&= \prod_{i=0}^{d-1} q^{r_i(|\Phi(G,T^i)|-|\Phi(G,T^{i+1})|)} \\
&= \exp_q\left(\sum_{i=0}^{d-1} r_i\left(|\Phi(G,T^i)|-|\Phi(G,T^{i+1})|\right)\right)
\end{aligned}
$$

It follows that

$$
c_0(\widetilde{\pi}_\psi) = \widehat{\mu}^{G(F)}_{X_\psi^*}(\gamma-1)\exp_q\left(\frac{1}{2}\sum_{i=0}^{d-1} r_i\left(|\Phi(G,T^i)|-|\Phi(G,T^{i+1})|\right)\right) \tag{4.26}
$$

Furthermore, by Proposition 4.5 we have

$$
\Phi(G,T^i) = \{\alpha\in\Phi(G,T) : \alpha|_{T^i}\neq 1\} = \left\{\alpha^E_{i,j} : i\not\equiv j \bmod [E^i:F]\right\}
$$

so that

$$
|\Phi(G,T^i)| = n^2 - n[E:E^i]
$$

It follows that we may rewrite (4.26) as

$$
c_0(\widetilde{\pi}_\psi) = \widehat{\mu}^{G(F)}_{X_\psi^*}(\gamma-1)\exp_q\left(\frac{n}{2}\sum_{i=0}^{d-1} r_i\left([E:E^{i+1}]-[E:E^i]\right)\right) \tag{4.27}
$$

In [17] the constant term $c_0(\widetilde{\pi}_\psi)$ is computed via different methods and is shown to be given by

$$
c_0(\widetilde{\pi}_\psi) = n\exp_q\left(\frac{n}{2}\sum_{i=0}^{d-1} r_i\left([E:E^{i+1}]-[E:E^i]\right)\right) \tag{4.28}
$$

Comparing (4.27) and (4.28) yields (4.25), as desired.

5 Computations for SL_ℓ, ℓ a Prime

In this section we will compute $L(\gamma, t)$ in the case of SL_ℓ for ℓ a prime and where the torus T is unramified using our character data from §4.4.

We modify our notation slightly. Let E be an unramified extension of F with $[E : F] = \ell$. Let $\widetilde{G} = \mathrm{GL}_\ell$ and $G = \mathrm{SL}_\ell$. Furthermore, we choose $\widetilde{T} \subset \widetilde{G}$ such that $\widetilde{T}(F) = E^\times$ and such that $x = x_{\widetilde{T}}$ lies in the standard alcove of $\mathcal{B}(\widetilde{G}, F)$. From §2.1 we have that $\widetilde{G}_x(F) = \widetilde{G}(\mathcal{O}_F)$ so that $G_x(F) = G(\mathcal{O})$.

We establish the following:

Theorem 5.1. *Let T be an unramified maximal torus of SL_ℓ. Then for $\gamma \in T(F)$ we have*

$$L(\gamma, t) = E_T + C_T q^{\min\{d(\gamma), d(t)\}\frac{\ell^2+\ell-2}{2}} + \frac{1}{|D_G(\gamma)|^{\frac{1}{2}}} \sum_{\beta \in \gamma^{Gal(E/F)}} \delta_\beta$$

For

$$E_T = \ell(-1)^\ell \left(|T(\mathfrak{f})| \left(\frac{q^{\frac{\ell^2-\ell}{2}} - 1}{q^{\frac{\ell^2+\ell-2}{2}} - 1} - 1 \right) - 1 \right)$$

and

$$C_T = \ell(-1)^\ell |T(\mathfrak{f})| \frac{q^{\frac{\ell^2-\ell}{2}} - 1}{q^{\frac{\ell^2+\ell-2}{2}} - 1}$$

5.1 Some Comments on Distributions and Principle Value Integrals

The standard Fourier transform $\mathcal{F} : L^1(T) \to \mathcal{C}(\widehat{T})$ from harmonic analysis, given by

$$(\mathcal{F}\varphi)(\psi) = \int_T \varphi(t)\psi^{-1}(t)dt$$

restricts to an isomorphism

$$\mathcal{F} : \mathcal{C}_c^\infty(T) \to \mathcal{C}_c^\infty(\widehat{T})$$

We write $\mathcal{F}(\varphi) = \widehat{\varphi}$.

Let $\mathcal{D}(T) = \mathrm{Hom}_\mathbb{C}(\mathcal{C}^\infty(T), \mathbb{C})$ and $\mathcal{D}(\widehat{T}) = \mathrm{Hom}_\mathbb{C}(\mathcal{C}_c^\infty(\widehat{T}), \mathbb{C})$ denote the spaces of distributions on T and $\widehat{T}$, respectively. For any $f \in \mathcal{C}^\infty(T)$ we define the distribution D_f via

$$\langle D_f, \varphi \rangle = \int_T f(t)\varphi(t)dt$$

We embed $\mathcal{C}^\infty(T) \hookrightarrow \mathcal{D}(T)$ via the map $f \mapsto D_f$. We similarly embed $\mathcal{C}^\infty(\widehat{T}) \hookrightarrow \mathcal{D}(\widehat{T})$.

For any function $D : \widehat{T} \to \mathbb{C}$ we may define a distribution $\widehat{D} \in \mathcal{D}(T)$ via the formula

$$\left\langle \widehat{D}, \varphi \right\rangle = \langle D, \widehat{\varphi} \rangle$$

The following fact underpins many of our calculations:

Proposition 5.2. *Let $D : \widehat{T} \to \mathbb{C}$ be a function. For $m \geq 0$ set*

$$K_m = \left\{ \psi \in \widehat{T} : d(\psi) \leq m \right\}$$

and define $E_m \in \mathcal{D}(T)$ via

$$\langle E_m, \varphi \rangle = \left\langle \widehat{\mathbf{1}_{K_m} D}, \varphi \right\rangle$$

Suppose there is some $m_0 \geq 0$ such that $E_m = E_{m_0}$ for all $m \geq m_0$. Then $\widehat{D} = E_{m_0}$. Moreover, if the function $f(t)$ defined via

$$f(t) = \int_{K_{m_0}} D(\psi)\psi(t^{-1}) d\psi \tag{5.1}$$

is such that $f \in \mathcal{C}^\infty(T)$ then $\widehat{D} = D_f$.

Proof. For $\varphi \in \mathcal{C}^\infty(T)$ let $m \geq m_0$ be such that $\widehat{\varphi}$ is supported on K_m. Now

$$\left\langle \widehat{D}, \varphi \right\rangle = \langle D, \widehat{\varphi} \rangle = \langle \mathbf{1}_{K_m} D, \widehat{\varphi} \rangle = \left\langle \widehat{\mathbf{1}_{K_m} D}, \varphi \right\rangle = \langle E_m, \varphi \rangle = \langle E_{m_0}, \varphi \rangle \tag{5.2}$$

That $D = D_f$ if $f \in \mathcal{C}^\infty(T)$ is immediate. □

As an immediate corollary of Proposition 5.2 we obtain

Corollary 5.3. *If $D(1) = 0$ and $\widehat{D} = D_f$ for some $f \in \mathcal{C}^\infty(T)$ we compute*

$$f(t) = \lim_{m \to \infty} \sum_{d=0}^{m} \sum_{\psi : d(\psi) = d} D(\psi)\psi(t^{-1})$$

Notably, we have

$$f(t) = \sum_{d=0}^{M} \sum_{\psi : d(\psi) = d} D(\psi)\psi(t^{-1})$$

if

$$\sum_{\psi : d(\psi) = d} D(\psi)\psi(t^{-1}) = 0$$

for $d > M$.

By Proposition 5.2 and the Fourier Inversion Theorem, for any $s \in T$ and $D_s(\psi) = \psi(s)$ we have

$$\widehat{D}_s = \delta_s \tag{5.3}$$

Similarly, for

$$D'_s(\psi) = \begin{cases} 0 & d(\psi) < d(s) \\ \psi(s) & \text{else} \end{cases} \tag{5.4}$$

we have

$$\widehat{D'}_s = \delta_s - |\{\psi : d(\psi) < d(s)\}|\mathbf{1}_{\{t:d(t)\geq d(s)\}} \tag{5.5}$$

Indeed, for

$$D''_s(\psi) = \begin{cases} \psi(s) & d(\psi) < d(s) \\ \\ 0 & d(\psi) \geq d(s) \end{cases}$$

we may write $D'_s = D_s - D''_s$ so that $\widehat{D'}_s = \widehat{D}_s - \widehat{D''}_s$ and where we easily compute

$$\widehat{D''}_s(t) = \begin{cases} 0 & d(t) < d(s) \\ \\ |\{\psi : d(\psi) < d(s)\}| & d(t) \geq d(s) \end{cases}$$

5.2 Calculation of $L(\gamma, t)$

To perform our calculations we use data from Theorem 4.11. We also compute the sizes of various subsets of $|T(\mathfrak{f})|$ which arise in our computations. The calculation of $L(\gamma, t)$ is drastically different in the cases $\gamma \notin T(F)^{\widetilde{G}(F)}$ and $\gamma \in T(F)^{\widetilde{G}(F)}$; we handle these cases separately.

Counting Characters

To compute $L(\gamma, t)$ we require the ability to count the sizes of various finite sets of characters which occur naturally in our calculations.

In this case note that T is defined over $\mathfrak{f}$ and $T(F) = T_0(F)$. Moreover, $\dim(T) = \ell - 1$. The Moy-Prasad isomorphism shows

$$|T_m(F)/T_{m^+}(F)| = |\mathfrak{t}_m(F)/\mathfrak{t}_{m^+}(F)| = |\mathfrak{t}(\mathfrak{f})| = q^{\ell-1} \tag{5.6}$$

for any $m > 0$, and therefore

$$\begin{aligned} \left|\left\{\psi \in \widehat{T(F)} : d(\psi) < r\right\}\right| &= |\left(T(F)/T_r(F)\right)^\wedge| \\ &= |\left(T(F)/T_r(F)\right)| \\ &= |T(F)/T_{0^+}(F)| \prod_{i=1}^{r-1} |T_i(F)/T_{i^+}(F)| \\ &= |T(\mathfrak{f})| q^{(r-1)(\ell-1)} \end{aligned} \tag{5.7}$$

for $r > 0$. It follows that

$$|\{\psi : d(\psi) = r\}| = \begin{cases} |T(\mathfrak{f})| - 1 & r = 0 \\ \\ |T(\mathfrak{f})|(q^{\ell-1} - 1)q^{(r-1)(\ell-1)} & r > 0 \end{cases} \tag{5.8}$$

Computing $L(\gamma,t)$: The Case $\gamma \notin {}^{\widetilde{G}(F)}T(F)$

Since $\Theta_\psi(\gamma)=0$ for $d(\psi)<d(\gamma)$ the sum defining $L(\gamma,t)$ is convergent. Let

$$a_{\mathcal{O}}(\gamma)=\ell(-1)^{\ell+r_{\mathcal{O}}}(r_{\mathcal{O}})!q^{d(\gamma)\frac{\dim(\mathcal{O})}{2}+e_{\mathcal{O}}(\gamma)}$$

and

$$b_{\mathcal{O}}=\frac{|\Phi_{\mathcal{O}}|}{2}+\ell-1$$

Now we compute

- For $d(t)\le d(\gamma)$ we have

$$\begin{aligned}
&L(\gamma,t)\\
=&\sum_{r=0}^{\infty}\sum_{\psi:d(\psi)=r}\Theta_\psi(\gamma)\psi(t^{-1})\\
=&\sum_{\mathcal{O}\le\mathcal{O}_\gamma}a_{\mathcal{O}}(\gamma)\left(\sum_{d=0}^{d(t)-1}|\{\psi:d(\psi)=d\}|q^{d\frac{|\Phi_{\mathcal{O}}|}{2}}+\sum_{\psi:d(\psi)=d(t)}\psi(t^{-1})q^{d(t)\frac{|\Phi_{\mathcal{O}}|}{2}}\right)\\
=&\sum_{\mathcal{O}\le\mathcal{O}_\gamma}a_{\mathcal{O}}(\gamma)\left(|T(\mathfrak{f})|-1+|T(\mathfrak{f})|(q^{\ell-1}-1)q^{\frac{|\Phi_{\mathcal{O}}|}{2}}\sum_{d=1}^{d(t)-1}q^{(d-1)b_{\mathcal{O}}}-q^{\frac{|\Phi_{\mathcal{O}}|}{2}}|T(\mathfrak{f})|q^{(d(t)-1)b_{\mathcal{O}}}\right)\\
=&\sum_{\mathcal{O}\le\mathcal{O}_\gamma}a_{\mathcal{O}}(\gamma)\left(|T(\mathfrak{f})|-1+|T(\mathfrak{f})|(q^{\ell-1}-1)q^{\frac{|\Phi_{\mathcal{O}}|}{2}}\frac{q^{(d(t)-1)b_{\mathcal{O}}}-1}{q^{b_{\mathcal{O}}}-1}-q^{\frac{|\Phi_{\mathcal{O}}|}{2}}|T(\mathfrak{f})|q^{(d(t)-1)b_{\mathcal{O}}}\right)\\
=&\sum_{\mathcal{O}\le\mathcal{O}_\gamma}a_{\mathcal{O}}(\gamma)\left(|T(\mathfrak{f})|\left(\frac{q^{\frac{|\Phi_{\mathcal{O}}|}{2}}-1}{q^{b_{\mathcal{O}}}-1}-1\right)-1-|T(\mathfrak{f})|\frac{q^{\frac{|\Phi_{\mathcal{O}}|}{2}}-1}{q^{b_{\mathcal{O}}}-1}q^{d(t)b_{\mathcal{O}}}\right)
\end{aligned}$$

- For $d(\gamma)<d(t)$ we have

$$\begin{aligned}
&L(\gamma,t)\\
=&\sum_{d=0}^{d(\gamma)-1}\sum_{\psi:d(\psi)=r}\Theta_\psi(\gamma)\psi(t^{-1})\\
=&\sum_{\mathcal{O}\le\mathcal{O}_\gamma}a_{\mathcal{O}}(\gamma)\sum_{d=0}^{d(\gamma)-1}|\{\psi:d(\psi)=d\}|q^{d\frac{|\Phi_{\mathcal{O}}|}{2}}\\
=&\sum_{\mathcal{O}\le\mathcal{O}_\gamma}a_{\mathcal{O}}(\gamma)\left(|T(\mathfrak{f})|-1+|T(\mathfrak{f})|(q^{\ell-1}-1)q^{\frac{|\Phi_{\mathcal{O}}|}{2}}\sum_{d=1}^{d(\gamma)-1}q^{(d-1)b_{\mathcal{O}}}\right)\\
=&\sum_{\mathcal{O}\le\mathcal{O}_\gamma}a_{\mathcal{O}}(\gamma)\left(|T(\mathfrak{f})|\left(\frac{q^{\frac{|\Phi_{\mathcal{O}}|}{2}}-1}{q^{b_{\mathcal{O}}}-1}-1\right)-1-|T(\mathfrak{f})|\left(\frac{q^{\frac{|\Phi_{\mathcal{O}}|}{2}}-1}{q^{b_{\mathcal{O}}}-1}-1\right)q^{(d(\gamma)-1)b_{\mathcal{O}}}\right)
\end{aligned}$$

Letting

$$E(\gamma)=\ell(-1)^{\ell+r_{\mathcal{O}}}(r_{\mathcal{O}}-1)!\left(|T(\mathfrak{f})|\left(\frac{q^{\frac{|\Phi_{\mathcal{O}}|}{2}}-1}{q^{\frac{|\Phi_{\mathcal{O}}|}{2}+\ell-1}-1}-1\right)-1\right)q^{d(\gamma)\frac{\dim(\mathcal{O})}{2}+e_{\mathcal{O}}(\gamma)}$$

and

$$C_{\mathcal{O}}(\gamma,t)=\begin{cases}\ell(-1)^{\ell+r_{\mathcal{O}}}(r_{\mathcal{O}}-1)!|T(\mathfrak{f})|\dfrac{q^{\frac{|\Phi_{\mathcal{O}}|}{2}}-1}{q^{\frac{|\Phi_{\mathcal{O}}|}{2}+\ell-1}-1}q^{d(\gamma)\frac{\dim(\mathcal{O})}{2}+e_{\mathcal{O}}(\gamma)} & d(t)<d(\gamma)\\[2ex] \ell(-1)^{\ell+r_{\mathcal{O}}}(r_{\mathcal{O}}-1)!|T(\mathfrak{f})|\left(\dfrac{q^{\frac{|\Phi_{\mathcal{O}}|}{2}}-1}{q^{\frac{|\Phi_{\mathcal{O}}|}{2}+\ell-1}-1}-1\right)q^{d(\gamma)\frac{\dim(\mathcal{O})}{2}+e_{\mathcal{O}}(\gamma)} & d(t)\geq d(\gamma)\end{cases}$$

we have

$$L(\gamma,t)=E(\gamma)+C(\gamma,t)q^{\min\{d(t),\lceil d(\gamma)-1\rceil\}} \tag{5.9}$$

The Case $\gamma\in{}^{\widetilde{G}(F)}T(F)$

We may, and shall, assume $\gamma\in T(F)$.

Our calculations above and Corollary 5.3 show that for $d(t)<d(\gamma)$ we have

$$L(\gamma,t)=\ell(-1)^{\ell}\left(|T(\mathfrak{f})|\left(\frac{q^{\frac{\ell^2-\ell}{2}}-1}{q^{\frac{\ell^2+\ell-2}{2}}-1}-1\right)-1\right)+\ell(-1)^{\ell}|T(\mathfrak{f})|\frac{q^{\frac{\ell^2-\ell}{2}}-1}{q^{\frac{\ell^2+\ell-2}{2}}-1}q^{d(t)\frac{\ell^2+\ell-2}{2}}$$

If $d(\gamma)<d(t)$ of $d(\gamma)=d(t)$ with $\gamma^{\sigma}\not\equiv t \bmod \varpi^{d(\gamma)}$ for any $\sigma\in\Gamma$ use Corollary 5.3 and computations above to compute

$$\begin{aligned}&L(\gamma,t)\\ =&\sum_{r=0}^{\infty}\sum_{\psi:d(\psi)=r}\Theta_{\psi}(\gamma)\psi(t^{-1})\\ =&\ell(-1)^{\ell}\left(\sum_{d=0}^{d(t)-1}|\{\psi:d(\psi)=d\}|q^{d\frac{\ell^2-\ell}{2}}+\sum_{\psi:d(\psi)=d(t)}\psi(t^{-1})q^{d(t)\frac{\ell^2-\ell}{2}}+\right)\\ =&\ell(-1)^{\ell}\left(|T(\mathfrak{f})|-1+|T(\mathfrak{f})|(q^{\ell-1}-1)q^{\frac{\ell^2-\ell}{2}}\sum_{d=1}^{d(t)-1}q^{(d-1)\frac{\ell^2+\ell-2}{2}}-q^{\frac{\ell^2-\ell}{2}}|T(\mathfrak{f})|q^{(d(t)-1)\frac{\ell^2+\ell-2}{2}}\right)\\ =&\ell(-1)^{\ell}\left(|T(\mathfrak{f})|-1+|T(\mathfrak{f})|(q^{\ell-1}-1)q^{\frac{\ell^2-\ell}{2}}\frac{q^{(d(t)-1)\frac{\ell^2+\ell-2}{2}}-1}{q^{\frac{\ell^2+\ell-2}{2}}-1}-q^{\frac{\ell^2-\ell}{2}}|T(\mathfrak{f})|q^{(d(t)-1)\frac{\ell^2+\ell-2}{2}}\right)\\ =&\ell(-1)^{\ell}\left(|T(\mathfrak{f})|\left(\frac{q^{\frac{\ell^2-\ell}{2}}-1}{q^{\frac{\ell^2+\ell-2}{2}}-1}-1\right)-1-|T(\mathfrak{f})|\frac{q^{\frac{\ell^2-\ell}{2}}-1}{q^{\frac{\ell^2+\ell-2}{2}}-1}q^{d(t)\frac{\ell^2+\ell-2}{2}}\right)\end{aligned}$$

It follows that in either of these cases we have

$$L(\gamma,t) = \ell(-1)^\ell \left(|T(\mathfrak{f})| \left(\frac{q^{\frac{\ell^2-\ell}{2}}-1}{q^{\frac{\ell^2+\ell-2}{2}}-1} - 1 \right) - 1 \right) + \ell(-1)^\ell |T(\mathfrak{f})| \frac{q^{\frac{\ell^2-\ell}{2}}-1}{q^{\frac{\ell^2+\ell-2}{2}}-1} q^{\min\{d(\gamma),d(t)\}\frac{\ell^2+\ell-2}{2}} \tag{5.10}$$

It remains to consider the case where $d(\gamma) = d(t)$ and $\gamma^\sigma \equiv t \bmod \varpi^{d(\gamma)}$ for some $\sigma \in \Gamma$. To do so, given differences in the character formulas, we must consider the cases $\ell = 2$ and ℓ odd separately.

Suppose that t and γ^σ are not conjugate for any $\sigma \in \Gamma$ and $\sigma_0 \in \Gamma$ is such that $\gamma^{\sigma_0} \equiv t \bmod \varpi^{d(\gamma)}$; without loss of generality we may assume $\sigma_0 = 1$. Then $d(\gamma^\sigma t^{-1}) = d(\gamma)$ for all $\sigma \neq 1$ and let $d(\gamma t^{-1}) = M > d(\gamma)$.

- ℓ odd

  In this case, our calculation differs from the case of $\gamma \not\equiv t \bmod \varpi^{d(\gamma)}$ as follows: we must replace the term

$$\sum_{\psi:d(\psi)=d(\gamma)} \frac{1}{|D_G(\gamma)|^{\frac{1}{2}}} \psi(\gamma t^{-1})$$

  which contributed the quantity

$$-\frac{1}{|D_G(\gamma)|^{\frac{1}{2}}} |\{\psi : d(\psi) < d(\gamma)\}| \tag{5.11}$$

  must be replaced with the term

$$\sum_{d=d(\gamma)}^{M} \sum_{\psi:d(\psi)=d} \frac{1}{|D_G(\gamma)|^{\frac{1}{2}}} \psi(\gamma t^{-1})$$

  We compute

$$\begin{aligned}
&\sum_{d=d(\gamma)}^{M} \sum_{\psi:d(\psi)=d} \frac{1}{|D_G(\gamma)|^{\frac{1}{2}}} \psi(\gamma t^{-1}) \\
=&\frac{1}{|D_G(\gamma)|^{\frac{1}{2}}} \left(\sum_{d=d(\gamma)}^{M-1} \sum_{\psi:d(\psi)=d} \psi(\gamma^{\sigma_0} t^{-1}) + \sum_{\psi:d(\psi)=M} \psi(\gamma^{\sigma_0} t^{-1}) \right) \\
=&\frac{1}{|D_G(\gamma)|^{\frac{1}{2}}} \left(\sum_{d=d(\gamma)}^{M-1} |\{\psi : d(\psi) = d\}| - |\{\psi : d(\psi) < M\}| \right) \\
=&\frac{1}{|D_G(\gamma)|^{\frac{1}{2}}} \left(|\{\psi : d(\psi) < M\}| - |\{\psi : d(\psi) < d(\gamma)\}| - |\{\psi : d(\psi) < M\}| \right) \\
=&-\frac{1}{|D_G(\gamma)|^{\frac{1}{2}}} |\{\psi : d(\psi) < d(\gamma)\}|
\end{aligned} \tag{5.12}$$

  Comparing (5.11) and (5.12) it follows that (5.10) continues to hold.

- $\ell = 2$

  In this case the term

$$-\sum_{\psi:d(\psi)=d(\gamma)} \frac{1}{|D_G(\gamma)|^{\frac{1}{2}}}\psi(\gamma t^{-1})$$

is to be replaced with

$$-\sum_{d=d(\gamma)}^{m} \sum_{\psi:d(\psi)=d} \frac{1}{|D_G(\gamma)|^{\frac{1}{2}}}(-1)^{d+d(\gamma)}\psi(\gamma t^{-1})$$

We compute the latter as

$$\begin{aligned}
&-\sum_{d=d(\gamma)}^{m} \sum_{\psi:d(\psi)=d} \frac{1}{|D_G(\gamma)|^{\frac{1}{2}}}\psi(\gamma t^{-1}) \\
&= -(-q)^{d(\gamma)}\left((q+1)(q-1)\sum_{d=d(\gamma)}^{M-1}(-1)^d q^{d-1} - (q+1)(-1)^M q^{M-1}\right) \\
&= -(-q)^{d(\gamma)}\left((-q-1)(q-1)\sum_{d=d(\gamma)}^{M-1}(-q)^{d-1} - (q+1)(-q)^{M-1}\right) \\
&= -(-q)^{d(\gamma)}\left((q-1)((-q)^{M-1} - (-q)^{d(\gamma)-1}) + (q+1)(-q)^{M-1}\right) \\
&= -(q-1)q^{2d(\gamma)-1} + 2(-1)^{d(\gamma)+M}q^{d(\gamma)+M}
\end{aligned}$$

Putting everything together, we obtain

$$L(\gamma,t) = 2(-1)^{d(\gamma)+M}q^{d(\gamma)+M}$$

We note that this is exactly what we desire; a quick calculation shows that this quantity is equal to

$$2\frac{\operatorname{sgn}(\operatorname{Tr}(\gamma)-\operatorname{Tr}(t))}{|\operatorname{Tr}(\gamma)-\operatorname{Tr}(t)|}$$

Now, suppose ℓ is odd and γ and t are stably conjugate; assume $\gamma = t$. It is in this case that we see that $L(\gamma,t)$ cannot be represented by a smooth function.

For $\sigma \in \Gamma$ define $D_{\gamma,\sigma}(\psi)$ via $D_{\gamma,\sigma}(\psi) = \Theta_\psi(\gamma^\sigma)$ so that

$$\Theta_\psi(\gamma) = \sum_{\sigma\in\Gamma} D_{\gamma,\sigma}(\psi) \tag{5.13}$$

For $\sigma \neq 1$, similarly to the calculation of (5.10) we obtain

$$\widehat{D}_{\gamma,\sigma}(\gamma) = (-1)^\ell\left(|T(\mathfrak{f})|\left(\frac{q^{\frac{\ell^2-\ell}{2}}-1}{q^{\frac{\ell^2+\ell-2}{2}}-1}-1\right)-1\right) + (-1)^\ell|T(\mathfrak{f})|\frac{q^{\frac{\ell^2-\ell}{2}}-1}{q^{\frac{\ell^2+\ell-2}{2}}-1}q^{d(\gamma)\frac{\ell^2+\ell-2}{2}} \tag{5.14}$$

Moreover, we have

$$D_{\gamma,1} = D^{\sim} + \frac{1}{|D_G(\gamma)|^{\frac{1}{2}}} D'_\gamma \tag{5.15}$$

for

$$D^{\sim}(\psi) = \begin{cases} \Theta_\psi(\gamma) & d(\psi) < d(\gamma) \\ \\ 0 & \text{else} \end{cases} \tag{5.16}$$

and D'_γ as defined in (5.4). Computing

$$\widehat{D}^{\sim}(t) = (-1)^\ell \left(|T(\mathfrak{f})| \left(\frac{q^{\frac{\ell^2-\ell}{2}} - 1}{q^{\frac{\ell^2+\ell-2}{2}} - 1} - 1 \right) - 1 \right) + (-1)^\ell |T(\mathfrak{f})| q^{\frac{\ell^2-\ell}{2}} \frac{q^{(d(\gamma)-1)\frac{\ell^2+\ell-2}{2}} - 1}{q^{\frac{\ell^2+\ell-2}{2}} - 1} \tag{5.17}$$

and $\widehat{D}'_\gamma$ as in (5.5) we obtain

$$\begin{aligned} &L(\gamma, t) \\ =&\ell(-1)^\ell \left(|T(\mathfrak{f})| \left(\frac{q^{\frac{\ell^2-\ell}{2}} - 1}{q^{\frac{\ell^2+\ell-2}{2}} - 1} - 1 \right) - 1 \right) + \ell(-1)^\ell |T(\mathfrak{f})| \frac{q^{\frac{\ell^2-\ell}{2}} - 1}{q^{\frac{\ell^2+\ell-2}{2}} - 1} q^{d(\gamma)\frac{\ell^2+\ell-2}{2}} + \frac{1}{|D_G(\gamma)|^{\frac{1}{2}}} \delta_\gamma \end{aligned}$$

6 The Case $\mathbf{SL}_n(F)$ for n Composite

Let E be an unramified extension of F with $[E:F] = n$. Let $\widetilde{G} = \mathrm{GL}_\ell$ and $G = \mathrm{SL}_\ell$. Furthermore, we choose $\widetilde{T} \subset \widetilde{G}$ such that $\widetilde{T}(F) = E^\times$ and such that $x = x_{\widetilde{T}}$ lies in the standard alcove of $\mathcal{B}(\widetilde{G}, F)$ so that $\widetilde{G}_x(F) = \widetilde{G}(\mathcal{O}_F)$ and $G_x(F) = G(\mathcal{O})$.

For $\psi \in \widehat{T(F)}$ with Howe factorization

$$\psi = \psi_0\psi_1 \cdot \ldots \cdot \psi_{d-1}\psi_d$$

set $E^\psi = E^{d-1}$. For $E \supset M \supset F$ we define

$$L^M(\gamma, t) = \sum_{\psi : E^\psi = M} \Theta_\psi(\gamma)\psi(t^{-1}) \tag{6.1}$$

so that

$$L(\gamma, t) = \sum_{E \supset M \supset F} L^M(\gamma, t) \tag{6.2}$$

The distributions $L^M(\gamma, t)$ may be computed by a sort of induction which relies on knowing formulas for analogous distributions for subextensions of M containing F.

We compute $L^M(\gamma, t)$ in the two simplest cases: $M = E$ and when $[M:E]$ is prime. We then give partial formulas for $L(\gamma, t)$ in the case where n is the product of two primes.

6.1 Decomposition of Character Groups

Let $M \supsetneq F$ be an extension of F contained in E and let $m = [M:F]$ and $\mathfrak{m}$ denote the residue field of M. We have an exact sequence

$$1 \;\to\; \ker(N_{E/M}) \;\to\; \ker(N_{E/F}) \;\overset{N_{E/M}}{\to}\; \ker(N_{M/F}) \;\to\; 1 \tag{6.3}$$

which gives rise to the dual sequence

$$1 \;\to\; \ker(N_{M/F})^\wedge \;\overset{\phi \mapsto \phi \circ N_{E/M}}{\to}\; \ker(N_{E/F})^\wedge \;\overset{\psi \mapsto \psi|_{\ker(N_{E/M})}}{\to}\; \ker(N_{E/M})^\wedge \;\to\; 1 \tag{6.4}$$

Let $T^M \subset T$ be the subtorus defined over F with $T^M(F) \simeq \ker(N_{M/F})$ and $S^M \subset T$ the subtorus defined over M with $S^M(M) \simeq \ker(N_{E/M})$. Then by (6.3) and (6.4) we have analogous sequences

$$1 \;\to\; S^M(M) \;\to\; T(F) \;\overset{N_{E/M}}{\to}\; T^M(F) \;\to\; 1 \tag{6.5}$$

with dual sequence

$$1 \;\to\; T^M(F)^\wedge \;\overset{\phi \mapsto \phi \circ N_{E/M}}{\to}\; T(F)^\wedge \;\overset{\psi \mapsto \psi|_{S^M(M)}}{\to}\; S^M(M)^\wedge \;\to\; 1 \tag{6.6}$$

Denote by f the map $\psi \mapsto \psi|_{S^M(M)}$. Letting

$$B_{M,r} = \left\{\psi \in T(F)^\wedge : d(\psi) = r, E^\psi = M\right\}$$

we see that

$$f(B_{M,r}) = \left\{\psi' \in S^M(M)^\wedge : d(\psi') < r\right\}$$

For $\psi \in S_{M,r}$ we may write $\psi = \psi'\psi''$ with $d(\psi') < r$ and $E^{\psi'} \supsetneq M$ and $\psi'' \in T^M(F)^\wedge$ with ψ'' strongly primitive and $d(\psi'') = r$. Let

$$B''_{M,r} = \left\{\phi \in (T^M_r(F)/T^M_{r^+}(F))^\wedge : \phi \text{ strongly primitive}\right\}$$

Then the map $F : B_{M,r} \to \left\{\psi' \in S^M(M)^\wedge : d(\psi') < r\right\} \times B''_{M,r}$ defined via

$$F(\psi) = (\psi'|_{S^M(M)}, \psi''|_{T^M(F)})$$

is surjective and $C_{M,r}$-to-one with

$$C_{M,r} = \left|\left(T^M(F)/T^M_r(F)\right)^\wedge\right| = \left|T^M(F)/T^M_r(F)\right| = |T^M(\mathfrak{f})|q^{(r-1)(m-1)} \tag{6.7}$$

For $H^M = C_G(T^M)$ we have $H^M \simeq \mathbf{Res}_{M/F}\mathrm{SL}_{\frac{n}{m}}$ wherein S^M is a maximal elliptic torus. We observe the following:

Proposition 6.1. *For $\psi = \psi'\psi''$ as above we compute*

$$\Theta^G_\psi(\gamma) = \frac{m}{n}(-1)^{\frac{n}{m}}\Theta^{H^M}_{\psi'}(\gamma_{<r})\Theta^G_{\psi''}(\gamma_{\geq r})$$

for any (ψ', ψ'') with $F(\psi) = \left(\psi'|_{S^M(M)}, \psi''|_{T^M(F)}\right)$.

We note that T^M is defined over $\mathfrak{f}$, S^M is defined over $\mathfrak{m}$ and moreover we compute that

$$T^M(\mathfrak{f})S^M(\mathfrak{m}) = \frac{q^m - 1}{q - 1} \cdot \frac{q^n - 1}{q^m - 1} = \frac{q^n - 1}{q - 1} = T(\mathfrak{f}) \tag{6.8}$$

We also require a few additional facts. Let $\mu : \mathbb{N} \to \{-1, 1, 0\}$ be the Möbius function defined via

$$\mu(n) = \begin{cases} 1 & \text{if } n \text{ is square-free with an even number of prime factors} \\ -1 & \text{if } n \text{ is square-free with an odd number of prime factors} \\ 0 & \text{else} \end{cases} \tag{6.9}$$

which satisfies the well-known property

$$\sum_{d|n} \mu(d) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{else} \end{cases} \tag{6.10}$$

Now we have

Lemma 6.2. *Let M be an extension of F with $E \supset M \supsetneq F$. Then for any t with $d(t) \leq r$ we have*

$$\sum_{\psi : E^\psi = M, d(\psi) = r} \psi(t^{-1}) = \mu([M:F]) \sum_{\psi : d(\psi) < r} \psi(t^{-1})$$

Proof. Assume the result holds for all $M \supsetneq L \supsetneq F$ and let $m = [M:F]$. We note that the set

$$\{\psi : d(\psi) < r\} \cup \{\psi : E^\psi \subset M, d(\psi) = r\}$$

is a group on which $\psi \mapsto \psi(t^{-1})$ is a non-trivial character. It follows that

$$\begin{aligned}
\sum_{\psi : E^\psi = M, d(\psi) = r} \psi(t^{-1}) &= - \sum_{\psi : d(\psi) < r} \psi(t^{-1}) - \sum_{M \supsetneq L \supsetneq F} \sum_{\psi : E^\psi = L, d(\psi) = r} \psi(t^{-1}) \\
&= \left(-1 - \sum_{d | m, d \neq 1, m} \mu(d) \right) \sum_{\psi : d(\psi) < r} \psi(t^{-1}) \\
&= (-1 + \mu(1) + \mu(m)) \sum_{\psi : d(\psi) < r} \psi(t^{-1}) \\
&= \mu(m) \sum_{\psi : d(\psi) < r} \psi(t^{-1})
\end{aligned}$$

□

We also require the following facts which are proved similarly to the above:

Lemma 6.3. *For $E \supset M \supsetneq F$ with $[M:F] = m$ we have*

a)
$$\left| \left\{ Y \in \mathfrak{t}(F) : Y^{\sigma^m} = Y, Y^{\sigma^k} \neq Y \text{ for } 1 \leq k < m \right\} \right| = \sum_{d|m} \mu\left(\frac{m}{d}\right) q^{d-1} \tag{6.11}$$

b) *For t with $d(t) = r$ we have*
$$\sum_{\psi'' \in B''_r} \psi(t^{-1}) = \mu(m) \tag{6.12}$$

6.2 Computation of $L^M(\gamma, t)$

We will partially compute the distributions $L^E(\gamma, t)$ in some cases. For simplicity we assume throughout that γ and t are good elements.

The Case $M = E$

We will compute $L^E(\gamma, t)$ in the case where γ and t are not too close to one another. We suppose $d(t) \neq d(\gamma)$ or $d(t) = d(\gamma)$ but $\gamma^\sigma \not\equiv t \bmod \varpi^{d(\gamma)}$ for any $\sigma \in \Gamma_{E/F}$.

For $A = \min\{d(t), d(\gamma)\}$ we compute

$$\begin{aligned}
&L^E(\gamma, t)\\
=&\sum_{r=0}^{\infty} \sum_{\psi : E^\psi = E, d(\psi) = r} \Theta_\psi(\gamma)\psi(t^{-1})\\
=&\sum_{r=0}^{A-1} \left|\left\{\psi : E^\psi = E, d(\psi) = r\right\}\right| nq^{r\frac{n^2-n}{2}} + \mu(n)\left|\{\psi : d(\psi) < r\}\right| nq^{A\frac{n^2-n}{2}}\\
=&n(-1)^n \left\{\psi : E^\psi = E, d(\psi) = 0\right\}\\
&+ n(-1)^n \left(q^{\frac{n^2-n}{2}} \sum_{r=1}^{A-1} |T(\mathfrak{f})| \left(\sum_{d|n} \mu\left(\frac{n}{d}\right) q^{d-1} \right) q^{(r-1)\frac{n^2+n-2}{2}} + \mu(n) q^{\frac{n^2-n}{2}} |T(\mathfrak{f})| q^{(A-1)\frac{n^2+n-2}{2}} \right)\\
=&n(-1)^n \left\{\psi : E^\psi = E, d(\psi) = 0\right\}\\
&- n(-1)^n |T(\mathfrak{f})| \frac{q^{\frac{n^2-n}{2}} \left(\sum_{d|n} \mu\left(\frac{n}{d}\right) q^{d-1} \right)}{q^{\frac{n^2+n-2}{2}} - 1}\\
&+ n(-1)^n |T(\mathfrak{f})| q^{\frac{n^2-n}{2}} \left(\frac{\sum_{d|n} \mu\left(\frac{n}{d}\right) q^{d-1}}{q^{\frac{n^2+n-2}{2}} - 1} + \mu(n) \right) q^{(A-1)\frac{n^2+n-2}{2}}
\end{aligned} \tag{6.13}$$

The Case $[E : M]$ is Prime

Denote by $P(\gamma; r)$ the quantity

$$P^M(\gamma; r) = \sum_{\psi' \in S^M(M)^\wedge : d(\psi') < r} \Theta_{\psi'}^{H^M}(\gamma_{<r}) \tag{6.14}$$

From §5 we have that

$$P^M(\gamma_{<r}; r) = E'_{S^M} + E_{S^M} (q^m)^{(r-1)\frac{(\frac{n}{m})^2 + \frac{n}{m} - 2}{2}} = E'_{S^M} + E_{S^M} q^{(r-1)\frac{\frac{n^2}{m} + n - 2m}{2}} \tag{6.15}$$

for

$$E_{S^M} = \frac{n}{m}(-1)^{\frac{n}{m}} |S^M(\mathfrak{m})| q^{\frac{\frac{n^2}{m} - n}{2}} \frac{q^{n-m} - 1}{q^{\frac{\frac{n^2}{m} + n - 2m}{2}} - 1} \tag{6.16}$$

and

$$E'_{S^M} = \frac{n}{m}(-1)^{\frac{n}{m}} \left(|S^M(\mathfrak{m})| q^{\frac{\frac{n^2}{m} - n}{2}} \frac{q^{n-m} - 1}{q^{\frac{\frac{n^2}{m} + n - 2m}{2}} - 1} - 1 \right) = E_{S^M} - \frac{n}{m}(-1)^{\frac{n}{m}} \tag{6.17}$$

For $d(t) < d(\gamma)$ we compute

$$
\begin{aligned}
&L^M(\gamma, t)\\
=&\sum_{r=0}^{\infty} \sum_{\psi \in S_{M,r}} \Theta_\psi(\gamma)\psi(t^{-1})\\
=&\sum_{\psi : d(\psi)=0, E^\psi = M} \Theta_\psi(\gamma)\psi(t^{-1})\\
&+\sum_{r=1}^{d(t)} C_{M,r} \sum_{\psi' \in S^M(M)^\wedge, d(\psi')<r} \sum_{\psi'' \in B''_{M,r}} \Theta_{\psi'\psi''}(\gamma)\psi'\psi''(t^{-1})\\
=&n(-1)^n \left\{\psi : E^\psi = M, d(\psi) = 0\right\}\\
&+\frac{m}{n}(-1)^{\frac{n}{m}} \sum_{r=1}^{d(t)} C_{M,r} \sum_{\psi'' \in B''_{M,r}} \Theta^G_{\psi''}(\gamma_{\geq r})\psi''(t^{-1}) \sum_{\psi' \in S^M(M)^\wedge, d(\psi')<r} \Theta^{H^M}_{\psi'}(\gamma_{<r})\psi'(t^{-1})\\
=&n(-1)^n \left\{\psi : E^\psi = M, d(\psi) = 0\right\}\\
&+m(-1)^{n+\frac{n}{m}} \sum_{r=1}^{d(t)-1} C_{M,r} \left(\sum_{d|m} \mu\left(\frac{m}{d}\right) q^{d-1}\right) q^{r\frac{n^2-nm}{2}} P^M(\gamma_{<r}, r)\\
&\qquad +m(-1)^{n+\frac{n}{m}} C_{M,d(t)} \sum_{\psi'' \in B''_{M,r}} q^{d(t)\frac{n^2-nm}{2}} \psi''(t^{-1}) P^M(\gamma_{<r}, d(t))\\
=&n(-1)^n \left\{\psi : E^\psi = M, d(\psi) = 0\right\}\\
&+m(-1)^{n+\frac{n}{m}} |T^M(\mathfrak{f})| q^{\frac{n^2-nm}{2}} \left(\sum_{d|m} \mu\left(\frac{m}{d}\right) q^{d-1}\right) \sum_{r=1}^{d(t)-1} q^{(r-1)\left(\frac{n^2-nm}{2}+m-1\right)} P^M(\gamma_{<r}, r)\\
&\qquad +\mu(m)m(-1)^{n+\frac{n}{m}} |T^M(\mathfrak{f})| q^{\frac{n^2-nm}{2}} q^{(d(t)-1)\left(\frac{n^2-nm}{2}+m-1\right)} P^M(\gamma_{<r}, d(t))
\end{aligned}
$$

Continuing on,

$$m(-1)^{n+\frac{n}{m}}|T^M(\mathfrak{f})|q^{\frac{n^2-nm}{2}}\left(\sum_{d|m}\mu\left(\frac{m}{d}\right)q^{d-1}\right)\sum_{r=1}^{d(t)-1}q^{(r-1)\left(\frac{n^2-nm}{2}+m-1\right)}P^M(\gamma_{<r},r)$$

$$+\mu(m)m(-1)^{n+\frac{n}{m}}|T^M(\mathfrak{f})|q^{\frac{n^2-nm}{2}}q^{(d(t)-1)\left(\frac{n^2-nm}{2}+m-1\right)}P^M(\gamma_{<r},d(t))$$

$$=n(-1)^n|T(\mathfrak{f})|q^{\frac{n^2+\frac{n^2}{m}-nm-n}{2}}\frac{q^{n-m}-1}{q^{\frac{\frac{n^2}{m}+n-2m}{2}}-1}\left(\sum_{d|m}\mu\left(\frac{m}{d}\right)q^{d-1}\right)\sum_{r=1}^{d(t)-1}q^{(r-1)\left(\frac{n^2-nm}{2}+m-1\right)}\left(1+q^{(r-1)\frac{\frac{n^2}{m}+n-2m}{2}}\right)$$

$$+\mu(m)n(-1)^n|T(\mathfrak{f})|q^{\frac{n^2+\frac{n^2}{m}-nm-n}{2}}\frac{q^{n-m}-1}{q^{\frac{\frac{n^2}{m}+n-2m}{2}}-1}q^{(d(t)-1)\left(\frac{n^2-nm}{2}+m-1\right)}\left(1+q^{(d(t)-1)\frac{\frac{n^2}{m}+n-2m}{2}}\right)$$

$$-n(-1)^n|T^M(\mathfrak{f})|q^{\frac{n^2-nm}{2}}\left(\left(\sum_{d|m}\mu\left(\frac{m}{d}\right)q^{d-1}\right)\sum_{r=1}^{d(t)-1}q^{(r-1)\left(\frac{n^2-nm}{2}+m-1\right)}+\mu(m)q^{(d(t)-1)\left(\frac{n^2-nm}{2}+m-1\right)}\right)$$

$$=n(-1)^n|T(\mathfrak{f})|q^{\frac{n^2+\frac{n^2}{m}-nm-n}{2}}\frac{q^{n-m}-1}{q^{\frac{\frac{n^2}{m}+n-2m}{2}}-1}\left(\sum_{d|m}\mu\left(\frac{m}{d}\right)q^{d-1}\right)$$

$$\cdot\left(\frac{q^{(d(t)-1)\left(\frac{n^2-nm}{2}+m-1\right)}-1}{q^{\frac{n^2-nm}{2}+m-1}-1}+\frac{q^{(d(t)-1)\frac{n^2+\frac{n^2}{m}-nm+n-2}{2}}-1}{q^{\frac{n^2+\frac{n^2}{m}-nm+n-2}{2}}-1}\right)$$

$$+\mu(m)n(-1)^n|T(\mathfrak{f})|q^{\frac{n^2+\frac{n^2}{m}-nm-n}{2}}\frac{q^{n-m}-1}{q^{\frac{\frac{n^2}{m}+n-2m}{2}}-1}q^{(d(t)-1)\left(\frac{n^2-nm}{2}+m-1\right)}\left(1+q^{(d(t)-1)\frac{\frac{n^2}{m}+n-2m}{2}}\right)$$

$$-n(-1)^n|T^M(\mathfrak{f})|q^{\frac{n^2-nm}{2}}\left(-\frac{\sum_{d|m}\mu\left(\frac{m}{d}\right)q^{d-1}}{q^{\frac{n^2-nm}{2}+m-1}-1}+\left(\frac{\sum_{d|m}\mu\left(\frac{m}{d}\right)q^{d-1}}{q^{\frac{n^2-nm}{2}+m-1}-1}+\mu(m)\right)q^{(d(t)-1)\left(\frac{n^2-nm}{2}+m-1\right)}\right)$$

$$=-n(-1)^n|T(\mathfrak{f})|q^{\frac{n^2+\frac{n^2}{m}-nm-n}{2}}\frac{q^{n-m}-1}{q^{\frac{\frac{n^2}{m}+n-2m}{2}}-1}\left(\sum_{d|m}\mu\left(\frac{m}{d}\right)q^{d-1}\right)\left(\frac{1}{q^{\frac{n^2-nm}{2}+m-1}-1}+\frac{1}{q^{\frac{n^2+\frac{n^2}{m}-nm+n-2}{2}}-1}\right)$$

$$+n(-1)^n|T^M(\mathfrak{f})|q^{\frac{n^2-nm}{2}}\frac{\sum_{d|m}\mu\left(\frac{m}{d}\right)q^{d-1}}{q^{\frac{n^2-nm}{2}+m-1}-1}$$

$$+n(-1)^n|T(\mathfrak{f})|q^{\frac{n^2+\frac{n^2}{m}-nm-n}{2}}\frac{q^{n-m}-1}{q^{\frac{\frac{n^2}{m}+n-2m}{2}}-1}\left(\frac{\sum_{d|m}\mu\left(\frac{m}{d}\right)q^{d-1}}{q^{\frac{n^2-nm}{2}+m-1}-1}+\mu(m)\right)q^{(d(t)-1)\left(\frac{n^2-nm}{2}+m-1\right)} \tag{6.18}$$

$$-n(-1)^n|T^M(\mathfrak{f})|q^{\frac{n^2-nm}{2}}\left(\frac{\sum_{d|m}\mu\left(\frac{m}{d}\right)q^{d-1}}{q^{\frac{n^2-nm}{2}+m-1}-1}+\mu(m)\right)q^{(d(t)-1)\left(\frac{n^2-nm}{2}+m-1\right)}$$

$$+n(-1)^n|T(\mathfrak{f})|q^{\frac{n^2+\frac{n^2}{m}-nm-n}{2}}\frac{q^{n-m}-1}{q^{\frac{\frac{n^2}{m}+n-2m}{2}}-1}\left(\frac{\sum_{d|m}\mu\left(\frac{m}{d}\right)q^{d-1}}{q^{\frac{n^2+\frac{n^2}{m}-nm+n-2}{2}}-1}+\mu(m)\right)q^{(d(t)-1)\frac{n^2+\frac{n^2}{m}-nm+n-2}{2}}$$

6.3 Examples: $n = \ell^2$ and $n = \ell_1\ell_2$

In this case our calculations can easily be made much more explicit since our computations in §5 give us enough information to fully compute the distributions $L^M(\gamma, t)$. For simplicity we suppose n is odd. Further, we suppose γ and t are good elements and that $0 < d(t) < d(\gamma)$.

The Case $n = \ell^2$

In this case we have two possibilities for M: either $M = E$ or $M = E'$ where $[E' : F] = \ell$. We have by (6.13) and (6.18) that

$$L^E(\gamma, t) = A_E + C_E q^{(d(t)-1)\frac{\ell^4+\ell^2-2}{2}}$$

and

$$L^{E'}(\gamma, t) = A_{E'} + B_{E'} q^{(d(t)-1)\left(\frac{\ell^4-\ell^3}{2}+\ell-1\right)} + C_{E'} q^{(d(t)-1)\frac{\ell^4+\ell^2-2}{2}}$$

for constants $A_E, C_E, A_{E'}, B_{E'}, C_{E'} \in \mathbb{C}$. Now

$$L(\gamma, t) = A_{\ell^2} + B_{\ell^2} q^{d(t)\left(\frac{\ell^4-\ell^3}{2}+\ell-1\right)} + C_{\ell^2} q^{d(t)\frac{\ell^4+\ell^2-2}{2}} \tag{6.19}$$

where we compute

$$B_{\ell^2} = -\ell^2 \left(|T(\mathfrak{f})| \left(\frac{q^{\frac{\ell^3-\ell^2}{2}} - 1}{q^{\frac{\ell^3+\ell^2-2\ell}{2}} - 1} - 1 \right) + |T^{E'}(\mathfrak{f})| \right) \cdot \frac{q^{\frac{\ell^4-\ell^3}{2}} - 1}{q^{\frac{\ell^4-\ell^3}{2}+\ell-1} - 1} \tag{6.20}$$

and

$$C_{\ell^2} = -\ell^2 |T(\mathfrak{f})| \left(\frac{q^{\frac{\ell^3-\ell^2}{2}} - 1}{q^{\frac{\ell^3+\ell^2-2\ell}{2}} - 1} - 1 \right) \cdot \frac{q^{\frac{\ell^4-\ell^3}{2}} - 1}{q^{\frac{\ell^4+\ell^2-2}{2}} - 1} \tag{6.21}$$

The Case $n = \ell_1\ell_2$

In this case we have three possibilities for M: either $M = E$ or $M = E'_i$ where $[E : E'_i] = \ell_i$ for $i = 1, 2$. Then

$$L^E(\gamma, t) = A_E + C_E q^{(d(t)-1)\frac{\ell^4+\ell^2-2}{2}}$$

and

$$L^{E'_i}(\gamma, t) = A_{E'_i} + B_{E'_i} q^{(d(t)-1)\left(\frac{\ell_1^2\ell_2^2-\ell_i^2\ell_j}{2}+\ell_i-1\right)} + C_{E'_i} q^{(d(t)-1)\left(\frac{\ell_1^2\ell_2^2+\ell_1\ell_2-2}{2}+\frac{\ell_i\ell_j^2-\ell_i^2\ell_j}{2}\right)}$$

for $\{i, j\} = \{1, 2\}$ and for constants $A_E, C_E \in \mathbb{C}$ and $A_{E'_i}, B_{E'_i}, C_{E'_i} \in \mathbb{C}$ for $i = 1, 2$. We then have

$$\begin{aligned} L(\gamma, t) = & A_{\ell_1\ell_2} + B^1_{\ell_1\ell_2} q^{d(t)\left(\frac{\ell_1^2\ell_2^2-\ell_1^2\ell_2}{2}+\ell_1-1\right)} + B^2_{\ell_1\ell_2} q^{d(t)\left(\frac{\ell_1^2\ell_2^2-\ell_1\ell_2^2}{2}+\ell_2-1\right)} \\ & + C_{\ell_1\ell_2} q^{d(t)\frac{\ell^4+\ell^2-2}{2}} + C^1_{\ell_1\ell_2} q^{d(t)\left(\frac{\ell_1^2\ell_2^2+\ell_1\ell_2-2}{2}+\frac{\ell_1\ell_2^2-\ell_1^2\ell_2}{2}\right)} + C^2_{\ell_1\ell_2} q^{d(t)\left(\frac{\ell_1^2\ell_2^2+\ell_1\ell_2-2}{2}-\frac{\ell_1\ell_2^2-\ell_1^2\ell_2}{2}\right)} \end{aligned} \tag{6.22}$$

for

$$B^i_{\ell_1\ell_2} = -\ell_1\ell_2 \left(|T(\mathfrak{f})| \left(\frac{q^{\frac{\ell_i\ell_j^2 - \ell_1\ell_2}{2}} - 1}{q^{\frac{\ell_i\ell_j^2 + \ell_1\ell_2 - 2\ell_i}{2}} - 1} - 1 \right) + |T^{E'_i}(\mathfrak{f})| \right) \cdot \frac{q^{\frac{\ell_1^2\ell_2^2 - \ell_i^2\ell_j}{2}} - 1}{q^{\frac{\ell_1^2\ell_2^2 - \ell_i^2\ell_j}{2} + \ell_i - 1} - 1} \tag{6.23}$$

as well as

$$C^i_{\ell_1\ell_2} = \ell_1\ell_2|T(\mathfrak{f})|q^{\ell_i - \ell_1\ell_2} \frac{q^{\ell_1\ell_2 - \ell_i} - 1}{q^{\frac{\ell_i\ell_j^2 + \ell_1\ell_2 - 2\ell_i}{2}} - 1} \cdot \frac{q^{\frac{\ell_1^2\ell_2^2 + \ell_i\ell_j^2 - \ell_i^2\ell_j + \ell_1\ell_2 - 2\ell_i}{2}} - 1}{q^{\frac{\ell_1^2\ell_2^2 + \ell_i\ell_j^2 - \ell_i^2\ell_j + \ell_1\ell_2 - 2}{2}} - 1} \tag{6.24}$$

and

$$C_{\ell_1\ell_2} = -\ell_1\ell_2|T(\mathfrak{f})|q^{-\ell_1\ell_2 + 1} \left(\frac{q^{\ell_1\ell_2 - 1} - q^{\ell_1 - 1} - q^{\ell_2 - 1} + 1}{q^{\frac{\ell_1^2\ell_2^2 + \ell_1\ell_2 - 2}{2}} - 1} + 1 \right) \tag{6.25}$$

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