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ASYMPTOTIC ANALYSIS FOR PORTFOLIO PROBLEMS UNDER FAST AND SLOW STOCHASTIC ENVIRONMENTS

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ABSTRACT

The main objective of this dissertation is to use the multi-scale asymptotic analysis to study two-asset portfolio optimization problems under fast and slow stochastic environments. We first review the stochastic control theory and the classical Merton portfolio problem. Then we introduce the multi-scale asymptotic analysis for single-asset portfolio optimization problem [Fouque and Hu, 2017-2020]. We rigorously establish that, in the multi-scale case, the Merton type zeroth order optimal strategy recovers the first order approximation of the associated problem value. The asymptotic analysis can be extended to a multi-asset scenario. We consider two-asset portfolio problems with full and partial information, respectively. Combining the filtering theory and the asymptotic analysis, we show that similar results about the zeroth order optimal strategy hold for both full information case and partial information case under different assumptions.

CHAPTER 1 INTRODUCTION

This dissertation is dedicated to extend the results from multi-scale asymptotic analysis for single-asset portfolio optimization problems to multi-asset scenarios.

Empirical studies imply that both slow-scale and fast-scale factors are present in the volatility of an underlying asset. The slow-varying factor in the volatility model is particularly important for long-term investment, because the fast-scale factor shows less presence as its effect is approximately averaged out in the long-term. On the other hand, to capture the volatility in a shorter time scale, the analysis of the fast-scale case is also required, where the return and the volatility of the underlying asset are fast-varying. Therefore, this dissertation focuses on the multi-scale model, where the dynamics of the underlying asset price are affected by two stochastic factors: the slow-scale factor and the fast-scale factor.

Meanwhile, asymptotic analysis has been developed over many years for option pricing problems, where efficient approximations are derived by regular and singular perturbation methods. In the context of portfolio optimization problems, we summarize recent papers by [Fouque and Hu, 2017-2020] on asymptotic analysis for nonlinear Merton problem under different stochastic environments. As Merton problem considers a single risky asset, we extend the results by setting up a multi-asset scenario under a multi-scale volatility model with full information. Furthermore, we consider the additional partial information case where the volatility factors are unobservable. Combining the filtering theory and the asymptotic analysis, we derive explicit formulas and give insights for the approximations of the optimal strategy with higher dimension up to the first order.

1.1 Thesis Outline

In Chapter 1 of this dissertation, we give a review of the stochastic control theory and the classical Merton problem. Starting with the introduction of stochastic control problem in Section 1.2, we discuss the well-known dynamical programming principle and the Hamilton-Jacobi-Bellman equation method in Section 1.3 and Section 1.4, respectively. Then Section 1.5 applies the stochastic control theory on the classical Merton problem, and derives the classical Merton equation for further use.

Chapter 2 introduces the asymptotic analysis for single-asset portfolio problems under different stochastic environments. In Section 2.2, we study the asymptotic analysis for a slow-scale model by deriving and analyzing the asymptotic expansion of the value function and the optimal strategy. Section 2.3 follows the same procedure for a fast-scale model, where an extra step of singular perturbation method is applied. Then we combine the previous two sections into a multi-scale scenario in Section 2.4. Then in Section 2.5, we present a rigorous proof of the main result about the approximation of the optimal strategy in the multi-scale case.

The main contribution of Chapter 3 is to extend the multi-scale asymptotic analysis for the single-asset portfolio problem to a two-asset scenario. Section 3.1 introduces the multi-scale model on two risky assets: one fast-scale and one slow-scale. We set up the portfolio optimization problem and derive the HJB equation in Section 3.2. The asymptotic analysis is applied in Section 3.3, where we derive the heuristic asymptotic expansion of the associated value function. Section 3.4 rigorously proves that the main result in the single-asset problem holds in the two-asset case. Finally, we compute and analyze the first order correction of the 2-dimensional optimal strategy in Section 3.5.

Chapter 4 considers the two-asset portfolio problem with partial information, where the volatility factors are considered to be unobservable. Section 4.1 applies the Kalman-Bucy filtering theory on the volatility factors. The filter asymptote is computed under the steady state assumption in Section 4.2. Following the same methodology in Chapter 3, we set up the two-asset portfolio problem with the filtered system and the filter asymptote, derive the asymptotic expansion of the value function, prove the performance of the zeroth order optimal strategy, and analyze the first order optimal strategy in Section 4.3, Section 4.4, Section 4.5, and Section 4.6, respectively.

1.2 Stochastic Control Problem

Portfolio optimization problems are typically viewed as stochastic control problems, where the desired optimal strategies correspond to the optimal controls. We first introduce the general background of continuous-time stochastic control over a finite horizon.

Consider a filtered probability space $(\Omega, \mathscr{F}, \mathbb{F} = (\mathscr{F}_t)_{t\in[0,T]}, \mathbb{P})$ satisfying the usual conditions, i.e. the filtration $(\mathscr{F}_t)$ is right-continuous and complete, and a d-dimensional Brownian motion W on Ω.

A *control process* $\alpha = (\alpha_s)_{s\in[0,T]}$ is a progressively measurable process with respect to $\mathbb{F}$ valued in a subset A of $\mathbb{R}^m$.

A *controlled diffusion process* is a process governed by the stochastic differential equation (SDE) valued in $\mathbb{R}^n$ of the following form:

$$dX_s^\alpha = b(s, X_s^\alpha, \alpha_s)ds + \sigma(s, X_s^\alpha, \alpha_s)dW_s,\ s \in [0,T], \tag{1.2.1}$$

where the drift function $b(t,x,a) : [0,T] \times \mathbb{R}^n \times A \to \mathbb{R}^n$ and the dispersion function $\sigma(t,x,a) : [0,T] \times \mathbb{R}^n \times A \to \mathbb{R}^{n\times d}$ are both deterministic functions that satisfy the following uniform Lipschitz condition: $\exists K \geq 0, \forall x, y \in \mathbb{R}^n, t \in [0,T], a \in A$,

$$\|b(t,x,a) - b(t,y,a)\| + \|\sigma(t,x,a) - \sigma(t,y,a)\| \leq K\|x - y\|. \tag{1.2.2}$$

Denote the set of *admissible controls* $\mathcal{A}$ as the set of all $\mathcal{F}_t$-progressively measurable and L^2-integrable control processes $\alpha = (\alpha_s)$ such that

$$\mathbb{E}\left[\int_0^T \|b(t,x,\alpha_t)\|^2 + \|\sigma(t,x,\alpha_t)\|^2 dt\right] < \infty, \tag{1.2.3}$$

where x is a fixed arbitrary value in the support of the diffusion.

The conditions (1.2.2) and (1.2.3) ensure that for all $\alpha \in \mathcal{A}$ and for any initial condition $(t,x) \in [0,T] \times \mathbb{R}^n$, the SDE (1.2.1) has a unique strong solution starting from x at time $s = t$. Denote $(X_s^{t,x})_{s\in[t,T]}$ as the unique solution with a.s. continuous paths. We also abuse the notation X_s^α with X_s when there is no confusion.

A *feedback control* is a control process adapting to the natural filtration generated by $\mathcal{F}^X$; a *Markovian control* is a control process in the form of $\alpha_s = h(s, X_s^{t,x})$ for some measurable function $h : [0,T] \times \mathbb{R}^n \to A$.

Now we set up the stochastic control problem. In general, consider a measurable (utility) function $U : \mathbb{R}^n \to \mathbb{R}$ such that U is lower-bounded, or U satisfies a quadratic growth condition:

$$|U(x)| \leq C(1 + |x|^2), \forall x \in \mathbb{R}^n, \tag{1.2.4}$$

for some constant C independent of x. Let X_s^α be a controlled diffusion process, which represents a self-financed wealth process in a portfolio problem setting. Define an objective function $J(t, x, \alpha)$ as the expected value of U at terminal time T with respect to control $\alpha \in \mathscr{A}$ when the position $X_t^\alpha = x$ at time t:

$$J(t, x, \alpha) = \mathbb{E}\left(U(X_T^\alpha) \,|\, X_t^\alpha = x\right). \tag{1.2.5}$$

The associated value function is defined by maximizing the objective function over all admissible control processes:

$$v(t, x) = \sup_{\alpha \in \mathscr{A}} J(t, x, \alpha). \tag{1.2.6}$$

A control process $\hat{\alpha} \in \mathscr{A}$ is called an *optimal control* if $v(t, x) = J(t, x, \hat{\alpha})$. Here we assume implicitly that the value function v is measurable. However, the measurability is not trivial, and we refer to measurable selection theorems in Dellacherie and Meyer [1975] for sufficient conditions.

1.3 Dynamic Programming Principle

The dynamic programming principle (DPP) is a fundamental principle in stochastic control theory. Generally speaking, the DPP can be applied on controlled Markov processes, including the controlled diffusion process described in the previous section. The DPP is formulated as follows:

Theorem 1.3 (Dynamic programming principle). Let X_s be a controlled diffusion process given by (1.2.1), and $v(t,x)$ be the associated value function that maximizes the expected utility of the terminal state $X_T^{t,x}$ over all admissible controls. For $(t,x) \in [0,T] \times \mathbb{R}^n$, we have

$$\begin{aligned} v(t,x) &= \sup_{\alpha \in \mathscr{A}} \sup_{\theta \in \mathscr{T}_{t,T}} \mathbb{E}\left(v(\theta, X_\theta^{t,x})\right), \\ &= \sup_{\alpha \in \mathscr{A}} \inf_{\theta \in \mathscr{T}_{t,T}} \mathbb{E}\left(v(\theta, X_\theta^{t,x})\right), \end{aligned} \tag{1.3.1}$$

where $\mathscr{T}_{t,T}$ denote the set of stopping times with values in $[t,T]$.

The DPP implies that the optimization in the stochastic control problem can be split in two parts: firstly, we can search for an optimal control from time θ given the state value $X_\theta^{t,x}$; then we maximize $\mathbb{E}\left(v(\theta, X_\theta^{t,x})\right)$ over controls on the time interval $[t,\theta]$.

Proof. 1. Given an admissible control $\alpha \in \mathscr{A}$, we have the Markovian structure

$$X_s^{t,x} = X_s^{\theta, X_\theta^{t,x}}, s \geq \theta,$$

for any stopping time $\theta \in \mathscr{T}_{t,T}$, because of the path-wise uniqueness of the flow of the SDE for X_s. Then by the law of iterated conditional expectation,

$$
\begin{aligned}
J(t,x,\alpha) &= \mathbb{E}\left(U(X_T^{t,x})\right) = \mathbb{E}\left(\mathbb{E}\left(U(X_T^{t,x})\,|\,\mathscr{F}_\theta\right)\right) \\
&= \mathbb{E}\left(U(X_T^{\theta,X_\theta^{t,x}})\right) = \mathbb{E}\left(J(\theta, X_\theta^{t,x}, \alpha)\right)
\end{aligned}
\tag{1.3.2}
$$

for any $\theta \in \mathscr{T}_{t,T}$. Since $J \le v$ and θ is arbitrary in $\mathscr{T}_{t,T}$,

$$
\begin{aligned}
J(t,x,\alpha) &\le \inf_{\theta\in\mathscr{T}_{t,T}} \mathbb{E}\left(v(\theta, X_\theta^{t,x})\right) \\
&\le \sup_{\alpha\in\mathscr{A}} \inf_{\theta\in\mathscr{T}_{t,T}} \mathbb{E}\left(v(\theta, X_\theta^{t,x})\right).
\end{aligned}
$$

Taking the supremum over all admissible controls on the left hand side, we obtain

$$
v(t,x) \le \sup_{\alpha\in\mathscr{A}} \inf_{\theta\in\mathscr{T}_{t,T}} \mathbb{E}\left(v(\theta, X_\theta^{t,x})\right). \tag{1.3.3}
$$

2. Fix some arbitrary control $\alpha \in \mathscr{A}$ and stopping time $\theta \in \mathscr{T}_{t,T}$. For any $\epsilon > 0$ and $\omega \in \Omega$, there exists an ϵ-optimal control $\alpha_{\epsilon,\omega}$ for $v(\theta(\omega), X_{\theta(\omega)}^{t,x}(\omega))$:

$$
v(\theta(\omega), X_{\theta(\omega)}^{t,x}(\omega)) - \epsilon \le J(\theta(\omega), X_{\theta(\omega)}^{t,x}(\omega), \alpha^{\epsilon,\omega}). \tag{1.3.4}
$$

Define the process

$$
\hat{\alpha}_s(\omega) = \begin{cases} \alpha_s(\omega), s \in [0,\theta(\omega)] \\ \alpha_s^{\epsilon,\omega}(\omega), s \in [\theta(\omega), T]. \end{cases}
$$

By the measurable selection theorem in Chapter 7 of Bertsekas and Shreve [1978], the process $\hat{\alpha}$ is progressively measurable and lies in $\mathscr{A}$. By the law of iterated conditional expectation in (1.3.2) and the ϵ-optimality in (1.3.4),

$$v(t,x) \geq J(t,x,\hat{\alpha}) = \mathbb{E}\left(J(\theta, X_\theta^{t,x}), \alpha^\epsilon\right) \geq \mathbb{E}\left(v(\theta, X_\theta^{t,x})\right) - \epsilon.$$

Since $\epsilon > 0$, $\alpha \in \mathscr{A}$ and $\theta \in \mathscr{T}_{t,T}$ are arbitrary, we obtain

$$v(t,x) \geq \sup_{\alpha\in\mathscr{A}} \sup_{\theta\in\mathscr{T}_{t,T}} \mathbb{E}\left(v(\theta, X_\theta^{t,x})\right). \tag{1.3.5}$$

By combining (1.3.3) and (1.3.5), we acquire the desired result.

Corollary 1.3. For any stopping time $\theta \in \mathscr{T}_{t,T}$, we have

$$v(t,x) = \sup_{\alpha\in\mathscr{A}} \mathbb{E}\left(v(\theta, X_\theta^{t,x})\right). \tag{1.3.6}$$

1.4 Hamilton-Jacobi-Bellman Equation

A partial differential equation (PDE), in particular, a nonlinear second order PDE, called Hamilton-Jacobi-Bellman (HJB) equation can be derived from a stochastic control problem by applying Itô's formula and the DPP. The HJB equation is the infinitesimal version of the DPP. It describes the local behavior of the associated value function as the stopping time θ in (1.3.6) goes to t. In this section, we only focus on formally deriving the HJB equation and obtaining an optimal feedback Markovian control.

Assume the value function $v(t,x)$ associated to the stochastic control problem given by (1.2.6) is smooth enough. In particular, $v(t,x) \in C^{1,2}([0,T) \times \mathbb{R}^m)$. Consider the time $\theta = t+h$ and a constant control $\alpha_s \equiv a$ for some $a \in A$. By the DPP in (1.3.1),

$$v(t,x) \geq \mathbb{E}\left(v(t+h, X^{t,x}_{t+h})\right) \tag{1.4.1}$$

where $X^{t,x}_{t+h}$ is the state system solution to (1.2.1) associated to the constant control. Since v is assumed to be smooth enough, we apply Itô's formula on v between t and $t+h$:

$$v(t+h, X^{t,x}_{t+h}) = v(t,x) + \int_t^{t+h} \left(\frac{\partial v}{\partial t} + \mathscr{L}^a v\right)(s, X^{t,x}_s)\ ds + \text{martingale}, \tag{1.4.2}$$

where $\mathscr{L}^a$ is the infinitesimal generator of the controlled diffusion process X_s with constant control $\alpha_s \equiv a$ in (1.2.1) defined by

$$\mathscr{L}^a u = b(t,x,a)\nabla_x u + \frac{1}{2}\mathrm{tr}\left(\sigma(t,x,a)\sigma(t,x,a)^T \nabla_x^2 u\right). \tag{1.4.3}$$

Substituting (1.4.2) into (1.4.1), we get

$$\mathbb{E}\left(\int_t^{t+h} \left(\frac{\partial v}{\partial t} + \mathscr{L}^a v\right)(s, X^{t,x}_s)\ ds\right) \leq 0. \tag{1.4.4}$$

Then we divide (1.4.4) by h and let h go to 0. By mean-value theorem,

$$\frac{\partial v}{\partial t}(t,x) + \mathscr{L}^a v(t,x) \leq 0.$$

Because $a \in A$ is arbitrary, we acquire the inequality

$$\frac{\partial v}{\partial t}(t,x) + \sup_{a \in A} \mathscr{L}^a v(t,x) \leq 0. \tag{1.4.5}$$

On the other hand, suppose $\hat{\alpha}$ is an optimal control, then by Corollary 1.3.2 in the previous section,

$$v(t,x) = \mathbb{E}\left(v(t+h, \hat{X}_{t+h}^{t,x})\right)$$

where $\hat{X}_{t+h}^{t,x}$ is the state system solution to (1.2.1) associated to $\hat{\alpha}$. Replacing "$\leq$" by "$=$" in the previous arguments, we get

$$\frac{\partial v}{\partial t}(t,x) + \mathscr{L}^{\alpha^*} v(t,x) = 0. \tag{1.4.6}$$

By combining (1.4.5) and (1.4.6), we obtain that v should satisfy the HJB equation:

$$\frac{\partial v}{\partial t}(t,x) + \sup_{a \in A} \mathscr{L}^a v(t,x) = 0, \tag{1.4.7}$$

for all $(t,x) \in [0,T) \times \mathbb{R}^n$, if the supremum in a is finite. From the definition of $v(t,x)$ in (1.2.5) and (1.2.6), the terminal condition associated to the HJB equation is

$$v(T,x) = U(x), \forall x \in \mathbb{R}^n. \tag{1.4.8}$$

As a byproduct, if we can find a measurable function $\hat{\alpha}(t,x)$ such that

$$\mathscr{L}^{\hat{\alpha}(t,x)} v(t,x) = \sup_{a \in A} \mathscr{L}^a v(t,x),$$

then we would get

$$\frac{\partial v}{\partial t}(t,x) + \mathscr{L}^{\hat{\alpha}(t,x)} v(t,x) = 0, v(T,x) = U(x).$$

By Feynman-Kac formula, $v(t,x) = \mathbb{E}\left(U(\hat{X}_T^{t,x})\right)$ where $\hat{X}_{t+h}^{t,x}$ is the state system solution to (1.2.1) starting at x at time t, with the control $\hat{\alpha}(t,x)$. This shows that $\hat{\alpha}(t,x)$ is an optimal feedback Markovian control.

Now we've presented the derivation of the HJB equation, and acquired an optimal feedback Markovian control. The next crucial step is to verify that, given a smooth solution to the HJB equation, this solution coincides with the value function. The sufficient conditions for the above-mentioned verification theorem vary from one case to another. Examples of a verification theorem can be found in Pham [2009] and Touzi [2013]. From now on, we compute the HJB equation and the optimal feedback Markovian control associated to a stochastic control problem without explanations.

1.5 Classical Merton Problem

The Merton problem of portfolio optimization in Merton [1969] has been studied extensively in various cases. Equipped with the PDE technique discussed in the previous section, now we present the classical Merton portfolio problem. The resulting classical Merton value function will play an important role in the next chapter when we construct the approximation to an optimal strategy in the asymptotic analysis.

Consider a risky asset modeled by the following geometric Brownian motion:

$$dS_t = \mu S_t dt + \sigma S_t dW_t, \tag{1.5.1}$$

where the growth rate μ, the volatility coefficient σ are constants and W_t is a standard Brownian motion in the filtered probability space $(\Omega, \mathscr{F}, \mathbb{F} = (\mathscr{F}_t)_{t\in[0,T]}, \mathbb{P})$. Let $(X_t, t \in [0,T])$ denote the wealth process over the investment period $[0,T]$ and π_t denote the amount of wealth held in the risky asset at time t, where the remaining held in a risk-free market at an interest rate r. With a continuous self-financing strategy, the wealth process X_t is given by

$$dX_t = \pi_t \frac{dS_t}{S_t} + r(X_t - \pi_t)dt = (rX_t + \pi_t(\mu - r))dt + \pi_t \sigma dW_t. \tag{1.5.2}$$

Without loss of generality, we take the interest rate $r = 0$ in (1.5.2) for simplicity. Then we have

$$dX_t = \pi_t \mu dt + \pi_t \sigma dW_t. \tag{1.5.3}$$

Assumption 1.5 (Utility function assumption). The utility function $U(x)$ on $\mathbb{R}^+$ is smooth, strictly increasing and strictly concave. $U(x)$ also satisfies the "Inada and asymptotic elasticity":

$$U'(0^+) = \infty, U'(\infty) = 0, \lim_{x\to\infty} x\frac{U'(x)}{U(x)} < 1.$$

From now on, we always assume Assumption 1.5 holds for utility functions. Define the classical Merton value function for $(t, x) \in [0,T] \times \mathbb{R}^+$ by

$$M(t, x; \lambda) = \sup_{\pi} \mathbb{E}\left(U(X_T) \,|\, X_t = x\right), \tag{1.5.4}$$

where $\lambda := \dfrac{\mu}{\sigma}$ is called the Sharpe ratio and the supremum is taken over all control processes $\pi = (\pi_t)$ that are $\mathscr{F}_t$-progressively measurable and L^2-integrable.

Applying the PDE method from the previous section to the above stochastic control problem, we conclude that M is the unique smooth, strictly increasing, strictly concave solution of the HJB equation

$$M_t - \frac{1}{2}\lambda^2 \frac{M_x^2}{M_{xx}} = 0, M(T, x; \lambda) = U(x), \tag{1.5.5}$$

where the M_t, M_x, and M_{xx} denote the partial derivatives of M. The optimal feedback Markovian control is given by

$$\pi_t^M = \frac{\lambda}{\sigma} R(t, x; \lambda), \tag{1.5.6}$$

where $R(t, x; \lambda)$ is the so-called risk-tolerance function associated with the classical Merton value function defined by

$$R(t, x; \lambda) := -\frac{M_x(t, x; \lambda)}{M_{xx}(t, x; \lambda)}, \tag{1.5.7}$$

which is well-defined and smooth in (t, x) because M is strictly concave and smooth in (t, x).

We end this chapter by defining several notations. Define the differential operators

$$D_k = R(t, x; \lambda)^k \frac{\partial^k}{\partial x^k}, k \in \mathbb{N}, \tag{1.5.8}$$

and the linear operator

$$\mathscr{L}_{t,x}(\lambda) = \frac{\partial}{\partial t} + \frac{1}{2}\lambda^2 R^2(t, x; \lambda)\frac{\partial^2}{\partial x^2} + \lambda^2 R(t, x; \lambda)\frac{\partial}{\partial x} = \frac{\partial}{\partial t} + \frac{1}{2}\lambda^2 D_2 + \lambda^2 D_1. \tag{1.5.9}$$

Observe that the HJB equation (1.5.5) can be re-written as

$$\mathscr{L}_{t,x}(\lambda)M = 0. \tag{1.5.5*}$$

Proposition 1.5 (Uniqueness of Merton equation). Let $\mathscr{L}_{t,x}(\lambda)$ be the operator defined in (1.5.9), and assume that the utility function $U(x)$ satisfies Assumption 1.5 and $U(0+) = 0$, then

$$\mathscr{L}_{t,x}(\lambda)u(t,x;\lambda) = 0, u(T,x;\lambda) = U(x), \tag{1.5.10}$$

has a unique non-negative solution.

Proof. First, observe that there exists a solution $u = M(t,x;\lambda)$ for (1.5.10). To show uniqueness, we define the injective transformation:

$$\begin{cases} \xi = -\log M_x(t,x;\lambda) + \frac{1}{2}\lambda^2(T-t), \\ t' = t\,. \end{cases}$$

Define $\omega(t',\xi;\lambda) = u(t,x;\lambda)$, then ω solves the heat equation

$$\mathscr{H}\omega := \omega_{t'} + \frac{1}{2}\lambda^2\omega_{\xi\xi} = 0, \omega(T,\xi;\lambda) = U(I(e^{-\xi})),$$

where $I : \mathbb{R}^+ \to \mathbb{R}^+$, $I(y) := (M_x)^{-1}(y)$ is the inverse function of M_x. Therefore, uniqueness of the non-negative solution follows from classical results for the heat equation. ∎

CHAPTER 2 ASYMPTOTIC ANALYSIS

This chapter presents the preliminary for the application chapters, and summarizes the key steps from a series of works studied by Fouque and Hu. We investigate the optimal portfolio problem under stochastic volatility models, where the volatility processes are characterized by the scales of fluctuation. Using the approach of asymptotic approximation developed in the book by [Fouque et al., 2011], we derive the corresponding asymptotic expansion of the value function and investigate the asymptotic optimal strategy for the portfolio problems with slow-scale, fast-scale, and multi-scale volatility models.

2.1 Overview

The goal of the following sections in this chapter is: (1) to derive the zeroth order and the first order asymptotic expansion of the value function for the incomplete markets portfolio optimization problem under multi-scale stochastic volatility model, and (2) to show that the zeroth order asymptotic approximation of the optimal strategy recovers the value function up to the first order. In order to tackle the multi-scale stochastic volatility model, we first study the Merton problems under single-scale case, including both slow-scale and fast-scale volatility.

In Section 2.2, we discuss the slow-scale stochastic volatility model discussed in [Fouque and Hu, 2017]. Assuming the return and volatility terms of the underlying asset are driven by a slow-scale stochastic process, we set up the Merton problem for the underlying asset and derive the corresponding HJB equation. With the assumption that the HJB problem has a smooth solution, we present the regular perturbation method that derives the asymptotic expansion of the value function that solves the HJB equation. Focusing on the zeroth and the first order expansions, we also check that the zeroth order optimal strategy recovers the value function up to the first order.

In Section 2.3, we present the case in [Hu, 2018] where the return and volatility of the asset are modeled by a fast-scale process. In addition to the process in Section 2.2, the fast-scale volatility model requires additional steps: the singular perturbation method is applied to derive efficient approximations of the value function for the HJB problem. Similar to the slow-scale model, the zeroth order optimal strategy in the fast-scale case also recovers the value function up to the first order.

Section 2.4 combine the methods from the previous two sections. In a multi-scale stochastic volatility model in [Fouque and Hu, 2020], the asymptotic expansion of the corresponding value function for the Merton problem will have a zeroth order term and two first order terms for slow-scale factor and fast-scale factor, respectively. Again, assuming the regularity of the value function that solves the HJB problem, we can derive the zeroth and the first order expansions, together with the zeroth order optimal strategy.

A more rigorous result for the multi-scale model is presented in Section 2.5. The regularity of the value function is not assumed. Instead, we enforce two sets of assumptions (in Appendix B) on the state processes and the utility function. Under these assumptions, we prove that the heuristic expansion up to the first order derived in Section 2.4 coincides with the rigorous first order approximation for the value function corresponding to the zeroth order optimal strategy. We present the proof in two detailed steps: (1) heuristic derivation, and (2) expansion justification. In fact, similar rigorous results also hold for the slow-scale model and the fast-scale model. We refer to [Fouque and Hu, 2017] and [Hu, 2018] for the detailed assumptions and proofs.

2.2 Slow-scale Stochastic Volatility Model

Stochastic Control Problem set-up: Consider the following dynamics of the underlying asset S_t and the slow-scale factor Z_t:

$$\begin{aligned} dS_t &= \mu(Z_t)S_t dt + \sigma(Z_t)S_t dW_t \\ dZ_t &= \delta c(Z_t)dt + \sqrt{\delta} g(Z_t) dW_t^Z \end{aligned} \tag{2.2.1}$$

where W_t and W_t^Z are two standard Brownian motions in a filtered probability space $(\Omega, \mathscr{F}, \mathscr{F}_t, \mathbb{P})$ with a correlation: $d\left\langle W, W^Z \right\rangle_t = \rho dt$ with $|\rho| < 1$. It is assumed that the process $Z_t = Z_{\delta t}^{(1)}$ in distribution, where $Z^{(1)}$ is a diffusion process with coefficients c and g. This re-scaling of time explains the $\sqrt{\delta}$ in the diffusion term in (2.2.1) We refer to [Fouque et al., 2011, Chapter 3] for more details on stochastic volatility time scales. Also, we require the coefficient $\mu(z)$ and $\sigma(z)$ to be differentiable.

Similar to the classical Merton problem, let $(X_t, t \in [0,T])$ denote the wealth process over the investment period $[0,T]$ and $\pi(t, x, z)$ denote the amount of wealth held in the underlying asset at time t, when the underlying asset price is x, and the level of the slow-scale factor is z, with the remaining held in a risk-free environment at an interest rate r. Assuming that the portfolio is under continuous self-financing setting and, without loss of generality, the risk-free interest rate r is zero, the wealth process X_t satisfies

$$dX_t = \pi(t, X_t, Z_t)\mu(Z_t)dt + \pi(t, X_t, Z_t)\sigma(Z_t)dW_t. \tag{2.2.2}$$

Now we consider the optimization problem with a utility function $U(x)$ satisfying Assumption 1.5. The slow-scale value function $V^\delta(t,x,z)$ is defined by

$$V^\delta(t,x,z) = \sup_\pi \mathbb{E}[U(X_T) \mid X_t = x, Z_t = z], \tag{2.2.3}$$

where the supremum is taken over all admissible strategies $\pi \in \mathscr{A}^\delta(t,x,z)$,

$$\mathscr{A}^\delta(t,x,z) = \{\pi : X_s \geq 0 \forall s \geq t, \text{ given } X_t = x, Z_t = z\}. \tag{2.2.4}$$

The HJB Equation: The HJB equation for V^δ is given by

$$V_t^\delta + \delta \mathscr{M}_Z V^\delta + \max_{\pi \in \mathscr{A}^\delta} \left(\frac{1}{2}\sigma(z)^2 \pi^2 V_{xx}^\delta + \pi \left(\mu(z) V_x^\delta + \sqrt{\delta}\rho g(z)\sigma(z) V_{xz}^\delta \right) \right) = 0,$$

$$V^\delta(T,x,z) = U(x), \tag{2.2.5}$$

where $\mathscr{M}_Z$ is the infinitesimal generator of the process $Z^{(1)}$:

$$\mathscr{M}_Z = \frac{1}{2} g(z)^2 \frac{\partial^2}{\partial z^2} + c(z)\frac{\partial}{\partial z}. \tag{2.2.6}$$

Rigorously speaking, V^δ only represents the viscosity solution of this HJB equation with terminal condition. However, in the current stage of heuristic derivation, we assume that V^δ is the unique classical solution of (2.2.5).

Assumption 2.2. The value function $V^\delta(t,x,z)$ is the unique smooth function of the HJB equation (2.2.5) that is strictly increasing and strictly concave in x for each $t \in [0,T)$ and $z \in \mathbb{R}$.

The optimal control of the maximization term in the HJB equation is given in the feedback form by

$$\pi^* = -\frac{\lambda(z)V_x^\delta}{\sigma(z)V_{xx}^\delta} - \frac{\sqrt{\delta}\rho g(z)V_{xz}^\delta}{\sigma(z)V_{xx}^\delta}, \tag{2.2.7}$$

where $\lambda(z) = \dfrac{\mu(z)}{\sigma(z)}$ is the Sharpe ratio. Plugging the optimal control in the HJB equation gives the following equation with terminal condition,

$$V_t^\delta + \delta\mathscr{M}_Z V^\delta - \frac{\left(\lambda(z)V_x^\delta + \sqrt{\delta}\rho g(z)V_{xz}^\delta\right)^2}{2V_{xx}^\delta} = 0,\ V^\delta(T,x,z) = U(x). \tag{2.2.8}$$

Asymptotic Expansion: Under Assumption 2.2, the goal is to find the zeroth and the first order approximation in an asymptotic expansion for the value function V^δ of the form

$$V^\delta(t,x,z) = \sum_{k=0}^{\infty}(\sqrt{\delta})^k v^{(k)}(t,x,z). \tag{2.2.9}$$

By letting $\delta = 0$ in (2.2.8) and (2.2.9), we obtain that the zeroth order approximation $v^{(0)}$ solves the HJB equation

$$v_t^{(0)} - \frac{1}{2}\lambda(z)^2\frac{(v_x^{(0)})^2}{v_{xx}^{(0)}} = 0,\ v^{(0)}(T,x,z) = U(x). \tag{2.2.10}$$

As a result, $v^{(0)}$ coincides with the constant parameter Merton value function

$$v^{(0)}(t,x,z) = M(t,x;\lambda(z)), \tag{2.2.11}$$

where the Merton value function $M(t,x;\lambda)$ is defined in Section 1.5 with the Sharpe ratio $\lambda(z) = \dfrac{\mu(z)}{\sigma(z)}$.

For the first order approximation, we take the order $\sqrt{\delta}$ terms after inserting the expansion (2.2.9) into the HJB equation (2.2.8)

$$\mathscr{L}_{t,x}(\lambda(z))v^{(1)} = \rho\lambda(z)g(z)\left(\frac{v_x^{(0)}v_{xz}^{(0)}}{v_{xx}^{(0)}}\right), v^{(1)}(T,x,z) = 0, \tag{2.2.12}$$

where $\mathscr{L}_{t,x}$ is the linear operator defined in (1.5.9). In order to construct the solution to (2.2.12) and explicitly compute the first order approximation, we require the following lemmas.

Lemma A.3. The classical Merton value function $M(t,x;\lambda)$ satisfies

$$\mathscr{L}_{t,x}(\lambda)D_1^k M(t,x;\lambda) = 0, \tag{A.6}$$

for all $k = 1,2,\cdots$.

Lemma A.4 ("Vega-Gamma" relationship). The Merton value function $M(t,x;\lambda)$ introduced in Section 1.5 satisfies

$$\frac{\partial M}{\partial \lambda} = -(T-t)\lambda\frac{M_x^2}{M_{xx}} = (T-t)\lambda D_1 M, \tag{A.7}$$

where R is the risk-tolerance function defined in (1.5.7).

The proofs of Lemma A.3 and Lemma A.4 require a commutation result (Lemma A.2). The detailed proof is presented in Appendix A. Now we use the "Vega-Gamma" relationship to derive an explicit expression for the first order approximation $v^{(1)}$ in terms of the zeroth order approximation $v^{(0)}$.

Proposition 2.2. The PDE problem with zero terminal condition (2.2.12) has a unique solution:

$$v^{(1)}(t,x,z) = \frac{1}{2}(T-t)^2 \rho \lambda^2(z) g(z) \lambda'(z) D_1^2 v^{(0)}. \tag{2.2.13}$$

Proof. Re-write the PDE (2.2.12) as:

$$\mathscr{L}_{t,x}(\lambda(z)) v^{(1)} = -\rho \lambda(z) g(z) D_1 v_z^{(0)}. \tag{2.2.14}$$

By the “Vega-Gamma” relationship (Lemma A.4) and the Chain Rule,

$$v_z^{(0)} = (T-t)\lambda(z)\lambda'(z) D_1 v^{(0)}. \tag{2.2.15}$$

Substituting (2.2.15) into (2.2.14), we have

$$\mathscr{L}_{t,x}(\lambda(z)) v^{(1)} = -(T-t)\rho \lambda^2(z) g(z) \lambda'(z) D_1^2 v^{(0)}. \tag{2.2.16}$$

We can verify that the expression (2.2.13) solves the PDE (2.2.16) with zero terminal condition:

$$\begin{aligned}
&\mathscr{L}_{t,x}(\lambda(z)) \left(\frac{1}{2}(T-t)^2 \rho \lambda^2(z) g(z) \lambda'(z) D_1^2 v^{(0)} \right) \\
&= -(T-t)\rho \lambda^2(z) g(z) \lambda'(z) D_1^2 v^{(0)} + \frac{1}{2}(T-t)^2 \rho \lambda^2(z) g(z) \lambda'(z) \mathscr{L}_{t,x}(\lambda(z)) D_1^2 v^{(0)} \\
&= -(T-t)\rho \lambda^2(z) g(z) \lambda'(z) D_1^2 v^{(0)},
\end{aligned}$$

because $v^{(0)}(t,x,z) = M(t,x;\lambda(z))$ in (2.2.11) and $\mathscr{L}_{t,x}(\lambda(z)) D_1^2 v^{(0)} = 0$ by Lemma A.3 in Appendix A.

The uniqueness of the solution (2.2.13) can be proved using a change of variable method and the uniqueness result of the heat equation. We refer to Lemma A.5 in Appendix A for a detailed example of the change of variable method. ∎

To summarize, we acquire the asymptotic expansion of the slow-scale value function V^δ up to the first order approximation:

$$\begin{aligned}V^\delta(t,x,z) &= v^{(0)}(t,x,z) + \sqrt{\delta}v^{(1)}(t,x,z) + \cdots \\ v^{(0)}(t,x,z) &= M(t,x;\lambda(z)) \\ v^{(1)}(t,x,z) &= \frac{1}{2}(T-t)^2\rho\lambda^2(z)g(z)\lambda'(z)D_1^2 v^{(0)}.\end{aligned}$$

Asymptotic Optimal Strategy: With the asymptotic expansion of the value function, we can discuss the zeroth order optimal control as the investing strategy. By the end of this section, we will see that the zeroth order strategy reproduce the optimal value function V^δ up to the first order $\sqrt{\delta}$ term. Recall that the slow-scale optimal feedback strategy is given by

$$\pi^* = -\frac{\lambda(z)V_x^\delta}{\sigma(z)V_{xx}^\delta} - \frac{\sqrt{\delta}\rho g(z)V_{xz}^\delta}{\sigma(z)V_{xx}^\delta}. \tag{2.2.7}$$

Inserting the zeroth order term $v^{(0)}$ in the expansion of V^δ into (2.2.7) gives the zeroth order strategy:

$$\pi^{(0)}(t,x,z) := \frac{\lambda(z)}{\sigma(z)}R(t,x;\lambda(z)), \tag{2.2.17}$$

which is the Merton strategy updated with the moving level z of the slow-scale factor Z_t. Thus, the wealth process follows:

$$dX_t = \pi_t^{(0)}\mu(Z_t)dt + \pi_t^{(0)}\sigma(Z_t)dW_t.$$

The corresponding value function $\tilde{V}^\delta(t,x,z) := \mathbb{E}\left(U(X_T)\,|\,X_t = x, Z_t = z\right)$ solves the linear PDE with terminal condition:

$$\tilde{V}_t^\delta + \delta \mathscr{M}_Z \tilde{V}^\delta + \frac{1}{2}\sigma(z)^2 (\pi^{(0)})^2 \tilde{V}_{xx}^\delta + \pi^{(0)}\left(\mu(z)\tilde{V}_x^\delta + \sqrt{\delta}\rho g(z)\sigma(z)\tilde{V}_{xz}^\delta\right) = 0,$$

$$\tilde{V}^\delta(T,x,z) = U(x),$$

where $\mathscr{M}_Z$ is defined in (2.2.6). Re-writing the PDE according to the power of $\sqrt{\delta}$, we have

$$\mathscr{L}_{t,x}(\lambda(z))\tilde{V}^\delta + \sqrt{\delta}\rho\lambda(z)g(z)D_1\frac{\partial}{\partial z}\tilde{V}^\delta + \delta\mathscr{M}_Z\tilde{V}^\delta = 0. \qquad (2.2.18)$$

Now we assume the asymptotic expansion of $\tilde{V}^\delta$:

$$\tilde{V}^\delta(t,x,z) = \tilde{v}^{(0)}(t,x,z) + \sqrt{\delta}\tilde{v}^{(1)}(t,x,z) + \delta\tilde{v}^{(2)} + \cdots.$$

Then we insert the expansion into the re-written PDE (2.2.18). By taking the constant and the $\sqrt{\delta}$ terms in the PDE (2.2.18), we obtain

$$\mathscr{L}_{t,x}(\lambda(z))\tilde{v}^{(0)} = 0,\, \tilde{v}^{(0)}(T,x) = U(x)$$

and

$$\mathscr{L}_{t,x}(\lambda(z))\tilde{v}^{(1)} = -\rho\lambda(z)g(z)D_1\tilde{v}_z^{(0)},\, \tilde{v}^{(1)}(T,x) = 0.$$

The uniqueness of (2.2.10) gives $\tilde{v}^{(0)} \equiv v^{(0)}$, and the uniqueness of (2.2.14) gives $\tilde{v}^{(1)} \equiv v^{(1)}$. Therefore, we can conclude that the zeroth order optimal strategy $\pi^{(0)}$ given by (2.2.17) recovers the optimal value function V^δ of the stochastic control problem up to order $\sqrt{\delta}$.

Further Study: In fact, by replacing Assumption 2.2 on regularity of the value function with Assumption B.1 and Assumption B.3 in Appendix B, we can conclude a stronger result that the residual function E defined by

$$E(t,x,z) := \tilde{V}^{\delta}(t,x,z) - v^{(0)}(t,x,z) - \sqrt{\delta}v^{(1)}(t,x,z),$$

has order δ. In other words, for all $(t,x,z) \in [0,T] \times \mathbb{R}^{+} \times \mathbb{R}$, there exists C such that $\left|E(t,x,z)\right| \leq \delta C$, where C may depend on (t,x,z) but independent of δ. The detailed proof can be found in [Fouque and Hu, 2017]. We will not present the proof since we will show the proof of a similar result for the multi-scale case in Section 2.5.

2.3 Fast-scale Stochastic Volatility Model

Stochastic Control Problem set-up: The stochastic control problem associated with the fast-scale volatility model is similar to the slow-scale case. Consider the following dynamics of the underlying asset S_t and the fast-scale factor Y_t:

$$\begin{aligned} dS_t &= \mu(Y_t)S_t dt + \sigma(Y_t)S_t dW_t \\ dY_t &= \frac{1}{\epsilon}b(Y_t)dt + \frac{1}{\sqrt{\epsilon}}a(Y_t)dW_t^Y \end{aligned} \tag{2.3.1}$$

where W_t and W_t^Y are two standard Brownian motions in a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ with a correlation: $d\left\langle W, W^Y \right\rangle_t = \rho dt$ with $|\rho| < 1$. In this case, since we want to view the fast-scale factor as mean-reversion of volatility, we assume that the process $Y_t = Y^{(1)}_{t/\epsilon}$ in distribution, where $Y^{(1)}$ is an ergodic diffusion process with unique ϵ-independent invariant distribution Φ. Hence, the re-scaling of time implies that the fast-scale factor Y also has unique invariant distribution Φ. Also, we assume that the coefficient $\mu(y)$ and $\sigma(y)$ to be differentiable.

Under a continuous self-financing trading with the risk-free interest rate $r = 0$, the wealth process $(X_t, t \in [0,T])$ satisfies

$$dX_t = \pi(t, X_t, Y_t)\mu(Y_t)dt + \pi(t, X_t, Y_t)\sigma(Y_t)dW_t, \tag{2.3.2}$$

where $\pi(t, X_t, Y_t)$ is the stochastic control, which is viewed as the investment strategy. The fast-scale value function $V^\epsilon(t, x, y)$ of the optimization problem is defined by

$$V^\epsilon(t, x, y) = \sup_\pi \mathbb{E}[U(X_T) \,|\, X_t = x, Y_t = y], \tag{2.3.3}$$

where the utility function $U(x)$ satisfies Assumption 1.5, and the supremum is taken over all admissible strategies $\pi \in \mathscr{A}^\epsilon(t, x, y)$,

$$\mathscr{A}^\epsilon(t, x, y) = \{\pi : X_s \geq 0 \forall s \geq t, \text{ given } X_t = x, Y_t = y\}. \tag{2.3.4}$$

The HJB Equation: The HJB equation for V^ϵ is given by

$$V_t^\epsilon + \frac{1}{\epsilon}\mathscr{M}_Y V^\epsilon + \max_{\pi \in \mathscr{A}^\epsilon}\left(\frac{1}{2}\sigma(y)^2\pi^2 V_{xx}^\epsilon + \pi\left(\mu(y)V_x^\epsilon + \frac{\rho a(y)\sigma(y)}{\sqrt{\epsilon}}V_{xy}^\epsilon\right)\right) = 0,$$

$$V^\epsilon(T, x, y) = U(x), \tag{2.3.5}$$

where $\mathscr{M}_Y$ is the infinitesimal generator of the process $Y^{(1)}$:

$$\mathscr{M}_Y = \frac{1}{2}a(y)^2\frac{\partial^2}{\partial y^2} + b(y)\frac{\partial}{\partial y}. \tag{2.3.6}$$

Similar to the slow-scale case, we make the following assumption on the solution V^ϵ of the HJB equation with the terminal condition (2.3.5).

Assumption 2.3. The value function $V^\epsilon(t,x,y)$ is the unique smooth function of the HJB equation (2.3.5) that is strictly increasing and strictly concave in x for each $t \in [0,T)$ and $y \in \mathbb{R}$.

The corresponding optimal feedback control in this case is given by

$$\pi^* = -\frac{\lambda(y)V_x^\epsilon}{\sigma(y)V_{xx}^\epsilon} - \frac{\rho a(y)V_{xy}^\epsilon}{\sqrt{\epsilon}\sigma(y)V_{xx}^\epsilon}, \tag{2.3.7}$$

where $\lambda(y) = \dfrac{\mu(y)}{\sigma(y)}$ is the Sharpe ratio. Plugging (2.3.7) in the HJB equation (2.3.5) gives the following equation with terminal condition,

$$V_t^\epsilon + \frac{1}{\epsilon}\mathscr{M}_Y V^\epsilon - \frac{\left(\lambda(z)V_x^\epsilon + \frac{\rho a(y)}{\sqrt{\epsilon}}V_{xy}^\epsilon\right)^2}{2V_{xx}^\epsilon} = 0,\ V^\epsilon(T,x,y) = U(x)\,. \tag{2.3.8}$$

Asymptotic Expansion: Under Assumption 2.3, we aim to find the zeroth and the first order terms in the asymptotic expansion for the value function V^ϵ in the fast-scale case:

$$V^\epsilon(t,x,y) = \sum_{k=0}^{\infty} (\sqrt{\epsilon})^k v^{(k)}(t,x,y). \tag{2.3.9}$$

Because there are negative power terms of ϵ in the HJB equation (2.3.8), we cannot apply the same method as in the slow-scale case. The computation of $v^{(0)}$ and $v^{(1)}$ in the fast-scale case requires additional steps.

We first show that $v^{(0)}$ and $v^{(1)}$ are independent of y. We insert the expansion (2.3.9) into the HJB equation (2.3.8) and collect the terms in successive powers of ϵ. The term with order ϵ^{-1} is given by

$$\mathscr{M}_Y v^{(0)} - \frac{1}{2}\rho^2 a(y)^2 \frac{(v_{xy}^{(0)})^2}{v_{xx}^{(0)}} = 0,\ v^{(0)}(T, x, y) = U(x).$$

Note that $\mathscr{M}_Y$ only takes derivatives in y and the terminal condition is independent of y. Hence, this equation is satisfied by $v^{(0)}(t, x)$ independent of y. Thus, $v_y^{(0)} = 0$. The term with order $\epsilon^{-\frac{1}{2}}$ is given by

$$\mathscr{M}_Y v^{(1)} = 0,\ v^{(1)}(T, x, y) = 0.$$

Similarly, we obtain that $v^{(1)} = v^{(1)}(t, x)$ is independent of y and $v_y^{(1)} = 0$.

To compute the zeroth order approximation $v^{(0)}(t, x)$, we collect the constant order terms after inserting (2.3.9) into (2.3.8):

$$v_t^{(0)} + \mathscr{M}_Y v^{(2)} - \frac{1}{2}\lambda(y)^2 \frac{(v_x^{(0)})^2}{v_{xx}^{(0)}} = 0. \tag{2.3.10}$$

This equation is a Poisson equation for $v^{(2)}$ with solvability condition requiring the Fredholm Alternative:

$$\langle v_t^{(0)} - \frac{1}{2}\lambda(y)^2 \frac{(v_x^{(0)})^2}{v_{xx}^{(0)}} \rangle = 0,$$

where $\langle \cdot \rangle$ denotes averaging with respect to Φ:

$$\langle f \rangle = \int f(y)\Phi(dy).$$

Define the constant square-averaged Sharpe ratio $\bar{\lambda}$ by

$$\bar{\lambda}^2 = \langle \frac{\mu(y)^2}{\sigma(y)^2} \rangle.$$

As $v^{(0)}$ is independent of y, the Fredholm Alternative matches the classical Merton problem (1.5.5) with constant Sharpe ratio $\bar{\lambda}$:

$$v_t^{(0)} - \frac{1}{2}\bar{\lambda}^2 \frac{(v_x^{(0)})^2}{v_{xx}^{(0)}} = 0.$$

With the terminal condition $v^{(0)}(T, x) = U(x)$, we acquire the zeroth order approximation of the value function:

$$v^{(0)}(t, x) = M(t, x; \bar{\lambda}), \tag{2.3.11}$$

where M is the classical Merton value function introduced in Section 1.5. Note that, with (2.3.11), the above Merton equation of $v^{(0)}$ can be re-written as $\mathscr{L}_{t,x}(\bar{\lambda})v^{(0)} = 0$.

For the first order approximation, we re-write (2.3.10) as

$$\begin{aligned} 0 = \mathscr{M}_Y v^{(2)} + \mathscr{L}_{t,x}(\lambda(y))v^{(0)} &= \mathscr{M}_Y v^{(2)} + \mathscr{L}_{t,x}(\lambda(y))v^{(0)} - \mathscr{L}_{t,x}(\bar{\lambda})v^{(0)} \\ &= \mathscr{M}_Y v^{(2)} + \left(\mathscr{L}_{t,x}(\lambda(y)) - \mathscr{L}_{t,x}(\bar{\lambda})\right) v^{(0)} \\ &= \mathscr{M}_Y v^{(2)} + \left(\lambda(y)^2 - \bar{\lambda}^2\right) \left(\frac{1}{2}D_2 + D_1\right) v^{(0)}, \end{aligned}$$

since $\mathscr{L}_{t,x}(\bar{\lambda})v^{(0)} = 0$. This Poisson equation for $v^{(2)}$ have the following solutions in $L^2(\Phi)$ (see [Fouque et al., 2011, Chapter 3]):

$$v^{(2)} = -\theta(y)\left(\frac{1}{2}D_2 + D_1\right)v^{(0)} + C(t,x) = -\frac{1}{2}\theta(y)D_1 v^{(0)} + C(t,x) \tag{2.3.12}$$

where $\theta(y)$ solves the ordinary differential equation in y variable:

$$\mathscr{M}_Y\theta = \lambda(y)^2 - \bar{\lambda}^2, \tag{2.3.13}$$

and $C(t,x)$ is a constant of integration in y that may depend on (t,x). Plugging (2.3.12) and $v_y^{(1)} = 0$ into the expansion of the HJB equation (2.3.8), we simplify the nonlinear term up to order $\sqrt{\epsilon}$:

$$\begin{aligned}
\frac{\left(\lambda(z)V_x^\epsilon + \frac{\rho a(y)}{\sqrt{\epsilon}}V_{xy}^\epsilon\right)^2}{2V_{xx}^\epsilon} &= \left(\lambda(y)(v_x^{(0)} + \sqrt{\epsilon}v_x^{(1)}) + \rho a(y)(v_{xy}^{(1)} + \sqrt{\epsilon}v_{xy}^{(2)})\right)^2 \frac{1}{2v_{xx}^{(0)}}\left(1 - \sqrt{\epsilon}\frac{v_{xx}^{(1)}}{v_{xx}^{(0)}}\right) + \cdots \\
&= \left(\lambda(y)(v_x^{(0)} + \sqrt{\epsilon}v_x^{(1)}) + \frac{1}{2}\sqrt{\epsilon}\rho a(y)\theta'(y)\frac{\partial}{\partial x}\frac{(v_x^{(0)})^2}{v_{xx}^{(0)}}\right)^2 \frac{1}{2v_{xx}^{(0)}}\left(1 - \sqrt{\epsilon}\frac{v_{xx}^{(1)}}{v_{xx}^{(0)}}\right) + \cdots \\
&= \cdots + \frac{\sqrt{\epsilon}}{2v_{xx}^{(0)}}\left(2\lambda(y)v_x^{(0)}\Big(\lambda(y)v_x^{(1)} + \frac{1}{2}\rho a(y)\theta'(y)\frac{\partial}{\partial x}\frac{(v_x^{(0)})^2}{v_{xx}^{(0)}}\Big) - \lambda(y)^2(v_x^{(0)})^2\frac{v_{xx}^{(1)}}{v_{xx}^{(0)}}\right) + \cdots \\
&= \cdots + \sqrt{\epsilon}\left(-\lambda(y)^2 D_1 v^{(1)} + \frac{1}{2}\rho\lambda(y)a(y)\theta'(y)D_1^2 v^{(0)} - \frac{1}{2}\lambda(y)^2 D_2 v^{(1)}\right) + \cdots.
\end{aligned}$$

Therefore, after re-writing and rearranging, we obtain the term with order $\sqrt{\epsilon}$ in the asymptotic expansion of the HJB equation (2.3.8):

$$\mathscr{M}_Y v^{(3)} + \mathscr{L}_{t,x}(\lambda(y))v^{(1)} - \frac{1}{2}\rho\lambda(y)a(y)\theta'(y)D_1^2 v^{(0)} = 0. \tag{2.3.14}$$

Equation (2.3.14) is a Poisson equation of $v^{(3)}$, and the solvability condition is

$$\mathscr{L}_{t,x}(\bar{\lambda})v^{(1)} = \frac{1}{2}\rho B D_1^2 v^{(0)}. \tag{2.3.15}$$

where we define the constant $B := \langle \lambda(y)a(y)\theta'(y)\rangle$.

Proposition 2.3. The PDE problem (2.3.15) with terminal condition $v^{(1)}(T, x) = 0$ has a unique solution:

$$v^{(1)}(t, x) = -(T - t)\frac{1}{2}\rho B D_1^2 v^{(0)}(t, x). \tag{2.3.16}$$

Proof. First, we compute that (2.3.16) satisfies (2.3.15):

$$\begin{aligned} \mathscr{L}_{t,x}(\bar{\lambda})\left(-(T - t)\frac{1}{2}\rho B D_1^2 v^{(0)}(t, x)\right) &= \frac{1}{2}\rho B D_1^2 v^{(0)} - (T - t)\frac{1}{2}\rho B \mathscr{L}_{t,x}(\bar{\lambda})D_1^2 v^{(0)} \\ &= \frac{1}{2}\rho B D_1^2 v^{(0)} \end{aligned}$$

because $v^{(0)}(t, x) = M(t, x; \bar{\lambda})$ and $\mathscr{L}_{t,x}(\bar{\lambda})D_1^2 M(t, x; \bar{\lambda}) = 0$ (Lemma A.3 in Appendix A). Also, the expression (2.3.16) satisfies the terminal condition $v^{(1)}(T, x) = 0$.

The uniqueness is proved with the same method as in the slow-scale case. We refer to Lemma A.5 in Appendix A for the detailed proof of uniqueness. ∎

To summarize, the asymptotic expansion of the fast-scale value function V^ϵ up to the first order approximation is given by:

$$
\begin{aligned}
V^{\epsilon}(t,x,y) &= v^{(0)}(t,x) + \sqrt{\epsilon}\, v^{(1)}(t,x) + \cdots \\
v^{(0)}(t,x) &= M(t,x;\bar{\lambda}\,) \\
v^{(1)}(t,x) &= -\,(T-t)\frac{1}{2}\rho B D_1^2 v^{(0)}(t,x)\,.
\end{aligned}
$$

Asymptotic Optimal Strategy: Now we discuss the zeroth order strategy for the fasts-scale volatility model. Similarly, it recovers the optimal value function V^{ϵ} up to the first order $\sqrt{\epsilon}$ term. The fast-scale optimal feedback strategy is given by (2.3.7):

$$
\pi^* = -\frac{\lambda(y)V_x^{\epsilon}}{\sigma(y)V_{xx}^{\epsilon}} - \frac{\rho a(y)V_{xy}^{\epsilon}}{\sqrt{\epsilon}\sigma(y)V_{xx}^{\epsilon}}. \tag{2.3.7}
$$

Hence, by plugging in $v^{(0)}(t,x) = M(t,x;\bar{\lambda}\,)$, we have the zeroth order strategy

$$
\pi^{(0)}(t,x,y) := \frac{\lambda(y)}{\sigma(y)} R(t,x;\bar{\lambda}\,). \tag{2.3.17}
$$

This is a hybrid Merton strategy in which the coefficient $\dfrac{\lambda(y)}{\sigma(y)}$ moves as the fast-scale factor Y_t moves, and the risk tolerance component R uses the constant averaged Sharpe ratio $\bar{\lambda}$. With the wealth process following:

$$
dX_t = \pi_t^{(0)}\mu(Y_t)dt + \pi_t^{(0)}\sigma(Y_t)dW_t,
$$

the corresponding value function $\tilde{V}^{\epsilon}(t,x,y) := \mathbb{E}\left(U(X_T)\,|\,X_t = x, Y_t = y\right)$ solves the linear PDE with terminal condition:

$$\tilde{V}^\epsilon_t + \frac{1}{\epsilon}\mathscr{M}_Y\tilde{V}^\epsilon + \frac{1}{2}\sigma(y)^2(\pi^{(0)})^2\tilde{V}^\epsilon_{xx} + \pi^{(0)}\left(\mu(y)\tilde{V}^\epsilon_x + \frac{1}{\sqrt{\epsilon}}\rho a(y)\sigma(y)\tilde{V}^\epsilon_{xy}\right) = 0,$$

$$\tilde{V}^\epsilon(T,x,y) = U(x),$$

where $\mathscr{M}_Y$ is defined in (2.3.6). By inserting $\pi^{(0)}$ from (2.3.17), we re-write the PDE as

$$\mathscr{L}_{t,x}(\lambda(y))\tilde{V}^\epsilon + \frac{1}{\sqrt{\epsilon}}\rho\lambda(y)a(y)D_1\frac{\partial}{\partial y}\tilde{V}^\epsilon + \frac{1}{\epsilon}\mathscr{M}_Y\tilde{V}^\epsilon = 0. \tag{2.3.18}$$

Assuming the asymptotic expansion of $\tilde{V}^\epsilon$:

$$\tilde{V}^\epsilon(t,x,y) = \tilde{v}^{(0)}(t,x,y) + \sqrt{\epsilon}\,\tilde{v}^{(1)}(t,x,y) + \epsilon\,\tilde{v}^{(2)} + \cdots,$$

we insert the expansion into (2.3.18) and collecting the terms in successive power of ϵ. Following the same computation process as in the fast-scale asymptotic expansion subsection, we are able to reproduce $\tilde{v}^{(0)}$ and $\tilde{v}^{(1)}$. In particular, the order ϵ^{-1} term and the order $\epsilon^{-\frac{1}{2}}$ term of the expanded equation gives that $\tilde{v}^{(0)}$ and $\tilde{v}^{(1)}$ are independent of y. The solvability condition of the Poisson equation (with respect to $\tilde{v}^{(2)}$) deduced from the constant order term is given by $\mathscr{L}_{t,x}\tilde{v}^{(0)} = 0$. With the terminal condition $\tilde{v}^{(0)}(T,x,y) = U(x)$, we acquire that $\tilde{v}^{(0)} \equiv v^{(0)}$ by uniqueness of Merton equation. Solving this Poisson equation deduced from the constant order term yields

$$\tilde{v}^{(2)} = -\frac{1}{2}\theta(y)D_1\tilde{v}^{(0)} + \tilde{C}(t,x), \tag{2.3.19}$$

where θ is a solution of the ordinary differential equation (2.3.13), and $\tilde{C}(t, x)$ is a constant of integration in y. Plugging (2.3.19) into the solvability condition of the Poisson equation (with respect to $\tilde{v}^{(3)}$) deduced from the order $\sqrt{\epsilon}$ term of the expansion, we acquire

$$\mathscr{L}_{t,x}(\bar{\lambda})\tilde{v}^{(1)} = \frac{1}{2}\rho\langle\lambda(y)a(y)\theta'(y)\rangle D_1^2\tilde{v}^{(0)},$$

with terminal condition $\tilde{v}^{(1)}(T, x, y) = 0$. By the uniqueness of Proposition 2.3, $\tilde{v}^{(1)} \equiv v^{(1)}$. Therefore, in the fast-scale case, the zeroth order optimal strategy $\pi^{(0)}$ given by (2.3.17) also recovers the optimal value function V^{ϵ} of the stochastic control problem up to order $\sqrt{\epsilon}$.

Further Study: Similar to the slow-scale case, by replacing Assumption 2.3 with Assumption B.1 and Assumption B.3 in Appendix B, we have a stronger conclusion that the residual function E defined by

$$E(t, x, y) := \tilde{V}^{\epsilon}(t, x, y) - v^{(0)}(t, x) - \sqrt{\epsilon}\, v^{(1)}(t, x),$$

has order ϵ. In other words, for all $(t, x, y) \in [0,T] \times \mathbb{R}^+ \times \mathbb{R}$, there exists C such that $\left|E(t, x, y)\right| \leq \epsilon C$, where C may depend on (t, x, y) but independent of ϵ. This result follows from the discussion in [Hu, 2018]. Again, we omit the proof of the fast-scale case since a multi-scale version will be presented in Section 2.5.

2.4 Multi-scale Stochastic Volatility Model

Stochastic Control Problem set-up: With the asymptotic analysis of slow-scale and fast-scale cases, we now discuss the two-factor multi-scale stochastic volatility model. Consider the following dynamics of the underlying asset S_t, the fast-scale factor Y_t, and the slow-scale factor Z_t:

$$
\begin{aligned}
dS_t &= \mu(Y_t, Z_t)S_t dt + \sigma(Y_t, Z_t)S_t dW_t \\
dY_t &= \frac{1}{\epsilon} b(Y_t)dt + \frac{1}{\sqrt{\epsilon}} a(Y_t)dW_t^Y \\
dZ_t &= \delta c(Z_t)dt + \sqrt{\delta} g(Z_t)dW_t^Z
\end{aligned}
\tag{2.4.1}
$$

where (W_t, W_t^Y, W_t^Z) are correlated Brownian motions in a filtered probability space $(\Omega, \mathscr{F}, \mathscr{F}_t, \mathbb{P})$ such that

$$
d\left\langle W, W^Y \right\rangle_t = \rho_Y dt, d\left\langle W, W^Z \right\rangle_t = \rho_Z dt, d\left\langle W^Y, W^Z \right\rangle_t = \rho_{YZ} dt,
$$

with positive definite constraints:

$$
|\rho_Y| < 1,\ |\rho_Z| < 1,\ |\rho_{YZ}| < 1, \text{ and } 1 + 2\rho_Y\rho_Z\rho_{YZ} - \rho_Y^2 - \rho_Z^2 - \rho_{YZ}^2 > 0.
$$

We assume that the process $Y_t = Y_{\epsilon t}^{(1)}$ and $Z_t = Z_{\delta t}^{(1)}$ in distribution, respectively, where $Y^{(1)}$ is a continuous diffusion process with a ϵ-free infinitesimal generator $\mathscr{M}_Y$ defined in (2.3.6) and $Z^{(1)}$ is a continuous diffusion process with a δ-free infinitesimal generator $\mathscr{M}_Z$ defined in (2.2.6). Also, we assume $Y^{(1)}$ is ergodic and equipped with a unique invariant distribution Φ, where $\langle \cdot \rangle$ denotes averaging with respect to Φ:

$$
\langle f \rangle = \int f(y)\Phi(dy).
$$

In the multi-scale case, under a continuous self-financing trading with the risk-free interest rate $r = 0$, the wealth process $(X_t, t \in [0,T])$ satisfies

$$dX_t = \pi(t, X_t, Y_t, Z_t)\mu(Y_t, Z_t)dt + \pi(t, X_t, Y_t, Z_t)\sigma(Y_t, Z_t)dW_t. \tag{2.4.2}$$

The multi-scale value function $V^{\epsilon,\delta}(t, x, y, z)$ of the optimization problem is defined by

$$V^{\epsilon,\delta}(t, x, y, z) = \sup_{\pi} \mathbb{E}[U(X_T) \,|\, X_t = x, Y_t = y, Z_t = z], \tag{2.4.3}$$

where the utility function $U(x)$ satisfies Assumption 1.5, and the supremum is taken over all admissible strategies $\pi \in \mathscr{A}^{\epsilon,\delta}(t, x, y, z)$,

$$\mathscr{A}^{\epsilon,\delta}(t, x, y, z) = \{\pi : X_s \geq 0 \forall s \geq t, \text{ given } X_t = x, Y_t = y, Z_t = z\}. \tag{2.4.4}$$

The HJB Equation: Applying dynamical programming, we acquire the HJB equation for the multi-scale value function $V^{\epsilon,\delta}$:

$$\left(\frac{\partial}{\partial t} + \frac{1}{\epsilon}\mathscr{M}_Y + \delta\mathscr{M}_Z + \sqrt{\frac{\delta}{\epsilon}}\mathscr{M}_{YZ}\right)V^{\epsilon,\delta} + \mathrm{NL}^{\epsilon,\delta} = 0,\ V^{\epsilon,\delta}(T, x, y, z) = U(x), \tag{2.4.5}$$

where $\mathscr{M}_Y$ is defined in (2.3.6), $\mathscr{M}_Z$ is defined in (2.2.6), $\mathscr{M}_{YZ}$ is defined by

$$\mathscr{M}_{YZ} := \rho_{YZ}a(y)g(z)\frac{\partial^2}{\partial y \partial z}, \tag{2.4.6}$$

and the nonlinear term $\mathrm{NL}^{\epsilon,\delta}$ is given by

$$\max_{\pi \in \mathscr{A}^{\epsilon,\delta}} \left(\frac{1}{2}\pi^2\sigma(y,z)^2V^{\epsilon,\delta}_{xx} + \pi\left(\mu(y,z)V^{\epsilon,\delta}_x + \frac{1}{\sqrt{\epsilon}}\rho_Y a(y)\sigma(y,z)V^{\epsilon,\delta}_{xy} + \sqrt{\delta}\rho_Z g(z)\sigma(y,z)V^{\epsilon,\delta}_{xz}\right)\right),$$

The optimal control of the maximization term in the HJB equation is given in the feedback form by

$$\pi^* = -\frac{\left(\lambda(y,z)V_x^{\epsilon,\delta} + \frac{1}{\sqrt{\epsilon}}\rho_Y a(y) V_{xy}^{\epsilon,\delta} + \sqrt{\delta}\rho_Z g(z) V_{xz}^{\epsilon,\delta}\right)}{\sigma(y,z)V_{xx}^{\epsilon,\delta}}, \tag{2.4.7}$$

where the Sharpe ratio is $\lambda(y,z) := \dfrac{\mu(y,z)}{\sigma(y,z)}$. Plugging the optimal control in the HJB equation gives the following equation with terminal condition,

$$\left(\frac{\partial}{\partial t} + \frac{1}{\epsilon}\mathcal{M}_Y + \delta\mathcal{M}_Z + \sqrt{\frac{\delta}{\epsilon}}\mathcal{M}_{YZ}\right)V^{\epsilon,\delta} - \frac{\left(\lambda V_x^{\epsilon,\delta} + \frac{\rho_Y a V_{xy}^{\epsilon,\delta}}{\sqrt{\epsilon}} + \sqrt{\delta}\rho_Z V_{xz}^{\epsilon,\delta}\right)^2}{2V_{xx}^{\epsilon,\delta}} = 0, \tag{2.4.8}$$

$$V^{\epsilon,\delta}(T,x,y,z) = U(x)\,.$$

Assumption 2.4. The value function $V^{\epsilon,\delta}(t,x,y,z)$ is the unique smooth solution of the HJB equation (2.4.8) that is strictly increasing and strictly concave in x for each $t \in [0,T)$ and $(y,z) \in \mathbb{R}^2$.

Assumption 2.4 is assumed in the derivation of the asymptotic expansion of $V^{\epsilon,\delta}$ in [Fouque et al., 2016], because $V^{\epsilon,\delta}$ is only generally identified as the viscosity solution of (2.4.8). However, in fact, the result do not reply on the regularity of $V^{\epsilon,\delta}$. Later in Section 2.5, we will present a more detailed and rigorous main result in [Fouque and Hu, 2020] about the asymptotic expansion of $V^{\epsilon,\delta}$, which is proved with a different set of assumptions from Assumption 2.4.

Asymptotic Expansion: Under Assumption 2.4, we expand $V^{\epsilon,\delta}$ in terms of both the fast-scale and the slow-scale approximation:

$$V^{\epsilon,\delta}(t,x,y,z) = v^{(0)} + \sqrt{\epsilon}v^{(1,0)} + \sqrt{\delta}v^{(0,1)} + \epsilon v^{(2,0)} + \delta v^{(0,2)} + \sqrt{\epsilon\delta}v^{(1,1)} + \cdots.$$

where the superscript of v stands for the power in $\sqrt{\epsilon}$ and $\sqrt{\delta}$, and $v^{(0)} := v^{(0,0)}$. We aim to combine Section 2.2 and Section 2.3 in order to derive $v^{(0)}$, $v^{(1,0)}$, and $v^{(0,1)}$.

By setting $\delta = 0$, we obtain that

$$V^{\epsilon,0} = v^{(0)} + \sqrt{\epsilon}v^{(1,0)} + \epsilon v^{(2,0)}\cdots, \tag{2.4.9}$$

satisfies the equation:

$$\left(\frac{\partial}{\partial t} + \frac{1}{\epsilon}\mathscr{M}_Y\right)V^{\epsilon,0} - \frac{\left(\lambda(y,z)V_x^{\epsilon,0} + \frac{1}{\sqrt{\epsilon}}\rho_Y a(y)V_{xy}^{\epsilon,0}\right)^2}{2V_{xx}^{\epsilon,0}} = 0 \tag{2.4.10}$$

with terminal condition $V^{\epsilon,0}(T,x,y,z) = U(x)$. Equation (2.4.10) is the same as (2.3.8) except that the Sharpe ratio λ depends on both volatility factors y and z. First, we define

$$\bar{\lambda}^2(z) := \langle \lambda^2(\,\cdot\,,z)\rangle.$$

Let the risk-tolerance function R and the differential operators D_1 be associated with $\bar{\lambda}(z)$. By the asymptotic expansion subsection in Section 2.3, we have $v_y^{(0)} = v_y^{(1,0)} = 0$ and

$$\begin{aligned} v^{(0)}(t,x,z) &= M(t,x;\bar{\lambda}(z)) \\ v^{(1,0)}(t,x,z) &= -(T-t)\frac{1}{2}\rho_Y B(z) D_1^2 v^{(0)}(t,x,z), \end{aligned}$$

where $B(z) = \langle \lambda(\,\cdot\,, z) a(\,\cdot\,) \dfrac{\partial \theta}{\partial y}(\,\cdot\,, z) \rangle$ and $\theta(y, z)$ is a solution of the following ordinary differential equation in y:

$$\mathscr{M}_Y \theta = \lambda^2(y, z) - \bar{\lambda}^2(z). \tag{2.4.11}$$

For the derivation of $v^{(0,1)}$, we first construct an expansion in power of $\sqrt{\delta}$:

$$V^{\epsilon,\delta} = V^{\epsilon,0} + \sqrt{\delta} V^{\epsilon,1} + \cdots, \tag{2.4.12}$$

where $V^{\epsilon,0}$ has the expansion (2.4.9) and $V^{\epsilon,1}$ has an expansion

$$V^{\epsilon,1} = v^{(0,1)} + \sqrt{\epsilon}\, v^{(1,1)} + \epsilon\, v^{(2,1)} + \cdots. \tag{2.4.13}$$

Inserting (2.4.12) into (2.4.8) and extracting the order $\sqrt{\delta}$ term, we obtain the following equation for $V^{\epsilon,1}$ with terminal condition:

$$\frac{\partial}{\partial t} V^{\epsilon,1} + \frac{1}{\epsilon} \mathscr{M}_Y V^{\epsilon,1} + \frac{1}{\sqrt{\epsilon}} \mathscr{M}_{YZ} V^{\epsilon,0} + \mathrm{NL}^{(1)} = 0,\ V^{\epsilon,1}(T, x, y, z) = 0, \tag{2.4.14}$$

where the nonlinear term $\mathrm{NL}^{(1)}$ is given by

$$\mathrm{NL}^{(1)} = -\frac{1}{V^{\epsilon,0}_{xx}} \left(\lambda V^{\epsilon,0}_x + \frac{\rho_Y a}{\sqrt{\epsilon}} V^{\epsilon,0}_{xy} \right) \left(\rho_Z g V^{\epsilon,0}_{xz} + \lambda V^{\epsilon,1}_x + \frac{\rho_Y a}{\sqrt{\epsilon}} V^{\epsilon,1}_{xy} \right) + \frac{1}{2} \left(\frac{\lambda V^{\epsilon,0}_x + \frac{\rho_Y a}{\sqrt{\epsilon}} V^{\epsilon,0}_{xy}}{V^{\epsilon,0}_{xx}} \right)^2 V^{\epsilon,1}_{xx}.$$

Note that we only need the first term in the expansion (2.4.13). Because the operator $\mathscr{M}_{YZ}$ takes a derivative in y and the first two terms in the expansion of $V^{\epsilon,0}$ do not depend on y, the term

$\epsilon^{-1/2}\mathscr{M}_{YZ}V^{\epsilon,0}$ has order $\sqrt{\epsilon}$ and does not play a role in deriving $v^{(0,1)}$. Additionally, the terms involving $V^{\epsilon,0}_{xy}$ in $\mathrm{NL}^{(1)}$ also have order $\sqrt{\epsilon}$ and are not involved in the derivation of $v^{(0,1)}$.

Now arrange equation (2.4.14) with respect to the power of ϵ. The order ϵ^{-1} term gives $\mathscr{M}_Y v^{(0,1)} = 0$, and the order $\epsilon^{-1/2}$ term gives $\mathscr{M}_Y v^{(1,1)} = 0$. Thus, $v^{(0,1)}$ and $v^{(1,1)}$ are independent of y. At the constant order, we have

$$\begin{aligned} 0 &= \mathscr{M}_Y v^{(2,1)} + v_t^{(0,1)} + \frac{1}{2}\lambda^2\frac{(v_x^{(0)})^2}{(v_{xx}^{(0)})^2}v_{xx}^{(0,1)} - \lambda^2\frac{v_x^{(0)}}{v_{xx}^{(0)}}v_x^{(0,1)} - \rho_Z g\lambda\frac{v_x^{(0)}v_{xz}^{(0)}}{v_{xx}^{(0)}} \\ &= \mathscr{M}_Y v^{(2,1)} + \mathscr{L}_{t,x}(\lambda(y,z))v^{(0,1)} - \rho_Z\lambda(y,z)g(z)\frac{v_x^{(0)}v_{xz}^{(0)}}{v_{xx}^{(0)}}. \end{aligned}$$

This is a Poisson equation for $v^{(2,1)}$ with solvability condition for $v^{(0,1)}$:

$$\mathscr{L}_{t,x}(\overline{\lambda}(z))v^{(0,1)} = \rho_Z\hat{\lambda}(z)g(z)\frac{v_x^{(0)}v_{xz}^{(0)}}{v_{xx}^{(0)}}, \tag{2.4.15}$$

where $\hat{\lambda}(z) = \langle\lambda(\,\cdot\,,z)\rangle$. Equation (2.4.15) is the same as (2.2.12) except that the Sharpe ratio is replaced by $\hat{\lambda}(z)$. By Proposition 2.2, we conclude that

$$v^{(0,1)}(t,x,z) = \frac{1}{2}(T-t)\rho_Z\hat{\lambda}(z)g(z)D_1 v_z^{(0)} = \frac{1}{2}(T-t)^2\rho_Z\hat{\lambda}(z)\overline{\lambda}(z)\overline{\lambda}'(z)g(z)D_1^2 v^{(0)}.$$

In summary, the asymptotic expansion of the multi-scale value function up to the first order is given by:

$$
\begin{aligned}
V^{\epsilon,\delta}(t,x,y,z) &= v^{(0)} + \sqrt{\epsilon}\,v^{(1,0)} + \sqrt{\delta}\,v^{(0,1)} + \cdots, \\
v^{(0)}(t,x,z) &= M(t,x;\bar{\lambda}(z)), \\
v^{(1,0)}(t,x,z) &= -(T-t)\frac{1}{2}\rho_Y B(z) D_1^2 v^{(0)}(t,x,z), \\
v^{(0,1)}(t,x,z) &= \frac{1}{2}(T-t)^2 \rho_Z \hat{\lambda}(z)\bar{\lambda}(z)\bar{\lambda}'(z) g(z) D_1^2 v^{(0)}.
\end{aligned}
\tag{2.4.16}
$$

Asymptotic Optimal Strategy: Recall that, in the multi-scale case, the optimal feedback control for the HJB equation (2.4.5) is given by (2.4.7):

$$
\pi^* = -\frac{\left(\lambda(y,z)V_x^{\epsilon,\delta} + \frac{1}{\sqrt{\epsilon}}\rho_Y a(y) V_{xy}^{\epsilon,\delta} + \sqrt{\delta}\rho_Z g(z) V_{xz}^{\epsilon,\delta}\right)}{\sigma(y,z)V_{xx}^{\epsilon,\delta}}. \tag{2.4.7}
$$

The corresponding zeroth order strategy is given by inserting the zeroth order approximation $v^{(0)}$ into (2.4.7) and taking the constant order term:

$$
\pi^{(0)}(t,x,y,z) = \frac{\lambda(y,z)}{\sigma(y,z)} R(t,x;\bar{\lambda}(z)). \tag{2.4.17}
$$

This is a moving Merton strategy with respect to the slow-scale factor Z, and a hybrid Merton strategy with respect to the fast-scale factor Y. Combining the asymptotic optimal strategy subsections in Section 2.2 and Section 2.3, we can deduce that $\pi^{(0)}$ in (2.4.17) recovers the multi-scale optimal value function $V^{\epsilon,\delta}$ up to orders $\sqrt{\epsilon}$ and $\sqrt{\delta}$.

In fact, we are able to show a main result (Theorem 2.5) about the performance of the zeroth order optimal strategy without Assumption 2.4. Instead, we have two sets of assumptions on the state processes (S_t, X_t, Y_t, Z_t) and on the utility function U, respectively. The details of the assumptions are described in Assumption B.1 and Assumption B.3 in Appendix B.

As a byproduct, we can collect the $\sqrt{\epsilon}$ and $\sqrt{\delta}$ terms to obtain the first order approximations of the optimal strategy:

$$\pi^{(1,0)}(t,x,y,z) = \frac{\rho_Y}{\sigma v_x^{(0)}} \left(\frac{1}{2}(T-t)\lambda B(D_1 + D_2) + \frac{1}{2} a\theta_y \mathbf{1} \right) D_1 D_2 v^{(0)} \tag{2.4.18}$$

and

$$\pi^{(0,1)}(t,x,y,z) = \frac{\rho_Z g}{\sigma v_x^{(0)}} \left(\frac{1}{2}(T-t)\lambda \hat{\lambda}(D_1 + D_2) + \mathbf{1} \right) D_1 v_z^{(0)}. \tag{2.4.19}$$

2.5 Multi-scale Zeroth Order Strategy Performance

In Section 2.5, we present the rigorous proof of the main result (Theorem I) for the multi-scale volatility model in Section 2.4 by assuming only Assumption B.1 and Assumption B.3. The main usage of these two sets of assumptions is to ensure the following Lemma B.2 about integrability of $v^{(0)}$ and Lemma B.4 about boundedness of derivatives of the risk-tolerance function R. The proofs of Lemma B.2 and Lemma B.4 are in Appendix B.

Lemma B.2. Under Assumption B.1 (iv) and (v), the process $v^{(0)}(t, X_t^{\pi^{(0)}}, Z_t)$ is in $L^2([0,T] \times \Omega)$ uniformly in (ϵ, δ): for all $(t,x,y,z) \in [0,T] \times \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}$,

$$\mathbb{E}_{(t,x,y,z)} \left(\int_0^T (v^{(0)}(s, X_s^{\pi^{(0)}}, Z_s)^2 \, ds \right) \leq C_3(T,x,y,z),$$

where $v^{(0)}(t,x,z) = M(t,x;\bar{\lambda}(z))$.

Lemma B.4. Under Assumption B.3 for the utility function $U(x)$, the risk-tolerance function $R(t,x;\bar{\lambda}(z))$ satisfies: for $0 \le j \le 6$, $\exists K_j > 0$ such that for all $(t,x,\bar{\lambda}(z)) \in [0,T) \times \mathbb{R}^+ \times \mathbb{R}$,

$$\left| R^j(t,x;\bar{\lambda}(z))(\partial_x^{j+1} R(t,x;\bar{\lambda}(z))) \right| \le K_j.$$

Or equivalently, for $1 \le j \le 7$, $\exists \tilde{K}_j > 0$ such that for all $(t,x,z) \in [0,T) \times \mathbb{R}^+ \times \mathbb{R}$,

$$\left| \partial_x^j R^j(t,x;\bar{\lambda}(z))) \right| \le \tilde{K}_j.$$

Moreover, RR_{xxz}, R^2R_{xxzz}, RR_{xzz}, RR_{xxzz}, and R^2R_{xxxzz} are uniformly bounded.

With Lemma B.2 and Lemma B.4, we aim to show an assertion that the multi-scale zeroth order strategy $\pi^{(0)}$ defined in (2.4.17) can reproduce the multi-scale value function $V^{\epsilon,\delta}$ defined in (2.4.3) up to the first order approximation.

Theorem I ($\pi^{(0)}$ performance). Let $v^{(0)}$, $v^{(1,0)}$, and $v^{(0,1)}$ be the zeroth and first order terms in the asymptotic expansion of the multi-scale value function $V^{\epsilon,\delta}$, which are given by (2.4.16). Let $\tilde{V}^{\epsilon,\delta}$ be the value function associated to the zeroth order optimal strategy:

$$\pi^{(0)}(t,x,y,z) = \frac{\lambda(y,z)}{\sigma(y,z)} R(t,x;\bar{\lambda}(z)).$$

Then

$$\tilde{V}^{\epsilon,\delta} = \mathbb{E}\left(U(X_T^{\pi^{(0)}}) \,|\, X_t^{\pi^{(0)}} = x, Y_t = y, Z_t = z \right),$$

where $X_t^{\pi^{(0)}}$ is given by (2.4.2) with $\pi(t,x,y,z) = \pi^{(0)}$. Define the residual function $E(t,x,y,z)$ by

$$E(t,x,y,z) := \tilde{V}^{\epsilon,\delta}(t,x,y,z) - v^{(0)}(t,x,z) - \sqrt{\epsilon}\, v^{(1,0)}(t,x,z) - \sqrt{\delta}\, v^{(0,1)}(t,x,z).$$

Under Assumption B.1 and Assumption B.3, the residual function $E(t,x,y,z)$ has order $\epsilon + \delta$, for all $(t,x,y,z) \in [0,T] \times \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}$. That is, $\left|E(t,x,y,z)\right| \leq (\epsilon + \delta)C$, where the constant C is independent of ϵ and δ.

Proof. The proof is split into two steps. Firstly, we identify the zeroth and first order terms of the assumed expansion form of $\tilde{V}^{\epsilon,\delta}$ with $v^{(0)}$, $v^{(1,0)}$, and $v^{(0,1)}$. Then the second step is to justify the residual function is $\mathcal{O}(\epsilon + \delta)$.

Step 1: (Heuristic derivation). The first step significantly relies on the derivation in the asymptotic expansion subsection in Section 2.4. By the martingale property, the value function $\tilde{V}^{\epsilon,\delta}$ associated to $\pi^{(0)}$ satisfies the HJB equation (2.4.5) where we take $\pi^{(0)}$ as the control:

$$\left(\mathscr{L}_0 + \frac{1}{\sqrt{\epsilon}} \mathscr{L}_Y + \frac{1}{\epsilon} \mathscr{M}_Y + \sqrt{\delta} \mathscr{L}_Z + \delta \mathscr{M}_Z + \sqrt{\frac{\delta}{\epsilon}} \mathscr{M}_{YZ} \right) \tilde{V}^{\epsilon,\delta} = 0 \tag{2.5.1}$$

where the operators $\mathscr{M}_Z$, $\mathscr{M}_Y$, $\mathscr{M}_{YZ}$ are defined by (2.2.6), (2.3.6), (2.4.6), and $\mathscr{L}_0$, $\mathscr{L}_Z$, $\mathscr{L}_Y$ are defined by:

$$\begin{aligned} &\mathscr{L}_0 = \partial_t + \mu(y,z)\pi^{(0)}\partial_x + \frac{1}{2}\sigma^2(y,z)(\pi^{(0)})^2\partial_x^2; \\ &\mathscr{L}_Y = \rho_Y \sigma(y,z) a(y) \pi^{(0)} \partial_x \partial_y; \\ &\mathscr{L}_Z = \rho_Z \sigma(y,z) g(z) \pi^{(0)} \partial_x \partial_z . \end{aligned} \tag{2.5.2}$$

The strategy is to apply the same method in Section 2.4. We aim to acquire an expansion as the following:

$$
\begin{aligned}
\tilde{V}^{\epsilon,\delta} &= \tilde{V}^{\epsilon,0} + \sqrt{\delta}\tilde{V}^{\epsilon,1} + \cdots \\
\tilde{V}^{\epsilon,0} &= \tilde{v}^{(0)} + \sqrt{\epsilon}\,\tilde{v}^{(1,0)} + \epsilon\,\tilde{v}^{(2,0)} + \epsilon^{3/2}\tilde{v}^{(3,0)} + \cdots \\
\tilde{V}^{\epsilon,1} &= \tilde{v}^{(0,1)} + \sqrt{\epsilon}\,\tilde{v}^{(1,1)} + \epsilon\,\tilde{v}^{(2,1)} + \cdots,
\end{aligned}
$$

where the superscript (i,j) indicates the power in $(\sqrt{\epsilon},\sqrt{\delta})$ respectively, and (0,0) is reduced to (0) for simplicity. Taking $\delta = 0$ first, we obtain the HJB problem:

$$
\left(\mathscr{L}_0 + \frac{1}{\sqrt{\epsilon}}\mathscr{L}_Y + \frac{1}{\epsilon}\mathscr{M}_Y\right)\tilde{V}^{\epsilon,0} = 0,\ \tilde{V}^{\epsilon,0}(T,x,y,z) = U(x). \tag{2.5.3}
$$

This equation is an identical to (2.3.18) except that $\lambda(y)$ is replaced by $\lambda(y,z)$. However, since there is no z-derivatives in (2.5.3), z can be viewed as a parameter in $\tilde{V}^{\epsilon,0}$. Therefore, we can apply the derivation and arguments in Section 2.3. By solving the Poisson equations and the corresponding solvability conditions, we deduce the following terms:

$$
\begin{aligned}
\tilde{v}^{(0)} &= v^{(0)} = M(t,x,\bar{\lambda}(z)), \\
\tilde{v}^{(2,0)} &= -\frac{1}{2}\theta(y,z)D_1\tilde{v}^{(0)} + C_1(t,x,z), \\
\tilde{v}^{(1,0)} &= v^{(1,0)} = -\frac{1}{2}(T-t)\rho_Y B(z)D_1^2 v^{(0)}, \\
\tilde{v}^{(3,0)} &= \frac{1}{2}(T-t)\theta(y,z)\rho_Y B(z)\left(\frac{1}{2}D_2 + D_1\right)D_1^2 v^{(0)} \\
&\quad + \frac{1}{2}\rho_Y\theta_1(y,z)D_1^2 v^{(0)} + C_2(t,x,z),
\end{aligned} \tag{2.5.4}
$$

where $\theta(y,z)$ is defined in (2.4.11), $\theta_1(y,z)$ is the solution to the ODE in y:

$$\mathscr{M}_Y\theta_1(y,z) = \lambda(y,z)a(y)\partial_y\theta(y,z) - \langle\lambda(\cdot,z)a(\cdot)\partial_y\theta(\cdot,z)\rangle, \tag{2.5.5}$$

and $C_1(t,x,z)$, $C_2(t,x,z)$ are constants of integration in y.

Now we aim to derive $v^{(0,1)}$ and $v^{(2,1)}$ by collecting terms of order $\sqrt{\delta}$ in (2.5.1):

$$\left(\mathscr{L}_0 + \frac{1}{\sqrt{\epsilon}}\mathscr{L}_Y + \frac{1}{\epsilon}\mathscr{M}_Y\right)\tilde{V}^{\epsilon,1} + \left(\mathscr{L}_Z + \frac{1}{\sqrt{\epsilon}}\mathscr{M}_{YZ}\right)\tilde{V}^{\epsilon,0} = 0, \tag{2.5.6}$$

with terminal condition $\tilde{V}^{\epsilon,1}(T,x,y,z) = 0$. This is exactly equation (2.4.14). Also, since there is no y-derivatives in the ϵ^{-1} and $\epsilon^{-1/2}$ terms, $\tilde{v}^{(0,1)}$ and $\tilde{v}^{(1,1)}$ are independent of y. Therefore, we can apply the derivation and arguments in Section 2.4. By solving the Poisson equation with respect to $\tilde{v}^{(2,1)}$ and the corresponding solvability condition with respect to $\tilde{v}^{(0,1)}$, we deduce the following terms:

$$\begin{aligned}
\tilde{v}^{(0,1)} &= v^{(0,1)}(t,x,z) = \frac{1}{2}(T-t)^2\rho_Z\hat{\lambda}(z)\bar{\lambda}(z)\bar{\lambda}'(z)g(z)D_1^2v^{(0)},\\
\tilde{v}^{(2,1)} &= -\theta(y,z)\frac{1}{2}(T-t)^2\rho_Z\hat{\lambda}\bar{\lambda}\bar{\lambda}'g\left(\frac{1}{2}D_2 + D_1\right)D_1^2v^{(0)}\\
&\quad -\theta_2(y,z)\rho_Z(T-t)\bar{\lambda}\bar{\lambda}'gD_1^2v^{(0)} + C_3(t,x,z),
\end{aligned} \tag{2.5.7}$$

where $\hat{\lambda}(z) = \langle\lambda(\cdot,z)\rangle$, $\theta_2(y,z)$ is the solution to the ODE in y:

$$\mathscr{M}_Y\theta_2(y,z) = \lambda(y,z) - \langle\lambda(\cdot,z)\rangle, \tag{2.5.8}$$

and $C_3(t,x,z)$ is a constant of integration in y.

To summarize the first step, we have identified the desired terms: $\tilde{v}^{(0)} = v^{(0)}$, $\tilde{v}^{(1,0)} = v^{(1,0)}$, and $\tilde{v}^{(0,1)} = v^{(0,1)}$. Also, the terms $\tilde{v}^{(2,0)}$, $\tilde{v}^{(3,0)}$, and $\tilde{v}^{(2,1)}$ are derived heuristically. Now we move on to the second step of justification.

Step 2: (Expansion justification). Recall that the goal is to show the residual function $E(t,x,y,z)$ has order higher than $(\sqrt{\epsilon}+\sqrt{\delta})$. Firstly, we analyze an auxiliary residual function $\tilde{E}(t,x,y,z)$ defined by

$$\tilde{E}(t,x,y,z) = \tilde{V}^{\epsilon,\delta} - v^{(0)} - \sqrt{\epsilon}\, v^{(1,0)} - \sqrt{\delta}\, v^{(0,1)} - \epsilon\, \tilde{v}^{(2,0)} - \epsilon^{3/2}\tilde{v}^{(3,0)} - \epsilon\sqrt{\delta}\,\tilde{v}^{(2,1)}, \quad (2.5.9)$$

where the expansion terms are defined in the previous step with $C_i(t,x,z) = 0$ for $i = 1,2,3$. Applying the infinitesimal generator of $(X_t^{\pi^{(0)}}, Y_t, Z_t)$ to $\tilde{E}$, we have

$$\begin{aligned}
&\left(\mathscr{L}_0 + \frac{1}{\sqrt{\epsilon}}\mathscr{L}_Y + \frac{1}{\epsilon}\mathscr{M}_Y + \sqrt{\delta}\mathscr{L}_Z + \delta\mathscr{M}_Z + \sqrt{\frac{\delta}{\epsilon}}\mathscr{M}_{YZ}\right)\tilde{E} \\
&+\mathscr{L}_0\left(\epsilon\,\tilde{v}^{(2,0)} + \epsilon^{3/2}\tilde{v}^{(3,0)} + \epsilon\sqrt{\delta}\,\tilde{v}^{(2,1)}\right) + \mathscr{L}_Y\left(\epsilon\,\tilde{v}^{(3,0)} + \sqrt{\epsilon\delta}\,\tilde{v}^{(2,1)}\right) \\
&+\sqrt{\delta}\mathscr{M}_{YZ}\left(\sqrt{\epsilon}\,\tilde{v}^{(2,0)} + \epsilon\,\tilde{v}^{(3,0)} + \sqrt{\epsilon\delta}\,\tilde{v}^{(2,1)}\right) \\
&+\delta\mathscr{M}_Z\left(v^{(0)} + \sqrt{\epsilon}\, v^{(1,0)} + \sqrt{\delta}\, v^{(0,1)} + \epsilon\,\tilde{v}^{(2,0)} + \epsilon^{3/2}\tilde{v}^{(3,0)} + \epsilon\sqrt{\delta}\,\tilde{v}^{(2,1)}\right) \\
&+\sqrt{\delta}\mathscr{L}_Z\left(\sqrt{\epsilon}\, v^{(1,0)} + \sqrt{\delta}\, v^{(0,1)} + \epsilon\,\tilde{v}^{(2,0)} + \epsilon^{3/2}\tilde{v}^{(3,0)} + \epsilon\sqrt{\delta}\,\tilde{v}^{(2,1)}\right) = 0,
\end{aligned}$$

with terminal condition $\tilde{E}(T,x,y,z) = -\epsilon\,\tilde{v}^{(2,0)}(T,x,y,z) - \epsilon^{3/2}\tilde{v}^{(3,0)}(T,x,y,z)$. After assembling the terms, we define the following:

$$\mathscr{R}^{(1)} = \mathscr{L}_0\left(\tilde{v}^{(2,0)} + \sqrt{\epsilon}\,\tilde{v}^{(3,0)} + \sqrt{\delta}\tilde{v}^{(2,1)}\right) + \mathscr{L}_Y\tilde{v}^{(3,0)} + \sqrt{\delta}\mathscr{L}_Z\tilde{v}^{(2,0)},$$

$$\mathscr{R}^{(2)} = \mathscr{M}_Z\left(v^{(0)} + \sqrt{\epsilon}\,v^{(1,0)} + \sqrt{\delta}v^{(0,1)} + \epsilon\,\tilde{v}^{(2,0)} + \epsilon^{3/2}\tilde{v}^{(3,0)} + \epsilon\sqrt{\delta}\,\tilde{v}^{(2,1)}\right) + \mathscr{L}_Z v^{(0,1)},$$

$$\mathscr{R}^{(3)} = \mathscr{L}_Y\tilde{v}^{(2,1)} + \mathscr{M}_{YZ}\left(\tilde{v}^{(2,0)} + \sqrt{\epsilon}\,\tilde{v}^{(3,0)} + \sqrt{\delta}\,\tilde{v}^{(2,1)}\right) + \mathscr{L}_Z\left(v^{(1,0)} + \epsilon\,\tilde{v}^{(3,0)} + \sqrt{\epsilon\delta}\tilde{v}^{(2,1)}\right).$$

By Feynman-Kac formula,

$$\begin{aligned}\tilde{E}(t,x,y,z) = \epsilon\mathbb{E}_{(t,x,y,z)}&\left(\int_t^T \mathscr{R}^{(1)}(s, X_s^{\pi^{(0)}}, Y_s, Z_s)\; ds\right)\\ &+\delta\mathbb{E}_{(t,x,y,z)}\left(\int_t^T \mathscr{R}_s^{(2)}\; ds\right) + \sqrt{\epsilon\delta}\mathbb{E}_{(t,x,y,z)}\left(\int_t^T \mathscr{R}_s^{(3)}\; ds\right)\\ &-\epsilon\mathbb{E}_{(t,x,y,z)}\left(\tilde{v}^{(2,0)}(T, X_T^{\pi^{(0)}}, Y_T, Z_T)\right)\\ &-\epsilon^{3/2}\mathbb{E}_{(t,x,y,z)}\left(\tilde{v}^{(3,0)}(T, X_T^{\pi^{(0)}}, Y_T, Z_T)\right)\end{aligned} \tag{2.5.10}$$

Now we estimate the bound of each expectations in (2.5.10). Straightforward computation gives that each expectation $\mathbb{E}_{(t,x,y,z)}\left(\int_t^T \mathscr{R}_s^{(i)}\; ds\right)$ is a sum of integrals of the form:

$$\mathbb{E}_{(t,x,y,z)}\left(\int_t^T h(Y_s, Z_s)\mathscr{D}v^{(0)}(s, X_s^{\pi^{(0)}}, Z_s)\; ds\right), \tag{2.5.11}$$

where $h(y,z)$ is at most polynomial growth, and $\mathscr{D}v^{(0)}$ takes derivatives of $v^{(0)}$. Note that different operators corresponds to different derivatives in $\mathscr{D}$:

$$
\begin{aligned}
\mathscr{L}_0, \mathscr{L}_Y, \mathscr{M}_Y &: D_1^2, D_1^3, D_1^4, D_2D_1, D_2D_1^2, D_2D_1^3, D_1D_2D_1^2, D_2^2D_1^2; \\
\mathscr{M}_{YZ} &: \partial_z D_1, \partial_z D_1^2, \partial_z D_1^3, \partial_z D_2 D_1^2; \\
\mathscr{M}_Z &: \partial_z D_1, \partial_z D_1^2, \partial_z D_1^3, \partial_z D_2 D_1^2, \partial_z, \partial_z^2, \partial_z^2 D_1, \partial_z^2 D_1^2 \partial_z^2 D_1^3, \partial_z^2 D_2 D_1^2; \\
\mathscr{L}_Z &: D_1 \partial_z D_1, D_1 \partial_z D_1^2, D_1 \partial_z D_1^3, D_1 \partial_z D_2 D_1^2 .
\end{aligned}
$$

A repeated use of the concavity of $v^{(0)}$ and Lemma B.4 guarantees that $\mathscr{D}v^{(0)}$ is bounded by

$$\left| \mathscr{D}v^{(0)}(t, x, z) \right| \leq k(z) v^{(0)}(t, x, z), \tag{2.5.12}$$

where $k(z)$ is some non-negative function with at most polynomial growth. A detailed example of the bound can be found in [Fouque and Hu, 2020, Section 3]. Using (2.5.12), we apply the Cauchy-Schwartz inequality on (2.5.11) to reduce each term to

$$\left(\mathbb{E}_{(t,y,z)} \int_t^T h^2(Y_s, Z_s) k^2(Z_s)\, ds \right)^{\frac{1}{2}} \left(\mathbb{E}_{(t,x,y,z)} \int_t^T v^{(0)}(s, X_s^{\pi^{(0)}}, Z_s)\, ds \right)^{\frac{1}{2}}. \tag{2.5.13}$$

By Assumption B.1, the first part of (2.5.13) is uniformly bounded in (ϵ, δ). By Lemma B.2, the second part of (2.5.13) is also uniformly bounded in (ϵ, δ). For the last two terms in (2.5.10), the boundedness follows from repeating the above argument with Assumption B.3 equation (B.1) in Appendix B. Thus, by bounding (2.5.10), we obtain a bound for $\tilde{E}$:

$$\left| \tilde{E}(t, x, y, z) \right| \leq (\epsilon + \delta + \sqrt{\epsilon\delta})\tilde{C} \leq (\epsilon + \delta)\tilde{C},$$

for any $(t, x, y, z) \in [0,T] \times \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}$, and $\tilde{C}$ is a constant independent of (ϵ, δ). Therefore,

$$
\begin{aligned}
\left|E(t,x,y,z)\right| &= \left|\tilde{V}^{\epsilon,\delta} - v^{(0)} - \sqrt{\epsilon}\, v^{(1,0)} - \sqrt{\delta}\, v^{(0,1)}\right| \\
&= \left|\tilde{E}(t,x,y,z) + \epsilon\,\tilde{v}^{(2,0)} + \epsilon^{3/2}\tilde{v}^{(3,0)} + \epsilon\sqrt{\delta}\,\tilde{v}^{(2,1)}\right| \\
&\leq \left|\tilde{E}(t,x,y,z)\right| + \left|\epsilon\,\tilde{v}^{(2,0)} + \epsilon^{3/2}\tilde{v}^{(3,0)} + \epsilon\sqrt{\delta}\,\tilde{v}^{(2,1)}\right| \\
&\leq (\epsilon + \delta)C,
\end{aligned}
$$

where $C(t,x,y,z)$ is a constant independent of (ϵ,δ). Hence, the residual function E has order $\epsilon + \delta$. ∎

CHAPTER 3 MULTI-SCALE TWO-ASSET PROBLEM

Motivated by recent work on multi-scale asymptotic analysis [Fouque and Hu, 2020], this chapter applies the multi-scale asymptotic analysis to a two-asset portfolio optimization problem with full information on the volatility factors. The framework includes two time-scaled assets where each logarithmic asset price is affected by a fast or slow varying stochastic volatility factor. For simplicity, we also assume that the risk-free interest rate is zero. The main goal is to understand the performance of the zeroth and the first order asymptotic optimal strategy for the two-asset portfolio problem.

First, we set up the two-asset portfolio problem for a general utility function. Following the stochastic optimal control approach in Chapter 1, we derive the associated HJB problem. Then we apply the multi-scale asymptotic analysis in Chapter 2 to derive the asymptotic expansions of the value function. Under certain assumptions, we give a rigorous proof on the performance of the two-dimensional zeroth order strategy that the strategy recovers the value function for the two-asset portfolio problem up to the first order. Finally, we also expand and analyze the two-dimensional optimal strategy up to the first order.

The main contribution of this chapter is extending the multi-scale single-asset portfolio problem in [Fouque and Hu, 2020] to a multi-asset scenario. The explicit formulas for the asymptotic approximations of the value function and the optimal strategy are derived for the two-asset problem. The extension not only proves a similar result about the performance of the zeroth order strategy in higher dimension, but also provides us intuition and insight about the influence of multi-scale volatility factors on the zeroth and the first order optimal strategy in a multi-asset portfolio.

3.1 Multi-scale Model on Two Assets

Consider two correlated assets $\tilde{S}^f$ and $\tilde{S}^s$ on a given probability space $(\Omega, \mathscr{F}, \mathbb{F} = (\mathscr{F}_t)_{t\in[0,T]}, \mathbb{P})$, where f represents the fast-scale asset and s represents the slow-scale asset. Suppose the dynamical system for the logarithmic asset prices S^f and S^s is given by

$$\begin{aligned} dS^f &:= \frac{d\tilde{S}^f_t}{\tilde{S}^f_t} = \mu_f(Y_t)dt + \sigma_f(Y_t)dW^f_t, \\ dS^s &:= \frac{d\tilde{S}^s_t}{\tilde{S}^s_t} = \mu_s(Z_t)dt + \sigma_s(Z_t)dW^s_t, \end{aligned} \tag{3.1.1}$$

where Y_t and Z_t are fast-scale volatility factor and slow-scale volatility factor:

$$\begin{aligned} dY_t &= \frac{1}{\epsilon}\mu_y(Y_t)dt + \frac{1}{\sqrt{\epsilon}}\sigma_y(Y_t)dW^y_t, \\ dZ_t &= \delta\mu_z(Z_t)dt + \sqrt{\delta}\sigma_z(Z_t)dW^z_t. \end{aligned} \tag{3.1.2}$$

The model is described by the coefficients μ_f, μ_s, μ_y, μ_z, σ_f, σ_s, σ_y, and σ_z. The small parameter ϵ and δ characterize the fast and slow variation of Y and Z. The correlation between W^f, W^s, W^y, and W^z is given by the correlation matrix

$$Co = \begin{pmatrix} 1, & \rho_{12}, & \rho_{13}, & \rho_{14} \\ \rho_{12}, & 1, & \rho_{23}, & \rho_{24} \\ \rho_{13}, & \rho_{23}, & 1, & \rho_{34} \\ \rho_{14}, & \rho_{24}, & \rho_{34}, & 1 \end{pmatrix},$$

where $0 \le \rho_{ij} < 1$ are defined by

$$
\begin{aligned}
dW_t^f dW_t^s &= \rho_{12} dt, \qquad & dW_t^f dW_t^y &= \rho_{13} dt, \qquad & dW_t^f dW_t^z &= \rho_{14} dt, \\
dW_t^s dW_t^y &= \rho_{23} dt, \qquad & dW_t^s dW_t^z &= \rho_{24} dt, \qquad & dW_t^y dW_t^z &= \rho_{34} dt\,.
\end{aligned}
$$

It is natural to assume that the following three correlations are small: (1) the correlation between the slow-scale asset S_t^s and the fast-scale factor Y_t, (2) the correlation between the fast-scale asset S_t^f and the slow-scale factor Z_t, and (3) the correlation between the two volatility factors Y_t and Z_t. From now on, we assume that $\rho_{14} = \rho_{23} = \rho_{34} = 0$ for simplicity.

Let Σ be a lower triangular matrix that satisfies the Cholesky decomposition $\Sigma\Sigma^\top = Co$. Hence, Σ can be computed explicitly as

$$
\Sigma = \begin{pmatrix}
1, & 0, & 0, & 0 \\
\rho_{12}, & \sqrt{1-\rho_{12}^2}, & 0, & 0 \\
\rho_{13}, & -\dfrac{\rho_{12}\rho_{13}}{\sqrt{1-\rho_{12}^2}}, & \dfrac{\sqrt{1-\rho_{12}^2-\rho_{13}^2}}{\sqrt{1-\rho_{12}^2}}, & 0 \\
0, & \dfrac{\rho_{24}}{\sqrt{1-\rho_{12}^2}}, & \dfrac{\rho_{12}\rho_{13}\rho_{24}}{\sqrt{1-\rho_{12}^2}\sqrt{1-\rho_{12}^2-\rho_{13}^2}}, & \dfrac{\sqrt{1-\rho_{12}^2-\rho_{13}^2-\rho_{24}^2+\rho_{13}^2\rho_{24}^2}}{\sqrt{1-\rho_{12}^2-\rho_{13}^2}}
\end{pmatrix},
$$

and W^f, W^s, W^y ,W^z can be expressed with a 4-dimensional Brownian motion:

$$\begin{pmatrix} W_t^f \\ W_t^s \\ W_t^y \\ W_t^z \end{pmatrix} = \Sigma \begin{pmatrix} B_t^1 \\ B_t^2 \\ B_t^3 \\ B_t^4 \end{pmatrix}.$$

Let $\mathbf{W}_t = \begin{pmatrix} B_t^1 \\ B_t^2 \end{pmatrix}$ and $\mathbf{B}_t = \begin{pmatrix} B_t^3 \\ B_t^4 \end{pmatrix}$. Then the dynamical system (3.1.1) for the logarithmic asset prices $\mathbf{S}_t = \begin{pmatrix} S_t^f \\ S_t^s \end{pmatrix}$ and the dynamical system (3.1.2) for volatility factors $\mathbf{Y}_t = \begin{pmatrix} Y_t \\ Z_t \end{pmatrix}$ can be re-written with $\mathbf{W}_t$ and $\mathbf{B}_t$:

$$\begin{cases} d\mathbf{S}_t = \mu_{\mathbf{S}}(\mathbf{Y}_t)dt + \tilde{\Sigma}_{11}(\mathbf{Y}_t)d\mathbf{W}_t, \\ d\mathbf{Y}_t = \mu_{\mathbf{Y}}(\mathbf{Y}_t)dt + \tilde{\Sigma}_{12}(\mathbf{Y}_t)d\mathbf{W}_t + \tilde{\Sigma}_{22}(\mathbf{Y}_t)d\mathbf{B}_t, \end{cases} \tag{3.1.3}$$

where the matrix functions $\mu_{\mathbf{S}}, \mu_{\mathbf{Y}}, \tilde{\Sigma}_{11}, \tilde{\Sigma}_{12}, \tilde{\Sigma}_{22}$ are defined by

$$\mu_{\mathbf{S}}(y,z) = \begin{pmatrix} \mu_f(y) \\ \mu_s(z) \end{pmatrix}, \mu_{\mathbf{Y}}(y,z) = \begin{pmatrix} \mu_y(y) \\ \mu_z(z) \end{pmatrix},$$

$$\tilde{\Sigma}_{11}(y,z) = \begin{pmatrix} \sigma_f(y), & 0 \\ 0, & \sigma_s(z) \end{pmatrix} \begin{pmatrix} 1, & 0 \\ \rho_{12}, & \sqrt{1-\rho_{12}^2} \end{pmatrix},$$

$$\tilde{\Sigma}_{12}(y,z) = \begin{pmatrix} \frac{1}{\sqrt{\epsilon}}\sigma_y(y), & 0 \\ 0, & \sqrt{\delta}\sigma_z(z) \end{pmatrix} \begin{pmatrix} \rho_{13}, & -\frac{\rho_{12}\rho_{13}}{\sqrt{1-\rho_{12}^2}} \\ 0, & \frac{\rho_{24}}{\sqrt{1-\rho_{12}^2}} \end{pmatrix},$$

$$\tilde{\Sigma}_{22}(y,z) = \begin{pmatrix} \frac{1}{\sqrt{\epsilon}}\sigma_y(y), & 0 \\ 0, & \sqrt{\delta}\sigma_z(z) \end{pmatrix} \begin{pmatrix} \frac{\sqrt{1-\rho_{12}^2-\rho_{13}^2}}{\sqrt{1-\rho_{12}^2}}, & 0 \\ \frac{\rho_{12}\rho_{13}\rho_{24}}{\sqrt{1-\rho_{12}^2}\sqrt{1-\rho_{12}^2-\rho_{13}^2}}, & \frac{\sqrt{1-\rho_{12}^2-\rho_{13}^2-\rho_{24}^2+\rho_{13}^2\rho_{24}^2}}{\sqrt{1-\rho_{12}^2-\rho_{13}^2}} \end{pmatrix}.$$

Here we assume that the process Y_t has a unique invariant distribution Ψ, and we use the notation $\langle \cdot \rangle$ in this chapter as the average of a function with respect to $\Psi(y)$:

$$\langle f \rangle = \int f(y)\Psi(dy). \tag{3.1.4}$$

3.2 Two-Asset Portfolio Problem

With the dynamical system (3.1.3), we can set up a two-asset portfolio problem for S^f and S^s with investment period $[0,T]$. Hence, the wealth process follows

$$dX_t^\pi = \pi_t^f dS_t^f + \pi_t^s dS_t^s = \pi_t^\top d\mathbf{S}_t, \tag{3.2.1}$$

where $\pi_t = \begin{pmatrix} \pi_t^f \\ \pi_t^s \end{pmatrix}$ represents the optimal portfolio allocation on $\mathbf{S}_t$. Recall that the risk-free interest rate is assumed to be zero. Hence, the risk-free market is not involved in (3.2.1).

Let $U : \mathbb{R} \to \mathbb{R}$ be the utility function that satisfies Assumption B.3 in Appendix B. Let $\mathscr{A}$ be the set of admissible strategies for π_t. We are interested in the terminal utility maximization problem over $\mathscr{A}$. Thus, we define the value function by

$$u^{\epsilon,\delta}(t,x,y,z) = \sup_{\pi \in \mathscr{A}} \mathbb{E}\left[U(X_T^\pi)\,|\,X_t^\pi = x, \mathbf{Y}_t = \overrightarrow{\mathbf{y}} := (y,z)\right]. \tag{3.2.2}$$

The corresponding HJB equation for the stochastic control problem is

$$u_t^{\epsilon,\delta} + \mathscr{L}_{t,\overrightarrow{\mathbf{y}}} u^{\epsilon,\delta} + \mathrm{NL}^{\epsilon,\delta} = 0, \tag{3.2.3}$$

with the terminal condition $u^{\epsilon,\delta}(T,x,y,z) = U(x)$, where the infinitesimal generator $\mathscr{L}_{t,\overrightarrow{\mathbf{y}}}$ is given by

$$\mathscr{L}_{t,\overrightarrow{\mathbf{y}}} = \frac{1}{\epsilon}\Big(\mu_y(y)\partial_y + \frac{1}{2}\sigma_y^2(y)\partial_y^2\Big) + \delta\Big(\mu_z(z)\partial_z + \frac{1}{2}\sigma_z^2(z)\partial_z^2\Big), \tag{3.2.4}$$

and the nonlinear term is given by

$$\mathrm{NL}^{\epsilon,\delta} = \sup_{\pi \in \mathscr{A}} \left[\pi^\top \mu_{\mathbf{S}}(\overrightarrow{\mathbf{y}}) u_x^{\epsilon,\delta} + \frac{1}{2}\pi^\top \tilde{\Sigma}_{11}(\overrightarrow{\mathbf{y}})\tilde{\Sigma}_{11}^\top(\overrightarrow{\mathbf{y}})\pi u_{xx}^{\epsilon,\delta} + \pi^\top \tilde{\Sigma}_{11}(\overrightarrow{\mathbf{y}})\tilde{\Sigma}_{12}^\top(\overrightarrow{\mathbf{y}})\nabla_{\overrightarrow{\mathbf{y}}} u_x^{\epsilon,\delta}\right].$$

Define the following matrix functions:

$$\Sigma_1(y,z) := \tilde{\Sigma}_{11}\tilde{\Sigma}_{11}^\top = \begin{pmatrix} \sigma_f^2(y), & \rho_{12}\sigma_f(y)\sigma_s(z) \\ \rho_{12}\sigma_f(y)\sigma_s(z), & \sigma_s^2(z) \end{pmatrix},$$

and

$$\Sigma_{12}(y,z) := \tilde{\Sigma}_{11}\tilde{\Sigma}_{12}^{\top} = \begin{pmatrix} \frac{1}{\sqrt{\epsilon}}\rho_{13}\sigma_f(y)\sigma_y(y), & 0 \\ 0, & \sqrt{\delta}\rho_{24}\sigma_s(z)\sigma_z(z) \end{pmatrix}.$$

Then we have the optimal control for the associated problem in the feedback form:

$$\pi_t^* = -\Sigma_1^{-1}(\overrightarrow{\mathbf{y}})\frac{\mu_{\mathbf{S}}(\overrightarrow{\mathbf{y}})u_x^{\epsilon,\delta} + \Sigma_{12}(\overrightarrow{\mathbf{y}})\nabla_{\overrightarrow{\mathbf{y}}}u_x^{\epsilon,\delta}}{u_{xx}^{\epsilon,\delta}}. \tag{3.2.5}$$

Plugging the optimal control (3.2.5) into (3.2.3), we obtain the HJB problem:

$$u_t^{\epsilon,\delta} + \mathscr{L}_{t,\overrightarrow{\mathbf{y}}}u^{\epsilon,\delta} - \frac{(\mu_{\mathbf{S}}u_x^{\epsilon,\delta} + \Sigma_{12}\nabla_{\overrightarrow{\mathbf{y}}}u_x^{\epsilon,\delta})^{\top}\Sigma_1^{-1}(\mu_{\mathbf{S}}u_x^{\epsilon,\delta} + \Sigma_{12}\nabla_{\overrightarrow{\mathbf{y}}}u_x^{\epsilon,\delta})}{2u_{xx}^{\epsilon,\delta}} = 0, \tag{3.2.6}$$

with the terminal condition $u^{\epsilon,\delta}(T,x,y,z) = U(x)$, where Σ_1^{-1} can also be explicitly expressed by

$$\Sigma_1^{-1}(y,z) = \begin{pmatrix} \frac{1}{\sigma_f^2(y)(1-\rho_{12}^2)}, & -\frac{\rho_{12}}{\sigma_f(y)\sigma_s(z)(1-\rho_{12}^2)} \\ -\frac{\rho_{12}}{\sigma_f(y)\sigma_s(z)(1-\rho_{12}^2)}, & \frac{1}{\sigma_s^2(z)(1-\rho_{12}^2)} \end{pmatrix}.$$

3.3 Asymptotic Expansion

Assumption 3.3. The value function $u^{\epsilon,\delta}(t,x,y,z)$ is the unique smooth solution of the HJB equation (3.2.6) that is strictly increasing and strictly concave in x for each $t \in [0,T)$ and $(y,z) \in \mathbb{R}\times\mathbb{R}$.

Similar to the derivations in Chapter 2, Assumption 3.3 guarantees the regularity of the value function $u^{\epsilon,\delta}$. Hence, we are able to assume an asymptotic expansion of $u^{\epsilon,\delta}$ with respect to

$(\sqrt{\epsilon}, \sqrt{\delta})$. Now we can apply the asymptotic analysis from Chapter 2 to the HJB problem (3.2.6). Under Assumption 3.3, we expand the value function $u^{\epsilon,\delta}$ in power of $(\sqrt{\epsilon}, \sqrt{\delta})$:

$$u^{\epsilon,\delta} = u^{(0)} + \sqrt{\epsilon}u^{(1,0)} + \sqrt{\delta}u^{(0,1)} + \epsilon u^{(2,0)} + \delta u^{(0,2)} + \sqrt{\epsilon\delta}u^{(1,1)} + \cdots \tag{3.3.1}$$

The $u^{(0)}$ approximation: By setting $\delta = 0$, we obtain that

$$u^{\epsilon,0} = u^{(0)} + \sqrt{\epsilon}u^{(1,0)} + \epsilon u^{(2,0)} \cdots \tag{3.3.2}$$

satisfies the HJB equation:

$$u_t^{\epsilon,0} + \frac{1}{\epsilon}\mathscr{M}_Y u^{\epsilon,0} - \frac{\lambda^2(y,z)(u_x^{\epsilon,0})^2 + \frac{2}{\sqrt{\epsilon}}a(y,z)u_x^{\epsilon,0}u_{xy}^{\epsilon,0} + \frac{1}{\epsilon}\frac{\rho_{13}^2\sigma_y^2}{(1-\rho_{12}^2)}(u_{xy}^{\epsilon,0})^2}{2u_{xx}^{\epsilon,0}} = 0, \tag{3.3.3}$$

with terminal condition $u^{\epsilon,0}(T,x,y,z) = U(x)$, where the differential operator $\mathscr{M}_Y$ is defined by

$$\mathscr{M}_Y = \mu_y(y)\partial_y + \frac{1}{2}\sigma_y^2(y)\partial_y^2, \tag{3.3.4}$$

and the functions $\lambda(y,z)$, $a(y,z)$ are defined by

$$\lambda^2(y,z) = \frac{\sigma_s^2(z)\mu_f^2(y) - 2\rho_{12}\sigma_f(y)\sigma_s(z)\mu_f(y)\mu_s(z) + \sigma_f^2(y)\mu_s^2(z)}{\sigma_f^2(y)\sigma_s^2(z)(1-\rho_{12}^2)}, \tag{3.3.5}$$

$$a(y,z) = \frac{\rho_{13}\sigma_y(y)}{(1-\rho_{12}^2)}\left(\frac{\mu_f(y)}{\sigma_f(y)} - \frac{\rho_{12}\mu_s(z)}{\sigma_s(z)}\right).$$

Inserting the expansion (3.3.2) into the HJB equation (3.3.3), we collect the terms in successive power of ϵ. The term with order ϵ^{-1} is given by

$$\mathscr{M}_Y u^{(0)} - \frac{1}{2}\frac{\rho_{13}^2\sigma_y^2}{(1-\rho_{12}^2)}\frac{(u_{xy}^{(0)})^2}{u_{xx}^{(0)}} = 0, u^{(0)}(T,x,y,z) = U(x).$$

Because $\mathscr{M}_Y$ only takes derivatives in y and the terminal condition is independent of y, $u^{(0)}$ is independent of y and $u_y^{(0)} = 0$. Then the term with order $\epsilon^{-\frac{1}{2}}$ is given by

$$\mathscr{M}_Y u^{(1,0)} = 0, u^{(1,0)}(T,x,y,z) = 0.$$

Similarly, $u^{(1,0)}$ is independent of y and $u_y^{(1,0)} = 0$. With $u_y^{(0)} = u_y^{(1,0)} = 0$, we collect the constant order terms:

$$\mathscr{M}_Y u^{(2,0)} + u_t^{(0)} - \frac{1}{2}\lambda^2(y,z)\frac{(u_x^{(0)})^2}{u_{xx}^{(0)}} = 0. \tag{3.3.6}$$

This is a Poisson equation of the same form as (2.3.10) in Chapter 2. Thus, we apply the same argument in Chapter 2 Section 2.3 to obtain that

$$u^{(0)}(t,x,z) = M(t,x;\bar{\lambda}(z)), \tag{3.3.7}$$

where M is the classical Merton value function with terminal condition $M(T,x,\bar{\lambda}(z)) = U(x)$ associated to the square-averaged Sharpe ratio $\bar{\lambda}(z)$ defined by

$$\bar{\lambda}^2(z) := \left\langle \lambda^2(\,\cdot\,,z)\right\rangle.$$

Recall that the notation $\langle\,\cdot\,\rangle$ is defined in (3.1.4).

The $u^{(2,0)}$ approximation: Define the risk-tolerance function and associated differential operators as:

$$R(t,x;\bar{\lambda}(z)) := -\frac{M_x(t,x;\bar{\lambda}(z))}{M_{xx}(t,x;\bar{\lambda}(z))},$$

$$D_k := R(t,x;\bar{\lambda}(z))^k \frac{\partial^k}{\partial x^k}, k = 1,2,\cdots.$$

As a byproduct from the Poisson equation (3.3.6), the argument in Chapter 2 Section 2.3 also derives that

$$u^{(2,0)} = -\frac{1}{2}\psi(y,z)D_1 u^{(0)} + C(t,x,z) \tag{3.3.8}$$

where $\psi(y,z)$ solves the ordinary differential equation in y variable:

$$\mathscr{M}_Y \psi = \lambda^2(y,z) - \bar{\lambda}^2(z), \tag{3.3.9}$$

and $C(t,x,z)$ is a constant of integration in y that may depend on (t,x,z). Note that ψ is a polynomial in y and z.

The $u^{(1,0)}$ approximation: With the expression (3.3.8) for $u^{(2,0)}$ and $u_y^{(0)} = u_y^{(1,0)} = 0$, the expansion of the nonlinear term up to order $\sqrt{\epsilon}$ in (3.3.3) simplifies to:

$$\begin{aligned}
&\frac{\lambda^2(y,z)(u_x^{\epsilon,0})^2 + \frac{2}{\sqrt{\epsilon}}a(y,z)u_x^{\epsilon,0}u_{xy}^{\epsilon,0} + \frac{1}{\epsilon}\frac{\rho_{13}^2\sigma_y^2}{(1-\rho_{12}^2)}(u_{xy}^{\epsilon,0})^2}{2u_{xx}^{\epsilon,0}} \\
&= \left(\lambda^2(y,z)(u_x^{(0)} + \sqrt{\epsilon}u_x^{(1,0)})^2 + 2\sqrt{\epsilon}a(y,z)u_x^{(0)}u_{xy}^{(2,0)}\right)\frac{1}{2u_{xx}^{(0)}}\left(1 - \sqrt{\epsilon}\frac{u_{xx}^{(1,0)}}{u_{xx}^{(0)}}\right) + \cdots \\
&= \cdots + \sqrt{\epsilon}\left(-\lambda^2(y,z)D_1 u^{(1,0)} + \frac{1}{2}a(y,z)\psi_y(y,z)D_1^2 u^{(0)} - \frac{1}{2}\lambda^2(y,z)D_2 u^{(1,0)}\right) + \cdots
\end{aligned}$$

Thus, the term with order $\sqrt{\epsilon}$ in the HJB equation (3.3.3) is given by

$$\mathscr{M}_Y u^{(3,0)} + u_t^{(1,0)} + \frac{1}{2}\lambda^2 D_2 u^{(1,0)} + \lambda^2 D_1 u^{(1,0)} - \frac{1}{2} a\psi_y D_1^2 u^{(0)} = 0. \tag{3.3.10}$$

The solvability condition for the above Poisson equation with respect to $u^{(3,0)}$ is the following Fredholm Alternative:

$$\mathscr{L}_{t,x}(\bar{\lambda}(z))u^{(1,0)} = \frac{1}{2}\left\langle a(\,\cdot\,,z)\psi_y(\,\cdot\,,z)\right\rangle D_1^2 u^{(0)}, \tag{3.3.11}$$

where the differential operator $\mathscr{L}_{t,x}$ is defined by

$$\mathscr{L}_{t,x}(\bar{\lambda}(z)) := \frac{\partial}{\partial_t} + \frac{1}{2}\bar{\lambda}^2(z)D_2 + \bar{\lambda}^2(z)D_1. \tag{3.3.12}$$

The above PDE problem (3.3.11) with terminal condition $u^{(1,0)}(T,x,z) = 0$ has the same form as (2.3.15) in Chapter 2 Section 3. By Proposition 2.3, we acquire the unique solution for $u^{(1,0)}$:

$$u^{(1,0)}(t,x,z) = -\frac{1}{2}(T-t)A(z)D_1^2 u^{(0)}(t,x,z), \tag{3.3.13}$$

where $A(z)$ is defined by

$$A(z) = \left\langle a(\,\cdot\,,z)\psi_y(\,\cdot\,,z)\right\rangle.$$

The $u^{(0,1)}$ approximation: In order to derive $u^{(0,1)}$, we construct an expansion in power of $\sqrt{\delta}$:

$$u^{\epsilon,\delta} = u^{\epsilon,0} + \sqrt{\delta}u^{\epsilon,1} + \cdots, \tag{3.3.14}$$

where $u^{\epsilon,0}$ has the expansion (3.3.2) and $u^{\epsilon,1}$ has an expansion

$$u^{\epsilon,1} = u^{(0,1)} + \sqrt{\epsilon}u^{(1,1)} + \epsilon u^{(2,1)} + \cdots.$$

Inserting the expansion (3.3.14) into the HJB equation (3.2.6), we are able to collect the order $\sqrt{\delta}$ term as an equation of $u^{\epsilon,1}$ with terminal condition:

$$u_t^{\epsilon,1} + \frac{1}{\epsilon}\mathscr{M}_Y u^{\epsilon,1} + \mathrm{NL}^{(1)} = 0. \tag{3.3.15}$$

The nonlinear term $\mathrm{NL}^{(1)}$ is given by

$$\begin{aligned}\mathrm{NL}^{(1)} = &-\frac{1}{u_{xx}^{\epsilon,0}}\Big[\lambda^2 u_x^{\epsilon,0}u_x^{\epsilon,1} + b(y,z)u_x^{\epsilon,0}u_{xz}^{\epsilon,0} + \frac{1}{\epsilon}\frac{\rho_{13}^2\sigma_y^2}{(1-\rho_{12}^2)}u_{xy}^{\epsilon,0}u_{xy}^{\epsilon,1} \\ &-\frac{1}{\sqrt{\epsilon}}\frac{\rho_{12}\rho_{13}\rho_{24}\sigma_y\sigma_z}{(1-\rho_{12}^2)}u_{xy}^{\epsilon,0}u_{xz}^{\epsilon,0} + \frac{1}{\sqrt{\epsilon}}a(y,z)(u_x^{\epsilon,0}u_{xy}^{\epsilon,1} + u_x^{\epsilon,1}u_{xy}^{\epsilon,0})\Big] \\ &+\frac{1}{2}\frac{u_{xx}^{\epsilon,1}}{(u_{xx}^{\epsilon,0})^2}\Big(\lambda^2(y,z)(u_x^{\epsilon,0})^2 + \frac{1}{\epsilon}\frac{\rho_{13}^2\sigma_y^2}{(1-\rho_{12}^2)}(u_{xy}^{\epsilon,0})^2 + \frac{2}{\sqrt{\epsilon}}a(y,z)u_x^{\epsilon,0}u_{xy}^{\epsilon,0}\Big),\end{aligned}$$

where $\lambda(y,z)$, $a(y,z)$ are defined in (3.3.5), and $b(y,z)$ is defined by

$$b(y,z) = \frac{\rho_{24}\sigma_z(z)}{(1-\rho_{12}^2)}\Big(\frac{\mu_s(z)}{\sigma_s(z)} - \frac{\rho_{12}\mu_f(y)}{\sigma_f(y)}\Big). \tag{3.3.16}$$

Now we repeat the same strategy as the previous derivation of $u^{(0)}$. Note that the first two terms in the expansion of $u^{\epsilon,0}$ are independent of y. As a result, the term with order ϵ^{-1} in the equation (3.3.15) is given by $\mathscr{M}_Y u^{(0,1)} = 0$. Hence, we have $u_y^{(0,1)} = 0$. Then the term with order $\epsilon^{-\frac{1}{2}}$ is $\mathscr{M}_Y u^{(1,1)} = 0$. Hence, we have $u_y^{(1,1)} = 0$. With $u_y^{(0,1)} = u_y^{(1,1)} = 0$, we collect the constant order terms:

$$\mathscr{M}_Y u^{(2,1)} + u_t^{(0,1)} + \frac{1}{2}\lambda^2 D_2 u^{(0,1)} + \lambda^2 D_1 u^{(0,1)} - b\,\frac{u_x^{(0)} u_{xz}^{(0)}}{u_{xx}^{(0)}} = 0. \tag{3.3.17}$$

This is a Poisson equation for $u^{(2,1)}$. The solvability condition gives the following Fredholm Alternative:

$$\mathscr{L}_{t,x}(\bar{\lambda}(z))u^{(0,1)} = \langle b(\,\cdot\,,z)\rangle\,\frac{u_x^{(0)} u_{xz}^{(0)}}{u_{xx}^{(0)}}. \tag{3.3.18}$$

With terminal condition $u^{(0,1)}(T,x,z) = 0$, the PDE problem (3.3.18) has the same form as (2.2.12) in Chapter 2 Section 2. By Proposition 2.2, we acquire the unique solution for $u^{(0,1)}$:

$$u^{(0,1)}(t,x,z) = \frac{1}{2}(T-t)B(z)D_1 u_z^{(0)}(t,x,z), \tag{3.3.19}$$

where $B(z)$ is defined by:

$$B(z) = \langle b(\,\cdot\,,z)\rangle.$$

Summary: Under Assumption 3.3 on the smoothness of the value function $u^{\epsilon,\delta}$, we have derived the asymptotic expansion of $u^{\epsilon,\delta}$ up to the first order:

$$\begin{aligned}
u^{\epsilon,\delta}(t,x,y,z) &= u^{(0)} + \sqrt{\epsilon}u^{(1,0)} + \sqrt{\delta}u^{(0,1)} + \cdots,\\
u^{(0)}(t,x,z) &= M(t,x;\bar{\lambda}(z)),\\
u^{(1,0)}(t,x,z) &= -\frac{1}{2}(T-t)A(z)D_1^2 u^{(0)}(t,x,z),\\
u^{(0,1)}(t,x,z) &= \frac{1}{2}(T-t)B(z)D_1 u_z^{(0)}(t,x,z)\,.
\end{aligned} \tag{3.3.20}$$

3.4 Zeroth Order Strategy Performance

In this section, we first heuristically derive the zeroth order optimal strategy $\pi^{(0)}$ for the HJB problem (3.2.6). Then we aim to show a rigorous result (Theorem II) about the performance of $\pi^{(0)}$ without the assumption on the smoothness of the value function $u^{\epsilon,\delta}$ (Assumption 3.3). The goal is to prove that, under a set of assumptions on the state processes and the utility function, the zeroth order optimal strategy can reproduce $u^{\epsilon,\delta}$ up to the first order correction. That is, the value function associated to $\pi^{(0)}$ takes the form $u^{(0)} + \sqrt{\epsilon}u^{(1,0)} + \sqrt{\delta}u^{(0,1)} + o(\sqrt{\epsilon} + \sqrt{\delta})$.

Recall that the optimal feedback strategy (3.2.5):

$$\pi_t^* = -\Sigma_1^{-1}\frac{\mu_{\mathbf{S}}u_x^{\epsilon,\delta} + \Sigma_{12}\nabla_{\overrightarrow{\mathbf{y}}}u_x^{\epsilon,\delta}}{u_{xx}^{\epsilon,\delta}},$$

where Σ_1, Σ_{12} are defined in Section 3.2, and $\overrightarrow{\mathbf{y}} = (y, z)$ is the vector that represents the states of the volatility factors. We expand the optimal strategy π^* in power of $(\sqrt{\epsilon}, \sqrt{\delta})$:

$$\pi^* = \pi^{(0)} + \sqrt{\epsilon}\pi^{(1,0)} + \sqrt{\delta}\pi^{(0,1)} + \cdots. \tag{3.4.1}$$

Inserting the expansion (3.4.1) of π^* and the expansion (3.3.1) of $u^{\epsilon,\delta}$ into the feedback form (3.2.5), we collect the constant power terms to obtain the zeroth order optimal strategy:

$$\pi^{(0)} = -\Sigma_1^{-1}(\overrightarrow{\mathbf{y}})\mu_{\mathbf{S}}(\overrightarrow{\mathbf{y}})\frac{u_x^{(0)}(t,x,z)}{u_{xx}^{(0)}(t,x,z)} = \Sigma_1^{-1}\mu_{\mathbf{S}}R(t,x;\bar{\lambda}(z)), \tag{3.4.2}$$

where $\bar{\lambda}(z)$ and $R(t,x;\bar{\lambda}(z))$ are defined in Section 3.3.

Now we present two sets of assumptions for the main theorem (Theorem II). Assumption 3.4 is a version of Assumption B.1 adapted to the two-asset portfolio problem, which is required to prove Lemma 3.4. For the proof of Lemma 3.4, we refer to the proof of Lemma B.2 in Appendix B. In addition, Assumption B.3 for the utility function U is required. Recall from Appendix B that Lemma B.4 can be proved under Assumption B.3.

Assumption 3.4. The following assumptions are on the state processes $(\mathbf{S}_t, X_t, Y_t, Z_t)$:

(i) For any starting points $(s, y, z) \in \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}$ and fixed (ϵ, δ), the system (3.1.1) has a unique strong solution $(\mathbf{S}_t, Y_t, Z_t)$. The coefficients $\mu_f(y)$, $\mu_s(z)$, $\mu_y(y)$, $\mu_z(z)$, $\sigma_f(y)$, $\sigma_s(z)$, $\sigma_y(y)$, and $\sigma_z(z)$ are in $C^3(\mathbb{R})$. The coefficients and their derivatives are at most polynomially growing.

(ii) The process $Y^{(1)}$ with infinitesimal generator $\mathscr{M}_Y$ defined in (3.3.4) is ergodic with a unique invariant distribution Φ, and admits moments of any order uniformly in $t \leq T$:

$$\sup_{t\leq T}\left(\mathbb{E}\,|\,Y_t^{(1)}|^k\right) \leq C(T,k).$$

The solution $\phi(y,z)$ of the Poisson equation $\mathscr{M}_Y\phi(y,z) = l(y,z)$ is assumed to be polynomial in y if $l(y,z)$ is polynomial in y.

(iii) The process $Z^{(1)}$ with infinitesimal generator $\mathscr{M}_Z$ defined by

$$\mathscr{M}_Z = \mu_z(z)\partial_z + \frac{1}{2}\sigma_z^2(z)\partial_z^2,$$

admits moments of any order uniformly in $t \leq T$:

$$\sup_{t\le T}\left(\mathbb{E}\,|Z_t^{(1)}|^k\right) \le C(T,k).$$

(iv) Observe that, for fixed $(t,z) \in [0,T] \times \mathbb{R}$, $u^{(0)}(t,x,z) = M(t,x;\overline{\lambda}(z))$ is a concave function with a linear upper bound. In particular, there exists a function $\overline{G}(z)$ such that

$$u^{(0)}(0,x,z) \le \overline{G}(z) + x.$$

Assume that the process $\overline{G}(Z_t)$ is in $L^2([0,T] \times \Omega)$ uniformly in δ:

$$\mathbb{E}_{(0,z)}\left(\int_0^T \overline{G}^2(Z_s)\,ds\right) \le C_1(T,z),$$

where $C_1(T,z)$ is independent of δ, and Z_s is the slow-scale factor with $Z_0 = z$.

(v) The wealth process $X_t^{\pi^{(0)}}$ associated with the zeroth order optimal strategy stays non-negative. Moreover, $X_t^{\pi^{(0)}}$ is in $L^2([0,T] \times \Omega)$ uniformly in (ϵ,δ):

$$\mathbb{E}_{(0,x,y,z)}\left(\int_0^T (X_s^{\pi^{(0)}})^2\,ds\right) \le C_2(T,x,y,z),$$

where $C_2(T,x,y,z)$ is independent of (ϵ,δ).

Lemma 3.4. Under Assumption 3.4 (iv) and (v), the process $u^{(0)}(t,X_t^{\pi^{(0)}},Z_t)$ is in $L^2([0,T] \times \Omega)$ uniformly in (ϵ,δ): for all $(t,x,y,z) \in [0,T] \times \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}$,

$$\mathbb{E}_{(t,x,y,z)}\left(\int_0^T (u^{(0)}(s,X_s^{\pi^{(0)}},Z_s)^2\,ds\right) \le C(T,x,y,z),$$

where $u^{(0)}(t,x,z) = M(t,x;\bar{\lambda}(z))$.

With the above Lemma 3.4 and Lemma B.3 in Appendix B, we present the proof for the main result about the performance of the zeroth order optimal strategy $\pi^{(0)}$ for the two-asset portfolio problem.

Theorem II (Vectorized $\pi^{(0)}$ performance). Let $u^{(0)}$, $u^{(1,0)}$, and $u^{(0,1)}$ be the zeroth and first order terms in the asymptotic expansion of the value function $u^{\epsilon,\delta}$, which are given by (3.3.20). Let $\tilde{u}^{\epsilon,\delta}$ be the value function associated to the zeroth order optimal strategy:

$$\pi^{(0)} = \Sigma_1^{-1}\mu_{\mathbf{S}} R(t,x;\bar{\lambda}(z)).$$

Note that $\pi^{(0)}$ is a vector. Then

$$\tilde{u}^{\epsilon,\delta}(t,x,y,z) = \mathbb{E}\left(U(X_T^{\pi^{(0)}}) \,|\, X_t^{\pi^{(0)}} = x, Y_t = y, Z_t = z \right),$$

where $X_t^{\pi^{(0)}}$ is given by (3.2.1) with $\pi_t = \pi^{(0)}$. Define the residual function $E(t,x,y,z)$ by

$$E(t,x,y,z) := \tilde{u}^{\epsilon,\delta}(t,x,y,z) - u^{(0)}(t,x,z) - \sqrt{\epsilon}u^{(1,0)}(t,x,z) - \sqrt{\delta}u^{(0,1)}(t,x,z).$$

Under Assumption 3.4 and Assumption B.3, the residual function $E(t,x,y,z)$ has order $\epsilon + \delta$, for all $(t,x,y,z) \in [0,T] \times \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}$. That is, $\left|E(t,x,y,z)\right| \leq (\epsilon+\delta)C$, where the constant C is independent of ϵ and δ.

Proof. The proof follows the two steps in the proof of Theorem I in Chapter 2 Section 2.5. Firstly, we identify the zeroth and first order terms of the assumed expansion form of $\tilde{u}^{\epsilon,\delta}$ with $u^{(0)}$, $u^{(1,0)}$, and $u^{(0,1)}$. Then the second step is to justify the residual function is $\mathcal{O}(\epsilon+\delta)$.

Step 1: (Heuristic derivation). By the martingale property of $\tilde{u}^{\epsilon,\delta}(t, X_t, Y_t, Z_t)$ and Itô's formula, $\tilde{u}^{\epsilon,\delta}$ solves the HJB equation:

$$\tilde{u}^{\epsilon,\delta}_t + \mathscr{L}_{t,\overrightarrow{\mathbf{y}}}\tilde{u}^{\epsilon,\delta} + (\pi^{(0)})^\top \mu_{\mathbf{S}}\tilde{u}^{\epsilon,\delta}_x + \frac{1}{2}(\pi^{(0)})^\top\Sigma_1\pi^{(0)}\tilde{u}^{\epsilon,\delta}_{xx} + (\pi^{(0)})^\top\Sigma_{12}\nabla_{\overrightarrow{\mathbf{y}}}\tilde{u}^{\epsilon,\delta}_x = 0,$$

where the infinitesimal generator $\mathscr{L}_{t,\overrightarrow{\mathbf{y}}}$ is defined by (3.2.4). The terminal condition is

$$\tilde{u}^{\epsilon,\delta}(T, x, y, z) = U(x).$$

We re-write the above HJB problem in the following form:

$$\left(\mathscr{L}_0 + \frac{1}{\sqrt{\epsilon}}\mathscr{L}_Y + \frac{1}{\epsilon}\mathscr{M}_Y + \sqrt{\delta}\mathscr{L}_Z + \delta\mathscr{M}_Z\right)\tilde{u}^{\epsilon,\delta} = 0,\ \tilde{u}^{\epsilon,\delta}(T, x, y, z) = U(x), \quad (3.4.3)$$

where the operators are defined by:

$$\begin{aligned}
\mathscr{L}_0 &= \partial_t + (\pi^{(0)})^\top\mu_{\mathbf{S}}(\overrightarrow{\mathbf{y}})\partial_x + \frac{1}{2}(\pi^{(0)})^\top\Sigma_1(\overrightarrow{\mathbf{y}})\pi^{(0)}\partial_x^2;\\
\mathscr{M}_Y &= \mu_y(y)\partial_y + \frac{1}{2}\sigma_y^2(y)\partial_y^2;\\
\mathscr{M}_Z &= \mu_z(z)\partial_z + \frac{1}{2}\sigma_z^2(z)\partial_z^2;\\
\mathscr{L}_Y &= \rho_{13}\sigma_f(y)\sigma_y(y)(1,0)\,\pi^{(0)}\partial_x\partial_y;\\
\mathscr{L}_Z &= \rho_{24}\sigma_s(z)\sigma_z(z)(0,1)\,\pi^{(0)}\partial_x\partial_z\,.
\end{aligned}$$

Now we aim to acquire a heuristic expansion of $\tilde{u}^{\epsilon,\delta}$ as the following:

$$\begin{aligned}
\tilde{u}^{\epsilon,\delta} &= \tilde{u}^{\epsilon,0} + \sqrt{\delta}\tilde{u}^{\epsilon,1} + \cdots \\
\tilde{u}^{\epsilon,0} &= \tilde{u}^{(0)} + \sqrt{\epsilon}\tilde{u}^{(1,0)} + \epsilon\tilde{u}^{(2,0)} + \epsilon^{3/2}\tilde{u}^{(3,0)} + \cdots \\
\tilde{u}^{\epsilon,1} &= \tilde{u}^{(0,1)} + \sqrt{\epsilon}\tilde{u}^{(1,1)} + \epsilon\tilde{u}^{(2,1)} + \cdots,
\end{aligned} \tag{3.4.5}$$

where the superscript (i,j) indicates the power in $(\sqrt{\epsilon},\sqrt{\delta})$ respectively, and (0,0) is reduced to (0) for simplicity.

Following the steps from Section 3.3, we take $\delta = 0$ and obtain the following HJB problem:

$$\left(\mathscr{L}_0 + \frac{1}{\sqrt{\epsilon}}\mathscr{L}_Y + \frac{1}{\epsilon}\mathscr{M}_Y\right)\tilde{u}^{\epsilon,0} = 0,\ \tilde{u}^{\epsilon,0}(T,x,y,z) = U(x). \tag{3.4.6}$$

We insert the expansion of $\tilde{u}^{\epsilon,0}$ in (3.4.5) into the above HJB equation (3.4.6) and collect the terms in power of $(\sqrt{\epsilon},\sqrt{\delta})$. The term corresponding to the power ϵ^{-1} is given by

$$\mathscr{M}_Y\tilde{u}^{(0)} = 0,\ \tilde{u}^{(0)}(T,x,y,z) = U(x).$$

Hence, $\tilde{u}^{(0)}_y = 0$. Then the term with order $\epsilon^{-\frac{1}{2}}$ is given by

$$\mathscr{M}_Y\tilde{u}^{(1,0)} = 0,\ \tilde{u}^{(1,0)}(T,x,y,z) = 0.$$

Hence, $\tilde{u}^{(1,0)}_y = 0$. With $\tilde{u}^{(0)}_y$ and $\tilde{u}^{(1,0)}_y$ independent of y, the constant order term is given by

$$\mathscr{M}_Y\tilde{u}^{(2,0)} + \tilde{u}^{(0)}_t + (\pi^{(0)})^\top\mu_{\mathbf{S}}\tilde{u}^{(0)}_x + \frac{1}{2}(\pi^{(0)})^\top\Sigma_1\pi^{(0)}\tilde{u}^{(0)}_{xx} = 0.$$

Plugging in $\pi^{(0)} = \Sigma_1^{-1}\overrightarrow{\mathbf{y}}R(t,x;\overline{\lambda}(z))$, the above Poisson equation can be re-written as

$$\mathscr{M}_Y \tilde{u}^{(2,0)} + \left(\frac{\partial}{\partial_t} + \frac{1}{2}\lambda^2(y,z)D_2 + \lambda^2(y,z)D_1\right)\tilde{u}^{(0)} = 0, \tag{3.4.7}$$

where $\lambda(y,z)$ is defined in (3.3.5). Thus, the solvability condition of the Poisson equation (3.4.7) (with respect to $\tilde{u}^{(2,0)}$) is the Merton equation:

$$\mathscr{L}_{t,x}(\overline{\lambda}(z))\tilde{u}^{(0)} = 0,$$

where $\mathscr{L}_{t,x}(\overline{\lambda}(z))$ is defined in (3.3.12). With the terminal condition

$$\tilde{u}^{(0)}(T,x,z) = U(x) = u^{(0)}(T,x,z),$$

we deduce that

$$\tilde{u}^{(0)} = u^{(0)}(t,x,z) = M(t,x;\overline{\lambda}(z)), \tag{3.4.8}$$

by uniqueness of Merton equation. As a result, the Poisson equation (3.4.7) is identical to (3.3.6) with zero terminal condition for $\tilde{u}^{(2,0)}$. Hence, we solve the Poisson equation (3.4.7) and deduce that

$$\tilde{u}^{(2,0)} = -\frac{1}{2}\psi(y,z)D_1 u^{(0)} + C_1(t,x,z), \tag{3.4.9}$$

where $C_1(t,x,z)$ is a constant of integration in y, and ψ is defined in (3.3.9). Also, with (3.4.8) and (3.4.9), the term with order $\sqrt{\epsilon}$ in (3.4.6) is identical to (3.3.10) with zero terminal condition for $\tilde{u}^{(1,0)}$:

$$\mathscr{M}_Y \tilde{u}^{(3,0)} + \tilde{u}_t^{(1,0)} + \frac{1}{2}\lambda^2 D_2 \tilde{u}^{(1,0)} + \mathscr{L}_Y \tilde{u}^{(2,0)} = 0. \tag{3.4.10}$$

By the argument in Section 3.3, we acquire the unique solution for $\tilde{u}^{(1,0)}$:

$$\tilde{u}^{(1,0)} = u^{(1,0)}(t,x,z) = -\frac{1}{2}(T-t)A(z)D_1^2u^{(0)}(t,x,z), \tag{3.4.11}$$

where $A(z)$ is defined by $A(z) = \langle a(\,\cdot\,,z)\psi_y(\,\cdot\,,z)\rangle$, where $a(y,z)$ is defined in Section 3.3. Plugging (3.4.11) back to the Poisson equation (3.4.10) and solving for $\tilde{u}^{(3,0)}$ gives

$$\begin{aligned}\tilde{u}^{(3,0)} = &\frac{1}{2}(T-t)\psi(y,z)A(z)\left(\frac{1}{2}D_2 + D_1\right)D_1^2u^{(0)} \\ &+\frac{1}{2}\psi_1(y,z)D_1^2u^{(0)} + C_2(t,x,z),\end{aligned} \tag{3.4.12}$$

where $C_2(t,x,z)$ is a constant of integration in y, and $\psi_1(y,z)$ is the solution to the ODE in y:

$$\mathscr{M}_Y\psi_1(y,z) = a(y,z)\psi_y(y,z) - A(z). \tag{3.4.13}$$

Next, we collect the $\sqrt{\delta}$ terms in the HJB equation (3.4.3):

$$\left(\mathscr{L}_0 + \frac{1}{\sqrt{\epsilon}}\mathscr{L}_Y + \frac{1}{\epsilon}\mathscr{M}_Y\right)\tilde{u}^{\epsilon,1} + \mathscr{L}_Z\tilde{u}^{\epsilon,0} = 0. \tag{3.4.14}$$

Again, because the zero terminal condition for $\tilde{u}^{\epsilon,1}$ is independent of y, we insert the expansion of $\tilde{u}^{\epsilon,1}$ and apply the same reasoning on the order ϵ^{-1} term and the order $\epsilon^{-1/2}$ terms successively to obtain that $\tilde{u}_y^{(0,1)} = \tilde{u}_y^{(1,1)} = 0$. Plugging $\pi^{(0)}$ into (3.4.14), we obtain the constant term in equation (3.4.14):

$$\mathscr{M}_Y\tilde{u}^{(2,1)} + \left(\frac{\partial}{\partial_t} + \frac{1}{2}\lambda^2D_2 + \lambda^2D_1\right)\tilde{u}^{(0,1)} + \mathscr{L}_Zu^{(0)} = 0, \tag{3.4.15}$$

where $\tilde{u}_y^{(1,1)} = 0$ and $\tilde{u}^{(0)} = u^{(0)}$ are used. The above Poisson equation (3.4.15) is identical to equation (3.3.17). The argument in Section 3.3 gives the unique solution for $\tilde{u}^{(0,1)}$:

$$\tilde{u}^{(0,1)} = u^{(0,1)}(t,x,z) = \frac{1}{2}(T-t)B(z)D_1 u_z^{(0)}(t,x,z), \tag{3.4.16}$$

where $B(z)$ is defined by $B(z) = \langle b(\cdot\,,z)\rangle$ and $b(y,z)$ is defined in (3.3.16). Plugging (3.4.16) back to the Poisson equation (3.4.15) and solving for $\tilde{u}^{(2,1)}$ gives

$$\tilde{u}^{(2,1)} = -\psi(y,z)\left(\frac{1}{2}D_2 + D_1\right)u^{(0,1)} - \psi_2(y,z)D_1 u_z^{(0)} + C_3(t,x,z),$$

where $C_3(t,x,z)$ is a constant of integration in y, and $\psi_2(y,z)$ is the solution to the ODE in y:

$$\mathscr{M}_Y \psi_2(y,z) = b(y,z) - B(z). \tag{3.4.17}$$

To express $\tilde{u}^{(2,1)}$ solely in $u^{(0)}$, we use (3.4.16) to obtain

$$\tilde{u}^{(2,1)} = -\frac{1}{2}(T-t)B(z)\psi(y,z)\left(\frac{1}{2}D_2 + D_1\right)D_1 u_z^{(0)} - \psi_2(y,z)D_1 u_z^{(0)} + C_3. \tag{3.4.18}$$

To summarize the first step, we have identified the desired terms: $\tilde{u}^{(0)} = u^{(0)}$, $\tilde{u}^{(1,0)} = u^{(1,0)}$, and $\tilde{u}^{(0,1)} = u^{(0,1)}$. The terms $\tilde{u}^{(2,0)}$, $\tilde{u}^{(3,0)}$, and $\tilde{u}^{(2,1)}$ are derived heuristically. Also note that ψ, ψ_1, ψ_2 are polynomials in y and z. Now we move on to the next step.

Step 2: (Expansion justification). Recall that the goal is to show the residual function $E(t,x,y,z)$ has order higher than $(\sqrt{\epsilon} + \sqrt{\delta})$. Firstly, we analyze an auxiliary residual function $\tilde{E}(t,x,y,z)$ defined by

$$\tilde{E}(t,x,y,z) = \tilde{u}^{\epsilon,\delta} - u^{(0)} - \sqrt{\epsilon}u^{(1,0)} - \sqrt{\delta}u^{(0,1)} - \epsilon\tilde{u}^{(2,0)} - \epsilon^{3/2}\tilde{u}^{(3,0)} - \epsilon\sqrt{\delta}\tilde{u}^{(2,1)}, \quad (3.4.19)$$

where the expansion terms are defined in the previous step with $C_i(t,x,z) = 0$ for $i = 1,2,3$. Applying the infinitesimal generator of $(X_t^{\pi^{(0)}}, Y_t, Z_t)$ to $\tilde{E}$, we have

$$\begin{aligned}
&\left(\mathscr{L}_0 + \frac{1}{\sqrt{\epsilon}}\mathscr{L}_Y + \frac{1}{\epsilon}\mathscr{M}_Y + \sqrt{\delta}\mathscr{L}_Z + \delta\mathscr{M}_Z\right)\tilde{E} \\
&+\mathscr{L}_0\left(\epsilon\tilde{u}^{(2,0)} + \epsilon^{3/2}\tilde{u}^{(3,0)} + \epsilon\sqrt{\delta}\tilde{u}^{(2,1)}\right) + \mathscr{L}_Y\left(\epsilon\tilde{u}^{(3,0)} + \sqrt{\epsilon\delta}\tilde{u}^{(2,1)}\right) \\
&+\delta\mathscr{M}_Z\left(u^{(0)} + \sqrt{\epsilon}u^{(1,0)} + \sqrt{\delta}u^{(0,1)} + \epsilon\tilde{u}^{(2,0)} + \epsilon^{3/2}\tilde{u}^{(3,0)} + \epsilon\sqrt{\delta}\tilde{u}^{(2,1)}\right) \\
&+\sqrt{\delta}\mathscr{L}_Z\left(\sqrt{\epsilon}u^{(1,0)} + \sqrt{\delta}u^{(0,1)} + \epsilon\tilde{u}^{(2,0)} + \epsilon^{3/2}\tilde{u}^{(3,0)} + \epsilon\sqrt{\delta}\tilde{u}^{(2,1)}\right) = 0,
\end{aligned}$$

with terminal condition $\tilde{E}(T,x,y,z) = -\epsilon\tilde{u}^{(2,0)}(T,x,y,z) - \epsilon^{3/2}\tilde{u}^{(3,0)}(T,x,y,z)$. The above HJB problem is similar to the HJB problem in the expansion justification step of Theorem I. Thus, the rest of this step exactly follows the proof of Theorem I. Define

$$\begin{aligned}
\mathscr{R}^{(1)} &= \mathscr{L}_0\left(\tilde{u}^{(2,0)} + \sqrt{\epsilon}\tilde{u}^{(3,0)} + \sqrt{\delta}\tilde{u}^{(2,1)}\right) + \mathscr{L}_Y\tilde{u}^{(3,0)} + \sqrt{\delta}\mathscr{L}_Z\tilde{u}^{(2,0)}, \\
\mathscr{R}^{(2)} &= \mathscr{M}_Z\left(u^{(0)} + \sqrt{\epsilon}u^{(1,0)} + \sqrt{\delta}u^{(0,1)} + \epsilon\tilde{u}^{(2,0)} + \epsilon^{3/2}\tilde{u}^{(3,0)} + \epsilon\sqrt{\delta}\tilde{u}^{(2,1)}\right) + \mathscr{L}_Z u^{(0,1)}, \\
\mathscr{R}^{(3)} &= \mathscr{L}_Y\tilde{u}^{(2,1)} + \mathscr{L}_Z\left(u^{(1,0)} + \epsilon\tilde{u}^{(3,0)} + \sqrt{\epsilon\delta}\tilde{u}^{(2,1)}\right).
\end{aligned}$$

By Feynman-Kac formula,

$$
\begin{aligned}
\tilde{E}(t,x,y,z) = \epsilon \mathbb{E}_{(t,x,y,z)} & \left(\int_t^T \mathscr{R}^{(1)}(s, X_s^{\pi^{(0)}}, Y_s, Z_s)\, ds \right) \\
& + \delta \mathbb{E}_{(t,x,y,z)} \left(\int_t^T \mathscr{R}_s^{(2)}\, ds \right) + \sqrt{\epsilon\delta} \mathbb{E}_{(t,x,y,z)} \left(\int_t^T \mathscr{R}_s^{(3)}\, ds \right) \\
& - \epsilon \mathbb{E}_{(t,x,y,z)} \left(\tilde{u}^{(2,0)}(T, X_T^{\pi^{(0)}}, Y_T, Z_T) \right) \\
& - \epsilon^{3/2} \mathbb{E}_{(t,x,y,z)} \left(\tilde{u}^{(3,0)}(T, X_T^{\pi^{(0)}}, Y_T, Z_T) \right)
\end{aligned} \tag{3.4.20}
$$

Now we estimate the bound of each expectations in (3.4.20). Straightforward computation gives that each expectation $\mathbb{E}_{(t,x,y,z)} \left(\int_t^T \mathscr{R}_s^{(i)}\, ds \right)$ is a sum of integrals of the form:

$$
\mathbb{E}_{(t,x,y,z)} \left(\int_t^T h(Y_s, Z_s) \mathscr{D} u^{(0)}(s, X_s^{\pi^{(0)}}, Z_s)\, ds \right), \tag{3.4.21}
$$

where $h(y,z)$ is at most polynomial growth, and $\mathscr{D}u^{(0)}$ takes derivatives of $u^{(0)}$. Note that different operators corresponds to different derivatives in $\mathscr{D}$:

$$
\begin{aligned}
\mathscr{L}_0, \mathscr{L}_Y, \mathscr{M}_Y &: D_1^2, D_1^3, D_1^4, D_2 D_1, D_2 D_1^2, D_2 D_1^3, D_1 D_2 D_1^2, D_2^2 D_1^2; \\
\mathscr{M}_Z &: \partial_z D_1, \partial_z D_1^2, \partial_z D_1^3, \partial_z D_2 D_1^2, \partial_z, \partial_z^2, \partial_z^2 D_1, \partial_z^2 D_1^2 \partial_z^2 D_1^3, \partial_z^2 D_2 D_1^2; \\
\mathscr{L}_Z &: D_1 \partial_z D_1, D_1 \partial_z D_1^2, D_1 \partial_z D_1^3, D_1 \partial_z D_2 D_1^2 .
\end{aligned}
$$

A repeated use of the concavity of $u^{(0)}$ and Lemma 3.4 guarantees that $\mathscr{D}u^{(0)}$ is bounded by

$$
\left| \mathscr{D} u^{(0)}(t,x,z) \right| \le k(z) u^{(0)}(t,x,z), \tag{3.4.22}
$$

where $k(z)$ is some non-negative function with at most polynomial growth. Using (3.4.22), we apply the Cauchy-Schwartz inequality on (3.4.21) to reduce each term to

$$\left(\mathbb{E}_{(t,y,z)}\int_t^T h^2(Y_s, Z_s)k^2(Z_s)\, ds\right)^{\frac{1}{2}}\left(\mathbb{E}_{(t,x,y,z)}\int_t^T \left(u^{(0)}(s, X_s^{\pi^{(0)}}, Z_s)\right)^2 ds\right)^{\frac{1}{2}}. \quad (3.4.23)$$

By Assumption 3.4, the first part of (3.4.23) is uniformly bounded in (ϵ, δ). By Lemma 3.4, the second part of (3.4.23) is also uniformly bounded in (ϵ, δ). For the last two terms in (3.4.20), the boundedness follows from repeating the above argument with Assumption B.3. Thus, by bounding (3.4.20), we obtain a bound for $\tilde{E}$:

$$\left|\tilde{E}(t, x, y, z)\right| \leq (\epsilon + \delta + \sqrt{\epsilon\delta})\tilde{C} \leq (\epsilon + \delta)\tilde{C},$$

for any $(t, x, y, z) \in [0,T] \times \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}$, and $\tilde{C}$ is a constant independent of (ϵ, δ). Therefore,

$$\begin{aligned}
\left|E(t, x, y, z)\right| &= \left|\tilde{u}^{\epsilon,\delta} - u^{(0)} - \sqrt{\epsilon}u^{(1,0)} - \sqrt{\delta}u^{(0,1)}\right| \\
&= \left|\tilde{E}(t, x, y, z) + \epsilon\tilde{u}^{(2,0)} + \epsilon^{3/2}\tilde{u}^{(3,0)} + \epsilon\sqrt{\delta}\tilde{u}^{(2,1)}\right| \\
&\leq \left|\tilde{E}(t, x, y, z)\right| + \left|\epsilon\tilde{u}^{(2,0)} + \epsilon^{3/2}\tilde{u}^{(3,0)} + \epsilon\sqrt{\delta}\tilde{u}^{(2,1)}\right| \\
&\leq (\epsilon + \delta)C,
\end{aligned}$$

where $C(t, x, y, z)$ is a constant independent of (ϵ, δ). Hence, the residual function E has order $\epsilon + \delta$. ∎

3.5 First Order Strategy and Practical Use

The optimal feedback strategy (3.2.5) can be further expanded up to orders $\sqrt{\epsilon}$ and $\sqrt{\delta}$. Recall the expansion

$$\pi^* = \pi^{(0)} + \sqrt{\epsilon}\pi^{(1,0)} + \sqrt{\delta}\pi^{(0,1)} + \cdots,$$

where $\pi^{(0)}$ is derived in (3.4.2):

$$\pi^{(0)} = \Sigma_1^{-1}(y,z)\begin{pmatrix}\mu_f(y)\\ \mu_s(z)\end{pmatrix} R(t,x;\bar{\lambda}(z)), \tag{3.5.1}$$

and $\Sigma_1^{-1}(y,z)$ is given in Section 3.1:

$$\Sigma_1^{-1}(y,z) = \begin{pmatrix} \dfrac{1}{\sigma_f^2(y)(1-\rho_{12}^2)}, & -\dfrac{\rho_{12}}{\sigma_f(y)\sigma_s(z)(1-\rho_{12}^2)} \\ -\dfrac{\rho_{12}}{\sigma_f(y)\sigma_s(z)(1-\rho_{12}^2)}, & \dfrac{1}{\sigma_s^2(z)(1-\rho_{12}^2)} \end{pmatrix}.$$

By collecting the first order terms when we insert the expansion of $u^{\epsilon,\delta}$ into the optimal feedback strategy (3.2.5), we obtain the fast-scale first order strategy:

$$\pi^{(1,0)} = \Sigma_1^{-1}\frac{1}{u_x^{(0)}}\left(\begin{pmatrix}\mu_f\\ \mu_s\end{pmatrix}\frac{1}{2}(T-t)A(D_1+D_2) + \begin{pmatrix}1\\ 0\end{pmatrix}\frac{1}{2}\psi_y\rho_{13}\sigma_f\sigma_y\mathbf{1}\right)D_1D_2u^{(0)}, \tag{3.5.2}$$

and the slow-scale first order strategy:

$$\pi^{(0,1)} = \Sigma_1^{-1}\frac{1}{u_x^{(0)}}\left(\begin{pmatrix}\mu_f\\ \mu_s\end{pmatrix}\frac{1}{2}(T-t)B(D_1+D_2) + \begin{pmatrix}0\\ 1\end{pmatrix}\rho_{24}\sigma_s\sigma_z\mathbf{1}\right)D_1u_z^{(0)}. \tag{3.5.3}$$

Here $u^{(0)}(t,x,z)$ is the Merton value function defined in (3.3.7); $A(z)$, $B(z)$, $\psi(y,z)$ are defined in Section 3.3.

For the practical understanding of the approximations for two-dimensional optimal strategy π^*, we first observe that the zeroth order approximation $\pi^{(0)}$ for the two-asset portfolio problem is similar to the single-asset case. It is a Merton type strategy in each entry, except for two points. First, the coefficient μ/σ^2 in the classical Merton strategy (1.5.6) is replaced by a rational function depending on the coefficients $\mu_f(y)$, $\mu_s(z)$, $\sigma_f(y)$, $\sigma_s(z)$ for both volatility factors. Second, the risk-tolerance function is associated to the squared-average of a special "Sharpe ratio", where the special "Shape ratio" $\lambda(y,z)$ is define in (3.3.5). Note that each entry of $\pi^{(0)}$ requires tracking both fast and slow volatility factors. For the first order approximations $\pi^{(1,0)}$ and $\pi^{(0,1)}$, each entry also depends on both volatility factors. In addition, they depend on the classical Merton value function and its derivatives with respect to the wealth state x and the slow-scale factor z. Therefore, the formulas (3.5.2) and (3.5.3) can be interpreted as expressions in terms of the classical Merton value function and its "Greeks".

Compared to the approximations for the optimal strategy in the single-asset case [Fouque, Sircar, and Zariphopoulou, 2016, Section 4.2], the formulas (3.5.1), (3.5.2) and (3.5.3) for $\pi^{(0)}$, $\pi^{(1,0)}$ and $\pi^{(0,1)}$ in the two-asset problem highlight the influence of the additional correlation ρ_{12} between the fast-scale asset and the slow-scale asset. To acquire a clearer comparison and a more straightforward intuition, it is worth computing the approximations when the two assets are uncorrelated. Plugging $\rho_{12}=0$ into (3.5.1), (3.5.2), and (3.5.3), we reduce the formulas for $\pi^{(0)}$, $\pi^{(1,0)}$ and $\pi^{(0,1)}$ to the following:

$$\pi^{(0)} = \begin{pmatrix} \frac{\mu_f(y)}{\sigma_f^2(y)} R(t,x;\bar{\lambda}_0(z)) \\ \frac{\mu_s(z)}{\sigma_s^2(z)} R(t,x;\bar{\lambda}_0(z)) \end{pmatrix};$$

$$\pi^{(1,0)} = \frac{1}{u_x^{(0)}} \begin{pmatrix} \frac{\rho_{13}}{\sigma_f(y)} \left(\frac{1}{2}(T-t) \frac{\mu_f(y)}{\sigma_f(y)} \left\langle \frac{\mu_f(\cdot)}{\sigma_f(\cdot)} \sigma_y(\cdot)\psi_y(\cdot,z) \right\rangle (D_1+D_2) + \frac{1}{2}\sigma_y(y)\psi_y(y,z)\mathbf{1} \right) \\ \frac{\rho_{13}}{\sigma_s(z)} \frac{1}{2}(T-t) \frac{\mu_s(z)}{\sigma_s(z)} \left\langle \frac{\mu_f(\cdot)}{\sigma_f(\cdot)} \sigma_y(\cdot)\psi_y(\cdot,z) \right\rangle (D_1+D_2) \end{pmatrix} D_1 D_2 u^{(0)};$$

$$\pi^{(0,1)} = \frac{1}{u_x^{(0)}} \begin{pmatrix} \frac{\rho_{24}\sigma_z(z)}{\sigma_f(y)} \frac{1}{2}(T-t) \frac{\mu_f(y)}{\sigma_f(y)} \frac{\mu_s(z)}{\sigma_s(z)} (D_1+D_2) \\ \frac{\rho_{24}\sigma_z(z)}{\sigma_s(z)} \left(\frac{1}{2}(T-t) \frac{\mu_s^2(z)}{\sigma_s^2(z)} (D_1+D_2) + \mathbf{1} \right) \end{pmatrix} D_1 u_z^{(0)};$$

where the risk-tolerance function R and the differential operators D_i $(i = 1,2)$ are associated to

$$\bar{\lambda}_0(z) = \sqrt{\left\langle \frac{\mu_f^2(y)}{\sigma_f^2(y)} \right\rangle + \frac{\mu_s^2(z)}{\sigma_s^2(z)}}. \tag{3.5.4}$$

Observe that each entry, $\pi_f^{(0)}$ and $\pi_s^{(0)}$, of $\pi^{(0)}$ matches the formula for the zeroth order strategy (2.4.17) in the single-asset problem with respect to the fast-scale asset S^f and the slow-scale asset S^s, respectively, except for the associated $\bar{\lambda}_0$. With the risk-tolerance function associated to $\bar{\lambda}_0$ defined in (3.5.4), the portfolio weight in each entry of $\pi^{(0)}$ increases when compared to the single-asset case. This coincides with the intuition that the two-asset portfolio benefits from diversified risk, since here we assume the two assets are uncorrelated. On the other hand, with a change of notation, we compare the above simplified first order strategies for the two-asset case with the formulas (2.4.18) and (2.4.19) from the single-asset problem in Chapter 2. Note that $\pi_f^{(1,0)}$ matches formula (2.4.18) for $\pi^{(1,0)}$ in a single-asset problem with the fast-scale asset S^f, except for the associated $\bar{\lambda}_0$, while $\pi_s^{(1,0)}$ consists of only the first term of (2.4.18) with an adjustment in coefficient. Similarly, $\pi_s^{(0,1)}$ matches (2.4.19) for $\pi^{(0,1)}$ in a single-asset problem with the slow-scale asset S^s, except for the associated $\bar{\lambda}_0$, while $\pi_f^{(0,1)}$ only has the adjusted first term of (2.4.19). Thus, we can consider the first term in formulas (2.4.18) and (2.4.19) as the principal correction part in the fast-scale and slow-scale first order strategies, respectively, which affects both assets when we extend the portfolio problem to a two-asset scenario. Meanwhile, the second term in (2.4.18) and (2.4.19) only appears in the first order strategies of the corresponding asset. To summarize, in the case where $\rho_{12} = 0$, we can intuitively view the zeroth and the first order optimal strategies in the two-asset problem as the following:

$$\begin{pmatrix} \pi_f^{(0)} \\ \pi_s^{(0)} \end{pmatrix} = \begin{pmatrix} \pi^{(0)} \text{ in single-asset problem w.r.t. } S^f \text{ with the special } \bar{\lambda}_0 \\ \pi^{(0)} \text{ in single-asset problem w.r.t. } S^s \text{ with the special } \bar{\lambda}_0 \end{pmatrix};$$

$$\begin{pmatrix} \pi_f^{(1,0)} \\ \pi_s^{(1,0)} \end{pmatrix} = \begin{pmatrix} \pi^{(1,0)} \text{ in single-asset problem w.r.t. } S^f \text{ with the special } \bar{\lambda}_0 \\ \text{adjusted principal correction part of } \pi^{(1,0)} \text{ in the above entry} \end{pmatrix};$$

$$\begin{pmatrix} \pi_f^{(0,1)} \\ \pi_s^{(0,1)} \end{pmatrix} = \begin{pmatrix} \text{adjusted principal correction part of } \pi^{(0,1)} \text{ in the below entry} \\ \pi^{(0,1)} \text{ in single-asset problem w.r.t. } S^s \text{ with the special } \bar{\lambda}_0 \end{pmatrix}.$$

Thus, compared to the optimal strategy in a single-asset problem, the main complexity in the formulas (3.5.1), (3.5.2), and (3.5.3) for the two-asset optimal strategy comes from a non-zero correlation $\rho_{12} \neq 0$.

In practice, the multi-scale two-asset model can be applied to the case where the difference between the time-scale of two correlated assets is significant. For example, we can consider the two-asset portfolio problem involving a short-term futures contract and a long-term futures contract. Given the coefficient functions for the model, the correlation matrix where $0 < \rho_{12} < 1$, and a general utility function satisfying Assumption B.3, we propose the zeroth order Merton type strategy (3.5.1) for the two-asset portfolio problem. Theorem II proved that, under Assumption 3.4, the two-dimensional zeroth order strategy $\pi^{(0)}$ recovers the optimal value function for the associated portfolio problem up to the first order approximations. Furthermore, if the small time-scale parameters ϵ and δ can be captured, we are able to adjust our strategy by computing the first order corrections strategy using formulas (3.5.2) and (3.5.3).

3.6 Conclusion

The asymptotic analysis for a multi-asset portfolio problem can be studied and reproduced. For the two-asset portfolio problem, we have derived explicit formulas for the heuristic asymptotic expansions of the associated value function, and have computed the 2-dimensional zeroth and the first order optimal strategies. The main result (Theorem II) proves that the zeroth order strategy rigorously recovers the value function up to the first order under certain assumptions. In addition, we have given intuitions and comments on the comparison between a special case of a two-asset scenario, where the two assets are uncorrelated, with the single-asset case. We conclude that, while the dependency on the variables is the same as the single-asset case, the 2-dimensional zeroth and first order optimal strategies highlight the importance of the correlation between the prices of the two assets.

Further studies can be done by considering an N-asset portfolio problem, where each asset is affected by its time-scaled volatility factor. We are able to apply the same procedure as in the two-asset case, except that the explicit formulas may not be in concise forms due to the large and tedious correlation matrix. On the other hand, in practice, the states of stochastic factors are difficult to capture with accuracy. As a result, in order to have a deeper understanding of the zeroth order strategy performance, we can either continue with numerical backtesting with real historical data, or introduce asymptotic analysis with partial information, where we assume the stochastic factors are unobservable. We will discuss the later option in the next chapter.

CHAPTER 4 MULTI-SCALE TWO-ASSET PROBLEM WITH PARTIAL INFORMATION

In this chapter, we consider the two-asset portfolio optimization problem with partial information on the volatility factors, where we assume the volatility factors are unobservable. We focus on a particular model where the drift and volatility coefficients for the assets are constant, and the unobservable volatility factors are mean-reverting processes with zero drift and constant volatility. Again, we assume that the risk-free interest rate is zero. Applying the Kalman-Bucy filtering theory, we can track the unobservable states of the volatility factors given the history of observable asset prices. We aim to derive the zeroth and the first order asymptotic optimal strategy for the two-asset portfolio problem with partial information, and to make a comparison with the full information case.

First, we use the Kalman-Bucy filtering theory to filter the unobservable factors. Then we explicitly compute the filter asymptote as the small parameters go to zero with the steady state assumption. With the filter asymptote, we are able to set up the two-asset portfolio problem, and derive the HJB problem. Next, we follow the same procedure as in Chapter 3 to compute the asymptotic expansions of the associated value function and the optimal strategy. Under the regularity assumption on the value functions, we prove that, in the partial information case, the zeroth order strategy also recovers the value function up to the first order. Finally, we expand optimal strategy for the two-asset portfolio problem with partial information up to the first order.

The main contribution of this chapter is extending the results about the asymptotic approximation of two-asset optimal strategy in Chapter 3 to a partial information scenario. We combine the filtering theory in [Xiong, 2008] with asymptotic analysis to explicitly derive the filter asymptote under the steady state assumption and the asymptotic expansion of the associated value function. In the partial information case, we reproduce the same result about the performance of the zeroth order strategy under the regularity assumption on the value functions. Lastly, by comparing the explicit formulas for the approximations of the optimal strategy, we are able to highlight the contribution from the filter asymptote.

4.1 Multi-scale Model on Two Assets with Partial Information

Consider two correlated assets $\tilde{S}^f$ and $\tilde{S}^s$ on a given probability space $(\Omega, \mathscr{F}, \mathbb{F} = (\mathscr{F}_t)_{t\in[0,T]}, \mathbb{P})$, where f represents the fast-scale asset and s represents the slow-scale asset. Suppose the dynamical system for the logarithmic asset prices S^f and S^s is given by

$$\begin{aligned} dS^f &:= \frac{d\tilde{S}^f_t}{\tilde{S}^f_t} = (\mu_f + Y_t)dt + \sigma_f dW^f_t, \\ dS^s &:= \frac{d\tilde{S}^s_t}{\tilde{S}^s_t} = (\mu_s + Z_t)dt + \sigma_s dW^s_t, \end{aligned} \tag{4.1.1}$$

where Y_t and Z_t are fast mean-reverting factor and slow mean-reverting factor:

$$\begin{aligned} dY_t &= -\frac{1}{\epsilon}Y_t dt + \frac{1}{\sqrt{\epsilon}}\sigma_y dW^y_t, \\ dZ_t &= -\delta Z_t dt + \sqrt{\delta}\sigma_z dW^z_t. \end{aligned} \tag{4.1.2}$$

Note that the coefficients μ_f, μ_s, σ_f, σ_s, σ_y, and σ_z in this chapter are known constants. The small parameter ϵ and δ characterize the fast and slow variation of the mean-reverting factors Y and Z. The correlation between W^f, W^s, W^y, W^z is given by the correlation matrix

$$Co = \begin{pmatrix} 1, & \rho_{12}, & \rho_{13}, & \rho_{14} \\ \rho_{12}, & 1, & \rho_{23}, & \rho_{24} \\ \rho_{13}, & \rho_{23}, & 1, & \rho_{34} \\ \rho_{14}, & \rho_{24}, & \rho_{34}, & 1 \end{pmatrix},$$

where $0 \leq \rho_{ij} < 1$ are defined by

$$\begin{aligned} dW_t^f dW_t^s &= \rho_{12} dt, \qquad & dW_t^f dW_t^y &= \rho_{13} dt, \qquad & dW_t^f dW_t^z &= \rho_{14} dt, \\ dW_t^s dW_t^y &= \rho_{23} dt, \qquad & dW_t^s dW_t^z &= \rho_{24} dt, \qquad & dW_t^y dW_t^z &= \rho_{34} dt\,. \end{aligned}$$

Similar to the assumptions on the correlation matrix in Chapter 3, we let $\rho_{14} = \rho_{23} = \rho_{34} = 0$ for simplicity. Then let Σ be a lower triangular matrix that satisfies the Cholesky decomposition $\Sigma\Sigma^\top = Co$. Hence, Σ can be computed explicitly as

$$\Sigma = \begin{pmatrix} 1, & 0, & 0, & 0 \\ \rho_{12}, & \sqrt{1-\rho_{12}^2}, & 0, & 0 \\ \rho_{13}, & -\frac{\rho_{12}\rho_{13}}{\sqrt{1-\rho_{12}^2}}, & \frac{\sqrt{1-\rho_{12}^2-\rho_{13}^2}}{\sqrt{1-\rho_{12}^2}}, & 0 \\ 0, & \frac{\rho_{24}}{\sqrt{1-\rho_{12}^2}}, & \frac{\rho_{12}\rho_{13}\rho_{24}}{\sqrt{1-\rho_{12}^2}\sqrt{1-\rho_{12}^2-\rho_{13}^2}}, & \frac{\sqrt{1-\rho_{12}^2-\rho_{13}^2-\rho_{24}^2+\rho_{13}^2\rho_{24}^2}}{\sqrt{1-\rho_{12}^2-\rho_{13}^2}} \end{pmatrix}.$$

Then we express W^f, W^s, W^y ,W^z with a 4-dimensional Brownian motion:

$$\begin{pmatrix} W_t^f \\ W_t^s \\ W_t^y \\ W_t^z \end{pmatrix} = \Sigma \begin{pmatrix} B_t^1 \\ B_t^2 \\ B_t^3 \\ B_t^4 \end{pmatrix}.$$

Now the dynamical system (4.1.1) for the asset prices $\mathbf{S}_t = \begin{pmatrix} S_t^f \\ S_t^s \end{pmatrix}$ and the dynamical system (4.1.2) for the mean-reverting factors $\mathbf{Y}_t = \begin{pmatrix} Y_t \\ Z_t \end{pmatrix}$ can be re-written with the 4-dimensional Brownian motion $\mathbf{W} = \begin{pmatrix} B^1 \\ B^2 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} B^3 \\ B^4 \end{pmatrix}$:

$$\begin{cases} d\mathbf{S}_t = (\mu_{\mathbf{S}} + \mathbf{Y}_t)dt + \tilde{\Sigma}_{11} d\mathbf{W}_t, \\ d\mathbf{Y}_t = -\begin{pmatrix} \frac{1}{\epsilon}, 0 \\ 0, \ \delta \end{pmatrix} \mathbf{Y}_t dt + \tilde{\Sigma}_{12} d\mathbf{W}_t + \tilde{\Sigma}_{22} d\mathbf{B}_t, \end{cases} \tag{4.1.3}$$

where $\mu_{\mathbf{S}} := \begin{pmatrix} \mu_f \\ \mu_s \end{pmatrix}$, and the matrices $\tilde{\Sigma}_{11}$, $\tilde{\Sigma}_{12}$, $\tilde{\Sigma}_{22}$ are defined by

$$\tilde{\Sigma}_{11} = \begin{pmatrix} \sigma_f, & 0 \\ 0, & \sigma_s \end{pmatrix} \begin{pmatrix} 1, & 0 \\ \rho_{12}, & \sqrt{1 - \rho_{12}^2} \end{pmatrix},$$

$$\tilde{\Sigma}_{12} = \begin{pmatrix} \frac{\sigma_y}{\sqrt{\epsilon}}, & 0 \\ 0, & \sqrt{\delta}\sigma_z \end{pmatrix} \begin{pmatrix} \rho_{13}, & -\frac{\rho_{12}\rho_{13}}{\sqrt{1-\rho_{12}^2}} \\ 0, & \frac{\rho_{24}}{\sqrt{1-\rho_{12}^2}} \end{pmatrix},$$

$$\tilde{\Sigma}_{22} = \begin{pmatrix} \frac{\sigma_y}{\sqrt{\epsilon}}, & 0 \\ 0, & \sqrt{\delta}\sigma_z \end{pmatrix} \begin{pmatrix} \frac{\sqrt{1-\rho_{12}^2-\rho_{13}^2}}{\sqrt{1-\rho_{12}^2}}, & 0 \\ \frac{\rho_{12}\rho_{13}\rho_{24}}{\sqrt{1-\rho_{12}^2}\sqrt{1-\rho_{12}^2-\rho_{13}^2}}, & \frac{\sqrt{1-\rho_{12}^2-\rho_{13}^2-\rho_{24}^2+\rho_{13}^2\rho_{24}^2}}{\sqrt{1-\rho_{12}^2-\rho_{13}^2}} \end{pmatrix}.$$

Under the partial information setting, the volatility factors Y and Z should be considered unobserved because they must be estimated from the observable market. We ought to filter the unobservable mean-reverting factors Y_t and Z_t based on the completion of the filtrations generated by the observable asset prices S^f and S^s. Let $\mathscr{G}$ denote the completion of the filtration $\mathscr{F}^{S^f} \vee \mathscr{F}^{S^s}$. We aim to acquire the following:

$$\hat{\mathbf{Y}}_t := \begin{pmatrix} \hat{Y}_t \\ \hat{Z}_t \end{pmatrix} := \begin{pmatrix} \mathbb{E}\left(Y_t | \mathscr{G}_t\right) \\ \mathbb{E}\left(Z_t | \mathscr{G}_t\right) \end{pmatrix}. \tag{4.1.4}$$

Note that the drifts of the observable asset prices $\mathbf{S}_t$ in (4.1.3) are Gaussian, which means that the Kalman-Bucy filter applies. Following the Kalman-Bucy filtering theory in [Xiong, 2008], we first define the covariance matrix $\Gamma^{\epsilon,\delta}(t)$:

$$\Gamma^{\epsilon,\delta}(t) := \begin{pmatrix} \mathbb{E}(Y_t - \hat{Y}_t)^2, & \mathbb{E}(Y_t - \hat{Y}_t)(Z_t - \hat{Z}_t) \\ \mathbb{E}(Y_t - \hat{Y}_t)(Z_t - \hat{Z}_t), & \mathbb{E}(Z_t - \hat{Z}_t)^2 \end{pmatrix}. \tag{4.1.5}$$

Then it follows that $\mathbf{S}_t = \begin{pmatrix} S_t^f \\ S_t^s \end{pmatrix}$ and $\hat{\mathbf{Y}}_t = \begin{pmatrix} \hat{Y}_t \\ \hat{Z}_t \end{pmatrix}$ satisfy the following equations:

$$\begin{cases} d\mathbf{S}_t = (\mu_{\mathbf{S}} + \hat{\mathbf{Y}}_t)dt + \tilde{\Sigma}_{11} d\nu_t, \\ d\hat{\mathbf{Y}}_t = -\begin{pmatrix} \frac{1}{\epsilon}, 0 \\ 0, \ \delta \end{pmatrix} \hat{\mathbf{Y}}_t dt + M^{\epsilon,\delta}(t) d\nu_t, \end{cases} \tag{4.1.6}$$

where $\nu_t = \tilde{\Sigma}_{11}^{-1}\mathbf{S}_t - \int_0^t \tilde{\Sigma}_{11}^{-1}(\mu_{\mathbf{S}} + \hat{\mathbf{Y}}_s)ds$ is a 2-dimensional Brownian motion adapted to $\mathcal{G}$, and the function $M^{\epsilon,\delta}(t)$ is defined by

$$M^{\epsilon,\delta}(t) = \tilde{\Sigma}_{12} + \Gamma^{\epsilon,\delta}(t)(\tilde{\Sigma}_{11}^{-1})^\top = \begin{pmatrix} \frac{1}{\sqrt{\epsilon}}\sigma_y \rho_{13}, & -\frac{1}{\sqrt{\epsilon}} \frac{\sigma_y \rho_{12}\rho_{13}}{\sqrt{1-\rho_{12}^2}} \\ 0, & \sqrt{\delta}\frac{\sigma_z \rho_{24}}{\sqrt{1-\rho_{12}^2}} \end{pmatrix} + \Gamma^{\epsilon,\delta}(t) \begin{pmatrix} \frac{1}{\sigma_f}, & \frac{-\rho_{12}}{\sigma_f\sqrt{1-\rho_{12}^2}} \\ 0, & \frac{1}{\sigma_s\sqrt{1-\rho_{12}^2}} \end{pmatrix}.$$

In addition, the covariance matrix $\Gamma^{\epsilon,\delta}(t)$ satisfies the Riccati equation:

$$\frac{d}{dt}\Gamma^{\epsilon,\delta}(t) = -\Gamma^{\epsilon,\delta}(t)\begin{pmatrix} \frac{1}{\epsilon},0 \\ 0,\delta \end{pmatrix} - \begin{pmatrix} \frac{1}{\epsilon},0 \\ 0,\delta \end{pmatrix}\Gamma^{\epsilon,\delta}(t) + \begin{pmatrix} \frac{\sigma_y^2}{\epsilon}, & 0 \\ 0, & \delta\sigma_z^2 \end{pmatrix} - M^{\epsilon,\delta}(t)M^{\epsilon,\delta}(t)^\top. \tag{4.1.7}$$

We refer to Appendix C and [Xiong, 2008] for more details of the Kalman-Bucy filter. Similar to Chapter 3, here we let the process $\hat{Y}_t$ have a unique invariant distribution $\hat{\Psi}$, and we use the notation $\langle \cdot \rangle$ in this chapter as the average of a function with respect to $\hat{\Psi}(y)$:

$$\langle f \rangle = \int f(y)\hat{\Psi}(dy). \tag{4.1.8}$$

In fact, $\hat{\Psi}$ is the density function of a normal distribution with zero mean and constant variance.

4.2 Steady State Filter Asymptote

In this section, we aim to derive an asymptotic expansion of the covariance matrix $\Gamma^{\epsilon,\delta}(t)$ in powers of $\sqrt{\epsilon}$ and $\sqrt{\delta}$ under the steady state assumption. The so-called steady state assumption is to assume the covariance matrix stays constant. That is, $\Gamma^{\epsilon,\delta}(t) = \Gamma^{\epsilon,\delta}(0)$ for all $t > 0$.

We aim for the long-term performance of an optimal strategy where $t \to \infty$, since the slow-scale factor Z_t shows its influence only when the investment period is long enough. A convergence result in [Guo and Yu, 2015] shows that, as $t \to \infty$, the solution $\Gamma^{\epsilon,\delta}(t)$ to the Riccati equation (4.1.7) converges to the minimal non-negative solution $\Gamma^{\epsilon,\delta}(0)$ to the corresponding algebraic Riccati equation, including the critical cases. After the derivation of the fast-scale small-ϵ filter asymptote in the general case, we import the steady state assumption to derive the slow-scale small-δ filter asymptote.

Proposition 4.2.A (Small-ϵ Filter Asymptote). The covariance matrix $\Gamma^{\epsilon,\delta}(t)$, $t > 0$, that solves the Riccati equation (4.1.7) can be expanded in powers of $\sqrt{\epsilon}$:

$$\Gamma^{\epsilon,\delta}(t) = \Gamma^{(0),\delta}(t) + \sqrt{\epsilon}\Gamma^{(1),\delta}(t) + o(\sqrt{\epsilon}).$$

The zeroth order term $\Gamma^{(0),\delta}(t)$ and the first order term $\Gamma^{(1),\delta}(t)$ are given by

$$\Gamma^{(0),\delta}(t) = \begin{pmatrix} \frac{\sigma_y^2}{2}\left(1 - \frac{\rho_{13}^2}{1-\rho_{12}^2}\right), & 0 \\ 0, & \Gamma_{22}^{(0),\delta}(t) \end{pmatrix} \tag{4.2.1}$$

and

$$\Gamma^{(1),\delta}(t) = \begin{pmatrix} -\frac{\sigma_y \rho_{13}}{\sigma_f(1-\rho_{12}^2)}\Gamma_{11}^{(0),\delta}, & \frac{\sigma_y \rho_{12}\rho_{13}}{1-\rho_{12}^2}\Big(\sigma_z \rho_{24}\sqrt{\delta} + \frac{\Gamma_{22}^{(0),\delta}(t)}{\sigma_s}\Big) \\ \frac{\sigma_y \rho_{12}\rho_{13}}{1-\rho_{12}^2}\Big(\sigma_z \rho_{24}\sqrt{\delta} + \frac{\Gamma_{22}^{(0),\delta}(t)}{\sigma_s}\Big), & 0 \end{pmatrix}, \quad (4.2.2)$$

where $\Gamma_{22}^{(0),\delta}(t)$ satisfies the Riccati equation

$$\begin{aligned} \frac{d}{dt}\Gamma_{22}^{(0),\delta}(t) = -2\delta\Gamma_{22}^{(0),\delta}(t) + \delta\sigma_z^2(1 - \frac{\rho_{24}^2}{1-\rho_{12}^2}) \\ -2\sqrt{\delta}\frac{\sigma_z\rho_{24}}{\sigma_s(1-\rho_{12}^2)}\Gamma_{22}^{(0),\delta}(t) - \frac{\big(\Gamma_{22}^{(0),\delta}(t)\big)^2}{\sigma_s^2(1-\rho_{12}^2)}, \end{aligned} \quad (4.2.3)$$

with initial condition $\Gamma_{22}^{(0),\delta}(0) = \mathbb{E}(Z_0 - \hat{Z}_0)^2$.

Proof. We use Itô isometry and Itô's product rule to compute each term of $\Gamma^{\epsilon,\delta}(t)$ from the definition (4.1.5). First, the filtering theory in [Xiong, 2008, Lemma 9.5] gives us that

$$\Gamma^{\epsilon,\delta}(t) = \begin{pmatrix} \mathbb{E}Y_t^2 - \mathbb{E}\hat{Y}_t^2, & \mathbb{E}Y_tZ_t - \mathbb{E}\hat{Y}_t\hat{Z}_t \\ \mathbb{E}Y_tZ_t - \mathbb{E}\hat{Y}_t\hat{Z}_t, & \mathbb{E}Z_t^2 - \mathbb{E}\hat{Z}_t^2 \end{pmatrix}.$$

To derive $\Gamma^{(0),\delta}(t)$, we compute the limit of $\Gamma^{\epsilon,\delta}(t)$ as $\epsilon \to 0$. For the fast state,

$$\mathbb{E}Y_t^2 = \mathbb{E}\Big(e^{-t/\epsilon}Y_0 + \frac{\sigma_y}{\sqrt{\epsilon}}\int_0^t e^{-(t-s)/\epsilon}dW_s^y\Big)^2 \to \frac{\sigma_y^2}{2}$$

and

$$
\begin{aligned}
\mathbb{E}\hat{Y}_t^2 &= \mathbb{E}\Big(e^{-t/\epsilon}\hat{Y}_0 + \int_0^t e^{-(t-s)/\epsilon} M_{11}^{\epsilon,\delta}(s) d\nu_s^1 + \int_0^t e^{-(t-s)/\epsilon} M_{12}^{\epsilon,\delta}(s) d\nu_s^2\Big)^2 \\
&= \mathbb{E}\Big(e^{-t/\epsilon}\hat{Y}_0 + \int_0^t e^{-(t-s)/\epsilon}\big(\frac{\rho_{13}\sigma_y}{\sqrt{\epsilon}} + \frac{\Gamma_{11}^{\epsilon,\delta}(s)}{\sigma_f}\big) d\nu_s^1 \\
&\quad + \int_0^t e^{-(t-s)/\epsilon}\big(- \frac{1}{\sqrt{\epsilon}} \frac{\rho_{12}\rho_{13}\sigma_y}{\sqrt{1-\rho_{12}^2}} - \frac{\rho_{12}\Gamma_{11}^{\epsilon,\delta}(s)}{\sigma_f\sqrt{1-\rho_{12}^2}} + \frac{\Gamma_{12}^{\epsilon,\delta}(s)}{\sigma_s\sqrt{1-\rho_{12}^2}}\big) d\nu_s^2\Big)^2 \\
&\to \frac{\sigma_y^2\rho_{13}^2}{2} + \frac{\sigma_y^2\rho_{12}^2\rho_{13}^2}{2(1-\rho_{12}^2)} = \frac{\sigma_y^2\rho_{13}^2}{2(1-\rho_{12}^2)}
\end{aligned}
$$

as $\epsilon \to 0$, where we use Itô isometry and the independence between ν^1 and ν^2. Thus, we have

$$
\Gamma_{11}^{(0),\delta}(t) = \frac{\sigma_y^2}{2}\Big(1 - \frac{\rho_{13}^2}{1-\rho_{12}^2}\Big).
$$

For the cross term, similar computation with Itô's product rule gives that

$$
\mathbb{E}Y_t Z_t = \mathbb{E}\Big(e^{-t(\epsilon^{-1}+\delta)} Y_0 Z_0 + \frac{\sigma_y}{\sqrt{\epsilon}} \int_0^t e^{-(t-s)(\epsilon^{-1}+\delta)} Z_s dW_s^y + \sqrt{\delta}\sigma_z \int_0^t e^{-(t-s)(\epsilon^{-1}+\delta)} Y_s dW_s^z\Big) \to 0
$$

and

$$
\begin{aligned}
\mathbb{E}\hat{Y}_t\hat{Z}_t &= \mathbb{E}\Big(e^{-t(\epsilon^{-1}+\delta)} Y_0 Z_0 + \int_0^t e^{-(t-s)(\epsilon^{-1}+\delta)} M_{11}^{\epsilon,\delta}(s)\hat{Z}_s d\nu_s^1 + \int_0^t e^{-(t-s)(\epsilon^{-1}+\delta)} M_{12}^{\epsilon,\delta}(s)\hat{Z}_s d\nu_s^2 \\
&\quad + \int_0^t e^{-(t-s)(\epsilon^{-1}+\delta)} M_{21}^{\epsilon,\delta}(s)\hat{Y}_s d\nu_s^1 + \int_0^t e^{-(t-s)(\epsilon^{-1}+\delta)} M_{22}^{\epsilon,\delta}(s)\hat{Y}_s d\nu_s^2 \\
&\quad + \int_0^t e^{-(t-s)(\epsilon^{-1}+\delta)}\big(M_{11}^{\epsilon,\delta}(s)M_{21}^{\epsilon,\delta}(s) + M_{12}^{\epsilon,\delta}(s)M_{22}^{\epsilon,\delta}(s)\big) ds\Big) \\
&\to 0
\end{aligned}
$$

as $\epsilon \to 0$. Thus, $\Gamma_{12}^{(0),\delta}(t) = \Gamma_{21}^{(0),\delta}(t) = 0$. For the slow state,

$$\mathbb{E}Z_t^2 = \mathbb{E}\Big(e^{-\delta t}Z_0 + \sqrt{\delta}\sigma_z\int_0^t e^{-\delta(t-s)}dW_s^z\Big)^2 = e^{-2\delta t}\mathbb{E}Z_0^2 + \delta\sigma_z^2\int_0^t e^{-2\delta(t-s)}ds$$

and

$$\begin{aligned}
\mathbb{E}\hat{Z}_t^2 &= \mathbb{E}\Big(e^{-\delta t}\hat{Z}_0 + \int_0^t e^{-\delta(t-s)}M_{21}^{\epsilon,\delta}(s)d\nu_s^1 + \int_0^t e^{-\delta(t-s)}M_{22}^{\epsilon,\delta}(s)d\nu_s^2\Big)^2 \\
&= \mathbb{E}\Big(e^{-\delta t}\hat{Z}_0 + \int_0^t e^{-\delta(t-s)}\big(\frac{1}{\sigma_f}\Gamma_{21}^{\epsilon,\delta}(s)\big)d\nu_s^1 \\
&\quad + \int_0^t e^{-\delta(t-s)}\big(\sqrt{\delta}\frac{\sigma_z\rho_{24}}{\sqrt{1-\rho_{12}^2}} - \frac{\rho_{12}\Gamma_{21}^{\epsilon,\delta}(s)}{\sigma_f\sqrt{1-\rho_{12}^2}} + \frac{\Gamma_{22}^{\epsilon,\delta}(s)}{\sigma_s\sqrt{1-\rho_{12}^2}}\big)d\nu_s^2\Big)^2 \\
&= e^{-2\delta t}\mathbb{E}\hat{Z}_0^2 + \int_0^t e^{-2\delta(t-s)}\big(\frac{1}{\sigma_f}\Gamma_{21}^{\epsilon,\delta}(s)\big)^2 ds + \int_0^t e^{-2\delta(t-s)}\big(\sqrt{\delta}\frac{\sigma_z\rho_{24}}{\sqrt{1-\rho_{12}^2}} + \frac{\Gamma_{22}^{\epsilon,\delta}(s)}{\sigma_s\sqrt{1-\rho_{12}^2}}\big)^2 ds \\
&\quad + \int_0^t e^{-2\delta(t-s)}\Big(\big(\frac{\rho_{12}\Gamma_{21}^{\epsilon,\delta}(s)}{\sigma_f\sqrt{1-\rho_{12}^2}}\big)^2 - 2\frac{\rho_{12}\Gamma_{21}^{\epsilon,\delta}(s)}{\sigma_f\sqrt{1-\rho_{12}^2}}\big(\sqrt{\delta}\frac{\sigma_z\rho_{24}}{\sqrt{1-\rho_{12}^2}} + \frac{\Gamma_{22}^{\epsilon,\delta}(s)}{\sigma_s\sqrt{1-\rho_{12}^2}}\big)\Big)ds
\end{aligned}$$

Combining the above two 2nd moments, we have that $\Gamma_{22}^{(0),\delta}(t) = \lim_{\epsilon\to 0}\Gamma_{22}^{\epsilon,\delta}(t)$ satisfies

$$\Gamma_{22}^{(0),\delta}(t) = e^{-2\delta t}\Gamma_{22}^{(0),\delta}(0) + \int_0^t e^{-2\delta(t-s)}\Big(\delta\sigma_z^2 - \big(\sqrt{\delta}\frac{\sigma_z\rho_{24}}{\sqrt{1-\rho_{12}^2}} + \frac{\Gamma_{22}^{\epsilon,\delta}(s)}{\sigma_s\sqrt{1-\rho_{12}^2}}\big)^2\Big)ds,$$

because $\Gamma_{21}^{\epsilon,\delta}(t) \to 0$ as $\epsilon \to 0$. This is exactly the Riccati equation (4.2.3) with a different form. The initial condition is $\Gamma_{22}^{(0),\delta}(0) = \mathbb{E}(Z_0 - \hat{Z}_0)^2$. This ends the computation for the formula (4.2.1).

Now we compute $\Gamma^{(1),\delta}(t)$ by taking the limit:

$$\lim_{\epsilon \to 0} \frac{1}{\sqrt{\epsilon}}(\Gamma^{\epsilon,\delta}(t) - \Gamma^{(0),\delta}(t)). \tag{4.2.4}$$

Using the same forms for variance and covariance that were used in the computation of $\Gamma^{(0),\delta}(t)$, we derive that all the terms are zero in the computation of the limit (4.3.4), except for the following terms:

$$-\frac{1}{\sqrt{\epsilon}}\left(\mathbb{E}\hat{Y}_t^2 - \frac{\sigma_y^2\rho_{13}^2}{2(1-\rho_{12}^2)}\right) \to -\frac{\sigma_y\rho_{13}}{\sigma_f}\Gamma_{11}^{(0),\delta} - \frac{\sigma_y\rho_{13}\rho_{12}^2}{\sigma_f(1-\rho_{12}^2)}\Gamma_{11}^{(0),\delta} = -\frac{\sigma_y\rho_{13}}{\sigma_f(1-\rho_{12}^2)}\Gamma_{11}^{(0),\delta};$$

$$-\frac{1}{\sqrt{\epsilon}}\mathbb{E}\hat{Y}_t\hat{Z}_t \to \frac{\sigma_y\rho_{12}\rho_{13}}{1-\rho_{12}^2}\left(\sigma_z\rho_{24}\sqrt{\delta} + \frac{\Gamma_{22}^{(0),\delta}(t)}{\sigma_s}\right).$$

This gives the formula (4.2.2) for $\Gamma^{(1),\delta}(t)$. ∎

Now we import the steady state assumption to derive the small-δ filter asymptote for $\Gamma_{22}^{\epsilon,\delta}(t)$. Thus, we assume that $\Gamma_{22}^{\epsilon,\delta}(t) = \Gamma_{22}^{\epsilon,\delta}(0)$ for all $t > 0$.

Proposition 4.2.B (Small-δ Filter Asymptote). Under the steady state assumption, the entry $\hat{\Gamma}_{22}^{(0),\delta} := \Gamma_{22}^{(0),\delta}(0)$ can be expanded in powers of $\sqrt{\delta}$:

$$\hat{\Gamma}_{22}^{(0),\delta} = \hat{\Gamma}_{22}^{(0)} + \sqrt{\delta}\,\hat{\Gamma}_{22}^{(0,1)} + o(\sqrt{\delta}).$$

The zeroth order term is $\hat{\Gamma}_{22}^{(0)} = 0$, and the first order term $\hat{\Gamma}_{22}^{(0,1)}$ is given by

$$\hat{\Gamma}_{22}^{(0,1)} = \sigma_z\sigma_s\big(\sqrt{1-\rho_{12}^2} - \rho_{24}\big). \tag{4.2.5}$$

Proof. The steady state assumption allows us to assume $\Gamma_{22}^{(0),\delta}(t) = \Gamma_{22}^{(0),\delta}(0)$ for all $t > 0$. Then the left-hand side of Riccati equation (4.2.3) becomes zero. Hence, we obtain the algebraic Riccati equation for $\hat{\Gamma}_{22}^{(0),\delta}$:

$$0 = -2\delta\hat{\Gamma}_{22}^{(0),\delta} + \delta\sigma_z^2\Big(1 - \frac{\rho_{24}^2}{1-\rho_{12}^2}\Big) - 2\sqrt{\delta}\,\frac{\sigma_z\rho_{24}}{\sigma_s(1-\rho_{12}^2)}\hat{\Gamma}_{22}^{(0),\delta} - \frac{\big(\hat{\Gamma}_{22}^{(0),\delta}\big)^2}{\sigma_s^2(1-\rho_{12}^2)}. \tag{4.2.6}$$

This is, in fact, a quadratic equation, which has a non-negative solution. Also, the non-negative solution $\hat{\Gamma}_{22}^{(0),\delta}$ is analytic in $\sqrt{\delta}$ by [Ran and Rodman, 1988], because the coefficients are analytic in $\sqrt{\delta}$. First, the equation (4.2.6) gives that $\hat{\Gamma}_{22}^{(0)} = 0$ as we let $\delta \to 0$. Then the first order approximation $\hat{\Gamma}_{22}^{(0,1)}$ is acquired by collecting the terms in (4.2.6) with order δ:

$$0 = \sigma_z^2\Big(1 - \frac{\rho_{24}^2}{1-\rho_{12}^2}\Big) - 2\frac{\sigma_z\rho_{24}}{\sigma_s(1-\rho_{12}^2)}\hat{\Gamma}_{22}^{(0,1)} - \frac{\big(\hat{\Gamma}_{22}^{(0,1)}\big)^2}{\sigma_s^2(1-\rho_{12}^2)}.$$

Solving the above quadratic equation, we obtain that $\hat{\Gamma}_{22}^{(0,1)} = \sigma_z\sigma_s\big(\sqrt{1-\rho_{12}^2} - \rho_{24}\big) \geq 0$. ∎

To summarize Proposition 4.2.A and Proposition 4.2.B, we define the following notation for the covariance matrix $\Gamma^{\epsilon,\delta}(t)$ under the steady state assumption:

$$\Gamma^{\epsilon,\delta}(t) = \Gamma^{\epsilon,\delta}(0) = \begin{pmatrix} \gamma_1, & \gamma_2 \\ \gamma_2, & \gamma_3 \end{pmatrix}, \tag{4.2.7}$$

where

$$\begin{aligned}
\gamma_1 &= \frac{\sigma_y^2}{2}(1 - \frac{\rho_{13}^2}{1-\rho_{12}^2}) - \frac{\sigma_y^3}{2\sigma_f}\frac{\rho_{13}}{1-\rho_{12}^2}(1 - \frac{\rho_{13}^2}{1-\rho_{12}^2})\sqrt{\epsilon} + o(\sqrt{\epsilon} + \sqrt{\delta}), \\
\gamma_2 &= \sigma_y\sigma_z\frac{\rho_{12}\rho_{13}}{\sqrt{1-\rho_{12}^2}}\sqrt{\epsilon}\sqrt{\delta} + o(\epsilon + \delta), \\
\gamma_3 &= \sigma_z\sigma_s(\sqrt{1-\rho_{12}^2} - \rho_{24})\sqrt{\delta} + o(\sqrt{\epsilon} + \sqrt{\delta}).
\end{aligned}$$

4.3 Two-Asset Portfolio Problem with Partial Information

With the filtered dynamic system (4.1.6), we can set up a two-asset portfolio problem with S^f and S^s with investment period $[0,T]$. Again, the interest rate is set to be zero. Hence, the wealth process follows

$$dX_t^\pi = \pi_t^f dS_t^f + \pi_t^s dS_t^s = \pi_t^\top d\mathbf{S}_t, \tag{4.3.1}$$

where $\pi_t = \begin{pmatrix} \pi_t^f \\ \pi_t^s \end{pmatrix}$ represents the optimal portfolio allocation on $\mathbf{S}_t$.

Let $U : \mathbb{R} \to \mathbb{R}$ be the utility function that satisfies Assumption B.3 in Appendix B. Let $\mathscr{A}$ be the set of admissible strategies for π_t. Define the value function by

$$u(t,x,y,z) = \sup_{\pi\in\mathscr{A}} \mathbb{E}\left(U(X_T^\pi)\,|\,X_t^\pi = x, \hat{\mathbf{Y}}_t = \overrightarrow{\mathbf{y}} := \begin{pmatrix} y \\ z \end{pmatrix}\right). \tag{4.3.2}$$

The corresponding HJB equation for the stochastic control problem is

$$u_t^{\epsilon,\delta} + \hat{\mathscr{L}}_{t,\overrightarrow{\mathbf{y}}} u^{\epsilon,\delta} + \hat{\mathrm{NL}}^{\epsilon,\delta} = 0, \tag{4.3.3}$$

with the terminal condition $u^{\epsilon,\delta}(T,x,y,z) = U(x)$, where the infinitesimal generator $\hat{\mathscr{L}}_{t,\overrightarrow{\mathbf{y}}}$ is given by

$$\hat{\mathscr{L}}_{t,\overrightarrow{\mathbf{y}}} = -\frac{1}{\epsilon} y \partial_y - \delta z \partial_z + \frac{1}{2} tr \left(M^{\epsilon,\delta}(t) M^{\epsilon,\delta}(t)^\top \nabla^2_{\overrightarrow{\mathbf{y}}} \right), \tag{4.3.4}$$

and the nonlinear term is given by

$$\hat{\mathrm{NL}}^{\epsilon,\delta} = \sup_{\pi \in \mathscr{A}} \left[\pi^\top (\mu_{\mathbf{S}} + \overrightarrow{\mathbf{y}}) u_x^{\epsilon,\delta} + \frac{1}{2} \pi^\top \tilde{\Sigma}_{11} \tilde{\Sigma}_{11}^\top \pi u_{xx}^{\epsilon,\delta} + \pi^\top (\tilde{\Sigma}_{11} \tilde{\Sigma}_{12}^\top + \Gamma^{\epsilon,\delta}(t)) \nabla_{\overrightarrow{\mathbf{y}}} u_x^{\epsilon,\delta} \right].$$

Define the following matrices

$$\Sigma_1 := \tilde{\Sigma}_{11} \tilde{\Sigma}_{11}^\top = \begin{pmatrix} \sigma_f^2, & \rho_{12}\sigma_f\sigma_s \\ \rho_{12}\sigma_f\sigma_s, & \sigma_s^2 \end{pmatrix},$$

and

$$\Sigma_{12} := \tilde{\Sigma}_{11} \tilde{\Sigma}_{12}^\top = \begin{pmatrix} \frac{1}{\sqrt{\epsilon}} \rho_{13}\sigma_f\sigma_y, & 0 \\ 0, & \sqrt{\delta}\rho_{24}\sigma_s\sigma_z \end{pmatrix}.$$

Then we have the optimal control for the associated problem in the feedback form:

$$\pi_t^* = -\Sigma_1^{-1} \frac{(\mu_{\mathbf{S}} + \overrightarrow{\mathbf{y}}) u_x^{\epsilon,\delta} + (\Sigma_{12} + \Gamma^{\epsilon,\delta}(t)) \nabla_{\overrightarrow{\mathbf{y}}} u_x^{\epsilon,\delta}}{u_{xx}^{\epsilon,\delta}}. \tag{4.3.5}$$

where Σ_1^{-1} can be expressed explicitly as

$$\Sigma_1^{-1} = \begin{pmatrix} \frac{1}{\sigma_f^2(1-\rho_{12}^2)}, & -\frac{\rho_{12}}{\sigma_f\sigma_s(1-\rho_{12}^2)} \\ -\frac{\rho_{12}}{\sigma_f\sigma_s(1-\rho_{12}^2)}, & \frac{1}{\sigma_s^2(1-\rho_{12}^2)} \end{pmatrix}.$$

Plugging the optimal control (4.3.5) into the nonlinear term of the HJB equation (4.3.3), we have the HJB equation

$$u_t^{\epsilon,\delta} + \hat{\mathscr{L}}_{t,\overrightarrow{\mathbf{y}}} u^{\epsilon,\delta} + \hat{\mathrm{NL}}_{op}^{\epsilon,\delta} = 0, \tag{4.3.6}$$

where the new nonlinear term $\hat{\mathrm{NL}}_{op}^{\epsilon,\delta}$ is given by

$$\hat{\mathrm{NL}}_{op}^{\epsilon,\delta} = -\frac{1}{2u_{xx}^{\epsilon,\delta}}\left((\mu_{\mathbf{S}} + \overrightarrow{\mathbf{y}})u_x^{\epsilon,\delta} + (\Sigma_{12} + \Gamma^{\epsilon,\delta}(t))\nabla_{\overrightarrow{\mathbf{y}}} u_x^{\epsilon,\delta}\right)^{\top} \Sigma_1^{-1}\left((\mu_{\mathbf{S}} + \overrightarrow{\mathbf{y}})u_x^{\epsilon,\delta} + (\Sigma_{12} + \Gamma^{\epsilon,\delta}(t))\nabla_{\overrightarrow{\mathbf{y}}} u_x^{\epsilon,\delta}\right).$$

The terminal condition remains $u^{\epsilon,\delta}(T,x,y,z) = U(x)$.

Now we import the steady state assumption in order to apply the filter asymptote from Section 4.2. Replacing the left-hand side of the matrix Riccati equation (4.1.7) by zero, we have

$$0 = \begin{pmatrix} -\frac{2}{\epsilon}\gamma_1 + \frac{\sigma_y^2}{\epsilon}, & -(\frac{1}{\epsilon}+\delta)\gamma_2 \\ -(\frac{1}{\epsilon}+\delta)\gamma_2, & -2\delta\gamma_3 + \delta\sigma_z^2 \end{pmatrix} - M^{\epsilon,\delta}(0)M^{\epsilon,\delta}(0)^{\top}. \tag{4.3.7}$$

We can use the above equation (4.3.7) to simplify the HJB equation (4.3.6):

$$
\begin{aligned}
0 = u_t - \frac{y}{\epsilon}u_y - \delta z u_z + \frac{1}{\epsilon}\Big(\frac{\sigma_y^2}{2} - \gamma_1\Big)u_{yy} - (\frac{1}{\epsilon} + \delta)\gamma_2 u_{yz} + \delta\Big(\frac{\sigma_z^2}{2} - \gamma_3\Big)u_{zz} \\
-\frac{1}{2u_{xx}^{\epsilon,\delta}}\left(\frac{P_1^1}{\sigma_f^2(1-\rho_{12}^2)} - \frac{2\rho_{12}P_1P_2}{\sigma_f\sigma_s(1-\rho_{12}^2)} + \frac{P_2^2}{\sigma_s^2(1-\rho_{12}^2)}\right),
\end{aligned}
\tag{4.3.8}
$$

where P_1, P_2 denote

$$
\begin{aligned}
P_1 &:= (\mu_f + y)u_x^{\epsilon,\delta} + \big((\frac{1}{\sqrt{\epsilon}}\rho_{13}\sigma_f\sigma_y + \gamma_1)u_{xy}^{\epsilon,\delta} + \gamma_2 u_{xz}^{\epsilon,\delta}\big), \\
P_2 &:= (\mu_s + z)u_x^{\epsilon,\delta} + \big((\sqrt{\delta}\rho_{24}\sigma_s\sigma_z + \gamma_3)u_{xz}^{\epsilon,\delta} + \gamma_2 u_{xy}^{\epsilon,\delta}\big)\,.
\end{aligned}
$$

By Proposition 4.2.A and Proposition 4.2.B, we have the following approximations, which will be used in the next section to derive the asymptotic expansion of the value function $u^{\epsilon,\delta}$:

$$
\begin{aligned}
&\frac{1}{\epsilon}\Big(\frac{\sigma_y^2}{2} - \gamma_1\Big) = \Big(\frac{1}{\epsilon}\frac{\rho_{13}^2\sigma_y^2}{2(1-\rho_{12}^2)} + \frac{1}{\sqrt{\epsilon}}\frac{\sigma_y^3\rho_{13}}{2\sigma_f(1-\rho_{12}^2)}(1 - \frac{\rho_{13}^2}{1-\rho_{12}^2})\Big)(1 + o(\sqrt{\epsilon} + \sqrt{\delta})) \\
&-\Big(\frac{1}{\epsilon} + \delta\Big)\gamma_2 = -\frac{\sqrt{\delta}}{\sqrt{\epsilon}}\frac{\rho_{12}\rho_{13}\sigma_y\sigma_z}{\sqrt{1-\rho_{12}^2}}(1 + o(\epsilon + \delta)), \\
&\delta\Big(\frac{\sigma_z^2}{2} - \gamma_3\Big) = \delta\frac{\sigma_z^2}{2} + o(\epsilon + \delta)\,.
\end{aligned}
\tag{4.3.9}
$$

4.4 Asymptotic Expansion

Assumption 4.4. The value function $u^{\epsilon,\delta}(t,x,y,z)$ is the unique smooth solution of the HJB equation (4.3.8) that is strictly increasing and strictly concave in x for each $t \in [0,T)$ and $(y,z) \in \mathbb{R} \times \mathbb{R}$.

Assumption 4.4 guarantees an asymptotic expansion of $u^{\epsilon,\delta}$ with respect to $(\sqrt{\epsilon},\sqrt{\delta})$:

$$u^{\epsilon,\delta} = u^{(0)} + \sqrt{\epsilon}u^{(1,0)} + \sqrt{\delta}u^{(0,1)} + \epsilon u^{(2,0)} + \delta u^{(0,2)} + \sqrt{\epsilon\delta}u^{(1,1)} + \cdots \qquad (4.4.1)$$

We follow the same procedure and notations as in Chapter 3 Section 3.3 to derive the asymptotic expansion of the value function in the partial information case. By plugging the expansion (4.4.1) into the HJB equation (4.3.8), we use the approximations (4.3.9) and collect the terms in powers of $\sqrt{\epsilon}$ and $\sqrt{\delta}$. Here we list out the key steps and equations, and refer to Chapter 3 Section 3.3. for some of the detailed arguments.

The $u^{(0)}$ approximation: By setting $\delta = 0$, we obtain that

$$u^{\epsilon,0} = u^{(0)} + \sqrt{\epsilon}u^{(1,0)} + \epsilon u^{(2,0)} \cdots \qquad (4.4.2)$$

satisfies the HJB equation:

$$\begin{aligned} 0 = u_t^{\epsilon,0} &- \frac{y}{\epsilon}u_y^{\epsilon,0} + \frac{1}{\epsilon}(\frac{\sigma_y^2}{2} - \gamma_1)u_{yy}^{\epsilon,0} - \frac{1}{2u_{xx}^{\epsilon,0}}\Big(\lambda^2(y,z)(u_x^{\epsilon,0})^2 \\ &+ \frac{2}{\sqrt{\epsilon}}\frac{\rho_{13}\sigma_y + \sqrt{\epsilon}\frac{\gamma_1}{\sigma_f}}{1-\rho_{12}^2}\Big(\frac{(\mu_f + y)}{\sigma_f} - \frac{\rho_{12}(\mu_s + z)}{\sigma_s}\Big)u_x^{\epsilon,0}u_{xy}^{\epsilon,0} \\ &+ \big(\frac{1}{\sqrt{\epsilon}}\rho_{13}\sigma_f\sigma_y + \gamma_1\big)^2(u_{xy}^{\epsilon,0})^2\Big), \end{aligned} \qquad (4.4.3)$$

with terminal condition $u^{\epsilon,0}(T,x,y,z) = U(x)$, where $\lambda(y,z)$ in this chapter is defined by

$$\lambda^2(y,z) = \frac{\sigma_s^2(\mu_f+y)^2 - 2\rho_{12}\sigma_f\sigma_s(\mu_f+y)(\mu_s+z) + \sigma_f^2(\mu_s+z)^2}{\sigma_f^2\sigma_s^2(1-\rho_{12}^2)}. \tag{4.4.4}$$

Now we insert the value function expansion (4.4.2), the approximation (4.3.9), and the filter asymptote (4.2.7) into the HJB equation (4.4.3), and collect the terms in powers of $\sqrt{\epsilon}$. The term with order ϵ^{-1} is given by

$$\mathscr{M}_Y u^{(0)} - \frac{1}{2}\frac{\rho_{13}^2\sigma_y^2}{(1-\rho_{12}^2)}\frac{(u_{xy}^{(0)})^2}{u_{xx}^{(0)}} = 0,\ u^{(0)}(T,x,y,z) = U(x),$$

where the differential operator $\mathscr{M}_Y$ in this chapter is defined as

$$\mathscr{M}_Y = -y\partial_y + \frac{\rho_{13}^2\sigma_y^2}{2(1-\rho_{12}^2)}\partial_{yy}. \tag{4.4.5}$$

This gives that $u_y^{(0)} = 0$. Then the term with order $\epsilon^{-\frac{1}{2}}$ is given by

$$\mathscr{M}_Y u^{(1,0)} = 0,\ u^{(1,0)}(T,x,y,z) = 0.$$

This gives that $u_y^{(1,0)} = 0$. With $u_y^{(0)} = u_y^{(1,0)} = 0$, we collect the constant order terms:

$$\mathscr{M}_Y u^{(2,0)} + u_t^{(0)} - \frac{1}{2}\lambda^2(y,z)\frac{(u_x^{(0)})^2}{u_{xx}^{(0)}} = 0. \tag{4.4.6}$$

Thus, the same solvability condition argument as in the previous chapters gives that

$$u^{(0)}(t,x,z) = M(t,x;\bar{\lambda}(z)), \tag{4.4.7}$$

where M is the classical Merton value function with terminal condition $M(T, x, z) = U(x)$ associated to the square-averaged Sharpe ratio $\overline{\lambda}(z)$ defined by

$$\overline{\lambda}^2(z) := \left\langle \lambda^2(\,\cdot\,, z) \right\rangle.$$

Recall that the notation $\langle \cdot \rangle$ in this chapter is defined in (4.1.8).

The $u^{(2,0)}$ approximation: Again, we define the risk-tolerance function and associated differential operators as:

$$R(t, x; \overline{\lambda}(z)) := -\frac{M_x(t, x; \overline{\lambda}(z))}{M_{xx}(t, x; \overline{\lambda}(z))},$$

$$D_k := R(t, x; \overline{\lambda}(z))^k \frac{\partial^k}{\partial x^k}, k = 1,2,\cdots.$$

A byproduct from the Poisson equation (4.4.6) is the $u^{(2,0)}$ approximation:

$$u^{(2,0)} = -\frac{1}{2}\psi(y, z) D_1 u^{(0)} + C(t, x, z) \tag{4.4.8}$$

where $\psi(y, z)$ solves the ordinary differential equation in y variable:

$$\mathcal{M}_Y \psi = \lambda^2(y, z) - \overline{\lambda}^2(z), \tag{4.4.9}$$

and $C(t, x, z)$ is a constant of integration in y that may depend on (t, x, z). Note that ψ is a polynomial in y and z.

The $u^{(1,0)}$ approximation: With the Poisson equation (4.4.6) and $u_y^{(0)} = u_y^{(1,0)} = 0$, we collect the term with order $\sqrt{\epsilon}$ in the HJB equation (4.4.3):

$$
\begin{aligned}
\mathscr{M}_Y u^{(3,0)} + u_t^{(1,0)} + \frac{1}{2}\lambda^2 D_2 u^{(1,0)} + \lambda^2 D_1 u^{(1,0)} & \\
+ \frac{\sigma_y^3}{2\sigma_f}\frac{\rho_{13}}{(1-\rho_{12}^2)}(1 - \frac{\rho_{13}^2}{1-\rho_{12}^2}) u_{yy}^{(2,0)} + a(y,z) D_1 u_y^{(2,0)} = 0, &
\end{aligned}
\tag{4.4.10}
$$

where $a(y,z)$ in this chapter is defined by

$$
a(y,z) = \frac{\rho_{13}\sigma_y}{1-\rho_{12}^2}\Big(\frac{(\mu_f + y)}{\sigma_f} - \frac{\rho_{12}(\mu_s + z)}{\sigma_s}\Big).
$$

Plugging the formula (4.4.8) for $u^{(2,0)}$ into the solvability condition for the above Poisson equation with respect to $u^{(3,0)}$, we have the Fredholm Alternative:

$$
\begin{aligned}
\mathscr{L}_{t,x}(\bar{\lambda}(z)) u^{(1,0)} = \frac{1}{2}\left\langle a(\,\cdot\,,z)\psi_y(\,\cdot\,,z)\right\rangle D_1^2 u^{(0)} & \\
+ \frac{1}{2}\frac{\sigma_y^3}{2\sigma_f}\frac{\rho_{13}}{(1-\rho_{12}^2)}(1 - \frac{\rho_{13}^2}{1-\rho_{12}^2})\left\langle \psi_{yy}(\,\cdot\,,z)\right\rangle D_1 u^{(0)}, &
\end{aligned}
\tag{4.4.11}
$$

where the differential operator $\mathscr{L}_{t,x}$ is defined by

$$
\mathscr{L}_{t,x}(\bar{\lambda}(z)) := \frac{\partial}{\partial_t} + \frac{1}{2}\bar{\lambda}^2(z) D_2 + \bar{\lambda}^2(z) D_1. \tag{4.4.12}
$$

By the argument in the previous chapters, the PDE problem (4.4.11) with terminal condition $u^{(1,0)}(T,x,z) = 0$ has the unique solution:

$$
u^{(1,0)}(t,x,z) = -\frac{1}{2}(T-t)\Big(A(z)D_1^2 + \hat{A}(z)D_1\Big)\, u^{(0)}(t,x,z), \tag{4.4.13}
$$

where $A(z)$ and $\hat{A}(z)$ are defined by

$$A(z) = \Big\langle a(\,\cdot\,,z)\psi_y(\,\cdot\,,z)\Big\rangle,$$

$$\hat{A}(z) = \frac{\sigma_y^3}{2\sigma_f}\frac{\rho_{13}}{(1-\rho_{12}^2)}(1-\frac{\rho_{13}^2}{1-\rho_{12}^2})\Big\langle \psi_{yy}(\,\cdot\,,z)\Big\rangle.$$

The $u^{(0,1)}$ approximation: To derive $u^{(0,1)}$, we construct an expansion in power of $\sqrt{\delta}$:

$$u^{\epsilon,\delta} = u^{\epsilon,0} + \sqrt{\delta}u^{\epsilon,1} + \cdots, \tag{4.4.14}$$

where $u^{\epsilon,0}$ has the expansion (4.4.2) and $u^{\epsilon,1}$ has an expansion

$$u^{\epsilon,1} = u^{(0,1)} + \sqrt{\epsilon}u^{(1,1)} + \epsilon u^{(2,1)} + \cdots.$$

Inserting the expansion (4.4.14) into the HJB equation (4.3.8), we are able to collect the order $\sqrt{\delta}$ term as an equation of $u^{\epsilon,1}$. Then, again, we collect the term in successive power of $\sqrt{\epsilon}$. Here we omit the tedious computations, and present the collected equations. The term with order $\sqrt{\delta}\epsilon^{-1}$ is given by

$$\mathscr{M}_Y u^{(0,1)} = 0.$$

Hence, we have $u_y^{(0,1)} = 0$. Then the term with order $\sqrt{\delta}\epsilon^{-\frac{1}{2}}$ is

$$\mathscr{M}_Y u^{(1,1)} = 0.$$

Hence, we have $u_y^{(1,1)} = 0$. With $u_y^{(0,1)} = u_y^{(1,1)} = 0$, we collect the term with order $\sqrt{\delta}$:

$$\mathscr{M}_Y u^{(2,1)} + u_t^{(0,1)} + \frac{1}{2}\lambda^2 D_2 u^{(0,1)} + \lambda^2 D_1 u^{(0,1)} - b(y,z)\frac{u_x^{(0)}u_{xz}^{(0)}}{u_{xx}^{(0)}} = 0. \tag{4.4.15}$$

where $b(y,z)$ in this chapter is defined by

$$b(y,z) = \frac{\rho_{24}\sigma_z}{1-\rho_{12}^2}\Big(\frac{(\mu_s+z)}{\sigma_s} - \frac{\rho_{12}(\mu_f+y)}{\sigma_f}\Big).$$

With terminal condition $u^{(0,1)}(T,x,z)=0$, the same argument in the previous chapters gives the unique solution for $u^{(0,1)}$:

$$u^{(0,1)}(t,x,z) = \frac{1}{2}(T-t)B(z)D_1 u_z^{(0)}(t,x,z), \tag{4.4.16}$$

where $B(z)$ is defined by:

$$B(z) = \langle b(\,\cdot\,,z)\rangle.$$

Summary: Under Assumption 4.4 on the smoothness of the value function $u^{\epsilon,\delta}$, we have derived the asymptotic expansion of $u^{\epsilon,\delta}$ up to the first order:

$$\begin{aligned}
u^{\epsilon,\delta}(t,x,y,z) &= u^{(0)} + \sqrt{\epsilon}u^{(1,0)} + \sqrt{\delta}u^{(0,1)} + \cdots, \\
u^{(0)}(t,x,z) &= M(t,x;\overline{\lambda}(z)), \\
u^{(1,0)}(t,x,z) &= = -\frac{1}{2}(T-t)\Big(A(z)D_1^2 + \hat{A}(z)D_1\Big)\,u^{(0)}(t,x,z), \\
u^{(0,1)}(t,x,z) &= \frac{1}{2}(T-t)B(z)D_1 u_z^{(0)}(t,x,z)\,.
\end{aligned} \tag{4.4.17}$$

4.5 Zeroth Order Strategy Performance

In this section, we derive the zeroth order optimal strategy for the portfolio problem in the partial information case. We also demonstrate that, if there exists an asymptotic expansion for the value function $\tilde{u}^{\epsilon,\delta}$ associated to the zeroth order optimal strategy, then $\tilde{u}^{\epsilon,\delta}$ recovers $u^{\epsilon,\delta}$ up to the first order expansions. Recall that the optimal feedback strategy (4.3.5):

$$\pi_t^* = -\Sigma_1^{-1}\frac{(\mu_{\mathbf{S}} + \overrightarrow{\mathbf{y}})u_x^{\epsilon,\delta} + (\Sigma_{12} + \Gamma^{\epsilon,\delta}(t))\nabla_{\overrightarrow{\mathbf{y}}} u_x^{\epsilon,\delta}}{u_{xx}^{\epsilon,\delta}}.$$

where Σ_1, Σ_{12} are defined in Section 4.3, and $\overrightarrow{\mathbf{y}} = \begin{pmatrix} y \\ z \end{pmatrix}$ is the vector that represents the states of $\hat{\mathbf{Y}}_t$. We continue with the steady state assumption that $\Gamma^{\epsilon,\delta}(t) = \Gamma^{\epsilon,\delta}(0)$ for all $t > 0$.

First, we expand the optimal strategy π^* in power of $(\sqrt{\epsilon}, \sqrt{\delta})$:

$$\pi^* = \pi^{(0)} + \sqrt{\epsilon}\pi^{(1,0)} + \sqrt{\delta}\pi^{(0,1)} + \cdots. \tag{4.5.1}$$

Inserting the expansion (4.5.1) of π^* and the expansion (4.4.1) of $u^{\epsilon,\delta}$ into the feedback form (4.3.5), we collect the constant power terms to obtain the zeroth order optimal strategy:

$$\pi^{(0)} = -\Sigma_1^{-1}(\mu_{\mathbf{S}} + \overrightarrow{\mathbf{y}})\frac{u_x^{(0)}(t,x,z)}{u_{xx}^{(0)}(t,x,z)} = \Sigma_1^{-1}(\mu_{\mathbf{S}} + \overrightarrow{\mathbf{y}})R(t,x;\bar{\lambda}(z)), \tag{4.5.2}$$

where $\bar{\lambda}(z)$ and $R(t,x;\bar{\lambda}(z))$ are defined in Section 4.4. Note that the zeroth order optimal strategy in the partial information case is the same as the zeroth order optimal strategy in the full information case in Chapter 3 Section 3.4.

Now we define the corresponding value function $\tilde{u}^{\epsilon,\delta}$ associated to $\pi^{(0)}$ by

$$\tilde{u}^{\epsilon,\delta}(t,x,y,z) = \mathbb{E}\left(U(X_T^{\pi^{(0)}})\,|\,X_t^{\pi^{(0)}} = x, \hat{Y}_t = y, \hat{Z}_t = z\right).$$

where $X_t^{\pi^{(0)}}$ is given by (4.3.1) with $\pi_t = \pi^{(0)}$. By the martingale property of $\tilde{u}^{\epsilon,\delta}(t, X_t, \hat{Y}_t, \hat{Z}_t)$ and Itô's formula, $\tilde{u}^{\epsilon,\delta}$ solves the HJB equation:

$$\tilde{u}_t^{\epsilon,\delta} + \hat{\mathscr{L}}_{t,\overrightarrow{\mathbf{y}}}\tilde{u}^{\epsilon,\delta} + \hat{\mathrm{N}}\mathrm{L}_{op}^{(0)} = 0,$$

with the terminal condition $\tilde{u}^{\epsilon,\delta}(t,x,y,z) = U(x)$, where $\hat{\mathscr{L}}_{t,\overrightarrow{\mathbf{y}}}$ is defined in Section 4.4, and $\hat{\mathrm{N}}\mathrm{L}_{op}^{(0)}$ is defined by

$$\hat{\mathrm{N}}\mathrm{L}_{op}^{(0)} = (\pi^{(0)})^\top(\mu_{\mathbf{S}} + \overrightarrow{\mathbf{y}})\tilde{u}_x^{\epsilon,\delta} + \frac{1}{2}(\pi^{(0)})^\top\Sigma_1\pi^{(0)}\tilde{u}_{xx}^{\epsilon,\delta} + (\pi^{(0)})^\top(\Sigma_{12} + \Gamma^{\epsilon,\delta}(0))\,\nabla_{\overrightarrow{\mathbf{y}}}\,\tilde{u}_x^{\epsilon,\delta}$$

Plugging $\Gamma^{\epsilon,\delta}(0)$ into the above HJB equation, we can re-write it as

$$\begin{aligned} 0 = \tilde{u}_t^{\epsilon,\delta} - \frac{y}{\epsilon}\tilde{u}_y^{\epsilon,\delta} - \delta z\tilde{u}_z^{\epsilon,\delta} + \frac{1}{\epsilon}\Big(\frac{\sigma_y^2}{2} - \gamma_1\Big)\tilde{u}_{yy}^{\epsilon,\delta} - (\frac{1}{\epsilon} + \delta)\gamma_2\tilde{u}_{yz}^{\epsilon,\delta} + \delta\Big(\frac{\sigma_z^2}{2} - \gamma_3\Big)\tilde{u}_{zz}^{\epsilon,\delta} \\ +(\pi^{(0)})^\top(\mu_{\mathbf{S}} + \overrightarrow{\mathbf{y}})\tilde{u}_x^{\epsilon,\delta} + \frac{1}{2}(\pi^{(0)})^\top\Sigma_1\pi^{(0)}\tilde{u}_{xx}^{\epsilon,\delta} + (\pi^{(0)})^\top(\Sigma_{12} + \Gamma^{\epsilon,\delta}(0))\,\nabla_{\overrightarrow{\mathbf{y}}}\,\tilde{u}_x^{\epsilon,\delta}, \end{aligned} \tag{4.5.3}$$

Assumption 4.5. The value function $\tilde{u}^{\epsilon,\delta}(t,x,y,z)$ is the unique smooth solution of the HJB equation (4.5.3) that is strictly increasing and strictly concave in x for each $t \in [0,T)$ and $(y,z) \in \mathbb{R}\times\mathbb{R}$.

Theorem III (Vectorized $\pi^{(0)}$ performance with regularity assumption). Let $\tilde{u}^{\epsilon,\delta}$ be the value function associated to the zeroth order optimal strategy:

$$\pi^{(0)} = \Sigma_1^{-1}(\mu_{\mathbf{S}} + \overrightarrow{\mathbf{y}})R(t,x;\overline{\lambda}(z)).$$

Under Assumption 4.5, $\tilde{u}^{\epsilon,\delta}$ has an asymptotic expansion

$$\tilde{u}^{\epsilon,\delta} = \tilde{u}^{(0)} + \sqrt{\epsilon}\tilde{u}^{(1,0)} + \sqrt{\delta}\tilde{u}^{(0,1)} + \epsilon\tilde{u}^{(2,0)} + \delta\tilde{u}^{(0,2)} + \sqrt{\epsilon\delta}\tilde{u}^{(1,1)} + o(\epsilon+\delta), \tag{4.5.4}$$

where the zeroth order and the first order approximations equal $u^{(0)}$, $u^{(1,0)}$, and $u^{(0,1)}$ given by (4.4.17). That is, $\tilde{u}^{(0)} = u^{(0)}$, $\tilde{u}^{(1,0)} = u^{(1,0)}$, and $\tilde{u}^{(0,1)} = u^{(0,1)}$.

Proof. Assumption 4.5 guarantees an asymptotic expansion of $\tilde{u}^{\epsilon,\delta}$ in powers of $(\sqrt{\epsilon}, \sqrt{\delta})$:

$$\tilde{u}^{\epsilon,\delta} = \tilde{u}^{(0)} + \sqrt{\epsilon}\tilde{u}^{(1,0)} + \sqrt{\delta}\tilde{u}^{(0,1)} + \epsilon\tilde{u}^{(2,0)} + \delta\tilde{u}^{(0,2)} + \sqrt{\epsilon\delta}\tilde{u}^{(1,1)} + o(\epsilon+\delta).$$

Then we repeat the procedure in Section 4.4. Inserting the expansion (4.5.4) and the filter asymptote for $\Gamma^{\epsilon,\delta}(0)$ into the HJB equation (4.5.3), we collect the terms in powers of $(\sqrt{\epsilon}, \sqrt{\delta})$.

The term with order ϵ^{-1}

$$\mathscr{M}_Y \tilde{u}^{(0)} = 0$$

where $\mathscr{M}_Y$ is defined in (4.4.5), gives that $\tilde{u}^{(0)}_y = 0$. Then the term with order $\epsilon^{-\frac{1}{2}}$

$$\mathscr{M}_Y \tilde{u}^{(1,0)} = 0$$

gives that $\tilde{u}^{(1,0)}_y = 0$. Using $\tilde{u}^{(0)}_y = \tilde{u}^{(1,0)}_y = 0$, we collect the term with constant power:

$$\mathscr{M}_Y \tilde{u}^{(2,0)} + \tilde{u}^{(0)}_t + (\pi^{(0)})^\top(\mu_{\mathbf{S}} + \overrightarrow{\mathbf{y}})\tilde{u}^{(0)}_v + \frac{1}{2}(\pi^{(0)})^\top \Sigma_1 \pi^{(0)} \tilde{u}^{(0)}_{xx} = 0. \tag{4.5.5}$$

Recall that $\pi^{(0)} = \Sigma_1^{-1}(\mu_{\mathbf{S}} + \overrightarrow{\mathbf{y}})R(t,x;\bar{\lambda}(z))$. A straightforward computation gives

$$(\mu_{\mathbf{S}} + \overrightarrow{\mathbf{y}})^\top \Sigma_1^{-1}(\mu_{\mathbf{S}} + \overrightarrow{\mathbf{y}}) = \lambda^2(y,z).$$

Hence, the solvability condition of the Poisson equation (4.5.5) is given by

$$\mathscr{L}_{t,x}(\bar{\lambda}(z))\tilde{u}^{(0)} = 0,$$

where we recall that $\mathscr{L}_{t,x}(\bar{\lambda}(z)) := \frac{\partial}{\partial_t} + \frac{1}{2}\bar{\lambda}^2(z)D_2 + \bar{\lambda}^2(z)D_1$. With the terminal condition

$$\tilde{u}^{(0)}(T,x,z) = U(x) = u^{(0)}(T,x,z), \tag{4.5.6}$$

we deduce that $\tilde{u}^{(0)} = u^{(0)}$ by uniqueness of the Merton equation. Solving the Poisson equation (4.5.5), we also obtain that $\tilde{u}^{(2,0)}$ differs from $u^{(2,0)}$ up to a constant:

$$\tilde{u}^{(2,0)} = -\frac{1}{2}\psi(y,z)D_1u^{(0)} + \tilde{C}(t,x,z), \tag{4.5.7}$$

where $\tilde{C}(t,x,z)$ is a constant of integration in y, and $\psi(y,z)$ is defined in (4.4.9). Plugging $\tilde{u}^{(0)} = u^{(0)}$ into the HJB equation, we collect the term with order $\sqrt{\epsilon}$:

$$\begin{aligned}\mathscr{M}_Y\tilde{u}^{(3,0)}&+\Big(\frac{\partial}{\partial_t} + \frac{1}{2}\lambda^2D_2 + \lambda^2D_1\Big)\tilde{u}^{(1,0)}\\ &+\frac{\sigma_y^3}{2\sigma_f}\frac{\rho_{13}}{1-\rho_{12}^2}(1-\frac{\rho_{13}^2}{1-\rho_{12}^2})\tilde{u}^{(2,0)}_{yy} + \begin{pmatrix}0, & \sigma_f\sigma_y\rho_{13}\end{pmatrix}\pi^{(0)}\tilde{u}^{(2,0)}_{xy} = 0.\end{aligned} \tag{4.5.8}$$

Another straightforward computation deduces

$$\begin{pmatrix}0, & \sigma_f\sigma_y\rho_{13}\end{pmatrix}\pi^{(0)}\tilde{u}^{(2,0)}_{xy} = a(y,z)D_1\tilde{u}^{(2,0)}_y.$$

Thus, the formula (4.5.7) for $\tilde{u}^{(2,0)}$ implies that the solvability condition for Poisson equation (4.5.8) is exactly equation (4.4.11). With the same zero terminal condition, the uniqueness of the solution gives $\tilde{u}^{(1,0)} = u^{(1,0)}$.

Now we derive $\tilde{u}^{(0,1)}$ by collecting the terms involving $\sqrt{\delta}$. The term with order $\sqrt{\delta}\epsilon^{-1}$

$$\mathscr{M}_Y\tilde{u}^{(0,1)} = 0$$

gives $\tilde{u}_y^{(0,1)} = 0$. Then the term with order $\sqrt{\delta}\epsilon^{-\frac{1}{2}}$

$$\mathcal{M}_Y \tilde{u}^{(1,1)} = 0$$

gives $\tilde{u}_y^{(1,1)} = 0$. Using $\tilde{u}_y^{(0,1)} = \tilde{u}_y^{(1,1)} = 0$, we collect the term with order $\sqrt{\delta}$:

$$\mathcal{M}_Y \tilde{u}^{(2,1)} + \left(\frac{\partial}{\partial_t} + \frac{1}{2}\lambda^2 D_2 + \lambda^2 D_1\right)\tilde{u}^{(0,1)} + \left(\sigma_s \sigma_z \sqrt{1-\rho_{12}^2},\ 0\right)\pi^{(0)}\tilde{u}_{xz}^{(0)} = 0. \quad (4.5.9)$$

Again, a straightforward computation shows

$$\left(\sigma_s \sigma_z \sqrt{1-\rho_{12}^2},\ 0\right)\pi^{(0)}\tilde{u}_{xz}^{(0)} = b(y,z)R(t,x;\bar{\lambda}(z))\tilde{u}_{xz}^{(0)}.$$

Thus, the Poisson equation (4.5.9) is exactly equation (4.4.15). With the same zero terminal condition, the uniqueness property in the argument from the previous chapters gives $\tilde{u}^{(0,1)} = u^{(0,1)}$.

Therefore, we can conclude the following: under Assumption 4.4 and Assumption 4.5, which ensures asymptotic expansions for both value function $u^{\epsilon,\delta}$ and $\tilde{u}^{\epsilon,\delta}$, the zeroth order optimal strategy $\pi^{(0)}$ given by (4.5.2) recovers the zeroth order approximation $u^{(0)}$ and the first order approximations $u^{(1,0)}$ and $u^{(0,1)}$ in the asymptotic expansion of $u^{\epsilon,\delta}$. ∎

4.6 First Order Strategy and Practical Use

Now we expand the optimal strategy (4.3.5) up to orders $\sqrt{\epsilon}$ and $\sqrt{\delta}$ by collecting the first order terms when we insert the expansion of $u^{\epsilon,\delta}$ into the optimal feedback strategy (4.3.5). Recall the expansion

$$\pi^* = \pi^{(0)} + \sqrt{\epsilon}\pi^{(1,0)} + \sqrt{\delta}\pi^{(0,1)} + \cdots,$$

where $\pi^{(0)}$ is derived in (4.5.2):

$$\pi^{(0)} = \Sigma_1^{-1} \begin{pmatrix} \mu_f + y \\ \mu_s + z \end{pmatrix} R(t, x; \bar{\lambda}(z)).$$

In the partial information case, we obtain the fast-scale first order strategy:

$$\pi^{(1,0)} = \Sigma_1^{-1} \frac{1}{u_x^{(0)}} \left(\begin{pmatrix} \mu_f + y \\ \mu_s + z \end{pmatrix} \frac{1}{2}(T-t)(D_1 + D_2)(AD_1 + \hat{A}\mathbf{1}) + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \frac{1}{2}\psi_y \rho_{13}\sigma_f \sigma_y D_1 \right) D_2 u^{(0)},$$

and the slow-scale first order strategy:

$$\pi^{(0,1)} = \Sigma_1^{-1} \frac{1}{u_x^{(0)}} \left(\begin{pmatrix} \mu_f + y \\ \mu_s + z \end{pmatrix} \frac{1}{2}(T-t)B(D_1 + D_2) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \sqrt{1-\rho_{12}^2}\sigma_s \sigma_z \mathbf{1} \right) D_1 u_z^{(0)}.$$

Here $u^{(0)}(t, x, z)$ is the Merton value function defined in (4.4.7); $A(z)$, $\hat{A}(z)$, $B(z)$, $\psi(y, z)$ are defined in Section 4.4.

For the two-asset portfolio problem with partial information, we observe that the zeroth order approximation $\pi^{(0)}$ has the same form of a Merton type strategy as in the full information case. Theorem III proves a similar result under a stronger regularity assumption that the zeroth order strategy $\pi^{(0)}$ for the two-asset problem with partial information recovers the optimal value function for the associated portfolio problem up to the first order approximations. Each entry of $\pi^{(0)}$ requires tracking both filtered volatility factors.

For the first order approximations, $\pi^{(1,0)}$ and $\pi^{(0,1)}$, in the partial information case, each entry also depends on both volatility factors, the classical Merton value function, and its derivatives with respect to the wealth state x and the slow-scale factor z. They can also be viewed as expressions in terms of the Merton strategy and the "Greeks" of the classical Merton value function. Compared to approximations for the full information case in Chapter 3, the filter for the unobservable processes affects $\pi^{(1,0)}$ and $\pi^{(0,1)}$ in two ways. First, because the first order expansion $u^{(1,0)}$ has an extra term that comes from the filter asymptote of $\mathbb{E}(Y_t - \hat{Y}_t)^2$ with order $\sqrt{\epsilon}$, the fast-scale first order strategy $\pi^{(1,0)}$ is changed accordingly. Second, the filter asymptote of $\mathbb{E}(Z_t - \hat{Z}_t)^2$ gives an extra term with order $\sqrt{\delta}$ that leads to a coefficient change in a term of the slow-asset portfolio $\pi_s^{(0,1)}$.

4.7 Conclusion

Given the history of asset prices, we are able to reproduce the asymptotic analysis for a multi-asset portfolio problem with partial information under the steady state assumption. The methodology in Chapter 3 can be applied to the two-asset portfolio problem after we used the Kalman-Bucy filtering theory to filter out the unobservable volatility factors. In the partial information setting, we have derived explicit formulas for the asymptotic expansions of the associated value function, and have computed the 2-dimensional zeroth and the first order optimal strategy. Under the regularity assumption, Theorem III proves a similar result to Theorem II that the zeroth order strategy in the partial information problem recovers the value function up to the first order. Compared to the full information case, the zeroth order optimal strategy remains the same, while we can observe the changes in the first order approximations of the optimal strategy brought by the filter asymptotes.

The coefficients of the asset prices and volatility factors can be generalized to functions depending on the volatility factors (see Chapter 3). The filtering theory can still be applied, while new methodology to compute the filter asymptote is required. In addition, as an alternative of steady state assumption, we may import other assumptions that can deduce a filter asymptote, for instance, the expert opinions in perturbation analysis [Fouque, Papanicolaou, Sircar, 2017].

APPENDIX A LEMMAS FOR MERTON PROBLEM

Recall the definitions (1.5.7), (1.5.8), and (1.5.9) from Section 1.5. Let $M(t, x; \lambda)$ denote the classical Merton value function. The associated risk-tolerance function $R(t, x; \lambda)$ is given by

$$R(t, x; \lambda) := -\frac{M_x(t, x; \lambda)}{M_{xx}(t, x; \lambda)}.$$

The differential operators D_k are defined by

$$D_k = R(t, x; \lambda)^k \frac{\partial^k}{\partial x^k}, k = 1,2,\cdots.$$

The linear operator $\mathscr{L}_{t,x}(\lambda)$ is defined by

$$\mathscr{L}_{t,x}(\lambda) = \frac{\partial}{\partial t} + \frac{1}{2}\lambda^2 R^2(t, x; \lambda)\frac{\partial^2}{\partial x^2} + \lambda^2 R(t, x; \lambda)\frac{\partial}{\partial x} = \frac{\partial}{\partial t} + \frac{1}{2}\lambda^2 D_2 + \lambda^2 D_1.$$

Lemma A.1 (Risk-tolerance equation). The risk-tolerance function $R(t, x; \lambda)$ satisfies the fast diffusion PDE:

$$R_t + \frac{1}{2}\lambda^2 R^2 R_{xx} = 0. \tag{A.1}$$

Proof. Note that R is smooth in (t, x). Differentiating the Merton equation $\mathscr{L}_{t,x}(\lambda)M = 0$ with respect to x yields

$$M_{tx} = \frac{1}{2}\lambda^2 R^2 M_{xxx} + \lambda^2 R R_x M_{xx}.$$

Because $RM_{xx} = -M_x$, we have $R^2 M_{xxx} = (R_x + 1)M_x$. Hence,

$$M_{tx} = \frac{1}{2}\lambda^2(R_x + 1)M_x - \lambda^2 R_x M_x = -\frac{1}{2}\lambda^2(R_x - 1)M_x. \tag{A.2}$$

Then we have

$$M_{txx} = -\frac{1}{2}\lambda^2(R_x - 1)M_{xx} - \frac{1}{2}\lambda^2 R_{xx} M_x. \tag{A.3}$$

Substituting (A.2) and (A.3) into the derivative of R with respect to t

$$R_t = -\frac{M_{tx}}{M_{xx}} + \frac{M_x}{(M_{xx})^2} M_{txx}$$

establishes (A.1). ∎

Lemma A.2 (Commutation result). The differential operator D_1 and the linear operator $\mathscr{L}_{t,x}(\lambda)$ commute when acting on smooth functions of (t, x):

$$\mathscr{L}_{t,x}(\lambda) D_1 = D_1 \mathscr{L}_{t,x}(\lambda). \tag{A.4}$$

Proof. Fix a smooth function $u(t, x)$. We compute

$$\begin{aligned} D_2 D_1 u - D_1 D_2 u &= R^2 \frac{\partial^2}{\partial x^2}(R u_x) - R \frac{\partial}{\partial x}(R^2 u_{xx}) \\ &= R^2 (R_{xx} u_x + 2 R_x u_{xx} + R u_{xxx}) - R(2 R R_x u_{xx} + R^2 u_{xxx}) \\ &= R^2 R_{xx} u_x \,. \end{aligned} \tag{A.5}$$

Using (A.5) and the risk-tolerance equation (A.1), we have

$$
\begin{aligned}
\mathscr{L}_{t,x}(\lambda)D_1u &= \left(\frac{\partial}{\partial t} + \frac{1}{2}\lambda^2 D_2 + \lambda^2 D_1\right) D_1 u \\
&= \frac{\partial}{\partial t}D_1 u + \frac{1}{2}\lambda^2 D_2 D_1 u + \lambda^2 D_1 D_1 u \\
&= (D_1\frac{\partial}{\partial t}u + R_t u_x) + \frac{1}{2}\lambda^2(D_1 D_2 u + R^2 R_{xx} u_x) + D_1 \lambda^2 u \\
&= D_1\left(\frac{\partial}{\partial t} + \frac{1}{2}\lambda^2 D_2 + \lambda^2 D_1\right) D_1 u + \left(R_t + \frac{1}{2}\lambda^2 R^2 R_{xx}\right) u_x \\
&= D_1 \mathscr{L}_{t,x}(\lambda) u \,. \blacksquare
\end{aligned}
$$

Lemma A.3. By Lemma A.2 and the Merton equation: $\mathscr{L}_{t,x}M(t,x;\lambda) = 0$, the classical Merton value function $M(t,x;\lambda)$ satisfies

$$\mathscr{L}_{t,x}(\lambda)D_1^k M(t,x;\lambda) = 0, \tag{A.6}$$

for all $k = 1,2,\cdots$.

Lemma A.4 ("Vega-Gamma" relationship). The classical Merton value function $M(t,x;\lambda)$ satisfies

$$\frac{\partial M}{\partial \lambda} = -(T-t)\lambda\frac{M_x^2}{M_{xx}} = (T-t)\lambda D_1 M. \tag{A.7}$$

Proof. Differentiating the Merton equation $\mathscr{L}_{t,x}(\lambda)M = 0$ with respect to λ gives

$$
\begin{aligned}
\mathscr{L}_{t,x}(\lambda)M_\lambda &= -\frac{1}{2}\left(\frac{\partial}{\partial\lambda}(\lambda R)^2\right)M_{xx} - \left(\frac{\partial}{\partial\lambda}(\lambda^2 R)\right)M_x \\
&= -\lambda D_2 M - 2\lambda D_1 M - (RM_{xx} + M_x)\lambda^2 R_\lambda \\
&= -\lambda D_2 M - 2\lambda D_1 M \\
&= \lambda\frac{M_x^2}{M_{xx}} = -\lambda D_1 M\,.
\end{aligned}
\tag{A.8}
$$

The terminal condition for the above PDE is $M_\lambda(T, x; \lambda) = 0$. The solution to the PDE (A.8) with zero terminal condition is given by $M_\lambda = (T-t)\lambda D_1 M$. The verification step uses Lemma A.3:

$$
\begin{aligned}
\mathscr{L}_{t,x}(\lambda)((T-t)\lambda D_1 M) &= \mathscr{L}_{t,x}(\lambda)(T\lambda D_1 M) - \mathscr{L}_{t,x}(\lambda)(t\lambda D_1 M) \\
&= T\lambda\mathscr{L}_{t,x}(\lambda)D_1 M - \left(\lambda D_1 M + t\lambda\mathscr{L}_{t,x}(\lambda)D_1 M\right) \\
&= -\lambda D_1 M\,.
\end{aligned}
$$

Therefore, (2.2.13) is established. ∎

Lemma A.5. The solution $v(t, x)$ to the following PDE is unique:

$$
\mathscr{L}_{t,x}(\lambda)v(t,x) = CD_1^2 M(t,x;\lambda),\ v(T,x) = 0, \tag{A.9}
$$

where C is a constant.

Proof. Define the new variable

$$
\xi = -\log M_x(t,x;\lambda) + \frac{1}{2}\lambda^2(T-t),
$$

which is a well-defined injective transformation, because the Merton value function M is strictly increasing and strictly concave. By making a change of variables $M(t,x;\lambda) = w^{(0)}(\hat{t},\xi)$, where $\hat{t} = t$, we have

$$\mathscr{L}_{t,x}(\lambda)M = \mathscr{H}(\lambda)w^{(0)} := \frac{\partial w^{(0)}}{\partial \hat{t}} + \frac{1}{2}\lambda^2 \frac{\partial^2 w^{(0)}}{\partial \xi^2}$$

where $\mathscr{H}$ denotes the backwards heat operator. Hence, $w^{(0)}$ is the solution to the backwards heat equation $\mathscr{H}w^{(0)} = 0$ with terminal condition depending on the terminal condition through the transformation ξ.

Fix a solution $v(t,x)$ to the PDE (A.9). Making the same change of variable to (A.9), we obtain that the solution $v(t,x) = w^{(1)}(\hat{t},\xi)$, where $w^{(1)}$ solves the heat equation with zero terminal condition:

$$\mathscr{H}(\lambda)w^{(1)} = C\frac{\partial^2}{\partial \xi^2}w^{(0)}, \; w^{(1)}(T,\xi) = 0. \tag{A.10}$$

The uniqueness of the solution $v(t,x)$ follows from the classical uniqueness result for the heat equation (A.10). ∎

APPENDIX B LEMMAS FOR ASYMPTOTIC ANALYSIS

Assumption B.1. The following assumptions are on the state processes (S_t, X_t, Y_t, Z_t):

(i) For any starting points (s, y, z) and fixed (ϵ, δ), the system (2.4.1) has a unique strong solution (S_t, Y_t, Z_t). The function $g(z)$ is in $C^2(\mathbb{R})$, and $\lambda(y, z)$ is in $C^3(\mathbb{R})$ in the z-variable. The coefficients and their derivatives $c(z)$, $a(y)$, $g(z)$, $g'(z)$, $g''(z)$, $\lambda_z(y, z)$, $\lambda_{zz}(y, z)$, $\lambda_{zzz}(y, z)$ are at most polynomially growing.

(ii) The process $Y^{(1)}$ with infinitesimal generator $\mathscr{M}_Y$ defined in (2.3.6) is ergodic with a unique invariant distribution Φ, and admits moments of any order uniformly in $t \leq T$:

$$\sup_{t \leq T} \left(\mathbb{E}\,|\,Y_t^{(1)}|^k \right) \leq C(T, k).$$

The solution $\phi(y, z)$ of the Poisson equation $\mathscr{M}_Y \phi(y, z) = l(y, z)$ is assumed to be polynomial in y if $l(y, z)$ is polynomial in y.

(iii) The process $Z^{(1)}$ with infinitesimal generator $\mathscr{M}_Z$ defined in (2.2.6) admits moments of any order uniformly in $t \leq T$:

$$\sup_{t \leq T} \left(\mathbb{E}\,|\,Z_t^{(1)}|^k \right) \leq C(T, k).$$

(iv) Observe that, for fixed $(t, z) \in [0,T] \times \mathbb{R}$, $v^{(0)}(t, x, z) = M(t, x; \bar{\lambda}(z))$ is a concave function with a linear upper bound. In particular, there exists a function $\overline{G}(z)$ such that

$$v^{(0)}(0,x, z) \leq \overline{G}(z) + x.$$

Assume that the process $\overline{G}(Z_t)$ is in $L^2([0,T] \times \Omega)$ uniformly in δ:

$$\mathbb{E}_{(0,z)}\left(\int_0^T \overline{G}^2(Z_s)\ ds \right) \leq C_1(T, z),$$

where $C_1(T, z)$ is independent of δ, and Z_s is the slow-scale factor with $Z_0 = z$.

(v) The wealth process $X_t^{\pi^{(0)}}$ associated with the zeroth order optimal strategy stays non-negative. Moreover, $X_t^{\pi^{(0)}}$ is in $L^2([0,T] \times \Omega)$ uniformly in (ϵ, δ):

$$\mathbb{E}_{(0,x,y,z)}\left(\int_0^T (X_s^{\pi^{(0)}})^2\ ds \right) \leq C_2(T, x, y, z),$$

where $C_2(T, x, y, z)$ is independent of (ϵ, δ).

Lemma B.2. Under Assumption B.1 (iv) and (v), the process $v^{(0)}(t, X_t^{\pi^{(0)}}, Z_t)$ is in $L^2([0,T] \times \Omega)$ uniformly in (ϵ, δ): for all $(t, x, y, z) \in [0,T] \times \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}$,

$$\mathbb{E}_{(t,x,y,z)}\left(\int_0^T (v^{(0)}(s, X_s^{\pi^{(0)}}, Z_s)^2\ ds \right) \leq C_3(T, x, y, z),$$

where $v^{(0)}(t, x, z) = M(t, x; \overline{\lambda}(z))$.

Proof. By a straight forward computation,

$$
\begin{aligned}
\mathbb{E}_{(t,x,z)}\left(\int_t^T \left(v^{(0)}(s, X_s^{\pi^{(0)}}, Z_s)\right)^2 ds\right) &\le \mathbb{E}_{(t,x,z)}\left(\int_t^T \left(v^{(0)}(0, X_s^{\pi^{(0)}}, Z_s)\right)^2 ds\right) \\
&\le \mathbb{E}_{(t,x,z)}\left(\int_t^T \left(\overline{G}(Z_s) + X_s^{\pi^{(0)}}\right)^2 ds\right) \\
&\le 2\mathbb{E}_{(t,x,z)}\left(\int_t^T \overline{G}^2(Z_s)\, ds\right) + 2\mathbb{E}_{(t,x,z)}\left(\int_t^T (X_s^{\pi^{(0)}})^2\, ds\right) \\
&\le 2C_1(T,z) + 2C_2(T,x,z) =: C_3(T,x,z)\,. \blacksquare
\end{aligned}
$$

Assumption B.3. The following assumptions are on the utility function $U(x)$:

(i) $U(x)$ satisfies Assumption 1.5.

(ii) $U(0+)$ is finite. Without loss of generality, assume $U(0+) = 0$.

(iii) Assume the risk tolerance function $R(x) := -\dfrac{U'(x)}{U''(x)}$ satisfies $R(0) = 0$, $R'(x) < \infty$, strictly increasing, and there exists $K \in \mathbb{R}^+$ such that for $x \ge 0, 2 \le i \le 7$,

$$
\left|\partial_x^i R^i(x)\right| \le K. \tag{B.1}
$$

(iv) Define $I : \mathbb{R}^+ \to \mathbb{R}^+$, $I(y) := (U')^{-1}(y)$ as the inverse function of $U'(x)$. Assume that there exists $\alpha, \kappa > 0$ such that $I(y)$ satisfies the polynomial growth condition: $I(y) \le \alpha + \kappa y^{-\alpha}$.

Lemma B.4. Under Assumption B.3 for the utility function $U(x)$, the risk-tolerance function $R(t, x; \overline{\lambda}(z))$ satisfies: for $0 \le j \le 6$, $\exists K_j > 0$ such that for all $(t, x, \overline{\lambda}(z)) \in [0,T) \times \mathbb{R}^+ \times \mathbb{R}$,

$$
\left|R^j(t, x; \overline{\lambda}(z))(\partial_x^{j+1} R(t, x; \overline{\lambda}(z)))\right| \le K_j.
$$

Or equivalently, for $1 \leq j \leq 7$, $\exists \tilde{K}_j > 0$ such that for all $(t, x, z) \in [0,T) \times \mathbb{R}^+ \times \mathbb{R}$,

$$\left| \partial_x^j R^j(t, x; \overline{\lambda}(z))) \right| \leq \tilde{K}_j.$$

Moreover, RR_{xxz}, R^2R_{xxzz}, RR_{xzz}, RR_{xxzz}, and R^2R_{xxxzz} are uniformly bounded.

Proof. For $j = 0,1$ with constant Sharpe ratio λ, the results follows [Källblad and Zariphopoulou, 2014, Proposition 14]. The generalization of the results to $\lambda(z)$ for $j = 0,1$ is presented in [Fouque and Hu, 2017, Appendix B]. The proof that generalizes the results to $j = 2,3,4,5,6$ is essentially using the comparison principle of heat equation repeatedly, where [Fouque and Hu, 2017, Appendix B] has a detailed example of the argument.

To prove the uniformly boundedness of the products of R and its derivatives, we successively differentiate the "Vega-Gamma" relationship from Lemma A.4 in Appendix A, and use the concavity of the classical Merton value function. Then the results is derived from a tedious and straightforward computation. ∎

APPENDIX C KALMAN-BUCY FILTERING

Kalman-Bucy filtering theory applies to a special filtering model, where the unobservable signal process is Gaussian and the observation function is linear. Consider a probability space $(\Omega, \mathscr{F}, \mathbb{F} = (\mathscr{F}_t)_{t\in[0,T]}, \mathbb{P})$ and some observation process $\mathbf{X}_t$ with filtration $(\mathscr{F}^X_t)_{t\in[0,T]}$ generated by $\mathbf{X}_t$. Suppose the observable process $\mathbf{X}_t$ and an unobservable Gaussian signal process $\mathbf{Y}_t$ are given by the following system

$$\begin{cases} d\mathbf{X}_t = (\tilde{h}_t + h_t\mathbf{Y}_t)dt + d\mathbf{W}_t, \mathbf{X}_0 = 0, \\ d\mathbf{Y}_t = (\tilde{b}_t + b_t\mathbf{Y}_t)dt + c_t d\mathbf{W}_t + \sigma_t d\mathbf{B}_t, \end{cases} \tag{C.1}$$

where $\mathbf{Y}_0$ is a normal random vector with mean $\hat{\mathbf{Y}}_0$ and covariance matrix $\Gamma_0 \in \mathbb{R}^d \times \mathbb{R}^d$, $(\mathbf{W}_t, \mathbf{B}_t)$ is a $(m+d)$-dimensional Brownian motion, the coefficients $\tilde{h}_t, h_t, \tilde{b}_t, b_t, c_t, \sigma_t$ are deterministic matrices of dimension $d\times 1$, $d\times d$, $d\times m$, $d\times d$, $m\times 1$, $m\times d$, respectively. Kalman-Bucy filtering allows us to filter out the signal process $\mathbf{Y}_t$ with respect to the filtration $\mathbb{F}$, i.e. compute $\hat{\mathbf{Y}}_t := \mathbb{E}\left(\mathbf{Y}_t \,|\, \mathscr{F}^X_t\right)$.

Theorem C.1 (Kalman-Bucy filtering). The filtered process $\hat{\mathbf{Y}}_t := \mathbb{E}\left(\mathbf{Y}_t \,|\, \mathscr{F}^X_t\right)$ can be expressed as

$$\hat{\mathbf{Y}}_t = \hat{\mathbf{Y}}_0 + \int_0^t (\tilde{b}_s + b_s\hat{\mathbf{Y}}_s)\, ds + \int_0^t (c_s + \Gamma_s h_s^\top)\, d\nu_s, \tag{C.2}$$

where $\nu_t = \mathbf{X}_t - \int_0^t (\tilde{h}_s + h_s \hat{\mathbf{Y}}_s)\, ds$ is a d-dimensional Brownian motion adapted to $(\mathscr{F}_t^X)_{t\in[0,T]}$,

and $\Gamma_t = (\Gamma_t^{ij}) = \left(\mathbb{E}(\mathbf{Y}_t^i \mathbf{Y}_t^j) - \mathbb{E}(\hat{\mathbf{Y}}_t^i \hat{\mathbf{Y}}_t^j)\right)$ is the covariance matrix for $\mathbf{Y}_t$ and $\hat{\mathbf{Y}}_t$ satisfying the matrix Riccati equation:

$$\frac{d}{dt}\Gamma_t = \Gamma_t b_t^\top + b_t \Gamma_t + c_t c_t^\top + \sigma_t \sigma_t^\top - (c_t + \Gamma_t h_t^\top)(c_t + \Gamma_t h_t^\top)^\top. \tag{C.3}$$

The proof can be found in [Xiong, 2008, Chapter 9], which is an application of Kushner-FKK equation to Gaussian linear system.

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