

Incorporating Preference Shocks and Liquidity Shocks into the Bond Pricing New Keynesian Model Appendix

Haomin Qin*

The University of Chicago

July 2021

*h Qin@uchicago.edu

1 Derivation for First Order Conditions in the Model with Preference Shocks

1.1 Intertemporal Condition

If the household was following an optimal plan to begin with, it must be the case that

$$U_{C,t}dC_t + E_t\{U_{C,t+1}dC_{t+1}\} = 0 \quad (\text{A.1.1})$$

- for any pair (dC_t, dC_{t+1}) satisfying

$$\frac{P_{t+1}}{R_{t+1}}dC_{t+1} = -P_t dC_t \quad (\text{A.1.2})$$

- Thus, we have

$$1 = E_t\left\{\frac{U_{C,t+1}}{U_{C,t}} \frac{P_t}{P_{t+1}}\right\}R_{t+1} \quad (\text{A.1.3})$$

1.2 Stochastic Discount Factor

Define the real stochastic discount factor $M_{t+1} \equiv \frac{U_{C,t+1}}{U_{C,t}}$, and the nominal stochastic discount factor, $M_{t+1}^{\$} \equiv \frac{M_{t+1}}{\Pi_{t+1}}$, we have

$$1 = E_t\left[M_{t+1} \frac{P_t}{P_{t+1}}\right]R_{t+1} = E_t[M_{t+1}^{\$}]R_{t+1} \quad (\text{A.1.4})$$

Note that, since

$$\text{real interest rates} = \frac{\text{nominal interest rates}}{\text{inflation rates}}$$

- and

$$\begin{aligned} \text{real interest rates} &= \frac{1}{\text{real discount factor}} \\ \text{nominal interest rates} &= \frac{1}{\text{nominal discount factor}} \end{aligned}$$

- Thus, we have

$$\text{nominal discount factor} = \frac{\text{real discount factor}}{\text{inflation rates}}$$

Solve for M_{t+1} , we have

$$M_{t+1} = \frac{U_{C,t+1}}{U_{C,t}} = \frac{\frac{\partial U_t}{\partial E_t[U_{t+1}^{1-\gamma}]^{\frac{1}{1-\gamma}}} \frac{\partial E_t[U_{t+1}^{1-\gamma}]^{\frac{1}{1-\gamma}}}{\partial U_{t+1}} \frac{\partial U_{t+1}}{\partial C_{t+1}}}{\frac{\partial U_t}{\partial C_t}} \quad (\text{A.1.5})$$

- where

$$\frac{\partial U_t}{\partial C_t} = \frac{\psi}{\psi - 1} (U_t^{\frac{\psi-1}{\psi}})^{\frac{1}{\psi-1}} (1-\beta)\lambda_t \frac{\psi-1}{\psi} (\bar{L}-L_t)^{1-\frac{1}{\psi}} C_t^{-\frac{1}{\psi}} = (1-\beta)\lambda_t U_t^{\frac{1}{\psi}} (C_t^*)^{1-\frac{1}{\psi}} C_t^{-1} \quad (\text{A.1.6})$$

$$\frac{\partial U_t}{\partial E_t[U_{t+1}^{1-\gamma}]^{\frac{1}{1-\gamma}}} = \beta U_t^{\frac{1}{\psi}} (E_t[U_{t+1}^{1-\gamma}]^{\frac{1}{1-\gamma}})^{-\frac{1}{\psi}} \quad (\text{A.1.7})$$

$$\frac{\partial E_t[U_{t+1}^{1-\gamma}]^{\frac{1}{1-\gamma}}}{\partial U_{t+1}} = \frac{1}{1-\gamma} E_t[U_{t+1}^{1-\gamma}]^{\frac{\gamma}{1-\gamma}} (1-\gamma) U_{t+1}^{-\gamma} = E_t[U_{t+1}^{1-\gamma}]^{\frac{\gamma}{1-\gamma}} U_{t+1}^{-\gamma} \quad (\text{A.1.8})$$

$$\frac{\partial U_{t+1}}{\partial C_{t+1}} = (1-\beta)\lambda_{t+1} U_{t+1}^{\frac{1}{\psi}} (C_{t+1}^*)^{1-\frac{1}{\psi}} C_{t+1}^{-1} \quad (\text{A.1.9})$$

- Thus we have

$$\begin{aligned} M_{t+1} &= \beta \left(\frac{\lambda_{t+1}}{\lambda_t} \right) \left(\frac{C_{t+1}^*}{C_t^*} \right)^{1-\frac{1}{\psi}} \left(\frac{C_{t+1}}{C_t} \right)^{-1} U_{t+1}^{\frac{1}{\psi}-\gamma} E_t[U_{t+1}^{1-\gamma}]^{\frac{\gamma-\frac{1}{\psi}}{1-\gamma}} \\ &= \beta \left(\frac{\lambda_{t+1}}{\lambda_t} \right) \left(\frac{C_{t+1}^*}{C_t^*} \right)^{\frac{1-\gamma}{\theta}} \left(\frac{C_{t+1}}{C_t} \right)^{-1} \left(\frac{U_{t+1}^{1-\gamma}}{E_t[U_{t+1}^{1-\gamma}]} \right)^{\frac{\theta-1}{\theta}} \end{aligned} \quad (\text{A.1.10})$$

1.3 Intratemporal Condition

Similarly, if the household was following an optimal plan to begin with, it must be the case that

$$U_{C,t} dC_t + U_{L,t} dL_t = 0 \quad (\text{A.1.11})$$

- for any pair (dC_t, dL_t) satisfying the budget constraint, i.e.

$$P_t dC_t = W_t dL_t \quad (\text{A.1.12})$$

- Thus, we have

$$\frac{W_t}{P_t} = -\frac{U_{L,t}}{U_{C,t}} = \frac{\tau C_t}{\bar{L} - L_t} \quad (\text{A.1.13})$$

2 Derivation for Bond Pricing Schemes

2.1 Bond Price

Following the standard asset pricing scheme, the time- t price of an n -period nominal bond, $P_t^{(n)\$}$, can be written as its expected price tomorrow discounted by the nominal pricing kernel

$$P_t^{(n)\$} = E_t[M_{t+1}^{\$} P_{t+1}^{(n-1)\$}] \quad (\text{A.2.1})$$

- The normalization gives $P_t^{(0)\$} = 1$, because the bond will pay 1 unit of consumption when it comes to maturity.

Assuming $M_{t+1}^{\$}$ is conditionally lognormally distributed. According to the theorem below,

Theorem:

If two random variables $X \sim N(m_X, \sigma_X^2)$, $Y = e^x \sim LN(m_Y, \sigma_Y^2)$, then for the mean of Y , we have $m_Y = e^{m_X + \frac{1}{2}\sigma_X^2}$, where m_X and σ_X^2 are the mean and variance of X respectively.

- The above equation can be written as

$$P_t^{(n)\$} = E_t[M_{t+1}^{\$} P_{t+1}^{(n-1)\$}] = e^{E_t\{\ln[M_{t+1}^{\$} P_{t+1}^{(n-1)\$}]\} + \frac{1}{2}\text{var}_t\{\ln[M_{t+1}^{\$} P_{t+1}^{(n-1)\$}]\}} \quad (\text{A.2.2})$$

- Take the logs on both sides, we have

$$p_t^{(n)\$} = E_t[p_{t+1}^{(n-1)\$} + m_{t+1}^{\$}] + \frac{1}{2}\text{var}_t[p_{t+1}^{(n-1)\$} + m_{t+1}^{\$}] \quad (\text{A.2.3})$$

- Since $p_t^{(0)\$} = 0$, after recursively substituting out prices, we have

$$p_t^{(n)\$} = E_t\left[\sum_{j=1}^n m_{t+j}^{\$}\right] + \frac{1}{2}\text{var}_t\left[\sum_{j=1}^n m_{t+j}^{\$}\right] \quad (\text{A.2.4})$$

2.2 Bond Yields

The nominal yields of the n -period bond are defined as

$$P_t^{(n)\$} \equiv (1 + y_t^{(n)\$})^{-n} \quad (\text{A.2.5})$$

- Written in logs, we have

$$\ln(1 + y_t^{(n)\$}) = -\frac{1}{n}p_t^{(n)\$} \quad (\text{A.2.6})$$

- As long as the nominal yields are small, we can make an approximation as

$$\ln(1 + y_t^{(n)\$}) \approx y_t^{(n)\$} \equiv -\frac{1}{n}p_t^{(n)\$} \quad (\text{A.2.7})$$

- Then we have

$$y_t^{(n)\$} = -\frac{1}{n}E_t\left[\sum_{j=1}^n m_{t+j}^{\$}\right] - \frac{1}{2n}\text{var}_t\left[\sum_{j=1}^n m_{t+j}^{\$}\right] \quad (\text{A.2.8})$$

- Similarly, the real yields of a n -period bond can be derived as

$$y_t^{(n)} = -\frac{1}{n}E_t\left[\sum_{j=1}^n m_{t+j}\right] - \frac{1}{2n}\text{var}_t\left[\sum_{j=1}^n m_{t+j}\right] \quad (\text{A.2.9})$$

3 Derivation for Nominal Bond Yields Decomposition

Using the definition for the excess bond returns, $rx_{t+1}^{(n)\$} = p_{t+1}^{(n-1)\$} - p_t^{(n)\$} - y_t^{(1)\$}$, we can recursively write

$$\begin{aligned}
-p_t^{(n)\$} &= [y_t^{(1)\$} + y_{t+1}^{(1)\$} + \cdots + y_{t+n-1}^{(1)\$}] + [(p_{t+1}^{(n-1)\$} - p_t^{(n)\$} - y_t^{(1)\$}) \\
&\quad + (p_{t+2}^{(n-2)\$} - p_{t+1}^{(n-1)\$} - y_{t+1}^{(1)\$}) + \cdots + (p_{t+n}^{(0)\$} - p_{t+n-1}^{(1)\$} - y_{t+n-1}^{(1)\$})] \\
&= [y_t^{(1)\$} + y_{t+1}^{(1)\$} + \cdots + y_{t+n-1}^{(1)\$}] + [rx_{t+1}^{(n)\$} + rx_{t+2}^{(n-1)\$} + \cdots + rx_{t+n}^{(1)\$}] \\
&= \sum_{i=1}^n y_{t+i-1}^{(1)\$} + \sum_{i=1}^n rx_{t+i}^{(n-i+1)\$}
\end{aligned} \tag{A.3.1}$$

Using the definition for nominal bond yields $y_t^{(n)\$} = -\frac{1}{n}p_t^{(n)\$}$, we can write

$$y_t^{(n)\$} = \frac{1}{n} \sum_{i=1}^n y_{t+i-1}^{(1)\$} + \frac{1}{n} \sum_{i=1}^n rx_{t+i}^{(n-i+1)\$} \tag{A.3.2}$$

Note that the one-period nominal yields equal to the real risk free rates plus expected inflation, we have

$$y_t^{(n)\$} = \frac{1}{n} \sum_{i=1}^n E_t(r_{f,t+i-1}) + \frac{1}{n} \sum_{i=1}^n E_t(\Pi_{t+i}) + \frac{1}{n} \sum_{i=1}^n E_t(rx_{t+i}^{(n-i+1)\$}) \tag{A.3.3}$$

The ex-ante decomposition also holds.

$$E_{t-1}[y_t^{(n)\$}] = \frac{1}{n} \sum_{i=1}^n E_{t-1}(r_{f,t+i-1}) + \frac{1}{n} \sum_{i=1}^n E_{t-1}(\Pi_{t+i}) + \frac{1}{n} \sum_{i=1}^n E_{t-1}(rx_{t+i}^{(n-i+1)\$}) \tag{A.3.4}$$

The difference of above two equations gives us

$$\varepsilon_{y^s,t}^{(n)} = \varepsilon_{r,t}^{(n)} + \varepsilon_{\pi,t}^{(n)} + \varepsilon_{tp^s,t}^{(n)} \tag{A.3.5}$$

4 Derivation for Equilibrium Conditions in the Model with Liquidity Shocks

4.1 Parent Households

The difference between the intertemporal condition for the parent households and Kung's Model with preference shocks is the expression for interest rates. Here R_t^s is used rather than the R_{t+1} . However, the logic is the same as section 1.1, Appendix.

$$1 = E_t \left\{ \frac{U_{C,t+1}}{U_{C,t}} \frac{P_t}{P_{t+1}} \right\} R_t^s \tag{A.4.1}$$

- As for the stochastic discount factor, here we can treat it as a special case of the model with preference shocks, where $\lambda_t = 1$, for any t , thus

$$M_{t+1} = \beta \left(\frac{C_{t+1}^*}{C_t^*} \right)^{\frac{1-\gamma}{\theta}} \left(\frac{C_{t+1}}{C_t} \right)^{-1} \left(\frac{U_{t+1}^{1-\gamma}}{E_t[U_{t+1}^{1-\gamma}]} \right)^{\frac{\theta-1}{\theta}} \quad (\text{A.4.2})$$

- The intratemporal condition stays the same.

$$\frac{W_t}{P_t} = -\frac{U_{L,t}}{U_{C,t}} = \frac{\tau C_t}{\bar{L} - L_t} \quad (\text{A.4.3})$$

4.2 Final Goods Production Firms

The final goods production firm use intermediate goods with a constant elasticity of substitution technology to produce final goods, and ν is the elasticity of substitution between intermediate goods.

$$Y_t = \left(\int_0^1 X_{i,t}^{\frac{\nu-1}{\nu}} di \right)^{\frac{\nu}{\nu-1}} \quad (\text{A.4.4})$$

- The profits are

$$\begin{aligned} & \max_{X_{i,t}} [Y_t P_t - \int_0^1 X_{i,t} P_{i,t} di] \\ & = \max_{X_{i,t}} \left[\left(\int_0^1 X_{i,t}^{\frac{\nu-1}{\nu}} di \right)^{\frac{\nu}{\nu-1}} P_t - \int_0^1 X_{i,t} P_{i,t} di \right] \end{aligned} \quad (\text{A.4.5})$$

- where the aggregate price level of final goods P_t is given by

$$P_t = \left[\int_0^1 P_{i,t}^{1-\nu} di \right]^{\frac{1}{1-\nu}} \quad (\text{A.4.6})$$

Take prices as given and maximize products by choosing optimal inputs $X_{i,t}$, we get the inverse demand schedule

$$P_{i,t} = P_t Y_t^{\frac{1}{\nu}} X_{i,t}^{-\frac{1}{\nu}} \quad (\text{A.4.7})$$

- Written in the isoelastic demand schedule

$$X_{i,t} = Y_t \left(\frac{P_{i,t}}{P_t} \right)^{-\nu} \quad (\text{A.4.8})$$

4.3 Intermediate Goods Production Firms

The dividends maximization problem for intermediate goods production firms is

$$\begin{aligned} \max_{P_{i,t}, I_{i,t}, S_{i,t}, K_{i,t+1}, N_{i,t+1}, L_{i,t}} V^{(i)}(P_{i,t-1}, K_{i,t}, N_{i,t}; \Upsilon_t) &= \frac{D_{i,t}}{P_t} + E_t[M_{t+1}V^{(i)}(P_{i,t}, K_{i,t+1}, N_{i,t+1}; \Upsilon_{t+1})] \\ \text{subject to} \quad K_{i,t+1} &= (1 - \delta_k)K_{i,t} + \Phi_k\left(\frac{I_{i,t}}{K_{i,t}}\right)K_{i,t} \\ \Phi_k\left(\frac{I_{i,t}}{K_{i,t}}\right) &= \frac{\alpha_{1,k}}{1 - \frac{1}{\xi_k}}\left(\frac{I_{i,t}}{K_{i,t}}\right)^{1 - \frac{1}{\xi_k}} + \alpha_{2,k} \\ N_{i,t+1} &= (1 - \delta_n)N_{i,t} + \Phi_n\left(\frac{S_{i,t}}{N_{i,t}}\right)N_{i,t} \\ \Phi_n\left(\frac{S_{i,t}}{N_{i,t}}\right) &= \frac{\alpha_{1,n}}{1 - \frac{1}{\xi_n}}\left(\frac{S_{i,t}}{N_{i,t}}\right)^{1 - \frac{1}{\xi_n}} + \alpha_{2,n} \\ P_{i,t} &= P_t Y_t^{\frac{1}{\nu}} [K_{i,t}^\alpha (A_t N_{i,t}^\eta N_t^{1-\eta} L_{i,t})^{1-\alpha}]^{1-\frac{1}{\nu}} \end{aligned} \quad (\text{A.4.9})$$

Dividends are given by

$$D_{i,t} = P_t Y_t^{\frac{1}{\nu}} [K_{i,t}^\alpha (A_t N_{i,t}^\eta N_t^{1-\eta} L_{i,t})^{1-\alpha}]^{1-\frac{1}{\nu}} - W_{i,t} L_{i,t} - P_t I_{i,t} - P_t S_{i,t} - P_t G(P_{i,t}, P_{i,t+1}; P_t, Y_t) \quad (\text{A.4.10})$$

where

$$G(P_{i,t}, P_{i,t+1}; P_t, Y_t) = \frac{\phi_R}{2} \left(\frac{P_{i,t}}{\Pi_{ss} P_{i,t-1}} - 1 \right)^2 Y_t \quad (\text{A.4.11})$$

and $\Upsilon_t \equiv [P_t, K_t, N_t, Y_t, A_t]$ is taken as given.

The Lagrangian can be written as

$$\begin{aligned} V^{(i)}(P_{i,t-1}, K_{i,t}, N_{i,t}; \Upsilon_t) &= F(K_{i,t}, N_{i,t}, L_{i,t}; A_t, N_t, Y_t) - \frac{W_{i,t}}{P_t} L_{i,t} - I_{i,t} - S_{i,t} - G(P_{i,t}, P_{i,t+1}; P_t, Y_t) \\ &+ E_t[M_{t+1}]V^{(i)}(P_{i,t}, K_{i,t+1}, N_{i,t+1}; \Upsilon_{t+1}) \\ &+ \Lambda_{i,t} \left\{ \frac{P_{i,t}}{P_t} - J(K_{i,t}, N_{i,t}, L_{i,t}; A_t, N_t, Y_t) \right\} \\ &+ Q_{i,k,t} \left\{ (1 - \delta_k)K_{i,t} + \Phi_k\left(\frac{I_{i,t}}{K_{i,t}}\right)K_{i,t} - K_{i,t+1} \right\} \\ &+ Q_{i,n,t} \left\{ (1 - \delta_n)N_{i,t} + \Phi_n\left(\frac{S_{i,t}}{N_{i,t}}\right)N_{i,t} - N_{i,t+1} \right\} \end{aligned} \quad (\text{A.4.12})$$

with auxiliary variables

$$J(K_{i,t}, N_{i,t}, L_{i,t}; A_t, N_t, Y_t) \equiv Y_t^{\frac{1}{\nu}} [K_{i,t}^\alpha (A_t N_{i,t}^\eta N_t^{1-\eta} L_{i,t})^{1-\alpha}]^{-\frac{1}{\nu}} \quad (\text{A.4.13})$$

$$F(K_{i,t}, N_{i,t}, L_{i,t}; A_t, N_t, Y_t) \equiv Y_t^{\frac{1}{\nu}} [K_{i,t}^\alpha (A_t N_{i,t}^\eta N_t^{1-\eta} L_{i,t})^{1-\alpha}]^{1-\frac{1}{\nu}} \quad (\text{A.4.14})$$

and Lagrangian multiplier $\Lambda_{i,t}$, $Q_{i,k,t}$, and $Q_{i,n,t}$. First-order conditions are

$$\begin{aligned}
\frac{\partial V^{(i)}}{\partial P_{i,t}} &= -G_{i,1,t} + E_t[M_{t+1}V_{p,t+1}^{(i)}] + \frac{\Lambda_{i,t}}{P_t} = 0 \\
\frac{\partial V^{(i)}}{\partial I_{i,t}} &= -1 + Q_{i,k,t}\Phi'_{i,k,t} = 0 \\
\frac{\partial V^{(i)}}{\partial S_{i,t}} &= -1 + Q_{i,n,t}\Phi'_{i,n,t} = 0 \\
\frac{\partial V^{(i)}}{\partial K_{i,t+1}} &= E_t[M_{t+1}V_{k,t+1}^{(i)}] - Q_{i,k,t} = 0 \\
\frac{\partial V^{(i)}}{\partial N_{i,t+1}} &= E_t[M_{t+1}V_{n,t+1}^{(i)}] - Q_{i,n,t} = 0 \\
\frac{\partial V^{(i)}}{\partial L_{i,t}} &= F_{i,l,t} - \frac{W_{i,t}}{P_t} - \Lambda_{i,t}J_{i,l,t} = 0
\end{aligned} \tag{A.4.15}$$

Envelope conditions are

$$\begin{aligned}
V_{p,t}^{(i)} &= -G_{i,2,t} \\
V_{k,t}^{(i)} &= F_{i,k,t} - \Lambda_{i,t}J_{i,k,t} + Q_{i,k,t}(1 - \delta_k - \frac{\Phi'_{i,k,t}I_{i,t}}{K_{i,t}} + \Phi_{i,k,t}) \\
V_{n,t}^{(i)} &= F_{i,n,t} - \Lambda_{i,t}J_{i,n,t} + Q_{i,n,t}(1 - \delta_n - \frac{\Phi'_{i,n,t}S_{i,t}}{N_{i,t}} + \Phi_{i,n,t})
\end{aligned} \tag{A.4.16}$$

The auxiliary variables in these conditions are

$$\begin{aligned}
G_{i,1,t} &= \phi_R \left(\frac{P_{i,t}}{\Pi_{ss} P_{i,t-1}} - 1 \right) \frac{Y_t}{\Pi_{SS} P_{i,t-1}} \\
G_{i,2,t} &= -\phi_R \left(\frac{P_{i,t}}{\Pi_{ss} P_{i,t-1}} - 1 \right) \frac{Y_t P_{i,t}}{\Pi_{SS} P_{i,t-1}^2} \\
\Phi_{i,k,t} &= \Phi_k \left(\frac{I_{i,t}}{K_{i,t}} \right) = \frac{\alpha_{1,k}}{1 - \frac{1}{\zeta_k}} \left(\frac{I_{i,t}}{K_{i,t}} \right)^{1 - \frac{1}{\zeta_k}} + \alpha_{2,k} \\
\Phi_{i,n,t} &= \Phi_n \left(\frac{S_{i,t}}{N_{i,t}} \right) = \frac{\alpha_{1,n}}{1 - \frac{1}{\zeta_n}} \left(\frac{S_{i,t}}{N_{i,t}} \right)^{1 - \frac{1}{\zeta_n}} + \alpha_{2,n} \\
\Phi'_{i,k,t} &= \alpha_{1,k} \left(\frac{I_{i,t}}{K_{i,t}} \right)^{-\frac{1}{\zeta_k}} \\
\Phi'_{i,n,t} &= \alpha_{1,n} \left(\frac{S_{i,t}}{N_{i,t}} \right)^{-\frac{1}{\zeta_n}} \\
F_{i,k,t} &= \frac{\alpha(1 - \frac{1}{\nu}) Y_t^{\frac{1}{\nu}} X_{i,t}^{1 - \frac{1}{\nu}}}{K_{i,t}} \\
F_{i,n,t} &= \frac{\eta(1 - \alpha)(1 - \frac{1}{\nu}) Y_t^{\frac{1}{\nu}} X_{i,t}^{1 - \frac{1}{\nu}}}{N_{i,t}} \\
F_{i,l,t} &= \frac{(1 - \alpha)(1 - \frac{1}{\nu}) Y_t^{\frac{1}{\nu}} X_{i,t}^{1 - \frac{1}{\nu}}}{L_{i,t}} \\
J_{i,k,t} &= \frac{-\left(\frac{\alpha}{\nu}\right) Y_t^{\frac{1}{\nu}} X_{i,t}^{-\frac{1}{\nu}}}{K_{i,t}} \\
J_{i,n,t} &= \frac{-\left(\frac{\eta(1-\alpha)}{\nu}\right) Y_t^{\frac{1}{\nu}} X_{i,t}^{-\frac{1}{\nu}}}{N_{i,t}} \\
J_{i,l,t} &= \frac{-\left(\frac{1-\alpha}{\nu}\right) Y_t^{\frac{1}{\nu}} X_{i,t}^{-\frac{1}{\nu}}}{L_{i,t}}
\end{aligned} \tag{A.4.17}$$

Combining above first-order conditions and envelope conditions gives

$$\frac{\Lambda_{i,t}}{P_t} = \phi_R \left(\frac{P_{i,t}}{\Pi_{ss} P_{i,t-1}} - 1 \right) \frac{Y_t}{\Pi_{SS} P_{i,t-1}} - E_t [M_{t+1} \phi_R \left(\frac{P_{i,t+1}}{\Pi_{ss} P_{i,t}} - 1 \right) \frac{Y_{t+1} P_{i,t+1}}{\Pi_{SS} P_{i,t}^2}] \tag{A.4.18}$$

$$Q_{i,k,t} = \frac{1}{\Phi'_{i,k,t}} \tag{A.4.19}$$

$$Q_{i,n,t} = \frac{1}{\Phi'_{i,n,t}} \tag{A.4.20}$$

$$\begin{aligned}
Q_{i,k,t} = E_t[M_{t+1}\{ & \frac{\alpha(1-\frac{1}{\nu})Y_{t+1}^{\frac{1}{\nu}}X_{i,t+1}^{1-\frac{1}{\nu}}}{K_{i,t+1}} \\
& + \frac{\Lambda_{i,t+1}(\frac{\alpha}{\nu})Y_{t+1}^{\frac{1}{\nu}}X_{i,t+1}^{-\frac{1}{\nu}}}{K_{i,t+1}}\}] + E_t[M_{t+1}Q_{i,k,t+1}(1-\delta_k - \frac{\Phi'_{i,k,t+1}I_{i,t+1}}{K_{i,t+1}} + \Phi_{i,k,t+1})]
\end{aligned} \tag{A.4.21}$$

$$\begin{aligned}
Q_{i,n,t} = E_t[M_{t+1}\{ & \frac{\eta(1-\alpha)(1-\frac{1}{\nu})Y_{t+1}^{\frac{1}{\nu}}X_{i,t+1}^{1-\frac{1}{\nu}}}{N_{i,t+1}} \\
& + \frac{\Lambda_{i,t+1}(\frac{\eta(1-\alpha)}{\nu})Y_{t+1}^{\frac{1}{\nu}}X_{i,t+1}^{-\frac{1}{\nu}}}{N_{i,t+1}}\}] + E_t[M_{t+1}Q_{i,n,t+1}(1-\delta_n - \frac{\Phi'_{i,n,t+1}S_{i,t+1}}{N_{i,t+1}} + \Phi_{i,n,t+1})]
\end{aligned} \tag{A.4.22}$$

$$\frac{W_{i,t}}{P_t} = \frac{(1-\alpha)(1-\frac{1}{\nu})Y_t^{\frac{1}{\nu}}X_{i,t}^{1-\frac{1}{\nu}}}{L_{i,t}} + \Lambda_{i,t}\frac{(1-\alpha)Y_t^{\frac{1}{\nu}}X_{i,t}^{-\frac{1}{\nu}}}{L_{i,t}} \tag{A.4.23}$$

4.4 Check for the Market Clearing Conditions

- Parent Households' Budget Constraints

$$P_t C_t + S_a t = W_t L_t + R_{t-1}^s S_{a,t-1} + D_t + D_t^{FI} + T_t - X_t - X_t^{FI} \tag{A.4.24}$$

- Child Households' Budget Constraints

$$P_t C_{b,t} + B_{t-1} R_{t-1}^b = B_t + X_t \tag{A.4.25}$$

- Production Firms' Dividends

$$D_t = P_t Y_t - W_t L_t - P_t I_t - P_t S_t - P_t G \tag{A.4.26}$$

- Financial Intermediaries' Dividends

$$D_{t+1}^{FI} = (R_t^b - R_t^s) B_t^{FI} + (R_t^{re} - R_t^s) R E_t + R_t^s X_t^{FI} \tag{A.4.27}$$

- Financial Intermediaries' Balance Sheet

$$B_t^{FI} + R E_t = S_a t + X_t^{FI} \tag{A.4.28}$$

- Central Bank's Balance Sheet

$$B_t^{cb} = R E_t \tag{A.4.29}$$

- Operating Surplus of Central Bank

$$T_{t+1} = R_t^b B_t^{cb} - R_t^{re} R E_t \tag{A.4.30}$$

- Bond Market Clearing Conditions

$$B_t = B_t^{FI} + B_t^{cb} \quad (\text{A.4.31})$$

Financial intermediaries' balance sheet, central bank's balance sheet, and bond market clearing conditions give

$$X_t^{FI} = B_t - Sa_t \quad (\text{A.4.32})$$

Financial intermediaries' balance sheet, and financial intermediaries' dividends, central bank's balance sheet, and bond market clearing conditions give

$$D_t^{FI} = R_{t-1}^b B_{t-1}^{FI} + R_{t-1}^{re} B_{t-1}^{cb} - R_{t-1}^s Sa_{t-1} \quad (\text{A.4.33})$$

Replace the X_t^{FI} , X_t , D_t , D_t^{FI} and T_t gives

$$\begin{aligned} P_t C_t + Sa_t = & W_t L_t + R_{t-1}^s Sa_{t-1} + [P_t Y_t - W_t L_t - P_t I_t - P_t S_t - P_t G] \\ & + [R_{t-1}^b B_{t-1}^{FI} + R_{t-1}^{re} B_{t-1}^{cb} - R_{t-1}^s Sa_{t-1}] \\ & + [R_{t-1}^b B_{t-1}^{cb} - R_{t-1}^{re} B_{t-1}^{cb}] - [P_t C_t + B_{t-1} R_{t-1}^b - B_t] - [B_t - Sa_t] \end{aligned} \quad (\text{A.4.34})$$

This is equivalent to

$$P_t C_t + P_t C_{b,t} + P_t I_t + P_t S_t + P_t G = P_t Y_t \quad (\text{A.4.35})$$

- Consistent with the goods market clearing condition

$$Y_t = C_t + C_{b,t} + I_t + S_t + G \quad (\text{A.4.36})$$

5 Equilibrium of the Model with Liquidity Shocks

The full set of equilibrium conditions includes 33+2i equations and 33+2i variables. i represents the maturity of bonds included in the model. I study the bonds up to five year maturities at quarterly frequency. Thus there will be 73 equations and 73 variables.¹

- Fourteen equations on the production sector
- Six equations on the households sector
- Two equations on the financial intermediaries sector
- Six equations on the central bank sector
- Two equations of the market clearing conditions
- Three equations of exogenous processes
- Forty equations from the bond pricing formula

¹Note that in the symmetric equilibrium, we can drop the sign i , because all intermediate firms make the same decisions.

$$L_t, d_t, \Pi_t, Q_{k,t}, Q_{n,t}, I_t, S_t, \Lambda_t, K_t, N_t, \Phi_{k,t}, \Phi_{n,t}, \Phi'_{k,t}, \Phi'_{n,t}, \\ U_t, C_t^*, w_t, C_t, M_t, C_{b,t}, sa_t, b_t^{FI}, R_t^s, QE_t, re_t, b_t^{cb}, R_t^d, R_t^{re}, Y_t, b_t, A_t, \Theta_t, \sigma_t$$

as well as

$$\{P_t^{\{(n)\$}\}, y_t^{\{(n)\$}\}\} \text{ for } n = 1, 2, \dots, 20$$

The lowercase letters are real variables adjusted by the aggregate price level,

$$d_t = \frac{D_t}{P_t} \\ w_t = \frac{W_t}{P_t} \\ sa_t = \frac{Sa_t}{P_t} \\ b_t^{FI} = \frac{B_t^{FI}}{P_t} \\ re_t = \frac{RE_t}{P_t} \\ b_t^{cb} = \frac{B_t^{cb}}{P_t} \\ b_t = \frac{B_t}{P_t}$$

5.1 Production

Production Technology

$$Y_t = K_t^\alpha (A_t L_t)^{1-\alpha} \quad (\text{A.5.1})$$

Production Dividends

$$d_t = Y_t - w_t L_t - I_t - S_t - \frac{\phi_R}{2} \left(\frac{\Pi_t}{\Pi_{ss}} - 1 \right)^2 Y_t \quad (\text{A.5.2})$$

Production Optimal Conditions

$$\Lambda_t = \phi_R \left(\frac{\Pi_t}{\Pi_{ss}} - 1 \right) \frac{Y_t \Pi_t}{\Pi_{SS}} - E_t [M_{t+1} \phi_R \left(\frac{\Pi_t}{\Pi_{ss}} - 1 \right) \frac{Y_{t+1} \Pi_{t+1}}{\Pi_{SS}}] \quad (\text{A.5.3})$$

$$Q_{k,t} = \frac{1}{\Phi'_{k,t}} \quad (\text{A.5.4})$$

$$Q_{n,t} = \frac{1}{\Phi'_{n,t}} \quad (\text{A.5.5})$$

$$Q_{k,t} = E_t[M_{t+1}\{\frac{\alpha(1-\frac{1}{\nu})Y_{t+1}^{\frac{1}{\nu}}X_{t+1}^{1-\frac{1}{\nu}}}{K_{t+1}} + \frac{\Lambda_{t+1}(\frac{\alpha}{\nu})Y_{t+1}^{\frac{1}{\nu}}X_{t+1}^{-\frac{1}{\nu}}}{K_{t+1}}\}] + E_t[M_{t+1}Q_{k,t+1}(1-\delta_k - \frac{\Phi'_{k,t+1}I_{t+1}}{K_{t+1}} + \Phi_{k,t+1})] \quad (\text{A.5.6})$$

$$Q_{n,t} = E_t[M_{t+1}\{\frac{\eta(1-\alpha)(1-\frac{1}{\nu})Y_{t+1}^{\frac{1}{\nu}}X_{t+1}^{1-\frac{1}{\nu}}}{N_{t+1}} + \frac{\Lambda_{t+1}(\frac{\eta(1-\alpha)}{\nu})Y_{t+1}^{\frac{1}{\nu}}X_{t+1}^{-\frac{1}{\nu}}}{N_{i,t+1}}\}] + E_t[M_{t+1}Q_{n,t+1}(1-\delta_n - \frac{\Phi'_{n,t+1}S_{t+1}}{N_{t+1}} + \Phi_{n,t+1})] \quad (\text{A.5.7})$$

$$w_t = \frac{(1-\alpha)(1-\frac{1}{\nu})Y_t^{\frac{1}{\nu}}X_t^{1-\frac{1}{\nu}}}{L_t} + \Lambda_t \frac{(\frac{1-\alpha}{\nu})Y_t^{\frac{1}{\nu}}X_t^{-\frac{1}{\nu}}}{L_t} \quad (\text{A.5.8})$$

Physical Capital Evolution

$$K_{t+1} = (1-\delta_k)K_t + \Phi_{k,t}K_t \quad (\text{A.5.9})$$

R&D Capital Evolution

$$N_{t+1} = (1-\delta_n)N_t + \Phi_{n,t}N_t \quad (\text{A.5.10})$$

Physical Capital Adjustment Cost

$$\Phi_{k,t} = \frac{\alpha_{1,k}}{1-\frac{1}{\zeta_k}} \left(\frac{I_t}{K_t}\right)^{1-\frac{1}{\zeta_k}} + \alpha_{2,k} \quad (\text{A.5.11})$$

R&D Capital Adjustment Cost

$$\Phi_{n,t} = \frac{\alpha_{1,n}}{1-\frac{1}{\zeta_n}} \left(\frac{S_t}{N_t}\right)^{1-\frac{1}{\zeta_n}} + \alpha_{2,n} \quad (\text{A.5.12})$$

Physical Capital Adjustment Cost Derivation

$$\Phi'_{k,t} = \alpha_{1,k} \left(\frac{I_t}{K_t}\right)^{-\frac{1}{\zeta_k}} \quad (\text{A.5.13})$$

R&D Capital Adjustment Cost Derivation

$$\Phi'_{n,t} = \alpha_{1,n} \left(\frac{S_t}{N_t}\right)^{-\frac{1}{\zeta_n}} \quad (\text{A.5.14})$$

5.2 Households

Definition for Utility and C_t^*

$$U_t = \{(1-\beta)(C_t^*)^{\frac{1-\gamma}{\theta}} + \beta(E_t[U_{t+1}^{1-\gamma}])^{\frac{1}{\theta}}\}^{\frac{\theta}{1-\gamma}} \quad (\text{A.5.15})$$

$$C_t^* = C_t(\bar{L} - L_t)^\tau \quad (\text{A.5.16})$$

Optimal Conditions for the Parent Households

$$w_t = \frac{\tau C_t}{\bar{L} - L_t} \quad (\text{A.5.17})$$

$$M_{t+1} = \beta \left(\frac{C_{t+1}^*}{C_t^*} \right)^{\frac{1-\gamma}{\theta}} \left(\frac{C_{t+1}}{C_t} \right)^{-1} \left(\frac{U_{t+1}^{1-\gamma}}{E_t[U_{t+1}^{1-\gamma}]} \right)^{\frac{\theta-1}{\theta}} \quad (\text{A.5.18})$$

$$1 = E_t[M_{t+1}\Pi_{t+1}^{-1}]R_t^s \quad (\text{A.5.19})$$

Child Households' Consumption

$$C_{b,t} = b_t \quad (\text{A.5.20})$$

5.3 Financial Intermediaries

Balance Sheet

$$b_t^{FI} + re_t = s_t + x^{FI} \quad (\text{A.5.21})$$

Credit constraints

$$b_t^{FI} = \Theta_t x^{FI} \quad (\text{A.5.22})$$

5.4 Central Bank

Central Banks' Policies

$$\ln\left(\frac{R_t^s}{R_{ss}^s}\right) = \rho_r \ln\left(\frac{R_{t-1}^s}{R_{ss}^s}\right) + (1 - \rho_r) \left[\rho_\pi \ln\left(\frac{\Pi_t}{\Pi_{ss}}\right) + \rho_y \ln\left(\frac{Y_t}{\hat{Y}_{ss}}\right) \right] + \sigma_\xi \xi_t \quad (\text{A.5.23})$$

$$\ln QE_t = (1 - \rho_q) \ln QE + \rho_q \ln QE_{t-1} + \sigma_q \varepsilon_{q,t} \quad (\text{A.5.24})$$

Liquidity Injection or Absorption

$$QE_t = b_t^{cb} \quad (\text{A.5.25})$$

Balance Sheet

$$b_t^{cb} = re_t \quad (\text{A.5.26})$$

Interest Rates Assumption

$$R_t^b = R_t^s \quad (\text{A.5.27})$$

$$R_t^{re} = R_t^s \quad (\text{A.5.28})$$

5.5 Market Clearing Conditions

Goods Markets Clearing

$$Y_t = C_t + C_{b,t} + S_t + I_t + \frac{\phi_R}{2} \left(\frac{\Pi_t}{\Pi_{ss}} - 1 \right)^2 Y_t \quad (\text{A.5.29})$$

Bond Markets Clearing

$$b_t = b_t^{FI} + b_t^{cb} \quad (\text{A.5.30})$$

5.6 Exogenous Processes

Productivity Shock

$$\ln A_t = (1 - \rho)a^* + \rho \ln A_{t-1} + \sigma_{t-1}\varepsilon_t, \quad \varepsilon_t \sim N(0, 1) \quad (\text{A.5.31})$$

Credit Shock

$$\ln \Theta_t = (1 - \rho_\theta) \ln \Theta + \rho_\theta \ln \Theta_{t-1} + \sigma_\theta \varepsilon_{\theta,t} \quad (\text{A.5.32})$$

Volatility Shock

$$\sigma_t^2 = \bar{\sigma}^2 + \lambda(\sigma_{t-1}^2 - \bar{\sigma}^2) + \sigma_e e_t, \quad e_t \sim N(0, 1) \quad (\text{A.5.33})$$

5.7 Bond Pricing

Nominal Bonds Prices

$$P_t^{(n)\$} = E_t[M_{t+1}^\$ P_{t+1}^{(n-1)\$}] \quad (\text{A.5.34})$$

Nominal Bonds Yields

$$y_t^{(n)\$} = -\frac{1}{n} \ln(P_t^{(n)\$}) \quad (\text{A.5.35})$$

The model is solved by detrending with N_t , so there would be 72 equations and 72 variables left.

6 Calibration for the Model with Liquidity Shocks

Table A.1, A.2, and A.3 report the calibration for parameters in the production sector, households sector, and (policy) exogenous processes respectively. Parameters in the production, households sector are calibrated the same as Kung (2015).² Thus, the model still has the same good features as the original model shown in Kung (2015), such as nice matches with the moments of macroeconomic variables and many characteristics of nominal yields. The parameters related to liquidity shocks and credit shocks are calibrated the same as Sims *et al.* (2021).

²I modified the value of τ to keep the steady-state total consumption the same as Kung (2015) after the introduction of liquidity shocks.

Parameter	Value	Description
α	0.33	Capital Share
ϕ_R	30	Magnitude of price adjustment costs
ν	6	Intermediate goods price elasticity
η	0.10	Degree of technological appropriability
δ_k	0.02	Physical capital deprecation rate
ζ_k	4.80	Elasticity of new physical capital investments relative to existing stock of physical capital
$\alpha_{1,k}$	0.442	Physical capital adjustment cost parameter 1
$\alpha_{2,k}$	-0.005	Physical capital adjustment cost parameter 2
δ_n	0.038	R&D capital deprecation rate
$\alpha_{1,n}$	0.370	R&D capital adjustment cost parameter 1
$\alpha_{2,n}$	-0.016	R&D capital adjustment cost parameter 2
ζ_n	3.3	Elasticity of new R&D capital investments relative to existing stock of R&D capital

Table A.1: Parameter Calibration in the Production Sector

Parameter	Value	Description
β	0.997	Stochastic Discount Factor
ψ	2	Elasticity of intertemporal substitution
γ	10	Risk Aversion
θ	-18	Self-defined parameter
τ	2.369	Weights of leisure in parent households' utility function
\bar{L}	3	Time endowment

Table A.2: Parameter Calibration in the Households Sector

Parameter	Value	Description
ρ	0.984	Persistence of productivity shock process
ρ_r	0.70	Persistence of monetary policy shock process
ρ_y	0.03	Sensitivity of interest rate to output
ρ_π	0.45	Sensitivity of interest rate to inflation
ρ_q	0.80	Persistence of liquidity shock process
ρ_t	0.80	Persistence of credit shock process
λ	0.997	Persistence of volatility shock process
σ_e	0.000	Volatility of volatility shock
σ_ξ	0.003	Volatility of monetary policy shock
σ_q	0.01	Volatility of liquidity shock
σ_t	0.01	Volatility of credit shock

Table A.3: Parameter Calibration in Exogenous Processes

7 Simulation Results for Alternative Calibration

Table A.4 reports the simulation results with different calibration for parameters. We could see that, the differences among different situations are negligible. This implies that the forecast error variance decomposition are most affected by the settings and structure of the model.

	Liquidity shocks situation 1	Liquidity shocks situation 2	Liquidity shocks situation 3	Preference shocks situation 1	Preference shocks situation 2
Maturity					
1	51.06%	50.84%	51.79%	113.86%	120.42%
2	66.99%	66.66%	66.62%	144.03%	147.84%
3	85.87%	85.48%	84.49%	163.10%	162.28%
4	105.08%	104.70%	102.98%	171.70%	166.96%
5	122.36%	122.04%	119.93%	174.32%	167.02%
6	136.64%	136.40%	134.20%	174.21%	165.39%
7	147.87%	147.71%	145.62%	173.05%	163.34%
8	156.48%	156.38%	154.52%	171.55%	161.34%
9	163.01%	162.97%	161.40%	170.03%	159.54%
10	167.99%	167.98%	166.73%	168.61%	157.97%
11	171.79%	171.80%	170.88%	167.31%	156.62%
12	174.73%	174.76%	174.15%	166.16%	155.47%
13	177.02%	177.06%	176.75%	165.13%	154.48%
14	178.83%	178.87%	178.86%	164.22%	153.63%
15	180.27%	180.31%	180.58%	163.41%	152.90%
16	181.44%	181.48%	182.00%	162.69%	152.27%
17	182.39%	182.42%	183.19%	162.05%	151.72%
18	183.17%	183.20%	184.21%	161.47%	151.25%
19	183.82%	183.85%	185.08%	160.96%	150.85%
20	184.37%	184.40%	185.84%	160.50%	150.50%

Table A.4: Nominal yields forecast error variance decomposition for alternative calibration

- Liquidity shocks situation 1: $\rho_q = \rho_t = 0.8$, steady-state QE is calibrated as $\frac{1}{10}$ of steady-state output, steady-state child consumption is calibrated as $\frac{1}{2}$ of steady-state consumption. (The simulation results reported in table 3).
- Liquidity shocks situation 2: $\rho_q = \rho_t = 0.7$, steady-state QE is calibrated as $\frac{1}{5}$ of steady-state output, steady-state child consumption is calibrated as $\frac{1}{2}$ of steady-state consumption.
- Liquidity shocks situation 3: $\rho_q = \rho_t = 0.5$, steady-state QE is calibrated as $\frac{3}{10}$ of steady-state output, steady-state child consumption is calibrated as $\frac{2}{3}$ of steady-state consumption.
- Preference shocks situation 1: $\rho_\lambda = 0.981$. (The simulation results reported in table 2).
- Preference shocks situation 2: $\rho_\lambda = 0.970$.

8 Impulsive Response Functions for Productivity Shocks and Monetary Policy Shocks

Figure A.1 and A.2 plot the impulsive response functions of macroeconomic variables and nominal yields when there is a positive liquidity shock and a positive monetary policy shock respectively. We could see that both shocks have no effects on the child consumption which are totally supported by private bonds. A positive productivity shock could decrease inflation, interest rates, and nominal yields across all maturities. A positive monetary policy shock would cause the temporary decrease of inflation and the temporary increase

of interest rates and nominal yields across all maturities. High interest rates increase the savings of parent households and stimulate their consumption. In general, the effects of productivity shocks are more persistent.

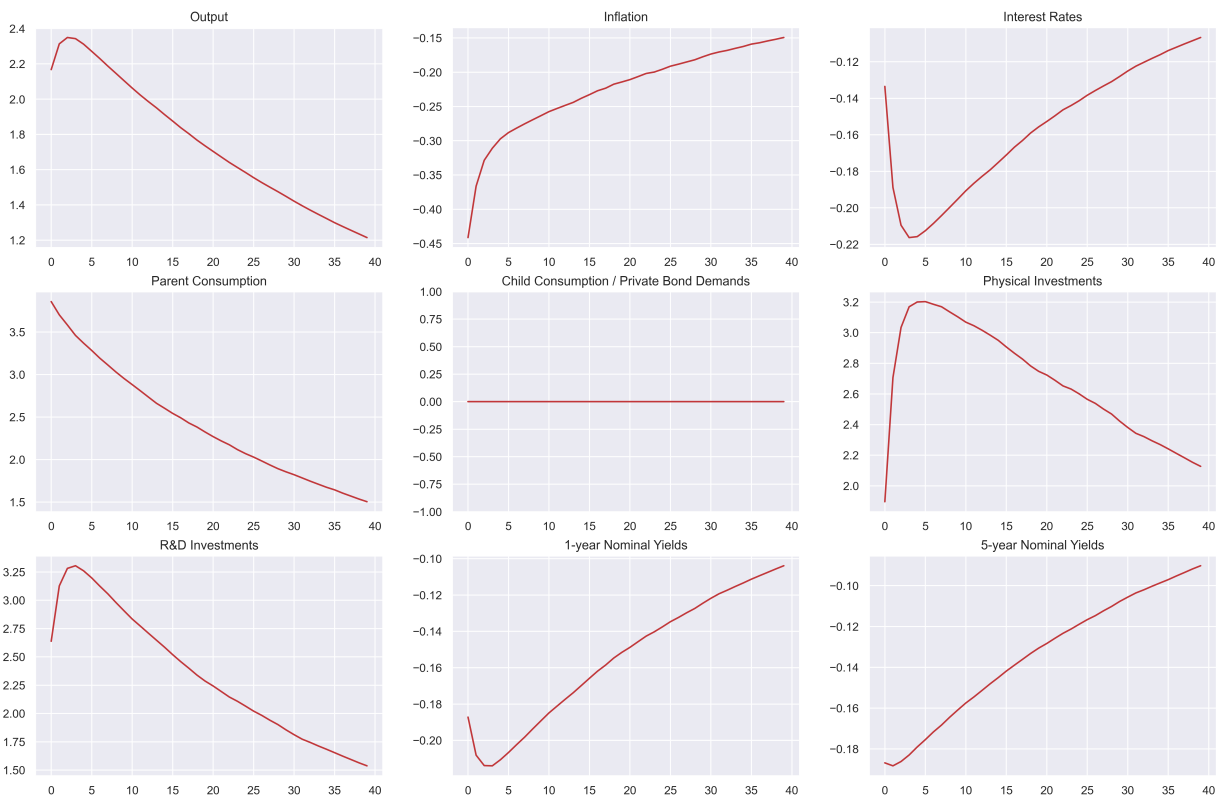


Figure A.1: Impulsive Response for a Positive Productivity Shock

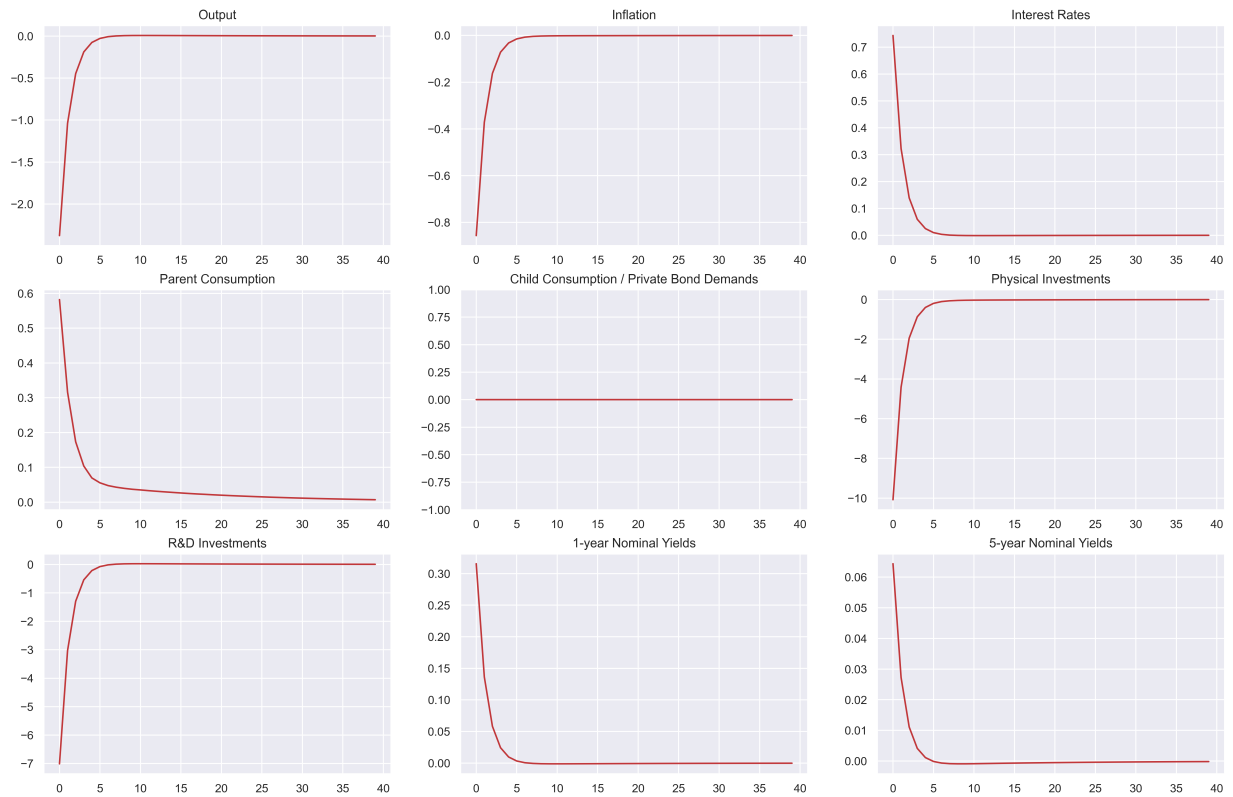


Figure A.2: Impulsive Response for a Positive Monetary Policy Shock

References

- KUNG, HOWARD. 2015. Macroeconomic linkages between monetary policy and the term structure of interest rates. *Journal of financial economics*, **115**(1), 42–57.
- SIMS, ERIC R., WU, JING CYNTHIA, & ZHANG, JI. 2021. The Four Equation New Keynesian Model. *Review of economics and statistics*, **Forthcoming**.