

# Supplemental Material for Dynamics of the unitary Bose gas near a narrow Feshbach resonance: universal coherent atom-molecule oscillations

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## S1. EXPERIMENTAL BACKGROUND

### A. Typical experimental parameters and characterization of the momentum distribution

For <sup>133</sup>Cs the resonance at  $B = 19.849$  G has a very small width of  $\Delta = 8.3$  mG. The experimental parameters are: the relative magnetic moment  $\delta\mu = 2\pi\hbar \times 0.76$  MHz/G, the atomic  $s$ -wave scattering length  $a_1 = 163a_0$  where  $a_0$  is Bohr radius. The molecular  $s$ -wave scattering length is  $a_2 = 220a_0$ . The mean BEC density is  $n = 3 \times 10^{13}$  cm<sup>-3</sup>. In this way the coupling constants used in the simulations can be obtained and they are given by  $g_1 = 3.28E_F/k_F^3$ ,  $g_2 \simeq 2.43E_F/k_F^3$  and  $\alpha \simeq 1.53E_F/k_F^{3/2}$ , when the cut-off is chosen to be  $a_1\Lambda = \pi/10$ . Here  $E_F = \hbar^2k_F^2/2m_1$ .

We extract experimentally from time of flight (TOF) measurements the particle number density distribution in a two-dimensional space given by  $n_{a/m}^{\text{TOF}}(x, y)$  which represents the atom and molecule particle number per pixel. Relating momentum and position variables  $\hbar\mathbf{k} = m\omega\mathbf{x}$ , we find that

$$\mathbf{k}_a = 0.193(\mu\text{m}^{-2}) \times \mathbf{x}_a, \quad \mathbf{k}_m = 0.387(\mu\text{m}^{-2}) \times \mathbf{x}_m. \quad (\text{S1})$$

Here  $\mathbf{k}_{a/m}$  represents the momentum of atoms or molecules while  $\mathbf{x}_{a/m}$  represents their positions. In this way one can use Eq. S1 to obtain a particle number distribution in momentum space,

$$N_{a/m}(k_x, k_y, t) = n_{a/m}^{\text{TOF}}(x, y, t) \times S_{\text{pixel}} = n_{a/m}^{\text{TOF}}(x, y, t) \times (0.605\mu\text{m})^2 \quad (\text{S2})$$

Here we use the area of a single pixel  $S_{\text{pixel}} = 0.605^2\mu\text{m}^2$ . The mean BEC density is  $n = 3 \times 10^{13}$  cm<sup>-3</sup>, which defines a momentum scale,

$$k_F = (6\pi^2n)^{1/3} \simeq 12.1\mu\text{m}^{-1}. \quad (\text{S3})$$

To visualize the 2D data as a 1D curve, we perform the angular average below to obtain a momentum distribution,

$$f_{a/m}(k_i, t) := \frac{1}{2\pi(k_i + \Delta/2)\Delta} \sum_{k_i \leq \sqrt{k_x^2 + k_y^2} < k_i + \Delta} N_{a/m}(k_x, k_y, t), \quad k_i = i\Delta. \quad (\text{S4})$$

Here  $f$  has the dimensions of area and  $\Delta = k_F/50$ . This allows us to obtain the momentum distribution as given in the main-text with the normalization

$$\int d^2k f_{a/m}(k, t) = N_{a/m} \quad (\text{S5})$$

### B. Alternative fit of the effective temperature data in Ref. [17].

In contrast to an earlier empirical fit to the effective temperature, in Fig 1 we use a sinusoidal function plus a linear increasing offset  $y(t) = A \sin(2\pi ft + \phi) + Bt + C$  to fit the coefficients (A,B,C) based on the error bars in the data. In this way we can replot the atomic and molecular temperature data presented in Fig.3b of Ref. [17]. We choose to fit the data points at  $t > 0.5$  ms when they start to oscillate. The oscillation frequencies are determined to be 2.0(2) kHz for atoms and 1.9(2) kHz for molecules, which are consistent with the universal 2 kHz oscillation frequency seen in other observables shown in the main text.

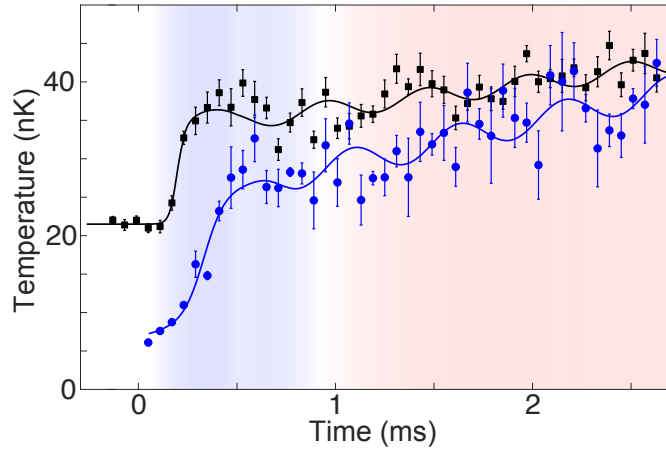


FIG. S1. Effective temperature of atoms (black points) and molecules (blue points) from Ref. [17]. Data points at  $t > 0.5$  ms are fit using the function  $y(t) = A \sin(2\pi ft + \phi) + Bt + C$ . The fit determines the oscillation frequency  $f$  to be 2.0(2) kHz for atoms and 1.9(2) kHz for molecules. The parts of the curves at  $t \leq 0.5$  ms are guides to the eye.

## S2. THEORETICAL ANALYSIS

### A. Two-dimensional momentum distribution

We theoretically compute the particle number distribution in momentum space,  $n(k)$ . When integrated over momentum this yields the density in each channel,  $n = \int \frac{d^3k}{(2\pi)^3} n(k)$ . Of interest is the integral along the  $z$ -axis representing a column integrated two dimensional distribution. We define

$$F_{a/m}(k_{\perp}) = \int \frac{dk_z}{2\pi} n_{a/m} \left( \sqrt{k_{\perp}^2 + k_z^2} \right) \quad (\text{S6})$$

Imposing a normalization condition  $(2\pi)^{-2} \int d^2k_{\perp} F_{a/m} = N_{a/m}/V$  yields the following relation between  $F_{a/m}$  and the variable  $f_{a/m}$ ,

$$\frac{V}{(2\pi)^2} F_{a/m}(k) = f_{a/m}(k) \iff \frac{N_0/n}{(2\pi)^2} F_{a/m}(k) = f_{a/m}(k), \quad (\text{S7})$$

which allows us to relate the theory to experimental data  $f_{a/m}$ . Here  $N_0$  is the initial particle number. The dimensionless quantity plot in Fig. 3c of the main-text is based on the relation:

$$\frac{k_F^2 f_{a/m}}{N_0} = (2\pi)^2 \frac{k_F^2}{n} F_{a/m}(k). \quad (\text{S8})$$

### B. Derivation of the Dynamical Equations

In this section, we discuss the 2-channel model and the associated dynamical equations. We start from the effective field theory of the Hamiltonian

$$H = \sum_{\sigma} \int d^d x \hat{\psi}_{\sigma}^{\dagger}(x) \left( -\frac{\nabla^2}{2m_{\sigma}} + \nu_{\sigma} \right) \hat{\psi}_{\sigma}(x) + \int d^d x \left( \sum_{\sigma} \frac{g_{\sigma}}{2} |\hat{\psi}_{\sigma}(x)|^4 - \alpha \hat{\psi}_{1}^{\dagger}(x) \hat{\psi}_{1}^{\dagger}(x) \hat{\psi}_2(x) - h.c. \right). \quad (\text{S9})$$

Here  $\sigma = 1, 2$  refers to the open channel atoms and closed-channel molecules,  $g_{\sigma}$  represents the background  $s$ -wave interaction in the open and closed channels and  $\alpha$  represents the Feshbach coupling strength between two channels. Here  $\nu_1 = 0$  and  $\nu_2 - \nu_1$  is related to the binding energy of molecules. This model allows us to address the kinematics of the open channel atoms and closed channel molecules, and the couplings between them. We note that with the appropriate parameters we are able to reproduce the resonance width and the 2-body scattering length of atoms.

To address the quantum dynamics we relate the fundamental dynamical variables in our model to the equal-time correlation functions of field operators. Here the one-point correlation function represents the condensate wave-function,

$$\xi_\sigma = \frac{1}{\sqrt{V}} \langle \hat{\psi}_\sigma(k=0) \rangle, \quad \sigma = 1, 2. \quad (\text{S10})$$

With this definition we deduce the *exact* equation of motion for condensates,

$$\begin{aligned} i \frac{d}{dt} \xi_1 &= 2g_1 n_1 \xi_1 + (g_1 x'_1 - 2\alpha \xi_2) \xi_1^* - 2\alpha f_{12} + g_1 f_{111} \\ i \frac{d}{dt} \xi_2 &= (\nu + 2g_2 n_2) \xi_2 + g_2 x'_2 \xi_2^* - 2\alpha x'_1 + g_2 f_{222} \end{aligned} \quad (\text{S11})$$

where we denote

$$f_{12} = \int d^3 q / (2\pi)^3 \langle \psi_1^\dagger(q) \psi_2'(q) \rangle, \quad f_{\sigma\sigma\sigma} = \int d^3 q_1 d^3 q_2 / (2\pi)^6 \langle \psi_\sigma^\dagger(q_1) \hat{\psi}'_\sigma(q_2) \hat{\psi}'_\sigma(q_1 - q_2) \rangle. \quad (\text{S12})$$

$$n_\sigma = \int d^3 q / (2\pi)^3 \langle \psi_\sigma^\dagger(q) \psi'_\sigma(q) \rangle, \quad x'_\sigma = \xi_\sigma^2 + \int d^3 q / (2\pi)^3 \langle \psi'_\sigma(q) \psi'_\sigma(-q) \rangle. \quad (\text{S13})$$

Here  $n_\sigma$  represents the number of particles in finite momentum states.

The correlation functions are given in terms of  $\hat{\psi}'_\sigma(k) = \hat{\psi}_\sigma(k) - \delta_{k,0} \xi_\sigma$  where we have introduced a 4-vector field operator,  $\Psi = (\psi'_1, \psi_1^\dagger, \psi'_2, \psi_2^\dagger)$ . One may define a general  $n$ -point correlation function below,

$$G_{i_1, i_2, \dots, i_n}(k_1, k_2, \dots, k_n) = \langle \hat{\Psi}_{i_1}(k_1) \hat{\Psi}_{i_2}(k_2) \dots \hat{\Psi}_{i_n}(k_n) \rangle. \quad (\text{S14})$$

Next we derive the dynamics of these correlation functions using the time evolution operator,  $\hat{U}(t) = \exp(-i\hat{H}t)$  based on the Hamiltonian in Eq. S9. This yields the following equations of motion:

$$i \frac{d}{dt} \hat{\Psi}_i(k) = L'_{i\beta}(k) \hat{\Psi}_\beta(k) + \sum_q Q_{i\beta\gamma} \hat{\Psi}_\beta(q) \hat{\Psi}_\gamma(k-q) + \sum_{q_1, q_2} C_{i\beta\gamma\delta} \hat{\Psi}_\beta(q_1) \hat{\Psi}_\gamma(q_2) \hat{\Psi}_\delta(k-q_1-q_2).$$

Here  $L'$  is 2 point vertex related to the quadratic part of Hamiltonian,  $Q$  is the 3-point vertex which includes the Feshbach coupling term while  $C$  is a 4-point vertex including background scatterings. From the bosonic commutators, one can readily derive the expression of  $L'$ ,

$$L' = \begin{pmatrix} \epsilon'_1(k) & g_1 \xi_1^2 - 2\alpha \xi_2 & -2\alpha \xi_1^* & 0 \\ -(g_1 \xi_1^2 - 2\alpha \xi_2)^* & -\epsilon'_1(-k) & 0 & 2\alpha \xi_1 \\ -2\alpha \xi_1 & 0 & \epsilon'_2(k) & g_2 \xi_2^2 \\ 0 & 2\alpha \xi_1^* & -(g_2 \xi_2^2)^* & -\epsilon'_2(-k) \end{pmatrix}, \quad \epsilon'_\sigma(k) = k^2/2m_i + \nu_\sigma \delta_{2,i} + 2g_i |\xi_i|^2. \quad (\text{S15})$$

Only a few values of  $Q_{\alpha\beta\gamma}$  are non-vanishing. These are given by

$$\begin{aligned} Q_{121} &= \frac{2g_1 \xi_1}{\sqrt{V}}, \quad Q_{111} = \frac{g_1 \xi_1^*}{\sqrt{V}}, \quad Q_{123} = -\frac{2\alpha}{\sqrt{V}}; \quad Q_{221} = -\frac{2g_1 \xi_1^*}{\sqrt{V}}, \quad Q_{222} = -\frac{g_1 \xi_1^*}{\sqrt{V}}, \quad Q_{214} = \frac{2\alpha}{\sqrt{V}}; \\ Q_{343} &= \frac{2g_2 \xi_2}{\sqrt{V}}, \quad Q_{333} = \frac{g_2 \xi_2^*}{\sqrt{V}}, \quad Q_{311} = -\frac{\alpha}{\sqrt{V}}; \quad Q_{443} = -\frac{2g_2 \xi_2^*}{\sqrt{V}}, \quad Q_{444} = -\frac{g_2 \xi_2}{\sqrt{V}}, \quad Q_{422} = \frac{\alpha}{\sqrt{V}}; \end{aligned} \quad (\text{S16})$$

Similarly, only a few values for  $C$  are non-zero,

$$C_{1211} = \frac{g_1}{V}, \quad C_{2221} = -\frac{g_1}{V}, \quad C_{3433} = \frac{g_2}{V}, \quad C_{4443} = -\frac{g_2}{V}. \quad (\text{S17})$$

This leads to the following heirarchy of equations,

$$\begin{aligned} i \frac{d}{dt} G_{i_1, i_2, \dots, i_n}(k_1, \dots, k_n) &= \sum_{j, \beta} L'_{i_j \beta}(k_j) G_{i_1, \dots, i_{j-1}, \beta, i_{j+1}, \dots, i_n}(k_1, \dots, k_n) \\ &+ \sum_{j, \beta \gamma} Q_{i_j \beta \gamma} \sum_q G_{i_1, \dots, i_{j-1}, \beta, \gamma, i_{j+1}, \dots, i_n}(k_1, \dots, k_{j-1}, q, k_j - q, k_{j+1}, \dots, k_n) \\ &+ \sum_{j, \beta \gamma \delta} C_{i_j \beta \gamma \delta} \sum_{q_1, q_2} G_{i_1, \dots, i_{j-1}, \beta, \gamma, \delta, i_{j+1}, \dots, i_n}(k_1, \dots, k_{j-1}, q_1, q_2, k_j - q_1 - q_2, k_{j+1}, \dots, k_n) \end{aligned} \quad (\text{S18})$$

which relates  $n$ -point correlations with  $n(L)$ ,  $n+1(Q)$ ,  $n+2(C)$  correlations.

Now consider a quench of a pure atomic condensate to unitarity. It is useful to consider the way in which the dynamics reflects a sequence of *propagation of information* in the correlation functions,

$$\xi_1 \longrightarrow \xi_2 \longrightarrow G_{\alpha\beta} \longrightarrow G_{\alpha\beta\gamma} \longrightarrow \dots \quad (\text{S19})$$

Correspondingly, we can consider a sequence of points on the time axis,

$$t = 0 \longrightarrow t_1 \longrightarrow t_2 \longrightarrow t_3 \longrightarrow \dots \quad (\text{S20})$$

Because of the small size of the coupling constant  $\alpha$ , one may have well-separated dynamical regimes

$$\text{I} = [0, t_1], \text{II} = [t_1, t_2], \text{III} = [t_2, t_3], \dots \quad (\text{S21})$$

We stress here the importance of the subscript  $i$  in this discussion. The  $i$  associated with  $t_i$  indicates the time interval where the  $i$ -point correlation is fully developed and becomes non-perturbative. For example, in the time interval associated with  $t_2$  we need only include 2-point correlation functions in modeling the dynamics.

We observe that with increasing time the atomic and molecular BEC's decay and the coherence of the dynamical system is significantly reduced after the time frame  $t_2$ . Since the coherence greatly enhances the speed of reaction/collision events, we expect that the time scale of each stage monotonically increases such that  $t_3 - t_2 \gg t_2 - t_1 \gg t_1$ . Therefore the time required to fully develop the 3-point correlation is much longer than the  $t_2$ , which is the order of 1ms. Within the experimentally accessible time scale ( $< 3\text{ms}$ ), one may conclude that the contribution of 3-point correlation to the dynamics is subleading compared to the 2-point correlation. As a consequence in modeling the dynamics we set 3-point correlations to be zero and decompose 4-point correlations into products of 2-point correlation functions. Note that 3-point correlations are indecomposable since the average of  $\psi'$  is zero by definition. This then leads to a closed form for the dynamical equations (now consisting of only one-point and two-point functions.)

Within this approximation, one can derive a closed set of dynamical equations that include one- and two-point correlation functions. In this scheme, the two-point correlation function,  $L'$ , receives corrections from the decomposition of four-point correlations. This leads to the expression for the dynamical matrix,  $L$ , which appears in Eq. 3 of the main text,

$$L = \begin{pmatrix} \epsilon_1(k) & g_1 x'_1 - 2\alpha\xi_2 & -2\alpha\xi_1^* & 0 \\ -(g_1 x'_1 - 2\alpha\xi_2)^* & -\epsilon_1(-k) & 0 & 2\alpha\xi_1 \\ -2\alpha\xi_1 & 0 & \epsilon_2(k) & g_2 x'_2 \\ 0 & 2\alpha\xi_1^* & -g_2 x'_2 & -\epsilon_2(-k) \end{pmatrix} \quad (\text{S22})$$

where  $\epsilon_i(k) = k^2/2m_i + \nu_r \delta_{2,i} + 2g_i n_i$ .

### C. Different stages in the dynamical evolution

After the quench of the atomic BEC, the next most immediate stage of the dynamical evolution involves the creation of a molecular condensate. To describe this we begin by considering the dynamics with only two condensates,  $\xi_1$  and  $\xi_2$ . We set  $g_1 = g_2 = 0$  for simplicity. Since our focus is on unitarity, we set the detuning to zero. This leads to the coupled equations below,

$$i\partial_t \xi_1 = -2\alpha\xi_2 \xi_1^*, \quad i\partial_t \xi_2 = -\alpha\xi_1^2. \quad (\text{S23})$$

One may use the constraint from the conservation of particle-number  $|\xi_1|^2 + 2|\xi_2|^2 = n$  to eliminate  $\xi_1$  and find a single equation for  $\xi_2$ ,

$$\partial_t^2 \xi_2 + 4\alpha^2 \xi_2 (n - |\xi_2|^2) = 0. \quad (\text{S24})$$

With the initial condition from the quenched dynamics  $\xi_2(0) = 0$ ,  $\partial_t \xi_2(0) = i\alpha n$ , the differential equation allows an instanton solution,

$$\xi_2 = i\sqrt{n} \tanh\left(\alpha\sqrt{2n}t\right). \quad (\text{S25})$$

This in turn sets a time scale for the growth of the molecular condensate,  $t_\alpha = \hbar/\alpha\sqrt{n}$ . Due to the narrow Feshbach resonance, this time is not particularly small and the time scale should be observable in experiment as it is of the order of 0.2ms. Such instanton phenomena are characterized by a sharp peak in the particle fraction of the molecular BEC as a function of time which can be seen in Fig. 2 of the maintext.

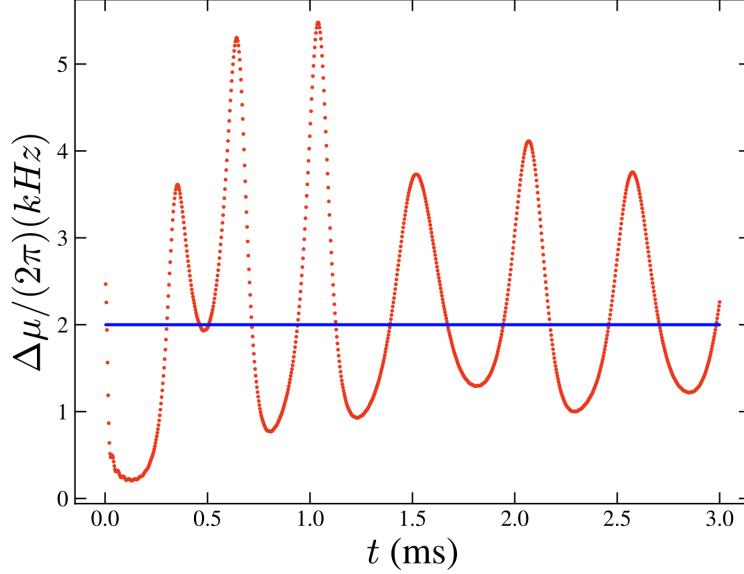


FIG. S2. The figure plots  $\Delta\mu$  (which is the time dependent energy level difference between the two condensates), as the function of time. Here  $\hbar = 1$ . Numerically we find that the average of  $\Delta\mu$  in the late time is around 2.1kHz.

#### D. Determining the Oscillation Frequency

In the main text, we note that a gap is dynamically generated between the atomic and molecular BEC. This gap originates from the energy level difference  $\Delta\mu = 2\Delta\mu_1 - \Delta\mu_2$  between the two condensates. Here,  $\Delta\mu_\sigma$  represents the self-energy correction. One may derive  $\Delta\mu_1$  from rewriting the condensate dynamics:

$$i\frac{d}{dt}\xi_1 = 2g_1n_1\xi_1(0) + (g_1x_1 - 2\alpha\xi_2)\xi_1^* - 2\alpha\xi_1\left(\text{Re}\frac{f_{12}}{\xi_1} + i\text{Im}\frac{f_{12}}{\xi_1}\right), \quad f_{12} = \int \frac{d^3k}{(2\pi)^3}f_{12}(k). \quad (\text{S26})$$

Similarly, one can derive  $\Delta\mu_2$ . Therefore one can obtain:

$$\Delta\mu_1 = -2\alpha\text{Re}\xi_1^{-1}\sum_k\langle\psi_1^\dagger(k)\psi_2(k)\rangle, \quad \Delta\mu_2 = -\alpha\text{Re}\xi_2^{-1}\sum_k\langle\psi_1(k)\psi_1(-k)\rangle. \quad (\text{S27})$$

These functions  $\Delta\mu$  are time-dependent and oscillate around a constant value. The physical energy gap should be determined by the time-average  $\langle\Delta\mu\rangle_t$ ; see Fig. S2.

The finite momentum particles will also oscillate at the same frequency as the condensates. Because they have a generally kinetic energy,  $\epsilon_k/\hbar \ll 2\text{kHz}$ , this will not appreciably alter the characteristic frequency.

#### E. Coherent Particle Flow Currents

There are four species of particles in the system. Let us denote them by

$$c_1, c_2 : \text{This labels the particles in the atomic and molecular BECs,} \quad (\text{S28})$$

$$n_1, n_2 : \text{This labels the finite momentum particles for atoms or molecules.} \quad (\text{S29})$$

Particles flow among all four species as plotted in Fig. S3. The equations governing the dynamics (and counted in terms of the number of atoms) are:

$$\frac{d}{dt}c_1 = -J_1 - J_2, \quad \frac{d}{dt}c_2 = J_1/2 - J_3, \quad \frac{d}{dt}n_1 = -J_2 + 2J_3, \quad \frac{d}{dt}n_2 = J_2. \quad (\text{S30})$$

The currents are controlled by the phases of condensates and of the various finite-momentum correlation functions. This leads to expressions for the various currents:

$$J_1 = 4\alpha c_1\sqrt{c_2}\sin(\phi_2 - 2\phi_1), \quad J_2 = 4\alpha|f_{12}|\sqrt{c_2}\sin(\theta_f - \phi_1), \quad J_3 = 2\alpha|x_1|\sqrt{c_2}\sin(\theta_x - \phi_2). \quad (\text{S31})$$

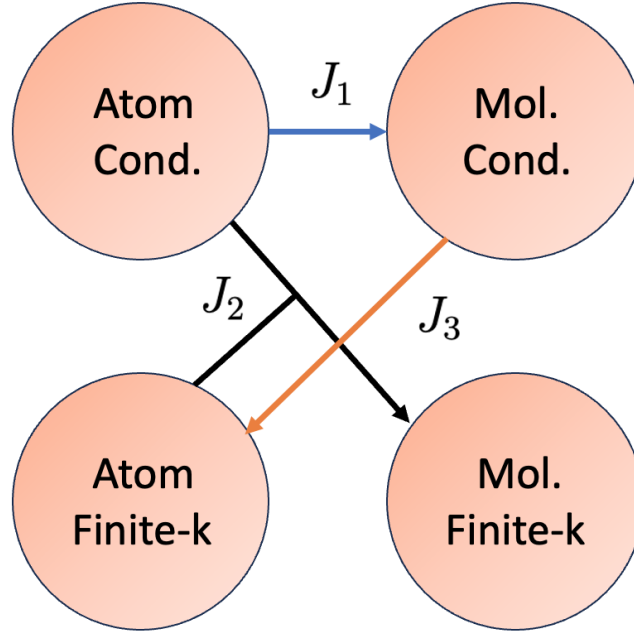


FIG. S3. There are three types of currents flow. (1)  $J_1$ : two cond. atoms form a molecule. (2)  $J_2$ : One cond. atom combine with one finite- $k$  atom, forming a finite momentum molecule. (3)  $J_3$ . One cond. Molecule dissociate into two finite- $k$  atoms.

Here  $\phi_1$  and  $\phi_2$  are phases of atomic and molecular condensates, and  $\theta_f$  and  $\theta_x$  are phases of the complex numbers  $f_{12}$  and  $x_1$ .

All finite momentum particles collectively determine these two phases. The net particle flow results from a competition between these three different current which mostly consist of low  $k$  states. At resonance, we find that  $|J_2| > |J_1| > 2|J_3|$  and that  $J_2$  has a different sign from  $J_1$  and  $J_3$  at least after 1ms. It then follows that the out-of-condensate molecules (represented by  $n_2$ ) are out-of-phase with the other three species of particles.

#### F. The generation of Atom-molecule complexes and molecule-molecule pairs

In this section, we expand the discussion surrounding Eq. 3 in the main-text to illustrate how the atomic and molecular pairs are generated through the 2-point correlation functions involving the Feshbach coupling,

$$\begin{aligned}
 i\partial_t n_1(k) &= -2\alpha \left( f_{12}(k)\xi_1^\dagger + x_1^*(k)\xi_2 - h.c. \right), & i\partial_t n_2(k) &= -2\alpha \left( f_{12}^*(k)\xi_1 - h.c. \right), \\
 \left( i\partial_t + \frac{k^2}{m} \right) x_1(k) &= -2\alpha \left( 2a_{12}(k)\xi_1^\dagger + (2n_1(k) + 1)\xi_2 \right), & \left( i\partial_t + \frac{k^2}{2m} \right) x_2(k) &= -2\alpha \left( a_{21}(k)\xi_1 + \xi_1 a_{12}(k) \right). \\
 \left( i\partial_t + \frac{3k^2}{4m} - \nu \right) a_{12}(k) &= -2\alpha \left( x_2(k)\xi_1^\dagger + \xi_2 f_{12}(k) + x_1(k)\xi_1 \right), & \left( i\partial_t - \frac{k^2}{4m} - \nu \right) f_{12}(k) &= -2\alpha \left( n_1(k)\xi_1 - n_2(k)\xi_1 - a_{12}(k)\xi_2^\dagger \right).
 \end{aligned}$$

Here  $a_{12}(k) = \langle \psi_1'(k)\psi_2'(-k) \rangle = a_{21}(-k)$ . In a more concrete example, assume that one starts from a molecular condensate, and considers the simultaneous dissociation of a molecular condensate ( $\xi_2$ ) into atom pairs ( $x_1$ ). Then this pair of atoms can combine with an atom to form an atom-molecule complex, ( $a_{12}$ ), which can then combine with an atom and further generate a molecule pair ( $x_2$ ).