SUPPLEMENTAL INFORMATION

Calculation of critical frequency

This analytic calculation of ω_c requires a set of simplifying assumptions: (i) We replace the sine function in Eq. 4 with a square wave so that f_w takes on only two discrete values $f_0 + A$ and $f_0 - A$, shown schematically in Fig. 4. (ii) We define a force f_c to be the value at which the second well just disappears, equal to minus the local maximum force from the strong springs, and assume $0 < f_0 - f_c \ll |f_c|$ so that the time-averaged potential is just barely stable, and $f_0 - f_c \ll |A|$ so that the potential is relatively flat (*i.e.*, variation in the force is small compared to its magnitude in the vicinity of the critical position, x_c, where the force from the strong springs is a maximum). Fig. 4 provides a graphical interpretation of these statements.

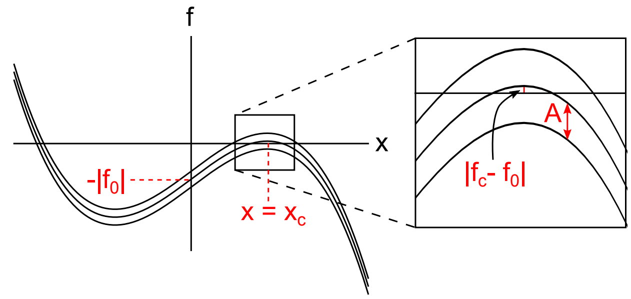

FIG. 4. Net force f on the middle node versus its position for $f_w = f_0 - A$, $f_w = f_0$, and $f_w = f_0 + A$ with $0 < f_0 - f_c \ll |A|$. Note in this case that f_0 and f_c are both negative; $|f_c|$ is just barely larger than $|f_0|$. Zeroes of the force correspond to local minima or maxima of the energy; inset shows the location of the energy minimum which is stable only during part of each cycle.

With the assumption of flatness in (ii), we can approximate the displacement of the middle node as the time spent in half of the cycle, $T/2$, times a characteristic velocity of the middle node $v \sim \langle f \rangle$, where $\langle f \rangle$ is the force averaged over time. The time average can be well approximated as a position average for low spatial variation in f, which is the case we are interested in here. Finally, we Taylor expand the force as a quadratic around x_c:

$$f(x) \approx (f_0 \pm A) - f_c - 3(x - x_c)^2 x_c. \tag{7}$$

We estimate the critical frequency by determining the size of a limit cycle centered at x_c and then finding the period of a cycle with that size. For such a cycle, assuming square wave forcing as specified in (i),

$$\int_{x_c-\delta/2}^{x_c+\delta/2} dx \Big[(f_0 + A) - f_c - 3(x - x_c)^2 x_c\Big]$$
$$= \int_{x_c+\delta/2}^{x_c-\delta/2} dx \Big[(f_0 - A) - f_c - 3(x - x_c)^2 x_c\Big]$$

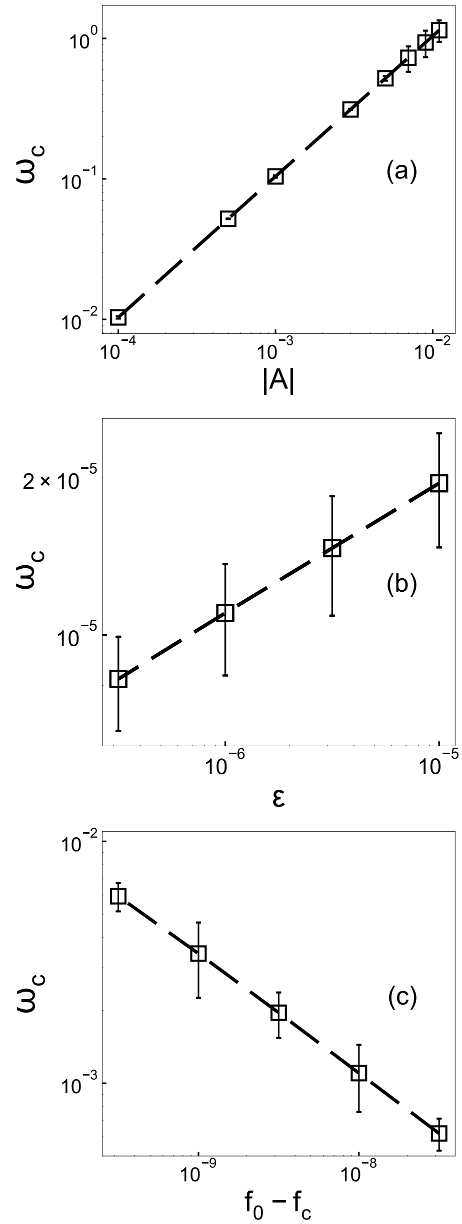

FIG. 5. The scaling of ω_c with (a) $|A|$, (b) ϵ, and (c) $f_0 - f_c$ as given in Eq. 8 (sinusoidal driving). We have used $\epsilon = 0.01$ and $f_0 = -0.00108$ in (a) where slope = 1; $A = -10^{-9}$ and $f_0 - f_c$ fixed at 10^{-11} in (b) where slope = 0.25; and $\epsilon = 10^{-4}$ and $A = -10^{-6}$ in (c) where slope = −0.50.

$$\Rightarrow \int_{x_c-\delta/2}^{x_c+\delta/2} dx \Big[f_0 - f_c - 3(x - x_c)^2 x_c\Big] = 0,$$

where δ is the size of the cycle. This can be solved for δ with the quadratic form of the force given in Eq. 7. Substituting in $x_c = \sqrt{2\epsilon/3}$, which can be found by setting the derivative of the force equal to zero, gives

$$\delta \sim \frac{(f_0 - f_c)^{1/2}}{\epsilon^{1/4}}.$$

To get a cycle of this size, we need the middle node to travel distance δ in a quarter cycle: $\delta = fT/4$, where T is the period and f is the characteristic force and hence determines the characteristic velocity v. By our assumption of flatness, the force near x_c at any point in the cycle is very close to $\pm A$, so we find

$$T_c \sim \frac{\delta}{f} \approx \frac{\delta}{A}$$

$$\omega_c = \frac{2\pi}{T_c} \sim \frac{|A|\epsilon^{1/4}}{(f_0 - f_c)^{1/2}}.$$

Recall that f_c is not a constant:

$$f_c = -f(x)|_{x=x_c} = -2\big(\frac{2}{3}\epsilon\big)^{3/2}$$

$$\omega_c \sim \frac{|A|\epsilon^{1/4}}{\big(f_0 + 2\big(\frac{2}{3}\epsilon\big)^{3/2}\big)^{1/2}}. \tag{8}$$

It is tempting to try to further reduce the equation to a true power law in ϵ by taking either f_0 or ϵ to zero. However, recall the assumptions required for this argument to hold: $0 < f_0 - f_c \ll |A|$, and $f_0 - f_c \ll |f_c| = 2(\frac{2}{3}\epsilon)^{3/2}$. To test the dependence of ω_c on ϵ in particular, we therefore fix the value of $f_0 - f_c$ by changing f_0 as ϵ, and hence f_c, changes. This limits the valid range of ϵ and A values for a given value of $f_0 - f_c$. However, Eq. 8 is a true scaling in the sense that it holds for arbitrarily small values of any given parameter — as long as other parameters are chosen to maintain the assumptions stated above.

Fig. 5 shows the numerical results for scaling of ω_c with A, ϵ, and $f_0 - f_c$ for a system driven by a sinusoidal weak force. The regime in which the scaling of ω_c with $|A|$ obeys Eq. 8 is much wider than the regime in which the scalings with ϵ and $f_0 - f_c$ are valid. Note that $|A|$ does not have to be smaller than any particular value for these scalings to work. This is the reason we were able to expand the scaling seen in Fig. 5a to much larger values of $|A|$ compared to the parameters used in Fig. 5(b,c). Similar scaling is observed when we use a square wave.

Cyclic motion in three-well systems

As described in the text, using a series of motions of the three-well boundary nodes shown in Fig. 3, we can lead the system quasistatically through the cycle $A \to B \to C \to A$. We can accomplish this using the two types of moves shown in that figure: one in which two nodes are pulled away from the center, and one in which one node is brought toward the center and another is moved azimuthally. The resulting energy diagrams for both types of moves are shown in Fig. 3(b-d).

Video 1 and Fig. 3 show the following set of steps:

1. Nodes T and R move outward. Starting from well A, this does nothing.
2. Node T moves toward the center and node R moves azimuthally toward node L. This moves the system from well A to well B.
3. Nodes L and R move outward. This moves the system from well B to well C.

Now these steps are repeated in reverse order. Note that each step had individually reversible boundary conditions, so repeating them in reverse order takes the boundaries back along the same path to the starting point;

1. Nodes L and R move outward. The system remains in well C.
2. Node T moves toward the center and node R moves azimuthally toward node L. The systems remains in well C.
3. Nodes T and R move outward. The system moves from well C back to well A.

We have therefore moved the boundaries "out" and "back" along the same path just as one might when shearing a solid out to a maximum strain and back to its original box shape. The result, unlike anything obtainable from a two-well system, is chiral in nature.

Three-rearrangement cycles in quasistatic hysterons

Generically, a pair of non-interacting hysterons will give rise to cycles with four rearrangements: each hysteron flips once as the strain is increased, and each flips once as the strain is decreased. If there is a degeneracy in the flipping strain, there will be (for example) two rearrangements as the strain is increased but only one as the strain is decreased. The chance of such a degeneracy is very small. However, this phenomenon can be made more robust by adding interactions. In that case, one hysteron flipping may trigger the other to flip. This is an example of an avalanche, and it can lead to cycles with an odd number of total rearrangements.