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TABLE OF CONTENTS

LIST OF FIGURES

LIST OF TABLES

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ABSTRACT

This dissertation mainly focuses on revenue management problems that focus on demand learning and sharing economy. It includes three major chapters.

In Chapter 1, we consider a platform that charges commission rates and subscription fees to sellers and buyers for facilitating transactions but does not directly control the transaction prices, which are endogenously determined. Buyers and sellers are divided into types, and we represent the compatibility between different types using a bipartite network. Traders are heterogeneous in terms of their valuations, and different types have possibly different value distributions. Buyers may have additional value for trading with some seller types. The platform chooses commissions/subscriptions to maximize its revenues. Two salient features of most online platforms are that they do not dictate the transaction prices, and use commissions/subscriptions for extracting revenues. We shed light on how these commissions/subscriptions should be set in networked markets. Using tools from convex optimization and combinatorial optimization, we obtain tractable methods for computing the optimal commissions/subscriptions and provide insights into the platform's revenues, buyer/seller surplus, and welfare. We provide a tractable convex optimization formulation to obtain the revenue-maximizing commissions/subscriptions, and establish that, typically, different types should be charged different commissions/subscriptions depending on their network positions. We establish that the latter result holds even when the traders on each side have identical value distributions, and in this setting we provide lower and upper bounds on the platform's revenues in terms of the supply-demand imbalance across the network. Motivated by simpler schemes used in practice, we show that the revenue loss can be unbounded when all traders on the same side are charged the same commissions/subscriptions, and bound the revenue loss in terms of the supply-demand imbalance across the network. Charging only buyers or only sellers leads to at least half of the optimal revenues, when different types on the same side can be charged differently. Our results highlight the suboptimality of commonly used payment schemes, and showcase the importance of accounting for the compatibility be-

tween different user types. Under mild assumptions, we establish that a revenue-maximizing platform achieves at least 2/3 of the maximum achievable social welfare.

In Chapter 2, we consider the markdown pricing problem of a firm that sells a product to a mixture of myopic and forward-looking customers. The firm faces an uncertainty about the customers' forward-looking behavior, arrival pattern, and valuations for the product, which we collectively refer to as the demand model. Over a multiperiod sales season, the firm sequentially marks down the product's price and makes demand observations to learn the underlying demand model. Because forward-looking customers create an intertemporal dependency, we identify that the keys to achieving good profit performance are: (i) judiciously accumulating information on the demand model, (ii) preserving the market size in early sales periods, and (iii) limiting the impact of the firm's learning on the forward-looking customers. Based on these, we construct and analyze markdown policies that exhibit near-optimal performance under a wide variety of forward-looking customer behaviors. Moreover, contrary to common intuition, we show that forward-looking customers can improve the performance of a learning policy: if the customers are forward-looking, the firm's profit loss due to demand model uncertainty can asymptotically vanish, whereas if the customers are myopic, the firm's profit loss is nonnegligible in the same asymptotic setting.

In Chapter 3, we consider a platform in which multiple sellers offer their products for sale over a time horizon of T periods. Each seller sets its own price. The platform collects a fraction of the sales revenue, and provides price-setting incentives to the sellers to maximize its own revenue. The demand for each seller's product is a function of all sellers' prices and some customer features. Initially, neither the platform nor the sellers know the demand function, but they can learn about it through sales observations: each seller observes its own sales, whereas the platform observes all sellers' sales as well as the customer feature information. In this setting, the platform faces a trade-off between exploiting its informational advantage and revealing information to facilitate demand learning. Measuring the platform's performance by comparing its expected revenue with the full-information optimal revenue, we design poli-

cies that enable the platform to judiciously manage information revelation and price-setting incentives. Perhaps surprisingly, a simple "do-nothing" policy does not always exhibit poor revenue performance and can perform exceptionally well under certain conditions. With a more conservative policy that reveals information to make price-setting incentives more effective, the platform can always protect itself from large revenue losses caused by demand model uncertainty. We develop a strategic reveal-and-incentivize policy that combines the benefits of the aforementioned policies, and thereby achieves asymptotically optimal revenue performance as T grows large.

CHAPTER 1

OPTIMAL COMMISSIONS AND SUBSCRIPTIONS IN NETWORKED MARKETS

1.1 Introduction

Platforms facilitating the exchange of goods and services between individuals are prevalent: one can purchase goods from others on eBay, arrange accommodation through Airbnb, find temporary projects/workers on online labor markets such as Upwork. The revenue models favored by these platforms vary. To facilitate the transactions, some of these platforms charge a commission (a percentage of the total transaction amount) to agents participating in a transaction, while others charge a subscription fee (a flat fee that users pay to gain access to the platform), or a combination of both. For example, most third-party sellers on Amazon pay a $39.99 monthly subscription fee plus per-item selling fees (which vary by category),[1] whereas Airbnb charges a 3% commission to the property owners (hosts) and a 0%-20% commission to the travelers (guests) whenever a property is rented. Hence, often the revenues of these platforms depend not only on the chosen commissions/subscriptions, but also on the prices at which buyers/sellers choose to transact.

However, most platforms do not dictate the transaction prices. Instead, buyers and sellers determine at which price the goods or services will be exchanged: hosts decide on the price per night for their properties on Airbnb, sellers set prices for their goods on Amazon, and freelancers set their hourly rates on online labor markets. These prices depend on seller/buyer characteristics as well as the amount of supply-demand in the market for comparable goods and services. For instance, on Airbnb the (reservation) value of potential guests (hosts) looking for short-term rentals (offering their properties) might depend on features such as the neighborhood and the number of bedrooms. Moreover, not all buyers and sellers on a platform are compatible with each other: a business traveler going to NYC is likely interested

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[1] Source: `https://services.amazon.com/selling/faq.html`

in renting a property in Manhattan and not in the Bronx, whereas a leisure traveler might be interested in either. The number of compatible hosts (guests) for each guest (host) type impacts the final transaction prices. Thus, the imbalance of supply-demand across different types complicates the choice of commissions and subscriptions for the platform.

The objective of this paper is to understand how platforms should design commission rates and subscription fees with the objective of maximizing their revenues in markets where not all buyers and sellers are compatible. Our contribution is threefold: (i) we introduce a stylized model capturing the main features of the platform's problem and characterize the revenue-optimal commissions/subscriptions (which exploit the compatibility structure), (ii) we study the impact of the compatibility structure on the surplus of market participants (buyers, sellers, and platform), and (iii) we analyze the implications of using even simpler commission/subscription schemes.

In our model, buyers and sellers are divided into finitely many types, each of which has some mass of infinitesimal agents. Not all buyer and seller types are compatible, and the compatibility between these types is represented by a bipartite network: nodes on one side correspond to buyer types, nodes on the other side correspond to seller types, and edges capture compatibility between different types of buyers and sellers. Each seller has a unit of good/service to offer, and each buyer demands at most one unit of the good/service from a compatible seller. These compatibilities can capture taste differences (e.g., a buyer may be interested only in the types of goods/services a subset of the sellers offer), geographical restrictions (e.g., being able to provide/receive services only in certain neighborhoods or cities), or other sources of mismatch (e.g., a mismatch between the desired and available skills in online labor markets). Buyers' valuations consist of an idiosyncratic term and a common term. The former term determines a buyer's valuation for transacting with *any* compatible seller and is drawn from a known (buyer) type-specific distribution. The latter term depends only on the type of the seller a buyer transacts with, and captures possible quality differences among seller types. Similarly, each seller has a reservation value for selling her good to *any* compatible buyer, which is also drawn from a (seller) type-specific distribution. Thus, the preferences of the buyers/sellers are summarized by the valuations for the good they demand/supply and by the compatibility network. The platform chooses the commissions and subscriptions, which are possibly type-specific. To capture the fact that the platform does not dictate the prices, we assume that, after the commissions and subscriptions are chosen by the platform, the transaction prices and equilibrium trades are determined *endogenously* in a competitive equilibrium.

We then provide a tractable optimization problem for obtaining the optimal commission rates and the subscription fees in the networked market described above. The problem of choosing revenue-maximizing commissions/subscriptions admits a natural nonconvex optimization formulation. We provide a convex relaxation of this formulation (which can thus be efficiently solved) and show that this relaxation is tight. Using the optimal dual solution of this relaxed problem, optimal commissions/subscriptions can be constructed in a tractable way. We establish that the optimal commissions/subscriptions are not unique, and the set of optimal commissions/subscriptions can be characterized by a system of linear inequalities. Using these inequalities we establish that it is always possible to maximize revenues by relying only on subscriptions and, in the absence of vertically differentiated sellers, the optimal revenues can also be achieved by using only commissions. Moreover, in general, optimal fees are type-dependent and both sides of the market must be charged. Notice that in the simpler setting where there is only one type of buyer and one type of seller, to maximize revenues it would suffice to charge payments only to one side. Thus, naively, the same result may be expected to hold in general settings as well. Our finding illustrates that taking into account the underlying compatibility network leads to significantly different insights, and is fundamental for a platform's commission/subscription design problem.

To isolate the impact of the network structure on the average surplus of different buyer/seller types, we study settings where traders on each side of the market share identical value distributions and sellers are not vertically differentiated. Intuitively, the buyer types that are the "most supply-constrained" are the ones whose average surplus is the lowest, the second most supply-constrained buyers have the second-lowest surplus, and so on. While this is intuitive, it is not a priori clear how to define the most supply-constrained types in a setting where not all buyers and sellers are compatible with each other. We make this intuition precise, by providing an appropriate measure of seller scarcity that takes into account the network structure. Our result allows us to rank buyer types according to their average surplus (and we show that a similar result applies to seller types as well). Moreover, by leveraging this result we also characterize how the revenues of the platform change as a function of the populations of seller/buyer types. In particular, we show that it is least profitable to expand the buyer types who have the lowest average surplus (as they are already the ones who are the most under-supplied), and that it is most profitable to expand the seller types whose average surplus is the highest.

We then explore the impact of the network structure on the revenues of the platform. We show that networks that satisfy a weighted variant of Hall's marriage condition (see, e.g., (1)) maximize the revenues among all networks with the same number of types, populations, and value distributions. Moreover, for any other network, the revenues of the platform can be lower bounded by measuring to what degree this condition is violated.

Motivated by the revenue schemes adopted by many real-world platforms, we study what happens if we restrict ourselves to using simpler commission/subscription schemes. We show that if we require using the same commissions/subscriptions for all buyers and similarly for all sellers, the revenue loss can be unbounded when agent types have heterogeneous value distributions. However, if all buyer types have the same value distribution and so do all seller types, then the revenue loss can be bounded in terms of the supply-demand imbalance induced by the network structure. We show that, in general, charging commissions/subscriptions only to one side of the market also leads to much lower revenues than optimal, even when different types on the same side are charged differently. This time, however, the revenue loss is not unbounded for heterogeneous distributions: it is possible to guarantee at least half of the optimal revenues by charging fees only to one side of the market. We illustrate these findings with an example motivated by Airbnb, which calibrates some of our model primitives with real data. It is worth highlighting that our observations are consistent with the trending practice of Amazon, Airbnb, and Alibaba, which charge heterogeneous commissions/subscriptions to sellers and buyers in their marketplaces.

While the focus of this paper is on the revenue maximization problem of the platform, it is also worth considering the welfare consequences of using revenue-maximizing commissions/subscriptions. Although we can show that the welfare under the revenue-maximizing commissions/subscriptions can be arbitrarily bad in general, we also establish that the welfare achieved by a platform using revenue-maximizing commissions/subscriptions is lower bounded under reasonable assumptions. In particular, if the value distributions of sellers and buyers are uniform, then the welfare under the optimal commissions/subscriptions is at least 75% of the maximum welfare. In addition, under Assumption 3 and mild assumptions on the value distributions, we show that the welfare induced by the optimal commissions/subscriptions is at least 66% of the optimal welfare.

Overall, our results shed light on the design of revenue-maximizing commissions/subscriptions for platforms and highlight the importance of explicitly taking into account the structure of the compatibility network for this purpose. We establish that doing so leads to qualitatively different insights into the optimal commission/subscription structures (e.g., it is no longer revenue-maximizing to have agents only on one side pay to use the platform). At the same time, the underlying network structure has a first-order impact on the revenues of the platform as well as the surplus of its users.

Select proofs are provided in the online appendix. Remaining technical details can be found in our e-companion.

1.2 Literature Review

The seminal papers by (2, 3) and (4) study how a platform should set commissions/subscriptions in two-sided markets by taking into account network externalities. This framework is later extended by (5) to multi-sided markets. In these papers, the payoff of each agent is given by an exogenously specified function of the number of participants in her own group as well as the other groups. By contrast, in our setting, buyers and sellers trade at endogenously determined prices that are influenced by the platform's choice of commissions/subscriptions. Hence, the network externalities do not admit a closed-form expression in terms of the number of traders who participate in the market. Moreover, the number of participants in different groups (types) are related through nontrivial equilibrium constraints. This has two important implications. First, the optimal commissions/subscriptions no longer admit a direct characterization in terms of first-order optimality conditions in the revenue optimization problem. Second, our equilibrium conditions add a matching element to the existing models of revenue maximization in platforms, thereby contributing to one of the main future research directions suggested by (5). In our setting, we establish that the problem of finding the revenue-maximizing commissions/subscriptions can be formulated in terms of the marginal traders of each buyer/seller type, which is consistent with a similar observation by (5). In contrast to the previous literature, we provide a tractable convex optimization formulation for obtaining the optimal commissions/subscriptions. Furthermore, our work explicitly considers a compatibility network, and sheds light on the dependence of the platform's revenues and the surplus of market participants on the network structure. In addition, we contribute to this literature by studying the limitations and scope of different commission/subscription schemes in networked markets.

Our paper closely relates to the models of buyer-seller networks; see, e.g., (6), (7), (8), (9), (10). In these models, each node of an underlying (bipartite) network corresponds to a trader, and the edges of the network encode which agents can trade with which other agents. A recent and growing literature has explored variants of these models in order to study intermediation and bargaining in networked systems (e.g., (11), (12), (13)), competition in networked Cournot markets (e.g., (14), (15), (16), (17)), inefficiencies due to barriers to trade in networked markets (e.g. (18), (19), (20)). In a recent paper, (21) provides a thorough review of this literature. Our paper complements these works by exploring how a platform can influence the trading outcome by appropriately designing commissions/subscriptions in a trading network. The recent literature has also explored other controls that platforms can use to improve their operations. For instance, (22) study how the platforms should control which sellers and buyers are visible to each other, and provides algorithms for the solutions of the induced decision problems. By contrast, in this paper we control only the commissions/subscriptions, and doing so leads to very different decision problems for the platform.

Our work is also related to the burgeoning literature in operations management that studies service platforms. A branch of this literature focuses on decentralized markets, and uses control levers other than pricing to influence the market outcomes (e.g., (23, 24, 25)). In our work, we also focus on a decentralized market but, in contrast to the aforementioned papers, we use commissions/subscriptions to study the platform's revenue maximization problem. (26) explore the design of commissions in a similar manner to our work, albeit in a setting with no underlying compatibility network – which plays a key role in our analysis and results. (27) study the performance of commission contracts for an on-demand platform. In their model, the platform chooses a commission contract that determines the wages that the sellers collect as a function of the prices that the customers pay. They show that there exists a linear contract that achieves 75% of the optimal profit. Among other application areas, this literature has also made a substantial impact on the operations of ride-sharing platforms (e.g., (28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38)). In our setting a key difference is that the transaction prices are determined endogenously, whereas this literature mainly assumes that these quantities are determined by the platform. Finally, (39) consider a dynamic setting where buyers and suppliers are also divided into types, and the value of the match between two agents depends on their types. In their setting, the platform has full control over matches, and its objective is to maximize welfare. By contrast, we study a static setting where matches occur in a decentralized fashion, and the platform seeks to maximize its revenues by optimally designing commissions/subscriptions.

1.3 Model and Preliminary Results

We consider a platform that provides a marketplace for buyers and sellers to trade with each other. Buyers and sellers are divided into finitely many types, denoted by $\mathcal{B} = \{1, \ldots, m\}$ and $\mathcal{S} = \{1, \ldots, n\}$, respectively. To capture taste differences or other trade frictions, we assume that not all types of buyers and sellers are compatible. Compatibility between types is represented using a *compatibility network*, i.e., an undirected bipartite graph $G(\mathcal{B} \cup \mathcal{S}, E)$, where the edge set E defines the potential trading opportunities between sellers and buyers. Without loss of generality, we consider networks in which each node has degree at least one. We denote the set of neighbors of type $i \in \mathcal{B} \cup \mathcal{S}$ in $G(\mathcal{B} \cup \mathcal{S}, E)$ by $N_E(i)$. Similarly, we denote by $N_E(X)$ the set of all neighbors of types in set $X \subseteq \mathcal{B} \cup \mathcal{S}$ that do not belong to X, i.e., $N_E(X) := \{j : (i,j) \in E, i \in X, j \notin X\}$.

Buyers/sellers are infinitesimal, and we denote the total mass of buyers of type $j \in \mathcal{B}$ by $b_j > 0$, and the total mass of sellers of type $i \in \mathcal{S}$ by $s_i > 0$. Each seller supplies an (infinitesimal) unit amount of an indivisible service or product and, similarly, each buyer seeks to purchase the same unit amount from a compatible seller. Thus, each buyer can transact with at most one seller, and vice versa.

The value a buyer of type j derives from transacting with a compatible seller of type i consists of two terms. The first term captures her idiosyncratic value for transacting with *any* compatible seller. We assume that buyers of the same type are heterogeneous in terms of this value component, and the cumulative distribution function of the idiosyncratic term is given by $F_{b_j} : [0, \bar{v}_{b_j}] \to [0,1]$ for type $j \in \mathcal{B}$ buyers. Here the upper bound $\bar{v}_{b_j}$ on the support is such that $\bar{v}_{b_j} \in \mathbb{R}_+ \cup \{\infty\}$, where with some abuse of notation $\bar{v}_{b_j} = \infty$

captures distributions with unbounded support. The second term captures the possible vertical differentiation among the sellers; i.e., some seller types may have higher quality, or the buyers may find their (service or product) offerings more attractive. In order to model vertical differentiation, we assume that all buyers (of all types) incur the same disutility from transacting with lower-quality sellers, and introduce parameters $\boldsymbol{c} = \{c_i\}_{i \in \mathcal{S}} \in \mathbb{R}_+^n$ to capture the disutility from transacting with different seller types. More precisely, using these parameters, we represent the value of a type-j buyer from transacting with a compatible type-i seller by $v_b - c_i$, where v_b is the idiosyncratic payoff term for the buyer that is distributed according to F_{b_j}.

We assume that sellers do not have preferences over different types of buyers with whom they are compatible. Thus, a seller's (reservation) value for providing service consists only of her idiosyncratic term, which is the same for *any* compatible buyer. We assume that the values of type-i sellers are distributed according to $F_{s_i} : [0, \bar{v}_{s_i}] \to [0, 1]$, where $\bar{v}_{s_i} \in \mathbb{R}_+ \cup \{\infty\}$.

We impose the following assumption on the value distributions $F_{s_i}(\cdot)$ and $F_{b_j}(\cdot)$, which we keep throughout the paper.

Assumption 1. *The value distributions are nonatomic. Furthermore, we assume that $F_{s_i}(v)$ and $F_{b_j}(v)$ are continuously differentiable and* strictly *increasing in $v \in (0, \bar{v}_{s_i})$ and $v \in (0, \bar{v}_{b_j})$ for all $i \in \mathcal{S}$ and $j \in \mathcal{B}$. Finally, we assume that (reservation) values have bounded means, i.e., $\int_0^{\bar{v}_{s_i}} 1 - F_{s_i}(x)dx < \infty, \int_0^{\bar{v}_{b_j}} 1 - F_{b_j}(x)dx < \infty$, for all $i \in \mathcal{S}, j \in \mathcal{B}$.*

Observe that under this assumption, functions F_{b_j} and F_{s_i} are invertible. For every $j \in \mathcal{B}$ and $i \in \mathcal{S}$, let $F_{b_j}^{-1} : [0, 1] \to [0, \bar{v}_{b_j}]$ and $F_{s_i}^{-1} : [0, 1] \to [0, \bar{v}_{s_i}]$ denote the corresponding inverse functions, i.e.,

$$F_{b_j}^{-1}(F_{b_j}(x)) = x \text{ for } x \in [0, \bar{v}_{b_j}], \text{ and } F_{s_i}^{-1}(F_{s_i}(x)) = x \text{ for } x \in [0, \bar{v}_{s_i}]. \tag{1.1}$$

To state our analytical results more conveniently, we extend the domains of the value distributions to $\mathbb{R}$: for every $j \in \mathcal{B}$ we let $F_{b_j}(v) = 1$ for $v \geq \bar{v}_{b_j}$ and $F_{b_j}(v) = 0$ for $v \leq 0$, and similarly for sellers. (Note that the ranges of the inverse functions $F_{b_j}^{-1}$ and $F_{s_i}^{-1}$ are respectively restricted to $[0, \bar{v}_{b_j}]$ and $[0, \bar{v}_{s_i}]$, and hence these functions are well defined despite the domain extension of F_{b_j} and F_{s_i}.)

The platform can charge fees to buyers and sellers for facilitating transactions. In particular, we assume that the platform chooses *commission* rates (a percentage of the total transaction price) and *subscription* fees (lump-sum transfers to access the market, which are independent of the transaction amount). We assume that these commission rates and subscription fees are identical for all agents with the same type, but we allow them to be different across types. Formally, given a trading network $G(\mathcal{B} \cup \mathcal{S}, E)$, we denote by $(\boldsymbol{\gamma}, \boldsymbol{\mu})$ with $\boldsymbol{\gamma}, \boldsymbol{\mu} \in \mathbb{R}^{|\mathcal{S}|+|\mathcal{B}|}$ the platform's *commission and subscription vectors*, where γ_i^s (γ_j^b) represents the commission rate and μ_i^s (μ_j^b) represents the subscription fee charged to type-i sellers (type-j buyers), respectively. We focus on nonnegative commissions/subscriptions, and denote the set of feasible commission rates by $\Gamma = \{\boldsymbol{\gamma} : \gamma_i^s \in [0,1], \gamma_j^b \in [0, \infty),\ \forall i \in \mathcal{S}, \forall j \in \mathcal{B}\}$, and the set of feasible subscription fees by $\mathcal{U} = \{\boldsymbol{\mu} : \mu_i^s, \mu_j^b \in [0, \infty),\ \forall i \in \mathcal{S}, \forall j \in \mathcal{B}\}$.

To illustrate the effect of these commissions/subscriptions on agents' utilities, suppose that a type-i seller offers her product/service at price p, and that a type-j buyer transacts with her. The buyer makes a payment of $p(1 + \gamma_j^b)$ to the platform for this transaction, and the seller receives $p(1 - \gamma_i^s)$. In addition, to transact through the platform the buyer (seller) makes a lump-sum transfer of μ_j^b (μ_i^s) to the platform. If the buyer has value v_b and the seller has (reservation) value v_s, their utilities as a result of this transaction are $v_b - p(1 + \gamma_j^b) - \mu_j^b - c_i$ and $p(1 - \gamma_i^s) - \mu_i^s - v_s$, respectively.

The buyers and sellers can choose not to participate in the platform, in which case their utility is normalized to zero. We assume that all buyers and sellers are utility-maximizing; i.e., given commissions/subscriptions and prices, the trade of any agent maximizes her utility. Note that utility-maximizing buyers and sellers trade only if doing so results in nonnegative utility.

In contrast to most of the recent literature on two-sided markets that either suppresses the role of prices or views prices as a decision of the platform, a novel feature of our model is that we allow prices to be formed endogenously. This is a key feature of many real-world platforms such as Airbnb, Upwork, eBay, etc., where the transaction prices are not dictated by the platform. To capture the endogenous nature of the prices formally, we focus on the competitive equilibria of the trading network, which we define next.

Definition 1. *Given a commission-subscription pair* $(\boldsymbol{\gamma}, \boldsymbol{\mu}) \in \Gamma \times \mathcal{U}$ *chosen by the platform, a* competitive equilibrium $(\boldsymbol{p}, \boldsymbol{x}, \boldsymbol{q}^s, \boldsymbol{q}^b)$ *is a tuple that consists of a* price vector $\boldsymbol{p} \in \mathbb{R}_+^{|\mathcal{S}|}$ *(where* p_i *denotes the price of the products/services offered by sellers of type* $i \in \mathcal{S}$*),* supply and demand vectors $\boldsymbol{q}^s \in \mathbb{R}^{|\mathcal{S}|}, \boldsymbol{q}^b \in \mathbb{R}^{|\mathcal{B}|}$ *(where* q_i^s/q_j^b *represent the total quantities supplied/demanded by type-i sellers/type-j buyers), and a* flow vector $\boldsymbol{x} \in \mathbb{R}^{|E|}$ *(where* x_{ij} *indicates the aggregate amount of products/services supplied by type-i sellers to type-j buyers), and that satisfies the following constraints:*

$$q_i^s = s_i F_{s_i}\Big((1-\gamma_i^s)p_i - \mu_i^s\Big), \quad \forall i \in \mathcal{S}, \tag{1.2a}$$

$$q_j^b = b_j\Big[1 - F_{b_j}\Big(\min_{i':(i',j)\in E}\{(1+\gamma_j^b)p_{i'} + c_{i'}\} + \mu_j^b\Big)\Big], \quad \forall j \in \mathcal{B}, \tag{1.2b}$$

$$q_i^s = \sum_{j':(i,j')\in E} x_{ij'}, \quad q_j^b = \sum_{i':(i',j)\in E} x_{i'j}, \quad \forall i \in \mathcal{S}, j \in \mathcal{B}, \tag{1.2c}$$

$$x_{ij} \geq 0, \quad \forall (i,j) \in E; \quad x_{ij} = 0, \quad \forall i \notin \underset{i':(i',j)\in E}{\arg\min}\,\{(1+\gamma_j^b)p_{i'} + c_{i'}\},\ j \in \mathcal{B}. \tag{1.2d}$$

We denote by $\mathcal{X}(\boldsymbol{\gamma}, \boldsymbol{\mu})$ *the set of competitive equilibria, i.e.,*

$$\mathcal{X}(\boldsymbol{\gamma}, \boldsymbol{\mu}) := \{(\boldsymbol{p}, \boldsymbol{x}, \boldsymbol{q}^s, \boldsymbol{q}^b) : \text{ Conditions (1.2a)–(1.2d) are satisfied}\}.$$

Condition (1.2a) states that, given price p_i, all type-i sellers who have nonnegative surplus from trading (i.e., who have values of at most $(1-\gamma_i^s)p_i - \mu_i^s$) will participate and transact in the market. Thus, the total mass of type-i sellers who participate and transact in the market is $q_i^s = s_i F_{s_i}\left((1-\gamma_i^s)p_i - \mu_i^s\right)$. Similarly, all buyers of type $j \in \mathcal{B}$ with nonnegative surplus from trading will participate in the market. It can be readily seen that a buyer receives the highest surplus from trading with a compatible seller of type i for which $(1+\gamma_j^b)p_i + c_i$ is minimized. Thus, it follows that buyers with values of at least $\min_{i':(i',j)\in E}\{(1+\gamma_j^b)p_{i'} + c_{i'}\} + \mu_j^b$ find it optimal to participate in the market. Hence, consistent with Condition (1.2b), the total mass of type-j buyers who participate and transact in the market is given by $q_j^b = b_j\big[1-F_{b_j}\big(\min_{i':(i',j)\in E}\{(1+\gamma_j^b)p_{i'}+c_{i'}\}+\mu_j^b\big)\big]$. Condition (1.2c) is the market-clearing condition: in equilibrium, there should be a feasible allocation of goods such that each seller who is willing to sell her product is able to do so, and each buyer who demands a product is able to buy it from compatible sellers with the lowest price. Finally, Condition (1.2d) ensures that the allocation of goods is consistent with buyers' preferences; i.e., buyers transact only with sellers who maximize their surplus.

Observe that here we abstract away information frictions and implicitly assume that buyers and sellers have full information about the prices available on the platform. In addition, we restrict attention to outcomes where all sellers of the same type offer the same price. Note that this is always the case in equilibrium: if sellers of the same type were to offer different prices, then all compatible buyers would demand goods from sellers who offer the lowest price. Thus, the sellers with higher prices would not find a buyer, and the market would not clear.

Next, we establish that, for any given commissions/subscriptions, an equilibrium exists and is essentially unique. That is, the mass of buyers (sellers) of each type transacting in any equilibrium is identical, and the equilibrium price of any seller type that is involved in some transactions is also identical, although there might exist several feasible flows that lead to the same outcome. We leverage this uniqueness result in Section 1.4, when studying the platform's revenue maximization problem.

Proposition 1. *For any* $(\boldsymbol{\gamma},\boldsymbol{\mu}) \in \Gamma \times \mathcal{U}$*, there exists a vector* $(\boldsymbol{p},\boldsymbol{x},\boldsymbol{q}^s,\boldsymbol{q}^b)$ *such that* $(\boldsymbol{p},\boldsymbol{x},\boldsymbol{q}^s,\boldsymbol{q}^b) \in \mathcal{X}(\boldsymbol{\gamma},\boldsymbol{\mu})$*. Furthermore, given* $(\boldsymbol{\gamma},\boldsymbol{\mu})$*, all competitive equilibria share the same supply-demand vector* $(\boldsymbol{q}^s,\boldsymbol{q}^b)$*. In addition, each seller type that transacts nonzero quantities* *in these equilibria always offers the same prices; i.e., the vector* $(p_i)_{i:q_i^s>0}$ *is the same in all equilibria.*

In Appendix 1.9, we show that for given commissions/subscriptions, an equilibrium can be constructed by solving a convex optimization problem. This result is analogous to the classic competitive equilibrium models where the allocation can be solved via a convex optimization problem and the price vector is determined by the corresponding dual variables; see, e.g., (6).

How should the platform choose commissions/subscriptions to maximize its revenues? We proceed by formulating the platform's revenue maximization problem:

$$\begin{aligned} V_{opt} \;=\; \max_{(\boldsymbol{\gamma},\boldsymbol{\mu},\boldsymbol{p},\boldsymbol{x},\boldsymbol{q}^s,\boldsymbol{q}^b)} \quad & \sum_{i,j:(i,j)\in E} (\gamma_i^s + \gamma_j^b) p_i x_{ij} + \sum_{i,j:(i,j)\in E} (\mu_i^s + \mu_j^b) x_{ij} && \text{(1.3a)} \\ s.t. \quad & (\boldsymbol{p},\boldsymbol{x},\boldsymbol{q}^s,\boldsymbol{q}^b) \in \mathcal{X}(\boldsymbol{\gamma},\boldsymbol{\mu}), && \text{(1.3b)} \\ & (\boldsymbol{\gamma},\boldsymbol{\mu}) \in \Gamma \times \mathcal{U}. && \text{(1.3c)} \end{aligned}$$

Consider a feasible solution $(\boldsymbol{\gamma},\boldsymbol{\mu},\boldsymbol{p},\boldsymbol{x},\boldsymbol{q}^s,\boldsymbol{q}^b)$ to this problem. Constraint (1.3c) ensures that $(\boldsymbol{\gamma},\boldsymbol{\mu})$ correspond to feasible commission-subscription vectors. Constraint (1.3b) implies that the tuple $(\boldsymbol{p},\boldsymbol{x},\boldsymbol{q}^s,\boldsymbol{q}^b)$ is an equilibrium under the vector of commissions and subscriptions $(\boldsymbol{\gamma},\boldsymbol{\mu})$. In the objective, the first term corresponds to the revenue obtained through the commissions in the aforementioned equilibrium, whereas the second term corresponds to the revenue due to subscription fees. In this problem, the objective is not concave in the decision variables and the feasible set is not convex.

Note that, in principle for given $(\boldsymbol{\gamma},\boldsymbol{\mu})$, the equilibria in $\mathcal{X}(\boldsymbol{\gamma},\boldsymbol{\mu})$ could lead to different revenues for the platform. We conclude this section with an immediate corollary of Proposition 1, which establishes that this is never the case.

Corollary 1. *For given* $(\boldsymbol{\gamma},\boldsymbol{\mu}) \in \Gamma \times \mathcal{U}$*, all* $(\boldsymbol{p},\boldsymbol{x},\boldsymbol{q}^s,\boldsymbol{q}^b) \in \mathcal{X}(\boldsymbol{\gamma},\boldsymbol{\mu})$ *yield the same revenue for the platform.*

Hereafter, we denote by $V(\boldsymbol{\gamma}, \boldsymbol{\mu})$ the platform's revenues achieved under commission-subscription pair[2] $(\boldsymbol{\gamma}, \boldsymbol{\mu})$. We refer to $(\boldsymbol{\gamma}, \boldsymbol{\mu})$ such that $V(\boldsymbol{\gamma}, \boldsymbol{\mu}) = V_{opt}$ as an optimal commission-subscription vector.

1.4 Revenue Maximization via Commissions and Subscriptions

The nonconvexity of the platform's revenue maximization problem poses a potential challenge to finding the optimal commissions/subscriptions. In this section, we first establish that despite this nonconvexity, under mild assumptions, the optimal commissions/subscriptions can still be obtained in a tractable way by solving a convex relaxation of the platform's revenue maximization problem (Section 1.4.1). We then explore the impact of the network structure on the surplus of different buyer/seller types and on the platform's revenues (Section 1.5).

1.4.1 Finding the Optimal Commissions and Subscriptions

Given an equilibrium $(\boldsymbol{p}, \boldsymbol{x}, \boldsymbol{q}^s, \boldsymbol{q}^b) \in \mathcal{X}(\boldsymbol{\gamma}, \boldsymbol{\mu})$, we refer to the trading buyer (seller) of type $j \in \mathcal{B}$ ($i \in \mathcal{S}$) with the lowest (highest) valuation as the *marginal buyer* (*seller*) of this type. It can be seen that the valuation of the type-j marginal buyer is given by

$$v_{b_j}^m := F_{b_j}^{-1}(1 - q_j^b/b_j), \tag{1.4}$$

and the valuation of the type-i marginal seller by

$$v_{s_i}^m := F_{s_i}^{-1}(q_i^s/s_i). \tag{1.5}$$

In equilibrium, marginal agents must have nonnegative surplus; otherwise they would prefer not to trade, thus violating the equilibrium conditions. Suppose that $c_i = 0$ for all $i \in \mathcal{S}$. Then, the nonnegativity of marginal type-j buyer's surplus implies that the net transfer from

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[2] It can be shown that $V(\boldsymbol{\gamma}, \boldsymbol{\mu})$ is not concave in its arguments, either. See Appendix 1.13 of (40) for an example.

her to the rest of the market (that is, to the sellers and the platform) must be at most equal to her value $v^m_{b_j}$. Similarly, the net transfer from the rest of the market to the marginal type-i seller must be at least $v^m_{s_i}$. Since all trading agents of the same type make the same payment, it follows that, in equilibrium, all trading type-j buyers pay at most $v^m_{b_j}$, and all trading type-i sellers receive at least $v^m_{s_i}$. These observations together with (1.4) and (1.5) imply that the platform's revenue is at most $\sum_{j\in\mathcal{B}} F^{-1}_{b_j}\left(1-\frac{q^b_j}{b_j}\right)q^b_j - \sum_{i\in\mathcal{S}} F^{-1}_{s_i}\left(\frac{q^s_i}{s_i}\right)q^s_i$. When $c_i > 0$, the same reasoning still applies but any buyer who transacts with a type-i seller pays c_i units less, as each such buyer experiences a disutility of c_i units from such a transaction. These observations suggest that in equilibrium $(\boldsymbol{p}, \boldsymbol{x}, \boldsymbol{q}^s, \boldsymbol{q}^b) \in \mathcal{X}(\boldsymbol{\gamma}, \boldsymbol{\mu})$ the revenue of the platform is upper bounded by[3,4]

$$h(\boldsymbol{q}^s, \boldsymbol{q}^b) := \sum_{j\in\mathcal{B}} F^{-1}_{b_j}\left(1-\frac{q^b_j}{b_j}\right)q^b_j - \sum_{i\in\mathcal{S}} F^{-1}_{s_i}\left(\frac{q^s_i}{s_i}\right)q^s_i - \sum_{i\in\mathcal{S}} c_i q^s_i. \tag{1.6}$$

We formalize this discussion (for arbitrary $\{c_i \geq 0\}_{i\in\mathcal{S}}$) in the proof of Theorem 1.

We provide our characterization of the optimal commissions/subscriptions under the following additional assumption on the value distributions, which we impose for the remainder of the paper.

Assumption 2. *The functions* $uF^{-1}_{b_j}(1-u)$ *and* $-uF^{-1}_{s_i}(u)$ *are strictly concave in* $u \in [0,1]$ *for all* $i \in \mathcal{S}$ *and for all* $j \in \mathcal{B}$.

Focusing momentarily on the buyers, it can be seen that the concavity of $uF^{-1}_{b_j}(1-u)$ is equivalent to the regularity of buyers' value distributions. Regularity is a standard assumption in mechanism design that renders optimal mechanism design problems tractable. Since

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[3] In fact, in general, the marginal sellers and buyers have zero surplus, and $h(\boldsymbol{q}^s, \boldsymbol{q}^b)$ captures the revenues of the platform. The only exception is when the value distributions are bounded and, for some $i \in \mathcal{S}$, the equilibrium price p_i for type-i sellers is higher than their maximum (reservation) value, which induces positive surplus for them under certain commissions/subscriptions. However, such outcomes are not revenue-maximizing.

[4] Without loss of generality, in case of $\bar{v}_{b_j} = \infty$, we define $F^{-1}_{b_j}(1 - q^b_j/b_j)q^b_j = 0$ for $q^b_j = 0$. In Lemma 4(ii) of (40), we prove that function $F^{-1}_{b_j}(1 - q^b_j/b_j)q^b_j$ is continuous at $q^b_j = 0$ under Assumptions 1 and 2.

in our setting the market has two sides, in addition to the regularity of buyers' valuations, we require sellers' (reservation) values to satisfy a closely related condition. Assumption 2 holds for a broad class of distributions, including uniform, exponential, generalized Pareto, Weibull, and (truncated) normal distributions.

Note that, under Assumption 2, $h(\boldsymbol{q}^s, \boldsymbol{q}^b)$ is a concave function of its argument. Exploiting this concavity, we next provide a convex relaxation of the revenue maximization problem in (1.3):

$$
\begin{aligned}
\tilde{V}_{opt} \;=\; \max_{\boldsymbol{x},\boldsymbol{q}^s,\boldsymbol{q}^b} \quad & \sum_{j\in\mathcal{B}} F_{b_j}^{-1}\left(1-\frac{q_j^b}{b_j}\right) q_j^b - \sum_{i\in\mathcal{S}} F_{s_i}^{-1}\left(\frac{q_i^s}{s_i}\right) q_i^s - \sum_{i\in\mathcal{S}} c_i q_i^s && \text{(1.7a)}\\
\text{s.t.} \quad & \sum_{i:(i,j)\in E} x_{ij} - q_j^b \;=\; 0, \quad \forall j \in \mathcal{B}, && \text{(1.7b)}\\
& q_i^s - \sum_{j:(i,j)\in E} x_{ij} \;=\; 0, \quad \forall i \in \mathcal{S}, && \text{(1.7c)}\\
& q_j^b \le b_j, \quad \forall j \in \mathcal{B}, && \text{(1.7d)}\\
& q_i^s \le s_i, \quad \forall i \in \mathcal{S}, && \text{(1.7e)}\\
& x_{ij} \;\ge\; 0, \quad \forall (i,j) \in E. && \text{(1.7f)}
\end{aligned}
$$

This relaxation is obtained by first replacing the objective of (1.3) with the upper bound $h(\boldsymbol{q}^s, \boldsymbol{q}^b)$ on revenues, and then relaxing constraint (1.3c)—$(\boldsymbol{\gamma}, \boldsymbol{\mu}) \in \Gamma\times\mathcal{U}$—as well as some of the equilibrium constraints in (1.3b). In particular, this formulation imposes the equilibrium flow constraint (1.2c) (plus nonnegativity), ensures that the equilibrium supply (demand) of each type is less than the mass of sellers (buyers) of the relevant type (as implied by (1.2a) and (1.2b)), and relaxes the remaining equilibrium constraints in (1.2). Given Assumption 2, problem (1.7) is a tractable convex optimization problem that can be interpreted as a min-cost network flow problem with convex edge costs captured in terms of $-F_{b_j}^{-1}(1-q_j^b/b_j)q_j^b$, $F_{s_i}^{-1}(q_i^s/s_i)q_i^s$, and $c_i q_i^s$.

Observe that by Assumption 2 the objective function is strictly concave in $(\boldsymbol{q^s}, \boldsymbol{q^b})$, and hence all optimal solutions to (1.7) share the same supply-demand vector $(\bar{\boldsymbol{q}}^{\boldsymbol{s}}, \bar{\boldsymbol{q}}^{\boldsymbol{b}})$. Also, note that after replacing the objective and relaxing the constraints, the resulting problem maximizes revenues by searching over supply-demand vectors $(\boldsymbol{q}^s, \boldsymbol{q}^b)$ and a flow vector $\boldsymbol{x}$, and neither the prices $\boldsymbol{p}$ nor the commissions/subscriptions $(\boldsymbol{\gamma}, \boldsymbol{\mu})$ appear in the new formulation. That is, the formulation does not guarantee that a feasible solution $(\boldsymbol{x}, \boldsymbol{q}^s, \boldsymbol{q}^b)$ coincides with the equilibrium flow, supply, and demand vectors under some commissions/subscriptions. We say that the tuple $(\boldsymbol{q}^{\boldsymbol{s}}, \boldsymbol{q}^{\boldsymbol{b}})$ is *implementable*, if there exist commissions/subscriptions $(\boldsymbol{\gamma}, \boldsymbol{\mu})$ and a price-flow vector $(\boldsymbol{p}, \boldsymbol{x})$ such that $(\boldsymbol{p}, \boldsymbol{x}, \boldsymbol{q}^{\boldsymbol{s}}, \boldsymbol{q}^{\boldsymbol{b}}) \in \mathcal{X}(\boldsymbol{\gamma}, \boldsymbol{\mu})$. While a priori it is not clear whether $(\bar{\boldsymbol{q}}^{\boldsymbol{s}}, \bar{\boldsymbol{q}}^{\boldsymbol{b}})$ associated with an optimal solution of problem (1.7) is implementable, our next result establishes that this is indeed the case and that our relaxation is tight.

Theorem 1. *Let* $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{q}}^{\boldsymbol{s}}, \bar{\boldsymbol{q}}^{\boldsymbol{b}})$ *denote an optimal solution to* (1.7). *Then:*

(i) *Optimization problem* (1.7) *is a tight relaxation of problem* (1.3)*, i.e., optimal solutions to* (1.3) *exist and they achieve an objective value of* $V_{opt} = \tilde{V}_{opt}$. *Moreover, any optimal solution to* (1.3) *must have supply-demand vectors equal to* $(\bar{\boldsymbol{q}}^{\boldsymbol{s}}, \bar{\boldsymbol{q}}^{\boldsymbol{b}})$.

(ii) *The optimal commission-subscription vector is not unique. Fix a commission-subscription vector* $(\boldsymbol{\gamma}, \boldsymbol{\mu}) \in \Gamma \times \mathcal{U}$ *with* $\gamma_i^s < 1$ *for all* $i \in \mathcal{S}$*, and let* $\tilde{\gamma}_i^s := \frac{1}{1-\gamma_i^s}$, $\tilde{\mu}_i^s := \frac{\mu_i^s}{1-\gamma_i^s}$ *for all* $i \in \mathcal{S}$*, and* $\tilde{\gamma}_j^b := \frac{1}{1+\gamma_j^b}$, $\tilde{\mu}_j^b := \frac{\mu_j^b}{1+\gamma_j^b}$ *for all* $j \in \mathcal{B}$. *Then* $(\boldsymbol{\gamma}, \boldsymbol{\mu})$ *is an optimal commission-subscription pair if and only if there exists a price vector* $\boldsymbol{p} \in \mathbb{R}_+^{|\mathcal{S}|}$ *that* *satisfies the following system of inequalities:*

$$p_i - \tilde{\mu}_i^s - F_{s_i}^{-1}\left(\frac{\bar{q}_i^s}{s_i}\right)\tilde{\gamma}_i^s \;=\; 0, \quad \forall i : \bar{q}_i^s > 0, \tag{1.8a}$$

$$p_i - \tilde{\mu}_i^s \;\leq\; 0, \quad \forall i : \bar{q}_i^s = 0, \tag{1.8b}$$

$$p_i + \tilde{\mu}_j^b + c_i\tilde{\gamma}_j^b - F_{b_j}^{-1}\left(1 - \frac{\bar{q}_j^b}{b_j}\right)\tilde{\gamma}_j^b \;=\; 0, \quad \forall (i,j) : \bar{x}_{ij} > 0, \tag{1.8c}$$

$$p_i + \tilde{\mu}_j^b + c_i\tilde{\gamma}_j^b - F_{b_j}^{-1}\left(1 - \frac{\bar{q}_j^b}{b_j}\right)\tilde{\gamma}_j^b \;\geq\; 0, \quad \forall (i,j) : \bar{x}_{ij} = 0, \tag{1.8d}$$

$$\tilde{\gamma}_i^s \;\geq\; 1, \quad \tilde{\mu}_i^s \;\geq\; 0, \quad p_i \;\geq\; 0, \quad \forall i \in \mathcal{S}, \tag{1.8e}$$

$$\tilde{\gamma}_j^b \;\geq\; 0, \quad \tilde{\gamma}_j^b \;\leq\; 1, \quad \tilde{\mu}_j^b \;\geq\; 0, \quad \forall j \in \mathcal{B}. \tag{1.8f}$$

In the proof of the first part of the theorem, we establish that the upper bound on the revenue provided in (1.6) gives precisely the revenues of the platform for $(\boldsymbol{q^s}, \boldsymbol{q^b}) = (\bar{\boldsymbol{q}}^{\boldsymbol{s}}, \bar{\boldsymbol{q}}^{\boldsymbol{b}})$. This in turn implies that in the special case where $c_i = 0$ for all $i \in \mathcal{S}$, under the optimal commissions/subscriptions all trading type-j buyers (type-i sellers) pay (receive) exactly $v_{b_j}^m$ ($v_{s_i}^m$), and the corresponding marginal agents have zero surplus. Moreover, the theorem provides a systematic approach to constructing optimal commissions/subscriptions. In particular, first an optimal solution to (1.7) can be obtained, and then a feasible solution to the system of inequalities in (1.8) can be constructed (e.g., by solving a linear program). Using the corresponding $\tilde{\boldsymbol{\gamma}}$ and $\tilde{\boldsymbol{\mu}}$, optimal commissions/subscriptions are obtained after a simple transformation: $\gamma_i^s = 1 - 1/\tilde{\gamma}_i^s$, $\gamma_j^b = 1/\tilde{\gamma}_j^b - 1$, $\mu_i^s = \tilde{\mu}_i^s(1-\gamma_i^s)$, and $\mu_j^b = \tilde{\mu}_j^b(1+\gamma_j^b)$.

Given any optimal solution $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{q}}^{\boldsymbol{s}}, \bar{\boldsymbol{q}}^{\boldsymbol{b}})$ to problem (1.7), Theorem 1 implies that vector $(\bar{\boldsymbol{q}}^{\boldsymbol{s}}, \bar{\boldsymbol{q}}^{\boldsymbol{b}})$ is unique. Consider a corresponding dual optimal vector $(\boldsymbol{\theta}^b, \boldsymbol{\theta}^s, \boldsymbol{\eta}^b, \boldsymbol{\eta}^s, \boldsymbol{\pi})$ to constraints (1.7b)–(1.7f) that maximizes $\sum_{j\in\mathcal{B}}[sgn(\bar{q}_j^b) - sgn(b_j - \bar{q}_j^b)](\theta_j^b + \eta_j^b) + \sum_{i\in\mathcal{S}}[-sgn(\bar{q}_i^s) + sgn(s_i - \bar{q}_i^s)](\theta_i^s - \eta_i^s)$, where $sgn(\cdot)$ is the sign function. It is worth noting that such a vector $(\boldsymbol{\theta}^b, \boldsymbol{\theta}^s, \boldsymbol{\eta}^b, \boldsymbol{\eta}^s, \boldsymbol{\pi})$ is unique (see Lemma 5 in Section 1.12.1 of (40)). We proceed by explicitly constructing some optimal commission-subscription pairs using this dual multiplier.

Corollary 2. *Given the optimal dual vector* $\{\theta_j^b\}_j$ *and* $\{\theta_i^s\}_i$ *above, an optimal commission-subscription pair* $(\boldsymbol{\gamma}, \boldsymbol{\mu}) \in \Gamma \times \mathcal{U}$ *can be obtained by*

(i) Setting $\boldsymbol{\gamma} = 0$, *and* $\mu_i^s = \theta_i^s - F_{s_i}^{-1}\left(\frac{\bar{q}_i^s}{s_i}\right)$, $\mu_j^b = F_{b_j}^{-1}\left(1 - \frac{\bar{q}_j^b}{b_j}\right) - \theta_j^b$ *for all* $i \in \mathcal{S}$, $j \in \mathcal{B}$.

(ii) Setting $\boldsymbol{\mu} = 0$, *and* $\gamma_i^s = 1 - F_{s_i}^{-1}\left(\frac{\bar{q}_i^s}{s_i}\right)/\theta_i^s$, $\gamma_j^b = F_{b_j}^{-1}\left(1 - \frac{\bar{q}_j^b}{b_j}\right)/\theta_j^b - 1$ *for all* $i \in \mathcal{S}$, $j \in \mathcal{B}$, *provided that* $c_i = 0$ *for all* $i \in \mathcal{S}$.

Corollary 2 shows that the revenue-maximizing outcome can always be implemented with subscriptions only. Hence, the platform does not need to consider more complicated schemes that involve both commissions and subscriptions. On the other hand, it is not always possible to maximize revenues only with commissions (see Section 1.14.2 in (40) for a counterexample). That said, Corollary 2 establishes that, in the absence of vertical differentiation among the seller types, the revenues can be maximized by relying only on commissions as well.[5] Interestingly, in both cases using the primal and dual optimal solutions of (1.7) yields the optimal commissions/subscriptions.

We also notice that the optimal payment structure can be type-dependent; i.e., different agent types, even on the same side, are exposed to different commissions/subscriptions. It is not clear what type of revenue loss should be expected if attention is restricted to offering identical commissions/subscriptions to all participants on one side of the market. We revisit this point in Section 1.6.

Figure 1.1 illustrates the optimal commissions/subscriptions characterized in Corollary 2 for a simple network. In this example, the platform finds it optimal to target different types with different commissions/subscriptions. It can also be seen that depending on their network

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[5] We emphasize that this case just provides a sufficient condition, and in more general settings it is still possible to rely on commissions for revenue maximization. For instance, consider a network with 2 seller types and 2 buyer types with unit populations. Let the value distributions be $F_{s_i}(v) = F_{b_j}(v) = v$ for $v \in [0, 1]$, $j \in \{1, 2\}$. We generated 10000 problem instances where each c_i is independently drawn from $U[0, 1]$. We then used Theorem 1(ii) to test whether the optimal revenues can be constructed using only commissions. When the underlying network is a complete bipartite graph, we observe that in 84.2% of the instances optimal revenues can be achieved using only commissions. When one of the edges is missing, this percentage is even higher: 91.8%.

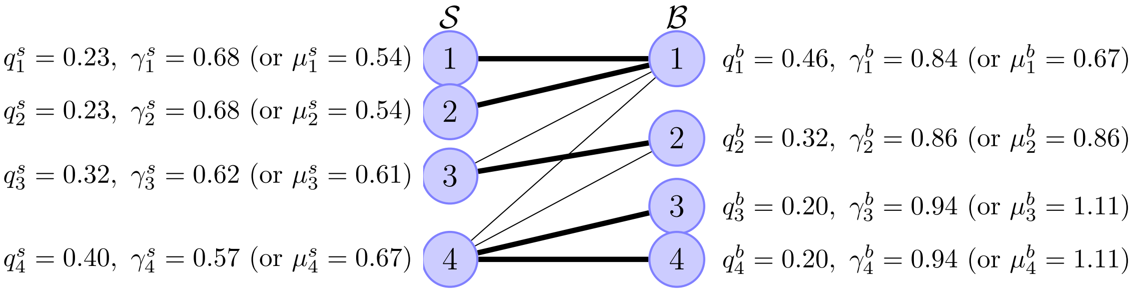

Figure 1.1: Consider a network with 4 buyer and 4 seller types, with value distributions $F_{s_i}(v) = F_{b_j}(v) = 1 - \exp(-v^2)$, where $i, j = 1, 2, 3, 4$ for all $v \in [0, \infty)$. Let $c_i = 0$ for all $i \in \mathcal{S}$. Denote the population vectors by $\boldsymbol{s} = (1, 1, 1, 1)$ and $\boldsymbol{b} = (2, 2, 2, 2)$. The revenue-maximizing commissions (or subscriptions) given in Corollary 2 as well as the corresponding mass of buyers/sellers of each type that trade are reported in the figure. The equilibrium prices for the sellers are $(p_1, p_2, p_3, p_4) = (0.8, 0.8, 0.99, 1.19)$. The flow associated with the highlighted edges is given by $x_{11} = x_{21} = 0.23$, $x_{32} = 0.32$, and $x_{43} = x_{44} = 0.20$. All other edges have zero flow (trade) under the optimal solution.

position some types may be involved in more trades than others, and not all compatible types of buyers/sellers trade.

Remark: The idea of finding the optimal transfers to the platform by formulating the platform's problem over allocations (i.e., the mass of different types of agents who participate in the market) appeared previously in (5). That paper, unlike ours, does not shed light on when these transfers can be tractably computed or offer a tractable approach to the construction of optimal transfers. Moreover, in our setting, due to market-clearing conditions the set of implementable allocations are further restricted through "flow constraints" (1.7b) and (1.7c), whereas no analogous restriction is present in (5). Finally, a key assumption in (5) is that the platform can choose a positive or negative transfer to any type of agent. This flexibility guarantees that, for any given allocation, there will exist transfers that implement this allocation, i.e., transfers such that the given allocation will indeed be the equilibrium allocation under those transfers. However, once the allowable commissions/subscriptions (and hence the transfers that can be charged by the platform) are restricted (e.g., to nonnegative values), it is no longer true that any allocation can be implemented, i.e., that given any allocation $(\boldsymbol{q^b}, \boldsymbol{q^s})$ one can find commissions/subscriptions $(\boldsymbol{\gamma}, \boldsymbol{\mu})$ such that $(\boldsymbol{q^b}, \boldsymbol{q^s})$ are the equilibrium demand/supply under $(\boldsymbol{\gamma}, \boldsymbol{\mu})$. (In fact, it can be shown that the set of implementable equilibrium supply-demand vectors constitutes a nonconvex subset of $\mathbb{R}^{|\mathcal{B}|+|\mathcal{S}|}$.) Theorem 1 implies that in order to find optimal commissions/subscriptions, the restriction to nonnegative transfers is without loss of optimality. This is because (1.7) remains a relaxation of (1.3) even if nonnegativity of transfers is not imposed in the latter problem. This observation also implies that, unlike in the literature on two-sided markets (see, e.g., (3)), in our setting the platform does not benefit from setting transfers below costs (which are assumed to be zero) on one side of the market.

1.5 Impact of the Network Structure

This section is devoted to understanding the impact of the compatibility network structure on the traders' surplus and on the platform's revenues, when the platform employs optimal commissions/subscriptions. In order to isolate the effect of the network structure, throughout this section we conduct our analysis under the following homogeneity assumption.

Assumption 3. *The value distributions are homogeneous on each side, i.e., $F_{s_i}(v) = F_s(v)$ and $F_{b_j}(v) = F_b(v)$ for all $v \in \mathbb{R}$, $j \in \mathcal{B}$, $i \in \mathcal{S}$, and the disutility vector is equal to zero, i.e., $c_i = 0$ for all*[6] *$i \in \mathcal{S}$.*

It is worth noting that we still allow for populations of different sizes for different types.

Agents' Surplus and Network Positions. We first focus on the impact of the compatibility network structure on the average surplus of the different buyer/seller types under revenue-optimal commissions and subscriptions. We start by characterizing the average equilibrium surplus of a trader type in terms of the valuation of its marginal agents. Recall that under a vector of optimal commissions and subscriptions, marginal buyers (sellers) have zero

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[6] The results of this section extend to settings with homogeneous taste parameters, i.e., $c_i = c$ for all $i \in \mathcal{S}$.

surplus in equilibrium (see the discussion after Theorem 1 for the case of $c_i = 0$ for all $i \in \mathcal{S}$). Moreover, all trading agents of the same type make the same payments to the platform as they purchase/sell the good at the same price and face the same commissions and subscriptions. In light of this, the equilibrium surplus of a buyer of type $j \in \mathcal{B}$ (seller of type $i \in \mathcal{S}$) with valuation v is given by $\max\{v - v^m_{b_j}, 0\}$, where $v^m_{b_j}$ ($v^m_{s_i}$) denotes the valuation of the marginal buyer of type j (seller of type i). Consequently, in equilibrium, the aggregate surplus of buyers of type $j \in \mathcal{B}$ can be given by $b_j \int_{v^m_{b_j}}^{\bar{v}_{b_j}} (v - v^m_{b_j}) dF_{b_j}(v)$, and that of sellers of type $i \in \mathcal{S}$ can be given by $s_i \int_0^{v^m_{s_i}} (v^m_{s_i} - v) dF_{s_i}(v)$. Finally, the average surplus of a type can be calculated by dividing its aggregate surplus by its population (e.g., $\int_{v^m_{b_j}}^{\bar{v}_{b_j}} (v - v^m_{b_j}) dF_b(v)$ for $j \in \mathcal{B}$). Notice that, by Theorem 1, all optimal commission/subscription schemes lead to the same $(\boldsymbol{q}^s, \boldsymbol{q}^b)$, which in turn implies that all optimal schemes induce the same marginal agents and thus the same surplus for any given agent.

It can be readily seen from the above discussion that if the valuation of the marginal buyer (seller) of a given type is large, then the corresponding average surplus is small (large). Therefore, in the rest of the subsection we focus on understanding the impact of the compatibility network structure on the valuations of the marginal agents of different types. It is worth noting that, while it is possible to efficiently compute the valuations of the marginal agents for a given problem instance by solving problem (1.7), a priori the impact of the network structure on the values of the marginal agents of different types is not clear. Our next result provides an answer to this question.

Theorem 2. *Suppose that Assumption 3 holds. Given a network $G(\mathcal{B} \cup \mathcal{S}, E)$ and a population profile $(\boldsymbol{s}, \boldsymbol{b})$, define $\mathcal{S}^{(0)} = \mathcal{S}$, $\mathcal{B}^{(0)} = \mathcal{B}$, and $E^{(0)} = E$. For $\tau = 1, 2, \ldots$, let $\mathcal{B}_\tau$ and $\mathcal{S}_\tau$ be iteratively defined as follows:*

$$\mathcal{B}_\tau = \operatorname*{arg\,min}_{B \subseteq \mathcal{B}^{(\tau-1)}} \left(\sum_{i \in N_{E^{(\tau-1)}}(B)} s_i \right) \Big/ \left(\sum_{j \in B} b_j \right) \qquad \textit{and} \qquad \mathcal{S}_\tau = N_{E^{(\tau-1)}}(\mathcal{B}_\tau),$$

where $\mathcal{B}^{(\tau)} = \mathcal{B}^{(\tau-1)} \backslash \mathcal{B}_\tau$, $\mathcal{S}^{(\tau)} = \mathcal{S}^{(\tau-1)} \backslash \mathcal{S}_\tau$, *and* $E^{(\tau)} = \Big\{(i,j) \in E : \; i \in \mathcal{S}^{(\tau)} \text{ and } j \in \mathcal{B}^{(\tau)}\Big\}$. *Here, the* arg min *operator stands for the largest set* B *that achieves the minimum, if there are multiple such sets.*[7] *If* $j_1 \in \mathcal{B}_{\tau_1}$, $j_2 \in \mathcal{B}_{\tau_2}$ *(similarly* $i_1 \in \mathcal{S}_{\tau_1}$, $i_2 \in \mathcal{S}_{\tau_2}$*) with* $\tau_1 \leq \tau_2$, *then the marginal agent of type* j_1 *(*i_1*) has a weakly higher valuation than the marginal agent of type* j_2 *(*i_2*) under optimal commissions/subscriptions.*[8]

The key quantity for ranking the valuations of the marginal agents of different types is the scarcity of sellers experienced by different types of buyers. In particular, to obtain our ranking, we first identify an induced subgraph that consists of a set of buyers and their neighboring sellers such that the ratio of the sellers' population to the buyers' population is the smallest. Intuitively, the buyers in this subgraph are those that are the most supply-constrained. As all seller (buyer) types have the same value distribution, under optimal commissions/subscriptions the seller (buyer) types in this subgraph correspond to those that have the highest (lowest) trade amount per unit of population, q_i^s/s_i (q_j^b/b_j). Consequently, the valuations of the marginal buyers and sellers in this subgraph are the largest, and the average surplus of the aforementioned buyers (sellers) is the smallest (largest). We then remove these buyers/sellers from the network, and repeat this procedure for the induced subnetwork in order to identify a subset of types whose marginal agents have the second-largest valuations, and so on. We illustrate Theorem 2 in the following simple example.

Example 1. *Consider the network in Figure 1.1, and recall that all seller types have a population of size one and all buyer types have a population of size two. The subset of buyers* B *for which the quantity* $\frac{\sum_{i \in N_E(B)} s_i}{\sum_{j \in B} b_j} = \frac{|N(B)|}{2|B|}$ *is the smallest is given by* $\mathcal{B}_1 = \{3, 4\}$.

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[7] We establish in the proof that the largest set is uniquely defined. This result follows from the fact that, if two sets achieve the minimum, then so does their union.

[8] In fact, in the appendix we prove a stronger version of this result, where when $\tau_1 < \tau_2$ the valuation of the marginal agent of type j_1 is strictly higher than that of the marginal agent of type j_2. A similar result holds on the seller side: there exists some $\bar{\tau}$ such that all seller types in $\mathcal{S}_\tau$ with $\tau \leq \bar{\tau}$ have marginal valuations that are equal to $\bar{v}_{s_i}$, i.e., the upper bound of the value distribution of sellers. For $\tau_1, \tau_2 > \bar{\tau}$, the marginal agents of different seller types in $\mathcal{S}_{\tau_1}$ and $\mathcal{S}_{\tau_2}$ admit a strict ranking. Here $\bar{\tau}$ is characterized in terms of the underlying value distributions. Moreover, except for cases where all sellers of a certain type trade under the optimal commissions/subscriptions (which necessitates bounded seller value distributions), we have that $\bar{\tau} = 0$, and a strict ranking of all seller types can be obtained.

Figure 1.1 shows that, under optimal commissions/subscriptions, we have that $q^b_3 = q^b_4 < q^b_j$ *for* $j \notin \mathcal{B}_1$*. Given that all buyer types have the same populations, this implies that the marginal agents of buyer types in* $\mathcal{B}_1$ *have the highest valuations. This observation is consistent with Theorem 2. Similarly,* $\mathcal{S}_1 = N(\mathcal{B}_1) = \{4\}$*, and* q^s_4 *is the highest among the seller types; thus, the valuation of the type-4 marginal seller is the highest.*

It is worth noting that the subgraphs constructed in Theorem 2 also shed light on which seller types have higher prices under revenue-maximizing commissions/subscriptions. We summarize this observation in the next result.

Proposition 2. *Suppose Assumption 3 holds. Consider the optimal commissions/subscriptions given in Corollary 2 (i) or (ii), and any competitive equilibrium* $(\boldsymbol{p}, \boldsymbol{x}, \boldsymbol{q}^s, \boldsymbol{q}^b)$ *induced under these commissions/subscriptions. The equilibrium prices are such that* $p_{i_1} \geq p_{i_2}$ *for any* $i_1 \in \mathcal{S}_{\tau_1}$*,* $i_2 \in \mathcal{S}_{\tau_2}$ *and* $\tau_2 \geq \tau_1$*, where the inequality is strict if* $\tau_1 \neq \tau_2$*.*

Intuitively, for small τ, many buyers compete with each other, thereby driving the prices higher. As τ increases, competition on the seller side intensifies (as there are fewer buyers per seller). As a result, the equilibrium price decreases in τ.

Remark: We conclude this section by pointing out that the ranking of network components obtained in Theorem 2 is closely related to optimization over polymatroids. For a nondecreasing submodular (set) function $\rho : 2^S \to \mathbb{R}$ defined on the subsets of some finite ground set S, the set given by $\mathcal{P}' = \{\mathbf{y} \in \mathbb{R}^{|S|}_+ : \sum_{i \in S'} y_i \leq \rho(S'), \forall S' \subset S\}$ is a *polymatroid*.[9] To see how the platform's revenue maximization problem in (1.7) is related to polymatroids, it is necessary to consider an alternative formulation of this problem.

Suppose that we try to identify an optimal solution to (1.7) in two steps. First, we assign a mass of compatible sellers to each buyer type so that a seller is allowed to trade only with the buyer type she is assigned to. This assignment is made in a way that ensures

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[9] ρ is a nondecreasing submodular function if $\rho(A) \leq \rho(B)$ for $A \subset B \subset S$, and $\rho(A) + \rho(B) \geq \rho(A \cup B) + \rho(A \cap B)$ for any $A, B \subset S$.

that the value distribution of the sellers assigned to each buyer is still $F_s(\cdot)$. The set $\mathcal{P} = \{\boldsymbol{q}^b : (\boldsymbol{x}, \boldsymbol{q}^s, \boldsymbol{q}^b)$ satisfies (1.7b), (1.7c), (1.7e), and (1.7f)$\}$, captures the feasible assignments of sellers to buyers that respect the underlying compatibility structure. Here, q_j^b is the mass of sellers assigned to type-j buyers, and we point out that not all of these sellers necessarily trade (as we do not impose (1.7d) in the definition of $\mathcal{P}$, and in principle the total mass of sellers assigned to type-j buyers may be greater than the population of this buyer type). Each buyer type together with the sellers assigned to it can now be viewed as a submarket. In the second step, we maximize the revenues within each submarket. The maximum revenue of the platform can be now expressed as the total of the revenues achieved from different submarkets.

Let $f(t)$ denote the maximum revenue the platform can achieve (by setting commissions/subscriptions optimally) in a network with a single type of buyer/seller, assuming that there are t units of sellers and 1 unit of buyers. Note that in the above construction if a submarket consists of t' units of sellers and b' units of buyers, the associated revenues can be expressed in terms of this function as $b' f(t'/b')$. Consequently, the revenue obtained under a seller assignment vector $\boldsymbol{y} \in \mathcal{P}$ (after we maximize revenues within each submarket) is given by $\bar{f}(\boldsymbol{y}) = \sum_{j \in \mathcal{B}} b_j f(y_j/b_j)$. In other words, the platform's problem is equivalent to maximizing $\bar{f}(\boldsymbol{y})$ over $\boldsymbol{y} \in \mathcal{P}$. On the other hand, by Assumption 2 the functions $f(\cdot)$ and $\bar{f}(\cdot)$ are concave. Moreover, by using Lemma 4.1 of (41), it can be shown that $\mathcal{P}$ can equivalently be represented as follows:

$$\mathcal{P} = \left\{ \boldsymbol{y} : \sum_{j \in B} y_j \leq \sum_{i \in N_E(B)} s_i, \forall B \subset \mathcal{B}, \quad y_j \geq 0, \forall j \in \mathcal{B} \right\}, \tag{1.9}$$

which immediately implies that $\mathcal{P}$ is a polymatroid. Hence, the platform's problem can equivalently be expressed as the problem of maximizing a concave function over a polymatroid.

Such problems admit a special structure, and their optimal solutions can be expressed in terms of the bases of the underlying polymatroid. We refer to $\mathbf{y} \in \mathcal{P}'$ as a base of a polymatroid $\mathcal{P}'$, if $\sum_{i \in S} y_i = \rho(S)$, where S again denotes the ground set. For $\mathbf{y} \in \mathbb{R}^{|S|}$, let $T(\mathbf{y})$ denote the vector obtained after the entries of $\mathbf{y}$ are arranged in increasing order. Given positive weights $\mathbf{w} = \{w_i\}_{i=1}^{|S|}$, a base $\mathbf{y}$ of $\mathcal{P}'$ is called *lexicographically optimal* with respect to $\mathbf{w}$, if $T(\{y_i/w_i\})$ is lexicographically greater than $T(\{\hat{y}_i/w_i\})$ for any base $\hat{y}$ of $\mathcal{P}'$. (42) shows that when a concave quadratic function is maximized over a polymatroid, the optimal solution corresponds to the base of this polymatroid with weight vectors related to the coefficients of the objective function. Similarly, in the proof of Theorem 2 we show that when the concave function $\bar{f}(\boldsymbol{y})$ is maximized over $\mathcal{P}$, the optimal solution is a lexicographically optimal base of $\mathcal{P}$ with weight vector $\boldsymbol{w} = \boldsymbol{b}$.[10] Moreover, by borrowing algorithmic ideas on the characterization of lexicographically optimal bases of polymatroids from (42), we show that the corresponding base can actually be explicitly expressed as $\tilde{y}_j = \frac{b_j}{\sum_{j' \in \mathcal{B}_\mathcal{T}} b_{j'}} \sum_{i' \in \mathcal{S}_\mathcal{T}} s_{i'}$ for $j \in \mathcal{B}_\mathcal{T}$, where $\mathcal{B}_\mathcal{T}$ and $\mathcal{S}_\mathcal{T}$ are defined as in Theorem 2. By exploiting this connection,[11] we obtain our ranking result given in Theorem 2.

Network Structure and Optimal Revenue. We next discuss the impact of the network structure on the platform's revenues. To obtain our results, we make use of the following variant of Hall's marriage condition.

Definition 2. *We say that a network* $G(\mathcal{B} \cup \mathcal{S}, E)$ *with seller/buyer populations* $(\boldsymbol{s}, \boldsymbol{b})$ *satisfies the* weighted Hall's marriage condition *if*

$$\sum_{i \in N_E(B)} s_i \geq \frac{\sum_{i \in \mathcal{S}} s_i}{\sum_{j \in \mathcal{B}} b_j} \sum_{j \in B} b_j \text{ for all } B \subset \mathcal{B}. \tag{1.10}$$

Similarly, we say that the network satisfies the ε-marriage condition *if for some* $\varepsilon \in [0,1]$ *we*

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[10] Hence, at the optimal solution all sellers are assigned to a buyer type. Note that this does not mean that all sellers trade, as the assignment merely identifies submarkets where subsets of buyers and sellers are allowed to trade, and within each submarket only a subset of agents may trade. Proposition 10 in Appendix 1.9 characterizes the mass of buyers/sellers who trade in each submarket, using the optimal solution of the aforementioned optimization problem.

[11] The relevant results of (41) and (42) are also replicated in Section (1.12.1).)

have that

$$\sum_{i \in N_E(B)} s_i \geq (1-\varepsilon) \frac{\sum_{i \in \mathcal{S}} s_i}{\sum_{j \in \mathcal{B}} b_j} \sum_{j \in B} b_j \text{ for all } B \subset \mathcal{B}.$$

Hall's marriage condition requires that $|N_E(B)| \geq |B|$ for all $B \subset \mathcal{B}$. A bipartite network (which has two sides with equal cardinality) admits a perfect matching if and only if it satisfies Hall's marriage condition (see, e.g., (1)). Intuitively, this condition implies that for any subset of nodes on one side, we have a sufficient number of nodes on the other side to "cover" them.

It can be seen that (1.10) is a weighted version of this condition. In particular, we assign a weight of $b_j / \sum_{\ell \in \mathcal{B}} b_\ell$ to each $j \in \mathcal{B}$ and a weight of $s_i / \sum_{\ell \in \mathcal{S}} s_\ell$ to each $i \in \mathcal{S}$, and require the total weight for buyers in B to be smaller than the total weight of corresponding sellers $N_E(B)$ on the other side of the market. In our setting, this condition ensures that for any set of buyers B, the ratio of total available supply to total demand of B (i.e., $\sum_{i \in N_E(B)} s_i / \sum_{j \in B} b_j$) is not smaller than the ratio of total supply to demand in the market (i.e., $\sum_{i \in \mathcal{S}} s_i / \sum_{j \in \mathcal{B}} b_j$). In other words, the supply in the market is "balanced," and no set of buyers in the market is excessively deprived of it.

Given buyer/seller types $\mathcal{B}$ and $\mathcal{S}$, and population profile $(\boldsymbol{s}, \boldsymbol{b})$, we denote by $V_{opt}(E, \boldsymbol{s}, \boldsymbol{b})$ the optimal revenue that can be obtained for the network $G(\mathcal{B} \cup \mathcal{S}, E)$, i.e., the optimal objective of (1.3) for this network. Furthermore, we denote by $V_{max}(\boldsymbol{s}, \boldsymbol{b})$ the maximum revenue that can be obtained by *any* edge set E, i.e.,

$$V_{max}(\boldsymbol{s}, \boldsymbol{b}) := \max_{E \subseteq \mathcal{B} \times \mathcal{S}} V_{opt}(E, \boldsymbol{s}, \boldsymbol{b}). \tag{1.11}$$

Using this notation, we provide a tight upper bound on the platform's revenues that holds independently of the network structure, and a lower bound on the revenues of a given network structure.

Theorem 3. *Suppose that Assumption 3 holds. Fix the set of sellers $\mathcal{S}$, the set of buyers*

$\mathcal{B}$, and the population profile $(\boldsymbol{s}, \boldsymbol{b})$. Let $s_0 = \sum_{i \in \mathcal{S}} s_i$ and $b_0 = \sum_{j \in \mathcal{B}} b_j$ denote the total populations of sellers and buyers, respectively. Then:

(i) $V_{max}(\boldsymbol{s}, \boldsymbol{b}) = b_0 \max_{r \leq \min\left\{1, \frac{s_0}{b_0}\right\}} \left[F_b^{-1}(1-r) - F_s^{-1}\left(\frac{r}{s_0/b_0}\right)\right] r$*. Moreover,* $V_{opt}(E, \boldsymbol{s}, \boldsymbol{b}) = V_{max}(\boldsymbol{s}, \boldsymbol{b})$ *if and only if* $G(\mathcal{B} \cup \mathcal{S}, E)$ *satisfies the weighted Hall's marriage condition.*

(ii) If $G = (\mathcal{B} \cup \mathcal{S}, E)$ *satisfies the* ε*-marriage condition, then* $V_{opt}(E, \boldsymbol{s}, \boldsymbol{b}) \geq (1-\varepsilon) V_{max}(\boldsymbol{s}, \boldsymbol{b})$*.*

Note that this result implies that *all networks with the same total population* (s_0, b_0) *of sellers and buyers* (and even with a possibly different number of seller/buyer types), admit the same upper bound on the revenues (given by $V_{max}(\boldsymbol{s}, \boldsymbol{b})$). Moreover, this upper bound is achieved (with appropriate commissions/subscriptions) when the weighted Hall's marriage condition holds. Intuitively, if the weighted Hall's marriage condition holds, then every subset of buyer types is exposed to a sufficient mass of sellers, and in this case the revenue the platform can achieve (under optimal commissions/subscriptions) is larger than the revenues achievable under any network structure. When this condition fails to hold, quantifying the extent of its "failure" in terms of the ε-marriage condition, yields a lower bound on the optimal revenue. Qualitatively, this result suggests that lower revenues can be expected in networks where supply is distributed less evenly, e.g., when some buyers have access to a limited supply while others have access to an abundant supply.

We next focus on a different question of practical relevance: if the platform could expand its market and increase the mass of traders of some type by a small amount (e.g., through ads), which types would improve the revenues most significantly and hence should be targeted? Interestingly, the ranking obtained in Theorem 2 provides an answer to this question.

Formally, let $\frac{\partial}{\partial s_i} V_{opt}(E, \boldsymbol{s}, \boldsymbol{b})$ denote the partial derivative of the platform's optimal revenues with respect to the size of the population of type-i sellers, whenever the revenue function $V_{opt}(\cdot)$ is differentiable with respect to s_i. Qualitatively, this quantity captures how sensitive the optimal revenue of the platform is to the size of the population of type-i sellers. Consider the set $D_s := \{\frac{\partial}{\partial s_i} V_{opt}(E, \boldsymbol{s}, \boldsymbol{b})\}_{i \in \mathcal{S}}$, which is the set of partial derivatives of the platforms optimal revenues with respect to the size of the population of sellers. For $\tau = 1, 2, \ldots$, we refer to the set of seller types with the τ^{th}-largest partial derivative values, i.e., $\{i \in \mathcal{S} : \frac{\partial}{\partial s_i} V_{opt}(E, \boldsymbol{s}, \boldsymbol{b})$ is among the τ^{th} largest$\}$, as the τ^{th}-*most* profitable seller types to expand. Analogously, we can define $\frac{\partial}{\partial b_j} V_{opt}(E, \boldsymbol{s}, \boldsymbol{b})$ for $j \in \mathcal{B}$, the set D_b, and the set of the τ^{th}-*least* profitable buyer types to expand.

Corollary 3. *Suppose that Assumption 3 holds. Fix population profile* $(\boldsymbol{s}, \boldsymbol{b})$ *and network* $G(\mathcal{B} \cup \mathcal{S}, E)$*. Suppose that* $V_{opt}(E, \boldsymbol{y}_1, \boldsymbol{y}_2)$ *is differentiable*[12] *with respect to* $\boldsymbol{y}_1, \boldsymbol{y}_2$ *at* $(\boldsymbol{y}_1, \boldsymbol{y}_2) = (\boldsymbol{s}, \boldsymbol{b})$*. Let* $\mathcal{S}_\tau$ *and* $\mathcal{B}_\tau$ *be defined as in Theorem 2. Then,* $\mathcal{S}_\tau$ *corresponds to the set of the* τ^{th}-most *profitable seller types to expand, and* $\mathcal{B}_\tau$ *corresponds to the set of the* τ^{th}-least *profitable buyer types to expand.*

Intuitively, if a buyer type is among the most supply-constrained, then a small increase in the population of the compatible seller types could lead to the most significant improvement in the revenues of the platform. Note that the aforementioned set of sellers is given precisely by $\mathcal{S}_1$. Corollary 3 supports this intuition, and shows that an (infinitesimal) increase in the population of sellers in $\mathcal{S}_1$ leads to the highest improvement in the revenues of the platform. The result further allows for ranking different seller types as well as buyer types in terms of how sensitive the revenues of the platform are to their population sizes.

Consider a network that satisfies the weighted Hall's marriage condition. It readily follows from (1.10) that for this network $\mathcal{B}_1 = \mathcal{B}$ and $\mathcal{S}_1 = \mathcal{S}$. Thus, Corollary 3 implies that for such networks, the impact of expanding any seller (similarly buyer) type (by an infinitesimal amount) on the revenues of the platform is identical. Similarly, Theorem 2 implies that

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[12] Notice that we assumed differentiability of $V_{opt}(E, \boldsymbol{y}_1, \boldsymbol{y}_2)$ at $(\boldsymbol{y}_1, \boldsymbol{y}_2) = (\boldsymbol{s}, \boldsymbol{b})$. This condition generically holds, and allows us to obtain an elementary proof of the claim by using the envelope theorem. Similar results can be obtained after relaxing the differentiability condition by using the sensitivity analysis ideas from convex optimization; see (43).

all seller (similarly buyer) types have the same average surplus. Recall that the weighted Hall's marriage condition captures when the supply in the market is balanced. Thus, these observations imply that when the supply and demand in the market are balanced, expanding any seller (buyer) type has the same impact on the platform's revenues, and all seller (buyer) types have the same average surplus (under the optimal commissions/subscriptions). However, when the supply and demand are not distributed in a balanced manner, different types end up with a different average surplus and the platform's revenues may be less/more sensitive to sizes of certain populations (as shown in Theorem 2 and Corollary 3).

We conclude this section by providing a result to compare compatibility networks in terms of the revenues that can be extracted from them.

Proposition 3. *Suppose that Assumption 3 holds, and that we have two networks* $G(\mathcal{B} \cup \mathcal{S}, E_1)$ *and* $G(\mathcal{B} \cup \mathcal{S}, E_2)$ *that differ only in their edge set, and let* $(\boldsymbol{s}, \boldsymbol{b})$ *be the seller/buyer populations. Suppose that the networks satisfy* $\sum_{i \in N_{E_1}(B)} s_i \geq \sum_{i \in N_{E_2}(B)} s_i$ *for all* $B \subset \mathcal{B}$*. Then,* $V_{opt}(E_1, \boldsymbol{s}, \boldsymbol{b}) \geq V_{opt}(E_2, \boldsymbol{s}, \boldsymbol{b})$*.*

Proposition 3 suggests a natural preorder on the network structures with the same seller/buyer populations, i.e., $G(\mathcal{B} \cup \mathcal{S}, E_1) \succeq G(\mathcal{B} \cup \mathcal{S}, E_2)$ if $\sum_{i \in N_{E_1}(B)} s_i \geq \sum_{i \in N_{E_2}(B)} s_i$ for all $B \subset \mathcal{B}$. Thus, our result states that the revenues in a larger network (in terms of this preorder) are larger. Intuitively, this result implies that if every subset of buyer types has access to more sellers in one of the networks, then the platform is able to extract more revenue from that network. This result is consistent with Theorem 3 since the network that is larger in terms of the partial order would satisfy the ε-marriage condition with a smaller ε. We illustrate this result in Figure 1.2.

We conclude this section, by exploring how the optimal revenues of the platform change as a function of the buyer's/seller's population sizes, the value distributions and the set of connections. In light of Theorem 1, such a characterization boils down to sensitivity analysis in (1.7). With some abuse of notation, we denote by $V_{opt}(\boldsymbol{s}, \boldsymbol{F_s}, \boldsymbol{b}, \boldsymbol{F_b}, E)$ the parameterized optimal revenue with respect to the population profile $\boldsymbol{s} = (s_1, \ldots, s_n)$ and $\boldsymbol{b} = (b_1, \ldots, b_m)$,

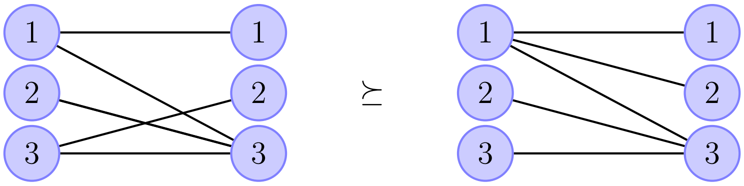

Figure 1.2: Denote the network on the left (right) by G_L (G_R) and its edge set by E_L (E_R). In both G_L and G_R, every node has a unit population, and all types have uniform valuations. It can be checked that $G_L \succeq G_R$. Proposition 3 implies that the platform can extract more revenue from the network on the left. In fact, despite the number of edges being the same, $V_{opt}(E_L, \boldsymbol{s}, \boldsymbol{b}) = \frac{3}{8} \geq \frac{1}{3} = V_{opt}(E_R, \boldsymbol{s}, \boldsymbol{b})$.

the value distribution function profile $\boldsymbol{F_s} = (F_{s_1}, \ldots, F_{s_n})$ and $\boldsymbol{F_b} = (F_{b_1}, \ldots, F_{b_m})$, and the edge set E of network $G(\mathcal{S} \cup \mathcal{B}, E)$. The following result shows that the platform's optimal revenues change monotonically with population sizes, value distributions (in the sense of first order stochastic dominance), and edge connections(in the sense of superset).

Lemma 1. *The parameterized optimal revenue* $V_{opt}(\boldsymbol{s}, \boldsymbol{F_s}, \boldsymbol{b}, \boldsymbol{F_b}, E)$ *satisfies the following properties:*

(i) Impact of the population size: Consider a buyer/seller population profile $(\tilde{\boldsymbol{b}}, \tilde{\boldsymbol{s}})$ *such that* $\tilde{\boldsymbol{b}} \geq \boldsymbol{b}$ *and* $\tilde{\boldsymbol{s}} \geq \boldsymbol{s}$*, where the inequalities are entrywise. Then,* $V_{opt}(\tilde{\boldsymbol{s}}, \boldsymbol{F_s}, \tilde{\boldsymbol{b}}, \boldsymbol{F_b}, E) \geq V_{opt}(\boldsymbol{s}, \boldsymbol{F_s}, \boldsymbol{b}, \boldsymbol{F_b}, E)$.

(ii) Impact of the value distribution: Consider a valuation profile $(\tilde{\boldsymbol{F}}_{\boldsymbol{b}}, \tilde{\boldsymbol{F}}_{\boldsymbol{s}})$ *such that* $\tilde{F}_{b_j}(v) \leq F_{b_j}(v)$ *for all* $v \geq 0$ *and* $j \in \mathcal{B}$*, and* $\tilde{F}_{s_i}(v) \geq F_{s_i}(v)$ *for all* $v \geq 0$ *and* $i \in \mathcal{S}$*. Then* $V_{opt}(\boldsymbol{s}, \tilde{\boldsymbol{F}}_{\boldsymbol{s}}, \boldsymbol{b}, \tilde{\boldsymbol{F}}_{\boldsymbol{b}}, E) \geq V_{opt}(\boldsymbol{s}, \boldsymbol{F_s}, \boldsymbol{b}, \boldsymbol{F_b}, E)$.

(iii) Impact of the edge set: Consider an edge set $\tilde{E} \subset \mathcal{S} \times \mathcal{B}$ *such that* $\tilde{E} \supset E$*. Then* $V_{opt}(\boldsymbol{s}, \boldsymbol{F_s}, \boldsymbol{b}, \boldsymbol{F_b}, \tilde{E}) \geq V_{opt}(\boldsymbol{s}, \boldsymbol{F_s}, \boldsymbol{b}, \boldsymbol{F_b}, E)$

The findings of this lemma are intuitive. The first part establishes that when the population size of any seller or buyer type in the system becomes larger (all else being equal), the platform is able to exploit this and extract higher revenues from the market. The second part shows that if the value distribution of a buyer type increases or the (reservation) value distribution of a seller type decreases (both in the first order stochastic dominance sense), then the revenues of the platform still increase. Note that the opposite effect of buyer/seller distributions should be expected. If a buyer type's distribution increases in the first order stochastic dominance sense, then the buyers have higher values for products/service on average and this allows the platform to extract more surplus. In contrast, if a seller type's distribution increases, then the reservation values become higher, and the sellers become more reluctant to trade. This forces the platform to give up revenues. The last part indicates that when the underlying trading network is larger, then the revenue increases. This monotonicity is due to the fact that there are no trade frictions in our current model. Thus, larger network gives the platform higher flexibility to guide the competitive equilibrium towards the one with higher revenue return.

1.6 Suboptimality of Simpler Commission/Subscription Schemes

The optimal commission/subscription scheme derived in Theorem 1 typically requires charging payments to both sides of the market and treating buyer/seller types differently in terms of their commissions/subscriptions. Consistent with these results, some platforms charge different commissions/subscriptions to different types of sellers. For instance, in Amazon and Alibaba (where third-party sellers also offer their products), commission rates depend on the types of products offered.

However, simpler commission/subscription schemes are also observed in practice. Until 2016, Upwork charged a 10% flat commission rate to freelancers and nothing to clients. Airbnb implements a slightly more complicated scheme by charging a flat commission rate of 3% to hosts and heterogeneous commissions rates of 0%–20% to guests.

Motivated by these examples, we explore how simpler and practically appealing schemes perform in terms of revenues relative to the optimal commissions/subscriptions. To that end, we first study a setting where the platform is restricted to charging the same commissions/subscriptions to all the agents on the same side of the market (Section 1.6.1). Then, we focus on settings where the platform can target different types with different commissions and subscriptions, but is restricted to charging payments only to one side of the market (Section 1.6.2). Finally, we illustrate our findings by focusing on a dataset that is representative of Airbnb bookings and explore how using different commission/subscription schemes impacts the revenues of the platform (Section 1.6.3).

1.6.1 Homogeneous Commissions and Subscriptions

We first consider the case where the platform charges the same commissions/subscriptions to all the agents on the same side of the market. Formally, we restrict attention to homogeneous subscription fees $\boldsymbol{\mu} = (\mu^s \mathbf{1}, \mu^b \mathbf{1})$ and homogeneous commission rates $\boldsymbol{\gamma} = (\gamma^s \mathbf{1}, \gamma^b \mathbf{1})$, and define the problem of maximizing revenues using homogeneous commissions/subscriptions as:

$$V_h = \max_{(\gamma^s \mathbf{1}, \gamma^b \mathbf{1}) \in \Gamma,\ (\mu^s \mathbf{1}, \mu^b \mathbf{1}) \in \mathcal{U}} V(\gamma^s \mathbf{1}, \gamma^b \mathbf{1}, \mu^s \mathbf{1}, \mu^b \mathbf{1}), \tag{1.12}$$

where, with some slight abuse of notation, $V(\gamma^s \mathbf{1}, \gamma^b \mathbf{1}, \mu^s \mathbf{1}, \mu^b \mathbf{1}) = V(\boldsymbol{\gamma}, \boldsymbol{\mu})$ denotes the platform's revenues under the aforementioned commissions/subscriptions. Similarly, we denote by $V_{h(sub)}$ and $V_{h(com)}$ the optimal revenue of the platform from using only homogeneous subscriptions (i.e., setting $\gamma^s = \gamma^b = 0$) and from using only homogeneous commissions (i.e., setting $\mu^s = \mu^b = 0$), respectively.

Recall that Section 1.4.1 obtains optimal commissions/subscriptions by solving the convex relaxation of the platform's revenue maximization problem given in (1.7). This relaxation is no longer tight when we restrict attention to homogeneous commissions/subscriptions, as the fee homogeneity restriction is nontrivial. That is, in general, it is not possible to construct homogeneous commissions/subscriptions that support the optimal solution of the aforementioned relaxation.

Despite not being able to use the convex optimization problem of Section 1.4.1, approximately optimal homogeneous commissions/subscriptions can be obtained in a tractable way by defining a grid over the four decision variables $(\mu^s, \mu^b, \gamma^s, \gamma^b)$, obtaining the equilibrium for any tuple of parameters by solving a convex optimization problem (see Appendix 1.9), and computing and comparing the corresponding revenues.

In light of Corollary 2, it may be appealing to search only over subscriptions or (when $c_i = 0$ for all $i \in \mathcal{S}$) only over commissions (as the corollary suggests that doing so is optimal when the platform is not restricted to using homogeneous commissions/subscriptions). Example 2 illustrates that this leads to strictly lower revenues for the platform in the context of homogeneous commissions/subscriptions.

Example 2. *Consider a bipartite network with three seller types and three buyer types, with the edge set given by* $E = \{(1,1), (1,2), (2,2), (2,3), (3,3)\}$. *The population profiles of sellers and buyers are* $\boldsymbol{s} = (7, 12, 20)$ *and* $\boldsymbol{b} = (70, 24, 20)$*, respectively. All seller types have a value distribution equal to* $F_s(v) = 1 - \exp\left(-\frac{v}{40}\right)$*, all buyer types have a value distribution equal to* $F_b(v) = 1 - \exp\left(-(\frac{v}{50})^5\right)$*, and we let* $\boldsymbol{c} = (0, 0, 0)$*. In this example, we obtain* $V_h/V_{opt} = 91.0\%$*. Furthermore, if we use only commissions we have that* $V_{h(com)}/V_{opt} = 90.8\%$*, and if we use only subscriptions we have that* $V_{h(sub)}/V_{opt} = 87.8\%$*.*

There are two important takeaways from the above example. First, homogeneous commissions/subscriptions are in general suboptimal and they can yield substantially lower revenues relative to the ones given in Theorem 1, even for simple instances satisfying Assumption 3. Second, V_h cannot be implemented using only commissions or only subscriptions and, moreover, the revenues obtained by using only commissions or only subscriptions are different.

Given that restricting attention to homogeneous commissions/subscriptions is not without loss of optimality, it is of interest to understand how large the revenue loss can be. Proposition 4 addresses this question, by showing that in general this revenue loss can be arbitrarily bad.

Proposition 4. *For any* $\varepsilon > 0$*, there exists a problem instance such that* $V_h/V_{opt} < \varepsilon$*.*

Intuitively, when the platform is able to charge heterogeneous commissions/subscriptions, it has some freedom to manage supply and demand differently for different submarkets. To see this, consider a network that consists of two seller types with identical value distributions and c_i's, and two connected components. Assume that the first seller type faces a set of buyers with larger value distributions (in the first-order stochastic dominance sense) relative to the set of buyers the other seller type faces. If the platform restricts attention to homogeneous commissions/subscriptions then either one component may end up with too much supply or the other may end up with too little supply, and both outcomes lead to a low aggregate revenue. By discriminating these seller types through the use of different commissions/subscriptions, the platform can induce desired supply levels in these different components and improve revenues.

In the proof of Proposition 4, we construct problem instances where the homogeneous commissions/subscriptions perform arbitrarily bad, by considering settings where different types have very different value distributions. One might ask whether a similar result could be obtained if all types on the same side share the same value distribution and all sellers share the same c_i's (i.e., if Assumption 3 holds). In fact, even though homogeneous commissions/subscriptions are still suboptimal in that setting (see Example 2 for an illustration), it is possible to obtain bounds on the revenue loss. In particular, by focusing on such problem instances, Proposition 5 identifies two settings where the revenue loss due to homogeneous commissions/subscriptions is bounded. This proposition as well as the rest of the results of this section focus on problem instances with a fixed network and a given population profile $(\boldsymbol{s}, \boldsymbol{b})$. As such, when we state our results we suppress the dependence of revenues on these quantities (i.e., we use V_{opt} and V_{max} as opposed to $V_{opt}(E, \boldsymbol{s}, \boldsymbol{b})$ and $V_{max}(\boldsymbol{s}, \boldsymbol{b})$).

Proposition 5. *Suppose that Assumption 3 holds.*

(i) If $F_s(v)$ is concave in $v \in [0, \bar{v}_s]$ and $F_b(v)$ is convex in $v \in [0, \bar{v}_b]$ with $\bar{v}_b < \infty$, then for any network $G(\mathcal{S} \cup \mathcal{B}, E)$, we have that $V_h \geq \frac{1}{2} V_{opt}$.

(ii) If the network satisfies the ε-marriage condition, then $V_h \geq (1-\varepsilon)V_{max}$.

This proposition provides lower bounds on the revenues achievable under the homogeneous commissions/subscriptions. In the first part, we show that at least half of the optimal revenue can be obtained when the value distributions of traders are well behaved (e.g., an exponential distribution for sellers, and a uniform one for buyers). Note that this bound is independent of the network structure. On the other hand, in the second part we do not make any assumption on the distributions but instead we provide a bound that depends on the network structure. In particular, the second part of the proposition implies that if the network satisfies the weighted Hall's marriage condition, then using homogeneous commissions/subscriptions is optimal. Moreover, when this condition does not hold, quantifying by how much it is violated (in terms of the ε-marriage condition) allows us to bound the revenue loss incurred by the platform. Note that, in this case, the bound is in terms of the upper bound V_{max} on revenues achievable for any network.

1.6.2 Charging Payments Only to One Side of the Market

We next restrict attention to commission/subscription schemes where the platform can charge only one side of the market, while possibly still discriminating agent types in terms of their commissions/subscriptions. Formally, let $\Gamma^s = [0,1]^n$, $\Gamma^b = [0,\infty)^m$, $\mathcal{U}^s = [0,\infty)^n$, and $\mathcal{U}^b = [0,\infty)^m$ denote the space of feasible seller commission rates, buyer commission rates, seller subscription fees, and buyer subscription fees, respectively. We denote by V_s (V_b) the maximum revenue that the platform can obtain by charging payments only to sellers (buyers). With some abuse of notation, these quantities are given as follows:

$$\begin{aligned} V_s &= \max_{(\boldsymbol{\gamma}^s,\boldsymbol{\mu}^s)\in\Gamma^s\times\mathcal{U}^s} V(\boldsymbol{\gamma}^s,\mathbf{0},\boldsymbol{\mu}^s,\mathbf{0}), \\ V_b &= \max_{(\boldsymbol{\gamma}^b,\boldsymbol{\mu}^b)\in\Gamma^b\times\mathcal{U}^b} V(\mathbf{0},\boldsymbol{\gamma}^b,\mathbf{0},\boldsymbol{\mu}^b). \end{aligned} \tag{1.13}$$

It is worthwhile to mention that, as in the case of homogeneous commissions/subscriptions, we cannot rely on the relaxation in (1.7) to solve this problem. In other words, having zero commissions/subscriptions on one side is a nontrivial restriction and, under this restriction, it is not always possible to find commissions/subscriptions that support the optimal solution of the aforementioned relaxation. Thus, for our numerical studies, we approximately characterize the revenue under one-sided extraction by constructing a grid of commissions/subscriptions and, for each commission-subscription vector in this grid, computing the equilibrium and the corresponding revenue.

We proceed by illustrating that charging payments only to one side of the market is not without loss of optimality. In particular, by focusing on a simple network structure, our next example shows that the revenue loss relative to the optimal commissions/subscriptions of Section 1.4.1 is nontrivial.

Example 3. *Consider a complete bipartite network with three seller types and three buyer types with population vectors* $\boldsymbol{s} = (1,1,1)$ *and* $\boldsymbol{b} = (2,2,2)$*, respectively. Let the value distributions for type-i sellers be* $F_{s_i}(v) = 1 - \exp\left(-(\frac{v}{\lambda_i^s})^{k_i^s}\right)$ *where* $\boldsymbol{k}^s = (3,2,1)$ *and* $\boldsymbol{\lambda}^s = (20,1,0.1)$*, let the value distributions for type-j buyers be* $F_{b_j}(v) = 1 - \exp\left(-(\frac{v}{\lambda_j^b})^{k_j^b}\right)$ *where* $\boldsymbol{k}^b = (3,2,1)$ *and* $\boldsymbol{\lambda}^b = (1,5,10)$*, and let* $\boldsymbol{c} = (0,0,0)$*. In this example we obtain* $V_b = 10.05$*,* $V_s = 8.84$*, and* $V_{opt} = 10.78$*, and hence* $\frac{V_s}{V_{opt}} = 81.9\%$ *and* $\frac{V_b}{V_{opt}} = 93.3\%$*.*

This example also illustrates why charging payments to both sides of the market is strictly better than charging payments only to one side. To see this suppose that the platform charges payments only to sellers, i.e., $\boldsymbol{\mu}^b = \boldsymbol{\gamma}^b = 0$. It can be shown that in this example type-3 buyers have larger valuations in expectation than the other two buyer types. Thus, when the platform does not charge any payments to buyers, too many type-3 buyers relative to other types demand to trade with sellers on the other side of the market. On the other hand, recall from Theorem 1 that at the revenue-maximizing outcome, the equilibrium supply and demand levels are uniquely defined. Hence, when the induced buyer demand is too high (under $\boldsymbol{\mu}^b = \boldsymbol{\gamma}^b = 0$), it may overshoot the optimal amount identified in Theorem

1, thereby leading to suboptimal revenues. By charging payments appropriately to the buyers, the platform can reduce the demand of type-3 buyers to the optimal levels. In other words, without charging commissions/subscriptions to both sides of the market, it may not be feasible to achieve optimal supply/demand levels. A similar argument also reveals the suboptimality of charging commissions/subscriptions only to the sellers.

In the previous subsection, we showed that when the platform is restricted to using homogeneous commissions/subscriptions, in general the revenue loss can be arbitrarily bad. We conclude this section by establishing that, in contrast to the case of homogeneous commissions/subscriptions, if the platform charges payments only to one side of the market, the revenue loss is always bounded. Furthermore, charging payments only to one side of the market is without loss of optimality when agents' valuations admit identical distributions and there is no vertical differentiation among sellers.

Proposition 6. *Suppose that the platform can charge payments only to one side of the market.*

(i) If the platform can choose whether to charge buyers or sellers, then at least half of the optimal revenue can be obtained, i.e., $\max\{V_s, V_b\} \geq \frac{1}{2} V_{opt}$.

(ii) If $F_{b_j}(v) = F_b(v)$ *for all* $j \in \mathcal{B}$*, we have that* $V_s = V_{opt}$*; if* $F_{s_i}(v) = F_s(v)$ *and* $c_i = 0$ *for all* $i \in \mathcal{S}$*, we have that* $V_b = V_{opt}$.

To establish the first part of the result we start with the optimal subscriptions in Corollary 2, and set payments of the buyers or the sellers equal to zero. We show that in at least one of the two cases, the induced subscription vector achieves at least half of the optimal revenues. We also point out that part (i) of the proposition does not make any assumptions on the network structure or the value distributions.

By contrast, the second part of the proposition focuses on settings where all agent types on one side of the market have the same value distributions, but allows for arbitrary distributions on the other side and for an arbitrary network structure. While in general charging payments to one side of the market leads to some revenue loss, part (ii) of Proposition 6 implies that this conclusion changes when one side has homogeneous value distributions. In this case, appropriate market prices induce the optimal trade levels on one side of the market, and the choice of commissions/subscriptions induces optimal trade levels on the other side, thereby eliminating any revenue loss.

1.6.3 Numerical Study

We now present an example motivated by Airbnb, an online marketplace that allows guests to rent homes for short stays from hosts. Focusing on some regions of the city of Chicago, we study the impact that different commission/subscription schemes have on the revenues of the platform. It is worth clarifying that our objective in this section is not to provide a detailed empirical study, which is beyond the scope of this paper. We recognize that Airbnb has a large-scale complex operation, and our model does not capture all features of this operation. However, we believe that this is a useful setting to illustrate the key ideas and findings presented so far in a realistic and nontrivial networked market. To this end, we remind the reader that the primitives of our model are the bipartite network that captures the compatibility between different buyer/seller types, the populations of these types, their value distributions, and the vector of disutilities for transacting with different seller types. Whenever possible, we use publicly available Airbnb data[13] to calibrate these primitives. We proceed by explaining how we obtained each primitive of the model.

First, we define the network. Agent types are characterized by geographical locations and the number of bedrooms of the supplied/demanded listings. In particular, we focus on three regions of Chicago: North Side (N), West Side (W), and Central (C) (see Appendix 1.17 in (40) for a map of neighborhoods contained in each region), which (according to the dataset)

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[13] In particular, we use the 2016–2017 listing data in Chicago obtained from "http://insideairbnb.com/get-the-data.html". The dataset has rich information on the host side, including listing locations, host information, room availability, listed price, cleaning fee, number of reviews, etc. Unfortunately, the dataset does not have any information on the guest side, and we make reasonable assumptions to construct the relevant primitives for the guests.

collectively contain more than 70% of the listings in the city. Within each region, we further restrict attention to listings for entire homes (as opposed to listings that offer a room in a home) that provide short-term rentals (the minimum length of stay requirement is no more than seven days). Each host (seller) type is defined by the region and the number of bedrooms in the listing: one bedroom (or studio), two bedrooms, three or more bedrooms. This gives us in total 9 host types, defined by the set $\mathcal{S} = \{N_1^s, N_2^s, N_3^s, C_1^s, C_2^s, C_3^s, W_1^s, W_2^s, W_3^s\}$. We also define a guest (buyer) type per location and number of bedrooms, which also gives us a total of nine types: $\mathcal{B} = \{N_1^b, N_2^b, N_3^b, C_1^b, C_2^b, C_3^b, W_1^b, W_2^b, W_3^b\}$. Here, N_2^s is the type that corresponds to hosts in the North Side whose listings have two bedrooms, and N_2^b is the type that corresponds to guests who target the North Side (for their stay) and require at least two bedrooms (a similar interpretation applies to the remaining types). We assume that each guest can be compatible only with host types whose listings have at least as many bedrooms as the guest requires. We also assume that guests who target the West Side or the North Side are compatible with the hosts in Central Chicago (which contains the main tourist attractions of the city). The resulting compatibility network is depicted in Figure 1.3.

The hosts can black out some dates and not offer their property to guests on these dates. The dataset contains information on the pre-rental nights available for each host (which is the total number of available unbooked nights in the 365 days following the days on which data was scraped). For each host type $i \in \mathcal{S}$, we aggregate the yearly pre-rental nights over all hosts of this type. We divide this quantity by 365 to obtain the (average daily) supply. Rounding this number to the closest integer, we define the host population s_i for the relevant type.[14]

The dataset also includes a single price for each listing. This is the host's default price, which the host may change for specific dates (such as holidays). Some hosts also require the guests to pay a cleaning fee (also included in the dataset), which is paid per stay (regardless

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[14] There were $1,161$ listings satisfying our criteria (involving location, rental duration requirements, etc.), and on average 84% of the nights were available on the hosts' calendar, yielding a total supply of roughly 972 units.

of the duration). We divide this quantity by the average stay length (which for Chicago is 3.3 days; see (44)) and add it to the listing's price. Airbnb keeps 3% of this sum as its commission from the hosts (see (45)); thus, we define the remainder as the net transfer to the host for a stay of one night. We assume that the reservation value of a host is equal to this net transfer. Furthermore, we assume that for each host of type $i \in \mathcal{S}$ the reservation values are drawn from an underlying Weibull distribution, $F_{s_i}(v) = 1 - \exp\Big(-\big(\frac{x}{\lambda_i^s}\big)^{k_i^s}\Big)$. For $i \in \mathcal{S}$, using the aforementioned net transfers as our observations, we estimate the parameters (λ_i, k_i) of this distribution using a maximum likelihood estimator. Columns 3–5 of Table 1.1 summarize the host distribution and population parameters obtained.

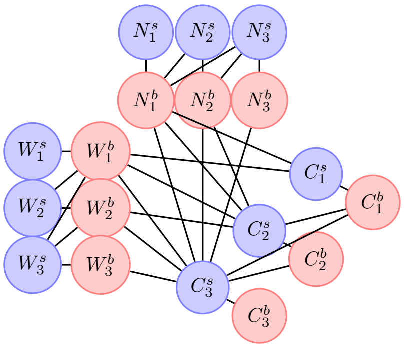

Figure 1.3: Chicago Compatibility Network

		Parameters						Eq. prices	
Loc	Rm	$\boldsymbol{s}$	$\boldsymbol{\lambda}^s$	$\boldsymbol{k}^s$	$\boldsymbol{b}$	$\boldsymbol{\lambda}^b$	$\boldsymbol{k}^b$	curr	opt
N	1	147	147	5.73	168	230	25	125.9	139.2
N	2	146	190	5.67	88	267	25	167.8	183.7
N	3	124	324	4.73	36	416	25	263.9	295.2
C	1	121	211	5.31	68	238	25	172.1	190.2
C	2	100	290	6.35	43	322	25	253.8	276.1
C	3	16	482	4.51	4	654	25	449.9	513.1
W	1	104	132	6.57	180	217	25	116.4	128.8
W	2	128	188	5.68	73	266	25	166.0	181.4
W	3	86	342	4.87	35	436	25	302.1	332.8

Table 1.1: Host and Guest Information

Next, we state our assumptions on the disutility vector, $\boldsymbol{c}$. First, the most valuable combination of property type and location is 3 (or more) bedroom apartments in Central Chicago, and therefore we set $c(C_3^s) = 0$. Second, we assume that a given property type in Central Chicago is preferred to the same property type in the North and West Sides by \$50, i.e., $c(N_x^s) - c(C_x^s) = 50$ and $c(W_x^s) - c(C_x^s) = 50$ for $x \in \{1, 2, 3\}$. Finally, we assume that customers prefer more rooms to fewer rooms, and that each additional room is valued at \$20, i.e., $c(X_1^s) - c(X_2^s) = 20$ and $c(X_1^s) - c(X_3^s) = 40$ for $X \in \{N, C, W\}$.

On the guest side, we similarly assume that the values come from a Weibull distribution $F_{b_j}(v) = 1 - \exp\left(-\left(\frac{x}{\lambda_j^b}\right)^{k_j^b}\right)$ for $j \in \mathcal{B}$. Due to limited availability of information on guests, we make two simplifying assumptions on the parameters of these distributions. First, we assume that the coefficient of variation for value distribution of guests is 5% across all types, which is equivalent to setting $k_j^b = 25$ for all $j \in \mathcal{B}$. Second, for $x \in \{1, 2, 3\}$ we assume that the expected value of a guest of type N_x^b (minus the cost $c(N_x^s)$) is respectively 0%, 10%, and 20% higher than the expected value of a host of type N_x^s. We make the same assumption for neighborhoods $\{C, W\}$ as well. For instance, the expected value of a guest who needs two bedrooms and targets the North Side (i.e., a guest of type N_2^b) minus $c(N_2^s)$ is 10% higher than the expected (reservation) value of hosts who have 2-bedroom listings in this area. Note that while for listings with 1 or 2 bedrooms the hotels may be a good substitute, for 3-bedroom listings this is less likely to be the case. Hence, we assume that the expected value of the guests relative to the reservation value of their hosts is higher for this type of listing (20%) than for other types of listings. Taken together, our assumptions pin down (λ_j^b, k_j^b) for all $j \in \mathcal{B}$.

The only part of the model that remains unspecified is the guest populations b_j for $j \in \mathcal{B}$. We solve for the guest populations such that, in the induced equilibrium, the number of stays in each region is consistent with the data. We do not have data on the exact number of stays in each region. However, the Airbnb data includes the number of reviews (made in a calendar year) for each of the listings. Following the model discussed in (46), we assume that 50% of the bookings result in a review and each property has at most 70% occupancy (in a year). Using these assumptions, we obtain an estimate of the total number of stays in each region per day. Our exploration on the Airbnb platform suggests that in Chicago, Airbnb charges about 14% of the price and cleaning fee to the guests as the platform's commission. Using this information, together with the estimated value distributions, we solve for the guest populations that match our estimated total number of stays in each region. This defines our guest population vector $\boldsymbol{b}$. The guest parameters are reported in columns 6–8.

Given the primitives of the model, we analyze the performance of different commission/subscription schemes. Since Airbnb relies exclusively on commissions, we set all subscription fees to zero. We consider three different settings: (i) a baseline model with homogeneous $\gamma^s = 3\%$ commission for the hosts and $\gamma^b = 14\%$ commission for the guests (which is consistent with the observed data), (ii) optimal homogeneous commissions for both sides, and (iii) optimal heterogeneous commissions for both sides. As Airbnb sets the guest commissions to be between 0% and 20% ((45)), in analyzing the second case, for both the host and guest sides, we search the commissions over a grid over $[0\%, 20\%]$ in a brute-force fashion and obtain the commission pair that induces the largest revenues for the platform. In the third case, we compute the optimal commission using the characterization provided in Theorem[15] 1 and, whenever the commission rate for a type is above 20%, we set it equal to 20%.

In the first case, we compute the (daily) revenue of the platform to be $V_{curr} = \$9.98 \times 10^3$. In the second case, we obtain the homogeneous commission rates $(\gamma^s, \gamma^b) = (15\%, 13\%)$, yielding a daily revenue of $V_{h(com)} = \$12.3 \times 10^3$. In the third case, we obtain the commission vectors

$$\begin{aligned}\boldsymbol{\gamma}^s &= (16\%, 17\%, 19\%, 17\%, 15\%, 20\%, 15\%, 17\%, 19\%),\\ \boldsymbol{\gamma}^b &= (4\%, 6\%, 17\%, 5\%, 7\%, 20\%, 4\%, 7\%, 12\%), \qquad (1.14)\end{aligned}$$

which achieve a daily revenue of $V_{opt} = \$12.8 \times 10^3$. The equilibrium prices in the baseline and optimal settings are reported in the last two columns of Table 1.1. It can be seen that the difference between prices in these two settings is of the order of 10%.

We point out that relative to the first scenario, charging optimal homogeneous commission

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[15] In this particular example, we show that the revenue-optimal solution can be implemented using commissions only; i.e., we employ a commission vector of $\boldsymbol{\gamma}$ such that $(\boldsymbol{\gamma}, \mathbf{0})$ satisfies the system of inequalities in part(ii) of Theorem 1.

rates improves the revenues by $\frac{V_{h(com)}}{V_{curr}} - 1 = 23.5\%$. Similarly, targeting different types with different commissions improves the revenues by $\frac{V_{opt}}{V_{curr}} - 1 = 29.5\%$.

In summary, in this example, using the optimal homogeneous commissions may lead to roughly 23% revenue improvement (relative to V_{curr}), while an additional 6% (relative to V_{curr}) can be obtained using heterogeneous commissions. We also point out that in the last two scenarios we analyzed, the commissions for the hosts are higher than those for the guests, contrary to what we see in current practice. We emphasize that there may be other considerations (e.g., competition with other short-term rental platforms) that incentivize platforms to offer lower commissions to hosts (despite inducing suboptimal revenues), that are beyond the scope of our model. Nevertheless, our observation suggests that from a purely revenue point of view, it may be worthwhile to consider charging higher commissions to the hosts and, at least in some regions, lower ones to the guests. [16]

1.7 Welfare and Revenue Maximization

We next analyze the welfare implications of the platform's choice of revenue-maximizing commissions/subscriptions. We start by formally defining the aggregate welfare (that is, the sum of the surpluses of the buyers, sellers, and the revenues of the platform).

Definition 3. *For any commission-subscription vector* $(\boldsymbol{\gamma}, \boldsymbol{\mu}) \in \Gamma \times \mathcal{U}$ *that induces competitive equilibrium* $(\boldsymbol{p}, \boldsymbol{x}, \boldsymbol{q}^s, \boldsymbol{q}^b) \in \mathcal{X}(\boldsymbol{\gamma}, \boldsymbol{\mu})$*, the associated welfare is given by:*

$$W(\boldsymbol{\gamma}, \boldsymbol{\mu}) := \sum_{j \in \mathcal{B}} b_j \int_{v^m_{b_j}}^{\bar{v}_{b_j}} v dF_{b_j}(v) - \sum_{i \in \mathcal{S}} s_i \int_0^{v^m_{s_i}} v dF_{s_i}(v) - \sum_{i \in \mathcal{S}} c_i q^s_i \tag{1.15}$$

where $v^m_{b_j}$ *and* $v^m_{s_i}$ *are as given in* (1.4) *and* (1.5)*, respectively.*

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[16] The results reported in this section are robust to the choice of $\boldsymbol{c}$. To verify this, we randomly generated 1,000 instances of the disutility vector $\boldsymbol{c}$ such that $c(C^s_3) = 0$, $c(N^s_x) - c(C^s_x) \sim U[30, 70]$, and $c(W^s_x) - c(C^s_x) \sim U[30, 70]$ for $x \in \{1, 2, 3\}$ and $c(X^s_1) - c(X^s_2) \sim U[0, 40]$, $c(X^2_2) - c(X^s_3) \sim U[0, 40]$ for $X \in \{N, C, W\}$. Consistent with the results of this section, we find that in these instances, relative to setting (i), setting (ii) results in a 22.4%–25.1% increase in revenues, and setting (iii) (i.e., using optimal commissions) results in a 28.1%–30.7% increase in revenues.

To see that (1.15) captures the welfare associated with $(\boldsymbol{\gamma}, \boldsymbol{\mu})$, first note that since the transfers from buyers to the sellers and the platform reduce the surplus of the former with the same amount they increase the surpluses of the latter two, it follows that the welfare can be equivalently expressed in terms of the total values of the buyers and sellers for the trades they participate in. Recall that for given $(\boldsymbol{\gamma}, \boldsymbol{\mu})$, all equilibria in $\mathcal{X}(\boldsymbol{\gamma}, \boldsymbol{\mu})$ share the same supply-demand vector $(\boldsymbol{q}^s, \boldsymbol{q}^b)$ (by Proposition 1). Hence, in all equilibria, the marginal agents have the same values $\{v_{b_j}^m, v_{s_i}^m\}$. Moreover, all buyers of type j (sellers of type i) who have values higher (lower) than those of the marginal agent of their type, trade. Consider the case where $c_i = 0$ for all $i \in \mathcal{S}$. The above observations imply that the aggregate value that buyers of type j derive from the received goods/services is given by $b_j \int_{v_{b_j}^m}^{\bar{v}_{b_j}} v dF_{b_j}(v)$. Similarly, the aggregate reservation value received by the sellers of type i who sell goods is given by $s_i \int_0^{v_{s_i}^m} v dF_{s_i}(v)$. Subtracting the aggregate reservation values of *all* seller types from the aggregate values of *all* buyer types yields the welfare function in (1.15) (with $c_i = 0$ for all $i \in \mathcal{S}$). When $c_i > 0$ for some $i \in \mathcal{S}$, it is necessary to take into account the disutility the buyers incur when they trade with certain types of sellers. This is captured through the $-\sum_{i \in \mathcal{S}} c_i q_i^s$ term in the welfare function.

We first establish that welfare is maximized when the platform does not charge any payments to buyers/sellers. This is because, in this case, there are no trade frictions due to commissions/subscriptions, and in a competitive equilibrium goods are allocated efficiently.

Proposition 7. *The maximum welfare is achieved by the commission-subscription vector* $(\boldsymbol{\gamma}, \boldsymbol{\mu}) = (\mathbf{0}, \mathbf{0})$, *i.e.,* $W_{opt} := W(\mathbf{0}, \mathbf{0}) \ \geq \ W(\boldsymbol{\gamma}, \boldsymbol{\mu})$ *for any* $(\boldsymbol{\gamma}, \boldsymbol{\mu}) \in \Gamma \times \mathcal{U}$.

Recall that, for any given commissions/subscriptions, a corresponding equilibrium can be obtained by solving a convex optimization program (see Appendix 1.9). Thus in light of Proposition 7 we conclude that the maximum welfare can be computed by solving this optimization problem for $(\boldsymbol{\gamma}, \boldsymbol{\mu}) = (\mathbf{0}, \mathbf{0})$, obtaining the valuations of the marginal agents in the equilibrium, and evaluating (1.15).

Since the platform maximizes its revenues by employing nonzero commissions/subscriptions, loss of social welfare can be expected under the optimal commissions and subscriptions. Formally, let $(\boldsymbol{\gamma}', \boldsymbol{\mu}')$ denote the revenue-maximizing commissions-subscriptions chosen by the platform, and let $W_r := \frac{W(\boldsymbol{\gamma}', \boldsymbol{\mu}')}{W_{opt}}$ be the ratio of the welfare associated with the revenue-maximizing and socially optimal solutions. Our next result provides bounds on this welfare ratio.

Proposition 8. *Suppose that $W_{opt} > 0$. We have the following:*

- *(i) For any $\epsilon > 0$, there exists a problem instance such that $W_r < \epsilon$.*
- *(ii) Suppose that the value distributions are uniform, i.e., $F_{s_i}(v) = \frac{v}{\bar{v}_{s_i}}$ for all $v \in [0, \bar{v}_{s_i}]$ and $F_{b_j}(v) = \frac{v}{\bar{v}_{b_j}}$ for all $v \in [0, \bar{v}_{b_j}]$. Then, for any network $G(\mathcal{S} \cup \mathcal{B}, E)$, we have $W_r \geq \frac{3}{4}$. Moreover, this lower bound is tight.*
- *(iii) Under Assumption 3, if $F_s(v)$ concave in $v \in [0, \bar{v}_s]$ and $F_b(v)$ convex in $v \in [0, \bar{v}_b]$, then for any network $G(\mathcal{S} \cup \mathcal{B}, E)$, we have $W_r \geq \frac{2}{3}$. Moreover, this lower bound is tight.*

This result assumes away the cases where the maximum achievable welfare is zero, since in those cases no trade takes place in both revenue-maximizing and welfare-maximizing outcomes. The first part of this result implies that in the worst case the welfare under revenue-maximizing commissions/subscriptions can be arbitrarily low (relative to the socially optimal level). The result is obtained by considering settings where different types have very different value distributions, and the platform maximizes its revenue by imposing high fees. These fees allow for adequately extracting revenue from the high-value buyers (or low-value sellers) but, at the same time, they deter the majority of buyers/sellers from trading, thereby inducing a low welfare. However, under additional assumptions on the value distributions, Proposition 8 establishes that the welfare achieved under revenue-maximizing commissions/subscriptions is not arbitrarily small. In particular, under uniform valuations, the revenue-maximizing commissions/subscriptions still yield at least 75% of the maximum achievable welfare.[17] If instead we have homogeneous value distributions that are convex for buyers and concave for sellers, the welfare ratio is at least 66.7%.

1.8 Conclusion

In this paper, we consider a platform that facilitates trade between buyers and sellers. Different types of buyers/sellers have different value distributions, and a bipartite network captures which buyer/seller types are compatible. The platform charges commission rates and subscription fees to the trading agents, but does not dictate the prices at which trades need to occur. That is, the prices are determined endogenously and are only indirectly influenced by the platform's choice of commissions/subscriptions. In this setting, we study how the platform should choose its commissions/subscriptions to maximize its revenues. We show that, in general, the platform finds it optimal to target different buyer/seller types with different commissions/subscriptions that depend on their value distributions and network positions. Furthermore, charging agents only on one side of the market, or using identical commissions/subscriptions for all agents on the same side (both common practices in real-world platforms), may result in substantial revenue loss. We characterize how the network structure impacts the surplus of different types and the revenues of the platform. We also provide bounds on the revenue loss due to using simpler commission/subscription schemes. Our results highlight the suboptimality of some commonly used payment schemes, and showcase the importance of understanding the compatibility between different user types present on the platform.

This paper opens up interesting future directions, of which we mention two. First, in this work we assume that the population sizes are deterministic and known to the platform. If there is only noisy information available on the population sizes, the platform's prob-

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[17] Although, we are not aware of any formal connection, this ratio is reminiscent of the well-known price of anarchy ratios in congestion games with linear edge costs (see, e.g., (47)).

lem of choosing commissions/subscriptions becomes a challenging, yet important problem. Second, many platforms impact the trades that take place not only by choosing commissions/subscriptions, but also by facilitating a search protocol that allows buyers to find compatible sellers. The design of such search protocols is likely to have a first-order impact on the trading outcome, and promises to be an exciting future research direction.

1.9 Auxiliary Results

In this section, we first establish that given commissions/subscriptions the corresponding competitive equilibrium can be obtained through the optimal primal/dual solutions of a convex optimization problem (see (1.16)). Then, we show that optimal solutions of both this problem and the revenue optimization problem (1.7) can be characterized by maximizing a concave function over a polymatroid. The solutions of such problems admit an interesting structure, which we leverage in the subsequent sections to establish our key findings.

We start by presenting our *equilibrium problem.* Given $(\boldsymbol{\gamma}, \boldsymbol{\mu}) \in \Gamma \times \mathcal{U}$ such that $\gamma_i^s < 1$ for all $i \in \mathcal{S}$, define functions $\tilde{F}_{s_i}^{-1} : [0,1] \to \left[\frac{\mu_i^s}{1-\gamma_i^s}, \frac{\bar{v}_{s_i}+\mu_i^s}{1-\gamma_i^s}\right]$ and $\tilde{F}_{b_j}^{-1} : [0,1] \to \left[\frac{-\mu_j^b}{1+\gamma_j^b}, \frac{-\mu_j^b+\bar{v}_{b_j}}{1+\gamma_j^b}\right]$ such that $\tilde{F}_{s_i}^{-1}(x) := \frac{F_{s_i}^{-1}(x)+\mu_i^s}{1-\gamma_i^s}$ and $\tilde{F}_{b_j}^{-1}(x) := \frac{F_{b_j}^{-1}(x)-\mu_j^b}{1+\gamma_j^b}$. Consider the following problem:[18]

$$
\begin{aligned}
\max_{\boldsymbol{x},\boldsymbol{q}^s,\boldsymbol{q}^b} \quad & \sum_{j\in\mathcal{B}} \int_0^{q_j^b} \tilde{F}_{b_j}^{-1}\left(1-\frac{z}{b_j}\right) dz - \sum_{i\in\mathcal{S}} \int_0^{q_i^s} \tilde{F}_{s_i}^{-1}\left(\frac{z}{s_i}\right) dz - \sum_{(i,j)\in E} \frac{c_i}{1+\gamma_j^b} x_{ij} && (1.16a)\\
\text{s.t.} \quad & \sum_{i:(i,j)\in E} x_{ij} \;=\; q_j^b, \quad \forall j \in \mathcal{B}, && (1.16b)\\
& \sum_{j:(i,j)\in E} x_{ij} \;=\; q_i^s, \quad \forall i \in \mathcal{S}, && (1.16c)\\
& q_j^b \;\le\; b_j \quad \forall j \in \mathcal{B}, && (1.16d)\\
& q_i^s \;\le\; s_i, \quad \forall i \in \mathcal{S}, && (1.16e)\\
& x_{ij} \;\ge\; 0, \quad \forall (i,j) \in E. && (1.16f)
\end{aligned}
$$

In Section 1.12.1 of (40), we prove that for any optimal solution $(\boldsymbol{x}, \boldsymbol{q}^s, \boldsymbol{q}^b)$ to problem (1.16), vector $(\boldsymbol{q}^s, \boldsymbol{q}^b)$ is unique. Moreover, among all possible dual optimal solutions $(\boldsymbol{\theta}^b, \boldsymbol{\theta}^s, \boldsymbol{\eta}^b, \boldsymbol{\eta}^s, \boldsymbol{\pi})$ corresponding to constraints (1.16b)–(1.16f), we consider the one that maximizes $\sum_{j\in\mathcal{B}}[sgn(q_j^b)-sgn(b_j-q_j^b)](\theta_j^b+\eta_j^b)+\sum_{i\in\mathcal{S}}[-sgn(q_i^s)+sgn(s_i-q_i^s)](\theta_i^s-\eta_i^s)$. Lemma

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[18] Observe that under Assumption 1, we have $\int_0^0 \tilde{F}_{b_j}^{-1}(1-z/b_j)dz = \lim_{q\to 0} \int_0^q \tilde{F}_{b_j}^{-1}(1-z/b_j)dz = 0$ and $\int_0^{b_j} \tilde{F}_{b_j}^{-1}(1-z/b_j)dz = \lim_{q\to b_j} \int_0^q \tilde{F}_{b_j}^{-1}(1-z/b_j)dz < \infty$.

5 in Section 1.12.1 of (40) establishes that such a dual optimal vector $(\boldsymbol{\theta}^b, \boldsymbol{\theta}^s, \boldsymbol{\eta}^b, \boldsymbol{\eta}^s, \boldsymbol{\pi})$ is unique. We refer to problem (1.16) as the *equilibrium problem*, since its solutions correspond to competitive equilibria.

Proposition 9. *For any* $(\boldsymbol{\gamma}, \boldsymbol{\mu})$*, the tuple* $(\boldsymbol{p}, \boldsymbol{x}, \boldsymbol{q}^s, \boldsymbol{q}^b)$ *constitutes a competitive equilibrium if and only if*

(i) $(\boldsymbol{x}, \boldsymbol{q}^s, \boldsymbol{q}^b)$ *is an optimal solution to the optimization problem* (1.16)*;*

(ii) *the price vector* $\boldsymbol{p}$ *satisfies* $p_i = \theta_i^s$ *for all* i *such that* $q_i^s > 0$ *and* $\max_{j:(i,j)\in E}\{\theta_j^b - \frac{c_i}{1+\gamma_j^b}\} \leq p_i \leq \theta_i^s$ *for all* i *such that* $q_i^s = 0$*, where* $(\boldsymbol{\theta}^b, \boldsymbol{\theta}^s, \boldsymbol{\eta}^b, \boldsymbol{\eta}^s, \boldsymbol{\pi})$ *is the unique special dual multiplier specified above.*

Next we introduce a class of optimization problems involving polymatroids. Let $g, h : [0,1] \to \mathbb{R} \cup \{-\infty, +\infty\}$ be functions such that $|g(r)|, |h(r)| < \infty$ for all $r \in [0,1)$. Suppose that these functions satisfy the following assumptions:

(1.91) $g(r)$ is continuously differentiable, strictly concave in $r \in (0,1)$, continuous at $r = 0$, and continuous at $r = 1$ if $g(1) > -\infty$.

(1.92) $h(r)$ is continuously differentiable, strictly convex in $r \in (0,1)$, continuous at $r = 0$, and continuous at $r = 1$ if $h(1) < \infty$. [19]

(1.93) $th(\frac{r}{t})$ is strictly decreasing in t for $r > 0$ and jointly convex in (r,t) $\in \left\{(r',t') :\ t' > 0, \frac{r'}{t'} \in [0,1],\ h(\frac{r'}{t'}) < \infty\right\}$.

(1.94) $g'(0) > h'(0) \geq 0$ and $g'(1) \leq 0$.

---

[19] Given Assumptions (1.91) - (1.92), we let $g'(r)$ and $h'(r)$ be the derivative of $g(r)$ and $h(r)$ evaluated at $r \in (0,1)$. With some abuse of notation, we let $g'(0) := \lim_{r\downarrow 0} g'(r)$, $g'(1) := \lim_{r\uparrow 1} g'(r)$, $h'(0) := \lim_{r\downarrow 0} h'(r)$, and $h'(1) := \lim_{r\uparrow 1} h'(r)$. Note that when $g'(0)$ and $h'(0)$ are finite, we can leverage the mean value theorem to show that they respectively correspond to the right derivative $g(r)$ and $h(r)$ at $r = 0$. Similarly, when $g'(1)$ and $h'(1)$ are finite, they correspond to the left derivative of $g(r)$ and $h(r)$ at $r = 1$.

Define a function $f : (0,\infty) \to \mathbb{R}$, and correspondence $\rho : (0,\infty) \rightrightarrows \mathbb{R}$, which (for $t > 0$) are given by

$$f(t) := \max_{r\in[0,\min\{1,t\}]} g(r) - th\left(\frac{r}{t}\right), \tag{1.17}$$

$$\rho(t) := \arg\max_{r\in[0,\min\{1,t\}]} g(r) - th\left(\frac{r}{t}\right). \tag{1.18}$$

We have the following properties for $f(\cdot), \rho(\cdot)$:

Lemma 2. *Suppose that Assumptions* (1.91)–(1.94) *hold. Then:*

(i) *$\rho(t)$ is a singleton for $t > 0$, and hence $\rho(\cdot)$ is a function. Moreover, $\rho(t)$ is strictly increasing in t. Define $t_0 = [g']^{-1}\left(h'(1)\right)$ with the convention $[g']^{-1}(x) = 0$ for $x \geq \sup_{x'\in(0,1)} g'(x')$ and $[g']^{-1}(x) = 1$ for $x \leq \inf_{x'\in(0,1)} g'(x')$. We have that $\rho(t)/t = 1$ for $t \in (0, t_0]$ and that $\rho(t)/t$ is strictly decreasing in t for $t > t_0$.*

(ii) *$f(t)$ is continuous, strictly increasing, and strictly concave in $(0,\infty)$. Furthermore, $\lim_{t\downarrow 0} f(t) = g(0)$.*

Using the first part of this lemma, when Assumptions (1.91)–(1.94) hold, we extend the domain of $f(\cdot)$ to include 0 and, in particular, we let $f(0) = \lim_{t\downarrow 0} f(t)$. Note that this extension ensures that $f(t)$ is still continuous, strictly increasing, and strictly concave for $t \geq 0$. In the remainder of the paper, in settings where Assumptions (1.91)–(1.94) hold, we always focus on $f(\cdot)$ with the extended domain, which enjoys these properties. Thus, with some abuse, when we invoke Lemma 2(ii), we conclude that $f(t)$ has the aforementioned properties for $t \geq 0$. Similar to $f(\cdot)$, we extend the domain of $\rho(\cdot)$ and, in particular, we let

$\rho(0) = \lim_{t \downarrow 0} \rho(t) = 0$. We consider the following optimization problem:

$$\max_{\boldsymbol{y}} \quad \sum_{j \in \mathcal{B}} b_j f\left(\frac{y_j}{b_j}\right) \tag{1.19a}$$

$$\text{s.t.} \quad \sum_{j \in B} y_j \leq \sum_{i \in N_E(B)} s_i, \quad \forall B \subset \mathcal{B}, \tag{1.19b}$$

$$y_j \geq 0, \quad \forall j \in \mathcal{B}. \tag{1.19c}$$

Observe that under Assumptions (1.91)–(1.94) this problem has a strictly concave objective (by Lemma 2) and its feasible set is the polymatroid $\mathcal{P}$ given in (1.9). Optimal solutions of this problem admit a special structure, which we exploit in our subsequent analysis:

Lemma 3. *The optimal solution to problem* (1.19) *is unique. Let* $\boldsymbol{y}^\star$ *be this solution. Then* $\boldsymbol{y}^\star$ *is the lexicographically optimal base for polymatroid* $\mathcal{P} = \{\boldsymbol{y} \geq \boldsymbol{0} : \sum_{j \in B} y_j \leq \sum_{i \in N_E(B)} s_i,\ \forall B \subset \mathcal{B}\}$ *with respect to weight vector* $\boldsymbol{b}$. *Moreover,* $y_j^\star > 0$ *for all* $j \in \mathcal{B}$.

Furthermore, the solution to this problem (for appropriately chosen $g(\cdot), h(\cdot)$) sheds light on the optimal solutions of the revenue optimization and equilibrium problems.

Proposition 10. *Suppose that Assumptions 1, 2, and 3 hold and consider the associated revenue optimization problem* (1.7). *Let* $f(\cdot)$ *be given as in* (1.17), *where* $g(r) := F_b^{-1}(1-r)r$ *and* $h(r) := F_s^{-1}(r)r$ *for* $r \in [0,1]$, *and consider the associated problem* (1.19).

(i) Assumptions (1.91)–(1.94) *hold.*

(ii) Let $\boldsymbol{y}^\star$ *be the optimal solution to* (1.19), *and* $\boldsymbol{r}^\star = \{r_j^\star\}_{j \in \mathcal{B}}$ *be such that* $r_j^\star = \rho(y_j^\star / b_j)$. *Any optimal solution* $(\boldsymbol{x}, \boldsymbol{q}^s, \boldsymbol{q}^b)$ *of the revenue optimization problem* (1.7) *is such that (i)* $q_j^b = r_j^\star b_j$ *for all* $j \in \mathcal{B}$, *and (ii)* $q_i^s = \frac{r_j^\star b_j}{y_j^\star} s_i$ *for all* $i \in \mathcal{S}$ *and* $j \in \mathcal{B}$ *such that* $x_{ij} > 0$.

(iii) The optimal objective value of (1.19) *is the optimal revenue in* (1.7), *i.e.,* $V_{opt} = \sum_{j \in \mathcal{B}} b_j f\left(\frac{y_j^\star}{b_j}\right)$.

Proposition 11. *Suppose that Assumptions 1, 2, and 3 hold. Suppose further that commissions/subscriptions* $(\boldsymbol{\gamma}, \boldsymbol{\mu})$ *are homogeneous, i.e.,* $\gamma_j^b = \gamma^b$, $\mu_j^b = \mu^b$, $\gamma_i^s = \gamma^s$, $\mu_i^s = \mu^s$ *for all* $i \in \mathcal{S}$ *and* $j \in \mathcal{B}$, *and consider the associated equilibrium problem* (1.16)*. Let* $f(\cdot)$ *be given as in* (1.17)*, where* $g(r) := \int_0^r \frac{1}{1+\gamma^b} F_b^{-1}(1-x) - \frac{\mu^b}{1+\gamma^b} dx$ *and* $h(r) := \int_0^r \frac{1}{1-\gamma^s} F_s^{-1}(x) + \frac{\mu^s}{1-\gamma^s} dx$ *for* $r \in [0, 1]$*, and consider the associated formulation* (1.19)*.*

- *(i) Assumptions* (1.91)*–*(1.93) *hold. If* $\mu^b + \frac{1+\gamma^b}{1-\gamma^s}\mu^s < F_b^{-1}(1)$*, then Assumption* (1.94) *also holds.*[20]
- *(ii) Let* $\boldsymbol{y}^\star$ *be the optimal solution to* (1.19)*, and* $\boldsymbol{r}^\star = \{r_j^\star\}_{j \in \mathcal{B}}$ *be such that* $r_j^\star = \rho(y_j^\star / b_j)$*. Any optimal solution* $(\boldsymbol{x}, \boldsymbol{q}^s, \boldsymbol{q}^b)$ *of the equilibrium problem* (1.16) *is such that (i)* $q_j^b = r_j^\star b_j$ *for all* $j \in \mathcal{B}$*, and (ii)* $q_i^s = \frac{r_j^\star b_j}{y_j^\star} s_i$ *for all* $i \in \mathcal{S}$ *and* $j \in \mathcal{B}$ *such that* $x_{ij} > 0$*.*
- *(iii) The optimal objective value of* (1.19) *is the same as the optimal objective value of problem* (1.16)*.*

1.10 Proofs of Results in Section 1.4

Proof of Theorem 2. As Assumptions 1, 2, and 3 hold, Proposition 10 implies that the solution of the revenue optimization problem (1.7) can be characterized in terms of the solution of (1.19), where $f(\cdot)$ is as in (1.17), and $g(r) := F_b^{-1}(1-r)r$ and $h(r) := F_s^{-1}(r)r$ for $r \in [0, 1]$. That is, by Proposition 10, let $\boldsymbol{y}^\star > 0$ be the unique optimal solution to this problem[21] and let $\boldsymbol{r}^\star = \{r_j^\star\}_{j \in \mathcal{B}}$ be such that $r_j^\star = \rho(y_j^\star / b_j)$ (where $\rho(\cdot)$ is as in (1.18)). By Lemma 3, it follows that $\boldsymbol{y}^\star$ is the lexicographically optimal base of the feasible set of (1.19) (which we denote by $\mathcal{P}$) with respect to weight vector $\boldsymbol{b}$.

To establish the claim, we proceed in two steps. First, we show that the lexicographically optimal base $\boldsymbol{y}^\star$ can be characterized in terms of the sets $\{\mathcal{S}_\tau, \mathcal{B}_\tau\}$ given in the theorem statement. Then, we relate this characterization to the the ranking of the marginal agents.

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[20] Note that $\mu^b + \frac{1+\gamma^b}{1-\gamma^s}\mu^s \geq F_b^{-1}(1)$ yields the trivial equilibrium where no one trades in the system.

[21] The problem of maximizing (Schur) concave functions over a polymatroid is also studied in (48). In that paper, the author provides analytical solutions to a class of polymatroid optimization problems, and establishes certain monotonicity properties of the optimal solutions.

<u>Step 1: Characterization of the lexicographically optimal base.</u> The construction in the theorem statement identifies a nonempty set $\mathcal{B}_\tau$ at each step and removes the elements of this set from $\mathcal{B}^{(\tau-1)}$. Since $\mathcal{B}$ is a finite set, it follows that in $\ell \le |\mathcal{B}|$ iterations sets $\{\mathcal{B}_\tau\}_{\tau=1}^{\ell}$ and $\{\mathcal{S}_\tau\}_{\tau=1}^{\ell}$ are constructed. Let the vector $\boldsymbol{t'} = \{t'_j\}_{j\in\mathcal{B}}$ be defined as $t'_j := \left(\sum_{i\in N_{E^{(\tau-1)}}(\mathcal{B}_\tau)} s_i\right)/\left(\sum_{k\in\mathcal{B}_\tau} b_k\right)$ for $j \in \mathcal{B}_\tau$ and $\tau = 1, \dots, \ell$. We claim that the vector $\boldsymbol{y'} := \{t'_j b_j\}_{j\in\mathcal{B}}$ is the unique lexicographically optimal base of polymatroid $\mathcal{P}$ under weight vector $\boldsymbol{b}$, i.e., $\boldsymbol{y'} = \boldsymbol{y}^\star$.

In order to establish this claim, we employ Theorem 3.1 in (42), which establishes the connection between vector $\boldsymbol{y'}$ and the lexicographically optimal base of polymatroid $\mathcal{P}$ with respect to weight vector $\boldsymbol{b}$. (For the reader's convenience, the statement of Theorem 3.1 in (42) is replicated in Theorem 4 in Section (1.12.1) of (40).) To this end, first define the vector $\boldsymbol{v}^a := (ab_j)_{j\in\mathcal{B}}$ for any $a \ge 0$. Second, let the vector $\boldsymbol{u}^a$ be a vector such that: (i) $u^a_j := ab_j$ if $0 \le a \le t'_j$, and (ii) $u^a_j := t'_j b_j$ if $a \ge t'_j$. It can be readily seen that $\boldsymbol{u}^a \le \boldsymbol{u}^{a'}$ (where the inequality is entrywise) for $0 \le a \le a'$. We claim that for any $a \ge 0$, $\boldsymbol{u}^a$ is a base of the polymatroid $\mathcal{P}_a = \{\boldsymbol{z} \in \mathcal{P} | \boldsymbol{z} \le \boldsymbol{v}^a\}$ (where once again the inequality is entrywise). Note that if this claim holds, then the above observations imply that the weight vector $\boldsymbol{b}$ and vector $\boldsymbol{t'}$ satisfy Conditions (3.1)–(3.5) of (42). Hence, Theorem 3.1 of (42) applies, and it implies that $\boldsymbol{y'}$ is the unique lexicographically optimal base of polymatroid $\mathcal{P}$ under weight vector $\boldsymbol{b}$.

Fix some $a \ge 0$. We complete the proof of Step 1 by establishing that $\boldsymbol{u}^a$ is a base of $\mathcal{P}_a$. Note that for any $B \subset \mathcal{B}$ we have that

$$\begin{aligned}\sum_{j\in B} u^a_j &\overset{(1)}{\le} \sum_{j\in B} t'_j b_j \overset{(2)}{=} \sum_{\tau=1}^{\ell} \sum_{j\in B\cap\mathcal{B}_\tau} t'_j b_j \overset{(3)}{\le} \sum_{\tau=1}^{\ell} \Bigg(\sum_{i\in N_{E^{(\tau-1)}}(\mathcal{B}_\tau)} s_i \Bigg) \frac{\sum_{j\in B\cap\mathcal{B}_\tau} b_j}{\sum_{j\in\mathcal{B}_\tau} b_j} \\ &\overset{(4)}{\le} \sum_{\tau=1}^{\ell} \sum_{i\in N_{E^{(\tau-1)}}(B\cap\mathcal{B}_\tau)} s_i = \sum_{i\in N_E(B)} s_i.\end{aligned} \tag{1.20}$$

Here, (1) uses the definition of $\boldsymbol{u}^a$, (2) follows since $\cup_{\tau=1}^{\ell}\mathcal{B}_\tau = \mathcal{B}$ and the sets $\{\mathcal{B}_\tau\}$ are disjoint, and (3) uses the definition of t'_j. Finally, (4) follows since the definition of $\mathcal{B}_\tau$ implies that $(\sum_{i\in N_{E(\tau-1)}(\mathcal{B}_\tau)} s_i)/(\sum_{j\in\mathcal{B}_\tau} b_j) \le (\sum_{i\in N_{E(\tau-1)}(B')} s_i)/(\sum_{j\in B'} b_j)$ for any $B' \subset \mathcal{B}_\tau$. The inequality (1.20) together with the fact that $\boldsymbol{u}^a \le \boldsymbol{v}^a$ (which holds by the construction of $\boldsymbol{u}^a, \boldsymbol{v}^a$) implies that $\boldsymbol{u}^a \in \mathcal{P}_a$.

Let $\bar{B}_1 = \{j|\ a \le t'_j\}$ and $\bar{B}_2 = \mathcal{B} \setminus \bar{B}_1$. Note that by the definition of $\{\mathcal{B}_\tau\}_{\tau=1}^{\ell}$ it follows that $t'_{j_1} < t'_{j_2}$ for $j_1 \in \mathcal{B}_{\tau_1}$ and $j_2 \in \mathcal{B}_{\tau_2}$ and $\tau_1, \tau_2 \in \{1, \dots, \ell\}$ such that $\tau_1 < \tau_2$. Moreover, $t'_{j_1} = t'_{j_2}$ for $j_1, j_2 \in \mathcal{B}_\tau$ for $\tau \in \{1, \dots, \ell\}$. These observations imply that $\bar{B}_1 = \cup_{\tau=\ell'+1}^{\ell} \mathcal{B}_\tau$ and $\bar{B}_2 = \cup_{\tau=1}^{\ell'} \mathcal{B}_\tau$ for some $\ell' \in \{1, \dots, \ell\}$. By the definition of $\boldsymbol{u}^a$, we have that $u_j^a = v_j^a$ for $j \in \bar{B}_1$. Moreover, for $\tau \le \ell'$ we have that $\mathcal{B}_\tau \subset \bar{B}_2$, and hence the definition of $\boldsymbol{u}^a$ and t'_j imply that $\sum_{k\in\mathcal{B}_\tau} u_k^a = \sum_{k\in\mathcal{B}_\tau} t'_k b_k = \sum_{i\in N_{E(\tau-1)}(\mathcal{B}_\tau)} s_i = \sum_{i\in\mathcal{S}_\tau} s_i$. Hence, we conclude that

$$\sum_{k\in\mathcal{B}} u_k^a = \sum_{k\in\bar{B}_1} v_k^a + \sum_{\tau=1}^{\ell'} \sum_{k\in\mathcal{S}_\tau} s_k. \tag{1.21}$$

Consider an arbitrary $z \in \mathcal{P}_a$, and observe that

$$\sum_{k\in\mathcal{B}} z_k = \sum_{k\in\bar{B}_1} z_k + \sum_{\tau=1}^{\ell'} \sum_{k\in\mathcal{B}_\tau} z_k \le \sum_{k\in B_1} v_k^a + \sum_{\tau=1}^{\ell'} \sum_{k\in N_E(\mathcal{B}_\tau)} s_k,$$

where we use $z_k \le v_k^a$ for $k \in \bar{B}_1$ and the fact that $\sum_{k\in\mathcal{B}_\tau} z_k \le \sum_{k\in N_E(\mathcal{B}_\tau)} s_k$ for any $z \in \mathcal{P}_a \subset \mathcal{P}$. Together with (1.21), this implies that $\boldsymbol{u}^a$ is a base of $\mathcal{P}_a$. Hence, we conclude that $\boldsymbol{y}' = \boldsymbol{y}^\star$.

Step 2: Characterization of marginal agents. By Proposition 10, we have that $\frac{q_j^b}{b_j} = r_j^\star = \rho(y_j^\star/b_j)$ for all $j \in \mathcal{B}$, which implies that the valuation of the marginal agent of type $j \in \mathcal{B}$ is given by $v_{b_j}^m = F_b^{-1}(1 - \frac{q_j^b}{b_j}) = F_b^{-1}(1 - \rho(y_j^\star/b_j))$. By Lemma 2, $\rho(\cdot)$ is a strictly increasing function, and hence the entries of the vector $\{\rho(y_j^\star/b_j)\}_{j\in\mathcal{B}}$ admit the same ranking as the entries of the vector $\{\frac{y_j^\star}{b_j}\}_{j\in\mathcal{B}}$ (note that, by Proposition 10, the assumptions in Lemma 2 are satisfied). Since $\boldsymbol{y}^\star = \boldsymbol{y}'$ by the definition of $\boldsymbol{y}'$, these observations imply that

$$v^m_{b_j} = v^m_{b_k} \quad \text{for } j,k \in \mathcal{B}_\tau \quad \text{with} \quad \tau \in \{1,\ldots,\ell\}, \tag{1.22}$$

$$v^m_{b_j} > v^m_{b_k} \quad \text{for } j \in \mathcal{B}_{\tau_1}, k \in \mathcal{B}_{\tau_2}, \quad \text{with} \quad \tau_1,\tau_2 \in \{1,\ldots,\ell\} \quad \text{and} \quad \tau_1 < \tau_2. \tag{1.23}$$

Thus, we have established that $j \in \mathcal{B}_\tau$ if and only if type j's marginal agent has the τth-highest value among the values of marginal agents of all buyer types.

We next establish the ranking result for the marginal agents of seller types. By Theorem 1, under any optimal commission-subscription pair $(\boldsymbol{\gamma}, \boldsymbol{\mu})$, the mass of agents who trade $(\boldsymbol{q}^b, \boldsymbol{q}^s)$, and therefore the marginal agents, are the same. Furthermore, by Proposition 6(ii), and the fact that Assumption 3 holds, it follows that there exists an optimal commission-subscription pair where $\boldsymbol{\gamma}^b = \boldsymbol{\mu}^b = 0$. Consider such an optimal commission-subscription pair, and note that since $\boldsymbol{\gamma}^b = \boldsymbol{\mu}^b = 0$, under this solution the price trading buyers of type $j \in \mathcal{B}$ pay is equal to $v^m_{b_j} = F_b^{-1}\left(1 - \frac{q^b_j}{b_j}\right)$.

We claim that buyers in $\mathcal{B}_\tau$ trade only with the sellers in $\mathcal{S}_\tau$, and vice versa. To see this, first focus on $\tau = 1$ and observe that buyers in $\mathcal{B}_1$ can trade only with sellers in $\mathcal{S}_1 = N_{E(0)}(\mathcal{B}_1)$, as they are not adjacent to any other seller type. Since the price the trading buyers in $\mathcal{B}_\tau$ pay is given by $v^m_{b_j}$ for any $j \in \mathcal{B}_\tau$ (by (1.22)), it follows by (1.23) that type $\mathcal{B}_1$ buyers trade at the highest prices. Thus, if sellers of type $\mathcal{S}_1$ trade, then they trade with buyers whose types belong to $\mathcal{B}_1$. On the other hand, by the construction in Proposition 10, it follows that all seller types involve in some trade at the revenue-maximizing solution. Hence, it follows that all seller types in $\mathcal{S}_1$ have nonzero trade with some buyer type in $\mathcal{B}_1$, and buyers in $\mathcal{B}_1$ trade only with sellers in $\mathcal{S}_1$ and vice versa. By induction, the same argument implies that buyers in $\mathcal{B}_\tau$ trade positive quantities with the sellers in $\mathcal{S}_\tau$. Thus, Proposition 10 implies that for any $i \in \mathcal{S}_\tau$ we have that $v^m_{s_i} = F_s^{-1}\left(\frac{q^s_i}{s_i}\right) = F_s^{-1}\left(\frac{\rho(y^\star_j/b_j)}{y^\star_j/b_j}\right)$ for some buyer type $j \in \mathcal{B}_\tau$. Moreover, since $y^\star_j = y'_j = t'_j b_j$, it follows that $v^m_{s_i} = F_s^{-1}\left(\frac{\rho(t'_j)}{t'_j}\right)$.

By Lemma 2, it follows that $\rho(t)/t = 1$ for $0 < t \leq t_0$, where t_0 is a constant that depends on the value distributions. Let $\mathcal{T} = \{\tau | t'_j \leq t_0 \text{ for } j \in \mathcal{B}_\tau \}$, and observe that by construction of $\{t'_j\}$ we have that $t'_j > 0$ for all $j \in \mathcal{B}$ and $\mathcal{T} = \{1, \ldots, \bar{\tau}\}$ for some $\bar{\tau} \in \mathbb{Z}$. Consider some $\tau \leq \bar{\tau}$ and $i \in \mathcal{S}_\tau$. It follows from the observations above that $v^m_{s_i} = F_s^{-1}(\rho(t'_j)/t'_j) = F_s^{-1}(1) = \bar{v}_{s_i}$. Next, consider $\tau > \bar{\tau}$ and $i \in S_\tau$. Observe that Lemma 2 also implies that $\rho(t)/t$ is strictly decreasing for $t > t_0$. Hence, for such τ we have that $v^m_{s_i} = F_s^{-1}\left(\frac{\rho(t'_j)}{t'_j}\right) < \bar{v}_{s_i}$ where $j \in \mathcal{B}_\tau$. Moreover, $v^m_{s_i}$ is strictly decreasing in τ. These observations together imply that

$$v^m_{s_i} = \bar{v}_{s_i} \quad \text{for } i \in \mathcal{S}_\tau, \quad \tau \leq \bar{\tau}, \tag{1.24}$$

$$v^m_{s_{i_1}} = v^m_{s_{i_2}} \quad \text{for } i_1, i_2 \in \mathcal{S}_\tau \quad \text{with} \quad \tau \in \{1, \ldots, \ell\}, \tag{1.25}$$

$$v^m_{s_{i_1}} > v^m_{s_{i_2}} \quad \text{for } i_1 \in \mathcal{S}_{\tau_1}, i_2 \in \mathcal{S}_{\tau_2}, \quad \text{with} \quad \tau_1, \tau_2 \in \{1, \ldots, \ell\}, \quad \tau_2 > \bar{\tau} \quad \text{and} \quad \tau_1 < \tau_2. \tag{1.26}$$

Hence the claim follows. □

Proof of Theorem 3. Proof of part (i). Let

$$f(t) := \max_{r \in [0, \min\{1,t\}]} \left[F_b^{-1}(1-r) - F_s^{-1}\left(\frac{r}{t}\right)\right] r \tag{1.27}$$

for $t > 0$; note that $f(\cdot)$ is defined as in the statement of Proposition 10. Let $\bar{V}(\boldsymbol{s}, \boldsymbol{b})$ be defined as $\bar{V}(\boldsymbol{s}, \boldsymbol{b}) := b_0 f\left(\frac{s_0}{b_0}\right)$ where s_0 and b_0 are defined as in the statement of the theorem. Therefore, we need to show that $V_{max}(\boldsymbol{s}, \boldsymbol{b}) = \bar{V}(\boldsymbol{s}, \boldsymbol{b})$.

We first prove that $V_{max}(\boldsymbol{s}, \boldsymbol{b}) \leq \bar{V}(\boldsymbol{s}, \boldsymbol{b})$. Fix any arbitrary network $G(\mathcal{S} \cup \mathcal{B}, E)$ and, by Proposition 10, let $\boldsymbol{y}^\star > 0$ be the unique optimal solution to Problem (1.19) when f is defined as in (1.27). Then,

$$\begin{aligned} V_{opt}(E, \boldsymbol{s}, \boldsymbol{b}) &\overset{(a)}{=} \left(\sum_{j\in\mathcal{B}} b_j\right) \sum_{j\in\mathcal{B}} \frac{b_j}{\sum_{j\in\mathcal{B}} b_j} f\left(\frac{y_j^\star}{b_j}\right) \overset{(b)}{\leq} \left(\sum_{j\in\mathcal{B}} b_j\right) f\left(\frac{\sum_{j\in\mathcal{B}} y_j^\star}{\sum_{j\in\mathcal{B}} b_j}\right) \\ &\overset{(c)}{\leq} \left(\sum_{j\in\mathcal{B}} b_j\right) f\left(\frac{\sum_{i\in\mathcal{S}} s_i}{\sum_{j\in\mathcal{B}} b_j}\right) \overset{(d)}{=} \bar{V}(\boldsymbol{s}, \boldsymbol{b}), \end{aligned} \tag{1.28}$$

where equality (a) follows from part (iii) of Proposition 10 (note that the assumptions in the statement of the theorem imply that the assumptions in Proposition 10 hold). Inequality (b) follows from the fact that f is strictly concave (Proposition 10 together with Lemma 2). Inequality (c) follows from the fact that f is strictly increasing and that $\boldsymbol{y}^\star$ is feasible and thus must satisfy $\sum_{j\in\mathcal{B}} y_j^\star \leq \sum_{i\in N(\mathcal{B})=\mathcal{S}} s_i$. Finally, equality (d) follows from the fact that, by definition, $\bar{V}(\boldsymbol{s}, \boldsymbol{b}) = b_0 f(\frac{s_0}{b_0})$. Therefore, we have established that $V_{opt}(E, \boldsymbol{s}, \boldsymbol{b}) \leq \bar{V}(\boldsymbol{s}, \boldsymbol{b})$ for any network $G(\mathcal{S} \cup \mathcal{B}, E)$ and, thus, we must have that $V_{max}(\boldsymbol{s}, \boldsymbol{b}) = \max_{E\subseteq\mathcal{B}\times\mathcal{S}} V_{opt}(E, \boldsymbol{s}, \boldsymbol{b}) \leq \bar{V}(\boldsymbol{s}, \boldsymbol{b})$.

Next, let $\boldsymbol{y}$ be defined as $y_j := b_j \frac{\sum_{i\in\mathcal{S}} s_i}{\sum_{j\in\mathcal{B}} b_j}$ for all $j \in \mathcal{B}$. As the function f is strictly concave (Lemma 2), inequality (b) in (1.28) holds by equality if and only if $\boldsymbol{y}^\star = \boldsymbol{y}$. Note that, in fact, such a y is feasible for Problem (1.19) with f given by (1.27) (i.e. it satisfies constraints (1.19b) and (1.19c)) if and only if the network satisfies the weighted Hall's marriage condition; this readily follows from Definition 2. Moreover, by the definition of $\boldsymbol{y}$, we have that $\sum_{j\in\mathcal{B}} y_j = \sum_{i\in\mathcal{S}} s_i$, and thus inequality (c) in Equation (1.28) holds with equality as well. Therefore, we have established that $V_{opt}(E, \boldsymbol{s}, \boldsymbol{b}) = \bar{V}(\boldsymbol{s}, \boldsymbol{b})$ if and only if network $G(\mathcal{S} \cup \mathcal{B}, E)$ satisfies the weighted Hall's marriage condition. Because a complete bipartite network satisfies the weighted Hall's marriage condition, we obtain that $V_{max}(\boldsymbol{s}, \boldsymbol{b}) = \max_{E\subseteq\mathcal{B}\times\mathcal{S}} V_{opt}(E, \boldsymbol{s}, \boldsymbol{b}) \geq \bar{V}(\boldsymbol{s}, \boldsymbol{b})$. This completes the proof of part (i).

Proof of part (ii): By Claim (ii) in Proposition 5 we have that, if a network $G(\mathcal{S} \cup \mathcal{B}, E)$ satisfies the ε-marriage condition, then $V_h(E, \boldsymbol{s}, \boldsymbol{b}) \geq (1-\varepsilon)V_{max}(\boldsymbol{s}, \boldsymbol{b})$ where V_h is the revenue derived from the optimal homogeneous commissions/subscriptions. Hence,

$V_{opt}(E, \boldsymbol{s}, \boldsymbol{b}) \geq V_h(E, \boldsymbol{s}, \boldsymbol{b}) \geq (1-\varepsilon)V_{max}(\boldsymbol{s}, \boldsymbol{b})$, as desired. $\square$

Proof of Proposition 3. Let $\boldsymbol{y}_1^\star, \boldsymbol{y}_2^\star$ be optimal solutions to problem (1.19) associated with networks $G(\mathcal{S} \cup \mathcal{B}, E_1)$ and $G(\mathcal{S} \cup \mathcal{B}, E_2)$, respectively, when f is defined as in the statement of Proposition 10. As $\sum_{i \in N_{E_1}(B)} s_i \geq \sum_{i \in N_{E_2}(B)} s_i$ for all $B \subset \mathcal{B}$, we have that $\boldsymbol{y}_2^\star$ is a feasible solution for the problem associated with network $G(\mathcal{S} \cup \mathcal{B}, E_1)$. Therefore, by Proposition 10, we have that $\boldsymbol{y}_1^\star > 0$, $\boldsymbol{y}_2^\star > 0$ and

$$V_{opt}(E_1, \boldsymbol{s}, \boldsymbol{b}) \;=\; \sum_{j \in \mathcal{B}} b_j f\left(\frac{(y_1^\star)_j}{b_j}\right) \;\geq\; \sum_{j \in \mathcal{B}} b_j f\left(\frac{(y_2^\star)_j}{b_j}\right) \;=\; V_{opt}(E_2, \boldsymbol{s}, \boldsymbol{b}),$$

which completes the proof. $\square$

1.11 Proofs of Results in Section 1.6

Proof of Proposition 5. Fix a network $G(\mathcal{S} \cup \mathcal{B}, E)$. To ease exposition, throughout the rest of the proof we omit the dependence on the network in the notation.

With a slight abuse of notation, let $\bar{V}_h(\mu^b)$ denote the revenue under homogeneous commissions/commissions as a function of the buyers' subscription fee μ^b, when μ^s, γ^b, and γ^s are all set to zero, and define $\bar{V}_h = \max_{\mu^b \geq 0} \bar{V}_h(\mu^b)$. Note that $V_h \geq \bar{V}_h$, and therefore it suffices to establish the results in parts (i) and (ii) using either $\bar{V}_h$ or $\bar{V}_h(\mu^b)$ for some $\mu^b \geq 0$. As $\bar{v}_b < \infty$, it suffices to consider $\mu^b \in [0, \bar{v}_b]$ as otherwise no trade will take place and $\bar{V}_h(\mu^b) = 0$ when $\mu^b > \bar{v}_b$.

Before proceeding to the proofs of parts (i) and (ii) of this proposition, we establish a couple of intermediate results that will be exploited later on.

$\underline{\text{Claim 1: } \bar{V}_h(\mu^b) = \sum_{j \in \mathcal{B}} \mu^b b_j r_j^\star(\mu^b) \text{ for } \mu^b \in [0, \bar{v}_b], \text{ where } r_j^\star(\mu^b) \text{ is defined as in (1.29).}}$ As a first step in our proof, we provide an expression for $\bar{V}_h(\mu^b)$. We start by finding the optimal equilibrium demand as a function of μ^b, which we denote by $\boldsymbol{q}^b(\mu^b)$. (With a slight abuse of notation, in what follows we define the relevant quantities as functions of μ^b to make this dependency explicit.)

Fix $\mu^b \in [0, \bar{v}_b)$. By Proposition 11(i), Assumptions (1.91)–(1.94) hold. Using Proposition 11(ii), we know that the equilibrium demands are $q^\star_j(\mu^b) = b_j r^\star_j(\mu^b)$ where

$$r^\star_j(\mu^b) = \underset{r \in [0, \min\{1, y^\star_j(\mu^b)/b_j\}]}{\arg\max} \int_0^r F_b^{-1}(1-x) - \mu^b - F_s^{-1}\left(\frac{x}{y^\star_j(\mu^b)/b_j}\right) dx, \tag{1.29}$$

and $\boldsymbol{y}^\star(\mu^b) > 0$ is the optimal solution to (1.19). By Lemma 3, $\boldsymbol{y}^\star$ is the lexicographically optimal base of polymatroid $\mathcal{P} = \{\boldsymbol{y} \geq \boldsymbol{0} : \sum_{j \in B} y_j \leq \sum_{i \in N_E(B)} s_i,\ \forall B \subset \mathcal{B}\}$, which is independent of μ^b. Therefore, we drop the dependency on μ^b in $y^\star(\mu^b)$, and define $t_j := \frac{y^\star_j}{b_j}$ for all $j \in \mathcal{B}$. Note that, as $y^\star_j > 0$, we have that $t_j > 0$. From expression (1.29), it follows that $r^\star_j(\bar{v}^b) = 0$ (corresponding to no trade in equilibrium). Therefore can use expression (1.29) for all $\mu^b \in [0, \bar{v}^b]$.

Notice that the value of $r^\star_j(\mu^b)$ can be found by solving a (strictly) concave maximization problem. Using the first-order optimality condition, we can express $r^\star_j(\mu^b)$ as

$$r^\star_j(\mu^b) = \max\left\{r : F_b^{-1}(1-r) - F_s^{-1}\left(\frac{r}{t_j}\right) \geq \mu^b,\ 0 \leq r \leq \min\{1, t_j\}\right\}, \qquad \forall \mu^b \in [0, \bar{v}_b]. \tag{1.30}$$

Finally, the revenue can be expressed as

$$\bar{V}_h(\mu^b) = \sum_{j \in \mathcal{B}} \mu^b_j q^\star_j(\mu^b) = \sum_{j \in \mathcal{B}} \mu^b b_j r^\star_j(\mu^b) \qquad \text{for } \mu^b \in [0, \bar{v}_b]. \tag{1.31}$$

Claim 2: The functions $\{\mu^b r^\star_j(\mu^b)\}$ are continuous, concave, and satisfy $0 \leq \mu^b r^\star_j(\mu^b) \leq f(t_j)$, for all $\mu^b \in$ where $f(\cdot)$ is defined as in (1.32). Moreover, these bounds are tight. We now derive some properties of the functions $\mu^b r^\star_j(\mu^b)$ that we will later use to bound $\bar{V}_h(\mu^b)/V_{opt}$. In what follows, we show that the functions $\mu^b r^\star_j(\mu^b)$ are continuous and concave in $\mu^b \in [0, \bar{v}_b]$ for every

$j \in \mathcal{B}$, and that $0 \leq r_j^\star(\mu^b) \leq f(t_j)$, where

$$f(t) := \max_{r \in [0, \min\{1,t\}]} \left[F_b^{-1}(1-r) - F_s^{-1}\left(\frac{r}{t}\right)\right] r \text{ for } t > 0. \tag{1.32}$$

Fix j. We first show that $r_j^\star(\mu^b)$ is continuous, weakly decreasing, and concave in $\mu^b \in [0, \bar{v}_b]$. Consider the expression in (1.30) and note that, the objective function r is jointly continuous in (r, μ^b) and that the function $F_b^{-1}(1-r) - F_s^{-1}(\frac{r}{t_j}) - \mu^b$ is also jointly continuous in (r, μ^b). By the maximum theorem on page 116 of (49), the value function $r_j^\star(\mu^b)$ is thus continuous in $\mu^b \in [0, \bar{v}_b]$. Moreover, the function $F_b^{-1}(1-r) - F_s^{-1}\left(\frac{r}{t_j}\right)$ is continuous and strictly decreasing in r (recall that $t_j > 0$ by definition). Thus, for any $\mu_1^b \geq \mu_2^b$ we have that $\{r : F_b^{-1}(1-r) - F_s^{-1}\left(\frac{r}{t_j}\right) \geq \mu_1^b,\ 0 \leq r \leq \min\{1, t_j\}\} \subset \{r : F_b^{-1}(1-r) - F_s^{-1}\left(\frac{r}{t_j}\right) \geq \mu_2^b,\ 0 \leq r \leq \min\{1, t_j\}\}$, which implies that $r_j^\star(\mu^b)$ is weakly decreasing in μ^b. Finally, for any $\mu_1^b, \mu_2^b \in [0, \bar{v}_b]$, let $\bar{r} := r_j^\star\left(\frac{1}{2}\mu_1^b + \frac{1}{2}\mu_2^b\right)$, $r_1 := r_j^\star\left(\mu_1^b\right)$, and $r_2 := r_j^\star\left(\mu_2^b\right)$. Given that functions $F_b^{-1}(1-r)$ and $-F_s^{-1}(r)$ are concave, we have that $F_b^{-1}\left(1 - \frac{(r_1+r_2)}{2}\right) - F_s^{-1}\left(\frac{r_1+r_2}{2t_j}\right) \geq \frac{1}{2}\left(F_b^{-1}(1-r_1) - F_s^{-1}(\frac{r_1}{t_j})\right) + \frac{1}{2}\left(F_b^{-1}(1-r_2) - F_s^{-1}(\frac{r_2}{t_j})\right) \geq \frac{1}{2}\mu_1^b + \frac{1}{2}\mu_2^b$. Moreover, by definition of r_1 and r_2, we have that $\frac{r_1+r_2}{2} \leq t_j$ and $\frac{r_1+r_2}{2} \leq 1$ because both $r_1, r_2 \leq \min\{1, t_j\}$. This implies that $\frac{r_1+r_2}{2} \in \{r : F_b^{-1}(1-r) - F_s^{-1}\left(\frac{r}{t_j}\right) \geq \frac{\mu_1^b + \mu_2^b}{2},\ r \leq t_j\}$. Thus, we have $\bar{r} \geq \frac{r_1+r_2}{2}$, and hence $r_j^\star(\mu^b)$ is a concave function in $\mu^b \in [0, \bar{v}_b]$. Therefore, we have established that $r_j^\star(\mu^b)$ is continuous, weakly decreasing, and concave, which implies that the function $\mu^b r_j^\star(\mu^b)$ is continuous and concave in $\mu^b \in (0, \bar{v}_b]$.

Next, we obtain tight lower and upper bounds on $\mu^b r_j^\star(\mu^b)$. To that end, we start by showing that $\max_{\mu^b \in [0, \bar{v}_b]} \mu^b r_j^\star(\mu^b) = f(t_j)$ for all $j \in \mathcal{B}$, where f is defined as in (1.32). Recall that we have defined t_j as $t_j = y_j^\star / b_j$. Let $\bar{\mu}^b := \arg\max_{\mu^b \in [0, \bar{v}_b]} \mu^b r_j^\star(\mu^b)$. From (1.30), we have that $\bar{\mu}^b \leq F_b^{-1}(1 - r_j^\star(\bar{\mu}^b)) - F_s^{-1}\left(\frac{r_j^\star(\bar{\mu}^b)}{t_j}\right)$. Therefore, $\bar{\mu}^b r_j^\star(\bar{\mu}^b) \leq [F_b^{-1}(1 - r_j^\star(\bar{\mu}^b)) - F_s^{-1}(\frac{r_j^\star(\bar{\mu}^b)}{t_j})] r_j^\star(\bar{\mu}^b) \leq \max_{r \in [0, \min\{1, t_j\}]} [F_b^{-1}(1-r) - F_s^{-1}(\frac{r}{t_j})] r = f(t_j)$. To show that this bound is tight, let $\bar{r} := \arg\max r \in [0, \min\{1, t_j\}] \left[F_b^{-1}(1-r) - F_s^{-1}\left(\frac{r}{t_j}\right)\right] r$, and consider $\mu^b = F_b^{-1}(1-\bar{r}) - F_s^{-1}(\frac{\bar{r}}{t_j})$. Notice that $\mu^b \in [0, \bar{v}_b]$ and, by (1.30), we have that

$r^\star_j(\mu^b) = \bar{r}$. Therefore, $\mu^b r^\star_j(\mu^b) = \mu^b \bar{r} = [F_b^{-1}(1-r) - F_s^{-1}(\frac{\bar{r}}{t_j})]\bar{r} = f(t_j)$. Finally, by (1.30), it is immediate to see that $r^\star_j(\mu^b)\mu^b \geq 0$ and that it is equal to zero if $\mu^b \in \{0, \bar{v}_b\}$. Therefore, we conclude that $0 \leq \mu^b r^\star_j(\mu^b) \leq f(t_j)$, and that both these bounds are tight.

Proof of part (i): $\frac{V_h}{V_{opt}} \geq \frac{1}{2}$. Define $\left(\tilde{b}_j\right)_{j\in\mathcal{B}}$ as $\tilde{b}_j := b_j f(t_j)$, and define $\tilde{r}^\star_j(\mu^b) := \frac{b_j}{\tilde{b}_j} r^\star_j(\mu^b)$ for $\mu^b \in [0, \bar{v}_b]$. Note that the function $\tilde{r}^\star_j(\mu^b)\mu^b$ is still concave in $\mu^b \in [0, \bar{v}_b]$ as $\tilde{b}_j$ is a positive constant. Moreover, by using the tight upper bound on $r^\star_j$, we conclude that $\max_{\mu^b\in[0,\bar{v}_b]} \tilde{r}^\star_j(\mu^b)\mu^b = 1$ for every $j \in \mathcal{B}$. Let the weight vector $\boldsymbol{w}$ be defined as $w_j := \frac{\tilde{b}_j}{\sum_{j'\in\mathcal{B}} \tilde{b}_{j'}}$ for all $j \in \mathcal{B}$. Then,

$$\begin{aligned}\frac{V_h}{V_{opt}} &\overset{(a)}{\geq} \frac{\max_{\mu^b\in[0,\bar{v}_b]} \sum_{j\in\mathcal{B}} b_j r^\star_j(\mu^b)\mu^b}{\sum_{j\in\mathcal{B}} b_j f(\frac{y^\star_j}{b_j})} \overset{(b)}{=} \frac{\max_{\mu^b\in[0,\bar{v}_b]} \sum_{j\in\mathcal{B}} \tilde{b}_j \tilde{r}^\star_j(\mu^b)\mu^b}{\sum_{j\in\mathcal{B}} \tilde{b}_j} \\ &\overset{(c)}{=} \max_{\mu^b\in[0,\bar{v}_b]} \sum_{j\in\mathcal{B}} w_j \tilde{r}^\star_j(\mu^b)\mu^b \overset{(d)}{\geq} \max_{\mu^b\in[0,\bar{v}_b]} \min_{j\in\mathcal{B}} \tilde{r}^\star_j(\mu^b)\mu^b, \end{aligned} \tag{1.33}$$

where inequality (a) follows from the fact that the numerator satisfies $V_h \geq \max_{\mu\in[0,\bar{v}_b]} \bar{V}_h(\mu) = \max_{\mu\in[0,\bar{v}_b]} \sum_{j\in\mathcal{B}} b_j r^\star_j(\mu^b)\mu^b$, and in the denominator we used Proposition 10(iii) to derive $V_{opt} = \sum_{j\in\mathcal{B}} b_j f(\frac{y^\star_j}{b_j})$. Equality (b) follows from the definition of $\tilde{\boldsymbol{b}}$ and, similarly, equality (c) follows from using the definition of $\tilde{\boldsymbol{w}}$. Finally, inequality (d) follows by noting that $\sum_{j\in\mathcal{B}} w_j = 1$, $w_j \geq 0$ for all $j \in \mathcal{B}$, which in turn implies that $\sum_{j\in\mathcal{B}} w_j \tilde{r}^\star_j(\mu^b)\mu^b \geq \min_{j\in\mathcal{B}} \tilde{r}^\star_j(\mu^b)\mu^b$ for all $\mu^b \in [0, \bar{v}_b]$.

Next, we provide a lower bound on $\max_{\mu^b\in[0,\bar{v}_b]} \min_{j\in\mathcal{B}} \tilde{r}^\star_j(\mu^b)\mu^b$. To ease notation, let $H_j(\mu^b) := \mu^b \tilde{r}^\star_j(\mu^b)$ for all $\mu^b \in [0, \bar{v}_b]$ and define two functions $H, G : [0, \bar{v}_b] \to [0, 1]$ where $H(\mu^b) := \min_{j\in\mathcal{B}} H_j(\mu^b)$ and $G(\mu^b) := \min\{\frac{\mu^b}{\bar{v}_b}, 1 - \frac{\mu^b}{\bar{v}_b}\}$ for $\mu^b \in [0, \bar{v}_b]$. By Claim 2 above, we have that $H_j(\mu^b)$ is continuous and concave for every $\mu^b \in [0, \bar{v}_b]$. We conclude that $H(\mu^b)$ is continuous and concave in $\mu^b \in [0, \bar{v}_b]$ because it is the minimum of a finite set of continuous and concave functions. Moreover, as each H_j is continuous and concave, it is differentiable almost everywhere, and thus H is differentiable almost everywhere. By Claim 2 above, we also have that $H(0) = H(\bar{v}_b) = 0$. In addition, note that $G(\mu^b)$ is a piecewise linear concave function, consisting of two pieces, and it is symmetric around $\mu^b = \frac{1}{2}\bar{v}_b$ where it achieves its peak value of $\frac{1}{2}$.

We want to show that $H\left(\frac{1}{2}\bar{v}_b\right) \geq G\left(\frac{1}{2}\bar{v}_b\right) = 1/2$; this will imply the desired bound as $\max_{\mu^b \in [0,\bar{v}_b]} \min_{j \in \mathcal{B}} \tilde{r}_j^\star(\mu^b)\mu^b \geq \min_{j \in \mathcal{B}} \tilde{r}_j^\star(\frac{1}{2}\bar{v}_b)\frac{1}{2}\bar{v}_b = H(\frac{1}{2}\bar{v}_b)$. In fact, we show a stronger result: we show that $H(\mu^b) \geq G(\mu^b)$ for all $\mu^b \in [0, \frac{1}{2}\bar{v}_b]$. (This trivially holds at $\mu^b = 0$ as $H(0) = G(0)$.)

Suppose on the contrary that there exists $\bar{\mu} \in (0, \frac{1}{2}\bar{v}_b]$ such that $H(\bar{\mu}) < G(\bar{\mu})$, and suppose that H is differentiable at $\bar{\mu}$. The latter is without loss of generality; as both H and G are continuous in $[0, \bar{v}_b]$ and H is differentiable almost everywhere. Next, we consider different cases which cover different possible values of $H'(\bar{\mu})$, and obtain contradictions in each case:

(1) If $H'(\bar{\mu}) > \frac{1}{\bar{v}_b}$, then by the concavity of H, we have that $H(0) \leq H(\bar{\mu}) + H'(\bar{\mu})(0 - \bar{\mu})$. As $H'(\bar{\mu})(0-\bar{\mu}) \leq -\frac{\bar{\mu}}{\bar{v}_b}$, we have that $H(\bar{\mu}) + H'(\bar{\mu})(0 - \bar{\mu}) < G(\bar{\mu}) - \frac{\bar{\mu}}{\bar{v}_b} = 0$, and thus $0 = H(0) < 0$, which is a contradiction.

(2) If $0 \leq H'(\bar{\mu}) \leq \frac{1}{\bar{v}_b}$, then, as we assumed that H is differentiable at $\bar{\mu}$, there exist $j_0 \in \mathcal{B}$ and $\epsilon > 0$ such that $H(\mu) = H_{j_0}(\mu)$ for every $\mu \in (\bar{\mu} - \epsilon, \bar{\mu} + \epsilon)$, and thus we must have $0 \leq H'_{j_0}(\bar{\mu}) \leq \frac{1}{\bar{v}_b}$. We argue that this contradicts $\max_{x \in [0,\bar{v}_b]} H_{j_0}(x) = 1$, which was established in Claim 2 above. To that end, we use the fact that the concavity of $H_{j_0}(\mu)$ implies that $H_{j_0}(\mu) \leq H_{j_0}(\bar{\mu}) + H'_{j_0}(\bar{\mu})(\mu - \bar{\mu})$, for any $\mu \in [0, \bar{v}_b]$. Then, for $\mu \in [0, \bar{\mu}]$, we have that $H_{j_0}(\mu) \leq H_{j_0}(\bar{\mu}) < G(\bar{\mu}) < 1$. Moreover, for any $\mu \in [\bar{\mu}, \bar{v}_b]$, we have that $H'_{j_0}(\bar{\mu})(\mu - \bar{\mu}) \leq \frac{1}{\bar{v}_b}(\mu - \bar{\mu})$, which implies that $H_{j_0}(\mu) < G(\bar{\mu}) + \frac{1}{\bar{v}_b}(\tilde{\mu} - \bar{\mu}) \leq \frac{\mu}{\bar{v}_b} \leq 1$. Therefore, we have that $H_{j_0}(\mu) < 1$ for every $\mu \in [0, \bar{v}_b]$, and thus $\max_{x \in [0,\bar{v}_b]} H_{j_0}(x) < 1$, which is a contradiction.

(3) If $-\frac{1}{\bar{v}_b} \leq H'(\bar{\mu}) < 0$, we follow an argument along the lines of the one in case (2). We define $j_0 \in \mathcal{B}$ as in (2), and then prove a contradiction to $\max_{\mu \in [0,\bar{v}_b]} H_{j_0}(\mu) = 1$. For

any $\mu \in [0, \bar{\mu}]$, we have that $H'_{j_0}(\bar{\mu})(\mu - \bar{\mu}) \leq -\frac{1}{\bar{v}_b}(\mu - \bar{\mu})$. As $H_{j_0}(\bar{\mu}) < G(\bar{\mu}) \leq 1 - \frac{\bar{\mu}}{\bar{v}_b}$, we have that $H_{j_0}(\mu) < (1 - \frac{\bar{\mu}}{\bar{v}_b}) - \frac{1}{\bar{v}_b}(\mu - \bar{\mu}) \leq 1$. For any $\mu \in [\bar{\mu}, \bar{v}_b]$, we have that $H'_{j_0}(\bar{\mu})(\mu - \bar{\mu}) \leq 0$, which implies that $H_{j_0}(\mu) \leq H_{j_0}(\bar{\mu}) < G(\bar{\mu}) < 1$. Therefore, we have that $H_{j_0}(\mu) < 1$ for every $\mu \in [0, \bar{v}_b]$, and thus $\max_{x \in [0, \bar{v}_b]} H_{j_0}(x) < 1$, which is a contradiction.

(4) If $H'(\bar{\mu}) < -\frac{1}{\bar{v}_b}$, the argument is similar to that in case (1). By the concavity of H, we have that $H(\bar{v}_b) \leq H(\bar{\mu}) + H'(\bar{\mu})(\bar{v}_b - \bar{\mu})$. Since $H'(\bar{\mu})(\bar{v}_b - \bar{\mu}) < -\frac{1}{\bar{v}_b}(\bar{v}_b - \bar{\mu}) \leq -1 + \frac{\bar{\mu}}{\bar{v}_b}$, this leads to a contradiction as $0 = H_{j_0}(\bar{v}_b) < G(\bar{\mu}) - 1 + \frac{\bar{\mu}}{\bar{v}_b} = -1 + 2\frac{\bar{\mu}}{\bar{v}_b} < 0$.

Therefore, we have established that $H(\mu^b) \geq G(\mu^b)$ for all $\mu^b \in [0, \frac{1}{2}\bar{v}_b]$. To conclude the proof, note that

$$\frac{V_h}{V_{opt}} \geq \max_{\mu^b \in [0, \bar{v}_b]} \min_{j \in \mathcal{B}} \tilde{r}^\star_j(\mu^b)\mu^b = \max_{\mu^b \in [0, \bar{v}_b]} H(\mu^b) \;\geq\; H\left(\frac{1}{2}\bar{v}_b\right) \;\geq\; G\left(\frac{1}{2}\bar{v}_b\right) \;=\; \frac{1}{2}, \tag{1.34}$$

where the first inequality follows from (1.33). Thus, we obtain that $V_h \geq \frac{1}{2}V_{opt}$, as desired.

Proof of Part (ii). Suppose that the network satisfies the ε-marriage condition for $\varepsilon \in [0, 1)$. (The claim holds trivially when $\varepsilon = 1$ because $V_h \geq 0$.) By part (i) of Theorem 3, we have that $V_{max} = (\sum_{j \in \mathcal{B}} b_j) f(\tilde{t})$, where $\tilde{t} := \frac{\sum_{i \in \mathcal{S}} s_i}{\sum_{j \in \mathcal{B}} b_j}$ and f is defined as in (1.32). Define $\tilde{r} := \arg\max_{r \in [0, \min\{1, \tilde{t}\}]} \left[F_b^{-1}(1 - r) - F_s^{-1}(\frac{r}{\tilde{t}})\right] r$, and define $\bar{\mu}^b := F_b^{-1}(1 - \tilde{r}) - F_s^{-1}(\frac{\tilde{r}}{\tilde{t}})$. To obtain the desired lower bound, we consider the revenue $\bar{V}(\bar{\mu}^b)$ defined in (1.31) and use lower bounds on $\frac{y^\star_j}{b^\star_j}$ and $r^\star_j(\bar{\mu}^b)$ (defined as in (1.30)) for all $j \in \mathcal{B}$.

Claim (ii)-1: $\frac{y^\star_j}{b^\star_j} \geq (1 - \varepsilon)\tilde{t}$ for all $j \in \mathcal{B}$. Let vector $\boldsymbol{y}^\star > 0$ be the optimal solution to problem (1.19) and, as before, let $t_j := \frac{y^\star_j}{b_j}$. We further let the distinct values of t_j's to be given by $0 < t_1 < t_2 \cdots < t_l$ and let $\bar{\mathcal{B}}_1 = \{j \in \mathcal{B} : \frac{y^\star_j}{b_j} = t_1\}$. This implies that

$$\frac{y^\star_j}{b_j} \;\geq\; t_1 \;=\; \frac{\sum_{j \in \bar{\mathcal{B}}_1} y^\star_j}{\sum_{j \in \bar{\mathcal{B}}_1} b_j} \overset{(e)}{=} \frac{\sum_{i \in N_E(\bar{\mathcal{B}}_1)} s_i}{\sum_{j \in \bar{\mathcal{B}}_1} b_j} \overset{(f)}{\geq} (1 - \varepsilon)\frac{\sum_{i \in \mathcal{S}} s_i}{\sum_{j \in \mathcal{B}} b_j} = (1 - \varepsilon)\tilde{t} \qquad \text{for every } j \in \mathcal{B}, \tag{1.35}$$

where equality (e) follows from the fact that, by Lemma 3, vector $\boldsymbol{y}^\star$ is a lexicographically optimal base in polymatroid $\mathcal{P} = \{\boldsymbol{y} \geq \mathbf{0} : \sum_{j \in B} y_j \leq \sum_{j \in B} b_j, \forall B \subset \mathcal{B}\}$ and, by the equivalence of item (i) and item (ii) of Theorem 3.2 in (42), we have that $\sum_{j \in \bar{\mathcal{B}}_1} y_j^\star = \sum_{i \in N_E(\bar{\mathcal{B}}_1)} s_i$. Equality (f) follows directly from the definition of the ε-marriage condition.

Claim (ii)-2: $r_j^\star(\bar{\mu}^b) \geq (1-\varepsilon)\tilde{r}$ for all $j \in \mathcal{B}$. Let $r_j^\star := r_j^\star(\bar{\mu}^b)$ where $r_j^\star(\bar{\mu}^b)$ is defined as in (1.30). We want to show $r_j^\star \geq (1-\varepsilon)\tilde{r}$ for all $j \in \mathcal{B}$. As the $F_b^{-1}(1-r) - F_s^{-1}\left(\frac{r}{t_j}\right)$ is decreasing in r, then we have that one of the constraints in (1.30) must be binding, that is, either $\bar{\mu}^b = F_b^{-1}(1-r_j^\star) - F_s^{-1}(\frac{r_j^\star}{y_j^\star/b_j})$ or $r_j^\star = \frac{y_j^\star}{b_j}$. If $r_j^\star = \frac{y_j^\star}{b_j}$, then $r_j^\star = \frac{y_j^\star}{b_j} \geq (1-\varepsilon)\tilde{t}$ follows from (1.35), and thus $\frac{1}{1-\varepsilon}\frac{r_j^\star}{\tilde{t}} \geq 1 \geq \frac{\tilde{r}}{\tilde{t}}$, where the last inequality follows from the definition of $\tilde{r}$. Hence, if $r_j^\star = \frac{y_j^\star}{b_j}$, we have that $r_j^\star \geq (1-\varepsilon)\tilde{r}$.

Suppose that $\bar{\mu}^b = F_b^{-1}(1-r_j^\star) - F_s^{-1}(\frac{r_j^\star}{y_j^\star/b_j})$. Recall that, by the definition of $\bar{\mu}^b$, we also have that $\bar{\mu}^b = F_b^{-1}(1-\tilde{r}) - F_s^{-1}(\frac{\tilde{r}}{\tilde{t}})$ and, therefore,

$$F_b^{-1}(1-r_j^\star) - F_s^{-1}\left(\frac{r_j^\star}{y_j^\star/b_j}\right) = F_b^{-1}(1-\tilde{r}) - F_s^{-1}(\frac{\tilde{r}}{\tilde{t}}). \tag{1.36}$$

We show that $r_j^\star \geq (1-\varepsilon)\tilde{r}$ by discussing the two subcases depending on whether $\frac{y_j^\star}{b_j} \geq \tilde{t}$ or $\frac{y_j^\star}{b_j} < \tilde{t}$. If $\frac{y_j^\star}{b_j} \geq \tilde{t}$, then (1.36) implies that $r_j^\star \geq \tilde{r} \geq (1-\varepsilon)\tilde{r}$. On the other hand, if $\frac{y_j^\star}{b_j} < \tilde{t}$, then (1.36) implies that $r_j^\star < \tilde{r}$. In this case, to show that $\frac{r_j}{y_j^\star/b_j} \geq \frac{\tilde{r}}{\tilde{t}}$, suppose on the contrary that $\frac{r_j^\star}{y_j^\star/b_j} < \frac{\tilde{r}}{\tilde{t}}$. Then, we have that $F_s^{-1}(\frac{r_j^\star}{y_j^\star/b_j}) < F_s^{-1}(\frac{\tilde{r}}{\tilde{t}})$. Given $r_j^\star < \tilde{r}$, we have that $F_b^{-1}(1-r_j^\star) > F_b^{-1}(1-\tilde{r})$. This leads to a contradiction as $\bar{\mu}^b = F_b^{-1}(1-r_j^\star) - F_s^{-1}(\frac{r_j^\star}{y_j^\star/b_j}) > F_b^{-1}(1-\tilde{r}) - F_s^{-1}(\frac{\tilde{r}}{\tilde{t}}) = \bar{\mu}^b$. Therefore, we have that $\frac{r_j^\star}{y_j^\star/b_j} \geq \frac{\tilde{r}}{\tilde{t}}$. As we established in Claim (ii)-1 that $\frac{y_j^\star}{b_j} \geq (1-\varepsilon)\tilde{t}$, we have that $\frac{1}{1-\varepsilon}\frac{r_j^\star}{\tilde{t}} \geq \frac{r_j^\star}{y_j^\star/b_j} \geq \frac{\tilde{r}}{\tilde{t}}$ or equivalently $r_j^\star \geq (1-\varepsilon)\tilde{r}$.

Finally, note that

$$
\begin{aligned}
V_h &\overset{(g)}{\geq} \bar{V}_h(\bar{\mu}^b) \overset{(h)}{=} \sum_{j\in\mathcal{B}} \bar{\mu}^b b_j r_j^\star(\bar{\mu}^b) \\
&\overset{(i)}{\geq} \sum_{j\in\mathcal{B}} \bar{\mu}^b b_j(1-\varepsilon)\tilde{r} \overset{(j)}{\geq} (1-\varepsilon)\left(\sum_{j\in\mathcal{B}} b_j\right)\left[F_b^{-1}(1-\tilde{r}) - F_s^{-1}\left(\frac{\tilde{r}}{\tilde{t}}\right)\right]\tilde{r} \overset{(k)}{=} (1-\varepsilon)V_{max},
\end{aligned}
\tag{1.37}
$$

where inequality (g) follows from the definition of $\bar{V}_h(\bar{\mu}^b)$, and equality (h) follows from (1.31). Inequality (i) follows from the fact $r_j \geq (1-\varepsilon)\tilde{r}$ for all $j \in \mathcal{B}$ (Claim (ii)-1). Inequality (j) follows from the fact that $\bar{\mu}^b = F_b^{-1}(1-\tilde{r}) - F_s^{-1}(\frac{\tilde{r}}{\tilde{t}})$. Equality (k) follows from the definition of V_{max}. This completes the proof that $V_h \geq (1-\varepsilon)V_{max}$. □

1.12 Supporting Results

In Section 1.12.1, we first summarize all of the supporting results that are very useful to establish the claims in this paper (including the results in Appendix 1.9). We provide the proof of the supporting results in Section 1.12.2. The proofs of results in Appendix 1.9 is provided in Section 1.12.3.

1.12.1 A General Optimization Framework and its Relationship to the Equilibrium, Revenue Optimization Problems

In this section, we introduce an optimization framework that unifies all of the optimization problems covered in this paper. We start by introducing the optimization framework, which consists of a general optimization problem (which we call the "Framework Problem") and some properties of its optimal solution. We then show how to formulate the equilibrium and revenue optimization optimization problems in using this framework. Finally, we show how, under the homogeneity assumption, the framework problem admits a polymatroid formulation.

The Framework Problem. Define functions $g_j : [0, b_j] \to \mathbb{R} \cup \{-\infty, \infty\}$ with $b_j \in \mathbb{R}_+$

for all $j \in \mathcal{B}$ and $h_i : [0, s_i] \to \mathbb{R} \cup \{-\infty, \infty\}$ with $s_i \in \mathbb{R}_+$ for all $i \in \mathcal{S}$. Let $\boldsymbol{w} \in \mathbb{R}_+^{|E|}$. Consider the following optimization problem

$$
\begin{aligned}
\max_{(\boldsymbol{x}, \boldsymbol{q}^s, \boldsymbol{q}^b)} \quad & \sum_{j \in \mathcal{B}} g_j(q_j^b) - \sum_{i \in \mathcal{S}} h_i(q_i^s) - \sum_{(i,j) \in E} w_{ij} x_{ij} && \text{(1.38a)} \\
\text{s.t.} \quad & \sum_{i:(i,j) \in E} x_{ij} \;=\; q_j^b \quad \forall j \in \mathcal{B}, && \text{(1.38b)} \\
& \sum_{j:(i,j) \in E} x_{ij} \;=\; q_i^s \quad \forall i \in \mathcal{S}, && \text{(1.38c)} \\
& q_j^b \;\leq\; b_j \quad \forall j \in \mathcal{B}, && \text{(1.38d)} \\
& q_i^s \;\leq\; s_i \quad \forall i \in \mathcal{S}, && \text{(1.38e)} \\
& x_{ij} \;\geq\; 0 \quad \forall (i,j) \in E. && \text{(1.38f)}
\end{aligned}
$$

We assume that the functions g_j and h_i satisfy the following set of properties:

(1.12.1-1) For all $j \in \mathcal{B}$, the function $g_j(q)$ is continuously differentiable and strictly concave in $q \in (0, b_j)$, and continuous at $q = 0$. Whenever $g_j(b_j) > -\infty$, we have that g_j is continuous at b_j;

(1.12.1-2) For all $i \in \mathcal{S}$, the function $h_i(q)$ is continuously differentiable and strictly convex in $q \in (0, s_i)$, and continuous at $q = 0$. Whenever $h_i(s_i) < \infty$, we have that h_i is continuous at $q = s_i$; [22]

(1.12.1-3) For all $j \in \mathcal{B}$, we have that $g_j'(0) > -\infty$, $g_j'(b_j) \leq 0$; for all $i \in \mathcal{S}$, we have that $h_i'(0) \in [0, \infty)$ and that $h_i'(s_i) > -\infty$;

(1.12.1-4) For all $j \in \mathcal{B}$, we have that $g_j(q) = b_j g\left(\frac{q}{b_j}\right)$; for all $i \in \mathcal{S}$, we have that

---

[22] Given Assumptions (1.12.1-1) - (1.12.1-2), we let $g_j'(q)$ be the derivative of $g_j(q)$ evaluated at $q \in (0, b_j)$. With some abuse of notation, we let $g_j'(0) = \lim_{q \downarrow 0} g_j'(q)$ and $g_j'(b_j) = \lim_{q \uparrow b_j} g_j'(q)$ for all $j \in \mathcal{B}$. Similarly, we let $h_i'(q)$ be the derivative of $h_i(q)$ evaluated at $q \in (0, s_i)$, and we let $h_i'(0) = \lim_{q \downarrow 0} h_i'(q)$ and $h_i'(s_i) = \lim_{q \uparrow s_i} h_i'(q)$ for all $i \in \mathcal{S}$. Note that when $g_j'(0)$ and $h_i'(0)$ are finite, by applying the mean value theorem, we can establish that they correspond to the right derivatives of $g_j(q)$ and $h_i(q)$ at $q = 0$. Similarly, when $g_j'(b_j)$ and $h_i'(s_i)$ are finite, they correspond to the left derivative of $g_j(q)$ at $q = b_j$ and the left derivative of $h_i(q)$ at $q = s_i$, respectively.

$w_{ij} = w_{ij'}$;

(1.12.1-5) For all $i \in \mathcal{S}$, we have that $h_i(q) = s_i h\left(\frac{q}{s_i}\right)$; for all $j \in \mathcal{B}$, we have that $w_{ij} = w_{i'j}$;

(1.12.1-6) For all $(i, j) \in E$, we have that $w_{ij} = 0$;

(1.12.1-7) For all $i \in \mathcal{S}$, we have that $h_i'(0) = 0$;

(1.12.1-8) For all $j \in \mathcal{B}$, we have that $g_j'(0) > 0$.

In the next result, we prove a list of properties related to optimization problem (1.38) that we will later use to establish the main results in the paper.

Proposition 12. *Consider the optimization problem defined in* (1.38)*, and suppose that Assumptions* (1.12.1-1) *and* (1.12.1-2) *hold. Then,*

(i) *There exists an optimal solution* $(\boldsymbol{x}, \boldsymbol{q}^s, \boldsymbol{q}^b)$ *to problem* (1.38)*;*

(ii) *Let* $(\boldsymbol{x}, \boldsymbol{q}^s, \boldsymbol{q}^b)$ *be any optimal solution to problem* (1.38)*. Define* $\partial g_j(q_j^b) := \{z \in \mathbb{R} | g_j(q') \leq g_j(q_j^b) + z(q' - q_j^b), \forall q' \in [0, b_j]\}$ *for all* $j \in \mathcal{B}$ *and* $\partial h_i(q_i^s) := \{z \in \mathbb{R} | h_i(q') \geq h_i(q_i^s) + z(q' - q_i^s), \forall q' \in [0, s_i]\}$ *for all* $i \in \mathcal{S}$*. There exists a finite dual multiplier vector* $(\boldsymbol{\theta}^b, \boldsymbol{\theta}^s, \boldsymbol{\eta}^b, \boldsymbol{\eta}^s, \boldsymbol{\pi})$ *associated with constraints* (1.38b)-(1.38f) *such that the primal-dual* *solution pair* $(\boldsymbol{x}, \boldsymbol{q}^s, \boldsymbol{q}^b)$ *and* $(\boldsymbol{\theta}^b, \boldsymbol{\theta}^s, \boldsymbol{\eta}^b, \boldsymbol{\eta}^s, \boldsymbol{\pi})$ *satisfies the KKT conditions*

$$\theta_j^b + \eta_j^b \;\in\; \partial g_j(q_j^b), \quad \forall j \in \mathcal{B}, \tag{1.39a}$$

$$\theta_i^s - \eta_i^s \;\in\; \partial h_i(q_i^s), \quad \forall i \in \mathcal{S}, \tag{1.39b}$$

$$\theta_j^b - \theta_i^s + \pi_{ij} - w_{ij} \;=\; 0, \quad \forall (i,j) \in E, \tag{1.39c}$$

$$q_j^b \;\le\; b_j \quad \perp \quad \eta_j^b \;\ge\; 0 \quad \forall j \in \mathcal{B}, \tag{1.39d}$$

$$q_i^s \;\le\; s_i \quad \perp \quad \eta_i^s \;\ge\; 0, \quad \forall i \in \mathcal{S}, \tag{1.39e}$$

$$x_{ij} \;\ge\; 0 \quad \perp \quad \pi_{ij} \;\ge\; 0, \quad \forall (i,j) \in E, \tag{1.39f}$$

$$\sum_{i:(i,j)\in E} x_{ij} \;=\; q_j^b, \quad \forall i \in \mathcal{B}, \tag{1.39g}$$

$$\sum_{j:(i,j)\in E} x_{ij} \;=\; q_i^s, \quad \forall j \in \mathcal{S}. \tag{1.39h}$$

Furthermore, every primal-dual solution pair $(\boldsymbol{x}, \boldsymbol{q}^s, \boldsymbol{q}^b)$ *and* $(\boldsymbol{\theta}^b, \boldsymbol{\theta}^s, \boldsymbol{\eta}^b, \boldsymbol{\eta}^s, \boldsymbol{\pi})$ *that satisfies the KKT conditions in* (1.39) *is optimal in problem* (1.38)*;*

(iii) *Suppose Assumption* (1.12.1-3) *holds. Let* $(\boldsymbol{x}, \boldsymbol{q}^s, \boldsymbol{q}^b)$ *be any optimal solution to problem* (1.38)*. Then* $\{g_j'(q_j^b) : j \in \mathcal{B}\}$ *and* $\{h_i'(q_i^s) : i \in \mathcal{S}\}$ *have finite values. Moreover, there exists a special dual multiplier vector* $(\boldsymbol{\theta}^b, \boldsymbol{\theta}^s, \boldsymbol{\eta}^b, \boldsymbol{\eta}^s, \boldsymbol{\pi})$ *associated with constraints*

(1.38b)-(1.38f) *such that*

$$
\begin{aligned}
g_j'(q_j^b) - \theta_j^b - \eta_j^b &= 0, \quad \forall j \in \mathcal{B}, && (1.40a)\\
h_i'(q_i^s) - \theta_i^s + \eta_i^s &= 0, \quad \forall i \in \mathcal{S}, && (1.40b)\\
\theta_j^b - \theta_i^s + \pi_{ij} - w_{ij} &= 0, \quad \forall (i,j) \in E, && (1.40c)\\
q_j^b \le b_j \quad &\perp \quad \eta_j^b \ge 0 \quad \forall j \in \mathcal{B}, && (1.40d)\\
q_i^s \le s_i \quad &\perp \quad \eta_i^s \ge 0, \quad \forall i \in \mathcal{S}, && (1.40e)\\
x_{ij} \ge 0 \quad &\perp \quad \pi_{ij} \ge 0, \quad \forall (i,j) \in E, && (1.40f)\\
\sum_{i:(i,j)\in E} x_{ij} &= q_j^b, \quad \forall i \in \mathcal{B}, && (1.40g)\\
\sum_{j:(i,j)\in E} x_{ij} &= q_i^s, \quad \forall j \in \mathcal{S}. && (1.40h)
\end{aligned}
$$

Furthermore, every primal-dual solution pair $(\boldsymbol{x}, \boldsymbol{q}^s, \boldsymbol{q}^b)$ *and* $(\boldsymbol{\theta}^b, \boldsymbol{\theta}^s, \boldsymbol{\eta}^b, \boldsymbol{\eta}^s, \boldsymbol{\pi})$ *for which* $\{g_j'(q_j^b) : j \in \mathcal{B}\}$ *and* $\{h_i'(q_i^s) : i \in \mathcal{S}\}$ *have finite values and that satisfies the conditions in* (1.40) *is optimal in problem* (1.38)*;* [23]

(iv) *All primal optimal solutions* $(\boldsymbol{x}, \boldsymbol{q}^s, \boldsymbol{q}^b)$ *to problem* (1.38) *share the same vector* $(\boldsymbol{q}^s, \boldsymbol{q}^b)$*;*

(v) *If Assumption* (1.12.1-3) *holds, then* $q_j^b < b_j$ *for all* $j \in \mathcal{B}$*;*

(vi) *If Assumption* (1.12.1-3) *holds, the dual optimal solution* $(\boldsymbol{\theta}^s, \boldsymbol{\theta}^b, \boldsymbol{\eta}^s, \boldsymbol{\eta}^b, \boldsymbol{\pi})$ *that satisfies* (1.40) *is unique;*

(vii) *Suppose Assumption* (1.12.1-3) *and Assumption* (1.12.1-4) *hold. Every optimal solution* $(\boldsymbol{x}, \boldsymbol{q}^s, \boldsymbol{q}^b)$ *satisfies the following: (a) if there exists* $i \in \mathcal{S}$ *such that* $x_{ij}, x_{ij'} > 0$, *then* $\frac{q_j^b}{b_j} = \frac{q_{j'}^b}{b_{j'}}$*; (b) if these exists* $i \in \mathcal{S}$ *such that* $x_{ij} > 0$ *and* $x_{ij'} = 0$, *then* $\frac{q_j^b}{b_j} \le \frac{q_{j'}^b}{b_{j'}}$*;*

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[23] Note that in part (ii), if $q_j^b = 0$, then $\partial g_j(q_j^b)$ is not necessarily a singleton. Similarly, $\partial h_i(q_i^s)$ is not necessarily a singleton if $q_i^s = 0$ or $q_i^s = s_i$. Part (iii) focuses on particular sub/supergradients, and uses them to define a special class of dual multipliers that satisfy the optimality conditions.

(viii) *Suppose Assumption* (1.12.1-3) *and Assumption* (1.12.1-5) *hold. Every optimal solution* $(\boldsymbol{x}, \boldsymbol{q}^s, \boldsymbol{q}^b)$ *satisfies the following: (a) if there exists* $j \in \mathcal{B}$ *such that* $x_{ij}, x_{i'j} > 0$, *then* $\frac{q_i^s}{s_i} = \frac{q_{i'}^s}{s_{i'}}$; *(b) if there exists* $j \in \mathcal{B}$ *such that* $x_{ij} > 0$ *and* $x_{i'j} = 0$, *then* $\frac{q_i^s}{s_i} \leq \frac{q_{i'}^s}{s_{i'}}$.

(ix) *Suppose Assumptions* (1.12.1-3), (1.12.1-6), *and* (1.12.1-8) *hold. Then the optimal solution* $\boldsymbol{q}^s$ *satisfies the following: (a) if Assumption* (1.12.1-7) *holds, then* $q_i^s > 0$ *for all* $i \in \mathcal{S}$; *(b) if Assumptions* (1.12.1-4) - (1.12.1-5) *hold and* $0 \leq h'(0) < g'(0)$, *then* $q_i^s > 0$ *for all* $i \in \mathcal{S}$;

The Equilibrium Problem and the Revenue Optimization Problem. We now show how to relate the framework problem (1.38) to the equilibrium problem (1.16) and the revenue optimization problem (1.7). The following lemma establishes how these two problems fit into framework problem (1.38).

Lemma 4. *(i) The equilibrium problem* (1.16) *can be formulated as an instance of the framework problem* (1.38) *by defining* $g_j(q) = \int_0^q \tilde{F}_{b_j}^{-1}\left(1 - \frac{x}{b_j}\right) dx$ *for all* $j \in \mathcal{B}$, $h_i(q) = \int_0^q \tilde{F}_{s_i}^{-1}\left(\frac{x}{s_i}\right) dx$ *for all* $i \in \mathcal{S}$, $w_{ij} = \frac{c_i}{1+\gamma_j^b}$ *for all* $(i,j) \in E$. *Moreover, under these definitions, if Assumption 1 holds, then Assumptions* (1.12.1-1) -(1.12.1-3) *hold as well. In addition, if* $\bar{v}_{b_j} > \mu_j^b$ *for all* $j \in \mathcal{B}$,[24] *then Assumption* (1.12.1-8) *also holds;*

(ii) The revenue optimization problem (1.7) *can be formulated as an instance of the framework problem* (1.38) *by defining* $g_j(q) = F_{b_j}^{-1}\left(1 - \frac{q}{b_j}\right) q$ *for all* $j \in \mathcal{B}$, $h_i(q) = F_{s_i}^{-1}\left(\frac{q}{s_i}\right) q$ *for all* $i \in \mathcal{S}$, $w_{ij} = c_i$ *for all* $(i,j) \in E$. *Moreover, under these definitions, if Assumption 1 - 2 hold, then Assumptions* (1.12.1-1) - (1.12.1-3) *and Assumptions* (1.12.1-7) - (1.12.1-8) *hold.*

Given that both the equilibrium problem (1.16) and the revenue optimization problem (1.7) are instances of the framework problem (1.38), let $(\boldsymbol{x}, \boldsymbol{q}^s, \boldsymbol{q}^b)$ be any optimal solution to (1.38). We consider a special dual optimal solution $(\boldsymbol{\theta}^b, \boldsymbol{\theta}^s, \boldsymbol{\eta}^b, \boldsymbol{\eta}^s, \boldsymbol{\pi})$ associated

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[24] If $\bar{v}_{b_j} \leq \mu_j^b$, this corresponds to a scenario where no trade happens at type-j buyers i.e. $q_j^b = 0$.

with constraints (1.38b) - (1.38f) that maximizes $\sum_{j\in\mathcal{B}}[sgn(q^b_j) - sgn(b_j - q^b_j)](\theta^b_j + \eta^b_j) + \sum_{i\in\mathcal{S}}[-sgn(q^s_i)+sgn(s_i-q^s_i)](\theta^s_i-\eta^s_i)$. The next result establishes that these dual multipliers correspond to the unique dual multiplier identified in (1.40).

Lemma 5. *In the equilibrium problem* (1.16) *and the revenue optimization problem* (1.7), *given any primal solution* $(\boldsymbol{x}, \boldsymbol{q}^s, \boldsymbol{q}^b)$, *let* $(\boldsymbol{\theta}^b, \boldsymbol{\theta}^s, \boldsymbol{\eta}^b, \boldsymbol{\eta}^s, \boldsymbol{\pi})$ *be a dual optimal solution that maximizes* $\sum_{j\in\mathcal{B}}[sgn(q^b_j) - sgn(b_j - q^b_j)](\theta^b_j + \eta^b_j) + \sum_{i\in\mathcal{S}}[-sgn(q^s_i) + sgn(s_i - q^s_i)](\theta^s_i - \eta^s_i)$. *Then,* $(\boldsymbol{x}, \boldsymbol{q}^s, \boldsymbol{q}^b)$ *and* $(\boldsymbol{\theta}^b, \boldsymbol{\theta}^s, \boldsymbol{\eta}^b, \boldsymbol{\eta}^s, \boldsymbol{\pi})$ *satisfy the conditions in* (1.40). *Moreover,* $(\boldsymbol{\theta}^b, \boldsymbol{\theta}^s, \boldsymbol{\eta}^b, \boldsymbol{\eta}^s, \boldsymbol{\pi})$ *is unique.*

A Polymatroid Formulation for the Framework Problem under Additional Assumptions. We next show that the framework problem (1.38) can be used to solve the optimization problem (1.19), which involves maximizing a concave objective function over a polymatroid. Problem (1.19) allows us to leverage existing results on optimizing convex functions over polymatroids, which plays a key role in the derivation of our results in Sections 1.4 and 1.6. To establish the connection between problem (1.38) and (1.19), we first focus on $f(\cdot)$ defined in (1.17) (by setting $f(0) = \lim_{t\downarrow 0} f(t)$), and consider the following optimization problem:

$$\max_{(\boldsymbol{y},\boldsymbol{z})} \quad \sum_{j\in\mathcal{B}} b_j f\left(\frac{y_j}{b_j}\right) \tag{1.41a}$$

$$\text{s.t.} \quad \sum_{i:(i,j)\in E} z_{ij} = y_j, \quad \forall j \in \mathcal{B} \tag{1.41b}$$

$$\sum_{j:(i,j)\in E} z_{ij} = s_i, \quad \forall i \in \mathcal{S} \tag{1.41c}$$

$$z_{ij} \geq 0, \quad \forall (i,j) \in E. \tag{1.41d}$$

Problem (1.41) can be used to obtain an optimal solution to problem (1.19), when function $f(\cdot)$ is continuous, strictly increasing and strictly concave. Our next result formalizes this claim.

Proposition 13. *When $f(\cdot)$ is continuous, strictly increasing, and strictly concave for $t \geq 0$, problem* (1.41) *and problem* (1.19) *share the same optimal vector $\boldsymbol{y}$.*

The next result establishes a connection between the framework problem (1.38) and problem (1.19).

Proposition 14. *Fix functions $g(\cdot)$ and $h(\cdot)$. Let functions $\{g_j(\cdot)\}_{j \in \mathcal{B}}$ and $\{h_i(\cdot)\}_{i \in \mathcal{S}}$ be defined as $g_j(q) := b_j g(\frac{q}{b_j})$ for all $j \in \mathcal{B}$ and $h_i(q) := s_i h(\frac{q}{s_i})$ for all $i \in \mathcal{S}$. Consider the corresponding framework problem* (1.38)*. Suppose that Assumptions* (1.12.1-1) - (1.12.1-6) *hold in this problem. Using $g(\cdot)$ and $h(\cdot)$, let $f(\cdot)$ be defined as in* (1.17) *i.e, $f(t) = \max_{r \in [0, \min\{1,t\}]} g(r) - th\left(\frac{r}{t}\right)$ for $t > 0$ with $f(0) = \lim_{t \downarrow 0} f(t)$. Consider an instance of problem* (1.41) *with function $f(\cdot)$. If function $f(\cdot)$ is continuous, strictly increasing and strictly concave for $t \geq 0$, then the following holds:*

(i) *let $(\boldsymbol{x}, \boldsymbol{q}^s, \boldsymbol{q}^b)$ be an optimal solution to problem* (1.38)*. Construct a tuple $(\boldsymbol{y}, \boldsymbol{z}, \boldsymbol{r})$ such that (1) $z_{ij} = \frac{x_{ij} s_i}{q_i^s}$ for all $(i,j) \in E$; (2) $r_j = \frac{q_j^b}{b_j}$ for all $j \in \mathcal{B}$; (3) $y_j = \frac{s_i}{q_i^s} q_j^b$ for all $j \in \mathcal{B}$ where $i : x_{ij} > 0$. Solution $(\boldsymbol{y}, \boldsymbol{z})$ is optimal in problem* (1.41)*, and solution r_j is optimal in problem* (1.17) *with $t = \frac{y_j}{b_j}$ for all $j \in \mathcal{B}$;*

(ii) *let $(\boldsymbol{y}, \boldsymbol{z})$ be an optimal solution to problem* (1.41) *and r_j be an optimal solution to problem* (1.17) *with $t = \frac{y_j}{b_j}$ for all $j \in \mathcal{B}$. Construct a tuple $(\boldsymbol{x}, \boldsymbol{q}^s, \boldsymbol{q}^b)$ such that (1) $q_j^b = r_j b_j$ for all $j \in \mathcal{B}$; (2) $q_i^s = \frac{r_j b_j}{y_j} s_i$ for all $i \in \mathcal{S}$ where $j : z_{ij} > 0$; (3) $x_{ij} = \frac{z_{ij} q_i^s}{s_i}$ for all $(i,j) \in E$. Solution $(\boldsymbol{x}, \boldsymbol{q}^s, \boldsymbol{q}^b)$ is optimal in problem* (1.38)*;*

(iii) *problem* (1.41) *and problem* (1.38) *share the same optimal objective values.*

Known Results on Polymatroid Optimization.. Recall that problem (1.19) is maximizing a concave objective function over a polymatroid. In this section, we state Theorem 3.1 and Theorem 3.2 of (42) by restricting attention to our polymatroid $\mathcal{P} = \{\boldsymbol{y} : \sum_{j \in B} y_j \leq \sum_{i \in N_E(B)} s_i, \forall B \subset \mathcal{B}, y_j \geq 0, \forall j \in \mathcal{B}\}$. We start by introducing some notation. Given a weight vector $\boldsymbol{b}$, define vector $\boldsymbol{v}_\alpha \in \mathbb{R}_+^{\mathcal{B}}$ for $\alpha \in \mathbb{R}_+$ by $\boldsymbol{v}_\alpha := (\alpha b_j)_{j\in\mathcal{B}}$ (see (3.1) of (42)). Vector $\boldsymbol{u} \geq 0$ is called a base of $\boldsymbol{v}$ with respect to polymatroid $\mathcal{P}$ if (i) $\boldsymbol{u}$ is an independent vector (i.e., a member) of $\mathcal{P}$, (ii) $\boldsymbol{u} \leq \boldsymbol{v}$, (iii) $\sum_{j\in\mathcal{B}} \boldsymbol{u}(j) = \min_{B\subset\mathcal{B}} \sum_{i\in N_E(B)} s_i + \sum_{j\in\mathcal{B}\setminus B} \boldsymbol{v}(j)$. Let $\boldsymbol{u}_\alpha$ be an independent vector of $\mathcal{P}$ satisfying the following two properties: $\boldsymbol{u}_\alpha$ is a base of $\boldsymbol{v}_\alpha$ with respect to $\mathcal{P}$ for any $\alpha \in \mathbb{R}_+$, and $\boldsymbol{u}_\alpha \leq \boldsymbol{u}_\beta$ (where the inequality is entrywise) for $0 \leq \alpha \leq \beta$ (see (3.2) - (3.3) of (42)). Let $(t'_j)_{j\in\mathcal{B}}$ be such that $\boldsymbol{u}_\alpha(j) = \alpha b_j$ for $0 \leq \alpha \leq t'_j$ and $\boldsymbol{u}_\alpha(j) = t'_j b_j$ for $t'_j \leq \alpha$ (see (3.4) - (3.5) of (42)). (42) establishes the following:

Theorem 4. *(Theorem 3.1 of (42)) Suppose that $(t'_j)_{j\in\mathcal{B}}$ satisfy the conditions above ((3.1) - (3.5) of (42)). The vector $\boldsymbol{y}^\star$ defined by $\boldsymbol{y}^\star = (t'_j b_j)_{j\in\mathcal{B}}$ is the unique lexicographically optimal base of polymatroid $\mathcal{P}$ with respect to a weight vector $\boldsymbol{b}$.*

Moreover, define $dep(\boldsymbol{y}, j) := \cap\{B \mid j \in B \subset \mathcal{B},\ \sum_{j\in B} y_j = \sum_{i\in N_E(B)} s_i\}$ for all $j \in \mathcal{B}$. (42) establishes that

Theorem 5. *(Theorem 3.2 of (42)) Let $\boldsymbol{y}$ be a base of $\mathcal{P}$ and $\boldsymbol{b}$ be a weight vector. Define $t'_j = y_j/b_j$ for all $j \in \mathcal{B}$ and let the distinct numbers of t'_j be given by $c'_1 < \cdots < c'_l$. Furthermore, define $\mathcal{B}_k \subset \mathcal{B}$ by $\mathcal{B}_k = \{j \in \mathcal{B} : t'_j \leq c'_k\}$ for $k \in \{1, \ldots, l\}$. Then the following three conditions are equivalent:*

(i) $\boldsymbol{y}$ is the lexicographically optimal base of $\mathcal{P}$ with respect to the weight vector $\boldsymbol{b}$;

(ii) $\sum_{j\in\mathcal{B}_k} y_j = \sum_{i\in N_E(\mathcal{B}_k)} s_i$ for $k \in \{1, \ldots, l\}$;

(iii) $\emptyset \neq dep(\boldsymbol{y}, j) \subset \mathcal{B}_k$ for $j \in \mathcal{B}_k$ and $k \in \{1, \ldots, l\}$.

We also use the results in (41) to establish a connection between the polyhedra $\mathcal{Q} = \{(\boldsymbol{y}, \boldsymbol{z}) : \sum_{i:(i,j)\in E} z_{ij} = y_j, \forall j \in \mathcal{B}, \sum_{j:(i,j)\in E} z_{ij} \leq s_i, \forall i \in \mathcal{S}, z_{ij} \geq 0, \forall (i,j) \in E\}$ and $\mathcal{P} = \{\boldsymbol{y} : \sum_{j\in B} y_j \leq \sum_{i\in N_E(B)} s_i, \forall B \subset \mathcal{B}, y_j \geq 0, \forall j \in \mathcal{B}\}$. Lemma 4.1 of (41) adapted to our setting implies the following:

Lemma 6. *Let $\boldsymbol{y}$ be a $|\mathcal{B}|$-tuple of nonnegative real numbers. A necessary and sufficient condition for the existence of $\boldsymbol{z}$ such that $(\boldsymbol{y}, \boldsymbol{z}) \in \mathcal{Q}$ is $\boldsymbol{y} \in \mathcal{P}$.*

1.12.2 Proofs of Results in Section 1.12.1

Proof of Proposition 12. We now proceed to prove each of the claims in the proposition. Recall that, for every claim in this proposition, we are assuming that Assumptions (1.12.1-1)-(1.12.1-2) holds.

Proof of claim (i). Let $\mathcal{Z}$ be the compact feasibility set characterized by constraints (1.38b) - (1.38f) in problem (1.38). We have that $\mathcal{Z} \neq \emptyset$ because $\mathbf{0} \in \mathcal{Z}$. Moreover, if the function $g_j(q)$ is continuous in $[0, b_j]$ for all $j \in \mathcal{B}$ and $h_i(q)$ is continuous in $[0, s_i]$ for all $i \in \mathcal{S}$ (recall that these functions are continuous in $[0, b_j)$ and $[0, s_j)$ by Assumptions (1.12.1-1)-(1.12.1-2)), then the objective function $\sum_{j\in\mathcal{B}} g_j(q_j^b) - \sum_{i\in\mathcal{S}} h_i(q_i^s) - \sum_{(i,j)\in E} w_{ij}x_{ij}$ is continuous in compact set $\mathcal{Z}$. By the Weierstrass extreme value theorem, an optimal solution $(\boldsymbol{x}, \boldsymbol{q}^s, \boldsymbol{q}^b)$ exists.

In the more general case, assume that not all functions g_j are continous at b_j or not all functions h_i are continuous at s_i, and define the following sets:

$$\begin{aligned} I &= \{i \in \mathcal{S} : h_i(s_i) < \infty\}, \quad I^c = \{i \in \mathcal{S} : i \notin I\}, \\ J &= \{j \in \mathcal{B} : g_j(b_j) > -\infty\}, \quad J^c = \{j \in \mathcal{B} : j \notin J\}. \end{aligned} \tag{1.42}$$

Using these sets, we can compactly express the "sup" version of problem (1.38) as follows.

$$\tilde{Y} \;=\; \sup_{(\bar{\boldsymbol{x}}, \bar{\boldsymbol{q}}^s, \bar{\boldsymbol{q}}^b)\in\mathcal{Z}} \left\{ \sum_{j\in J} g_j\left(\bar{q}_j^b\right) + \sum_{j\in J^c} g_j\left(\bar{q}_j^b\right) - \sum_{i\in I} h_i\left(\bar{q}_i^s\right) - \sum_{i\in I^c} h_i\left(\bar{q}_i^s\right) - \sum_{(i,j)\in E} w_{ij}\bar{x}_{ij} \right\}. \tag{1.43}$$

Next, we will show that $\tilde{Y}$ is upper bounded, i.e. $\tilde{Y} < \infty$. For $j \in \mathcal{B}$, define

$$K_j^b \;:=\; \max\left\{ \max_{q\in[0,\frac{1}{2}b_j]} g_j(q), \quad g_j\left(\frac{1}{2}b_j\right) + \max_{q\in[\frac{1}{2}b_j, b_j]} g_j'\left(\frac{1}{2}b_j\right)\left(q - \frac{1}{2}b_j\right) \right\}. \tag{1.44}$$

By Assumption (1.12.1-1), function $g_j(\cdot)$ is concave and differentiable in $q \in (0, b_j)$ and continuous at $q = 0$; therefore, K_j^b is well defined and, moreover, we have

$$g_j(q) \ \leq \ K_j^b < \infty \text{ for all } q \in [0, b_j],\ j \in \mathcal{B}. \tag{1.45}$$

Similarly, for $i \in \mathcal{S}$ define

$$K_i^s \ := \ \min \left\{ \min_{q \in [0, \frac{1}{2} s_i]} h_i(q), h_i\left(\frac{1}{2} s_i\right) + \min_{q \in [\frac{1}{2} s_i, s_i]} h_i'\left(\frac{1}{2} s_i\right)\left(q - \frac{1}{2} s_i\right) \right\}. \tag{1.46}$$

By Assumption (1.12.1-2), function $h(\cdot)$ is convex and differentiable in $q \in (0, s_i)$ and continuous at $q = 0$; therefore K_i^s is well defined and, moreover, we have

$$h_i(q) \ \geq \ K_i^s > -\infty \text{ for all } q \in [0, s_i],\ i \in \mathcal{S}. \tag{1.47}$$

Finally, note that

$$\sum_{(i,j) \in E} w_{ij} \bar{x}_{ij} \geq 0 \tag{1.48}$$

for any feasible $\bar{\boldsymbol{x}}$, as we have assumed $w_{ij} \geq 0$. Putting together (1.45), (1.47) and (1.48), it is easy to see that the objective function of problem (1.43) satisfies

$$\tilde{Y} \ \leq \ \sum_{j \in \mathcal{B}} K_j^b - \sum_{i \in \mathcal{S}} K_i^s \ < \ \infty. \tag{1.49}$$

By definition of the set J^c, we have that for any $j \in J^c$, function $g_j(q)$ is continuous in $[0, b_j)$ and $\lim_{q \uparrow b_j} g_j(q) = -\infty$; thus, there exists $\bar{b}_j < b_j$ such that for all $q \geq \bar{b}_j$, $g_j(q) \leq -3|\tilde{Y}| - Y_0$ for some $Y_0 > 0$. Similarly, for any $i \in I^c$, function $h_i(q)$ is continuous in $q \in [0, s_i)$ with $\lim_{q \uparrow s_i} h_i(q) = \infty$, so there exists $\bar{s}_i < s_i$ such that for all $q \geq \bar{s}_i$, $h_i(q) \geq 3|\tilde{Y}| + Y_0$. As a result, any feasible solution $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{q}}^s, \bar{\boldsymbol{q}}^b) \in \mathcal{Z}$ in which there exists $i \in I^c$ with $\bar{q}_i^s \geq \bar{s}_i$ or $j \in J^c$ with $\bar{q}_j^b \geq \bar{b}_j$ cannot be optimal as, otherwise, the objective function value would satisfy $\tilde{Y} \le \sum_{j\in J} g_j(\bar{q}_j^b) + \sum_{j\in J^c} g_j(\bar{q}_j^b) - \sum_{i\in I} h_i(\bar{q}_i^s) - \sum_{i\in I^c} h_i(\bar{q}_i^s) \le -3|\tilde{Y}| + \tilde{Y} - Y_0 \le -2|\tilde{Y}| - Y_0 < \tilde{Y}$.

Thus, we can construct $\mathcal{Z}_0$, a compact subset of $\mathcal{Z}$, such that we know that the optimal solution will be in $\mathcal{Z}_0$ as:

$$\mathcal{Z}_0 = \mathcal{Z} \cap \left\{ (\bar{\boldsymbol{x}}, \bar{\boldsymbol{q}}^s, \bar{\boldsymbol{q}}^b) : \bar{q}_i^s \le \bar{s}_i, \forall i \in I^c, \bar{q}_j^b \le \bar{b}_j, \forall j \in J^c \right\}, \tag{1.50}$$

Thus, we can find the optimal solution to problem (1.43) by solving

$$\sup_{(\bar{\boldsymbol{x}}, \bar{\boldsymbol{q}}^s, \bar{\boldsymbol{q}}^b)\in\mathcal{Z}_0} \left\{ \sum_{j\in J} g_j(\bar{q}_j^b) + \sum_{j\in J^c} g_j(\bar{q}_j^b) - \sum_{i\in I} h_i\left(\bar{q}_i^s\right) - \sum_{i\in I^c} h_i\left(\bar{q}_i^s\right)) - \sum_{(i,j)\in E} w_{ij}\bar{x}_{ij} \right\}. \tag{1.51}$$

In problem (1.51), we have $\mathcal{Z}_0 \neq \emptyset$ because $\mathbf{0} \in \mathcal{Z}_0$. Moreover, the objective function is continuous in $\mathcal{Z}_0$. Thus, by the Weierstrass extreme value theorem, problem (1.51) has an optimal solution $(\boldsymbol{x}, \boldsymbol{q}^s, \boldsymbol{q}^b)$. Since any solution vector with $(\boldsymbol{x}, \boldsymbol{q}^s, \boldsymbol{q}^b) \notin \mathcal{Z}_0$ is not optimal, this completes the proof that an optimal solution to optimization problem (1.38) exists.

Remark 1. *When we discuss the properties of the optimal solutions to problem* (1.38)*, unless otherwise stated, we implicitly limit the feasible region of problem* (1.38) *to the compact subset* $\mathcal{Z}_0$ *in* (1.50) *where the objective function* (1.38a) *is continuous.*

Proof of claim (ii). We divide the proof into the following arguments.

(ii)-1: the strong Slater's condition. We first show that the strong Slater's condition is satisfied, that is, there exists a feasible solution in the relative interior of the feasible region (a feasible solution in which all inequality constraints in (1.38) are satisfied strictly). Let vector $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{q}}^s, \hat{\boldsymbol{q}}^b)$ be defined as

$$\hat{x}_{ij} := \frac{1}{2|E|} \min\{\boldsymbol{s}, \boldsymbol{b}\} \text{ for all } (i,j) \in E, \tag{1.52a}$$

$$\hat{q}_i^s := \sum_{j:(i,j)\in E} \hat{x}_{ij} \text{ for all } i \in \mathcal{S}, \tag{1.52b}$$

$$\hat{q}_j^b := \sum_{i:(i,j)\in E} \hat{x}_{ij} \text{ for all } j \in \mathcal{B}. \tag{1.52c}$$

First, note that (1.52b) and (1.52c) ensure that equality constraints (1.38b) - (1.38c) are satisfied. Second, by construction, $\hat{x}_{ij} > 0$ for all $(i,j) \in E$, and $\boldsymbol{s} > 0$, $\boldsymbol{b} > 0$, the constraint (1.38f) is strictly satisfied for all $(i,j) \in E$. Finally, note that

$$\hat{q}_i^s = \sum_{j:(i,j)\in E} \hat{x}_{ij} = |N_E(i)| \frac{1}{2|E|} \min\{\boldsymbol{s}, \boldsymbol{b}\} \leq \frac{1}{2} \min\{\boldsymbol{s}, \boldsymbol{b}\} < s_i.$$

A similar argument shows that $\hat{q}_j^b < b_j$. These observations jointly imply that the strong Slater's condition holds.

<u>(ii)-2: alternative reformulation of the problem.</u> We first define the negative of the objective function of problem (1.38) compactly as

$$f_0(\boldsymbol{x}, \boldsymbol{q}^s, \boldsymbol{q}^b) \ := \ -\Big[\sum_{j\in\mathcal{B}} g_j(q_j^b) - \sum_{i\in\mathcal{S}} h_i(q_i^s) - \sum_{(i,j)\in E} w_{ij} x_{ij} \Big], \tag{1.53}$$

where we let the domain of function f_0 be

$$dom(f_0) \ := \ \{(\boldsymbol{x}, \boldsymbol{q}^s, \boldsymbol{q}^b) : \text{constraints (1.38b) - (1.38f) are satisfied, } f_0(\boldsymbol{x}, \boldsymbol{q}^s, \boldsymbol{q}^b) < \infty\}. \tag{1.54}$$

Recall from part (i) that there exists an optimal solution to problem (1.38) with finite objective value. Letting $C = dom(f_0)$, we can express problem (1.38) compactly as

$$\min_{(\boldsymbol{x},\boldsymbol{q}^s,\boldsymbol{q}^b)\in C} \quad f_0(\boldsymbol{x},\boldsymbol{q}^s,\boldsymbol{q}^b). \tag{1.55}$$

We next argue that this formulation is a special case of the framework of (50), and in particular, it satisfies the Assumptions (a) - (c) in page 273 of (50).

Assumption (a): We have $C = dom(f_0)$ by construction. We then establish that f_0 is proper convex in C. To establish proper convexity, we need to establish three things: (i) $f_0(\boldsymbol{x},\boldsymbol{q}^s,\boldsymbol{q}^b) > -\infty$ for all $(\boldsymbol{x},\boldsymbol{q}^s,\boldsymbol{q}^b) \in C$; (ii) there exists $(\hat{\boldsymbol{x}},\hat{\boldsymbol{q}}^s,\hat{\boldsymbol{q}}^b) \in C$ such that $f_0(\hat{\boldsymbol{x}},\hat{\boldsymbol{q}}^s,\hat{\boldsymbol{q}}^b) < \infty$; (iii) $f_0(\boldsymbol{x},\boldsymbol{q}^s,\boldsymbol{q}^b)$ is convex in C. We first show that $f_0(\boldsymbol{x},\boldsymbol{q}^s,\boldsymbol{q}^b) > -\infty$ for all $(\boldsymbol{x},\boldsymbol{q}^s,\boldsymbol{q}^b) \in C$. Recall from condition (1.45) in part (i) that there exists $K_j^b < \infty$ such that $g_j(q_j^b) \le K_j^b$ for all $q_j^b \in [0, b_j]$. Similarly, there exists $K_i^s > -\infty$ such that $h_i(q_i^s) \ge K_i^s > -\infty$ for all $q_i^s \in [0, s_i]$ (from (1.47)). Moreover, $w_{ij}x_{ij}$ is finite for all $x_{ij} \in \mathbb{R}$. These observations jointly imply that $f_0(\boldsymbol{x},\boldsymbol{q}^s,\boldsymbol{q}^b) > -\infty$ for all $(\boldsymbol{x},\boldsymbol{q}^s,\boldsymbol{q}^b) \in C$. Next, we establish that there exists at least one $(\hat{\boldsymbol{x}},\hat{\boldsymbol{q}}^s,\hat{\boldsymbol{q}}^b) \in C$ such that $f_0(\hat{\boldsymbol{x}},\hat{\boldsymbol{q}}^s,\hat{\boldsymbol{q}}^b) < \infty$. Let $(\hat{\boldsymbol{x}},\hat{\boldsymbol{q}}^s,\hat{\boldsymbol{q}}^b)$ be the vector constructed in (1.52). Since $\hat{q}_j^b \in (0, b_j)$ for all $j \in \mathcal{B}$, it readily follows from Assumption (1.12.1-1) that $g_j(\hat{q}_j^b)$ is finite. Similarly, given $\hat{q}_i^s \in (0, s_i)$ for all $i \in \mathcal{S}$, it follows that $h_i(\hat{q}_i^s)$ is finite (Assumption (1.12.1-2)). These observations together with $\hat{x}_{ij} \in (0, \frac{1}{|E|}\min\{\boldsymbol{s},\boldsymbol{b}\})$ for all $(i,j) \in E$ imply that $f_0(\hat{\boldsymbol{x}},\hat{\boldsymbol{q}}^s,\hat{\boldsymbol{q}}^b) < \infty$. It can be readily checked that Assumption (1.12.1-1) ensures that $g_j(q)$ is concave in $\{q \in [0, b_j] : g_j(q) > -\infty\}$. Similarly, Assumption (1.12.1-2) ensures that $h_i(q)$ is convex in $\{q \in [0, s_i] : h_i(q) < \infty\}$. Noting that $w_{ij}x_{ij}$ is a linear function, we conclude that $f_0(\boldsymbol{x},\boldsymbol{q}^s,\boldsymbol{q}^b)$ is convex for $(\boldsymbol{x},\boldsymbol{q}^s,\boldsymbol{q}^b) \in C$. Thus, $f_0(\boldsymbol{x},\boldsymbol{q}^s,\boldsymbol{q}^b)$ is proper convex in C.

Assumption (b) and (c): The constraint functions in (1.38b) - (1.38f) , which are defined in the domain of $\mathbb{R}^{|\mathcal{S}|+|\mathcal{B}|+|E|}$, are all affine. Hence, it immediately follows that these constraint functions are proper convex in $\mathbb{R}^{|\mathcal{S}|+|\mathcal{B}|+|E|}$. Moreover, since $C \subset \mathbb{R}^{|\mathcal{S}|+|\mathcal{B}|+|E|}$, we conclude that Assumption (b) and (c) holds.

(ii)- 2: necessity of the KKT conditions. Consider the Lagrangian associated with problem (1.55)(equivalently problem (1.38)), which can be expressed as:

$$
\begin{aligned}
\mathcal{L}(\boldsymbol{x}, \boldsymbol{q}^s, \boldsymbol{q}^b, \boldsymbol{\theta}^s, \boldsymbol{\theta}^b, \boldsymbol{\eta}^s, \boldsymbol{\eta}^b, \boldsymbol{\eta}^s, \boldsymbol{\pi}) \;:=\; & -\sum_{j\in\mathcal{B}} \left[g_j(q_j^b) - \theta_j^b \Big(q_j^b - \sum_{i:(i,j)\in E} x_{ij} \Big) - \eta_j^b \left(q_j^b - b_j \right) \right] \\
& - \sum_{i\in\mathcal{S}} \left[-h_i(q_i^s) + \theta_i^s \Big(q_i^s - \sum_{j:(i,j)\in E} x_{ij} \Big) - \eta_i^s \left(q_i^s - s_i \right) \right] \\
& - \sum_{(i,j)\in E} \Big(- w_{ij} x_{ij} + \pi_{ij} x_{ij} \Big). \qquad (1.56)
\end{aligned}
$$

By the claim in part (i) of the proposition, there exists an optimal solution $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{q}}^s, \bar{\boldsymbol{q}}^b)$ to problem (1.38) (equivalently to problem (1.55)) with finite objective value i.e., $f_0(\bar{\boldsymbol{x}}, \bar{\boldsymbol{q}}^s, \bar{\boldsymbol{q}}^b) > -\infty$. Moreover, consider the vector $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{q}}^s, \hat{\boldsymbol{q}}^b)$ in (1.52). Since $\hat{q}_j^b \in (0, b_j)$ for all $j \in \mathcal{B}$, $\hat{q}_i^s \in (0, s_i)$ for all $i \in \mathcal{S}$, and $\hat{x}_{ij} \in (0, \frac{1}{|E|} \min\{\boldsymbol{s}, \boldsymbol{b}\})$ for all $(i,j) \in E$, there exists a open subset $\tilde{C} \subset C$ with $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{q}}^s, \hat{\boldsymbol{q}}^b) \in \tilde{C}$ such that $f_0(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{q}}^s, \tilde{\boldsymbol{q}}^b) < \infty$ for all $(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{q}}^s, \tilde{\boldsymbol{q}}^b) \in \tilde{C}$ (by Assumption (1.12.1-1) - (1.12.1-2) and the fact that $f_0(\hat{\boldsymbol{x}}, \hat{\boldsymbol{q}}^s, \hat{\boldsymbol{q}}^b) < \infty$). This observation implies that $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{q}}^s, \hat{\boldsymbol{q}}^b)$ is in the relative interior of C, i.e., $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{q}}^s, \hat{\boldsymbol{q}}^b) \in ri(C)$. Thus, it readily follows that $ri(C) \neq \emptyset$. Since the strong Slater's condition holds, by Theorem 28.2 of (50), there exists a Kuhn-Tucker vector $(\bar{\boldsymbol{\theta}}^b, \bar{\boldsymbol{\theta}}^s, \bar{\boldsymbol{\eta}}^b, \bar{\boldsymbol{\eta}}^s, \bar{\boldsymbol{\pi}})$ such that $\mathcal{L}(\bar{\boldsymbol{x}}, \bar{\boldsymbol{q}}^s, \bar{\boldsymbol{q}}^b, \bar{\boldsymbol{\theta}}^b, \bar{\boldsymbol{\theta}}^s, \bar{\boldsymbol{\eta}}^b, \bar{\boldsymbol{\eta}}^s, \bar{\boldsymbol{\pi}}) = f_0(\bar{\boldsymbol{x}}, \bar{\boldsymbol{q}}^s, \bar{\boldsymbol{q}}^b)$ (defined in page 274 of (50)).

To establish conditions (1.39a) - (1.39c), we let $\tilde{g}_j(q) := -g_j(q) + \bar{\theta}_j^b q + \bar{\eta}_j^b q$ for any $q \in [0, b_j]$ and $j \in \mathcal{B}$, $\tilde{h}_i(q) := h_i(q) - \bar{\theta}_i^s q + \bar{\eta}_i^s q$ for any $q \in [0, s_i]$ and $i \in \mathcal{S}$, and $\tilde{l}_{ij}(x) := (-\bar{\theta}_j^b + \bar{\theta}_i^s + w_{ij} - \bar{\pi}_{ij})x$ for any $x \geq 0$ and $(i,j) \in E$. At the optimal primal solution $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{q}}^s, \bar{\boldsymbol{q}}^b)$, from the finiteness of $\{g_j(\bar{q}_j^b) : j \in \mathcal{B}\}$, $\{h_i(\bar{q}_i^s) : i \in \mathcal{S}\}$ and $(\bar{\boldsymbol{\theta}}^b, \bar{\boldsymbol{\theta}}^s, \bar{\boldsymbol{\eta}}^b, \bar{\boldsymbol{\eta}}^s, \bar{\boldsymbol{\pi}})$, it readily follows that $\{\tilde{g}_j(\bar{q}_j^b) : j \in \mathcal{B}\}$, $\{\tilde{h}_i(\bar{q}_i^s) : i \in \mathcal{S}\}$ have finite values. We define $\partial\tilde{g}_j(\bar{q}_j^b) := \{z \in \mathbb{R} | \tilde{g}_j(q') \geq \tilde{g}_j(\bar{q}_j^b) + z(q' - \bar{q}_j^b), \forall q' \in [0, b_j]\}$ for all $j \in \mathcal{B}$, $\partial\tilde{h}_i(\bar{q}_i^s) := \{z \in \mathbb{R} | \tilde{h}_i(q') \geq \tilde{h}_i(\bar{q}_i^s) + z(q' - \bar{q}_i^s), \forall q' \in [0, s_i]\}$ for all $i \in \mathcal{S}$, and $\partial\tilde{l}_{ij}(\bar{x}_{ij}) := \{-\bar{\theta}_j^b + \bar{\theta}_i^s + w_{ij} - \bar{\pi}_{ij}\}$ for all $(i,j) \in E$.

Before proceeding, we verify the conditions in Theorem 23.8 of (50). Recall that the vector $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{q}}^s, \hat{\boldsymbol{q}}^b)$ in (1.52) satisfies that $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{q}}^s, \hat{\boldsymbol{q}}^b) \in ri(C)$. Moreover, since the constraints functions in (1.38b) - (1.38f) are affine functions defined on the domain of $\mathbb{R}^{|\mathcal{S}|+|\mathcal{B}|+|E|}$, it readily follows that $(\hat{\boldsymbol{x}}, \hat{\boldsymbol{q}}^s, \hat{\boldsymbol{q}}^b) \in \mathbb{R}^{|\mathcal{S}|+|\mathcal{B}|+|E|}$. Thus, Theorem 28.3(c) and Theorem 23.8 of (50) jointly imply that $0 \in \partial \tilde{g}_j(\bar{q}^b_j)$ for all $j \in \mathcal{B}$, $0 \in \partial \tilde{g}_j(\bar{q}^s_i)$ for all $i \in \mathcal{S}$, and $0 \in \partial \tilde{l}_{ij}(\bar{x}_{ij})$ for all $(i,j) \in E$.

We next verify condition (1.39a). For all $j \in \mathcal{B}$, we first define $\partial g_j(\bar{q}^b_j) := \{z \in \mathbb{R} | g_j(q') \leq g_j(\bar{q}^b_j) + z(q' - \bar{q}^b_j), \forall q' \in [0, b_j]\}$. Since $0 \in \partial \tilde{g}_j(\bar{q}^b_j)$, by using the definition of $\partial \tilde{g}_j(\bar{q}^b_j)$, we obtain that $\tilde{g}_j(q') \geq \tilde{g}_j(\bar{q}^b_j)$ for all $q' \in [0, b_j]$. From the definition of $\tilde{g}_j(q^b_j)$, it follows that $-g_j(q') + \bar{\theta}^b_j q' + \bar{\eta}^b_j q' \geq -g_j(\bar{q}^b_j) + \theta^b_j \bar{q}^b_j + \bar{\eta}^b_j \bar{q}^b_j$ for all $q' \in [0, b_j]$, i.e., $\bar{\theta}^b_j + \bar{\eta}^b_j \in \partial g_j(\bar{q}^b_j)$. Thus, condition (1.39a) holds.

In condition (1.39b), for all $i \in \mathcal{S}$, we define $\partial h_i(\bar{q}^s_i) := \{z \in \mathbb{R} | h_i(q') \geq h_i(\bar{q}^s_i) + z(q' - \bar{q}^s_i), \forall q' \in [0, s_i]\}$. From $0 \in \partial \tilde{h}_i(\bar{q}^s_i)$, by using the definition of $\partial \tilde{h}_i(\bar{q}^s_i)$, we obtain that $\tilde{h}_i(q') \geq \tilde{h}_i(\bar{q}^s_i)$ for all $q' \in [0, s_i]$. From the definition of $\tilde{h}_i(\bar{q}^s_i)$, it follows that $h_i(q') - \bar{\theta}^s_i q' + \bar{\eta}^s_i q' \geq h_i(q^s_i) - \bar{\theta}^s_i q^s_i + \bar{\eta}^s_i q^s_i$ for all $q' \in [0, s_i]$. Thus, we conclude that $\bar{\theta}^s_i - \bar{\eta}^s_i \in \partial h_i(\bar{q}^s_i)$ and condition (1.39b) holds.

Condition (1.39c) follows immediately from $0 \in \partial \tilde{l}_{ij}(\bar{x}_{ij})$ and the definition that $\partial \tilde{l}_{ij}(\bar{x}_{ij}) = \{-\bar{\theta}^b_j + \bar{\theta}^s_i + w_{ij} - \bar{\pi}_{ij}\}$.

Similarly, conditions (1.39d) - (1.39f) and conditions (1.39g) - (1.39h) follow respectively from part (a) and part (b) in Theorem 28.3 of (50).

(ii)- 3: sufficiency of the KKT conditions. Let $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{q}}^s, \bar{\boldsymbol{q}}^b)$ and $(\bar{\boldsymbol{\theta}}^b, \bar{\boldsymbol{\theta}}^s, \bar{\boldsymbol{\eta}}^b, \bar{\boldsymbol{\eta}}^s, \bar{\boldsymbol{\pi}})$ be finite vectors that satisfy the KKT conditions in (1.39). The optimality follows directly from Theorem 28.3 of (50).

Proof of claim (iii). We divide the arguments into the following steps:

(iii)-1 necessity of the conditions By claim (i) of the proposition, there exists an optimal solution $(\boldsymbol{x}, \boldsymbol{q}^s, \boldsymbol{q}^b)$ to problem (1.38) with a finite objective value. It readily follows that $\{g_j(q^b_j) : j \in \mathcal{B}\}$ and $\{h_i(q^s_i) : i \in \mathcal{S}\}$ are finite. By claim (ii) of the proposition, there exists a dual optimal solution $(\boldsymbol{\theta}^b, \boldsymbol{\theta}^s, \boldsymbol{\eta}^b, \boldsymbol{\eta}^s, \boldsymbol{\pi})$ that satisfies conditions (1.39a) - (1.39h). We pick any pair of the primal-dual optimal solutions $(\boldsymbol{x}, \boldsymbol{q}^s, \boldsymbol{q}^b)$ and $(\boldsymbol{\theta}^b, \boldsymbol{\theta}^s, \boldsymbol{\eta}^b, \boldsymbol{\eta}^s, \boldsymbol{\pi})$.

We define the index sets $J_1 = \{j \in \mathcal{B} : q_j^b \in (0, b_j)\}$, $J_2 = \{j \in \mathcal{B} : q_j^b = 0\}$, and $J_3 = \{j \in \mathcal{B} : q_j^b = b_j\}$. Similarly, we define the index sets $I_1 = \{i \in \mathcal{S} : q_i^s \in (0, s_i)\}$, $I_2 = \{i \in \mathcal{S} : q_i^s = 0\}$ and $I_3 = \{i \in \mathcal{S} : q_i^s = s_i\}$.

Given $(\boldsymbol{x}, \boldsymbol{q}^s, \boldsymbol{q}^b)$, we initially set $(\tilde{\boldsymbol{\theta}}^b, \tilde{\boldsymbol{\theta}}^s, \tilde{\boldsymbol{\eta}}^b, \tilde{\boldsymbol{\eta}}^s, \tilde{\boldsymbol{\pi}}) := (\boldsymbol{\theta}^b, \boldsymbol{\theta}^s, \boldsymbol{\eta}^b, \boldsymbol{\eta}^s, \boldsymbol{\pi})$, and make local adjustments to $(\tilde{\boldsymbol{\theta}}^b, \tilde{\boldsymbol{\theta}}^s, \tilde{\boldsymbol{\eta}}^b, \tilde{\boldsymbol{\eta}}^s, \tilde{\boldsymbol{\pi}})$ for all $i \in \mathcal{S}$ and $j \in \mathcal{B}$. We show that the adjustments preserve conditions (1.39a) - (1.39h) and moreover, ensure conditions (1.40a) - (1.40b):

(1) for each $j \in J_1$, it follows from Assumption (1.12.1-1) that $g_j(q)$ is continuously differentiable in $(0, b_j)$. Thus, we have $\partial g_j(q_j^b) = \{g_j'(q_j^b)\}$ and condition (1.39a) implies that $\tilde{\theta}_j^b + \tilde{\eta}_j^b = g_j'(q_j^b)$;

(2) for each $j \in J_2$ such that $q_j^b = 0$, condition (1.39d) implies that $\eta_j^b = 0$. Moreover, we have $\theta_j^b \in \partial g_j(0)$ (by condition (1.39a)). We claim that $g_j'(0)$ is finite for this j. Recall that $g_j'(0) > -\infty$ (Assumption (1.12.1-3)). To prove that $g_j'(0) < \infty$, we assume towards a contradiction that $g_j'(0) = \infty$. We claim that this implies $\partial g_j(0) = \emptyset$, which contradicts with the fact that $\theta_j^b \in \partial g_j(0)$. Under the assumption that $g_j'(0) = \infty$, suppose that there exists $z_0 \in \partial g_j(0)$ such that $|z_0| < \infty$. By the continuity and the strict decreasingness of $g_j'(q)$ in $(0, b_j)$ (Assumption (1.12.1-1)) and the fact that $g_j'(0) = \lim_{q \downarrow 0} g_j'(q) = \infty$, it follows that there exists some $q_0 \in (0, b_j)$ such that $g_j'(q_0) \geq z_0$. Since $g_j(q)$ is continuous in $[0, q_0]$ and differentiable in $(0, q_0)$, by the mean value theorem, there exists $q_1 \in (0, q_0)$ such that $\frac{g_j(q_0) - g_j(0)}{q_0 - 0} = g_j'(q_1)$. By the continuous differentiability and the strict concavity of $g_j(q)$ in $(0, b_j)$ (Assumption (1.12.1-1)), we obtain that $g_j'(q_1) > g_j'(q_0) \geq z_0$. This implies that $\frac{g_j(q_0) - g_j(0)}{q_0 - 0} > z_0$, thereby leading to a contradiction with $z_0 \in \partial g_j(0)$. Thus, $g_j'(0) = \infty$ implies that $\partial g_j(0) = \emptyset$. Given the fact that $\theta_j^b \in \partial g_j(0)$, we have $g_j'(0) < \infty$. From the finiteness of $g_j(0)$ and $g_j'(0)$, the continuously differentiability and the strict concavity of $g_j(q)$ in $q \in (0, b_j)$, continuity of $g_j(q)$ at $q = 0$, the definition $\partial g_j(0) = \{z \in \mathbb{R} | g_j(q') \leq g_j(0) + z(q' - 0), \forall q' \in [0, b_j]\}$, and the fact that $\theta_j^b \in \partial g_j(0)$, it follows immediately that

$\theta_j^b \geq g_j'(0)$.

Let $\delta := \theta_j^b - g_j'(0) \geq 0$. Given the current profile of $(\tilde{\boldsymbol{\theta}}^b, \tilde{\boldsymbol{\theta}}^s, \tilde{\boldsymbol{\eta}}^b, \tilde{\boldsymbol{\eta}}^s, \tilde{\boldsymbol{\pi}})$ where $(\tilde{\theta}_j^b, \tilde{\eta}_j^b) = (\theta_j^b, \eta_j^b)$, we proceed by letting $\tilde{\pi}_{ij} := \tilde{\pi}_{ij} + \delta$ for all $i : (i,j) \in E$ and $\tilde{\theta}_j^b := \tilde{\theta}_j^b - \delta$. Among the conditions in (1.39) that are impacted, it can be readily checked that conditions (1.39a) and (1.39c) are still satisfied for this j and for any $i : (i,j) \in E$. Since $q_j^b = 0$, we have $x_{ij} = 0$ for all $i : (i,j) \in E$. Thus, condition (1.39f) still holds for all $i : (i,j) \in E$. These observations jointly imply that the adjustment of vector $(\tilde{\boldsymbol{\theta}}^b, \tilde{\boldsymbol{\theta}}^s, \tilde{\boldsymbol{\eta}}^b, \tilde{\boldsymbol{\eta}}^s, \tilde{\boldsymbol{\pi}})$ in this step preserves conditions (1.39a) - (1.39h). Moreover, the adjustment ensures that $\tilde{\theta}_j^b + \tilde{\eta}_j^b = g_j'(q_j^b)$ for this j;

(3) for each $j \in J_3$ such that $q_j^b = b_j$, condition (1.39a) implies that $\theta_j^b + \eta_j^b \in \partial g_j(q_j^b)$. We claim that $g_j'(b_j)$ is finite. By Assumption (1.12.1-3), we have $g_j'(b_j) \leq 0$. To establish $g_j'(b_j) > -\infty$, we assume towards a contradiction that $g_j'(b_j) = -\infty$. We claim the assumption would imply $\partial g_j(b_j) = \emptyset$, which is a contradiction to $\theta_j^b + \eta_j^b \in \partial g_j(q_j^b)$. To establish the claim, under the assumption that $g_j'(b_j) = -\infty$, suppose that there exists $z_0 \in \partial g_j(b_j)$ such that $|z_0| < \infty$. By the continuity and the strict decreasingness of $g_j'(q)$ in $(0, b_j)$, and the fact that $g_j'(b_j) = \lim_{q \uparrow b_j} g_j'(q) = -\infty$, it follows that there exists some $q_0 \in (0, b_j)$ such that $g_j'(q_0) \leq z_0$. Since $g_j(q)$ is continuous in $[q_0, b_j]$ and differentiable in (q_0, b_j), by the mean value theorem, there exists $q_1 \in (q_0, b_j)$ such that $\frac{g_j(b_j) - g_j(q_0)}{b_j - q_0} = g_j'(q_1)$. By the continuous differentiability and the strict concavity of $g_j(q)$ in $(0, b_j)$, we obtain that $g_j'(q_1) < g_j'(q_0) \leq z_0$. This implies that $\frac{g_j(b_j) - g_j(q_0)}{b_j - q_0} < z_0$, thereby leading to a contradiction with $z_0 \in \partial g_j(b_j)$. Thus, we conclude that $g_j'(b_j) = -\infty$ implies $\partial g_j(b_j) = \emptyset$. By the fact that $\theta_j^b + \eta_j^b \in \partial g_j(q_j^b)$, it follows that $g_j'(b_j)$ is finite. Given the finiteness of $g_j(b_j)$ and $g_j'(b_j)$, the continuously differentiability and the strict concavity of $g_j(q)$ in $q \in (0, b_j)$, the continuity of $g_j(q)$ at $q = b_j$, the definition $\partial g_j(b_j) = \{z \in \mathbb{R} | g_j(q') \leq g_j(b_j) + z(q' - b_j), \forall q' \in [0, b_j]\}$ and $\theta_j^b + \eta_j^b \in \partial g_j(b_j)$, we obtain that $\theta_j^b + \eta_j^b \leq g_j'(b_j)$.

Let $\delta := g_j'(b_j) - (\theta_j^b + \eta_j^b) \geq 0$. Given the current profile of $(\tilde{\boldsymbol{\theta}}^b, \tilde{\boldsymbol{\theta}}^s, \tilde{\boldsymbol{\eta}}^b, \tilde{\boldsymbol{\eta}}^s, \tilde{\boldsymbol{\pi}})$ where $(\tilde{\theta}_j^b, \tilde{\eta}_j^b) = (\theta_j^b, \eta_j^b)$ for this j, we let $\tilde{\eta}_j^b := \tilde{\eta}_j^b + \delta$. Among the conditions in (1.39) that are impacted, it can be readily checked that condition (1.39a) is still satisfied for this j.

Moreover, since $q_j^b = b_j$, condition (1.39d). These observations imply that the adjustment of vector $(\tilde{\boldsymbol{\theta}}^b, \tilde{\boldsymbol{\theta}}^s, \tilde{\boldsymbol{\eta}}^b, \tilde{\boldsymbol{\eta}}^s, \tilde{\boldsymbol{\pi}})$ in this step preserves the conditions (1.39a) - (1.39h). Moreover, the adjustment ensures that $\tilde{\theta}_j^b + \tilde{\eta}_j^b = g_j'(q_j^b)$ trivially holds.

(4) for each $i \in I_1$, it follows from Assumption (1.12.1-2) that $h_i(q)$ is continuously differentiable in $(0, s_i)$. Thus, we have $\partial h_i(q_i^s) = \{h_i'(q_i^s)\}$ and condition (1.39b) implies that $\tilde{\theta}_i^s + \tilde{\eta}_i^s = h_i'(q_i^s)$.

(5) for each $i \in I_2$ such that $q_i^s = 0$, condition (1.39e) implies that $\eta_i^s = 0$. Recall that $h_i'(0) \in [0, \infty)$ (by Assumption (1.12.1-3)). Given the finiteness of $h_i(0)$ and $h_i'(0)$, the continuous differentiability and the strict convexity of $h_i(q)$ in $q \in (0, s_i)$, the continuity of $h_i(q)$ at $q = 0$, the definition $\partial h_i(0) = \{z \in \mathbb{R} | h_i(q') \geq h_i(0) + z(q' - 0), \forall q' \in [0, s_i]\}$, and the fact that $\theta_i^s - \eta_i^s \in \partial h_i(q_i^s)$ and $\eta_i^s = 0$, it follows that $\theta_i^s \leq h_i'(0)$.

Let $\delta := h_i'(0) - \theta_i^s$. Given the current profile of $(\tilde{\boldsymbol{\theta}}^b, \tilde{\boldsymbol{\theta}}^s, \tilde{\boldsymbol{\eta}}^b, \tilde{\boldsymbol{\eta}}^s, \tilde{\boldsymbol{\pi}})$ where we have $(\tilde{\theta}_i^s, \tilde{\eta}_i^s) = (\theta_i^s, \eta_i^s)$ for this particular i, we let $\tilde{\theta}_i^s := \tilde{\theta}_i^s + \delta$ and $\tilde{\pi}_{ij} := \tilde{\pi}_{ij} + \delta$. Among the conditions in (1.39) that are impacted, it can be readily checked that condition (1.39b) and (1.39c) hold for this i and for any $j : (i, j) \in E$. Since $q_i^s = 0$, we have $x_{ij} = 0$ for all $j : (i, j) \in E$, which implies that condition (1.39f) still holds. Thus, the adjustments of vector $(\tilde{\boldsymbol{\theta}}^b, \tilde{\boldsymbol{\theta}}^s, \tilde{\boldsymbol{\eta}}^b, \tilde{\boldsymbol{\eta}}^s, \tilde{\boldsymbol{\pi}})$ in this step preserves conditions (1.39a) - (1.39h). Moreover, the adjustment also ensures that $\tilde{\theta}_i^s - \tilde{\eta}_i^s = h_i'(q_i^s)$.

(6) for each $i \in I_3$ such that $q_i^s = s_i$, condition (1.39b) implies that $\tilde{\theta}_i^s - \tilde{\eta}_i^s \in \partial h_i(q_i^s)$. Before proceeding, given $q_i^s = s_i$, we show that $h_i'(s_i)$ is finite. Recall that $h_i'(s_i) > -\infty$ (Assumption (1.12.1-3)). To establish that $h_i'(s_i) < \infty$, we assume towards a contradiction that $h_i'(s_i) = \infty$. We claim that this would imply $\partial h_i(s_i) = \emptyset$, which is a contradiction with the fact that $\tilde{\theta}_i^s - \tilde{\eta}_i^s \in \partial h_i(s_i)$. Under the assumption that $h_i'(s_i) = \infty$, suppose that there exists some $z_0 \in \partial h_i(s_i)$ such that $z_0 < \infty$. By the continuity and the strict increasingness of $h_i'(q)$ in $(0, s_i)$, and the fact that $h_i'(s_i) = \lim_{q \uparrow s_i} h_i'(q) = \infty$, there exists $q_0 \in (0, s_i)$ such that $h_i'(q_0) \geq z_0$. Since $h_i(q)$ is continuous in $[q_0, s_i]$ and differentiable in (q_0, s_i), by the mean value theorem, there exists $q_1 \in (q_0, s_i)$ such that $\frac{h_i(s_i) - h_i(q_0)}{s_i - q_0} = h_i'(q_1)$. By the continuous

differentiability and the strict convexity of $h_i(q)$ in $q \in (0, s_i)$, we obtain that $h_i'(q_1) > h_i'(q_0) \geq z_0$. It follows that $\frac{h_i(s_i) - h_i(q_0)}{s_i - q_0} > z_0$ or equivalently $h_i(q_0) < h_i(s_i) + z_0(q_0 - s_i)$, thereby leading to a contradiction with $z_0 \in \partial h_i(s_i)$. Thus, we conclude that $h_i'(s_i) = \infty$ implies $\partial h_i(s_i) = \emptyset$. Since $\tilde{\theta}_i^s - \tilde{\eta}_i^s \in \partial h_i(s_i)$, it immediately follows that $h_i'(s_i)$ is finite. From the continuous differentiability and the strict convexity of $h_i(q)$ in $q \in (0, s_i)$, the continuity of $h_i(q)$ at $q = s_i$, the definition $\partial h_i(s_i) = \{z \in \mathbb{R} | h_i(q') \geq h_i(s_i) + z(q' - s_i), \forall q' \in [0, s_i]\}$, and the fact that $\theta_i^s - \eta_i^s \in \partial h_i(s_i)$, we obtain that $\theta_i^s - \eta_i^s \geq h_i'(s_i)$.

Let $\delta := \theta_i^s - \eta_i^s - h_i'(s_i)$. Given the current profile of $(\tilde{\boldsymbol{\theta}}^b, \tilde{\boldsymbol{\theta}}^s, \tilde{\boldsymbol{\eta}}^b, \tilde{\boldsymbol{\eta}}^s, \tilde{\boldsymbol{\pi}})$ where we have $(\tilde{\theta}_i^s, \tilde{\eta}_i^s) = (\theta_i^s, \eta_i^s)$ for this particular i, we let $\tilde{\eta}_i^s := \tilde{\eta}_i^s + \delta$. Among the conditions in (1.39) that are impacted, it can be readily checked that condition (1.39b) is satisfied for this i. Moreover, since $q_i^s = s_i$, condition (1.39d) still holds. As a summary, we obtain that the updated vector $(\tilde{\boldsymbol{\theta}}^b, \tilde{\boldsymbol{\theta}}^s, \tilde{\boldsymbol{\eta}}^b, \tilde{\boldsymbol{\eta}}^s, \tilde{\boldsymbol{\pi}})$ still satisfies conditions (1.39a) - (1.39h). Moreover, the adjustment ensures that $\tilde{\theta}_i^s + \tilde{\eta}_i^s = g_i'(q_i^s)$.

In summary of the adjustments in (1) - (6), the adjusted dual optimal vector $(\tilde{\boldsymbol{\theta}}^b, \tilde{\boldsymbol{\theta}}^s, \tilde{\boldsymbol{\eta}}^b, \tilde{\boldsymbol{\eta}}^s, \tilde{\boldsymbol{\pi}})$ satisfies conditions (1.39a) - (1.39h) and (1.40a) - (1.40b). Thus, we conclude that it satisfies (1.40a) - (1.40h).

(iii)- 2: sufficiency of the conditions. Let $(\boldsymbol{x}, \boldsymbol{q}^s, \boldsymbol{q}^b)$ and $(\boldsymbol{\theta}^b, \boldsymbol{\theta}^s, \boldsymbol{\eta}^b, \boldsymbol{\eta}^s, \boldsymbol{\pi})$ be a pair of solutions such that $\{g_j'(\bar{q}_j^b) : j \in \mathcal{B}\}$ and $\{h_i'(\bar{q}_i^s) : i \in \mathcal{S}\}$ have finite values and the conditions in (1.40) are satisfied in problem (1.38). It immediately follows that $(\boldsymbol{x}, \boldsymbol{q}^s, \boldsymbol{q}^b)$ and $(\boldsymbol{\theta}^b, \boldsymbol{\theta}^s, \boldsymbol{\eta}^b, \boldsymbol{\eta}^s, \boldsymbol{\pi})$ also satisfy the KKT conditions in (1.39a) - (1.39h). The sufficiency of the conditions in (1.40) readily follows from part (ii) of the proposition statement.

Proof of claim (iv). To ease notation, denote the objective function for problem (1.38) as $f(\boldsymbol{x}, \boldsymbol{q}^s, \boldsymbol{q}^b) := \sum_{j \in \mathcal{B}} g_j(q_j^b) - \sum_{i \in \mathcal{S}} h_i(q_i^s) - \sum_{(i,j) \in E} w_{ij} x_{ij}$. According to Remark 1, it is without loss of optimality to consider the compact convex subset $\mathcal{Z}_0$ defined in (1.50) where the objective function $f(\boldsymbol{x}, \boldsymbol{q}^s, \boldsymbol{q}^b)$ is continuous. Moreover, by Assumptions (1.12.1-1) –(1.12.1-2), f is strictly concave in $(\boldsymbol{q}^s, \boldsymbol{q}^b)$ for every $(\boldsymbol{q}^s, \boldsymbol{q}^b)$ with $q_i^s \in (0, s_i)$ for all $i \in \mathcal{S}$ and $q_j^b \in (0, b_j)$ for all $j \in \mathcal{B}$, and f is linear in $\boldsymbol{x}$. Together with the continuity of f in $\mathcal{Z}_0$ and the fact that $\mathcal{Z}_0$ is compact, we conclude that f is strictly concave in $\mathcal{Z}_0$. Suppose, towards a contradiction, that there exists two optimal solution vectors $(\boldsymbol{x}_0, \boldsymbol{q}_0^s, \boldsymbol{q}_0^b) \in \mathcal{Z}_0$ and $(\boldsymbol{x}_1, \boldsymbol{q}_1^s, \boldsymbol{q}_1^b) \in \mathcal{Z}_0$ with $(\boldsymbol{q}_0^s, \boldsymbol{q}_0^b) \neq (\boldsymbol{q}_1^s, \boldsymbol{q}_1^b)$. Pick any $\lambda \in (0,1)$ and let $(\boldsymbol{x}_\lambda, \boldsymbol{q}_\lambda^s, \boldsymbol{q}_\lambda^b) = \lambda(\boldsymbol{x}_1, \boldsymbol{q}_1^s, \boldsymbol{q}_1^b) + (1-\lambda)(\boldsymbol{x}_0, \boldsymbol{q}_0^s, \boldsymbol{q}_0^b)$. Since set $\mathcal{Z}_0$ is compact, we have $(\boldsymbol{x}_\lambda, \boldsymbol{q}_\lambda^s, \boldsymbol{q}_\lambda^b) \in \mathcal{Z}_0$, and thus, feasible in problem (1.38). By Jensen's inequality, the objective function satisfies $f(\boldsymbol{q}_\lambda^b, \boldsymbol{q}_\lambda^s, \boldsymbol{x}_\lambda) > \lambda[f(\boldsymbol{q}_1^b, \boldsymbol{q}_1^s, \boldsymbol{x}_1)] + (1-\lambda)[f(\boldsymbol{q}_0^b, \boldsymbol{q}_0^s, \boldsymbol{x}_0)]$, which is a contradiction that $(\boldsymbol{x}_0, \boldsymbol{q}_0^s, \boldsymbol{q}_0^b)$ and $(\boldsymbol{x}_1, \boldsymbol{q}_1^s, \boldsymbol{q}_1^b)$ are optimal. As a result, for any pair of optimal solutions $(\boldsymbol{x}_0, \boldsymbol{q}_0^s, \boldsymbol{q}_0^b)$ and $(\boldsymbol{x}_1, \boldsymbol{q}_1^s, \boldsymbol{q}_1^b)$ we must have $(\boldsymbol{q}_0^s, \boldsymbol{q}_0^b) = (\boldsymbol{q}_1^s, \boldsymbol{q}_1^b)$ as desired.

Proof of claim (v). Let $(\boldsymbol{x}, \boldsymbol{q}^s, \boldsymbol{q}^b)$ and $(\boldsymbol{\theta}^b, \boldsymbol{\theta}^s, \boldsymbol{\eta}^b, \boldsymbol{\eta}^s, \boldsymbol{\pi})$ respectively be a primal and a dual optimal solution to problem (1.38) that satisfies the conditions in (1.40). Suppose, towards a contradiction, that $q_j^b = b_j$. Therefore, there must exist an $i \in \mathcal{S}$ for which $i : x_{ij} > 0$ and thus $q_i^s > 0$. Then, we have

$$0 \overset{(a)}{\geq} g_j'(b_j) \overset{(b)}{=} \theta_j^b + \eta_j^b \overset{(c)}{\geq} \theta_j^b \overset{(d)}{=} \theta_i^s + w_{ij} \overset{(e)}{=} h_i'(q_i^s) + \eta_i^s + w_{ij} \overset{(f)}{>} h_i'(0) \overset{(g)}{\geq} 0, \tag{1.57}$$

where (a) follows directly from Assumption (1.12.1-3). Step (b) follows directly from the condition in (1.40a). The inequality in (c) follows from the fact that $\eta_j^b \geq 0$ by condition (1.40d). In step (d), given that $x_{ij} > 0$, we have $\pi_{ij} = 0$ by condition (1.40f). From $\theta_j^b - \theta_i^s + \pi_{ij} - w_{ij} = 0$ in condition (1.40c), we obtain $\theta_j^b = \theta_i^s + w_{ij}$. Step (e) follows directly from condition (1.40b). In step (f), we have $\eta_i^s \geq 0$ by condition (1.40e) and $h_i'(q_i^s) > h_i'(0)$ by strict convexity of $h_i(\cdot)$ in Assumption (1.12.1-2). By Proposition 12(iii), $h_i'(q_i^s)$ is finite. Step (g) follows directly from Assumption (1.12.1-3). Thus, we have $q_j^b < b_j$ for all $j \in \mathcal{B}$.

Proof of claim (vi). Let $(\boldsymbol{x}, \boldsymbol{q}^s, \boldsymbol{q}^b)$ and $(\boldsymbol{\theta}^b, \boldsymbol{\theta}^s, \boldsymbol{\eta}^b, \boldsymbol{\eta}^s, \boldsymbol{\pi})$ respectively be a primal and dual optimal solution to problem (1.38) that satisfies the conditions in (1.40).

We first show the uniqueness of $\boldsymbol{\eta}^b$. By Claim (v), we have $q_j^b < b_j$ for all $j \in \mathcal{B}$. The, by the complementarity condition in (1.40d), we must have $\eta_j^b = 0$ for all $j \in \mathcal{B}$.

Next, we show that $\boldsymbol{\theta}^b$ is unique. For that, we use the fact that $\eta^b_j = 0$ together with condition (1.40a) imply that $\theta^b_j = g'_j(q^b_j)$ for all $j \in \mathcal{B}$. The uniqueness of θ^b_j then follows from the uniqueness of q^b_j in established in Claim (iv).

Next, we prove the uniqueness of $(\boldsymbol{\theta}^s, \boldsymbol{\eta}^s)$. Fix $i \in \mathcal{S}$. If $q^s_i = 0$, then by condition (1.40e) we have $\eta^s_i = 0$, which further implies from condition (1.40b) that $\theta^s_i = h'_i(q^s_i)$. The uniqueness of (η^s_i, θ^s_i) then follows from the uniqueness of q^s_i in Claim (iv). If $q^s_i > 0$, then we can find $j \in \mathcal{B}$ such that $x_{ij} > 0$ and thus, $\pi_{ij} = 0$ by condition (1.40f). Using condition (1.40c), this implies $\theta^s_i + w_{ij} = \theta^b_j$. The uniqueness of θ^s_i then follows from the uniqueness of θ^b_j. By condition (1.40b), we have that $\eta^s_i = -h'_i(q^s_i) + \theta^s_i$, and thus the uniqueness of η^s_i follows from the uniqueness of θ^s_i and of q^s_i (established in Claim (iv)).

Finally, the uniqueness of $\boldsymbol{\pi}$ follows directly from condition (1.40c) and the uniqueness of $(\boldsymbol{\theta}^s, \boldsymbol{\theta}^b)$.

Proof of claim (vii). We prove the two parts of this claim.

Part (a): suppose that $x_{ij}, x_{ij'} > 0$. Then, by the complementarity condition (1.40f), we have $\pi_{ij} = \pi_{ij'} = 0$. By Assumption (1.12.1-4), we have $w_i = w_{ij} = w_{ij'}$. Thus, by condition (1.40c), we obtain

$$\theta^b_j \;=\; \theta^s_i + w_i \;=\; \theta^b_{j'}. \tag{1.58}$$

As $g_j(q) = b_j g\left(\frac{q}{b_j}\right)$ by Assumption (1.12.1-4), we then must have $g'_j(q) = g'\left(\frac{q}{b_j}\right)$. Suppose, towards a contradiction, that $\frac{q^b_j}{b_j} \neq \frac{q^b_{j'}}{b_{j'}}$. Assume, without loss of generality, that $\frac{q^b_j}{b_j} > \frac{q^b_{j'}}{b_{j'}}$. By feasibility, we have that $1 \geq \frac{q^b_j}{b_j} > \frac{q^b_{j'}}{b_{j'}}$. Therefore, the complementarity condition (1.40d) implies

$$\eta^b_j \;\geq\; 0 \;=\; \eta^b_{j'}. \tag{1.59}$$

From (1.58) and (1.59), we obtain that $\theta^b_j + \eta^b_j \geq \theta^b_{j'} + \eta^b_{j'}$. Using condition (1.40a), we have that $\theta_j^b + \eta_j^b = g'\left(\frac{q_j^b}{b_j}\right)$ and $\theta_{j'}^b + \eta_{j'}^b = g'\left(\frac{q_{j'}^b}{b_{j'}}\right)$. From (1.58) and (1.59), we further obtain

$$g'\left(\frac{q_j^b}{b_j}\right) \geq g'\left(\frac{q_{j'}^b}{b_{j'}}\right). \tag{1.60}$$

However, from $\frac{q_j^b}{b_j} > \frac{q_{j'}^b}{b_{j'}}$ and the concavity of function $g(\cdot)$ (Assumption (1.12.1-1)), we obtain

$$g'\left(\frac{q_j^b}{b_j}\right) < g'\left(\frac{q_{j'}^b}{b_{j'}}\right), \tag{1.61}$$

which is a contradiction. Thus, we must have $\frac{q_j^b}{b_j} = \frac{q_{j'}^b}{b_{j'}}$.

Part (b): suppose that $x_{ij} > 0$ and $x_{ij'} = 0$. By the complementarity condition (1.40f), we have $\pi_{ij} = 0 \leq \pi_{ij'}$. Condition (1.40c) implies $\theta_j^b = \theta_i^s + w_i \geq \theta_{j'}^b$ (recall that by Assumption (1.12.1-4), we have $w_i = w_{ij} = w_{ij'}$). Assume, towards a contradiction that, $\frac{q_j^b}{b_j} > \frac{q_{j'}^b}{b_{j'}}$. We can then repeat the same argument as in part (a) to reach a contradiction between $g'\left(\frac{q_j^b}{b_j}\right) \geq g'\left(\frac{q_{j'}^b}{b_{j'}}\right)$ and $g'\left(\frac{q_j^b}{b_j}\right) < g'\left(\frac{q_{j'}^b}{b_{j'}}\right)$. Thus, we must have $\frac{q_j^b}{b_j} \leq \frac{q_{j'}^b}{b_{j'}}$, as desired.

Proof of claim (viii). We prove the two parts of this claim.

Part (a): suppose that $x_{ij}, x_{i'j} > 0$. Then, by complementarity condition (1.40f), we have $\pi_{ij} = \pi_{i'j} = 0$. By Assumption (1.12.1-5), we have $w_j = w_{ij} = w_{i'j}$. From condition (1.40c), we obtain

$$\theta_i^s \;=\; \theta_j^b - w_j \;=\; \theta_{i'}^s, \tag{1.62}$$

By Assumption (1.12.1-5), we have $h_i(q) = s_i h(\frac{q}{s_i})$, and thus $h_i'(q) = h'(\frac{q}{s_i})$. Suppose towards contradiction that $\frac{q_i^s}{s_i} \neq \frac{q_{i'}^s}{s_{i'}}$. Without loss of generality, we assume $\frac{q_i^s}{s_i} > \frac{q_{i'}^s}{s_{i'}}$. By feasibility, we have $1 \geq \frac{q_i^s}{s_i} > \frac{q_{i'}^s}{s_{i'}}$. Thus, the complementary condition (1.40e) implies

$$\eta_i^s \;\geq\; 0 \;=\; \eta_{i'}^s. \tag{1.63}$$

From (1.62) and (1.63), we obtain that $\theta_i^s - \eta_i^s \leq \theta_{i'}^s - \eta_{i'}^s$. Using condition (1.40b), we have that $\theta_i^s - \eta_i^s = h'\left(\frac{q_i^s}{s_i}\right)$ and $\theta_{i'}^s - \eta_{i'}^s = h'\left(\frac{q_{i'}^s}{s_{i'}}\right)$. By (1.62) and (1.63), we further obtain

$$h'\left(\frac{q_i^s}{s_i}\right) \leq h'\left(\frac{q_{i'}^s}{s_{i'}}\right). \tag{1.64}$$

However, with $\frac{q_i^s}{s_i} > \frac{q_{i'}^s}{s_{i'}}$ and the strict convexity of function $h(\cdot)$ in Assumption (1.12.1-2), we deduce that

$$h'\left(\frac{q_i^s}{s_i}\right) > h'\left(\frac{q_{i'}^s}{s_{i'}}\right), \tag{1.65}$$

which is a contradiction. Thus, we must have $\frac{q_i^s}{s_i} = \frac{q_{i'}^s}{s_{i'}}$.

Part (b): suppose that $x_{ij} > 0$ and $x_{i'j} = 0$. By condition (1.40f), we have $\pi_{ij} = 0 \leq \pi_{i'j}$ which further implies $\theta_i^s = \theta_j^b - w_j \leq \theta_{i'}^s$ by condition (1.40c) (recall that by Assumption (1.12.1-5), we have $w_j = w_{ij} = w_{i'j}$). Suppose, towards a contradiction, that $\frac{q_i^s}{s_i} > \frac{q_{i'}^s}{s_{i'}}$ and repeat the same argument as part (a) to achieve a contradiction between $h'\left(\frac{q_i^s}{s_i}\right) \leq h'\left(\frac{q_{i'}^s}{s_{i'}}\right)$ and $h'(\frac{q_i^s}{s_i}) > h'\left(\frac{q_{i'}^s}{s_{i'}}\right)$. The contradiction implies that $\frac{q_i^s}{s_i} \leq \frac{q_{i'}^s}{s_{i'}}$, as desired.

Proof of claim (ix). Let $(\boldsymbol{x}, \boldsymbol{q}^s, \boldsymbol{q}^b)$ and $(\boldsymbol{\theta}^b, \boldsymbol{\theta}^s, \boldsymbol{\eta}^b, \boldsymbol{\eta}^s, \boldsymbol{\pi})$ respectively be a primal optimal and a dual optimal solution to problem (1.38) that satisfies the conditions in (1.40). We prove the two parts of this claim.

Part (a): by Assumption (1.12.1-6), we have $w_{ij} = 0$ for all $(i,j) \in E$. We start by showing $g_j'(q_j^b) > 0$ for all $j \in \mathcal{B}$. Suppose, towards a contradiction, that there exists $j \in \mathcal{B}$ such that $g_j'(q_j^b) \leq 0$. From $g_j'(0) > 0$ (Assumption (1.12.1-8)), we know that $q_j^b \neq 0$ and thus $q_j^b > 0$. Then, we can find i such that $x_{ij} > 0$ and thus $q_i^s > 0$. Given $h_i'(0) = 0$ (Assumption

(1.12.1-7)) and the strict convexity of $h_i(\cdot)$ over $(0, s_i)$ (Assumption (1.12.1-2)), we obtain $h_i'(q_i^s) > h_i'(0) = 0$. This would lead to the following contradiction

$$0 \overset{(h)}{<} h_i'(q_i^s) + \eta_i^s \overset{(i)}{=} \theta_i^s \overset{(j)}{=} \theta_j^b \overset{(k)}{=} g_j'(q_j^b) - \eta_j^b \overset{(l)}{=} g_j'(q_j^b) \overset{(m)}{\leq} 0, \tag{1.66}$$

where step (h) follows from $h_i'(q_i^s) > 0$ and $\eta_i^s \geq 0$ (the complementary condition (1.40e)). Step (i) follows directly from condition (1.40b). In step (j), given $x_{ij} > 0$, we have $\pi_{ij} = 0$ by the complementary condition (1.40f), which further implies $\theta_i^s = \theta_j^b$ by condition (1.40c). Step (k) follows directly from condition (1.40a). In step (l), given $q_j^b < b_j$ established in Claim (v), we have $\eta_j^b = 0$ by the complementary condition (1.40d). Step (m) follows from the assumption in this contradiction argument. By the contradiction in (1.66), we obtain

$$g_j'(q_j^b) > 0 \quad \forall j \in \mathcal{B}. \tag{1.67}$$

Next, we prove $q_i^s > 0$ for all $i \in \mathcal{S}$. Suppose towards a contradiction that there exists $i \in \mathcal{S}$ with $q_i^s = 0$. Pick any j with $(i, j) \in E$. Then, we can derive the following contradiction

$$0 \overset{(n)}{=} h_i'(0) \overset{(o)}{=} \theta_i^s \overset{(p)}{\geq} \theta_j^b \overset{(q)}{=} g_j'(q_j^b) - \eta_j^b \overset{(r)}{>} 0, \tag{1.68}$$

where step (n) follows directly from Assumption (1.12.1-7). In step (o), given that $q_i^s = 0$, we first obtain $\eta_i^s = 0$ by the complementary condition (1.40e). Using $\eta_i^s = 0$, step (o) follows from $h_i'(q_i^s) - \theta_i^s + \eta_i^s = 0$ in condition (1.40b). The inequality in step (p) follows from $\theta_i^s - \theta_j^b = \pi_{ij}$ (condition (1.40c)) where we have $\pi_{ij} \geq 0$ (condition (1.40f)). Step (q) follows directly from condition (1.40a). In step (r), we obtain $\eta_j^b = 0$ by condition (1.40d) given $q_j^b < b_j$ in item (v). We also obtain $g_j'(q_j^b) > 0$ from (1.67). The contradiction in (1.68) implies that $q_i^s > 0$ for all $i \in \mathcal{S}$.

Part (b): when Assumptions (1.12.1-4) - Assumptions (1.12.1-5) hold, we have $g_j(q) = b_j g(\frac{q}{b_j})$ and $h_i(q) = s_i h(\frac{q}{s_i})$, which implies that $g_j'(q) = g'(\frac{q}{b_j})$ and $h_i'(q) = h'(\frac{q}{s_i})$. With the assumption $g'(0) > h'(0) \geq 0$, we suppose towards contradiction that there exists $i \in \mathcal{S}$ such that $q_i^s = 0$. For all j such that $(i,j) \in E$, we deduce that

$$h'(0) \overset{(s)}{=} \theta_i^s \overset{(t)}{\geq} \theta_j^b \overset{(u)}{=} g'\left(\frac{q_j^b}{b_j}\right) - \eta_j^b \overset{(v)}{=} g'\left(\frac{q_j^b}{b_j}\right), \tag{1.69}$$

where step (s) follows from condition (1.40b) given $\eta_i^s = 0$ (the complementary condition (1.40e)). Step (t) follows from $\theta_i^s - \theta_j^b = \pi_{ij}$ (condition (1.40c)) and $\pi_{ij} \geq 0$ (condition (1.40f)). Step (u) follows from condition (1.40a). Step (v) follows from $q_j^b < b_j$ (Claim (v)) and $\eta_j^b = 0$ (the complementary condition (1.40d)).

We discuss all possibilities on q_j^b to achieve a contradiction. If $q_j^b > 0$, then we can find $i_0 : x_{i_0 j} > 0$ and $q_{i_0}^s > 0$. By the complementary condition (1.40f), we have $\pi_{i_0 j} = 0$. By condition (1.40c), this further implies $\theta_{i_0}^s = \theta_j^b$. From condition (1.40b), we obtain $\theta_{i_0}^s = h'(\frac{q_{i_0}^s}{s_{i_0}}) + \eta_{i_0}^s$ with $\eta_{i_0}^s \geq 0$ (condition (1.40e)). Combining $\theta_j^b = \theta_{i_0}^s = h'(\frac{q_{i_0}^s}{s_{i_0}})$ with step (s) - (t) in (1.69), we have $h'(0) \geq h'(\frac{q_{i_0}^s}{s_{i_0}}) + \eta_{i_0}^s$, which contradicts to the strict convexity of function h given $q_{i_0}^s > 0$. If $q_j^b = 0$, then by (1.69), we have $h'(0) \geq g'(0)$ which contradicts $g'(0) > h'(0)$. Thus, we obtain $q_i^s > 0$ for all $i \in \mathcal{S}$.

Proof of Lemma 4. Proof of Claim (i) One can easily verify that the equilibrium problem (1.16) can be formulated as an instance of the framework problem (1.38) by defining

$$g_j(q) = \int_0^q \tilde{F}_{b_j}^{-1}\left(1 - \frac{x}{b_j}\right) dx \quad \text{where} \quad \tilde{F}_{b_j}^{-1}(q) = \frac{F_{b_j}^{-1}(q) - \mu_j^b}{1 + \gamma_j^b} \quad \forall j \in \mathcal{B}, \tag{1.70a}$$

$$h_i(q) = \int_0^q \tilde{F}_{s_i}^{-1}\left(\frac{x}{s_i}\right) dx \quad \text{where} \quad \tilde{F}_{s_i}^{-1}(q) = \frac{F_{s_i}^{-1}(q) + \mu_i^s}{1 - \gamma_i^s} \quad \forall i \in \mathcal{S}. \tag{1.70b}$$

and $w_{ij} = \frac{c_i}{1+\gamma_j^b}$ for all $(i,j) \in E$. We next verify that, under these definitions, Assumptions (1.12.1-1) - (1.12.1-3) in problem (1.16). Moreover, if $\bar{v}_{b_j} > \mu_j^b$, then Assumption (1.12.1-8) holds.

Verifying Assumption (1.12.1-1) in problem (1.16): We start by showing that $g_j(q)$ is continuously differentiable in $(0, b_j)$. For any $q \in (0, b_j)$, we pick $t_1 \in (0, q)$ and rewrite g_j as:

$$g_j(q) = \int_0^{t_1} \tilde{F}_{b_j}^{-1}\left(1 - \frac{x}{b_j}\right) dx + \int_{t_1}^{q} \tilde{F}_{b_j}^{-1}\left(1 - \frac{x}{b_j}\right) dx. \tag{1.71}$$

By (1.70a), the first term in (1.71) satisfies $\int_0^{t_1} \tilde{F}_{b_j}^{-1}(1-\frac{x}{b_j})dx = \frac{1}{1+\gamma_j^b}\int_0^{t_1} F_{b_j}^{-1}(1-\frac{x}{b_j})-\mu_j^b dx$. By using a change of variables, we have $\int_0^{b_j} F_{b_j}^{-1}(1-\frac{x}{b_j})dx = b_j \int_0^{\infty}(1-F_{b_j}(v))dv < \infty$, where the last inequality follows from Assumption 1. Thus, the first term in (1.71) is a finite constant term.

Consider the second term in (1.71). Since the function $\tilde{F}_{b_j}^{-1}(1-\frac{q}{b_j})$ is continuously differentiable in the compact set $[t_1, t_2]$ for every $t_2 \in (q, b_j)$; this follows from the fact that, by Assumption 1, F_{b_j} is continuously differentiable and strictly increasing in $(0, \bar{v}_{b_j})$ and so, by the inverse function theorem, $F_{b_j}^{-1}$ and $\tilde{F}_{b_j}^{-1}$ are continuously differentiable in $(0, b_j)$. By the fundamental theorem of calculus, function $g_j(q)$ is differentiable at $q \in [t_1, t_2] \subset (0, b_j)$ for every $t_2 \in (q, b_j)$. Moreover, the fundamental theorem of calculus also suggests that the derivative function satisfies $g_j'(q) = \tilde{F}_{b_j}^{-1}(1-\frac{q}{b_j})$, which is continuous. Since q is picked arbitrarily within $(0, b_j)$, we obtain that $q_j(q)$ is continuously differentiable in $(0, b_j)$.

We next show that $g_j(q)$ is strictly concave. Given that F_{b_j} is differentiable and strictly increasing in $(0, \bar{v}_{b_j})$ (Assumption 1), we obtain that $\tilde{F}_{b_j}^{-1}$ is strictly increasing (the inverse function theorem). Thus, the derivative function $g_j'(q) = \tilde{F}_{b_j}^{-1}(1-\frac{q}{b_j})$ is strictly decreasing in $q \in (0, b_j)$, which implies that function $g_j(q)$ is strictly concave in $(0, b_j)$.

We also need to verify the continuity of $g_j(q)$ at $q = 0$. Considering the case when $\bar{v}_{b_j} < \infty$, we have that

$$0 \;\le\; \lim_{q \downarrow 0} \int_0^{q} F_{b_j}^{-1}\left(1 - \frac{x}{b_j}\right) dx \;\le\; \lim_{q \downarrow 0} \bar{v}_{b_j} q \;=\; 0. \tag{1.72}$$

From (1.70a), we have $\lim_{q\downarrow 0}\int_0^q \tilde{F}_{b_j}^{-1}\left(1-\frac{x}{b_j}\right)dx = 0$.

When $\bar{v}_{b_j}=\infty$, consider (v,q) such that $F_{b_j}(v) = 1-\frac{q}{b_j}$. Then,

$$\int_0^q F_{b_j}^{-1}\left(1-\frac{x}{b_j}\right)dx = b_j v\left(1-F_{b_j}(v)\right) + b_j\int_v^\infty \left(1-F_{b_j}(x)\right)dx. \tag{1.73}$$

Moreover, the first term on the right hand side of (1.73) satisfies

$$\begin{aligned} v\left(1-F_{b_j}(v)\right) &= v\int_v^\infty F'_{b_j}(x)dx \le \int_0^\infty xF'_{b_j}(x)dx - \int_0^v xF'_{b_j}(x)dx \\ &= \int_0^\infty \left(1-F_{b_j}(x)\right)dx - \int_0^v \left(1-F_{b_j}(x)\right)dx \end{aligned} \tag{1.74}$$

Thus, we deduce that

$$\begin{aligned} &\lim_{q\downarrow 0}\int_0^q \tilde{F}_{b_j}^{-1}\left(1-\frac{x}{b_j}\right)dx \\ &\overset{(a)}{=} \lim_{v\to\infty}\left[\frac{b_j}{1+\gamma_j^b}v\left(1-F_{b_j}(v)\right) + \frac{b_j}{1+\gamma_j^b}\int_v^\infty \left(1-F_{b_j}(x)\right)dx\right] - \left[\lim_{q\downarrow 0}\frac{1}{1+\gamma_j^b}\int_0^q \mu_j^b dx\right] \\ &\overset{(b)}{\le} \frac{2b_j}{1+\gamma_j^b}\left[\int_0^\infty \left(1-F_{b_j}(x)\right)dx - \lim_{v\to\infty}\int_0^v \left(1-F_{b_j}(x)\right)dx\right] - \left[\lim_{q\downarrow 0}\int_0^q \frac{1}{1+\gamma_j^b}\mu_j^b dx\right] \\ &\overset{(c)}{=} 0, \end{aligned} \tag{1.75}$$

where step (a) follows from (1.70a) and (1.73). In step (b), the first term follows directly from (1.74) and $\int_v^\infty \left(1-F_{b_j}(x)\right)dx = \int_0^\infty \left(1-F_{b_j}(x)\right)dx - \int_0^v \left(1-F_{b_j}(x)\right)dx$ based on the finiteness of $\int_0^\infty \left(1-F_{b_j}(x)\right)dx$ in Assumption 1. In step (c), the first term is zero because of the finite expectation assumption of $F_{b_j}(\cdot)$ in Assumption 1. The second term, $\lim_{q\downarrow 0}\frac{1}{1+\gamma_j^b}\int_0^q \mu_j^b dx$, is also equal to zero.

To show $\lim_{q\downarrow 0}\int_0^q \tilde{F}_{b_j}^{-1}(1-\frac{x}{b_j})dx \ge 0$, we consider the expression in step (a) of (1.75). Since $\frac{b_j}{1+\gamma_j^b}v\left(1-F_{b_j}(v)\right) + \frac{b_j}{1+\gamma_j^b}\int_v^\infty \left(1-F_{b_j}(x)\right)dx \ge 0$ and $\lim_{q\downarrow 0}\frac{1}{1+\gamma_j^b}\int_0^q \mu_j^b dx = 0$, we obtain that

$$\lim_{q\downarrow 0}\int_0^q \tilde{F}_{b_j}^{-1}\left(1-\frac{x}{b_j}\right)dx \geq 0. \tag{1.76}$$

Thus, from (1.70a), (1.75) and (1.76), we conclude that $g_j(q)$ is continuous at $q=0$ with $g_j(0)=0$.

To show the continuity of $g_j(q)$ at $q=b_j$, we deduce that

$$\begin{aligned} g_j(b_j) &\overset{(d)}{=} \frac{1}{1+\gamma_j^b}\int_0^{b_j} F_{b_j}^{-1}\left(1-\frac{x}{b_j}\right)dx - \frac{\mu_j^b}{1+\gamma_j^b}b_j \\ &\overset{(e)}{=} \frac{b_j}{1+\gamma_j^b}\int_0^{\bar{v}_{b_j}}\Big(1-F_{b_j}(x)\Big)dx - \frac{\mu_j^b b_j}{1+\gamma_j^b} \overset{(f)}{\in} \left[-\frac{\mu_j^b b_j}{1+\gamma_j^b},\infty\right), \end{aligned} \tag{1.77}$$

where step (d) follows from (1.70a). Step (e) follows from a change of variables. Step (f) follows from the finite expectation in Assumption 1. Thus, the limiting point $g_j(b_j)$ is well-defined.

<u>Verifying Assumption (1.12.1-2) in problem (1.16):</u> We start showing that $h_i(q)$ is differentiable in $(0,s_i)$. Similar to the verification of Assumption (1.12.1-1), for any $q\in(0,s_i)$, we can pick $t_1\in(0,q)$ such that we can rewrite $h_i(q)$ as:

$$h_i(q)=\int_0^q \tilde{F}_{s_i}^{-1}\left(\frac{x}{s_i}\right)dx=\int_0^{t_1}\tilde{F}_{s_i}^{-1}\left(\frac{x}{s_i}\right)dx+\int_{t_1}^q \tilde{F}_{s_i}^{-1}\left(\frac{x}{s_i}\right)dx. \tag{1.78}$$

By (1.70b), the first term in (1.78) satisfies $\int_0^{t_1}\tilde{F}_{s_i}^{-1}(\frac{x}{s_i})dx=\frac{1}{1-\gamma_i^s}\int_0^{t_1}F_{s_i}^{-1}(\frac{x}{s_i})+\mu_i^s dx$. By changing variable, we obtain that $\int_0^{s_i}F_{s_i}^{-1}(\frac{x}{s_i})dx = s_i\int_0^{\infty}(1-F_{s_i}(v))dv<\infty$ where the inequality follows from the finite expectation of F_{s_i} in Assumption 1. Thus, the first term in (1.78) is a finite constant.

By Assumption 1, F_{s_i} is continuous differentiable. By the inverse function theorem, $F_{s_i}^{-1}(\frac{q}{s_i})$ and $\tilde{F}_{s_i}^{-1}(\frac{q}{s_i})$ are continuously differentiable in $q\in[t_1,t_2]$ for any $t_2\in(q,s_i)$. By the fundamental theorem of calculus, function $h_i(q)$ is differentiable in $q\in[t_1,t_2]\subset(0,s_i)$

with the derivative function defined as $h_i'(q) = \tilde{F}_{s_i}^{-1}(\frac{q}{s_i})$ which is continuous. Since q is picked arbitrarily in $(0, s_i)$, we conclude that function $h_i(q)$ is continuously differentiable in $q \in (0, s_i)$.

To establish the convexity of function $h_i(q)$, note that by Assumption 1, F_{s_i} is differentiable and strictly increasing, which implies that $F_{s_i}^{-1}$ is differentiable and strictly increasing (the inverse function theorem). Thus, $\tilde{F}_{s_i}^{-1}$ is strictly increasing. Thus, as the derivative function $h_i'(q) = \tilde{F}_{s_i}^{-1}(\frac{q}{s_i})$ is strictly increasing in $q \in (0, s_i)$, we conclude that function $h_i(q)$ is strictly convex in $(0, s_i)$.

To establish the continuity of $h_i(q)$ at $q = 0$, it is sufficient to show that the limiting point is well-defined. Given (1.70b), we obtain that $\lim_{q\downarrow 0} \int_0^q F_{s_i}^{-1}(\frac{x}{s_i})dx = 0$ and $\lim_{q\downarrow 0} \int_0^q \mu_i^s dx = 0$. Thus, $\lim_{q\downarrow 0} \int_0^q \tilde{F}_{s_i}^{-1}(\frac{x}{s_i})dx = 0$.

We next verify the continuity at $q = s_i$ by deducing that

$$\begin{aligned} h_i(s_i) \;=\; \int_0^{s_i} \tilde{F}_{s_i}^{-1}\left(\frac{x}{s_i}\right) dx \;&\overset{(g)}{=}\; \frac{1}{1-\gamma_i^s}\int_0^{s_i} F_{s_i}^{-1}\left(\frac{x}{s_i}\right) dx + \frac{\mu_i^s}{1-\gamma_i^s} s_i \\ &\overset{(h)}{=}\; \frac{s_i}{1-\gamma_i^s}\int_0^{\bar{v}_{s_i}} \left(1 - F_{s_i}(x)\right) dx + \frac{\mu_i^s}{1-\gamma_i^s} s_i \;\overset{(i)}{\in}\; (0,\infty), \end{aligned} \tag{1.79}$$

where step (g) follows directly from (1.70b). Step (h) follows from the change of variables. Step (i) follows from the finite expectation of distribution F_{s_i} in Assumption 1.

Verifying Assumption (1.12.1-3) in problem (1.16): From (1.70), it readily follows that

$$\begin{aligned} \lim_{q\uparrow b_j} g_j'(q) \;&=\; \lim_{q\uparrow b_j} \tilde{F}_{b_j}^{-1}\left(1-\frac{q}{b_j}\right) \;=\; \lim_{q\uparrow b_j} \frac{1}{1+\gamma_j^b} F_{b_j}^{-1}\left(1-\frac{q}{b_j}\right) - \frac{\mu_j^b}{1+\gamma_j^b} \;\in\; (-\infty, 0], \quad \forall j \in \mathcal{B}, \\ \lim_{q\downarrow 0} g_j'(q) \;&=\; \lim_{q\downarrow 0} \tilde{F}_{b_j}^{-1}\left(1-\frac{q}{b_j}\right) \;=\; \lim_{q\downarrow 0} \frac{1}{1+\gamma_j^b} F_{b_j}^{-1}\left(1-\frac{q}{b_j}\right) - \frac{\mu_j^b}{1+\gamma_j^b} \;>\; -\infty, \quad \forall j \in \mathcal{B}, \\ \lim_{q\downarrow 0} h_i'(q) \;&=\; \lim_{q\downarrow 0} \tilde{F}_{s_i}^{-1}\left(\frac{q}{s_i}\right) = \lim_{q\downarrow 0} \frac{1}{1-\gamma_i^s} F_{s_i}^{-1}\left(\frac{q}{s_i}\right) + \frac{\mu_i^s}{1-\gamma_i^s} \;\in\; [0,\infty), \quad \forall i \in \mathcal{S}, \\ \lim_{q\uparrow s_i} h_i'(q) \;&=\; \lim_{q\uparrow s_i} \tilde{F}_{s_i}^{-1}\left(\frac{q}{s_i}\right) = \lim_{q\uparrow s_i} \frac{1}{1-\gamma_i^s} F_{s_i}^{-1}\left(\frac{q}{s_i}\right) + \frac{\mu_i^s}{1-\gamma_i^s} \;>\; -\infty, \quad \forall i \in \mathcal{S}. \end{aligned} \tag{1.80}$$

Verifying Assumption (1.12.1-8) in problem (1.16): If $\bar{v}_{b_j} > \mu_j^b$ for all $j \in \mathcal{B}$, we deduce that

$$\lim_{q \downarrow 0} g_j'(q) \;=\; \lim_{q \downarrow 0} \tilde{F}_{b_j}^{-1}\left(1 - \frac{q}{b_j}\right) \;=\; \lim_{q \downarrow 0} \frac{1}{1+\gamma_j^b} F_{b_j}^{-1}\left(1 - \frac{q}{b_j}\right) - \frac{\mu_j^b}{1+\gamma_j^b} \;=\; \frac{\bar{v}_{b_j} - \mu_j^b}{1+\gamma_j^b} \;>\; 0. \tag{1.81}$$

Proof of Claim (ii). One can easily verify that the revenue optimization problem (1.7) can be formulated as an instance of the framework problem where

$$g_j(q) = F_{b_j}^{-1}\left(1 - \frac{q}{b_j}\right) q, \quad \forall q \in [0, b_j],\ j \in \mathcal{B}, \tag{1.82a}$$

$$h_i(q) = F_{s_i}^{-1}\left(\frac{q}{s_i}\right) q, \quad \forall q \in [0, s_i],\ i \in \mathcal{S}, \tag{1.82b}$$

and $w_{ij} = c_i$ for all $(i,j) \in E$. We next verify that under these definitions, Assumptions (1.12.1-1) - (1.12.1-2), Assumption (1.12.1-3) and Assumptions (1.12.1-7) - (1.12.1-8) hold.

Verifying Assumption (1.12.1-1) in problem (1.7): F_{b_j} is continuously differentiable in $(0, \bar{v}_{b_j})$ by Assumption 1, which implies that $F_{b_j}^{-1}$ is continuously differentiable in $(0,1)$ (by the inverse function theorem). Given (1.82a), we then obtain that $g_j(q)$ is continuously differentiable in $(0, b_j)$ by Assumption 1.

Using the expression in (1.82a), the concavity of $g_j(q)$ follows directly from Assumption 2.

To establish the continuity of $g_j(q)$ at $q = 0$, we note that

$$0 \overset{(i)}{\leq} \lim_{q \downarrow 0} g_j(q) \;=\; \lim_{q \downarrow 0} F_{b_j}^{-1}\left(1 - \frac{q}{b_j}\right) q \overset{(j)}{\leq} \lim_{q \downarrow 0} \int_0^q F_{b_j}^{-1}\left(1 - \frac{x}{b_j}\right) dx \overset{(k)}{=} 0, \tag{1.83}$$

where step (i) follows from the non-negativity of $F_{b_j}^{-1}(1 - \frac{q}{b_j})$ in $(0, b_j)$, which is derived from the non-negative support of distribution function F_{b_j} in Assumption 1. In step (j), given the differentiability and the strictly increasing property of F_{b_j} (Assumption 1), we obtain that $F_{b_j}^{-1}$ is strictly increasing by the inverse function theorem. Thus, function $F_{b_j}^{-1}(1 - \frac{q}{b_j})$ is strictly decreasing in $(0, b_j)$ and step (j) holds. Step (k) follows as a special case of (1.75) and (1.76) with $\gamma_j^b = \mu_j^b = 0$. Thus, $g_j(q)$ is continuous at $q = 0$.

To show the continuity of $g_j(q)$ at $q = b_j$, we directly use (1.82a) to deduce that $g_j(b_j) = \lim_{q\uparrow b_j} g_j(q) = \lim_{q\uparrow b_j} F_{b_j}^{-1}\left(1 - \frac{q}{b_j}\right) q = 0$.

<u>Verifying Assumption (1.12.1-2) in problem (1.7):</u> By Assumption 1, F_{s_i} is continuously differentiable in $(0, \bar{v}_{s_i})$, which implies that $F_{s_i}^{-1}$ is continuous differentiable in $(0, 1)$ (by the inverse function theorem). Thus, based on (1.82b), $h_i(q)$ is continuously differentiable in $(0, s_i)$.

Using the expression in (1.82b), the convexity of $h_i(q)$ follows from Assumption 2.

To establish the continuity of $h_i(q)$ at $q = 0$, by (1.82), we obtain that $h_i(0) = \lim_{q\downarrow 0} h_i(q) = \lim_{q\downarrow 0} F_{s_i}^{-1}(\frac{q}{s_i})q = 0$. For the continuity of $h_i(q)$ at $q = s_i$, when $\bar{v}_{s_i} = \infty$, we obtain that $\lim_{q\uparrow s_i} h_i(q) = \lim_{q\uparrow s_i} F_{s_i}^{-1}(\frac{q_i^s}{s_i})q = \bar{v}_{s_i} s_i = \infty$. In comparison, when $\bar{v}_{s_i} < \infty$, we obtain that $\lim_{q\uparrow s_i} h_i(q) = \lim_{q\uparrow s_i} F_{s_i}^{-1}(\frac{q_i^s}{s_i})q = \bar{v}_{s_i} s_i < \infty$. Thus, whenever $h_i(s_i) < \infty$, $h_i(q)$ is continuous at $q = s_i$.

<u>Verifying Assumption (1.12.1-3) in problem (1.7):</u> Recall first that $g_j(q)$ is continuous differentiable in $(0, b_j)$. Moreover, $g_j(q)$ is continuous at $q = 0$ and $q = b_j$ with $g_j(0) = g_j(b_j) = 0$. By the Rolle's theorem, there exists $q_0 \in (0, b_j)$ such that $g_j'(q_0) = 0$. By the strict concavity of $g_j(\cdot)$ in $(0, b_j)$, we have $g_j'(q) \leq 0$ for all $q \geq q_0$. Thus, $\lim_{q\uparrow b_j} g_j'(q) \leq g_j'(q_0) = 0$. We also have $g_j'(q) \geq 0$ for all $q \leq q_0$, which implies that $g_j'(0) = \lim_{q\downarrow 0} g_j'(q) \geq g_j'(q_0) = 0 > -\infty$.

Regarding $h_i'(0)$, we show that

$$0 \;=\; \lim_{q\downarrow 0}\left[2F_{s_i}^{-1}\left(\frac{q}{s_i}\right) - F_{s_i}^{-1}\left(\frac{q}{2s_i}\right)\right] \;\overset{(l)}{\leq}\; \lim_{q\downarrow 0} h_i'(q) \;\overset{(m)}{\leq}\; \lim_{q\downarrow 0}\left[2F_{s_i}^{-1}\left(\frac{2q}{s_i}\right) - F_{s_i}^{-1}\left(\frac{q}{s_i}\right)\right] \;=\; 0, \tag{1.84}$$

where in step (l) and (m), by the convexity of $h_i(q)$, we consider the supporting hyperplanes and obtain $h'(q) \geq \frac{h(q) - h(\frac{1}{2}q)}{\frac{1}{2}q}$ in step (l) and $h_i'(q) \leq \frac{h_i(2q) - h_i(q))}{q}$ in step (m). Based on

(1.84), we obtain that $\lim_{q\downarrow 0} h_i'(q) = 0$.

By the continuous differentiability and the strict increasiness of $h_i(q)$ in $q \in (0, s_i)$, we have $h_i'(q) > 0$ for all $q \in (0, s_i)$, which implies that $h_i'(s_i) = \lim_{q\uparrow s_i} \geq h_i'(q) \geq 0 \geq -\infty$.

<u>Verifying Assumption (1.12.1-7) in problem (1.7):</u> We can directly apply the expression in (1.84) to obtain that $\lim_{q\downarrow 0} h_i'(q) = 0$.

<u>Verifying Assumption (1.12.1-8) in problem (1.7):</u> Recall that function $g_j(q) = F_{b_j}^{-1}\left(1 - \frac{q}{b_j}\right) q$ is a strictly concave differentiable function on $(0, b_j)$. Moreover, $g_j(q)$ is continuous at $q = 0$ and $q = b_j$ with $g(0) = g(b_j) = 0$. By the Rolle's theorem, there exists q_0 with $g_j'(q_0) = 0$. By strict concavity of $g_j(\cdot)$, the derivative function satisfies $g_j'(q) > 0$ for all $q < q_0$. Thus, we have $\lim_{q\downarrow 0} g_j'(q) > g_j'(\frac{1}{2}q_0) \geq g_j'(q_0) = 0$. □

Proof of Lemma 5. Recall from Lemma 4 that both the equilibrium problem (1.16) and the revenue optimization problem (1.7) are instances of the framework problem (1.38) where Assumption (1.12.1-3) holds. By Proposition 12(i) and Proposition 12(ii), there exists an optimal primal-dual pair that satisfies the KKT conditions in (1.39). Let $(\boldsymbol{x}, \boldsymbol{q}^s, \boldsymbol{q}^b)$ be any primal optimal solution to problem (1.38). Let $(\boldsymbol{\theta}^b, \boldsymbol{\theta}^s, \boldsymbol{\eta}^b, \boldsymbol{\eta}^s, \boldsymbol{\pi})$ be any corresponding dual optimal solution associated with constraints (1.38b) - (1.38f) that maximizes $\sum_{j\in\mathcal{B}}[sgn(q_j^b) - sgn(b_j - q_j^b)](\theta_j^b + \eta_j^b) + \sum_{i\in\mathcal{S}}[-sgn(q_i^s) + sgn(s_i - q_i^s)](\theta_i^s - \eta_i^s)$.

The finiteness of the optimal objective value in problem (1.38) implies that $\{g_j(q_j^b) : j \in \mathcal{B}\}$ and $\{h_i(q_i^s) : i \in \mathcal{S}\}$ all have finite values. By Proposition 12(iii), $\{g_j'(q_j^b) : j \in \mathcal{B}\}$ and $\{h_i'(q_i^s) : i \in \mathcal{S}\}$ also have finite values. Moreover, it follows from Proposition 12(iii), there exists another dual vector $(\bar{\boldsymbol{\theta}}^b, \bar{\boldsymbol{\theta}}^s, \bar{\boldsymbol{\eta}}^b, \bar{\boldsymbol{\eta}}^s, \bar{\boldsymbol{\pi}})$ that satisfies conditions (1.40a) - (1.40b). By Proposition 12(vi), $(\bar{\boldsymbol{\theta}}^b, \bar{\boldsymbol{\theta}}^s, \bar{\boldsymbol{\eta}}^b, \bar{\boldsymbol{\eta}}^s, \bar{\boldsymbol{\pi}})$ is unique.

For any $j \in \mathcal{B}$, suppose first that $q_j^b \in (0, b_j)$. It can be readily checked that $[sgn(\bar{q}_j^b) - sgn(b_j - \bar{q}_j^b)](\theta_j^b + \eta_j^b) = 0$. From the continuously differentiability of g_j in $(0, b_j)$ (by Assumption (1.12.1-1)), we have $\partial g_j(q_j^b) = \{g_j'(q_j^b)\}$. From $\theta_j^b + \eta_j^b \in \partial g_j(q_j^b)$ (condition

(1.39a)) and $\bar{\theta}_j^b + \bar{\eta}_j^b = g_j'(q_j^b)$ (condition (1.40a)), it follows that

$$\text{if } q_j^b \in (0, b_j), \text{ then } \theta_j^b + \eta_j^b \ = \ g_j'(q_j^b) \ = \ \bar{\theta}_j^b + \bar{\eta}_j^b. \tag{1.85}$$

Suppose instead that $q_j^b = 0$. It immediately follows that $[sgn(q_j^b) - sgn(b_j - q_j^b)](\theta_j^b + \eta_j^b) = -(\theta_j^b + \eta_j^b)$. Recall that $g_j(0)$ and $g_j'(0)$ have finite values given the optimal solution is such that $q_j^b = 0$. By condition (1.39a), we have $\theta_j^b + \eta_j^b \in \partial g_j(0)$. From the continuous differentiability and the strict concavity of $g_j(q)$ in $q \in (0, b_j)$, the continuity at $q = 0$ (Assumption (1.12.1-1)), and the fact that $\partial g_j(0) = \{z \in \mathbb{R} | g_j(q') \le g_j(0) + z(q' - 0), \forall q' \in [0, b_j]\}$, it follows that $\theta_j^b + \eta_j^b \ \ge \ g_j'(0)$. Together with the fact that $\bar{\theta}_j^b + \bar{\eta}_j^b = g_j'(0)$ (by condition (1.40a)), this implies that

$$\text{if } q_j^b = 0, \text{ then } \theta_j^b + \eta_j^b \ \ge \ g_j'(0) \ = \ \bar{\theta}_j^b + \bar{\eta}_j^b. \tag{1.86}$$

Suppose next that $q_j^b = b_j$, which implies $[sgn(q_j^b) - sgn(b_j - q_j^b)](\theta_j^b + \eta_j^b) = \theta_j^b + \eta_j^b$. Recall that $g_j(b_j)$ and $g_j'(b_j)$ have finite values given the optimal solution $q_j^b = b_j$. Given that $g_j(q)$ is continuously differentiable and strictly concave in $q \in (0, b_j)$, continuous at $q = b_j$ (Assumption (1.12.1-1)), $\partial g_j(b_j) = \{z \in \mathbb{R} | g_j(q') \le g_j(b_j) + z(q' - b_j), \forall q' \in [0, b_j]\}$ and $\theta_j^b + \eta_j^b \in \partial g_j(b_j)$ (by condition (1.39a)) imply that $\theta_j^b + \eta_j^b \le g_j'(b_j)$. Together with the fact $\bar{\theta}_j^b + \bar{\eta}_j^b = g_j'(b_j)$ (condition (1.40a)), this observation implies that

$$\text{if } q_j^b = b_j, \text{ then } \theta_j^b + \eta_j^b \ \le \ g_j'(b_j) \ = \ \bar{\theta}_j^b + \bar{\eta}_j^b. \tag{1.87}$$

From (1.85), (1.86), and (1.87), we conclude that $\sum_{j \in \mathcal{B}} [sgn(q_j^b) - sgn(b_j - q_j^b)](\theta_j^b + \eta_j^b) \le \sum_{j \in \mathcal{B}} [sgn(q_j^b) - sgn(b_j - q_j^b)](\bar{\theta}_j^b + \bar{\eta}_j^b)$.

Similarly, for any $i \in \mathcal{S}$, suppose first that $q_i^s \in (0, s_i)$. It can be readily verified that $[-sgn(q_i^s) + sgn(s_i - q_i^s)](\theta_i^s - \eta_i^s) = 0$. From the continuously differentiability of h_i in $(0, s_i)$ (by Assumption (1.12.1-2)), we have $\partial h_i(q_i^s) = \{h_i'(q_i^s)\}$. From $\theta_i^s - \eta_i^s \in \partial h_i(q_i^s)$ (condition

(1.39b)) and $\bar{\theta}_i^s - \bar{\eta}_i^s = h_i'(q_i^s)$ (condition (1.40b)), we have that

$$\text{if } q_i^s \in (0, s_i), \text{ then } \theta_i^s - \eta_i^s \; = \; h_i'(q_i^s) \; = \; \bar{\theta}_i^s - \bar{\eta}_i^s. \tag{1.88}$$

Suppose instead that $q_i^s = 0$. It can be immediately verified that $[-sgn(q_i^s) + sgn(s_i - q_i^s)](\theta_i^s - \eta_i^s) = \theta_i^s - \eta_i^s$. Recall that $h_i(0)$ and $h_i'(0)$ have finite values given the optimal solution $q_i^s = 0$. Since $h_i(q)$ is continuously differentiable and strictly convex in $q \in (0, s_i)$, and continuous at $q = 0$ (Assumption (1.12.1-2)), it follows from $\partial h_i(0) = \{z \in \mathbb{R} | h_i(q') \geq h_i(0) + z(q' - 0), \forall q' \in [0, s_i]\}$ and $\theta_i^s - \eta_i^s \in \partial h_i(q_i^s)$ (condition (1.39b)) that $\theta_i^s - \eta_i^s \leq h_i'(0)$. Given $\bar{\theta}_i^s - \bar{\eta}_i^s = h_i'(0)$ (condition (1.40b)), we obtain that

$$\text{if } q_i^s = 0, \text{ then } \theta_i^s - \eta_i^s \; \leq \; h_i'(0) \; = \; \bar{\theta}_i^s - \bar{\eta}_i^s. \tag{1.89}$$

Suppose next that $q_i^s = s_i$, which implies $[-sgn(q_i^s) + sgn(s_i - q_i^s)](\theta_i^s - \eta_i^s) = -(\theta_i^s - \eta_i^s)$. Recall that $h_i(s_i)$ and $h_i'(s_i)$ have finite values given the optimal solution $q_i^s = s_i$. Similar to the arguments above, since $h_i(q)$ is continuously differentiable and strictly convex in $q \in (0, s_i)$ and continuous at $q = s_i$ (Assumption (1.12.1-2)), from $\partial h_i(s_i) = \{z \in \mathbb{R} | h_i(q') \geq h_i(s_i) + z(q' - s_i), \forall q' \in [0, s_i]\}$ and $\theta_i^s - \eta_i^s \in \partial h_i(s_i)$ (condition (1.39b)), we have $\theta_i^s - \eta_i^s \geq h_i'(s_i)$. Together with $\bar{\theta}_i^s - \bar{\eta}_i^s = h_i'(s_i)$ (condition (1.40b)), it follows that

$$\text{if } q_i^s = s_i, \text{ then } \theta_i^s - \eta_i^s \; \geq \; h_i'(s_i) \; = \; \bar{\theta}_i^s - \bar{\eta}_i^s. \tag{1.90}$$

In summary of (1.88), (1.89), and (1.90), we conclude that $\sum_{i \in \mathcal{S}}[-sgn(q_i^s) + sgn(s_i - q_i^s)](\theta_i^s - \eta_i^s) \leq \sum_{i \in \mathcal{S}}[-sgn(q_i^s) + sgn(s_i - q_i^s)](\bar{\theta}_i^s - \bar{\eta}_i^s)$.

Given the primal optimal solution $(\boldsymbol{x}, \boldsymbol{q}^s, \boldsymbol{q}^b)$, since $(\boldsymbol{\theta}^b, \boldsymbol{\theta}^s, \boldsymbol{\eta}^b, \boldsymbol{\eta}^s, \boldsymbol{\pi})$ maximizes $\sum_{j \in \mathcal{B}}[sgn(q_j^b) - sgn(b_j - q_j^b)](\theta_j^b + \eta_j^b) + \sum_{i \in \mathcal{S}}[-sgn(q_i^s) + sgn(s_i - q_i^s)](\theta_i^s - \eta_i^s)$, combining the observations above, we obtain that $\theta_j^b + \eta_j^b = \bar{\theta}_j^b + \bar{\eta}_j^b = g_j'(q_j^b)$ for all $j \in \mathcal{B}$ and $\theta_i^s - \eta_i^s = \bar{\theta}_i^s - \bar{\eta}_i^s = h_i'(q_i^s)$ for all $i \in \mathcal{S}$. Since $(\boldsymbol{\theta}^b, \boldsymbol{\theta}^s, \boldsymbol{\eta}^b, \boldsymbol{\eta}^s, \boldsymbol{\pi})$ satisfies conditions (1.39c) - (1.39h), it readily follows that $(\boldsymbol{\theta}^b, \boldsymbol{\theta}^s, \boldsymbol{\eta}^b, \boldsymbol{\eta}^s, \boldsymbol{\pi})$ satisfies conditions (1.40c) - (1.40h). Thus, we conclude that the primal-dual pair $(\boldsymbol{x}, \boldsymbol{q}^s, \boldsymbol{q}^b)$ and $(\boldsymbol{\theta}^b, \boldsymbol{\theta}^s, \boldsymbol{\eta}^b, \boldsymbol{\eta}^s, \boldsymbol{\pi})$ satisfies conditions (1.40a) - (1.40h). By Proposition 12(vi), $(\boldsymbol{\theta}^b, \boldsymbol{\theta}^s, \boldsymbol{\eta}^b, \boldsymbol{\eta}^s, \boldsymbol{\pi})$ is unique. □

Proof of Proposition 13. We first establish that, when function $f(\cdot)$ is strictly increasing, it is without loss of optimality to relax constraint $\sum_{j:(i,j)\in E} z_{ij} = s_i$ to $\sum_{j:(i,j)\in E} z_{ij} \leq s_i$ for all $i \in \mathcal{S}$ in constraint (1.41c) of problem (1.41). In this relaxation problem, suppose towards contradiction that there exists an optimal solution $(\boldsymbol{y}, \boldsymbol{z})$ where there exists $i_0 \in \mathcal{S}$ such that $\sum_{j:(i_0,j)\in E} z_{i_0 j} < s_{i_0}$. We can pick any $j_0 : (i_0, j_0) \in E$ and strictly increase the values of $z_{i_0 j_0}$ and y_{j_0} by $\delta \in (0, s_{i_0} - \sum_{j:(i_0,j)\in E} z_{i_0 j})$ such that $z_{i_0 j_0} := z_{i_0 j_0} + \delta$ and $y_{j_0} := y_{j_0} + \delta$ without violating any constraint in the relaxation problem. Given the strictly increasing property of $f(\cdot)$, this would strictly increase the optimal objective value of the relaxation problem, thereby leading to a contradiction with the optimality of $(\boldsymbol{y}, \boldsymbol{z})$. Thus, we can relax constraint (1.41c) to $\sum_{j:(i,j)\in E} z_{ij} \leq s_i$ for all $i \in \mathcal{S}$ in problem (1.41). In the remaining of the proof, we abuse terminology and refer to the relaxation of (1.41) we just introduced as problem (1.41).

By applying Lemma 4.1 of (41) adapted to our setting, we establish that any solution vector $(\boldsymbol{y}, \boldsymbol{z})$ is feasible in problem (1.41) if and only if $\boldsymbol{y}$ is feasible in problem (1.19). Given that the objective functions of these two problems are the same, we conclude that problem (1.41) and problem (1.19) share the same optimal solution vector $\boldsymbol{y}$ and the same optimal objective values. □

Proof of Proposition 14. We denote the optimal objective value for problem (1.38) by Y^f_{opt} and the optimal objective value for problem (1.41) by Y^a_{opt}. We divide the proof of this claim into the following steps. First, we show that the given an optimal solution $(\boldsymbol{x}, \boldsymbol{q}^s, \boldsymbol{q}^b)$ to problem (1.38), we leverage the solution mapping in (i) to obtain a feasible solution $(\boldsymbol{y}, \boldsymbol{z})$ to problem (that is, the mapping is well-defined) and, moreover, that (1.41) that satisfies $Y^f_{opt} \leq \sum_{j\in\mathcal{B}} b_j f(\frac{y_j}{b_j}) \leq Y^a_{opt}$. Then, given an optimal solution $(\boldsymbol{y}, \boldsymbol{z})$ to problem (1.41), we leverage the solution mapping in (ii) to obtain a well-defined feasible solution $(\boldsymbol{x}, \boldsymbol{q}^s, \boldsymbol{q}^b)$ to problem (1.38) that satisfies $Y^a_{opt} \leq \sum_{j\in\mathcal{B}} b_j g(\frac{q^b_j}{b_j}) - \sum_{i\in\mathcal{S}} s_i h(\frac{q^s_i}{s_i}) \leq Y^f_{opt}$. Finally, we conclude that $Y^a_{opt} = Y^f_{opt}$ and that the solution mappings in (i) and (ii) connect the optimal solutions.

Step 1: *Let* $(\boldsymbol{x}, \boldsymbol{q}^s, \boldsymbol{q}^b)$ *be an optimal solution to problem* (1.38)*, and let* $(\boldsymbol{y}, \boldsymbol{z}, \boldsymbol{r})$ *be as constructed by* ((i)) *in the proposition statement. Then,* $(\boldsymbol{y}, \boldsymbol{z})$ *is a well-defined feasible solution to problem* (1.41)*, and solution* r_j *is well-defined feasible solution to problem* (1.17) *with* $t = \frac{y_j}{b_j}$ *for all* $j \in \mathcal{B}$*.*

To establish that the construction of $(\boldsymbol{y}, \boldsymbol{z}, \boldsymbol{r})$ in ((i)) of the proposition statement is well-defined, given Assumptions (1.12.1-1) -(1.12.1-6) and the strict concavity of $f(\cdot)$, we first show that $q^s_i > 0$ for all $i \in \mathcal{S}$ and $q^b_j > 0$ for all $j \in \mathcal{B}$. Under Assumptions (1.12.1-1) - (1.12.1-6), we establish the following claim:

$$\text{if there exists } i_0 \in \mathcal{S} \text{ with } q^s_{i_0} = 0 \text{ or } j_0 \in \mathcal{B} \text{ with } q^b_{j_0} = 0, \text{ then } g'(0) \leq h'(0). \tag{1.91}$$

By Assumption (1.12.1-4), it follows from $g_j(q) = b_j g(\frac{q}{b_j})$ that $g'_j(q) = g'(\frac{q}{b_j})$. Similarly, by Assumption (1.12.1-5), it follows from $h_i(q) = s_i h(\frac{q}{s_i})$ that $h'_i(q) = h'(\frac{q}{s_i})$. From Assumption (1.12.1-6), we have $w_{ij} = 0$ for all $(i, j) \in E$.

Suppose there exists i_0 such that $q^s_{i_0} = 0$. Using the conditions in (1.40a) to (1.40f), we first establish that $q^b_j = 0$ for all j such that $(i_0, j) \in E$. Suppose, towards a contradiction, that there exists j with $(i_0, j) \in E$ such that $q^b_j > 0$. By the complementary condition (1.40e), $q^s_{i_0} = 0$ implies $\eta^s_{i_0} = 0$, which further implies that $h'(0) = \theta^s_{i_0}$ by condition (1.40b). Given $q^b_j > 0$, we pick any i_1 with $x_{i_1 j} > 0$ and thus $q^s_{i_1} > 0$. By the complementary condition (1.40f), $x_{i_0 j} = 0$ and $x_{i_1 j} > 0$ imply that $\pi_{i_0 j} \geq 0 = \pi_{i_1 j}$, which further implies that $\theta^s_{i_0} \geq \theta^b_j = \theta^s_{i_1}$ by condition (1.40c). Given that $\eta^s_{i_1} \geq 0$ (follows from condition (1.40e)), then we have that $\theta^s_{i_1} \geq h'(\frac{q^s_{i_1}}{s_{i_1}})$ by condition (1.40b). These observations jointly imply that $h'(0) = \theta^s_{i_0} \geq \theta^s_{i_1} \geq h'\left(\frac{q^s_{i_1}}{s_{i_1}}\right)$. However, by the differentiability and the strict convexity of $h(\cdot)$ (Assumption (1.12.1-2)), $h'(\cdot)$ is strictly increasing, which implies $h'(0) < h'\left(\frac{q_{i_1}^s}{s_{i_1}}\right)$ given $q_{i_1}^s > 0$, thereby leading to a contradiction. Thus, we establish that $q_j^b = 0$ for all j such that $(i_0, j) \in E$.

Next, given $q_{i_0}^s = 0$ and $q_j^b = 0$ for all j with $(i_0, j) \in E$, we establish that $g'(0) \leq h'(0)$. Pick any j with $(i_0, j) \in E$. Given $q_j^b = 0$, we obtain that $\eta_j^b = 0$ (by the complementary condition (1.40d)), which further implies $\theta_j^b = g'(0)$ (by condition (1.40a)). Moreover, as $\theta_{i_0}^s - \theta_j^b = \pi_{i_0 j}$ by condition (1.40c) and $\pi_{i_0 j} \geq 0$ by condition (1.40f), it follows that $\theta_{i_0}^s \geq \theta_j^b$. As $h'(0) = \theta_{i_0}^s$ (condition (1.40b)), we conclude that $g'(0) \leq h'(0)$.

Similarly, suppose that there exists $j_0 \in \mathcal{B}$ with $q_{j_0}^b = 0$. We establish that $q_i^s = 0$ for all i such that $(i, j_0) \in E$ by using an argument very similar to the one above. Suppose towards contradiction that there exists i with $(i, j_0) \in E$ with $q_i^s > 0$. We pick any j_1 with $x_{ij_1} > 0$ such that $q_{j_1}^b > 0$. Given $q_{j_1}^b < b_{j_1}$ (Proposition 12(v)) and $q_{j_0}^b = 0$, it follows that $\eta_{j_1}^b = 0$ and $\eta_{j_0}^b = 0$ (the complementary condition (1.40d)). From condition (1.40a), we obtain $g'(0) = \theta_{j_0}^b$ and $g'(\frac{q_{j_1}^b}{b_{j_1}}) = \theta_{j_1}^b$. Moreover, given $x_{ij_0} = 0$ and $x_{ij_1} > 0$, we deduce that $\pi_{ij_0} \geq 0 = \pi_{ij_1}$ (by the complementary condition (1.40f)), which further implies that $\theta_{j_0}^b \leq \theta_i^s = \theta_{j_1}^b$ (condition (1.40c)). These observations jointly imply that $g'(0) = \theta_{j_0}^b \leq \theta_{j_1}^b = g'(\frac{q_{j_1}^b}{b_{j_1}})$. However, as $g'(\cdot)$ is strictly decreasing (Assumption (1.12.1-1)), given that $q_{j_1}^b > 0$, it follows that $g'(0) > g'(\frac{q_{j_1}^b}{b_{j_1}})$, thereby leading to a contradiction. Thus, we conclude that $q_i^s = 0$ for all i such that $(i, j) \in E$.

Next, given $q_{j_0}^b = 0$ and $q_i^s = 0$ for all i with $(i, j_0) \in E$, we establish $g'(0) \leq h'(0)$. Similar to above, pick any i such that $(i, j_0) \in E$. From $h'(0) = \theta_i^s$ (condition (1.40b)), $\theta_{j_0}^b = g'(0)$ (condition (1.40a)), and $\theta_i^s \geq \theta_{j_0}^b$ (condition (1.40c)), we obtain that $g'(0) \leq h'(0)$.

In summary, we conclude that condition (1.91) holds.

To establish that $q_i^s > 0$ for all $i \in \mathcal{S}$ and $q_j^b > 0$ for all $j \in \mathcal{B}$, suppose towards contradiction that if there exists i with $q_i^s = 0$ or j with $q_j^b = 0$. By condition (1.91), we have $g'(0) \leq h'(0)$. In problem (1.17) given any $t > 0$, by using the first order condition, we obtain that the optimal solution satisfies $r = 0$ and $f(t) = g(0) - th(0)$, thereby leading to a contradiction with the strictly concavity of function $f(t)$. Thus, given that $f(\cdot)$ is strictly concave, we conclude that

$$q_i^s > 0,\ \forall i \in \mathcal{S}, \quad q_j^b > 0,\ \forall j \in \mathcal{B}. \tag{1.92}$$

It is also worth noting that, when setting $y_j = \frac{s_i}{q_i^s} q_j^b$ for all $j \in \mathcal{B}$, we will obtain the same value for y_j regardless of which i we pick as long as $x_{ij} > 0$. This follows from Proposition 12(viii)). In fact, by Proposition 12(viii), it follows that

$$\frac{y_j}{q_j^b} = \frac{q_i^s}{s_i}, \quad \forall i : x_{ij} > 0. \tag{1.93}$$

In summary of (1.92) and (1.93), the construction of $(\boldsymbol{y}, \boldsymbol{z}, \boldsymbol{r})$ in (i) of the proposition statement is well-defined.

We let vector $(\boldsymbol{y}, \boldsymbol{z}, \boldsymbol{r})$ be defined in (i) of the proposition statement. To prove $Y_{opt}^f \leq \sum_{j \in \mathcal{B}} b_j f(\frac{y_j}{b_j}) \leq Y_{opt}^a$, we first establish that the constructed vector $(\boldsymbol{y}, \boldsymbol{z})$ is feasible in optimization problem (1.41) and r_j is feasible in problem (1.17) given $t = \frac{y_j}{b_j}$ for all $j \in \mathcal{B}$.

To establish the feasibility of $(\boldsymbol{y}, \boldsymbol{z})$ in problem (1.41), we check constraints (1.41b) - (1.41d):

(1) Constraint (1.41b): for all $j \in \mathcal{B}$, given $z_{ij} = \frac{x_{ij} s_i}{q_i^s}$ (by the construction of z_{ij}), we have $\sum_{i:(i,j) \in E} z_{ij} = \sum_{i:(i,j) \in E} \frac{s_i}{q_i^s} x_{ij}$. Next, we deduce that $\sum_{i:(i,j) \in E} \frac{s_i}{q_i^s} x_{ij} = \sum_{i:(i,j) \in E} \frac{y_j}{q_j^b} x_{ij}$ given $\frac{y_j}{q_j^b} = \frac{s_i}{q_i^s}$ for any $i : x_{ij} > 0$ (condition (1.93)). Aggregating x_{ij} for $\sum_{i:(i,j) \in E} x_{ij} = q_j^b$ (by constraint (1.38b)), we obtain that $\sum_{i:(i,j) \in E} \frac{y_j}{q_j^b} x_{ij} = \frac{y_j}{q_j^b} q_j^b = y_j$. Thus, constraint (1.41b) holds.

(2) Constraint (1.41c): for all $i \in \mathcal{S}$, given $z_{ij} = \frac{x_{ij} s_i}{q_i^s}$ (by the construction of z_{ij}) and $q_i^s = \sum_{j:(i,j) \in E} x_{ij}$ (constraint (1.38c)), we deduce that $\sum_{j:(i,j) \in E} z_{ij} = \sum_{j:(i,j) \in E} \frac{s_i x_{ij}}{q_i^s} = \frac{s_i q_i^s}{q_i^s} = s_i$. Thus, constraint (1.41c) holds.

(3) Constraint (1.41d): for all $(i,j) \in E$, given $z_{ij} = \frac{x_{ij} s_i}{q_i^s}$ (by the construction of z_{ij}),

$x_{ij} \geq 0$ (constraint (1.38f)), and $q_i^s > 0$ (condition (1.92)), we establish that constraint (1.41d) holds.

Next, to establish that the constructed solution $r_j = \frac{q_j^b}{b_j}$ is feasible in problem (1.17) given $t = \frac{y_j}{b_j}$ for all $j \in \mathcal{B}$, we verify constraint $r_j \in [0, \min\{1, \frac{y_j}{b_j}\}]$:

(1) Constraint $0 \leq r_j \leq 1$: given $r_j = \frac{q_j^b}{b_j}$ (by the construction of r_j) and $q_j^b \leq b_j$ (constraint (1.38d)), we have $r_j \leq 1$. Moreover, from $q_j^b = \sum_{i:(i,j)\in E} x_{ij}$ (constraint (1.38b)) and $x_{ij} \geq 0$ (constraint (1.38f)), we establish that $r_j \geq 0$.

(2) Constraint $r_j \leq \frac{y_j}{b_j}$: recall that $r_j = \frac{q_j^b}{b_j}$ (by the construction of r_j), $y_j = \frac{s_i}{q_i^s} q_j^b$ (by the construction of y_j) and $q_i^s \leq s_i$ (constraint (1.38e)). Thus, we deduce that $r_j = \frac{q_j^b}{b_j} = \frac{q_i^s}{s_i}\frac{y_j}{b_j} \leq \frac{y_j}{b_j}$.

<u>Step 2:</u> $Y_{opt}^f \leq Y_{opt}^a$. To show that $Y_{opt}^f \leq \sum_{j\in\mathcal{B}} b_j f\left(\frac{y_j}{b_j}\right) \leq Y_{opt}^a$, we deduce that

$$\begin{aligned}
Y_{opt}^f &\overset{(a)}{=} \sum_{j\in\mathcal{B}} b_j g\left(\frac{q_j^b}{b_j}\right) - \sum_{i\in\mathcal{S}} s_i h\left(\frac{q_i^s}{s_i}\right) \\
&\overset{(b)}{=} \sum_{j\in\mathcal{B}} b_j g\left(\frac{q_j^b}{b_j}\right) - \sum_{j\in\mathcal{B}}\sum_{i:(i,j)\in E} h\left(\frac{q_i^s}{s_i}\right) s_i \frac{x_{ij}}{q_i^s} \\
&= \sum_{j\in\mathcal{B}} b_j g\left(\frac{q_j^b}{b_j}\right) - \sum_{j\in\mathcal{B}}\sum_{i:x_{ij}>0} h\left(\frac{q_i^s}{s_i}\right) s_i \frac{x_{ij}}{q_i^s} \\
&\overset{(c)}{=} \sum_{j\in\mathcal{B}} b_j g\left(\frac{q_j^b}{b_j}\right) - \sum_{j\in\mathcal{B}}\sum_{i:x_{ij}>0} h\left(\frac{q_j^b}{y_j}\right) \frac{y_j}{q_j^b} x_{ij} \\
&\overset{(d)}{=} \sum_{j\in\mathcal{B}} b_j \Big[g\left(\frac{q_j^b}{b_j}\right) - \frac{y_j}{b_j} h\left(\frac{q_j^b}{b_j}\frac{b_j}{y_j}\right) \Big] \\
&\overset{(e)}{\leq} \sum_{j\in\mathcal{B}} b_j f\left(\frac{y_j}{b_j}\right) \overset{(f)}{\leq} Y_{opt}^a,
\end{aligned} \tag{1.94}$$

where step (a) follows from the definition of Y_{opt}^f as the optimal objective value in problem (1.38) where $g_j(q_j^b) = b_j g(\frac{q_j^b}{b_j})$ (Assumption (1.12.1-4)), $h_i(q_i^s) = s_i h(\frac{q_i^s}{s_i})$ (Assumption

(1.12.1-5)) and $w_{ij} = 0$ (Assumption (1.12.1-6)). Step (b) follows from $\sum_{j:(i,j)\in E} x_{ij} = q_i^s$ (constraint (1.38c)). In step (c), by condition (1.93), we can replace $\frac{q_i^s}{s_i}$ with $\frac{q_j^b}{y_j}$. In step (d), for each $j \in \mathcal{B}$, we aggregate x_{ij} and implement the equation $\sum_{i:(i,j)\in E} x_{ij} = q_j^b$ (constraint (1.38b)). Step (e) follows from the optimality of value function $f(\cdot)$ in problem (1.17) given $t = \frac{y_j}{b_j}$ for all $j \in \mathcal{B}$. Step (f) follows from the optimality of Y_{opt}^a in problem (1.41). In summary, we conclude that $Y_{opt}^f \le \sum_{j\in\mathcal{B}} b_j f(\frac{y_j}{b_j}) \le Y_{opt}^a$.

Step 3: let $(\boldsymbol{y}, \boldsymbol{z})$ be an optimal solution to problem (1.41) and r_j be an optimal solution to problem (1.17) with $t = \frac{y_j}{b_j}$ for all $j \in \mathcal{B}$. We also let $(\boldsymbol{x}, \boldsymbol{q}^s, \boldsymbol{q}^b)$ be as constructed in (ii) in the proposition statement. Then $(\boldsymbol{x}, \boldsymbol{q}^s, \boldsymbol{q}^b)$ is a well-defined feasible solution to problem (1.38).

To establish that the construction in (ii) of the proposition statement is well-defined, we need to show that the construction $q_i^s = s_i \frac{r_j b_j}{y_j}$ is valid for all $i \in \mathcal{S}$. We first establish the following claim:

$$\text{for all } i \in \mathcal{S}, \text{ if } z_{ij_1}, z_{ij_2} > 0, \text{ then } \frac{y_{j_1}}{b_{j_1}} = \frac{y_{j_2}}{b_{j_2}} \text{ and } r_{j_1} = r_{j_2}. \tag{1.95}$$

Define the Lagrangian in problem (1.41) as

$$\begin{aligned}\mathcal{L}(\bar{\boldsymbol{z}}, \bar{\boldsymbol{y}}, \bar{\boldsymbol{\theta}}, \bar{\boldsymbol{\eta}}, \bar{\boldsymbol{\pi}}) := \sum_{j\in\mathcal{B}} b_j f\Big(\frac{\bar{y}_j}{b_j}\Big) + \sum_{j\in\mathcal{B}} \bar{\theta}_j \Big(\sum_{i:(i,j)\in E} \bar{z}_{ij} - \bar{y}_j\Big) \\ - \sum_{i\in\mathcal{S}} \bar{\eta}_i \Big(\sum_{j:(i,j)\in E} \bar{z}_{ij} - s_i\Big) + \sum_{(i,j)\in E} \bar{\pi}_{ij} \bar{z}_{ij}.\end{aligned} \tag{1.96}$$

We define the superdifferential of function $f(\cdot)$ evaluated at t as $\partial f(t) = \{z \in \mathbb{R} \mid f(t_0) \le f(t) + z(t_0 - t),\ \forall t_0 > 0\}$. We further define $\partial_+ f(t) = \max\{\partial f(t)\}$ and $\partial_- f(t) = \min\{\partial f(t)\}$. Based on the definition, we show that

$$\text{if } \frac{y_{j_1}}{b_{j_1}} < \frac{y_{j_2}}{b_{j_2}}, \text{ then } \partial_- f\left(\frac{y_{j_1}}{b_j}\right) > \partial_+ f\left(\frac{y_{j_2}}{b_j}\right). \tag{1.97}$$

From the strict concavity of function $f(\cdot)$, we can pick $t_1, t_2 \in \mathbb{R}$ with $\frac{y_{j_1}}{b_{j_1}} < t_1 < t_2 < \frac{y_{j_2}}{b_{j_2}}$ such that (1) $f'(t_1)$ and $f'(t_2)$ exist by the almost-everywhere differentiability of $f(\cdot)$, and (2) $f'(t_1) > f'(t_2)$. By the strict concavity of $f(\cdot)$, we deduce that $\partial_- f(\frac{y_{j_1}}{b_j}) \geq \frac{f(y_{j_1}/b_{j_1}) - f(t_1)}{y_{j_1}/b_{j_1} - t_1} \geq f'(t_1)$ and $f'(t_2) \geq \frac{f(y_{j_2}/b_{j_2}) - f(t_2)}{y_{j_2}/b_{j_2} - t_2} \geq \partial_+ f(\frac{y_{j_2}}{b_j})$, which leads to $\partial_- f(\frac{y_{j_1}}{b_j}) > \partial_+ f(\frac{y_{j_2}}{b_j})$. Thus, condition (1.97) holds.

Letting $(\boldsymbol{\theta}, \boldsymbol{\eta}, \boldsymbol{\pi})$ be any dual optimal solution to problem (1.41), we obtain the following subset of KKT conditions to facilitate our proof:

$$\theta_j \in \partial f\left(\frac{y_j}{b_j}\right), \quad \forall j \in \mathcal{B}, \tag{1.98a}$$

$$\theta_j - \eta_i + \pi_{ij} = 0, \quad \forall (i,j) \in E, \tag{1.98b}$$

$$z_{ij} \geq 0 \quad \perp \quad \pi_{ij} \geq 0, \quad \forall (i,j) \in E. \tag{1.98c}$$

For all $i \in \mathcal{S}$, if $z_{ij_1}, z_{ij_2} > 0$, we deduce from condition (1.98c) that $\pi_{ij_1} = \pi_{ij_2} = 0$, which further implies that $\theta_{j_1} = \eta_i = \theta_{j_2}$ (condition (1.98b)). Based on condition (1.98a), we have $\theta_{j_1} \in \partial f\left(\frac{y_{j_1}}{b_{j_1}}\right)$ and $\theta^b_{j_2} \in \partial f\left(\frac{y_{j_2}}{b_{j_2}}\right)$ where $\theta_{j_1} = \theta_{j_2}$. Thus, it follows that $\frac{y_{j_1}}{b_{j_1}} = \frac{y_{j_2}}{b_{j_2}}$ (condition (1.97)). In problem (1.17), recall that r_{j_1} is the optimal solution given $t = \frac{y_{j_1}}{b_{j_1}}$ and r_{j_2} is the optimal solution given $t = \frac{y_{j_2}}{b_{j_2}}$. Given the strict concavity of the objective function in problem (1.17) (Assumption (1.12.1-1) - (1.12.1-2)) and linear constraints $r \in [0, \min\{1, t\}]$ where $t = \frac{y_{j_1}}{b_{j_1}} = \frac{y_{j_2}}{b_{j_2}}$, we obtain that $r_{j_1} = r_{j_2}$. Thus, we conclude that condition (1.95) holds. These observations imply that for all $i \in \mathcal{S}$, the construction $q^s_i = s_i \frac{r_j b_j}{y_j}$ is valid for any selected $j : z_{ij} > 0$. In fact, we have

$$\frac{q^s_i}{s_i} = \frac{r_j b_j}{y_j}, \quad \forall j : z_{ij} > 0. \tag{1.99}$$

We also establish that

$$y_j > 0, \quad \forall j \in \mathcal{B}. \tag{1.100}$$

Suppose towards contradiction that there exists $j_0 \in \mathcal{B}$ such that $y_{j_0} = 0$. This implies that $z_{ij} = 0$ for all i with $(i,j) \in E$. Pick any $i : (i,j) \in E$. Given $\sum_{j:(i,j)\in E} z_{ij} = s_i$ (constraint (1.41d)), we can further pick $j_1 : z_{ij_1} > 0$ such that $y_{j_1} > 0$. By condition (1.98c), we have $\pi_{ij_0} \geq 0 = \pi_{ij_1}$, which further implies that $\theta_{j_0} \leq \eta_i = \theta_{j_1}$ (condition (1.98b)). From $\theta_{j_0} \in \partial f(0)$, $\theta_{j_1} \in \partial f(\frac{y_{j_1}}{b_{j_1}})$ (condition (1.98a)) and $\theta_{j_0} \leq \theta_{j_1}$, we reach a contradiction with condition (1.97).

These observations jointly imply that the construction of $(\boldsymbol{x}, \boldsymbol{q}^s, \boldsymbol{q}^b)$ in (ii) of the proposition statement is well-defined.

Let $(\boldsymbol{x}, \boldsymbol{q}^s, \boldsymbol{q}^b)$ be the solution vector in (ii) of the proposition statement. To establish $Y^a_{opt} \leq \sum_{j\in\mathcal{B}} b_j g(\frac{q^b_j}{b_j}) - \sum_{i\in\mathcal{S}} s_i h(\frac{q^s_i}{s_i}) \leq Y^f_{opt}$, we first show that $(\boldsymbol{x}, \boldsymbol{q}^s, \boldsymbol{q}^b)$ is feasible in problem (1.38) by verifying constraints (1.38b) - (1.38f):

(1) Constraint (1.38b): for all $j \in \mathcal{B}$, given $x_{ij} = z_{ij}\frac{q^s_i}{s_i}$ (by construction), we first establish that $\sum_{i:(i,j)\in E} x_{ij} = \sum_{i:(i,j)\in E} z_{ij}\frac{q^s_i}{s_i}$. From condition (1.99), it follows that $\sum_{i:(i,j)\in E} z_{ij}\frac{q^s_i}{s_i} = (\sum_{i:(i,j)\in E} z_{ij})\frac{r_j b_j}{y_j}$. From $\sum_{i:(i,j)\in E} z_{ij} = y_j$ (constraint (1.41b)), we obtain that $(\sum_{i:(i,j)\in E} z_{ij})\frac{r_j b_j}{y_j} = r_j b_j$. We also have $r_j b_j = q^b_j$ (by the construction of q^b_j). These observations jointly imply that $\sum_{i:(i,j)\in E} x_{ij} = q^b_j$. Thus, constraint (1.38b) holds.

(2) Constraint (1.38c): for all $i \in \mathcal{S}$, given $x_{ij} = z_{ij}\frac{q^s_i}{s_i}$ (by construction) and $\sum_{j:(i,j)\in E} z_{ij} = s_i$ (constraint (1.41c)), we establish that $\sum_{j:(i,j)\in E} x_{ij} = \sum_{j:(i,j)\in E} \frac{z_{ij} q^s_i}{s_i} = q^s_i$. Thus, constraint (1.38c) holds.

(3) Constraint (1.38d): for all $j \in \mathcal{B}$, by using $q^b_j = r_j b_j$ (the construction of q^b_j) and $r_j \leq 1$ (the constraint in problem (1.17)), we obtain that $\frac{q^b_j}{b_j} = r_j \leq 1$. Thus, constraint (1.38d) holds.

(4) Constraint (1.38e): recall that $q^s_i = s_i \frac{r_j b_j}{y_j}$ for all $i \in \mathcal{S}$ (by the construction of q^s_i) and that $r_j \leq \frac{y_j}{b_j}$ (the constraint in problem (1.17) given $t = \frac{y_j}{b_j}$), it follows that $\frac{q^s_i}{s_i} = \frac{r_j b_j}{y_j} \leq 1$. Thus, constraint (1.38e) holds.

(5) Constraint (1.38f): for all $(i,j) \in E$, from $x_{ij} = z_{ij}\frac{q_i^s}{s_i}$ (the construction of x_{ij}) and $z_{ij} \geq 0$ (constraint (1.41d)), we conclude that constraint (1.38f) holds.

Summarizing, we conclude that the constructed solution $(\boldsymbol{x}, \boldsymbol{q}^s, \boldsymbol{q}^b)$ in (ii) of the proposition statement is feasible in problem (1.38).

Step 4: $Y^a_{opt} \leq Y^f_{opt}$. We verify $Y^a_{opt} \leq \sum_{j\in\mathcal{B}} b_j g(\frac{q_j^b}{b_j}) - \sum_{i\in\mathcal{S}} s_i h(\frac{q_i^s}{s_i}) \leq Y^f_{opt}$ by showing that

$$\begin{aligned} Y^a_{opt} &\overset{(g)}{=} \sum_{j\in\mathcal{B}} b_j f\left(\frac{y_j}{b_j}\right) \overset{(h)}{=} \sum_{j\in\mathcal{B}} b_j \Big[g\left(\frac{q_j^b}{b_j}\right) - \frac{y_j}{b_j} h\left(\frac{q_j^b}{b_j}\frac{b_j}{y_j}\right) \Big] \\ &\overset{(i)}{=} \sum_{j\in\mathcal{B}} b_j g\left(\frac{q_j^b}{b_j}\right) - \sum_{j\in\mathcal{B}} \sum_{i:(i,j)\in E} h\left(\frac{q_j^b}{y_j}\right) z_{ij} \\ &\overset{(j)}{=} \sum_{j\in\mathcal{B}} b_j g\left(\frac{q_j^b}{b_j}\right) - \sum_{i\in\mathcal{S}} \sum_{j:z_{ij}>0} h\left(\frac{q_i^s}{s_i}\right) z_{ij} \\ &\overset{(k)}{=} \sum_{j\in\mathcal{B}} b_j g\left(\frac{q_j^b}{b_j}\right) - \sum_{i\in\mathcal{S}} s_i h\left(\frac{q_i^s}{s_i}\right) \overset{(l)}{\leq} Y^f_{opt}, \end{aligned} \tag{1.101}$$

where step (g) follows from the optimality of $(\boldsymbol{y}, \boldsymbol{z})$ in problem (1.41). Step (h) follows from the optimality of r_j in problem (1.17) given $t = \frac{y_j}{b_j}$ for all $j \in \mathcal{B}$ where $r_j = \frac{q_j^b}{b_j}$ (the construction of q_j^b). Step (i) follows from $\sum_{i:(i,j)\in E} z_{ij} = y_j$ (constraint (1.41b)). In step (j), given $q_j^b = r_j b_j$ (the construction of q_j^b), we first obtain that $\frac{q_j^b}{y_j} = \frac{r_j b_j}{y_j}$. Moreover, given that $\frac{q_i^s}{s_i} = \frac{r_j b_j}{y_j}$ for all j with $z_{ij} > 0$ (condition (1.99)), we can replace $\frac{q_j^b}{y_j}$ with $\frac{q_i^s}{s_i}$ in step (j). Step (k) follows from $\sum_{j:(i,j)\in E} z_{ij} = s_i$ (constraint (1.41c)). Step (l) follows from the optimality of the objective value Y^f_{opt} in problem (1.38) given that $g_j(q) = b_j g(\frac{q}{b_j})$ (Assumption (1.12.1-4)) and $h_i(q) = s_i h(\frac{q}{s_i})$ (Assumption (1.12.1-5)).

Thus, we conclude that $Y^a_{opt} \leq \sum_{j\in\mathcal{B}} b_j g(\frac{q_j^b}{b_j}) - \sum_{i\in\mathcal{S}} s_i h(\frac{q_i^s}{s_i}) \leq Y^f_{opt}$.

Step 5: conclusion of the claim. By (1.94) and (1.101), we conclude that $Y^f_{opt} = Y^a_{opt}$, which implies that claim (iii) holds. Moreover, this implies that the inequalities in step (e) and step (f) of (1.94) and the inequality in step (l) of (1.101) are all tight. Thus, solution mappings in ((i)) and ((ii)) establish the connections of the optimal solutions between problem (1.38) and problem (1.41). □

1.12.3 Proofs of Results in Appendix 1.9

Proof of Proposition 9. We proceed in two steps. The first step establishes necessity and the second step establishes sufficiency.

Step 1: necessity. Given any $(\boldsymbol{\gamma}, \boldsymbol{\mu}) \in \Gamma \times \mathcal{U}$, let $(\boldsymbol{p}, \boldsymbol{x}, \boldsymbol{q}^s, \boldsymbol{q}^b)$ be any competitive equilibrium satisfying the equilibrium expressions (1.2a) - (1.2d). Recall the definitions $\tilde{F}_{s_i}^{-1}(x) = \frac{F_{s_i}^{-1}(x)+\mu_i^s}{1-\gamma_i^s}$ and $\tilde{F}_{b_j}^{-1}(x) = \frac{F_{b_j}^{-1}(x)-\mu_j^b}{1+\gamma_j^b}$. The equilibrium has the following properties:

(1) on the seller side, by the equilibrium expression (1.2a), we have

$$\begin{aligned}
&\text{if } (1-\gamma_i^s)p_i - \mu_i^s \leq 0, \text{ then } \frac{q_i^s}{s_i} = 0, \quad \tilde{F}_{s_i}^{-1}\left(\frac{q_i^s}{s_i}\right) = \tilde{F}_{s_i}^{-1}(0) \geq p_i; \\
&\text{if } 0 < (1-\gamma_i^s)p_i - \mu_i^s < \bar{v}_{s_i}, \text{ then } 0 < \frac{q_i^s}{s_i} < 1, \quad \tilde{F}_{s_i}^{-1}\left(\frac{q_i^s}{s_i}\right) = p_i; \\
&\text{if } (1-\gamma_i^s)p_i - \mu_i^s \geq \bar{v}_{s_i}, \text{ then } \frac{q_i^s}{s_i} = 1, \quad \tilde{F}_{s_i}^{-1}\left(\frac{q_i^s}{s_i}\right) = \tilde{F}_{s_i}^{-1}(1) \leq p_i.
\end{aligned} \tag{1.102}$$

(2) on the buyer side, given that $\min_{i:(i,j)\in E}\{(1+\gamma_j^b)p_i + c_i\} + \mu_j^b \geq 0$ for all $j \in \mathcal{B}$, by the equilibrium expression (1.2b), we have

$$\begin{aligned}
&\text{if } \min_{i:(i,j)\in E}\{(1+\gamma_j^b)p_i + c_i\} + \mu_j^b < \bar{v}_{b_j}, \text{ then } \frac{q_j^b}{b_j} > 0, \quad \tilde{F}_{b_j}^{-1}\left(1 - \frac{q_j^b}{b_j}\right) = \min_{i:(i,j)\in E}\{p_i + \frac{c_i}{1+\gamma_j^b}\}; \\
&\text{if } \min_{i:(i,j)\in E}\{(1+\gamma_j^b)p_i + c_i\} + \mu_j^b \geq \bar{v}_{b_j}, \text{ then } \frac{q_j^b}{b_j} = 0, \\
&\quad \tilde{F}_{b_j}^{-1}\left(1 - \frac{q_j^b}{b_j}\right) = \tilde{F}_{b_j}^{-1}(1) \leq \min_{i:(i,j)\in E}\{p_i + \frac{c_i}{1+\gamma_j^b}\}.
\end{aligned} \tag{1.103}$$

By Lemma 4(i), the equilibrium problem (1.16) can be formulated as an instance of the framework problem (1.38) with $g_j(q) = \int_0^q \tilde{F}_{b_j}^{-1}(1 - \frac{x}{b_j})dx$ for all $j \in \mathcal{B}$ and $h_i(q) = \int_0^q \tilde{F}_{s_i}^{-1}(\frac{x}{s_i})dx$ for all $i \in \mathcal{S}$ and $w_{ij} = \frac{c_i}{1+\gamma_j^b}$ for all $(i,j) \in E$, where Assumptions (1.12.1-1)-(1.12.1-3) hold. By Assumption 1, the distribution functions F_{b_j} and F_{s_i} are continuously differentiable and strictly increasing respectively in $(0, \bar{v}_{b_j})$ and $(0, \bar{v}_{s_i})$, which implies that $F_{b_j}^{-1}$ and $F_{s_i}^{-1}$ are continuously differentiable in $(0,1)$(the inverse function theorem). Thus, $\tilde{F}_{b_j}^{-1}$ and $\tilde{F}_{s_i}^{-1}$ are also continuously differentiable in $(0,1)$. In the framework problem (1.38), given $g_j(q) = \int_0^q \tilde{F}_{b_j}^{-1}(1 - \frac{x}{b_j})dx$ and $h_i(q) = \int_0^q \tilde{F}_{s_i}^{-1}(\frac{x}{s_i})dx$, we obtain that $g_j'(q) = \tilde{F}_{b_j}^{-1}(1 - \frac{q}{b_j})$ for $q \in (0, b_j)$ and $h_i'(q) = \tilde{F}_{s_i}^{-1}(\frac{q}{s_i})$ for $q \in (0, s_i)$.

Next, given the framework problem (1.38), we verify that $(\boldsymbol{p}, \boldsymbol{x}, \boldsymbol{q}^s, \boldsymbol{q}^b)$ satisfies conditions (i) and (ii) in the proposition statement.

Step 1-1: condition (i). We first argue that $(\boldsymbol{x}, \boldsymbol{q}^s, \boldsymbol{q}^b)$ is a feasible solution to problem (1.38). Given $F_{s_i}(v) \leq 1$ and $F_{b_j}(v) \leq 1$ for all $v \in \mathbb{R}$, $q_i^s \leq s_i$ in constraint (1.38e) follows from the equilibrium supply expression (1.2a), and $q_j^b \leq b_j$ in constraint (1.38d) follows from the equilibrium demand expression (1.2b). The equilibrium flow expressions (1.2c)-(1.2d) guarantee constraints (1.38b), (1.38c), and (1.38f). Thus, we establish that $(\boldsymbol{x}, \boldsymbol{q}^s, \boldsymbol{q}^b)$ is feasible in problem (1.38).

From the equilibrium tuple $(\boldsymbol{p}, \boldsymbol{x}, \boldsymbol{q}^s, \boldsymbol{q}^b)$, we construct $(\boldsymbol{\theta}^b, \boldsymbol{\theta}^s, \boldsymbol{\eta}^b, \boldsymbol{\eta}^s, \boldsymbol{\pi})$ as follows:

- $\boldsymbol{\theta}^b$: for all $j \in \mathcal{B}$, we set $\theta_j^b = \tilde{F}_{b_j}^{-1}(1 - \frac{q_j^b}{b_j})$;
- $\boldsymbol{\theta}^s$: for all $i \in \mathcal{S}$, if $(1 - \gamma_i^s)p_i - \mu_i^s \leq 0$, then we set $\theta_i^s = \tilde{F}_{s_i}^{-1}(\frac{q_i^s}{s_i})$; otherwise, we set $\theta_i^s = p_i$;
- $\boldsymbol{\eta}^b$: for all $j \in \mathcal{B}$, we set $\eta_j^b = 0$;
- $\boldsymbol{\eta}^s$: for all $i \in \mathcal{S}$, if $(1 - \gamma_i^s)p_i - \mu_i^s \leq \bar{v}_{s_i}$, then we set $\eta_i^s = 0$; otherwise, we set $\eta_i^s = p_i - \tilde{F}_{s_i}^{-1}(\frac{q_i^s}{s_i})$;
- $\boldsymbol{\pi}$: for all $(i,j) \in E$, if $x_{ij} > 0$, then we set $\pi_{ij} = 0$; otherwise, we set $\pi_{ij} = \theta_i^s - \theta_j^b + \frac{c_i}{1+\gamma_j^b}$.

By Assumption 1, (1.102) and (1.103), it follows that vector $(\boldsymbol{\theta}^b, \boldsymbol{\theta}^s, \boldsymbol{\eta}^b, \boldsymbol{\eta}^s, \boldsymbol{\pi})$ in this construction has finite values. To establish that solution $(\boldsymbol{x}, \boldsymbol{q}^s, \boldsymbol{q}^b)$ and $(\boldsymbol{\theta}^b, \boldsymbol{\eta}^b, \boldsymbol{\theta}^s, \boldsymbol{\eta}^s, \boldsymbol{\pi})$ are optimal in problem (1.38), it is sufficient to verify conditions (1.40a) - (1.40f) (Proposition 12(iii)):

(1) condition (1.40a): for all $j \in \mathcal{B}$, since $g_j'(q_j^b) = \tilde{F}_{b_j}^{-1}(1-\frac{q_j^b}{b_j})$, by construction of (θ_j^b, η_j^b), we have $\theta_j^b = \tilde{F}_{b_j}^{-1}(1-\frac{q_j^b}{b_j})$ and $\eta_j^b = 0$, which implies that condition (1.40a) is satisfied;

(2) condition (1.40b): for all $i \in \mathcal{S}$, given $h_i'(q_i^s) = \tilde{F}_{s_i}^{-1}(\frac{q_i^s}{s_i})$, we consider the following three cases: (2-i) if $(1-\gamma_i^s)p_i - \mu_i^s \leq 0$, by construction, we have $\theta_i^s = \tilde{F}_{s_i}^{-1}(\frac{q_i^s}{s_i})$ and $\eta_i^s = 0$, which readily imply (1.40b); (2-ii) if $0 < (1-\gamma_i^s)p_i - \mu_i^s < \bar{v}_{s_i}$, then condition (1.40b) holds because we have $\theta_i^s = p_i$ (by the construction of θ_i^s), $p_i = \tilde{F}_{s_i}^{-1}(\frac{q_i^s}{s_i})$ (condition (1.102)), and $\eta_i^s = 0$ (by the construction of η_i^s); (2-iii) if $(1-\gamma_i^s)p_i - \mu_i^s \geq \bar{v}_{s_i}$, again by the construction of (η_i^s, θ_i^s), we have $\tilde{F}_{s_i}^{-1}(\frac{q_i^s}{s_i}) = p_i - \eta_i^s$ and $p_i^s = \theta_i^s$, which imply that $\tilde{F}_{s_i}^{-1}(\frac{q_i^s}{s_i}) = \theta_i^s - \eta_i^s$. Thus, condition (1.40b) holds ;

(3) condition (1.40c): for all $(i,j) \in E$, we discuss two cases: (3-i) if $x_{ij} > 0$, then we have $q_j^b > 0$, which implies that $\min_{i:(i,j)\in E}\{(1+\gamma_j^b)p_i + c_i\} + \mu_j^b < \bar{v}_{b_j}$ and $\tilde{F}_{b_j}^{-1}(1-\frac{q_j^b}{b_j}) = \min_{i':(i',j)\in E}\{p_{i'} + \frac{c_i}{1+\gamma_j^b}\}$ (condition (1.103)). By the construction of θ_j^b, we have $\theta_j^b = \tilde{F}_{b_j}^{-1}(1-\frac{q_j^b}{b_j}) = \min_{i':(i',j)\in E}\{p_{i'} + \frac{c_i}{1+\gamma_j^b}\}$. Similarly, since $x_{ij} > 0$, we have $q_i^s > 0$, which implies that $(1-\gamma_i^s)p_i - \mu_i^s > 0$ (condition (1.102)). This further implies that $p_i = \theta_i^s$ (by the construction of θ_i^s). By the equilibrium expression (1.2d), $x_{ij} > 0$ implies that $\min_{i':(i',j)\in E}\{p_{i'} + \frac{c_{i'}}{1+\gamma_j^b}\} = p_i + \frac{c_i}{1+\gamma_j^b}$. These observations jointly imply that $\theta_j^b = \theta_i^s + \frac{c_i}{1+\gamma_j^b}$. Moreover, when $x_{ij} > 0$, we have $\pi_{ij} = 0$ by construction. Thus, we obtain that $\theta_j^b - \theta_i^s + \pi_{ij} - \frac{c_i}{1+\gamma_j^b} = 0$, and condition (1.40c) follows; (3-ii) if $x_{ij} = 0$, then condition (1.40c) follows from the construction of π_{ij};

(4) condition (1.40d): for all $j \in \mathcal{B}$, recall that we have $\frac{q_j^b}{b_j} \leq 1$ (since $(\boldsymbol{x}, \boldsymbol{q}^s, \boldsymbol{q}^b)$ is feasible in problem (1.38)). Moreover, since $\eta_j^b = 0$ by construction, condition (1.40d) holds;

(5) condition (1.40e): for all $i \in \mathcal{S}$, recall that we have $\frac{q_i^s}{s_i} \leq 1$ (since $(\boldsymbol{x}, \boldsymbol{q}^s, \boldsymbol{q}^b)$ is feasible in problem (1.38)). We discuss the following two cases: (5-i) if $(1-\gamma_i^s)p_i - \mu_i^s \leq \bar{v}_{s_i}$,

since $\eta_i^s = 0$ by construction, condition (1.40e) holds; (5-ii) if $(1-\gamma_i^s)p_i - \mu_i^s > \bar{v}_{s_i}$, given $q_i^s = s_i$ and $\tilde{F}_{s_i}^{-1}(\frac{q_i^s}{s_i}) \leq p_i$ (condition (1.102)), by construction, we have $\eta_i^s = p_i - \tilde{F}_{s_i}^{-1}(\frac{q_i^s}{s_i}) \geq 0$, which implies that $\eta_i^s(q_i^s - s_i) = 0$. Thus, condition (1.40e) holds;

(6) condition (1.40f): for all $(i,j) \in E$, recall that we have verified $x_{ij} \geq 0$ (constraint (1.38f)). Given the construction of π_{ij}, we discuss two cases: (4-i) if $x_{ij} > 0$, then we have $\pi_{ij} = 0$ by construction, and condition (1.40d) follows from $\pi_{ij}x_{ij} = 0$; (4-ii) if $x_{ij} = 0$, given $\theta_j^b = \tilde{F}_{b_j}^{-1}(1 - \frac{q_j^b}{b_j})$ by construction, from condition (1.103), we obtain that $\tilde{F}_{b_j}^{-1}(1 - \frac{q_j^b}{b_j}) \leq \min_{i':(i',j)\in E}\{p_{i'} + \frac{c_{i'}}{1+\gamma_j^b}\}$. Thus, $\theta_j^b \leq \min_{i':(i',j)\in E}\{p_{i'} + \frac{c_{i'}}{1+\gamma_j^b}\}$. Note that if $q_i^s = 0$, we have $(1-\gamma_i^s)p_i - \mu_i^s \leq 0$ (by condition (1.102)) and $\theta_i^s = \tilde{F}_{s_i}^{-1}(0)$ (by the construction of θ_i^s). Thus, we have $\theta_i^s = \tilde{F}_{s_i}^{-1}(0) \geq p_i$ (by condition (1.102)). If $q_i^s > 0$, then we have $\theta_i^s = p_i$(by the construction of θ_i^s). As a result, our construction of θ_i^s satisfies $\theta_i^s \geq p_i$. These observations imply that $\pi_{ij} = \theta_i^s - \theta_j^b + \frac{c_i}{1+\gamma_j^b} \geq p_i + \frac{c_i}{1+\gamma_j^b} - \min_{i':(i',j)\in E}\{p_{i'} + \frac{c_{i'}}{1+\gamma_j^b}\} \geq 0$. With $x_{ij} = 0$, we establish $x_{ij}\pi_{ij} = 0$. Thus, condition (1.40f) holds;

(7) condition (1.40g) and (1.40h): they follow directly from the equilibrium condtion (1.2c);

Using conditions (1.40a) - (1.40f) and Proposition 12(iii), we conclude that $(\boldsymbol{x}, \boldsymbol{q}^s, \boldsymbol{q}^b)$ is an optimal solution to problem (1.16) and $(\boldsymbol{\theta}^b, \boldsymbol{\theta}^s, \boldsymbol{\eta}^b, \boldsymbol{\eta}^s, \boldsymbol{\pi})$ is a corresponding dual optimal solution.

Thus, it follows that if $(\boldsymbol{p}, \boldsymbol{x}, \boldsymbol{q}^s, \boldsymbol{q}^b)$ is a competitive equilibrium, then condition (i) of the proposition holds.

Step 1-2: condition (ii). To prove this claim, we first recall from the previous step that $(\boldsymbol{x}, \boldsymbol{q}^s, \boldsymbol{q}^b)$ is a primal optimal solution and $(\boldsymbol{\theta}^b, \boldsymbol{\theta}^s, \boldsymbol{\eta}^b, \boldsymbol{\eta}^s, \boldsymbol{\pi})$ is a corresponding dual optimal solution to problem (1.38) that satisfies the conditions in (1.40). Recall from Lemma 4(i) that the equilibrium problem (1.16) can be formulated as an instance of the framework problem (1.38) where Assumptions (1.12.1-1) -(1.12.1-3) hold. By Lemma 5, any dual optimal solution $(\bar{\boldsymbol{\theta}}^b, \bar{\boldsymbol{\theta}}^s, \bar{\boldsymbol{\eta}}^b, \bar{\boldsymbol{\eta}}^s, \bar{\boldsymbol{\pi}})$ associated with constraints (1.38b) - (1.38f) that maximizes

$\sum_{j \in \mathcal{B}}[sgn(q_j^b) - sgn(b_j - q_j^b)](\bar{\theta}_j^b + \bar{\eta}_j^b) + \sum_{i \in \mathcal{S}}[-sgn(q_i^s) + sgn(s_i - q_i^s)](\bar{\theta}_i^s - \bar{\eta}_i^s)$ is unique, and satisfies the conditions in (1.40). By Proposition 12(vi), the constructed dual optimal vector $(\boldsymbol{\theta}^b, \boldsymbol{\theta}^s, \boldsymbol{\eta}^b, \boldsymbol{\eta}^s, \boldsymbol{\pi})$ that satisfies the conditions in (1.40) is the unique vector. Thus, it follows that $(\bar{\boldsymbol{\theta}}^b, \bar{\boldsymbol{\theta}}^s, \bar{\boldsymbol{\eta}}^b, \bar{\boldsymbol{\eta}}^s, \bar{\boldsymbol{\pi}}) = (\boldsymbol{\theta}^b, \boldsymbol{\theta}^s, \boldsymbol{\eta}^b, \boldsymbol{\eta}^s, \boldsymbol{\pi})$.

To establish condition (ii) in the proposition statement, for all $i \in \mathcal{S}$, we discuss three cases:

(1) If $0 < q_i^s < s_i$, then we have $p_i = \tilde{F}_{s_i}^{-1}(\frac{q_i^s}{s_i})$ (condition (1.102)). Moreover, $q_i^s < s_i$ also implies that $\eta_i^s = 0$ by condition (1.40e). With $h_i'(q_i^s) = \tilde{F}_{s_i}^{-1}(\frac{q_i^s}{s_i})$, this implies that $\tilde{F}_{s_i}^{-1}(\frac{q_i^s}{s_i}) = \theta_i^s$ (condition (1.40b)). With $\tilde{F}_{s_i}^{-1}(\frac{q_i^s}{s_i}) = p_i$ (condition (1.102)), we obtain $p_i = \theta_i^s$;

(2) If $q_i^s = s_i$, then we can pick any $j : x_{ij} > 0$ such that $q_j^b > 0$. By condition (1.103), we obtain that $\min_{i:(i,j) \in E}\{(1 + \gamma_j^b)p_i + c_i\} + \mu_j^b < \bar{v}_{b_j}$ and $\min_{i':(i',j) \in E}\{p_{i'} + \frac{c_{i'}}{1+\gamma_j^b}\} = \tilde{F}_{b_j}^{-1}(1 - \frac{q_j^b}{b_j})$. Recall that $q_j^b < b_j$ from Proposition 12 (v), which implies that $\eta_j^b = 0$ (by condition (1.40f)). Given $g_j'(q_j^b) = \tilde{F}_{b_j}^{-1}(1 - \frac{q_j^b}{b_j})$, we have $\tilde{F}_{b_j}^{-1}(1 - \frac{q_j^b}{b_j}) = \theta_j^b$ (by condition (1.40a)). Furthermore, we have $\pi_{ij} = 0$ by condition (1.40f) and $\theta_i^s + \frac{c_i}{1+\gamma_j^b} = \theta_j^b$ by condition (1.40c). Given $x_{ij} > 0$, the equilibrium expression (1.2d) implies $p_i + \frac{c_i}{1+\gamma_j^b} = \min_{i':(i',j) \in E}\{p_{i'} + \frac{c_{i'}}{1+\gamma_j^b}\}$. Thus, we obtain that $p_i + \frac{c_i}{1+\gamma_j^b} = \min_{i':(i',j) \in E}\{p_{i'} + \frac{c_{i'}}{1+\gamma_j^b}\} = \tilde{F}_{b_j}^{-1}(1 - \frac{q_j^b}{b_j}) = \theta_j^b = \theta_i^s + \frac{c_i}{1+\gamma_j^b}$, which suggests that $p_i = \theta_i^s$;

(3) If $q_i^s = 0$, then by condition (1.40e), we have $\eta_i^s = 0$. We prove $p_i \leq \theta_i^s$ by contradiction. Suppose that $p_i > \theta_i^s$. Since $h_i'(q_i^s) = \tilde{F}_{s_i}^{-1}(\frac{q_i^s}{s_i})$ (condition (1.40b)), we have $\tilde{F}_{s_i}^{-1}(\frac{q_i^s}{s_i}) = \tilde{F}_{s_i}^{-1}(0) = \theta_i^s$. However, we have $p_i \leq \tilde{F}_{s_i}^{-1}(0)$ (condition (1.102)). These observations imply $F_{s_i}^{-1}(0) = \theta_i^s < p_i \leq F_{s_i}^{-1}(0)$, thereby leading to a contradiction. Thus, we conclude that $p_i \leq \theta_i^s$.

Next, we prove $\max_{j:(i,j) \in E}\{\theta_j^b - \frac{c_i}{1+\gamma_j^b}\} \leq p_i$ for all $j : (i,j) \in E$. Recall that $\min_{i':(i',j) \in E}\{p_{i'} + \frac{c_{i'}}{1+\gamma_j^b}\} \geq \tilde{F}_{b_j}^{-1}(1 - \frac{q_j^b}{b_j})$ (condition (1.103)), and $g_j'(q_j^b) = \tilde{F}_{b_j}^{-1}(1 - \frac{q_j^b}{b_j})$. Thus, it follows that $\tilde{F}_{b_j}^{-1}(1 - \frac{q_j^b}{b_j}) = \theta_j^b$, since $g_j'(q_j^b) = \theta_j^b + \eta_j^b$ (condition 1.103) and $\eta_j^b = 0$

(using condition (1.40d) and the fact that $q^b_j < b_j$ by Proposition 12(v)). This implies that $p_i + \frac{c_i}{1+\gamma^b_j} \geq \min_{i':(i',j)\in E}\{p_{i'} + \frac{c_{i'}}{1+\gamma^b_j}\} \geq \theta^b_j$. Thus, we obtain $p_i \geq \max_{j:(i,j)\in E}\{\theta^b_j - \frac{c_i}{1+\gamma^b_j}\}$.

Thus, it follows that if $(\boldsymbol{p}, \boldsymbol{x}, \boldsymbol{q}^s, \boldsymbol{q}^b)$ is a competitive equilibrium, then condition (ii) of the proposition holds.

Step 2: sufficiency. Let vector $(\boldsymbol{p}, \boldsymbol{x}, \boldsymbol{q}^s, \boldsymbol{q}^b)$ and $(\boldsymbol{\theta}^b, \boldsymbol{\theta}^s, \boldsymbol{\eta}^b, \boldsymbol{\eta}^s, \boldsymbol{\pi})$ satisfy conditions (i) and (ii) in the proposition statement. Together, these tuples satisfy conditions (1.40a) - (1.40f) (by Lemma 5). Define $\tilde{F}_{s_i} : \mathbb{R} \cup \{-\infty, \infty\} \to [0,1]$ by $\tilde{F}_{s_i}(v) = F_{s_i}((1-\gamma^s_i)v - \mu^s_i)$ for all $i \in \mathcal{S}$ and $\tilde{F}_{b_j} : \mathbb{R} \cup \{-\infty, \infty\} \to [0,1]$ by $\tilde{F}_{b_j}(v) = F_{b_j}((1+\gamma^b_j)v + \mu^b_j)$. This definition is consistent with the inverse function definition $\tilde{F}^{-1}_{s_i}(x) = \frac{F^{-1}_{s_i}(x)+\mu^s_i}{1-\gamma^s_i}$ and $\tilde{F}^{-1}_{b_j}(x) = \frac{F^{-1}_{b_j}(x)-\mu^b_j}{1+\gamma^b_j}$.

To establish that $(\boldsymbol{p}, \boldsymbol{x}, \boldsymbol{q}^s, \boldsymbol{q}^b) \in \mathcal{X}(\boldsymbol{\gamma}, \boldsymbol{\mu})$, we verify the equilibrium conditions (1.2a) - (1.2d):

(1) to establish the equilibrium expression (1.2a), for all $i \in \mathcal{S}$, we consider the following two cases: (1)-(i) if $q^s_i = 0$, then we have $\eta^s_i = 0$ (condition (1.40e)). Given $h'_i(q^s_i) = \tilde{F}^{-1}_{s_i}(\frac{q^s_i}{s_i})$, by condition (1.40b), we also have $\tilde{F}^{-1}_{s_i}(0) = \tilde{F}^{-1}_{s_i}(\frac{q^s_i}{s_i}) = \theta^s_i$. From condition (ii) of the proposition statement, we conclude $\theta^s_i \geq p_i$. Given that function F_{s_i} is increasing and nonnegative in $\mathbb{R}$ (Assumption 1), it follows that $\tilde{F}_{s_i}$ is increasing and has nonnegative values in $\mathbb{R}$. These observations imply that $0 \leq \tilde{F}_{s_i}(p_i) \leq \tilde{F}_{s_i}(\theta_i) = \tilde{F}_{s_i}(\tilde{F}^{-1}_{s_i}(0)) = 0$. From the definition $\tilde{F}_{s_i}(v) = F_{s_i}((1-\gamma^s_i)v - \mu^s_i)$, it also follows that $F_{s_i}((1-\gamma^s_i)p_i - \mu^s_i) = \tilde{F}_{s_i}(p_i) = 0 = \frac{q^s_i}{s_i}$. Thus, the equilibrium expression (1.2a) holds; (1)-(ii) if $q^s_i > 0$, by using $h'_i(q^s_i) = \tilde{F}^{-1}_{s_i}(\frac{q^s_i}{s_i})$, and the fact that $\theta^s_i = \tilde{F}^{-1}_{s_i}(\frac{q^s_i}{s_i}) + \eta^s_i$ (condition (1.40b)) and $\eta^s_i \geq 0$ (condition (1.40e)), we obtain $\theta^s_i \geq \tilde{F}^{-1}_{s_i}(\frac{q^s_i}{s_i})$. Moreover, $q^s_i > 0$ implies that $p_i = \theta^s_i$ (condition (ii) of the proposition statement). Given $q^s_i > 0$, we also have $\frac{q^s_i}{s_i} \geq \tilde{F}_{s_i}(\theta^s_i)$, which follows from $\tilde{F}^{-1}_{s_i}(\frac{q^s_i}{s_i}) = p_i = \theta^s_i$ when $q^s_i < s_i$ and from $\tilde{F}_{s_i}(\theta^s_i) \leq 1$ when $q^s_i = s_i$. Thus, we conclude that $\frac{q^s_i}{s_i} \geq \tilde{F}_{s_i}(\theta^s_i) = \tilde{F}_{s_i}(p_i) \geq \tilde{F}_{s_i}(\tilde{F}^{-1}_{s_i}(\frac{q^s_i}{s_i})) = \frac{q^s_i}{s_i}$. Given the definition $\tilde{F}_{s_i}(v) = F_{s_i}((1-\gamma^s_i)v - \mu^s_i)$, it follows that

$F_{s_i}((1-\gamma_i^s)p_i - \mu_i^s) = \tilde{F}_{s_i}(p_i) = \frac{q_i^s}{s_i}$, i.e., the equilibrium expression (1.2a) holds;

(2) to establish the equilibrium expression (1.2b), for all $j \in \mathcal{B}$, we discuss the following two cases: (2)-(i) if $q_j^b = 0$, we have $\eta_j^b = 0$ by condition (1.40f). Given $g_j'(q_j^b) = \tilde{F}_{b_j}^{-1}(1-\frac{q_j^b}{b_j})$, we also have $\tilde{F}_{b_j}^{-1}(1) = \tilde{F}_{b_j}^{-1}(1-\frac{q_j^b}{b_j}) = \theta_j^b$ by condition (1.40a). Next we establish that $\min_{i:(i,j)\in E}\{p_i + \frac{c_i}{1+\gamma_j^b}\} \geq \theta_j^b$. For all $i : (i,j) \in E$, if $q_i^s = 0$, then we have $p_i + \frac{c_i}{1+\gamma_j^b} \geq \theta_j^b$ by condition (ii) of the proposition statement; and if $q_i^s > 0$, then we have $p_i = \theta_i^s$ by condition (ii) of the proposition statement. Using $\theta_i^s + \frac{c_i}{1+\gamma_j^b} - \pi_{ij} = \theta_j^b$ (condition (1.40c)) and $\pi_{ij} \geq 0$ (condition (1.40f)), we obtain $p_i + \frac{c_i}{1+\gamma_j^b} \geq \theta_j^b$. Thus, we have $\min_{i:(i,j)\in E}\{p_i + \frac{c_i}{1+\gamma_j^b}\} \geq \theta_j^b$. Given that function F_{b_j} is increasing and $F_{b_j}(v) \leq 1$ for all $v \in \mathbb{R}$ (Assumption 1), we obtain that $\tilde{F}_{b_j}$ is strictly increasing with $\tilde{F}_{b_j}(v) \leq 1$ for all $v \in \mathbb{R}$. This further implies that $1 \geq \tilde{F}_{b_j}(\min_{i:(i,j)\in E}\{p_i + \frac{c_i}{1+\gamma_j^b}\}) \geq \tilde{F}_{b_j}(\theta_j^b) = \tilde{F}_{b_j}(\tilde{F}_{b_j}^{-1}(1)) = 1$. Equivalently, we have $F_{b_j}(\min_{i:(i,j)\in E}\{(1+\gamma_j^b)p_i + c_i\} + \mu_j^b) = \tilde{F}_{b_j}(\min_{i:(i,j)\in E}\{p_i + \frac{c_i}{1+\gamma_j^b}\}) = 1 = 1 - \frac{q_j^b}{b_j}$. Thus, the equilibrium expression (1.2b) holds; (2)-(ii) if $q_j^b > 0$, we obtain $q_j^b < b_j$ (Proposition 12(v)), $\eta_j^b = 0$ (condition (1.40f)), and $\tilde{F}_{b_j}^{-1}(1-\frac{q_j^b}{b_j}) = \theta_j^b$ (condition (1.40a)). This further implies that $\tilde{F}_{b_j}(\theta_j^b) = \tilde{F}_{b_j}(\tilde{F}_{b_j}^{-1}(1-\frac{q_j^b}{b_j})) = 1 - \frac{q_j^b}{b_j}$. Pick any $i : x_{ij} > 0$, and we obtain $\pi_{ij} = 0$ (condition (1.40f)) and $\theta_i^s + \frac{c_i}{1+\gamma_j^b} = \theta_j^b$ (condition (1.40c)). Given $p_i = \theta_i^s$ by condition (ii) of the proposition statement, we have $p_i + \frac{c_i}{1+\gamma_j^b} = \theta_j^b$. Furthermore, recall that we have $p_{i'} + \frac{c_{i'}}{1+\gamma_j^b} \geq \theta_j^b$ for all $i' : x_{i'j} = 0$ (condition (ii) of the proposition statement). Thus, we have $i \in \arg\min_{i':(i',j)\in E}\{p_{i'} + \frac{c_{i'}}{1+\gamma_j^b}\}$, which implies that $\tilde{F}_{b_j}(\min_{i:(i,j)\in E}\{p_i + \frac{c_i}{1+\gamma_j^b}\}) = \tilde{F}_{b_j}(p_i + \frac{c_i}{1+\gamma_j^b}) = \tilde{F}_{b_j}(\theta_i^s + \frac{c_i}{1+\gamma_j^b}) = \tilde{F}_{b_j}(\theta_j^b) = 1 - \frac{q_j^b}{b_j}$, or equivalently $F_{b_j}(\min_{i:(i,j)\in E}\{(1+\gamma_j^b)p_i + c_i\} + \mu_j^b) = \tilde{F}_{b_j}(\min_{i:(i,j)\in E}\{p_i + \frac{c_i}{1+\gamma_j^b}\}) = 1 - \frac{q_j^b}{b_j}$. Thus, the equilibrium expression (1.2b) holds;

(3) the equilibrium expression (1.2c) follows directly from constraints (1.38b) and (1.38c) in the framework problem (1.38);

(4) We proceed by establishing the equilibrium expression (1.2d) for any $(i,j) \in E$. Not that $x_{ij} \geq 0$ follows from constraint (1.38f) in the framework problem (1.38).

Also note that, for any $(i,j) \in E$, we have $\theta_i^s + \frac{c_i}{1+\gamma_j^b} - \theta_j^b = \pi_{ij} \geq 0$ by condition (1.40c) and (1.40d). Fix $j \in \mathcal{B}$, if there exists $i : x_{ij} > 0$, then we have $\pi_{ij} = 0$ (condition (1.40f)), which further implies $\theta_i^s + \frac{c_i}{1+\gamma_j^b} = \theta_j^b$ (condition (1.40f)). Thus, if $x_{ij} > 0$, we have $\theta_j^b = \theta_i^s + \frac{c_i}{1+\gamma_j^b} = \min_{i':(i',j)\in E}\{\theta_{i'}^s + \frac{c_{i'}}{1+\gamma_j^b}\}$. Moreover, with $x_{ij} > 0$, we have $q_i^s > 0$, which implies $p_i = \theta_i^s$ by condition (ii) of the proposition statement. Thus, it follows that $p_i + \frac{c_i}{1+\gamma_j^b} = \theta_i^s + \frac{c_i}{1+\gamma_j^b} = \min_{i':(i',j)\in E}\{\theta_{i'}^s + \frac{c_{i'}}{1+\gamma_j^b}\} = \theta_j^b$.

We next establish that $\theta_j^b \leq \min_{i':(i',j)\in E}\{p_{i'} + \frac{c_{i'}}{1+\gamma_j^b}\}$. For $i : (i,j) \in E$ such that $q_i^s = 0$, condition (ii) of the proposition statement implies that $\theta_j^b \leq p_i + \frac{c_i}{1+\gamma_j^b}$; for $i : (i,j) \in E$ such that $q_i^s > 0$, we have $p_i^s = \theta_i^s$, which suggests that $\theta_j^b \leq \theta_i^s + \frac{c_i}{1+\gamma_j^b} = p_i + \frac{c_i}{1+\gamma_j^b}$. Thus, we have $\theta_j^b \leq \min_{i':(i',j)\in E}\{p_{i'} + \frac{c_{i'}}{1+\gamma_j^b}\}$.

These observations imply that if there exists $(i,j) \in E$ with $x_{ij} > 0$, then

$$\begin{aligned}\min_{i':(i',j)\in E}\{p_{i'} + \frac{c_{i'}}{1+\gamma_j^b}\} &\leq p_i + \frac{c_i}{1+\gamma_j^b} \\ &= \min_{i':(i',j)\in E}\{\theta_{i'}^s + \frac{c_{i'}}{1+\gamma_j^b}\} \;=\; \theta_j^b \;\leq\; \min_{i':(i',j)\in E}\{p_{i'} + \frac{c_{i'}}{1+\gamma_j^b}\},\end{aligned} \tag{1.104}$$

which implies that $i \in \arg\min_{i':(i',j)\in E}\{(1+\gamma_j^b)p_{i'}^s + c_{i'}\}$. Thus, the equilibrium condition (1.2d) is also satisfied.

In summary, the equilibrium conditions (1.2a) - (1.2d) hold for $(\boldsymbol{p}, \boldsymbol{x}, \boldsymbol{q}^s, \boldsymbol{q}^b)$ that satisfy the requirements in (i) and (ii) of the proposition statement. Thus, we conclude that $(\boldsymbol{p}, \boldsymbol{x}, \boldsymbol{q}^s, \boldsymbol{q}^b) \in \mathcal{X}(\boldsymbol{\gamma}, \boldsymbol{\mu})$. □

Proof of Lemma 2. We proceed in the following seven steps. The first step establishes that an optimal solution $\rho(t)$ to problem (1.18) exists and is unique for $t > 0$. The second step shows that function $\rho(t)$ strictly increases in t for $t > 0$. The third step shows that $\frac{\rho(t)}{t}$ strictly decreases in t when $t > t_0$ and $\frac{\rho(t)}{t} = 1$ for $t \leq t_0$. These three steps establish claim (i) of the lemma statement. The fourth step establishes that the value function $f(t)$ is continuous in $t > 0$. In the fifth step, we prove that the optimal objective value $f(t)$ is strictly increasing in $t > 0$. The sixth step shows that $f(t)$ is strictly concave in $t > 0$. In the seventh step, we prove that $f(t)$ satisfies $\lim_{t \downarrow 0} f(t) = g(0)$. The last four steps establish claim (ii) of the lemma statement.

Step 1: $\rho(t)$ is well-defined. Given $t > 0$, problem (1.18) can be formulated as an instance of the framework problem (1.38) where (1-i) $g_j(q) = b_j g(\frac{q}{b_j})$ for all $j \in \mathcal{B}$, $h_i(q) = s_i h(\frac{q}{s_i})$ for all $i \in \mathcal{S}$ and $w_{ij} = 0$ for all $(i, j) \in E$, (1-ii) $b_j = 1$ for all $j \in \mathcal{B}$ and $s_i = t$ for all $i \in \mathcal{S}$, and (1-iii) $|\mathcal{B}| = |\mathcal{S}| = 1$.

In this instance, it is clear that Assumption (1.91) is equivalent to Assumption (1.12.1-1), and Assumption (1.92) is equivalent to Assumption (1.12.1-2). Thus, for any $t > 0$, by Proposition 12(i), an optimal solution $r^\star$ to problem (1.18) exists. Moreover, by Proposition 1.38(iv), any optimal solution $r^\star$ to problem (1.18) is unique given $t > 0$. Equivalently, we conclude that $\rho(t)$ is well-defined for all $t > 0$.

Step 2: $\rho(t)$ is strictly increasing in $t > 0$. In problem (1.18) given any $t > 0$, recall that the objective function $g(r) - th(\frac{r}{t})$ is continuously differentiable and strictly concave in $r \in (0, 1)$ (Assumptions (1.91) - (1.92)). As before, we define $g'(r)$ and $h'(r)$ for $r \in \{0, 1\}$ in terms of the relevant limits, i.e., $g'(0) = \lim_{r \downarrow 0} g'(r)$ and similarly for the rest. Moreover, by Assumption (1.94), we apply the first order optimality condition to problem (1.18) and obtain that

$$\rho(t) = \max \left\{ r \in [0, \min\{1, t\}] : g'(r) - h'\left(\frac{r}{t}\right) \geq 0 \right\}. \tag{1.105}$$

Using once more the fact that $g'(0) - h'(0) > 0$ (Assumption (1.94)), it immediately follows that for any $t > 0$, we have:

$$\rho(t) \ > \ 0. \tag{1.106}$$

Next, we establish $\rho(t) < 1$. Note that if $t < 1$, then constraint $r \in [0, \min\{1, t\}]$ directly implies $\rho(t) < 1$. Suppose instead that $t \geq 1$, and observe that

$$g'(1) - h'\left(\frac{1}{t}\right) \overset{(a)}{<} g'(1) - h'(0) \overset{(b)}{\leq} 0, \tag{1.107}$$

where step (a) follows from the strict increasing property of $h'(r)$ in $r \in (0, 1)$ (Assumption (1.92)). Step (b) follows from $g'(1) \leq 0$ and $h'(0) \geq 0$ (Assumption (1.94)). Based on condition (1.107), we conclude that by starting at $r = 1$ and decreasing r, the objective value of problem (1.18) can be increased. Thus, we obtain

$$\rho(t) \ < \ 1. \tag{1.108}$$

By (1.106) and (1.108), the optimality condition (1.105) can be further simplified to

$$\rho(t) = \max\left\{r \in [0, 1] : g'(r) - h'\left(\frac{r}{t}\right) \geq 0, \ r \leq t\right\}. \tag{1.109}$$

By the continuous differentiability and the strict concavity of $g(r)$ and $-h(r)$ in $(0, 1)$ (Assumption (1.91) - (1.92)), we obtain that $g'(r) - h'(\frac{r}{t})$ is continuous and decreasing in $r \in (0, 1)$. From condition (1.109), we deduce that

$$\text{either } g'(\rho(t)) - h'\left(\frac{\rho(t)}{t}\right) = 0 \text{ or } \rho(t) = t. \tag{1.110}$$

We next prove that $\rho(t)$ is strictly increasing for $t > 0$. We prove the claim by contradiction. Suppose that for some $0 < t_1 < t_2$, we have $\rho(t_1) \geq \rho(t_2)$. Recall that $h'(r)$ is strictly increasing in r (Assumption (1.92)). Moreover, by (1.106) and (1.108), we have $\rho(t_1), \rho(t_2) \in$

$(0, 1)$ and by (1.109), we have $\rho(t_1) \le t_1$. Thus, given $t_2 > t_1 > 0$, it follows that

$$g'(\rho(t_1)) - h'\left(\frac{\rho(t_1)}{t_2}\right) > g'(\rho(t_1)) - h'\left(\frac{\rho(t_1)}{t_1}\right). \tag{1.111}$$

Since $g'(r) - h'(\frac{r}{t})$ is strictly decreasing in r for any $r \in [0, \min\{1, t\}]$ (Assumption (1.91) - (1.92)) and $\rho(t_2) \le \rho(t_1) \le t_1 < t_2$, we also deduce that

$$g'(\rho(t_2)) - h'\left(\frac{\rho(t_2)}{t_2}\right) \ge g'(\rho(t_1)) - h'\left(\frac{\rho(t_1)}{t_2}\right). \tag{1.112}$$

On the other hand, (1.109), (1.111) and (1.112) collectively imply that

$$g'(\rho(t_2)) - h'\left(\frac{\rho(t_2)}{t_2}\right) > g'(\rho(t_1)) - h'\left(\frac{\rho(t_1)}{t_1}\right) \ \ge \ 0. \tag{1.113}$$

With $\rho(t_2) \le \rho(t_1)$ and $\rho(t_1) \le t_1 < t_2$, we have $\rho(t_2) < t_2$. This observation,together with the fact that $g'(\rho(t_2)) - h'\left(\frac{\rho(t_2)}{t_2}\right) > 0$ (by (1.113)) implies that $\rho(t_2)$ does not satisfy the optimality condition (1.110). Thus, we obtain a contradiction and conclude that $\rho(t_1) < \rho(t_2)$ whenever $0 < t_1 < t_2$.

Step 3: $\frac{\rho(t)}{t} = 1$ for $t \le t_0$ and $\frac{\rho(t)}{t}$ is strictly decreasing for $t > t_0$. By definition (see claim (i) of the lemma statement), we have $t_0 = [g']^{-1}(h'(1))$. By Assumption (1.94), we have $g'(1) \le 0 \le h'(0) < g'(0)$. Since $h'(r)$ is strictly increasing in r (Assumption (1.92)). It follows that $h'(1) > h'(0) \ge 0$. By definition of $[g']^{-1}$ in the lemma statement and the fact that $g'(r)$ is strictly decreasing in $r \in (0, 1)$ (Assumption (1.91)), it follows that there are two possible cases for t_0: if $h'(1) \in (0, g'(0))$, then $t_0 = [g']^{-1}(h'(1)) \in (0, 1)$; if $h'(1) \ge g'(0)$, then $t_0 = 0$. Note that in both cases, we have $g'(t_0) - h'(1) \le 0$.

We proceed by proving the claims for $t \le t_0$ and $t > t_0$.

First consider $t \le t_0$. Note that given $t > 0$, in this case we necessarily have $t_0 > 0$, and hence $g'(t_0) = h'(1)$. Since $g'(r)$ is strictly decreasing in $r \in (0, 1)$ (Assumption (1.91)), we obtain that $g'(t) \ge g'(t_0) = h'(1)$. Given that $g'(r) - h'\left(\frac{r}{t}\right)$ is strictly decreasing in r

(Assumption (1.91) - (1.92)), it follows that

$$g'(r) - h'\left(\frac{r}{t}\right) > g'(t) - h'(1) \geq 0, \quad \forall r \in [0, t). \tag{1.114}$$

We prove $\rho(t) = t$ by contradiction. Suppose that $\rho(t) < t$. From the optimality condition (1.110), we have $g'(\rho(t)) - h'\left(\frac{\rho(t)}{t}\right) = 0$, which contradicts condition (1.114) given $\rho(t) < t$. Thus, we conclude that $\rho(t) = t$ for all $t \leq t_0$.

Next, suppose $t > t_0$, since $g'(r)$ is strictly decreasing in r(Assumption (1.91)), we note that

$$g'(t) - h'(1) < g'(t_0) - h'(1) \leq 0. \tag{1.115}$$

We proceed by showing that $\rho(t) < t$ for all $t > t_0$. By (1.109), we have $\rho(t) \leq t$. Suppose for a contradiction that $\rho(t) = t$. Observe that condition (1.109) implies that $g'(\rho(t)) - h'\left(\frac{\rho(t)}{t}\right) = g'(t) - h'(1) \geq 0$, which leads to a contradiction to condition (1.115). Thus, it follows that

$$\rho(t) < t. \tag{1.116}$$

Finally, we establish that $\frac{\rho(t)}{t}$ is strictly decreasing in t for $t > t_0$. Suppose for a contradiction that for some t_1, t_2 such that $t_0 < t_1 < t_2$, we have $\frac{\rho(t_1)}{t_1} \leq \frac{\rho(t_2)}{t_2}$. This observation together with the fact that $\rho(t_1) < \rho(t_2)$ (which follows by step 2), the strict decreasingness of $g'(r)$ in $r \in (0, 1)$(Assumption (1.91)) and $-h'(r)$ in $r \in (0, 1)$(Assumption (1.92)), implies that $g'(\rho(t_1)) - h'\left(\frac{\rho(t_1)}{t_1}\right) > g'(\rho(t_2)) - h'\left(\frac{\rho(t_2)}{t_2}\right)$. By condition (1.109), we have $g'(\rho(t_2)) - h'\left(\frac{\rho(t_2)}{t_2}\right) \geq 0$. Thus,

$$g'(\rho(t_1)) - h'\left(\frac{\rho(t_1)}{t_1}\right) > 0. \tag{1.117}$$

Moreover from $\frac{\rho(t_1)}{t_1} \le \frac{\rho(t_2)}{t_2}$ and $\rho(t_2) < t_2$ (condition (1.116)), we have $\rho(t_1) \le \frac{t_1}{t_2}\rho(t_2) < t_1$, which implies that

$$\rho(t_1) < t_1. \tag{1.118}$$

However, conditions (1.117) and (1.118) imply that the optimality condition (1.110) cannot be satisfied. Thus, we obtain a contradiction and conclude that $\frac{\rho(t_1)}{t_1} > \frac{\rho(t_2)}{t_2}$ for all $t_0 < t_1 < t_2$.

Step 4: value function $f(t)$ is continuous in $t > 0$. To prove the continuity of $f(t)$ in $t > 0$, it is sufficient to show that for any $\kappa_0 > 0$, $f(t)$ is continuous in $t \ge \kappa_0$.

Fix any $\kappa_0 > 0$, we set $X = \{t : t \ge \kappa_0\}$, $Y = \{r : r \ge 0\}$ and $\Gamma(t) = \{r : 0 \le r \le 1, r \le t\} \subset Y$. In what follows, we apply the Maximum theorem to establish that $f(t)$ is continuous in t (see page116 of (49)).

First observe that we have $0 \in \Gamma(t)$ for all $t \in X$, and $\Gamma(t)$ is non-empty for all $t \in X$. We claim that $\Gamma(t)$ is both upper hemicontinuous and lower hemicontinuous in t for all $t \in X$. To show the upper hemicontinuity of $\Gamma(t)$, we establish that given any sequence $\{(r_k, t_k) \in X \times Y : k = 1, 2, \dots\}$ that satisfies $r_k \in \Gamma(t_k)$, $\lim_{k\to\infty} r_k = r$, and $\lim_{k\to\infty} t_k = t$, we have $r \in \Gamma(t)$. Since $[0, 1]$ is a compact set, we have $r_k \in [0, 1]$ (by $r_k \in \Gamma(t_k)$), and $r = \lim_{k\to\infty} r_k \in [0, 1]$. Moreover, we also have $r_k \le t_k$ (by $r_k \in \Gamma(t_k)$), which implies that $r = \lim_{k\to\infty} r_k \le \lim_{k\to\infty} t_k = t$. Thus, it follows that $r \in \Gamma(t)$, and we conclude that $\Gamma(t)$ is upper hemicontinuous at t for all $t \in X$.

To establish the lower hemicontinuity of $\Gamma(t)$, we establish that given any $r \in \Gamma(t)$ and sequence $\{t_k \in X : k = 1, 2, \dots\}$ such that $\lim_{k\to\infty} t_k = t$, there exists a sequence $\{r_k : k = 1, 2, \dots\}$ such that $r_k \in \Gamma(t_k)$ for all k and $\lim_{k\to\infty} r_k = r$. Let $r_k = \min\{r, t_k\}$ for all k. Since $r \in \Gamma(t)$, it follows that $r \le 1$. Similarly, since $t_k \in X$, we have $t_k > 0$ and since $r \in \Gamma(t)$, we have $r \ge 0$. Using the definition of r_k, those observations collectively imply that $r_k \in [0, 1]$ for all k. Moreover, since $r_k = \min\{r, t_k\}$, it follows that $r_k \le t_k$ for all k. Thus, we conclude that $r_k \in \Gamma(t_k)$ for all k. Moreover, given $r \leq t$ (by $r \in \Gamma(t)$) and $\lim_{k\to\infty} t_k = t$, we have $\lim_{k\to\infty} r_k = \lim_{k\to\infty} \min\{r, t_k\} = r$. Thus, it follows that $\Gamma(t)$ is lower hemicontinuous at t for all $t \in X$.

Note that the function $g(r) - th(\frac{r}{t})$ is not defined for $r \geq \min\{1, t\}$. We next construct functions $\hat{g}(r)$ and $\hat{h}(r)$ such that $\hat{g}(r) - t\hat{h}(\frac{r}{t})$ is well-defined and continuous in $Y = \{r : r \geq 0\}$ and $f(t) = \max_{r\in[0,\min\{1,t\}]} \hat{g}(r) - t\hat{h}(\frac{r}{t})$ for all $t \in X$.

Fix any $K_0 \in \mathbb{R}_{++}$. If $g(1) = -\infty$, we claim that $\lim_{r\uparrow 1} g'(r) = -\infty$. Suppose $\lim_{r\uparrow 1} g'(r) = M$ for some $M > -\infty$. By Assumption (1.94), we have $\lim_{r\uparrow 1} g'(r) = M \leq 0$. Given that $g'(r)$ is strictly decreasing in $r \in (0, 1)$ (Assumption (1.91)), we have $g'(r) \geq M$ for $r \in [\frac{1}{2}, 1]$, which further implies that $\lim_{r\uparrow 1} g(r) = \lim_{r\uparrow 1} g(\frac{1}{2}) + \int_{\frac{1}{2}}^{r} g'(x)dx \geq g(\frac{1}{2}) + \frac{1}{2}M > -\infty$, thereby leading to a contradiction to $\lim_{r\uparrow 1} g(r) = -\infty$. Thus, it follows that $\lim_{r\uparrow 1} g'(r) = -\infty$. Let $\bar{r}_1 \in (0, 1)$ be a constant such that

$$g'(r) \leq -K_0 < 0, \quad \forall r > \bar{r}_1. \tag{1.119}$$

If $g(1) > -\infty$, set $\bar{r}_1 = 1$. We define a continuous function $\hat{g} : Y \to \mathbb{R}$ as follows:

$$\hat{g}(r) = \begin{cases} g(r) & \text{if } r \leq \bar{r}_1; \\ g(\bar{r}_1) & \text{if } r > \bar{r}_1. \end{cases} \tag{1.120}$$

Similarly, if $h(1) = \infty$, we claim that $\lim_{r\uparrow 1} h'(r) = \infty$. Suppose towards a contradiction that $\lim_{r\uparrow 1} h'(r) = M$ for some $M < \infty$. Since $h'(r)$ is strictly increasing in $r \in (0, 1)$ (Assumption (1.92)) and $h'(0) \geq 0$ (Assumption (1.94)), we have $h'(r) \leq M$ for all $r \in [0, 1]$, which implies that $\lim_{r\uparrow 1} h(r) = \lim_{r\uparrow 1} h(\frac{1}{2}) + \int_{\frac{1}{2}}^{r} h'(x)dx \leq h(\frac{1}{2}) + \frac{1}{2}M < \infty$, thereby leading to a contradiction to $\lim_{r\uparrow 1} h(r) = \infty$. Thus, it follows that $\lim_{r\uparrow 1} h'(r) = \infty$. Since $\rho(t)$ is strictly increasing in $t \in X$ (established in step 2), $\rho(t) > 0$ for all $t \in X$ (by condition (1.106)), and $g'(\cdot)$ is strictly decreasing in $r \in (0, 1)$ (Assumption (1.91)), we have

$t \geq \rho(t) \geq \rho(\kappa_0) > 0$, as well as,

$$g'(\rho(t)) \ \leq \ g'(\rho(\kappa_0)) \ < \ g'(0) \ \leq \ \infty, \quad \forall t \in X. \tag{1.121}$$

Let $\bar{r}_2 \in (0, 1)$ be a constant such that

$$h'(r) \geq g'(\rho(\kappa_0)) + K_0, \quad \forall r \geq \bar{r}_2. \tag{1.122}$$

If instead $h(1) < \infty$, set $\bar{r}_2 = 1$. We define a continuous function $\hat{h} : Y \to \mathbb{R}$ as follows:

$$\hat{h}(r) = \begin{cases} h(r) & \text{if } r \leq \bar{r}_2; \\ h(\bar{r}_2) & \text{if } r > \bar{r}_2. \end{cases} \tag{1.123}$$

Recall that $\rho(t)$ is the optimal solution to problem (1.17) for any $t \in X$. We claim that $\rho(t) \leq \bar{r}_1$ and $\frac{\rho(t)}{t} \leq \bar{r}_2$ for all $t \in X$. Suppose towards contradiction that there exists $t \in X$ such that $\rho(t) > \bar{r}_1$. From $g'(\rho(t)) \leq -K_0$ (condition (1.119)) and $h'(\frac{\rho(t)}{t}) \geq 0$ (Assumption (1.92) and (1.94) given $\frac{\rho(t)}{t} \geq 0$), it follows that

$$g'(\rho(t)) - h'\left(\frac{\rho(t)}{t}\right) \ \leq \ -K_0 - h'\left(\frac{\rho(t)}{t}\right) \ < \ 0, \tag{1.124}$$

thereby leading to a contradiction with the optimality condition (1.109).

Similarly, suppose towards contradiction that there exists $t > 0$ such that $\frac{\rho(t)}{t} > \bar{r}_2$. From $g'(\rho(t)) \leq g'(\rho(\kappa_0))$ (condition (1.121)) and $h'\left(\frac{\rho(t)}{t}\right) \geq K_0 + g'(\rho(\kappa_0))$ (condition (1.122)), we deduce that

$$g'(\rho(t)) - h'\left(\frac{\rho(t)}{t}\right) \ \leq \ g'(\rho(\kappa_0)) - K_0 - g'(\rho(\kappa_0)) \ \leq \ -K_0 \ < \ 0, \tag{1.125}$$

thereby leading to a contradiction with the optimality condition (1.109).

These observations jointly imply that, for all $t \in X$, instead of problem (1.17), $f(t)$ can be defined as follows:

$$f(t) \;=\; \max_{r\in[0,\min\{1,t\}]} \hat{g}(r) - t\hat{h}\left(\frac{r}{t}\right) \;=\; \max_{r\in\Gamma(t)} \hat{g}(r) - t\hat{h}\left(\frac{r}{t}\right). \tag{1.126}$$

Given the continuity of $\hat{g}(r) - t\hat{h}\left(\frac{r}{t}\right)$ in $r \in Y$ (by definition of $\hat{g}$ and $\hat{h}$), the upper and lower hemicontinuity of $\Gamma(t)$ in $t \in X$, and the fact that $\Gamma(t) \neq \emptyset$ for all $t \in X$, it follows by the Maximum theorem (see page 116 of (49)) that $f(t)$ is continuous in t for $t \geq \kappa_0$. Since κ_0 is picked arbitrarily in $(0,\infty)$, we conclude that $f(t)$ is continuous in t for $t > 0$.

Step 5: $f(t)$ is strictly increasing in $t > 0$. For any $0 < t_1 < t_2$, we let $\rho(t_1)$ and $\rho(t_2)$ be defined as in (1.18). Note that $0 < \rho(t_1) < \rho(t_2)$ (by step 2), Thus, it follows that

$$\begin{aligned} f(t_1) &= g(\rho(t_1)) - t_1 h\left(\frac{\rho(t_1)}{t_1}\right) \\ &\overset{(c)}{<} g(\rho(t_1)) - t_2 h\left(\frac{\rho(t_1)}{t_2}\right) \\ &\overset{(d)}{\leq} g(\rho(t_2)) - t_2 h\left(\frac{\rho(t_2)}{t_2}\right) \;=\; f(t_2). \end{aligned} \tag{1.127}$$

Note that we have $0 < \rho(t_1) \leq t_1 < t_2$, which implies that $h(\frac{\rho(t_1)}{t_2}) < \infty$ since $h(r) < \infty$ for $r < 1$. Step (c) follows from this observation and the fact that $th(\frac{r}{t})$ in strictly decreasing in $t > 0$ for $r > 0$ (Assumption (1.93)). Step (d) follows from the optimality of $\rho(t_2)$ in problem (1.17) with $t = t_2$.

Step 6: $f(t)$ is strictly concave in $t > 0$. For any $t_0, t_1 > 0$ with $t_0 \neq t_1$, Let $r_1 = \rho(t_1)$ and $r_0 = \rho(t_0)$. Since $\rho(t)$ is strictly increasing in $t > 0$ (by step 2), we obtain that $\rho(t_0) \neq \rho(t_1)$. For any $\theta \in (0,1)$, we set $t_\theta = \theta t_1 + (1-\theta)t_0$, $r_\theta = \theta r_1 + (1-\theta)r_0$, and $\rho(t_\theta) = \arg\max_{r\in[0,\min\{1,t_\theta\}]} g(r) - t_\theta h(\frac{r}{t_\theta})$. By the construction of r_θ, we have $r_\theta \in [0, \min\{1, t_\theta\}]$. Since $t_1, t_0 > 0$, it follows that $r_1, r_0 \in (0,1)$ (by condition (1.106) and (1.108)), which further implies that $r_\theta \in (0,1)$. By the strict concavity of $g(\cdot)$ in $(0,1)$

(Assumption (1.91)), we apply the Jensen's inequality to obtain that

$$g(r_\theta) \;>\; \theta g(r_1) + (1-\theta) g(r_0). \tag{1.128}$$

Next, we establish that $(r_\theta, t_\theta), (r_1, t_1), (r_0, t_0) \in \{(r', t') : t' > 0, \frac{r'}{t'} \in [0,1],\ h(\frac{r'}{t'}) < \infty\}$. Recall that $t_x > 0$ and $r_x \in [0, \min\{1, t_x\}]$ for $x \in \{0, 1, \theta\}$. It is sufficient to prove that $h(\frac{r_x}{t_x}) < \infty$ for all $x \in \{0, 1, \theta\}$. If $h(1) < \infty$, by Assumption (1.92), $h(r)$ is continuous in $r \in [0,1]$, so the claim holds. Suppose $h(1) = \infty$. Since $f(t)$ is continuous (and bounded) at $t = t_1$ and $t = t_0$, and $g(r_0), g(r_1) < \infty$ given $r_0, r_1 \in (0,1)$, it readily follows that $r_0 = \rho(t_0) < t_0$ and $r_1 = \rho(t_1) < t_1$, which further implies that $r_\theta = \theta r_1 + (1-\theta) r_0 < \theta t_1 + (1-\theta) t_0 = t_\theta$. Since $h(r)$ is continuous in $r \in [0,1)$ (by Assumption (1.92)), we conclude that $h(\frac{r_x}{t_x}) < \infty$ for all $x \in \{0, 1, \theta\}$. By the joint convexity of $th(\frac{r}{t})$ in $(r,t) \in \{(r', t') : t' > 0, \frac{r'}{t'} \in [0,1],\ h(\frac{r'}{t'}) < \infty\}$ (Assumption (1.93)), it follows from the Jensen's inequality that

$$t_\theta h\left(\frac{r_\theta}{t_\theta}\right) \;\le\; \theta t_1 h\left(\frac{r_1}{t_1}\right) + (1-\theta) t_0 h\left(\frac{r_0}{t_0}\right). \tag{1.129}$$

. We conclude that

$$\begin{aligned} f(t_\theta) &= g(\rho(t_\theta)) - t_\theta h\left(\frac{\rho(t_\theta)}{t_\theta}\right) \\ &\overset{(e)}{\ge} g(r_\theta) - t_\theta h\left(\frac{r_\theta}{t_\theta}\right) \\ &\overset{(f)}{>} \Big[\theta g(r_1) + (1-\theta) g(r_0)\Big] - \left[\theta t_1 h\left(\frac{r_1}{t_1}\right) + (1-\theta) t_0 h\left(\frac{r_0}{t_0}\right)\right] \\ &\overset{(g)}{=} \theta f(t_1) + (1-\theta) f(t_0), \end{aligned} \tag{1.130}$$

where step (e) follows from the optimality of $\rho(t_\theta)$ in problem (1.17) given $t = t_\theta$. Step (f) follows from (1.128) and (1.129). Step (g) follows from the optimality of r_1 (given $r_1 = \rho(t_1)$) in problem (1.17) for $t = t_1$ and the optimality of r_0 (given $r_0 = \rho(t_0)$) in problem (1.17) for

$t = t_0$. Since θ, t_1, t_2 are arbitrary, from (1.130) we conclude that $f(t)$ is strictly concave in t for $t > 0$.

Step 7: $\lim_{t\downarrow 0} f(t) = g(0)$. From $0 \leq \rho(t) \leq t$, it readily follows that

$$\lim_{t\downarrow 0} \rho(t) = 0. \tag{1.131}$$

Let t_h be a constant such that $t_h = h(\frac{1}{2}) + \min_{r\in[0,1]} h'(\frac{1}{2})(r - \frac{1}{2})$. By the strictly convexity of $h(r)$ in $r \in (0, 1)$ (Assumption (1.92)), it follows that $h(r) \geq h(\frac{1}{2}) + h'(\frac{1}{2})(r - \frac{1}{2}) \geq t_h$ for all $r \in [0, 1]$. Thus,

$$f(t) \leq g(\rho(t)) - tt_h, \quad \forall t \in (0, 1]. \tag{1.132}$$

Moreover, if $t < 1$, it is clear that $r = \frac{1}{2}t$ is feasible solution to problem (1.17). Thus, we conclude that

$$f(t) \geq g\left(\frac{t}{2}\right) - th\left(\frac{1}{2}\right), \quad \forall t \in (0, 1). \tag{1.133}$$

By (1.131), the upper bound in (1.132) satisfies $\lim_{t\downarrow 0} g(\rho(t)) - tt_h = g(0)$ and the lower bound in (1.133) satisfies $\lim_{t\downarrow 0} g(\frac{t}{2}) - th(\frac{1}{2}) = g(0)$. Thus, we conclude that $\lim_{t\downarrow 0} f(t) = g(0)$. $\square$

Proof of Lemma 3. Let $\boldsymbol{y}^\star$ be any optimal solution to problem (1.19). We first prove that $\boldsymbol{y}^\star$ is a lexicographically optimal base using the framework provided by (42).

Given Assumptions (1.91) - (1.94), by Lemma 2(ii), function $f(t)$ is continuous and strictly concave in $t \geq 0$. We define the superdifferential of function $f(\cdot)$ evaluated at $t \geq 0$ as $\partial f(t) = \{z \in \mathbb{R} \mid f(t_0) \leq f(t) + z(t_0 - t), \ \forall t_0 \geq 0\}$. We further define $\partial_+ f(t) = \max\{\partial f(t)\}$ and $\partial_- f(t) = \min\{\partial f(t)\}$.

In the optimal solution $\boldsymbol{y}^\star$, we let the distinct values of $\{\frac{y_j^\star}{b_j}\}_{j\in\mathcal{B}}$ be given by $t_1' < t_2' \cdots <$

t'_l. Given that $\boldsymbol{y}^\star$ is a feasible solution to problem (1.19), it follows that $\boldsymbol{y}^\star$ an element (or independent vector) of $\mathcal{P} = \{\boldsymbol{y} : \ \sum_{j\in B} y_j \le \sum_{i\in N_E(B)} s_i, \forall B \subset \mathcal{B}, \quad y_j \ge 0, \forall j \in \mathcal{B}\}$ (by (2.5) of (42)). By Lemma 2.3 of (42), we have $dep(\boldsymbol{y}^\star, j) = \cap\{B \mid j \in B \subset \mathcal{B},\ \sum_{j\in B} y^\star_j = \sum_{i\in N_E(B)} s_i\}$. By convention, we say $dep(\boldsymbol{y}^\star, j) = \emptyset$ if $\sum_{j'\in B} y^\star_{j'} < \sum_{i\in N_E(B)} s_i$ for all $B \subset \mathcal{B}$ such that $j \in B$. Note that for $dep(\boldsymbol{y}^\star, j) \ne \emptyset$, $dep(\boldsymbol{y}^\star, j)$ is the set of all elements $j' \in \mathcal{B}$ such that for some $d > 0$, the vector $\boldsymbol{y'} \in R_+^{|\mathcal{B}|}$ defined by $\boldsymbol{y'} = \boldsymbol{y}^\star + d\mathcal{X}_j - d\mathcal{X}_{j'}$ is an independent vector of $\mathcal{P}$ (by (2.19) of (42)).

We claim that $dep(\boldsymbol{y}^\star, j) \ne \emptyset$ for all $j \in \mathcal{B}$. Suppose towards a contradiction that there exists $j \in \mathcal{B}$ such that $dep(\boldsymbol{y}^\star, j) = \emptyset$. Then $\sum_{j'\in B} y_{j'} < \sum_{i\in N_E(B)} s_i$ for all $B \subset \mathcal{B}$ such that $j \in B$. Since $f(t)$ is strictly increasing in $t \ge 0$ (by Lemma 2(ii)), it follows another feasible solution can be obtained by increasing $y^\star_j$ and this solution improves the objective function, thereby leading to a contradiction. Thus, we have

$$dep(\boldsymbol{y}^\star, j) \ne \emptyset, \quad \forall j \in \mathcal{B}. \tag{1.134}$$

Next, we claim that for all $j_1, j_2 \in \mathcal{B}$, if $\frac{y^\star_{j_1}}{b_{j_1}} < \frac{y^\star_{j_2}}{b_{j_2}}$, then $j_2 \notin dep(\boldsymbol{y}^\star, j_1)$. Suppose not, and for some $j_1, j_2 \in \mathcal{B}$ with $\frac{y^\star_{j_1}}{b_{j_1}} < \frac{y^\star_{j_2}}{b_{j_2}}$, we have $j_2 \in dep(\boldsymbol{y}^\star, j_1)$.

Note that $\frac{y^\star_{j_1}}{b_{j_1}} < \frac{y^\star_{j_2}}{b_{j_2}}$ implies that $\partial_- f(\frac{y^\star_{j_1}}{b_j}) > \partial_+ f(\frac{y^\star_{j_2}}{b_j}) > 0$. To see this, first observe that by the strict concavity of $f(t)$ in $t \ge 0$ (by Lemma 2(ii)), we can pick $t_1, t_2 \in \mathbb{R}$ satisfying $\frac{y^\star_{j_1}}{b_{j_1}} < t_1 < t_2 < \frac{y^\star_{j_2}}{b_{j_2}}$ such that (1) $f'(t_1)$ and $f'(t_2)$ exist by the almost-everywhere differentiability of $f(\cdot)$, and (2) $f'(t_1) > f'(t_2)$. Moreover, the strict concavity of $f(t)$ in $t \ge 0$ also implies that $\partial_- f(\frac{y^\star_{j_1}}{b_j}) \ge \frac{f(y^\star_{j_1}/b_{j_1}) - f(t_1)}{y^\star_{j_1}/b_{j_1} - t_1} \ge f'(t_1)$ and $f'(t_2) \ge \frac{f(y^\star_{j_2}/b_{j_2}) - f(t_2)}{y^\star_{j_2}/b_{j_2} - t_2} \ge \partial_+ f(\frac{y^\star_{j_2}}{b_j})$. Thus, $\partial_- f(\frac{y^\star_{j_1}}{b_j}) > \partial_+ f(\frac{y^\star_{j_2}}{b_j})$ holds. Since $f(t)$ is is strictly increasing in $t \ge 0$ (by Lemma 2(ii)), we obtain that $\partial_+ f(\frac{y^\star_{j_2}}{b_j}) > 0$.

Next, we define a collection of sets $\mathcal{J} = \{B \subset \mathcal{B} : \sum_{j\in B} y^\star_j = \sum_{i\in N_E(B)} s_i\}$. Pick $\delta > 0$ small enough such that the following two conditions are satisfied: (1) $\delta < \min_{B \notin \mathcal{J}}\{\sum_{i\in N_E(B)} s_i - \sum_{j\in B} y_j\}$, and (2) $\partial_- f(\frac{y^\star_{j_1} + \delta}{b_j}) > \partial_+ f(\frac{y^\star_{j_2} - \delta}{b_j}) > 0$. We construct a new vector $\bar{\boldsymbol{y}}$ such that

$\bar{y}_{j_1} := y^\star_{j_1} + \delta$, $\bar{y}_{j_2} := y^\star_{j_2} - \delta$, and $\bar{y}_j = y^\star_j$ for all $j \neq j_1, j_2$. Given that $j_2 \in \mathit{dep}(\boldsymbol{y}^\star, j_1)$ and that δ is small, we have $\bar{\boldsymbol{y}} \in \mathcal{P}$.

Define function $H(\boldsymbol{y}) = \sum_{j\in\mathcal{B}} b_j f\left(\frac{y_j}{b_j}\right)$. By the strict concavity of function $f(t)$ for $t \geq 0$, it follows that $H(\boldsymbol{y})$ is strictly concave, and hence $H(\boldsymbol{y}^\star) < H(\bar{\boldsymbol{y}}) + \boldsymbol{z}^\mathsf{T}(\boldsymbol{y}^\star - \bar{\boldsymbol{y}})$ for any $\boldsymbol{z}$ satisfying $z_j \in \partial f(\frac{\bar{y}_j}{b_j})$ for all $j \in \mathcal{B}$. Since $\boldsymbol{z}^\mathsf{T}(\boldsymbol{y}^\star - \bar{\boldsymbol{y}}) = -\delta z_{j_1} + \delta z_{j_2} \leq -\delta\partial_- f(\frac{y^\star_{j_1}+\delta}{b_j}) + \delta\partial_+ f(\frac{y^\star_{j_2}-\delta}{b_j}) < 0$, we obtain that $H(\boldsymbol{y}^\star) < H(\bar{\boldsymbol{y}})$, which is a contradiction to the optimality of $\boldsymbol{y}^\star$. Thus,

$$\text{if } \frac{y^\star_{j_1}}{b_{j_1}} < \frac{y^\star_{j_2}}{b_{j_2}}, \text{ then } j_2 \notin \mathit{dep}(\boldsymbol{y}^\star, j_1). \tag{1.135}$$

For $k = 1, \ldots, l$, define $\bar{\mathcal{B}}^{(k)} = \{j \in \mathcal{B} \mid \frac{y^\star_j}{b_j} \leq t'_k\}$. Based on (1.134) and (1.135), it follows that

$$\emptyset \neq \mathit{dep}(\boldsymbol{y}^\star, j) \subset \bar{\mathcal{B}}^{(k)},\ \forall j \in \bar{\mathcal{B}}^{(k)},\ k = 1, \ldots, l. \tag{1.136}$$

Using the equivalence of item (i) and (iii) in Theorem 3.2 of (42), we conclude that $\boldsymbol{y}^\star$ is a lexicographically optimal base of $\mathcal{P}$ with respect to weight vector $\boldsymbol{b}$.

By Theorem 3.1 of (42), the lexicographically optimal base is unique. Since we have established that an arbitrary optimal solution $\boldsymbol{y}^\star$ is the lexicographically optimal base of $\mathcal{P}$ with respect to weight vector $\boldsymbol{b}$, it follows that problem (1.19) admits a unique optimal solution.

To establish that $y^\star_j > 0$ for all $j \in \mathcal{B}$, we suppose towards a contradiction that there exists $j \in \mathcal{B}$ such that $y^\star_j = 0$. By (1.134), $\mathit{dep}(\boldsymbol{y}^\star, j) \neq \emptyset$. Thus, there exists $j' \in \mathit{dep}(\boldsymbol{y}^\star, j)$. By Theorem 4.1 of (42) and the subsequent discussion, $\boldsymbol{y}^\star$ is not a lexicographically optimal base with respect to weight vector $\boldsymbol{b}$, thereby leading to a contradiction. Thus, we conclude that $y^\star_j > 0$ for all $j \in \mathcal{B}$. □

Proof of Proposition 10. Proof of claim (i). By Lemma 4(ii), the revenue optimization problem (1.7) can be formulated as an instance of the framework problem (1.38) where $g_j(q) = F_{b_j}^{-1}(1-\frac{q}{b_j})q$ for all $j \in \mathcal{B}$, $h_i(q) = F_{s_i}^{-1}(\frac{q}{s_i})q$ for all $i \in \mathcal{S}$ and $w_{ij} = c_i$ for all $(i,j) \in E$. Moreover, Assumptions (1.12.1-1)-(1.12.1-3) and Assumptions (1.12.1-7)-(1.12.1-8) hold. From $F_{b_j}(\cdot) = F_b(\cdot)$, $F_{s_i}(\cdot) = F_s(\cdot)$, and $c_i = 0$ (Assumption 3), we establish that $g_j(q) = b_j g(\frac{q}{b_j})$ for all $j \in \mathcal{B}$ and $h_i(q) = s_i h(\frac{q}{s_i})$ for all $i \in \mathcal{S}$ where $g(r) = F_b^{-1}(1-r)r$ and $h(r) = F_s^{-1}(r)r$. Thus, Assumptions (1.12.1-4) - (1.12.1-6) also hold.

Verifying Assumptions (1.91) - (1.92) in claim (i): Given $g_j(q) = b_j g(\frac{q}{b_j})$ and $h_i(q) = s_i h(\frac{q}{s_i})$, Assumption (1.91) - (1.92) follow from Assumptions (1.12.1-1) - (1.12.1-2).

Verifying Assumption (1.93) in claim (i): Note that by the continuous differentiability and the strict increasingness of function $F_s(\cdot)$ in $(0, \bar{v}_s)$ (Assumption 1), it follows that $F_s^{-1}(\cdot)$ is continuously differentiable and strictly increasing in $(0,1)$ (the inverse function theorem). Moreover, $F_s^{-1}(r)$ is continuous at $r = 0$ and at $r = 1$ if $\bar{v}_s < \infty$. Given $th(\frac{r}{t}) = F_s^{-1}(\frac{r}{t})r$, it follows that $th(\frac{r}{t})$ is strictly decreasing in t for $t \geq r$ and $r > 0$.

We next verify the joint convexity of $th(\frac{r}{t})$ by discussing the following two cases: if $\bar{v}_s < \infty$, then $h(1) = F_s^{-1}(1) < \infty$. Following the notation of (51), we let $dom(h) = [0,1]$, and $dom(th(\frac{r}{t})) = \{(r',t') : t' > 0, \frac{r'}{t'} \in [0,1]\} = \{(r',t') : t' > 0, \frac{r'}{t'} \in [0,1], h(\frac{r'}{t'}) < \infty\}$. Since $h(r)$ is continuous at $r = 0$ and $r = 1$ (by Assumption (1.92) established above) and convex in $r \in (0,1)$, it follows that it is convex in $dom(h)$. Using these observations, we conclude that $th(\frac{r}{t})$ is jointly convex in $(r,t) \in \{(r',t') : t' > 0, \frac{r'}{t'} \in [0,1], h(\frac{r'}{t'}) < \infty\}$ (by the discussion in page 89 of (51)). Suppose instead that $\bar{v}_s = \infty$. Then $h(1) = F_s^{-1}(1) = \infty$. We let $dom(h) = [0,1)$ and $dom(th(\frac{r}{t})) = \{(r',t') : t' > 0, \frac{r'}{t'} \in [0,1)\} = \{(r',t') : t' > 0, \frac{r'}{t'} \in [0,1], h(\frac{r'}{t'}) < \infty\}$. Since $h(r)$ is convex in $r \in (0,1)$ and continuous at $r = 0$ (by Assumption (1.92)), it follows that h is convex in $dom(h)$. Using this observation and once again the discussion in page 89 of (51), we conclude that $th(\frac{r}{t})$ is jointly convex in $(r,t) \in \{(r',t') : t' > 0, \frac{r'}{t'} \in [0,1], h(\frac{r'}{t'}) < \infty\}$. Thus, it follows that Assumption (1.93) holds.

Verifying Assumption (1.94) in claim (i): Note that by Assumption (1.12.1-7), we have

$h_i'(0) = 0$ for all $i \in \mathcal{S}$. By Assumption (1.12.1-3), we have $g_j'(b_j) \leq 0$ for all $j \in \mathcal{B}$. By Assumption (1.12.1-8), we have $g_j'(0) > 0$ for all $j \in \mathcal{B}$. Since $g_j(q) = b_j g(\frac{q}{b_j})$ for all $j \in \mathcal{B}$ and $h_i(q) = s_i h(\frac{q}{s_i})$ for all $i \in \mathcal{S}$, we obtain $g_j'(q) = g'(\frac{q}{b_j})$ and $h_i'(q) = h'(\frac{q}{s_i})$, which further implies that $g'(0) > h'(0) = 0$ and $g'(1) \leq 0$. Thus, Assumption (1.94) holds.

Proof of claim (ii). We plan to leverage Proposition 13 and Proposition 14 to prove this claim. Before proceeding, we first verify the conditions of Proposition 13 and Proposition 14 for $f(\cdot)$, $g(\cdot)$ and $h(\cdot)$ as in the statement of the proposition. Recall that Assumptions (1.12.1-1) - (1.12.1-6) hold in the instance of the framework problem (1.38) associated with problem (1.7) (see the first part of the proof). Using the first part of the proposition and applying Lemma 2(ii), we conclude that $f(t)$ is continuous, strictly increasing, and strictly concave for $t \geq 0$. These observations imply that the conditions in Proposition 13 and Proposition 14 are satisfied.

Proposition 14 establishes that problem (1.41) is equivalent to problem (1.38) in the sense that the optimal solutions of one can be mapped to the other as specified in the proposition. By Proposition 13, we also have that problem (1.41) and problem (1.19) share the same optimal solution vector $\boldsymbol{y}^\star$, which is unique (by Lemma 3). Thus, given the mappings in part (i) of Proposition 14, claim (ii) readily follows.

Proof of claim (iii). Let $f(\cdot)$, $g(\cdot)$ and $h(\cdot)$ be defined as in the statement of the proposition, and let $\{g_j, h_i\}$ be as given in part (i). Consider the formulations of (1.38) and (1.41) with these functions. By Proposition 13, problem (1.41) and problem (1.19) share the same optimal solution vector $\boldsymbol{y}^\star$, which implies that they also share the same optimal objective value $\sum_{j \in \mathcal{B}} b_j f(\frac{y_j^\star}{b_j})$. By Proposition 14(iii), it follows that the optimal objective value of (1.41) is the same as that of problem (1.38). Since problem (1.38) is an instance of the latter problem (with $\{g_j, h_i\}$ given in part (i)), using the fact that the optimal objective value of problem (1.7) is V_{opt} (by Theorem 1), we conclude that $V_{opt} = \sum_{j \in \mathcal{B}} b_j f(\frac{y_j^\star}{b_j})$. □

Proof of Proposition 11. Proof of claim(i). By Lemma 4(i), the equilibrium problem

(1.16) can be formulated as an instance of the framework problem (1.38) where $g_j(q) = \int_0^q \tilde{F}_{b_j}^{-1}(1 - \frac{x}{b_j})dx$ for all $j \in \mathcal{B}$, $h_i(q) = \int_0^q \tilde{F}_{s_i}^{-1}(\frac{x}{s_i})dx$ for all $i \in \mathcal{S}$, and $w_{ij} = \frac{c_i}{1+\gamma_j^b}$ for all $(i,j) \in E$. Moreover, Assumptions (1.12.1-1)- (1.12.1-3) hold in this instance of the framework problem (1.38). Given the definition of $\tilde{F}_{b_j}^{-1}$ and $\tilde{F}_{s_i}^{-1}$, we have $g_j(q) = \int_0^q \frac{1}{1+\gamma_j^b} F_{b_j}^{-1}(1 - \frac{x}{b_j}) - \frac{\mu_j^b}{1+\gamma_j^b} dx$ for all $j \in \mathcal{B}$ and $h_i(q) = \int_0^q \frac{1}{1-\gamma_i^s} F_{s_i}^{-1}(\frac{x}{s_i}) + \frac{\mu_i^s}{1-\gamma_i^s}$ for all $i \in \mathcal{S}$. From $F_{b_j}(\cdot) = F_b(\cdot)$, $F_{s_i}(\cdot) = F_s(\cdot)$, $c_i = 0$ (Assumption 3), $\gamma_j^b = \gamma^b$, $\gamma_i^s = \gamma^s$, $\mu_j^b = \mu^b$, and $\mu_i^s = \mu^s$ (by the assumptions of the proposition), it follows that $g_j(q) = b_j g(\frac{q}{b_j})$, $h_i(q) = s_i h(\frac{q}{s_i})$ and $w_{ij} = c_i = 0$ where $g(r) = \int_0^r \frac{1}{1+\gamma^b} F_b^{-1}(1-x) - \frac{\mu^b}{1+\gamma^b} dx$ and $h(r) = \int_0^r \frac{1}{1-\gamma^s} F_s^{-1}(x) + \frac{\mu^s}{1-\gamma^s} dx$ as given in the statement of the proposition. These observations imply that, for the aforementioned instance problem (1.38), Assumptions (1.12.1-4) - (1.12.1-6) hold.

<u>Verifying Assumptions (1.91)- (1.92) in claim (i):</u> Given that $g_j(q) = b_j g(\frac{q}{b_j})$ and $h_i(q) = s_i h(\frac{q}{s_i})$, Assumptions (1.91) - (1.92) follow from Assumptions (1.12.1-1) - (1.12.1-2).

<u>Verifying Assumption (1.93) in claim (i):</u> Note that $th(\frac{r}{t}) = t\int_0^{r/t} \frac{1}{1-\gamma^s} F_s^{-1}(x) + \frac{\mu^s}{1-\gamma^s} dx$ by definition. By a change of variables, it follows that $t\int_0^{r/t} \frac{1}{1-\gamma^s} F_s^{-1}(x) + \frac{\mu^s}{1-\gamma^s} dx = \int_0^r \frac{1}{1-\gamma^s} F_s^{-1}(\frac{x}{t}) + \frac{\mu^s}{1-\gamma^s} dx$. By Assumption 1, the distribution function $F_s(v)$ is continuously differentiable and strictly increasing in $v \in (0, \bar{v}_s)$, which implies that $F_s^{-1}(r)$ is continuously differentiable and strictly increasing in $r \in (0,1)$ (by the inverse function theorem). Moreover, $F_s^{-1}(r)$ is continuous at $r = 0$ and at $r = 1$ if $\bar{v}_s < \infty$. Since $th(\frac{r}{t}) = \int_0^r \frac{1}{1-\gamma^s} F_s^{-1}(\frac{x}{t}) + \frac{\mu^s}{1-\gamma^s} dx$, it follows that $th(\frac{r}{t})$ is strictly decreasing in t for any $t \geq r$ and $r > 0$.

We next verify the joint convexity of $th(\frac{r}{t})$ by discussing the following two cases: if $\bar{v}_s < \infty$, then $h(1) = F_s^{-1}(1) < \infty$. Following the notation of (51), we let $dom(h) = [0,1]$, and $dom(th(\frac{r}{t})) = \{(r', t') : t' > 0, \frac{r'}{t'} \in [0,1]\} = \{(r', t') : t' > 0, \frac{r'}{t'} \in [0,1], h(\frac{r'}{t'}) < \infty\}$. Note that since $h(r)$ is continuous at $r = 0$ and $r = 1$ (by Assumption (1.92) established above) and convex in $r \in (0,1)$, it follows that it is convex in $dom(h)$. Using these observations, we conclude that $th(\frac{r}{t})$ is jointly convex in $(r,t) \in \{(r', t') : t' > 0, \frac{r'}{t'} \in [0,1], h(\frac{r'}{t'}) < \infty\}$ (by the discussion in page 89 of (51)). Suppose instead that $\bar{v}_s = \infty$. Then $h(1) = F_s^{-1}(1) = \infty$. We let $dom(h) = [0,1)$, and $dom(th(\frac{r}{t})) = \{(r',t') : t' > 0, \frac{r'}{t'} \in [0,1)\} = \{(r',t') : t' > 0, \frac{r'}{t'} \in [0,1], h(\frac{r'}{t'}) < \infty\}$. Since $h(r)$ is convex in $r \in (0,1)$ and continuous at $r = 0$ (by Assumption (1.92)), it follows that h is convex in $dom(h)$. Using this observation (and once again the discussion in page 89 of (51)), we conclude that $th(\frac{r}{t})$ is jointly convex in $(r,t) \in \{(r',t') : t' > 0, \frac{r'}{t'} \in [0,1], h(\frac{r'}{t'}) < \infty\}$. Thus, it follows that Assumption (1.93) holds.

Verifying Assumption (1.94) in claim (i): Observe that $g'(r) = \frac{1}{1+\gamma^b} F_b^{-1}(1-r) - \frac{\mu^b}{1+\gamma^b}$ and $h'(r) = \frac{1}{1-\gamma^s} F_s^{-1}(r) + \frac{\mu^s}{1-\gamma^s}$. By assumption $\mu^b + \frac{1+\gamma^b}{1-\gamma^s}\mu^s < F_b^{-1}(1)$, this observation implies that $g'(0) - h'(0) = \frac{1}{1+\gamma^b} F_b^{-1}(1) - \frac{\mu^b}{1+\gamma^b} - \frac{\mu^s}{1-\gamma^s} > 0$. Since $g_j'(q) = g'(\frac{q}{b_j})$ and $h_i'(q) = h'(\frac{q}{s_i})$, Assumption (1.12.1-3) implies that $g'(1) \leq 0$ and $h'(0) \geq 0$. Thus, we conclude that Assumption (1.94) also holds.

Proof of claim (ii). We plan to apply Proposition 13 and Proposition 14 to establish this claim. Before proceeding, we first establish the conditions in Proposition 13 and Proposition 14. Recall that Assumptions (1.12.1-1) -(1.12.1-6) hold in the instance of the framework problem (1.38) associated with the equilibrium problem (1.16) (see the first part of the proof). Using the first part of the proposition and applying Lemma 2(ii), we establish that $f(\cdot)$ is continuous, strictly increasing, and strictly concave for $t \geq 0$. These observations jointly imply that the conditions in Proposition 13 and Proposition 14 are satisfied.

Proposition 14 establishes that problem (1.41) is equivalent to problem (1.38) in the sense that the optimal solutions of one can be mapped to the optimal solutions of the other as specified by the proposition statement. Moreover, Proposition 13 establishes that problem (1.41) and problem (1.19) share the same optimal solution vector $\boldsymbol{y}^\star$. Thus, given the mapping in Proposition 14(i), we conclude that claim (ii) of the proposition statement holds.

Proof of claim (iii). Let $f(\cdot)$, $g(\cdot)$, and $h(\cdot)$ be defined as in the statement of the proposition, and let $\{g_j, h_i\}$ be as given in part (i). Consider the formulations of problem (1.38) and problem (1.41) with these functions. By Proposition 13, problem (1.41) and problem (1.19) share the same optimal solution vector $\boldsymbol{y}^\star$, which implies that they also share the same optimal objective value $\sum_{j\in\mathcal{B}} b_j f(\frac{y_j^\star}{b_j})$. By Proposition 14(iii), it follows that the optimal objective value of problem (1.41) is the same as that of problem (1.38). Since problem (1.16) is an instance of the latter problem (with $\{g_j, h_i\}$ given in part (i)), these observations imply that problem (1.16) and problem (1.19) have the same objective value, as claimed. $\square$

1.13 Proofs of Results in Section 1.3

We provide the proofs of the results from Section 1.3 in Section 1.13. In Section 1.13, we illustrate that the revenue maximization problem (1.3) of the platform has an objective that is not concave.

Proofs

Proof of Proposition 1. We prove the result in the following two steps.

Step 1: existence of competitive equilibrium. By Lemma 4(i), the equilibrium problem (1.16) can be formulated as an instance of the framework problem (1.38) where Assumptions (1.12.1-1) - (1.12.1-3) hold. By applying Proposition 12 (i), we establish that there exists an optimal solution $(\boldsymbol{x}, \boldsymbol{q}^s, \boldsymbol{q}^b)$ to the equilibrium problem (1.16). Moreover, Proposition 12(iii) implies that there exists a dual optimal multipliers $(\boldsymbol{\theta}^b, \boldsymbol{\theta}^s, \boldsymbol{\eta}^b, \boldsymbol{\eta}^s, \boldsymbol{\pi})$ that satisfies the conditions in (1.40) such that $\theta_i^s \geq \theta_j^b - w_{ij} = \theta_j^b - \frac{c_i}{1+\gamma_j^b}$ for all $(i,j) \in E$. Thus, there exists a vector $\boldsymbol{p}$ such that $\theta_i^s \geq p_i \geq \theta_j^b - \frac{c_i}{1+\gamma_j^b}$. Proposition 9 implies that this $\boldsymbol{p}$ together with the optimal solution $(\boldsymbol{x}, \boldsymbol{q}^s, \boldsymbol{q}^b)$ constitutes a competitive equilibrium. Thus, we conclude that a competitive equilibrium exists.

Step 2: uniqueness. We apply Proposition 12 (iv) to establish that any optimal solution vector $(\boldsymbol{x}, \boldsymbol{q}^s, \boldsymbol{q}^b)$ to the equilibrium problem (1.16) has the same $(\boldsymbol{q}^s, \boldsymbol{q}^b)$. By Proposition 12(vi), we also obtain that the optimal dual vector $(\boldsymbol{\theta}^s, \boldsymbol{\theta}^b)$ corresponding to constraints (1.16c) - (1.16b) is unique. Given that $p_i = \theta_i^s$ for all $i : q_i^s > 0$ (Proposition 9(ii)), we establish that price vector $(p_i)_{i:q_i^s>0}$ is unique. $\square$

Proof of Corollary 1. Consider any commission-subscription profile $(\boldsymbol{\gamma}, \boldsymbol{\mu}) \in \Gamma \times \mathcal{U}$, and any associated competitive equilibrium $(\boldsymbol{p}, \boldsymbol{x}, \boldsymbol{q}^s, \boldsymbol{q}^b) \in \mathcal{X}(\boldsymbol{\gamma}, \boldsymbol{\mu})$. At this equilibrium, the platform's revenues can be expressed as follows:

$$\begin{aligned}
&\sum_{(i,j)\in E} (\gamma_i^s + \gamma_j^b) p_i x_{ij} + \sum_{(i,j)\in E} (\mu_i^s + \mu_j^b) x_{ij} \\
&\quad \overset{(a)}{=} \sum_{j\in\mathcal{B}} \sum_{i:x_{ij}>0} \left[(1+\gamma_j^b)p_i + c_i + \mu_j^b\right] x_{ij} - \sum_{i\in\mathcal{S}} \sum_{j:x_{ij}>0} \left[(1-\gamma_i^s)p_i - \mu_i^s\right] x_{ij} - \sum_{(i,j):x_{ij}>0} c_i x_{ij}, \\
&\quad \overset{(b)}{=} \sum_{j\in\mathcal{B}} \left[\min_{i:(i,j)\in E} \left\{(1+\gamma_j^b)p_i + c_i\right\} + \mu_j^b\right] \Big(\sum_{i:x_{ij}>0} x_{ij} \Big) \\
&\qquad - \sum_{i\in\mathcal{S}} \left[(1-\gamma_i^s)p_i - \mu_i^s\right] \Big(\sum_{j:x_{ij}>0} x_{ij} \Big) - \sum_{(i,j):x_{ij}>0} c_i x_{ij}, \\
&\quad \overset{(c)}{=} \sum_{j\in\mathcal{B}} \left[\min_{i:(i,j)\in E} \left\{(1+\gamma_j^b)p_i + c_i\right\} + \mu_j^b\right] q_j^b - \sum_{i\in\mathcal{S}} \left[(1-\gamma_i^s)p_i - \mu_i^s\right] q_i^s - \sum_{i\in\mathcal{S}} c_i q_i^s,
\end{aligned} \tag{1.137}$$

where step (a) follows by adding and subtracting terms $\sum_{(i,j):x_{ij}>0} p_i x_{ij}$ and $\sum_{(i,j):x_{ij}>0} c_i x_{ij}$, and rearranging the expressions. In step (b), we use the fact that, $x_{kj} > 0$ implies that $k \in \min_{i:(i,j)\in E}\{(1+\gamma_j^b)p_i + c_i\}$ by the equilibrium expression (1.2d). In step (c), we use the fact that $\sum_{i:(i,j)\in E} x_{ij} = q_j^b$ and $\sum_{j:(i,j)\in E} x_{ij} = q_i^s$ (by the equilibrium expression (1.2c)).

By Proposition 1, the equilibrium supply-demand vector $(\boldsymbol{q}^s, \boldsymbol{q}^b)$ and the equilibrium price vector $(p_i)_{i:q_i^s>0}$ are unique. Thus, (1.137) implies that for any $(\boldsymbol{\gamma}, \boldsymbol{\mu}) \in \Gamma \times \mathcal{U}$, all equilibria in $\mathcal{X}(\boldsymbol{\gamma}, \boldsymbol{\mu})$ induce the same revenue, and the claim follows. $\square$

Nonconcavity of the Revenue Function

We leverage the equilibrium problem (1.16) to illustrate the nonconvexity of the platform's revenue function $V(\boldsymbol{\gamma}, \boldsymbol{\mu})$ provided that the induced revenue for any competitive equilibrium is unique (Corollary 1).

Example 4. *Consider a network with one seller type and one buyer type. Let the value distribution be* $F_s(v) = F_b(v) = 1 - \exp(-v)$ *for* $v \geq 0$ *and the population profile be* $s = b = 1$ *with* $c = 0$. *Consider the following three commission-subscription vectors* $(\gamma^s, \gamma^b, \mu^s, \mu^b) \in \{(0,0,1,1),(0,0,2,2),(0,0,3,3)\}$. *Equilibria for these commissions-subscriptions can be obtained by solving problem* (1.16). *The corresponding revenues can be computed using* (1.3a) *and is given by* $V(0,0,1,1) = 0.24$, $V(0,0,2,2) = 0.07$ *and* $V(0,0,3,3) = 0.01$. *Note that since* $V(0,0,2,2) < \frac{1}{2}V(0,0,1,1) + \frac{1}{2}V(0,0,3,3)$, *it follows that* $V(\gamma^s, \gamma^b, \mu^s, \mu^b)$ *is not concave in* $(\gamma^s, \gamma^b, \mu^s, \mu^b)$. □

1.14 Proofs of Results in Section 1.4

1.14.1 Proofs of Results in Section 1.4.1.

Proof of Theorem 1. Proof of claim (i). By Lemma 4(ii), the revenue optimization problem (1.7) can be formulated as an instance of the framework problem (1.38) where we define $g_j(q) = F_{b_j}^{-1}\left(1 - \frac{q}{b_j}\right) q$ for all $j \in \mathcal{B}$, $h_i(q) = F_{s_i}^{-1}\left(\frac{q}{s_i}\right) q$ for all $i \in \mathcal{S}$, and $w_{ij} = c_i$ for all $(i,j) \in E$. Moreover, also by Lemma 4(ii), Assumptions (1.12.1-1) - (1.12.1-3) hold.

We prove claim (i) of this theorem using the following three steps. In step (i-1), we establish that $V_{opt} \leq \tilde{V}_{opt}$ i.e., the optimal solution to problem (1.7) provides an upper bound to the optimal revenue in problem (1.3). In step (i-2), we construct a feasible solution to (1.3) that achieves a value of $\tilde{V}_{opt}$, which allows us to conclude that $V_{opt} = \tilde{V}_{opt}$. In step (i-3), we show that any optimal solution to problem (1.3) must have the same supply/demand vectors.

Step (i-1): $V_{opt} \leq \tilde{V}_{opt}$ Let $(\boldsymbol{\gamma}, \boldsymbol{\mu}) \in \Gamma \times \mathcal{U}$ feasible commission-subscription profile, and let $(\boldsymbol{p}, \boldsymbol{x}, \boldsymbol{q}^s, \boldsymbol{q}^b)$ be any induced competitive equilibrium that satisfies $(\boldsymbol{p}, \boldsymbol{x}, \boldsymbol{q}^s, \boldsymbol{q}^b) \in \mathcal{X}(\boldsymbol{\gamma}, \boldsymbol{\mu})$.

By the equilibrium expressions (1.2a) - (1.2b), we first deduce that

$$\begin{aligned}
\Big[\min_{i':(i',j)\in E}\{(1+\gamma_j^b)p_{i'}+c_{i'}\}+\mu_j^b\Big]q_j^b &\overset{(a)}{=} F_{b_j}^{-1}\left(1-\frac{q_j^b}{b_j}\right)q_j^b, \quad \forall j\in\mathcal{B},\\
\Big[(1-\gamma_i^s)p_i-\mu_i^s\Big]q_i^s &\overset{(b)}{\geq} F_{s_i}^{-1}\left(\frac{q_i^s}{s_i}\right)q_i^s, \quad \forall i\in\mathcal{S}, \qquad (1.138)
\end{aligned}$$

where equality (a) follows because: if $q_j^b=0$, then we must have $\infty > \min_{i':(i',j)\in E}\{(1+\gamma_j^b)p_{i'}+c_{i'}\}+\mu_j^b \geq \bar{v}_{b_j} = F_{b_j}^{-1}(1-\frac{q_j^b}{b_j}) \geq 0$, and both sides of the expressions equal to zero; if $q_j^b>0$, given that $q_j^b<b_j$ (Proposition 12 (v)), we establish that $1-\frac{q_j^b}{b_j}\in(0,1)$, which, from the equilibrium expression (1.2b), further implies the equality $\min_{i':(i',j)\in E}\{(1+\gamma_j^b)p_{i'}+c_{i'}\}+\mu_j^b = F_{b_j}^{-1}\big(1-\frac{q_j^b}{b_j}\big)$. Thus, the equality in step (a) holds.

Inequality (b) follows by similar arguments. If $q_i^s=0$, then we have $-\infty<(1-\gamma_i^s)p_i-\mu_i^s\leq 0 = F_{s_i}^{-1}(\frac{q_i^s}{s_i})$, and both sides equal to zero. If $q_i^s\in(0,s_i]$, we have $(1-\gamma_i^s)p_i-\mu_i^s\geq F_{s_i}^{-1}\big(\frac{q_i^s}{s_i}\big)$ (the equilibrium expression (1.2a)), which further leads to the inequality in step (b).

In Corollary 1, we show that all equilibria in $\mathcal{X}(\boldsymbol{\gamma},\boldsymbol{\mu})$ yield the same revenue, and thus $V(\boldsymbol{\gamma},\boldsymbol{\mu})$ is well-defined. Thus, we can obtain the following upper bound for the platform's revenue function:

$$\begin{aligned}
V(\boldsymbol{\gamma}, \boldsymbol{\mu}) &= \sum_{(i,j)\in E} (\gamma_i^s + \gamma_j^b) p_i x_{ij} + (\mu_i^s + \mu_j^b) x_{ij} \\
&\overset{(c)}{=} \sum_{(i,j)\in E} [(1+\gamma_j^b)p_i + c_i + \mu_j^b] x_{ij} - \sum_{(i,j)\in E} [(1-\gamma_i^s)p_i - \mu_i^s] x_{ij} - \sum_{(i,j)\in E} c_i x_{ij} \\
&\overset{(d)}{=} \left[\sum_{j\in\mathcal{B}} \Big(\min_{i':(i',j)\in E} \{(1+\gamma_j^b)p_{i'} + c_{i'}\} + \mu_j^b \Big) \Big(\sum_{i:(i,j)\in E} x_{ij} \Big) \right] \\
&\qquad - \left[\sum_{i\in\mathcal{S}} \Big((1-\gamma_i^s)p_i - \mu_i^s \Big) \Big(\sum_{j:(i,j)\in E} x_{ij} \Big) \right] - \sum_{(i,j)\in E} c_i x_{ij} \\
&= \left[\sum_{j\in\mathcal{B}} \Big(\min_{i':(i',j)\in E} \{(1+\gamma_j^b)p_{i'} + c_{i'}\} + \mu_j^b \Big) q_j^b \right] \\
&\qquad - \left[\sum_{i\in\mathcal{S}} \Big((1-\gamma_i^s)p_i - \mu_i^s \Big) q_i^s \right] - \sum_{(i,j)\in E} c_i x_{ij} \\
&\overset{(e)}{\le} \sum_{j\in\mathcal{B}} F_{b_j}^{-1}\Big(1 - \frac{q_j^b}{b_j}\Big) q_j^b - \sum_{i\in\mathcal{S}} F_{s_i}^{-1}\Big(\frac{q_i^s}{s_i}\Big) q_i^s - \sum_{i\in\mathcal{S}} c_i q_i^s \;=\; h(\boldsymbol{q}^s, \boldsymbol{q}^b), \qquad (1.139)
\end{aligned}$$

where step (c) follows from adding and subtracting $\sum_{(i,j)\in E} p_i x_{ij}$ and $\sum_{(i,j)\in E} c_i x_{ij}$. The equality in (d) follows from the fact that, at equilibrium, if $x_{ij} > 0$, then $(1+\gamma_j^b)p_i + c_i = \min_{i':(i',j)\in E}\{(1+\gamma_j^b)p_{i'} + c_{i'}\}$ (the equilibrium expression (1.2d)). Step (e) follows from (1.138). Given that $(\boldsymbol{p}, \boldsymbol{x}, \boldsymbol{q}^s, \boldsymbol{q}^b) \in \mathcal{X}(\boldsymbol{\gamma}, \boldsymbol{\mu})$, we verify that $(\boldsymbol{x}, \boldsymbol{q}^s, \boldsymbol{q}^b)$ is feasible in problem (1.38). Since $F_{s_i}(v) \le 1$ for all v, the equilibrium expression (1.2a) implies that $q_i^s \le s_i$ for all $i \in \mathcal{S}$. Thus, constraint (1.38e) is satisfied. With $F_{b_j}(v) \le 1$ for all v, the equilibrium expression (1.2b) implies that $q_j^b \le b_j$ for all $j \in \mathcal{B}$. Thus, constraint (1.38d) is satisfied. The equilibrium expression (1.2c) - (1.2d) ensure that constraints (1.38c), (1.38b),(1.38f) hold. These observations jointly imply that $(\boldsymbol{x}, \boldsymbol{q}^s, \boldsymbol{q}^b)$ is feasible in problem (1.7).

To conclude, we highlight that the arguments in this proof mimic the intuition provided in the main text where we (1) replace the objective function (1.3a) of problem (1.3) with $h(\boldsymbol{q}^s, \boldsymbol{q}^b)$, (2) drop $(\boldsymbol{\gamma}, \boldsymbol{\mu}) \in \Gamma \times \mathcal{U}$ in constraint (1.3c), and (3) relax constraint (1.3b) with constraints (1.7b) - (1.7f). This allows us to formally establish that $V_{opt} \le \tilde{V}_{opt}$.

Step (i-2): $V_{opt} = \tilde{V}_{opt}$. We now construct a feasible solution to (1.3) whose value is equal to $\tilde{V}_{opt}$. Let $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{q}}^s, \bar{\boldsymbol{q}}^b)$ be a primal optimal solution to the framework problem (1.38) with $g_j(q) := F_{b_j}^{-1}\left(1 - \frac{q}{b_j}\right) q$, $h_i(q) := F_{s_i}^{-1}\left(\frac{q}{s_i}\right) q$ and $w_{ij} := c_i$ (recall that this problem is equivalent to the revenue optimization problem). Let $(\boldsymbol{\theta}^b, \boldsymbol{\theta}^s, \boldsymbol{\eta}^b, \boldsymbol{\eta}^s, \boldsymbol{\pi})$ be a corresponding dual optimal solution that satisfies the conditions in (1.40) (these dual variables correspond to constraints (1.38b) - (1.38f), respectively). Recall from Proposition 12(vi) that the dual optimal solution $(\boldsymbol{\theta}^b, \boldsymbol{\theta}^s, \boldsymbol{\eta}^b, \boldsymbol{\eta}^s, \boldsymbol{\pi})$ is unique. Define the price vector $\boldsymbol{p}$ as

$$p_i \ := \ \theta_i^s, \quad \forall i \in \mathcal{S}. \tag{1.140}$$

We consider a subscription-only implementation in which we set the commission rates as $\boldsymbol{\gamma} = \boldsymbol{0}$ and then construct the following subscription fees

$$\mu_i^s = \theta_i^s - F_{s_i}^{-1}\left(\frac{\bar{q}_i^s}{s_i}\right), \quad \forall i \in \mathcal{S}, \tag{1.141a}$$

$$\mu_j^b = F_{b_j}^{-1}\left(1 - \frac{\bar{q}_j^b}{b_j}\right) - \theta_j^b, \quad \forall j \in \mathcal{B}. \tag{1.141b}$$

By construction, the commission profile $\boldsymbol{\gamma} = \boldsymbol{0} \in \Gamma$ and thus is feasible. We next verify that the constructed subscription profile in (1.141) satisfies $\boldsymbol{\mu} \in \mathcal{U}$.

For all $i \in \mathcal{S}$, by using Assumption 1, we obtain that F_{s_i} is continuously differentiable and strictly increasing in $(0, \bar{v}_{s_i})$, and by the implicit function theorem it follows that $F_{s_i}^{-1}$ is continuously differentiable and strictly increasing in $(0,1)$. Proposition 12(iii) implies that $h_i'(\bar{q}_i^s)$ is finite. Recall that $h_i'(q)$ is continuous in $(0, s_i)$ (Assumption (1.12.1-2)), and $h_i'(0) = \lim_{q \downarrow 0} h_i'(q)$, $h_i'(s_i) = \lim_{q \uparrow s_i} h_i'(q)$ (by definition). Thus, there exists a sequence $\{q_k \in (0, s_i) : k = 1, 2, \dots\}$ such that $\lim_{k\to\infty} q_k = \bar{q}_i^s$ and $\lim_{k\to\infty} h_i'(q_k) = h_i'(\bar{q}_i^s)$. Given

$h_i(q) = F_{s_i}^{-1}(\frac{q}{s_i})q$, we deduce that

$$\begin{aligned}\theta_i^s \;&=\; h_i'(\bar{q}_i^s) + \eta_i^s \;=\; \lim_{k\to\infty} h_i'(q_k) + \eta_i^s \\ &=\; \lim_{k\to\infty} \left[F_{s_i}^{-1}\right]'\left(\frac{q_k}{s_i}\right)\frac{q_k}{s_i} + F_{s_i}^{-1}\left(\frac{q_k}{s_i}\right) + \eta_i^s \;\geq\; F_{s_i}^{-1}\left(\frac{\bar{q}_i^s}{s_i}\right), \quad \forall i \in \mathcal{S}.\end{aligned} \tag{1.142}$$

Based on (1.141a) and (1.142), we have $\mu_i^s \geq 0$ for all $i \in \mathcal{S}$.

For all $j \in \mathcal{B}$, we have $\bar{q}_j^b < b_j$ (Proposition 12(v)), which, by the complementarity condition (1.40d), implies that $\eta_j^b = 0$. Given condition (1.40a), it follows that $g_j'(\bar{q}_j^b) = \theta_j^b$.

By Assumption 1, F_{b_j} is continuously differentiable and strictly increasing in $(0, \bar{v}_{b_j})$, and together with the implicit function theorem it follows that $F_{b_j}^{-1}$ is continuously differentiable and strictly increasing in $(0,1)$. Proposition 12(iii) implies that $g_j'(\bar{q}_j^b)$ is finite. Recall that $g_j'(q)$ is continuous in $(0, b_j)$ (Assumption (1.12.1-1)), and $g_j'(0) = \lim_{q\downarrow 0} g_j'(q)$, $g_j'(b_j) = \lim_{q\uparrow b_j} g_j'(q)$ (by definition). Thus, there exists a sequence $\{q_k \in (0, b_j) : k = 1, 2, \dots\}$ such that $\lim_{k\to\infty} q_k = \bar{q}_j^b$ and $\lim_{k\to\infty} g_j'(q_k) = g_j'(\bar{q}_j^b)$. Then

$$\begin{aligned}\theta_j^b \;&=\; g_j'(\bar{q}_j^b) \;=\; \lim_{k\to\infty} g_j'(q_k) \\ &=\; \lim_{k\to\infty} -\left[F_{b_j}^{-1}\right]'\left(1-\frac{q_k}{b_j}\right)\frac{q_k}{b_j} + F_{b_j}^{-1}\left(1-\frac{q_k}{b_j}\right) \;\leq\; F_{b_j}^{-1}\left(1-\frac{\bar{q}_j^b}{b_j}\right), \quad \forall j \in \mathcal{B}.\end{aligned} \tag{1.143}$$

From (1.141b) and (1.143), it follows that $\mu_j^b \geq 0$, and we can conclude that $(\boldsymbol{\mu}^s, \boldsymbol{\mu}^b) \in \mathcal{U}$.

Next, we verify that $(\boldsymbol{p}, \bar{\boldsymbol{x}}, \bar{\boldsymbol{q}}^s, \bar{\boldsymbol{q}}^b)$ satisfies the equilibrium conditions (1.2a)-(1.2d) under the commissions-subscriptions vector $(\mathbf{0}, \boldsymbol{\mu})$, where $\boldsymbol{p}$ and $\boldsymbol{\mu}$ are as constructed above. To establish expression (1.2a), for each $i \in \mathcal{S}$, we have that

$$s_i F_{s_i}\Big(p_i - \mu_i^s\Big) \overset{(f)}{=} s_i F_{s_i}\Big(\theta_i^s - \mu_i^s\Big) \overset{(g)}{=} s_i F_{s_i}\Big(F_{s_i}^{-1}\Big(\frac{\bar{q}_i^s}{s_i}\Big)\Big) \;=\; \bar{q}_i^s, \tag{1.144}$$

where (f) follows from $p_i = \theta_i^s$ (the construction of p_i in (1.140)) and step (g) follows from $\mu_i^s = \theta_i^s - F_{s_i}^{-1}(\frac{\bar{q}_i^s}{s_i})$ (construction of μ_i^s in (1.141a)).

We show that condition (1.2b) holds for each $j \in \mathcal{B}$ by discussing two cases. First, if $\bar{q}_j^b > 0$, for all i with $(i, j) \in E$, we have $p_i + c_i \geq \theta_j^b$, which is derived from $\theta_i^s + c_i - \theta_j^b = \pi_{ij}$ (condition (1.40c), $\pi_{ij} \geq 0$ (condition (1.40f)) and $p_i = \theta_i^s$ (the construction of p_i in (1.140)). Pick any i with $\bar{x}_{ij} > 0$. By condition (1.40f), we have $\pi_{ij} = 0$, which implies that $\theta_i^s + c_i = \theta_j^b$ by condition (1.40c). Thus, we have $p_i + c_i = \theta_j^b$ (the construction of p_i in (1.140)). In summary of the arguments above, we obtain that that $\min_{i':(i',j)\in E}\{p_{i'} + c_{i'}\} = p_i + c_i = \theta_j^b$, which allows us to further deduce that

$$\begin{aligned} b_j\Big[1 - F_{b_j}\Big(\min_{i':(i',j)\in E}\{p_{i'} + c_{i'}\} + \mu_j^b\Big)\Big] &\overset{(h)}{=} b_j\Big[1 - F_{b_j}\big(\theta_j^b + \mu_j^b\big)\Big] \\ &\overset{(i)}{=} b_j\Big[1 - F_{b_j}\Big(F_{b_j}^{-1}\Big(1 - \frac{\bar{q}_j^b}{b_j}\Big)\Big)\Big] \;=\; \bar{q}_j^b, \qquad (1.145) \end{aligned}$$

where step (h) follows from $\min_{i':(i',j)\in E}\{p_{i'} + c_{i'}\} = \theta_j^b$. Step (i) follows from the construction of μ_j^b in (1.141b). Thus, the equilibrium expression (1.2b) holds.

Second, if $\bar{q}_j^b = 0$, then we have $\bar{x}_{ij} = 0$ for all i with $(i, j) \in E$. Recall that $\theta_i^s + c_i - \theta_j^b = \pi_{ij} \geq 0$ for all i with $(i, j) \in E$ (condition (1.40c) and (1.40f)), and thus we establish that $\theta_j^b \leq \min_{i:(i,j)\in E}\{\theta_i^s + c_i\}$. Given that $p_i = \theta_i^s$ (construction of p_i in (1.140)), we obtain that $\theta_j^b \leq \min_{i:(i,j)\in E}\{p_i + c_i\}$. Thus, we can deduce that

$$\begin{aligned} 0 &\overset{(j)}{\leq} b_j\Big[1 - F_{b_j}\Big(\min_{i':(i',j)\in E}\{p_{i'} + c_{i'}\} + \mu_j^b\Big)\Big] \overset{(k)}{\leq} b_j\Big[1 - F_{b_j}\Big(\theta_j^b + \mu_j^b\Big)\Big] \\ &\overset{(l)}{=} b_j\Big[1 - F_{b_j}\Big(F_{b_j}^{-1}\Big(1 - \frac{\bar{q}_j^b}{b_j}\Big)\Big)\Big] \overset{(m)}{=} 0, \qquad (1.146) \end{aligned}$$

where step (j) follows from the fact that $F_{b_j}(v) \in [0, 1]$ for all $v \in \mathbb{R}$ (Assumption 1). To establish step (k), recall that $\theta_j^b \leq \min_{i:(i,j)\in E}\{\theta_i^s + c_i\}$; together with the increasing property of F_{b_j} (Assumption 1), we obtain the inequality. Step (l) follows from the construction of

μ_j^b in (1.141b). Step (m) follows from the fact that, in this case, we assumed $\bar{q}_j^b = 0$. Thus, we establish that the equilibrium expression (1.2b) holds.

The equilibrium constraint (1.2c) is guaranteed to hold by constraints (1.38b) and (1.38c).

Finally, we show that the equilibrium expression (1.2d) holds. Note that constraint (1.38f) guarantees that $\bar{x}_{ij} \geq 0$ for all $(i,j) \in E$. Moreover, for any $(i,j) \in E$ with $\bar{x}_{ij} > 0$, recall that we have established $p_i + c_i = \min_{i':(i',j)\in E}\{p_{i'} + c_{i'}\}$ when verifying that the equilibrium expression (1.2b) hold. Thus, we have $i \in \arg\min_{i':(i',j)\in E}\{p_{i'} + c_{i'}\}$, which guarantees the equilibrium expression (1.2d).

In summary, the equilibrium expressions (1.2a) - (1.2d) are satisfied and thus we have $(\boldsymbol{p}, \bar{\boldsymbol{x}}, \bar{\boldsymbol{q}}^s, \bar{\boldsymbol{q}}^b) \in \mathcal{X}(\boldsymbol{0}, \boldsymbol{\mu})$. Next, we show that the revenue induced by the subscription vector $\boldsymbol{\mu} \in \mathcal{U}$ satisfies

$$\begin{aligned}
V(\boldsymbol{0}, \boldsymbol{\mu}) &\overset{(m)}{=} \sum_{(i,j)\in E} (p_i + c_i + \mu_j^b)\bar{x}_{ij} - \sum_{(i,j)\in E} (p_i - \mu_i^s)\bar{x}_{ij} - \sum_{(i,j)\in E} c_i \bar{x}_{ij} \\
&\overset{(n)}{=} \left[\sum_{j\in\mathcal{B}} \sum_{i:\bar{x}_{ij}>0} (\theta_j^b + \mu_j^b)\bar{x}_{ij}\right] - \left[\sum_{i\in\mathcal{S}} \sum_{j:\bar{x}_{ij}>0} (\theta_i^s - \mu_i^s)\bar{x}_{ij}\right] - \sum_{(i,j):\bar{x}_{ij}>0} c_i \bar{x}_{ij} \\
&\overset{(o)}{=} \left[\sum_{j\in\mathcal{B}} (\theta_j^b + \mu_j^b)\bar{q}_j^b\right] - \left[\sum_{i\in\mathcal{S}:\bar{q}_i^s>0} (\theta_i^s - \mu_i^s)\bar{q}_i^s\right] - \sum_{i\in\mathcal{B}} c_i \bar{q}_i^s \\
&\overset{(p)}{=} \sum_{j\in\mathcal{B}} F_{b_j}^{-1}\left(1 - \frac{\bar{q}_j^b}{b_j}\right)\bar{q}_j^b - \sum_{i\in\mathcal{S}} F_{s_i}^{-1}\left(\frac{\bar{q}_i^s}{s_i}\right)\bar{q}_i^s - \sum_{i\in\mathcal{S}} c_i \bar{q}_i^s \;=\; h(\bar{\boldsymbol{q}}^s, \bar{\boldsymbol{q}}^b) \overset{(q)}{=} \tilde{V}_{opt},
\end{aligned} \tag{1.147}$$

where step (m) follows from reorganizing the revenue expression $V(\boldsymbol{0}, \boldsymbol{\mu}) = \sum_{(i,j)\in E}(\mu_i^s + \mu_j^b)\bar{x}_{ij}$ by adding and subtracting the terms $\sum_{(i,j)\in E} p_i \bar{x}_{ij}$ and $\sum_{(i,j)\in E} c_i \bar{x}_{ij}$. In step (n), we first have $p_i = \theta_i^s$ for all $i \in \mathcal{S}$ (by the construction of p_i in (1.140)). Whenever $\bar{x}_{ij} > 0$, we have $\pi_{ij} = 0$ (condition (1.40f)) and $\theta_j^b = \theta_i^s + c_i$ (condition (1.40c)). Thus, for all $j \in \mathcal{B}$, we replace $p_i + c_i$ with θ_j^b in the first component and then replace p_i with θ_i^s in the second component. In step (o), we use the fact that, by the equilibrium conditions, we have

$\sum_{i:\bar{x}_{ij}>0} \bar{x}_{ij} = \bar{q}^b_j$ and $\bar{x}_{ij}$ by $\sum_{j:\bar{x}_{ij}>0} \bar{x}_{ij} = \bar{q}^s_i$. Step (p) follows from the construction of subscription vector $\boldsymbol{\mu}$ in (1.141). Step (q) follows from the optimality of $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{q}}^s, \bar{\boldsymbol{q}}^b)$.

As $(\mathbf{0}, \boldsymbol{\mu}) \in \Gamma \times \mathcal{U}$, we deduce that $V_{opt} \geq V(\mathbf{0}, \boldsymbol{\mu}) = \tilde{V}_{opt}$. Thus, we conclude that $V_{opt} = \tilde{V}_{opt}$.

Step (i-3): $(\bar{\boldsymbol{q}}^s, \bar{\boldsymbol{q}}^b)$ as the unique optimal equilibrium supply-demand vector. Recall that, by Proposition 12(iv), for any optimal solution $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{q}}^s, \bar{\boldsymbol{q}}^b)$ to problem (1.38), vector $(\bar{\boldsymbol{q}}^s, \bar{\boldsymbol{q}}^b)$ is unique. Moreover, by the upper bound in (1.139), we have $V(\boldsymbol{\gamma}, \boldsymbol{\mu}) \leq h(\boldsymbol{q}^s, \boldsymbol{q}^b) \leq h(\bar{\boldsymbol{q}}^s, \bar{\boldsymbol{q}}^b) = V_{opt}$ for all $(\boldsymbol{\gamma}, \boldsymbol{\mu}) \in \Gamma \times \mathcal{U}$ and all $(\boldsymbol{p}, \boldsymbol{x}, \boldsymbol{q}^s, \boldsymbol{q}^b) \in \mathcal{X}(\boldsymbol{\gamma}, \boldsymbol{\mu})$.

Let $(\boldsymbol{\gamma}, \boldsymbol{\mu})$ be any revenue optimal commission-subscription profile. By the proof in step 2, must have $V(\boldsymbol{\gamma}, \boldsymbol{\mu}) = h(\bar{\boldsymbol{q}}^s, \bar{\boldsymbol{q}}^b) = V_{opt}$. This further implies that $h(\bar{\boldsymbol{q}}^s, \bar{\boldsymbol{q}}^b) = h(\boldsymbol{q}^s, \boldsymbol{q}^b)$ for all $(\boldsymbol{p}, \boldsymbol{x}, \boldsymbol{q}^s, \boldsymbol{q}^b) \in \mathcal{X}(\boldsymbol{\gamma}, \boldsymbol{\mu})$. As all equilibria in $\mathcal{X}(\boldsymbol{\gamma}, \boldsymbol{\mu})$ share the same vector $(\bar{\boldsymbol{q}}^s, \bar{\boldsymbol{q}}^b)$ (by Proposition 1) and the uniqueness of $(\bar{\boldsymbol{q}}^s, \bar{\boldsymbol{q}}^b)$, we conclude that $(\boldsymbol{q}^s, \boldsymbol{q}^b) = (\bar{\boldsymbol{q}}^s, \bar{\boldsymbol{q}}^b)$.

Summarizing the arguments above, we conclude that claim (i) holds.

Proof of claim (ii). Let $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{q}}^s, \bar{\boldsymbol{q}}^b)$ be an optimal solution to problem (1.38). We prove claim (ii) in three steps. In step (ii-1), we prove that the optimal commission-subscription vector is not unique. In step (ii-2), we prove the the necessity of condition (1.8) and in step (ii - 3), we prove the sufficiency of condition (1.8).

Step (ii-1): the optimal commission-subscription vector is not unique. Consider the construction of $\boldsymbol{p}$ and $\boldsymbol{\mu}$) in (1.140) and (1.141), respectively. We have verified that $(\boldsymbol{p}, \bar{\boldsymbol{x}}, \bar{\boldsymbol{q}}^s, \bar{\boldsymbol{q}}^b) \in \mathcal{X}(\mathbf{0}, \mathbf{0}, \boldsymbol{\mu}^s, \boldsymbol{\mu}^b)$.

Now consider an alternative commission-subscription profile $(\boldsymbol{\gamma}^s, \mathbf{0}, \mathbf{0}, \boldsymbol{\mu}^b)$ where $\boldsymbol{\gamma}^s$ satisfies $\gamma^s_i p_i = \mu^s_i$ for all $i \in \mathcal{S}$. If there exists $i \in \mathcal{S}$ with $p_i = 0$, then given that $\mu^s_i \geq 0$, we have $p_i - \mu^s_i \leq 0$, which implies that $\bar{q}^s_i = s_i F_{s_i}(p_i - \mu^s_i) = 0$. In this case, we would have $0 = p_i = \theta^s_i = \mu^s_i = 0$ (by the construction of p_i in (1.140) and μ^s_i in (1.141a)). Thus, we could pick any $\gamma^s_i \in [0, 1)$ to make sure that $\gamma^s_i p_i = \mu^s_i$.

Next, we want to show that $(\boldsymbol{p}, \bar{\boldsymbol{x}}, \bar{\boldsymbol{q}}^s, \bar{\boldsymbol{q}}^b) \in \mathcal{X}(\boldsymbol{\gamma}^s, \mathbf{0}, \mathbf{0}, \boldsymbol{\mu}^b)$. Note that, since $\boldsymbol{\mu}^b$ and $(\boldsymbol{p}, \bar{\boldsymbol{x}}, \bar{\boldsymbol{q}}^s, \bar{\boldsymbol{q}}^b)$ haven't changed, the equilibrium constraints (1.2b) - (1.2d) still hold. To establish the equilibrium constraint (1.2a), we deduce that

$$F_{s_i}((1-\gamma_i^s)p_i) \overset{(r)}{=} F_{s_i}(p_i - \mu_i^s) \overset{(s)}{=} \frac{\bar{q}_i^s}{s_i}, \quad \forall i \in \mathcal{S}, \tag{1.148}$$

where (r) follows from the construction $\gamma_i^s p_i = \mu_i^s$ and (s) follows from (1.144).

Moreover, for all $i \in \mathcal{S}$, since $(1-\gamma_i^s)p_i = p_i - \mu_i^s = F_{s_i}^{-1}\left(\frac{\bar{q}_i^s}{s_i}\right)$ (condition (1.140) and (1.141a)), the inequality in step (b) of (1.138) is tight, which further implies that step (e) of (1.139) is tight. Thus, we have that $V(\boldsymbol{\gamma}^s, \mathbf{0}, \mathbf{0}, \boldsymbol{\mu}^b) = h(\bar{\boldsymbol{q}}^s, \bar{\boldsymbol{q}}^b) = V_{opt}$ and therefore the optimal commission-subscription vector is not unique.

Step (ii-2): necessity. Let $(\boldsymbol{\gamma}, \boldsymbol{\mu}) \in \Gamma \times \mathcal{U}$ be an optimal commission-subscription profile, and let $\boldsymbol{p}$ be such that $(\boldsymbol{p}, \bar{\boldsymbol{x}}, \bar{\boldsymbol{q}}^s, \bar{\boldsymbol{q}}^b) \in \mathcal{X}(\boldsymbol{\gamma}, \boldsymbol{\mu})$. By the optimality of $(\boldsymbol{\gamma}, \boldsymbol{\mu})$ we must have that

$$V(\boldsymbol{\gamma}, \boldsymbol{\mu}) \;=\; h(\bar{\boldsymbol{q}}^s, \bar{\boldsymbol{q}}^b) \;=\; V_{opt}. \tag{1.149}$$

Given that $\gamma_i^s < 1$ for all $i \in \mathcal{S}$ (by the theorem statement), let the vector $(\tilde{\boldsymbol{\gamma}}, \tilde{\boldsymbol{\mu}})$ be defined as:

$$\tilde{\gamma}_i^s := \frac{1}{1-\gamma_i^s}, \quad \tilde{\mu}_i^s := \frac{\mu_i^s}{1-\gamma_i^s}, \quad \forall i \in \mathcal{S}, \tag{1.150a}$$

$$\tilde{\gamma}_j^b := \frac{1}{1+\gamma_j^b}, \quad \tilde{\mu}_j^b := \frac{\mu_j^b}{1+\gamma_j^b}, \quad \forall j \in \mathcal{B}. \tag{1.150b}$$

Below we verify that conditions (1.8a) - (1.8f) are satisfied:

1. Condition (1.8a): for all $i : \bar{q}_i^s > 0$, in order for (1.149) to hold, the inequality in step (b) of (1.138) must be tight i.e., $(1-\gamma_i^s)p_i - \mu_i^s \;=\; F_{s_i}^{-1}\left(\frac{\bar{q}_i^s}{s_i}\right)$. By using the change of notation in (1.150a), we obtain that $p_i - \tilde{\mu}_i^s - F_{s_i}^{-1}(\frac{\bar{q}_i^s}{s_i})\tilde{\gamma}_i^s = 0$. Thus, condition (1.8a) holds.

2. Condition (1.8b), for all i with $\bar{q}_i^s = 0$, by the equilibrium condition (1.2a), we have

$s_i F_{s_i}((1-\gamma_i^s)p_i - \mu_i^s) = \bar{q}_i^s = 0$, which suggests that $(1-\gamma_i^s)p_i - \mu_i^s \leq F_{s_i}^{-1}\left(\frac{\bar{q}_i^s}{s_i}\right) = 0$. With the change of notation in (1.150a), we have $p_i - \tilde{\mu}_i^s \ \leq \ 0$ in condition (1.8b).

3. Condition (1.8c): for all $(i,j) \in E$ with $\bar{x}_{ij} > 0$, we have $\bar{q}_j^b > 0$. By Proposition 12(v), we have that $\bar{q}_j^b < b_j$. By the equilibrium contraint (1.2b), we have $\min_{i':(i',j)\in E}(1+\gamma_j^b)p_{i'} + c_{i'} + \mu_j^b = F_{b_j}^{-1}(1-\frac{\bar{q}_j^b}{b_j})$. Moreover, by the equilibrium constraint (1.2d), we have $(1+\gamma_j^b)p_i + c_i = \min_{i':(i',j)\in E}(1+\gamma_j^b)p_{i'} + c_{i'}$. This implies that $(1+\gamma_j^b)p_i + c_i + \mu_j^b = F_{b_j}^{-1}(1-\frac{\bar{q}_j^b}{b_j})$. By using the change of notation in (1.150b), we obtain $p_i + \tilde{\mu}_j^b + c_i\tilde{\gamma}_j^b - F_{b_j}^{-1}(1-\frac{\bar{q}_j^b}{b_j})\tilde{\gamma}_j^b = 0$ in condition (1.8c).

4. Condition (1.8d): for all $(i,j) \in E$ with $\bar{x}_{ij} = 0$, we first have $(1+\gamma_j^b)p_i + c_i \geq \min_{i':(i',j)\in E}(1+\gamma_j^b)p_{i'} + c_{i'} \geq 0$. When either $\bar{q}_j^b = 0$ or $\bar{q}_j^b \geq 0$, we obtain that $\min_{i':(i',j)\in E}(1+\gamma_j^b)p_{i'} + c_{i'} + \mu_j^b \geq F_{b_j}^{-1}(1-\frac{\bar{q}_j^b}{b_j})$ by the equilibrium constraint in (1.2b), which further implies that $(1+\gamma_j^b)p_i + c_i + \mu_j^b \geq F_{b_j}^{-1}(1-\frac{\bar{q}_j^b}{b_j})$. Using the notation in (1.150b), we obtain the inequality $p_i + \tilde{\mu}_j^b + c_i\tilde{\gamma}_j^b - F_{b_j}^{-1}(1-\frac{\bar{q}_j^b}{b_j})\tilde{\gamma}_j^b \geq 0$ in (1.8d).

5. Condition (1.8e): given $\tilde{\gamma}_i^s = \frac{1}{1-\gamma_i^s}$ and $\tilde{\mu}_i^s = \frac{\mu_i^s}{1-\gamma_i^s}$ in (1.150a), with $\gamma_i^s \in [0,1)$ and $\mu_i^s \geq 0$, we obtain $\tilde{\gamma}_i^s \geq 1$ and $\tilde{\mu}_i^s \geq 0$. Moreover, from $(\boldsymbol{p}, \bar{\boldsymbol{x}}, \bar{\boldsymbol{q}}^s, \bar{\boldsymbol{q}}^b) \in \mathcal{X}(\boldsymbol{\gamma}, \boldsymbol{\mu})$, we have $\boldsymbol{p} \in \mathbb{R}_+^n$ (Definition 1). Thus, condition (1.8e) holds.

6. Condition (1.8f): by $\tilde{\gamma}_j^b = \frac{1}{1+\gamma_j^b}$ and $\tilde{\mu}_j^b = \frac{\mu_j^b}{1+\gamma_j^b}$ in (1.150b), we use $\gamma_j^b \geq 0$ and $\mu_j^b \geq 0$ to deduce that $0 < \tilde{\gamma}_j^b \leq 1$ and $\tilde{\mu}_j^b \geq 0$. Thus, condition (1.8f) holds.

Summarizing the arguments (1.8a) - (1.8f), we complete the proof of the necessity step.

Step (ii-3): sufficiency. Suppose there exists $(\tilde{\boldsymbol{\gamma}}, \tilde{\boldsymbol{\mu}})$ and $\boldsymbol{p}$ that satisfy conditions (1.8). We want to prove that the commission-subscription profile $(\boldsymbol{\gamma}, \boldsymbol{\mu})$ satisfying $\tilde{\gamma}_i^s = \frac{1}{1-\gamma_i^s}$, $\tilde{\mu}_i^s = \frac{\mu_i^s}{1-\gamma_i^s}$ for all $i \in \mathcal{S}$ and $\tilde{\gamma}_j^b = \frac{1}{1+\gamma_j^b}$, $\tilde{\mu}_j^b = \frac{\mu_j^b}{1+\gamma_j^b}$ for all $j \in \mathcal{B}$ would satisfy $(\boldsymbol{\gamma}, \boldsymbol{\mu}) \in \Gamma \times \mathcal{U}$ and $(\boldsymbol{p}, \bar{\boldsymbol{x}}, \bar{\boldsymbol{q}}, \bar{\boldsymbol{q}}^b) \in \mathcal{X}(\boldsymbol{\gamma}, \boldsymbol{\mu})$ and, moreover, $(\boldsymbol{\gamma}, \boldsymbol{\mu})$ is an optimal vector of commissions-subscriptions.

We start by showing that $(\boldsymbol{\gamma}, \boldsymbol{\mu}) \in \Gamma \times \mathcal{U}$. By condition (1.8e), we have $0 < \frac{1}{\tilde{\gamma}_i^s} \leq 1$ for all $i \in \mathcal{S}$, which implies that $1 - \gamma_i^s \in (0, 1]$. Using that $\tilde{\mu}_i^s \geq 0$, we obtain that $\mu_i^s = \tilde{\mu}_i^s(1 - \gamma_i^s) \geq 0$. For all $j \in \mathcal{B}$, from condition (1.8f), we have $\frac{1}{\tilde{\gamma}_j^b} \geq 1$, which implies that $1 + \gamma_j^b \geq 1$ or, equivalently, $\gamma_j^b \geq 0$. With $\tilde{\mu}_j^b \geq 0$, we have $\mu_j^b = \tilde{\mu}_j^b(1 + \gamma_j^b) \geq 0$. Thus, we can conclude that $(\boldsymbol{\gamma}, \boldsymbol{\mu}) \in \Gamma \times \mathcal{U}$.

Next, we verify $(\boldsymbol{p}, \bar{\boldsymbol{x}}, \bar{\boldsymbol{q}}^s, \bar{\boldsymbol{q}}^b) \in \mathcal{X}(\boldsymbol{\gamma}, \boldsymbol{\mu})$ by checking that the equilibrium conditions (1.2a) - (1.2d) are satisfied:

1. Condition (1.2a): for all $i \in \mathcal{S}$, with $\tilde{\gamma}_i^s = \frac{1}{1-\gamma_i^s}$ and $\tilde{\mu}_i^s = \frac{\mu_i^s}{1-\gamma_i^s}$, we have that $F_{s_i}\Big((1 - \gamma_i^s)p_i - \mu_i^s\Big) = F_{s_i}(\frac{1}{\tilde{\gamma}_i^s}p_i - \frac{1}{\tilde{\gamma}_i^s}\tilde{\mu}_i^s)$. If $\bar{q}_i^s > 0$, by condition (1.8a), we obtain that $F_{s_i}(\frac{1}{\tilde{\gamma}_i^s}p_i - \frac{1}{\tilde{\gamma}_i^s}\tilde{\mu}_i^s) = F_{s_i}(F_{s_i}^{-1}(\frac{\bar{q}_i^s}{s_i})) = \frac{\bar{q}_i^s}{s_i}$. Thus, $F_{s_i}\Big((1 - \gamma_i^s)p_i - \mu_i^s\Big) = \frac{\bar{q}_i^s}{s_i}$; if $\bar{q}_i^s = 0$, by condition (1.8b), we have $p_i - \tilde{\mu}_i^s \leq 0$. As F_{s_i} is increasing and $F_{s_i}(v) \in [0, 1]$ for all $v \in \mathbb{R}$ (Assumption 1), we have that $0 \leq F_{s_i}(\frac{1}{\tilde{\gamma}_i^s}p_i - \frac{1}{\tilde{\gamma}_i^s}\tilde{\mu}_i^s) \leq F_{s_i}(0) = 0$. Therefore, we obtain that $F_{s_i}\Big((1 - \gamma_i^s)p_i - \mu_i^s\Big) = \frac{\bar{q}_i^s}{s_i}$ and (1.2a) holds.

2. Condition (1.2b): for all $(i, j) \in E$, with $\tilde{\gamma}_j^b = \frac{1}{1+\gamma_j^b}$ and $\tilde{\mu}_j^b = \frac{\mu_j^b}{1+\gamma_j^b}$, condition (1.8c) implies that $(1+\gamma_j^b)p_i + c_i + \mu_j^b = \frac{1}{\tilde{\gamma}_j^b}(p_i + \tilde{\mu}_j^b + c_i\tilde{\gamma}_j^b)$. Next, for all $j \in \mathcal{B}$, we discuss the following two possibilities: if $\bar{q}_j^b > 0$, we pick any i with $\bar{x}_{ij} > 0$, and condition (1.8c) implies that $\frac{1}{\tilde{\gamma}_j^b}(p_i + \tilde{\mu}_j^b + c_i\tilde{\gamma}_j^b) = F_{b_j}^{-1}(1 - \frac{\bar{q}_j^b}{b_j})$. Moreover, for any $i_2 : \bar{x}_{i_2 j} = 0$, condition (1.8d) implies that $\frac{1}{\tilde{\gamma}_j^b}(p_{i_2} + \tilde{\mu}_j^b + c_{i_2}\tilde{\gamma}_j^b) \geq F_{b_j}^{-1}(1 - \frac{\bar{q}_j^b}{b_j})$. Thus, this allows us to conclude that $\frac{1}{\tilde{\gamma}_j^b}(p_i + \tilde{\mu}_j^b + c_i\tilde{\gamma}_j^b) = \min_{i':(i',j)\in E} \frac{1}{\tilde{\gamma}_j^b}(p_{i'} + \tilde{\mu}_j^b + c_{i'}\tilde{\gamma}_j^b) = F_{b_j}^{-1}(1 - \frac{\bar{q}_j^b}{b_j})$ or equivalently $(1+\gamma_j^b)p_i + c_i + \mu_j^b = \min_{i':(i',j)\in E}\{(1+\gamma_j^b)p_{i'} + c_{i'}\} + \mu_j^b = F_{b_j}^{-1}(1 - \frac{\bar{q}_j^b}{b_j})$, which allows us to conclude that $F_{b_j}(\min_{i:(i,j)\in E}\{(1 + \gamma_j^b)p_i + c_i\} + \mu_j^b) = F_{b_j}(F_{b_j}^{-1}(1 - \frac{\bar{q}_j^b}{b_j})) = 1 - \frac{\bar{q}_j^b}{b_j}$. If $\bar{q}_j^b = 0$, then we have $\bar{x}_{ij} = 0$ for all i such that $(i, j) \in E$. By condition (1.8d), we have that $\min_{i:(i,j)\in E} \frac{1}{\tilde{\gamma}_j^b}(p_i + \tilde{\mu}_j^b + c_i\tilde{\gamma}_j^b) \geq F_{b_j}^{-1}(1 - \frac{\bar{q}_j^b}{b_j})$ or equivalently $\min_{i':(i',j)\in E} \frac{1}{\tilde{\gamma}_j^b}(p_{i'} + \tilde{\mu}_j^b + c_{i'}\tilde{\gamma}_j^b) \geq F_{b_j}^{-1}(1 - \frac{\bar{q}_j^b}{b_j})$. As F_{b_j} is increasing and $F_{b_j}(v) \in [0, 1]$ for all $v \in \mathbb{R}$ (Assumption 1), we deduce that $1 \geq F_{b_j}(\min_{i:(i,j)\in E}\{(1 + \gamma_j^b)p_i + c_i\} +$

$\mu_j^b) \geq F_{b_j}^{-1}(F_{b_j}(1-\frac{\bar{q}_j^b}{b_j})) = 1$. Thus, $F_{b_j}(\min_{i:(i,j)\in E}\{(1+\gamma_j^b)p_i + c_i\}) = 1 = 1 - \frac{\bar{q}_j^b}{b_j}$. In both cases, we conclude that (1.2b) holds.

3. Condition(1.2c): given that $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{q}}^s, \bar{\boldsymbol{q}}^b)$ satisfy constraints (1.38d) and (1.38e), we conclude that (1.2c) holds.

4. Condition(1.2d): the constraint (1.38f) implies that $\bar{x}_{ij} \geq 0$. While verifying the equilibrium condition (1.2b) we established that, if $\bar{x}_{ij} > 0$, then $(1+\gamma_j^b)p_i + c_i + \mu_j^b = \min_{i':(i',j)\in E}\{(1+\gamma_j^b)p_{i'} + c_{i'}\} + \mu_j^b$. Thus, (1.2d) holds.

As the equilibrium conditions (1.2a) - (1.2d) are satisfied, we conclude that $(\boldsymbol{p}, \bar{\boldsymbol{x}}, \bar{\boldsymbol{q}}, \bar{\boldsymbol{q}}^b) \in \mathcal{X}(\boldsymbol{\gamma}, \boldsymbol{\mu})$.

Finally, we argue that $(\boldsymbol{\gamma}, \boldsymbol{\mu})$ is optimal. By condition (1.8a), for any i such that $\bar{q}_i^s > 0$, since $\tilde{\gamma}_i^s = \frac{1}{1-\gamma_i^s}$ and $\tilde{\mu}_i^s = \frac{\mu_i^s}{1-\gamma_i^s}$, we have that $(1-\gamma_i^s)p_i - \mu_i^s = \frac{1}{\tilde{\gamma}_i^s}p_i - \frac{1}{\tilde{\gamma}_i^s}\tilde{\mu}_i^s = F_{s_i}^{-1}(\frac{\bar{q}_i^s}{s_i})$. Thus, it follows that $[(1-\gamma_i^s)p_i - \mu_i^s]\bar{q}_i^s = F_{s_i}^{-1}(\frac{\bar{q}_i^s}{s_i})\bar{q}_i^s$. By condition (1.8b), for any i with $\bar{q}_i^s = 0$, we have $[(1-\gamma_i^s)p_i - \mu_i^s]\bar{q}_i^s = 0 = F_{s_i}^{-1}(\frac{\bar{q}_i^s}{s_i})\bar{q}_i^s$. Since $(\boldsymbol{p}, \bar{\boldsymbol{x}}, \bar{\boldsymbol{q}}, \bar{\boldsymbol{q}}^b) \in \mathcal{X}(\boldsymbol{\gamma}, \boldsymbol{\mu})$, we establish that the inequality in step (b) of (1.138) is tight, which further implies that step (e) of (1.139) is tight. Thus, we conclude that $V(\boldsymbol{\gamma}, \boldsymbol{\mu}) = h(\bar{\boldsymbol{q}}^s, \bar{\boldsymbol{q}}^b) = V_{opt}$.

This completes the the proof of claim (ii). □

Proof of Corollary 2. By Lemma 4(ii), the revenue optimization problem (1.7) can be viewed as an instance of the framework problem (1.38) where Assumptions (1.12.1-1) -(1.12.1-3) and Assumptions (1.12.1-7) - (1.12.1-8) hold. Let $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{q}}^s, \bar{\boldsymbol{q}}^b)$ be an optimal solution to problem (1.7) and let $(\boldsymbol{\theta}^b, \boldsymbol{\theta}^s, \boldsymbol{\eta}^b, \boldsymbol{\eta}^s, \boldsymbol{\pi})$ be a corresponding dual optimal solution in the corollary statement, associated to constraints (1.38b) - (1.38f). By Lemma 5, $(\boldsymbol{\theta}^b, \boldsymbol{\theta}^s, \boldsymbol{\eta}^b, \boldsymbol{\eta}^s, \boldsymbol{\pi})$ is the unique dual optimal vector that satisfies the conditions in (1.40).

Claim (i) of the Corollary statement has been established in step (i-2) in the proof of Theorem 1, when we constructed a feasible solution to (1.3) that is optimal.

To establish claim (ii), consider the price vector $\boldsymbol{p}$ defined as

$$p_i := \theta_i^s, \quad \forall i \in \mathcal{S}. \tag{1.151}$$

and define the commission-only implementation where we set the subscriptions to be zero, i.e., $\boldsymbol{\mu} = \mathbf{0}$ and the commissions as

$$\gamma_i^s := 1 - \frac{1}{\theta_i^s} F_{s_i}^{-1}\left(\frac{\bar{q}_i^s}{s_i}\right) \quad \forall i \in \mathcal{S}, \tag{1.152a}$$

$$\gamma_j^b := \frac{1}{\theta_j^b} F_{b_j}^{-1}\left(1 - \frac{\bar{q}_j^b}{b_j}\right) - 1 \quad \forall j \in \mathcal{B}. \tag{1.152b}$$

We first establish that $(\boldsymbol{\gamma}^s, \boldsymbol{\gamma}^b) \in \Gamma$. Given that $c_i = 0$ for all $i \in \mathcal{S}$, Assumption (1.12.1-6) in the framework problem (1.38) holds. Note that $\theta_i^s = h_i'(\bar{q}_i^s) + \eta_i^s$ (condition (1.40b)) and $\eta_i^s \geq 0$ (condition (1.40e). By using Assumption 1, we obtain that F_{s_i} is continuously differentiable and strictly increasing in $(0, \bar{v}_{s_i})$, and by the inverse function theorem it follows that $F_{s_i}^{-1}$ is continuously differentiable and strictly increasing in $(0, 1)$. Proposition 12(iii) implies that $h_i'(\bar{q}_i^s)$ is finite. Recall that $h_i'(q)$ is continuous in $(0, s_i)$ (Assumption (1.12.1-2)), and $h_i'(0) = \lim_{q \downarrow 0} h_i'(q)$, $h_i'(s_i) = \lim_{q \uparrow s_i} h_i'(q)$ (by definition). Thus, there exists a sequence $\{q_k \in (0, s_i) : k = 1, 2, \dots\}$ such that $\lim_{k\to\infty} q_k = \bar{q}_i^s$ and $\lim_{k\to\infty} h_i'(q_k) = h_i'(\bar{q}_i^s)$. Given that $h_i(q) = F_{s_i}^{-1}(\frac{q}{s_i})q$, we deduce that

$$\begin{aligned}\theta_i^s &= h_i'(\bar{q}_i^s) + \eta_i^s = \lim_{k\to\infty} h_i'(q_k) + \eta_i^s \\ &= \lim_{k\to\infty} \left[F_{s_i}^{-1}\right]'\left(\frac{q_k}{s_i}\right)\frac{q_k}{s_i} + F_{s_i}^{-1}\left(\frac{q_k}{s_i}\right) + \eta_i^s \geq F_{s_i}^{-1}\left(\frac{\bar{q}_i^s}{s_i}\right), \quad \forall i \in \mathcal{S}.\end{aligned} \tag{1.153}$$

Since $\bar{q}_i^s > 0$ (Proposition 12(ix)) and θ_i^s is finite (Proposition 12(iii)), it follows that $F_{s_i}^{-1}\left(\frac{\bar{q}_i^s}{s_i}\right) > 0$. From (1.152a), (1.153) and $F_{s_i}^{-1}\left(\frac{\bar{q}_i^s}{s_i}\right) > 0$, we conclude that $1 - \gamma_i^s = \frac{1}{\theta_i^s} F_{s_i}^{-1}\left(\frac{\bar{q}_i^s}{s_i}\right) \in [0, 1)$ and thus $\gamma_i^s \in [0, 1)$.

Next, we show that $\gamma_j^b \geq 0$ for all $j \in \mathcal{B}$. Given that $\bar{q}_j^b < b_j$ (Proposition 12(v)) and $\eta_j^b = 0$ (condition (1.40d)) for all $j \in \mathcal{B}$, it follows that $\theta_j^b = g_j'(\bar{q}_j^b)$ (by condition (1.40a)). By Assumption 1, F_{b_j} is continuously differentiable and strictly increasing in $(0, \bar{v}_{b_j})$, and by the implicit function theorem it follows that $F_{b_j}^{-1}$ is continuously differentiable and strictly increasing in $(0,1)$. Proposition 12(iii) implies that $g_j'(\bar{q}_j^b)$ is finite. Recall that $g_j'(q)$ is continuous in $(0, b_j)$ (Assumption (1.12.1-1)), and $g_j'(0) = \lim_{q\downarrow 0} g_j'(q)$, $g_j'(b_j) = \lim_{q\uparrow b_j} g_j'(q)$ (by definition). Thus, there exists a sequence $\{q_k \in (0, b_j) : k = 1, 2, \dots\}$ such that $\lim_{k\to\infty} q_k = \bar{q}_j^b$ and $\lim_{k\to\infty} g_j'(q_k) = g_j'(\bar{q}_j^b)$. It follows that

$$\begin{aligned} \theta_j^b \;=\; g_j'(\bar{q}_j^b) \;&=\; \lim_{k\to\infty} g_j'(q_k) \\ &=\; \lim_{k\to\infty} -\left[F_{b_j}^{-1}\right]' \left(1 - \frac{q_k}{b_j}\right)\frac{q_k}{b_j} + F_{b_j}^{-1}\left(1 - \frac{q_k}{b_j}\right) \;\leq\; F_{b_j}^{-1}\left(1 - \frac{\bar{q}_j^b}{b_j}\right), \quad \forall j \in \mathcal{B}. \end{aligned} \tag{1.154}$$

All that remains is to show that $\theta_j^b > 0$, which we do by contradiction. Suppose that $\theta_j^b \leq 0$. Recall that $g_j(q) = F_{b_j}^{-1}\left(1 - \frac{q}{b_j}\right) q$ in this instance of the framework problem (1.38) associated to the revenue optimization problem (1.7). Given condition (1.154), it follows that $g_j'(\bar{q}_j^b) \leq 0$. From $g_j'(0) > 0$ (Assumption (1.12.1-8)) and the strict decreasing property of $g_j'(\cdot)$ (Assumption (1.12.1-1)), this further implies that $\bar{q}_j^b > 0$. Furthermore, we can find i with $\bar{x}_{ij} > 0$. Given that $h_i'(0) = 0$ (Assumption (1.12.1-7)) and the strictly increasing property of $h_i'(\cdot)$ (Assumption (1.12.1-2)), it follows that $h_i'(\bar{q}_i^s) > 0$. This implies that there exists $\delta \in (0, \bar{x}_{ij})$ such that we can decrease $(\bar{q}_j^b, \bar{q}_i^s, \bar{x}_{ij})$ simultaneously by δ such that $\bar{x}_{ij} := \bar{x}_{ij} - \delta$, $\bar{q}_i^s := \bar{q}_i^s - \delta$, and $\bar{q}_j^b := \bar{q}_j^b - \delta$. This would strictly increase the objective value, thereby leading to a contradiction with the optimality of $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{q}}^s, \bar{\boldsymbol{q}}^b)$. These observations imply that $\theta_j^b > 0$ for all $j \in \mathcal{B}$. Thus, we conclude that $1 + \gamma_j^b = \frac{1}{\theta_j^b} F_{b_j}^{-1}\left(1 - \frac{\bar{q}_j^b}{b_j}\right) \geq 1$, or correspondingly $\gamma_j^b \geq 0$. This allows us to conclude that $(\boldsymbol{\gamma}^s, \boldsymbol{\gamma}^b) \in \Gamma$, as desired.

Next, we verify that $(\boldsymbol{p}, \bar{\boldsymbol{x}}, \bar{\boldsymbol{q}}^s, \bar{\boldsymbol{q}}^b) \in \mathcal{X}(\boldsymbol{\gamma}, \mathbf{0})$ by showing that the equilibrium conditions (1.2a) - (1.2d) are satisfied:

(1) Condition (1.2a): for each $i \in \mathcal{S}$, we deduce that

$$s_i F_{s_i}\Big((1-\gamma_i^s)p_i\Big) \overset{(a)}{=} s_i F_{s_i}\Big((1-\gamma_i^s)\theta_i^s\Big) \overset{(b)}{=} s_i F_{s_i}\left(F_{s_i}^{-1}\left(\frac{\bar{q}_i^s}{s_i}\right)\right) = \bar{q}_i^s, \tag{1.155}$$

where step (a) follows from $p_i = \theta_i^s$ in (1.151). Step (b) follows from the construction of γ_i^s in (1.152a). Thus, the equilibrium condition (1.2a) holds.

(2) Condition (1.2b): for all $j \in \mathcal{B}$, we start from $\theta_i^s \geq \theta_j^b$ for all i with $(i,j) \in E$, which is obtained from $\theta_i^s - \theta_j^b + \pi_{ij} = 0$ (condition (1.40c)) and $\pi_{ij} \geq 0$ (condition (1.40d)). For each $j \in \mathcal{B}$, we discuss two cases. If $\bar{q}_j^b > 0$, then for all i with $\bar{x}_{ij} > 0$, we have $\pi_{ij} = 0$ (condition (1.40d)) and $\theta_j^b = \theta_i^s$ (condition (1.40c)). Thus, it follows that

$$\theta_j^b = \theta_i^s = \min_{i':(i',j)\in E}\{\theta_{i'}^s\}, \quad \forall i : \bar{x}_{ij} > 0. \tag{1.156}$$

Then, condition (1.2b) is satisfied as

$$\begin{aligned} b_j\Big[1 - F_{b_j}\Big((1+\gamma_j^b)\min_{i':(i',j)\in E}\{p_{i'}\}\Big)\Big] &\overset{(c)}{=} b_j\Big[1 - F_{b_j}\Big((1+\gamma_j^b)\min_{i':(i',j)\in E}\{\theta_{i'}^s\}\Big)\Big] \\ &\overset{(d)}{=} b_j\Big[1 - F_{b_j}\Big((1+\gamma_j^b)\theta_j^b\Big)\Big] \\ &\overset{(e)}{=} b_j\Big[1 - F_{b_j}\Big(F_{b_j}^{-1}\Big(1 - \frac{\bar{q}_j^b}{b_j}\Big)\Big)\Big] = \bar{q}_j^b, \end{aligned} \tag{1.157}$$

where step (c) follows from $p_i = \theta_i^s$ (the construction of p_i in (1.140)). Step (d) follows from condition (1.156). Step (e) follows from $(1+\gamma_j^b)\theta_j^b = F_{b_j}^{-1}(1-\frac{\bar{q}_j^b}{b_j})$ (the construction of γ_j^b in (1.152b)).

If $\bar{q}_j^b = 0$, we have $\bar{x}_{ij} = 0$ for all i with $(i,j) \in E$. Recall that $\theta_i^s \geq \theta_j^b$ for all i with $(i,j) \in E$ (condition (1.40c) and (1.40f)) and $p_i = \theta_i^s$ (the construction of p_i in (1.140)). Then, it follows that $\theta_j^b \leq \min_{i:(i,j)\in E}\{\theta_i^s\} = \min_{i:(i,j)\in E}\{p_i\}$. Since $(1+\gamma_j^b)\theta_j^b = F_{b_j}^{-1}(1-\frac{q_j^b}{b_j})$ (the construction of γ_j^b in (1.152b)), we have that $(1 +$

$\gamma_j^b) \min_{i:(i,j)\in E}\{p_i\} \geq (1+\gamma_j^b)\theta_j^b = F_{b_j}^{-1}(1-\frac{q_j^b}{b_j}) = F_{b_j}^{-1}(1)$, which further implies that

$$b_j\Big[1 - F_{b_j}\Big((1+\gamma_j^b)\min_{i':(i',j)\in E}\{p_{i'}\}\Big)\Big] = b_j\Big[1 - F_{b_j}\Big(F_{b_j}^{-1}\big(1\big)\Big)\Big] = 0 = \bar{q}_j^b. \quad (1.158)$$

Thus, we conclude that condition (1.2b) holds.

(3) Condition (1.2c): constraints (1.38b) and (1.38c) guarantee that (1.2c) is satisfied.

(4) Condition (1.2d): for all $(i,j) \in E$, $\bar{x}_{ij} \geq 0$ follows from constraint (1.38f). Moreover, if $\bar{x}_{ij} > 0$, we have established that $i \in \arg\min_{i':(i',j)\in E}\{\theta_{i'}^s\} = \arg\min_{i':(i',j)\in E}\{p_{i'}\}$ (when verifying condition (1.2b)). Thus, the equilibrium expression (1.2d) holds.

Therefore, we conclude that $(\boldsymbol{p}, \bar{\boldsymbol{x}}, \bar{\boldsymbol{q}}^s, \bar{\boldsymbol{q}}^b) \in \mathcal{X}(\boldsymbol{\gamma}, \boldsymbol{0})$.

To establish the optimality of $(\boldsymbol{\gamma}^s, \boldsymbol{\gamma}^b)$, recall the upper bound in step (e) of (1.139) in the proof of Theorem 1. For all $i \in \mathcal{S}$, given that $(1-\gamma_i^s)p_i = F_{s_i}^{-1}(\frac{\bar{q}_i^s}{s_i})$ (by the construction of γ_i^s in (1.152a)), step (b) of (1.138) (in the proof of Theorem 1) is tight, which further implies that step (e) of (1.139) (in the proof of Theorem 1) is also tight. Thus, we conclude that $V(\boldsymbol{\gamma}, \boldsymbol{\mu}) = h(\bar{\boldsymbol{q}}^s, \bar{\boldsymbol{q}}^b) = V_{opt}$. □

Proof of Proposition 2. We prove the claim for the optimal subscription fees defined in Corollary 2(i). The same arguments can be applied to establish the result for the optimal commission rates defined in Corollary 2(ii).

Let $\boldsymbol{\mu}$ be the optimal subscription fees defined in Corollary 2(i), and $(\boldsymbol{p}, \boldsymbol{x}, \boldsymbol{q}^s, \boldsymbol{q}^b)$ be any competitive equilibrium induced by $(\boldsymbol{0}, \boldsymbol{\mu})$. By Theorem 1(i), it readily follows that $(\boldsymbol{q}^s, \boldsymbol{q}^b)$ is the unique optimal supply/demand vector for the revenue optimization problem (1.7). We divide the proof arguments into the following steps.

Step 1: positivity of the equilibrium supply and the uniqueness of the price vector. We first establish that the equilibrium supply vector satisfies that $q_i^s > 0$ for all $i \in \mathcal{S}$. By Lemma 4(ii), the revenue optimization problem (1.7) can be viewed as an instance of the framework problem (1.38) where Assumptions (1.12.1-1) -(1.12.1-3) and Assumptions (1.12.1-7)

- (1.12.1-8) hold. Given that $F_{b_j}(v) = F_b(v)$, $F_{s_i}(v) = F_s(v)$ and $c_{ij} = 0$ (Assumption 3), we have $g_j(q) = b_j g(\frac{q}{b_j})$ with $g(r) = F_b^{-1}(1-r)r$, $h_i(q) = s_i h(\frac{q}{s_i})$ with $h(r) = F_s^{-1}(r)r$, and $w_{ij} = c_i = 0$. This implies that Assumptions (1.12.1-4) - (1.12.1-6) hold. Thus, by Proposition 12(ix), we establish that $q_i^s > 0$ for all $i \in \mathcal{S}$. Combining Proposition 9 (ii) with the observation that $\boldsymbol{q}^s > \boldsymbol{0}$, we obtain that equilibrium price vector $\boldsymbol{p}$ is unique.

Step 2: the connection with the dual optimal solution. Let $(\boldsymbol{\theta}^b, \boldsymbol{\theta}^s, \boldsymbol{\eta}^b, \boldsymbol{\eta}^s, \boldsymbol{\pi})$ be a corresponding dual optimal solution specified in Corollary 2, associated with constraints (1.38b) - (1.38f). By Lemma 5, $(\boldsymbol{\theta}^b, \boldsymbol{\theta}^s, \boldsymbol{\eta}^b, \boldsymbol{\eta}^s, \boldsymbol{\pi})$ is the unique dual optimal vector that satisfies the conditions in (1.40). Moreover, Step 1, together with (1.140) and the subsequent arguments in Theorem 1, implies that the unique equilibrium price induced by the optimal subscription fees $\boldsymbol{\mu}$ is given as $\boldsymbol{p} = \boldsymbol{\theta}^s$.

Step 3: the ranking of the equilibrium price vector. Let $\{(\mathcal{S}_\tau, \mathcal{B}_\tau) : \tau = 1, \ldots, \ell\}$ be the network components constructed in Theorem 2, where we established that all buyers in $\mathcal{B}_\tau$ only trade with sellers in $\mathcal{S}_\tau$. For any $i_1 \in \mathcal{S}_{\tau_1}$, $i_2 \in \mathcal{S}_{\tau_2}$ with $\tau_2 \geq \tau_1$, based on the observations that $q_{i_1}^s > 0$ and $q_{i_2}^s > 0$, we can pick $j_1 \in \mathcal{B}_{\tau_1}$ with $x_{i_1 j_1} > 0$ and $j_2 \in \mathcal{B}_{\tau_2}$ with $x_{i_2 j_2} > 0$. We conclude the claim by deducing that

$$p_{i_1} \overset{(a)}{=} \theta_{i_1}^s \overset{(b)}{=} \theta_{j_1}^b \overset{(c)}{=} g'\left(\frac{q_{j_1}^b}{b_{j_1}}\right) \overset{(d)}{\geq} g'\left(\frac{q_{j_2}^b}{b_{j_2}}\right) \overset{(e)}{=} \theta_{j_2}^b \overset{(f)}{=} \theta_{i_2}^s \overset{(g)}{=} p_{i_2}, \tag{1.159}$$

where step (a) and step (g) follow directly from the conclusion in step 2 of this proof. In step (b) and step (f), we use the fact that $x_{i_1 j_1}, x_{i_2 j_2} > 0$, to obtain that $\pi_{i_1 j_1} = \pi_{i_2 j_2} = 0$ (by condition (1.40f)). Together with $w_{i_1 j_1} = w_{i_2 j_2} = 0$ (by Assumption 3) and condition (1.40c), we obtain equalities (b) and (f). In step (c) and step (e), we have that $q_{j_1}^b < b_{j_1}$ and $q_{j_2}^b < b_{j_2}$ (by Proposition 12(v)), which implies that $\eta_{j_1}^b = \eta_{j_2}^b = 0$ (by condition (1.40d)). From condition (1.40a), equalities (c) and (e) readily follow. In (d), we use the fact that $F_b^{-1}(r)$ is strictly increasing in $r \in (0,1)$ (by Assumption 1), $F_b^{-1}(1-\frac{q_{j_1}^b}{b_{j_1}}) = \bar{v}_{b_{j_1}}^m \geq \bar{v}_{b_{j_2}}^m = F_b^{-1}(1-\frac{q_{j_2}^b}{b_{j_2}})$ (by the inequalities (1.22) - (1.23) in Theorem 2), and $\frac{q_{j_1}^b}{b_{j_1}}, \frac{q_{j_2}^b}{b_{j_2}} \in (0,1)$. Note that these observations readily imply that $\frac{q^b_{j_1}}{b_{j_1}} \leq \frac{q^b_{j_2}}{b_{j_2}}$. Given that $g'(r)$ is strictly decreasing in $r \in (0,1)$ (by Assumption (1.12.1-1)), inequality (d) immediately follows. When $\tau_1 \neq \tau_2$, the strict inequality (1.23) in Theorem 2 and the fact that $F_b^{-1}(r)$ is strictly increasing in $r \in (0,1)$ imply that $\frac{q^b_{j_1}}{b_{j_1}} < \frac{q^b_{j_2}}{b_{j_2}}$ and hence inequality (d) is strict. □

Proof of Corollary 3. By Lemma 4(ii), the revenue optimization problem (1.7) can be formulated as an instance of the framework problem (1.38) where $g_j(q) = F_{b_j}^{-1}(1 - \frac{q}{b_j})q$ for all $j \in \mathcal{B}$, $h_i(q) = F_{s_i}^{-1}(\frac{q}{s_i})q$ for all $i \in \mathcal{S}$ and $w_{ij} = c_i$ for all $(i,j) \in E$. Moreover, Assumptions (1.12.1-1) - (1.12.1-3), and Assumptions (1.12.1-7) - (1.12.1-8) hold. Given $F_{b_j}(v) = F_b(v)$, $F_{s_i}(v) = F_s(v)$ and $c_{ij} = 0$ (Assumption 3), we have $g_j(q) = b_j g(\frac{q}{b_j})$ with $g(r) = F_b^{-1}(1-r)r$, $h_i(q) = s_i h(\frac{q}{s_i})$ with $h(r) = F_s^{-1}(r)r$, and $w_{ij} = c_i = 0$. This implies that Assumptions (1.12.1-4) - (1.12.1-6) hold. From Proposition 10(i), it follows that Assumptions (1.91) - (1.94) also hold.

We let $(\boldsymbol{x}, \boldsymbol{q}^s, \boldsymbol{q}^b)$ be an optimal solution to problem (1.38) and $(\boldsymbol{\theta}^s, \boldsymbol{\theta}^b, \boldsymbol{\eta}^s, \boldsymbol{\eta}^b, \boldsymbol{\pi})$ be the corresponding dual optimal solution to constraints (1.38b) - (1.38f) that satisfies the conditions in (1.40).

Step 1: the Lagrangian of the framework problem. In the framework problem (1.38), we define the corresponding Lagrangian as

$$\begin{aligned}\mathcal{L}(\bar{\boldsymbol{x}}, \bar{\boldsymbol{q}}^s, \bar{\boldsymbol{q}}^b, \bar{\boldsymbol{\theta}}^s, \bar{\boldsymbol{\theta}}^b, \bar{\boldsymbol{\eta}}^s, \bar{\boldsymbol{\eta}}^b, \bar{\boldsymbol{\pi}}, \boldsymbol{s}, \boldsymbol{b}) \;:=\; & \sum_{j \in \mathcal{B}} \left[b_j g\left(\frac{\bar{q}^b_j}{b_j}\right) - \bar{\theta}^b_j \Big(\bar{q}^b_j - \sum_{i:(i,j)\in E} \bar{x}_{ij} \Big) - \bar{\eta}^b_j \left(\bar{q}^b_j - b_j \right) \right] \\ & + \sum_{i \in \mathcal{S}} \left[-s_i h\left(\frac{\bar{q}^s_i}{s_i}\right) + \theta^s_i \Big(\bar{q}^s_i - \sum_{j:(i,j)\in E} \bar{x}_{ij} \Big) - \bar{\eta}^s_i \left(\bar{q}^s_i - s_i \right) \right] \\ & + \sum_{(i,j)\in E} \Big(-w_{ij}\bar{x}_{ij} + \pi_{ij}\bar{x}_{ij} \Big). \qquad (1.160)\end{aligned}$$

For all $i \in \mathcal{S}$, we deduce from (1.160) that

$$\frac{\partial}{\partial s_i} V_{opt}(\boldsymbol{s}, \boldsymbol{b}) \;=\; -h\left(\frac{q^s_i}{s_i}\right) + \frac{q^s_i}{s_i} h'\left(\frac{q^s_i}{s_i}\right) + \eta^s_i. \qquad (1.161)$$

Here, since $V_{opt}(\boldsymbol{s}, \boldsymbol{b})$ is differentiable at s_i (by the assumption of the corollary), we apply the envelope theorem to problem (1.38) to obtain the right-hand-side expression. By Proposition 12(iii), $h'(\frac{q_i^s}{s_i})$ has a finite value, and it follows that $h(\frac{q_i^s}{s_i})$ is also finite at the optimal solution.

Similarly, for any $j \in \mathcal{B}$, we deduce that

$$\frac{\partial}{\partial b_j} V_{opt}(\boldsymbol{s}, \boldsymbol{b}) \ = g\left(\frac{q_j}{b_j}\right) - \frac{q_j}{b_j} g'\left(\frac{q_j}{b_j}\right). \tag{1.162}$$

Since $V_{opt}(\boldsymbol{s}, \boldsymbol{b})$ is differentiable at b_j (by the assumption of the corollary), the right-hand-side expression follows from the envelope theorem. By Proposition 12(iii), $g'(\frac{q_j^b}{b_j})$ is finite, and it follows that $g(\frac{q_j}{b_j})$ is also finite at the optimal solution.

Let $\{(\mathcal{S}_\tau, \mathcal{B}_\tau) : \tau = 1, \ldots, l\}$ be the set of components constructed in Theorem 2. Furthermore, let $\bar{\tau}$ be defined as in the proof of Theorem 2 (in the paragraph before (1.24)). Recall that trades only happen between the sellers in $\mathcal{S}_\tau$ and the buyers in $\mathcal{B}_\tau$ for $\tau \in \{1, 2, \ldots, l\}$.

Step 2: the ranking of $\frac{\partial}{\partial s_i} V_{opt}(\boldsymbol{s}, \boldsymbol{b})$. Before proceeding, we establish that function $H(t) = -h(t) + th'(t)$ is strictly increasing for $t \in (0, 1)$. For any $t_1, t_2 \in (0, 1)$ with $t_1 > t_2$, since $h(t)$ is continuously differentiable and strict convex in $t \in (0, 1)$ (Assumption (1.92)), we apply the mean value theorem to obtain that there exists $h_0 \in [h'(t_1), h'(t_2)]$ such that $h(t_1) - h(t_2) = h_0(t_1 - t_2)$, which further implies that $H(t_1) - H(t_2) = -h(t_1) + h(t_2) + t_1 h'(t_1) - t_2 h'(t_2) = t_1(h'(t_1) - h_0) + t_2(h_0 - h'(t_2)) > 0$. Thus, function $H(t)$ is strictly increasing for $t \in \{t' : t' \in [0, 1], |H(t')| < \infty\}$.

case 2-(i): if there exists $\tau \in \{1, \ldots, l\}$ such that $\tau \leq \bar{\tau}$. For any $i \in \mathcal{S}_\tau$, recalling (1.24) of Theorem 2 that $q_i^s = s_i$, we can pick any j such that $x_{ij} > 0$ and $j \in \mathcal{B}_\tau$. Based on

(1.161), we further deduce that

$$\begin{aligned}\frac{\partial}{\partial s_i}V_{opt}(\boldsymbol{s},\boldsymbol{b}) &= -h\left(\frac{q_i^s}{s_i}\right)+\frac{q_i^s}{s_i}h'\left(\frac{q_i^s}{s_i}\right)+\eta_i^s \\ &\overset{(a)}{=} -h\left(\frac{q_i^s}{s_i}\right)+\frac{q_i^s}{s_i}h'\left(\frac{q_i^s}{s_i}\right)+\frac{q_i^s}{s_i}\eta_i^s \\ &\overset{(b)}{=} -h\left(\frac{q_i^s}{s_i}\right)+\frac{q_i^s}{s_i}\theta_i^s \overset{(c)}{=} -h\left(1\right)+\theta_j^b \overset{(d)}{=} -h(1)+g'\left(\frac{q_j^b}{b_j}\right), \end{aligned} \tag{1.163}$$

where step (a) follows from $\eta_i^s(q_i^s - s_i) = 0$ (the complementary condition (1.40e)). Step (b) follows from $h'(\frac{q_i^s}{s_i}) - \theta_i^s + \eta_i^s = 0$ (condition (1.40b)). In step (c), we first implement $\frac{q_i^s}{s_i} = 1$. Since $x_{ij} > 0$, we have $\pi_{ij} = 0$ (the complementary condition (1.40f)), which further suggests that $\theta_i^s = \theta_j^b$ (condition (1.40c)). In step (d), from $q_j^b < b_j$ (Proposition 12(v)), we obtain that $\eta_j^b = 0$ (the complementary condition (1.40d)). By condition (1.40a), we have $\theta_j^b = g'(\frac{q_j^b}{b_j})$.

For any $i_1 \in \mathcal{S}_{\tau_1}$ and $i_2 \in \mathcal{S}_{\tau_2}$ with $\tau_1 < \tau_2 \le \bar{\tau}$. Pick any $j_1 : x_{i_1 j_1} > 0$ such that $j_1 \in \mathcal{B}_{\tau_1}$ and $j_2 : x_{i_2 j_2} > 0$ such that $j_2 \in \mathcal{B}_{\tau_2}$. Recall from (1.23) of Theorem 2 that $\frac{q_{j_1}^b}{b_{j_1}} < \frac{q_{j_2}^b}{b_{j_2}}$. By the strict decreasingness of $g'(\cdot)$ in $(0,1)$ (Assumption (1.92)), we obtain that $g'(\frac{q_{j_1}^b}{b_{j_1}}) > g'(\frac{q_{j_2}^b}{b_{j_2}})$. By (1.163), it follows that $\frac{\partial}{\partial s_{i_1}}V_{opt}(\boldsymbol{s},\boldsymbol{b}) > \frac{\partial}{\partial s_{i_2}}V_{opt}(\boldsymbol{s},\boldsymbol{b})$;

<u>case 2-(ii):</u> if ther exists $\tau \in \{1, 2, \ldots, l\}$ such that $\tau > \bar{\tau}$. For all $i \in \mathcal{S}_\tau$, recall from the discussion before (1.24) of Theorem 2 that $q_i^s < s_i$. This implies that $\eta_i^s = 0$(the complementary condition (1.40e)).

Hence, for any $i_1 \in \mathcal{S}_{\tau_1}$ and $i_2 \in \mathcal{S}_{\tau_2}$ where $\bar{\tau} < \tau_1 < \tau_2$. Condition (1.26) of Theorem 2 establishes that $\frac{q_{i_1}^s}{s_{i_1}} > \frac{q_{i_2}^s}{s_{i_2}}$. From the strict increasingness of $H(t)$ in $t \in \{t' : t' \in [0,1], |H(t')| < \infty\}$, (1.161), and $\eta_{i_1}^s = \eta_{i_2}^s = 0$, we conclude that $\frac{\partial}{\partial s_{i_1}}V_{opt}(\boldsymbol{s},\boldsymbol{b}) > \frac{\partial}{\partial s_{i_2}}V_{opt}(\boldsymbol{s},\boldsymbol{b})$;

<u>case 2-(iii):</u> to complete the analysis, if there exists $i_1 \in \mathcal{S}_{\tau_1}$ and $i_2 \in \mathcal{S}_{\tau_2}$ where $\tau_1 \le \bar{\tau} < \tau_2$. Combining the observations above, from $q_{i_1}^s = s_{i_1}$, $\eta_{i_1}^s \ge 0$, $q_{i_2}^s < s_{i_2}$, $\eta_{i_2}^s = 0$, the strict increasingness of $H(t)$ in $t \in \{t' : t' \in [0,1], |H(t')| < \infty\}$, and (1.161), it follows that

$\frac{\partial}{\partial s_{i_1}} V_{opt}(\boldsymbol{s}, \boldsymbol{b}) = H(1) + \eta^s_{i_1} \geq H(1) > H(\frac{q^s_{i_2}}{s_{i_2}}) = \frac{\partial}{\partial s_{i_2}} V_{opt}(\boldsymbol{s}, \boldsymbol{b})$.

Combining these observations, we conclude that $\frac{\partial}{\partial s_i} V_{opt}(\boldsymbol{s}, \boldsymbol{b})$ is the τ^{th} highest if and only if $i \in \mathcal{S}_\tau$.

Step 3: the ranking of $\frac{\partial}{\partial b_j} V_{opt}(\boldsymbol{s}, \boldsymbol{b})$. From (1.162), we first establish that function $G(t) = g(t) - tg'(t)$ is strictly increasing in $t \in (0, 1)$. For any $t_1, t_2 \in (0, 1)$ where $t_1 > t_2$, we show that $G(t_1) > G(t_2)$. By the continuous differentiability and the strict concavity property of function $g(t)$ in $(0, 1)$(Assumption (1.91)), we can apply the mean value theorem to obtain that there exists $g_0 \in [g'(t_1), g'(t_2)]$ such that $g(t_1) - g(t_2) = g_0(t_1 - t_2)$. This implies that $G(t_1) - G(t_2) = g(t_1) - g(t_2) - t_1 g'(t_1) + t_2 g'(t_2) = t_1(g_0 - g'(t_1)) + t_2 g'(t_2) - g_0) > 0$. Thus, $G(t)$ is strictly increasing in $t \in \{t' \in [0, 1], |G(t')| < \infty\}$.

For any $j_1 \in \mathcal{B}_{\tau_1}$ and $j_2 \in \mathcal{B}_{\tau_2}$ with $\tau_1 < \tau_2$, recall from (1.23) of Theorem 2 that $\frac{q^b_{j_1}}{b_{j_1}} < \frac{q^b_{j_2}}{b_{j_2}}$. From (1.162) and the strict increasingness of $G(t)$ in $t \in \{t' \in [0, 1], |G(t')| < \infty\}$, we conclude that $\frac{\partial}{\partial b_{j_1}} V_{opt}(\boldsymbol{s}, \boldsymbol{b}) = G(\frac{q^b_{j_1}}{b_{j_1}}) < G(\frac{q^b_{j_2}}{b_{j_2}}) = \frac{\partial}{\partial b_{j_2}} V_{opt}(\boldsymbol{s}, \boldsymbol{b})$.

In summary, we conclude that $\frac{\partial}{\partial b_j} V_{opt}(\boldsymbol{s}, \boldsymbol{b})$ is the τ^{th} lowest if and only if $j \in \mathcal{B}_\tau$. $\square$

1.14.2 Counter Example for Commission-only Implementation.

Consider a complete bipartite network with 2 seller types and 2 buyer types. Let the population profiles be given by $\boldsymbol{s} = (5.6, 3.8)$ and $\boldsymbol{b} = (4.2, 5.8)$, and the value distribution be defined as $F_{s_i}(v) = v$ for all $i \in \{1, 2\}$ and $F_{b_j}(v) = \lambda_{b_j} v$ for $j \in \{1, 2\}$ with $(\lambda_{b_1}, \lambda_{b_2}) = (0.7, 0.2)$. Let the cost vector be $(c_1, c_2) = (0.9, 0.7)$.

The revenue optimization problem (1.7) can then be expressed as

$$\begin{aligned}
\max_{(\boldsymbol{q}^s,\boldsymbol{q}^b,\boldsymbol{x})} \quad & \frac{1}{0.7}\left(1-\frac{q_1^b}{4.2}\right)q_1^b + \frac{1}{0.2}\left(1-\frac{q_2^b}{5.8}\right)q_2^b - \frac{q_1^s}{5.6}q_1^s - \frac{q_2^s}{3.8}q_2^s \\
& - 0.9(x_{11}+x_{12}) - 0.7(x_{21}+x_{22}) \\
& x_{11}+x_{12} = q_1^s, \quad x_{21}+x_{22} = q_2^s, \quad x_{11}+x_{21} = q_1^b, \quad x_{12}+x_{22} = q_2^b, \\
& q_1^s \le 5.6, \quad q_2^s \le 3.8, \quad q_1^b \le 4.2, \quad q_2^b \le 5.8, \\
& x_{11}, x_{12}, x_{21}, x_{22} \ge 0. \qquad (1.164)
\end{aligned}$$

Solving for the revenue optimal solution (1.7) (or, equivalently, (1.164)), we obtain the following primal optimal solution $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{q}}^s, \bar{\boldsymbol{q}}^b)$ and dual optimal solution $(\boldsymbol{\theta}^s, \boldsymbol{\theta}^b, \boldsymbol{\eta}^s, \boldsymbol{\eta}^b, \boldsymbol{\pi})$:

$$\begin{aligned}
(\bar{q}_1^s, \bar{q}_2^s) &= (1.15, 1.16), \quad (\bar{q}_1^b, \bar{q}_2^b) = (0.17, 2.14), \\
(\bar{x}_{11}, \bar{x}_{21}, \bar{x}_{12}, \bar{x}_{22}) &= (0.09, 0.08, 1.06, 1.08), \\
(\theta_1^s, \theta_2^s) &= (0.41, 0.61), \quad (\theta_1^b, \theta_2^b) = (1.31, 1.31), \\
(\eta_1^s, \eta_2^s) &= (0, 0), \quad (\eta_1^b, \eta_2^b) = (0, 0), \\
(\pi_{11}, \pi_{21}, \pi_{12}, \pi_{22}) &= (0, 0, 0, 0). \qquad (1.165)
\end{aligned}$$

Suppose towards contradiction that there exists $(\boldsymbol{\gamma}, \boldsymbol{0}) \in \Gamma \times \mathcal{U}$ and $\boldsymbol{p} \in \mathbb{R}_+^n$ such that $(\boldsymbol{p}, \bar{\boldsymbol{x}}, \bar{\boldsymbol{q}}^s, \bar{\boldsymbol{q}}^b) \in \mathcal{X}(\boldsymbol{\gamma}, \boldsymbol{0})$. By Theorem 1(ii), defining $\tilde{\gamma}_i^s := \frac{1}{1-\gamma_i^s}$, $\tilde{\mu}_i^s := \frac{\mu_i^s}{1-\gamma_i^s} = 0$ for all $i \in \mathcal{S}$, and $\tilde{\gamma}_j^b := \frac{1}{1+\gamma_j^b}$ and $\tilde{\mu}_j^b := \frac{\mu_j^b}{1+\gamma_j^b} = 0$, vector $(\boldsymbol{p}, \tilde{\boldsymbol{\gamma}}^b)$ must satisfy conditions (1.8c) and (1.8f).

With $(\bar{q}_1^b, \bar{q}_2^b) = (0.17, 2.14)$ and $(c_1, c_2) = (0.9, 0.7)$, conditions (1.8c) and (1.8f) correspond to

$$p_1 \ \ge\ 0, p_2 \ \ge\ 0, \tilde{\gamma}_1^b \ >\ 0, \tilde{\gamma}_2^b \ >\ 0 \qquad (1.166a)$$

$$p_1 - 0.47\tilde{\gamma}_1^b = 0, \quad p_2 - 0.67\tilde{\gamma}_1^b = 0, \quad p_1 - 2.26\tilde{\gamma}_2^b = 0, \quad p_2 - 2.46\tilde{\gamma}_2^b = 0, \qquad (1.166b)$$

To show a contradiction, we use $p_1 - 2.26\nu_2^b = 0$ and $p_2 - 2.46\nu_2^b = 0$ to obtain that

$$p_2 = 1.09p_1. \tag{1.167}$$

From $p_1 - 0.47\nu_1^b = 0$ and $p_2 - 0.67\nu_1^b = 0$, we obtain

$$p_2 = 1.43p_1. \tag{1.168}$$

Thus, the only solution that can possibly satisfy (1.166) is $(p_1, p_2, \tilde{\gamma}_1^b, \tilde{\gamma}_2^b) = (0, 0, 0, 0)$, which contradicts the fact that $\tilde{\gamma}_1^b, \tilde{\gamma}_2^b > 0$. Therefore, there is no $(\boldsymbol{\gamma}, \mathbf{0}) \in \Gamma \times \mathcal{U}$ and $\boldsymbol{p} \in \mathbb{R}_+^n$ such that $(\boldsymbol{p}, \bar{\boldsymbol{x}}, \bar{\boldsymbol{q}}^s, \bar{\boldsymbol{q}}^b) \in \mathcal{X}(\boldsymbol{\gamma}, \mathbf{0})$.

1.15 Proofs of Results in Section 1.6

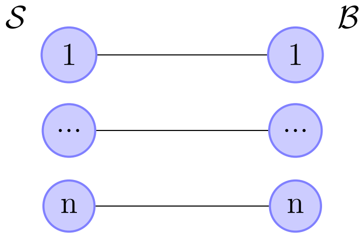
Proof of Proposition 4. The proof we will proceed as follows. We first define a class of networks, indexed by n; the n^{th} network in the class will consist of n buyer types and n seller types. Let $V_{opt}(n)$ denote the optimal revenue of the n^{th} network, and let $V_h(n)$ denote the optimal revenue *under homogeneous commissions-subscriptions* of the the n^{th} network. We will show that $V_h(n)/V_{opt}(n) \in \mathcal{O}(1/n)$; we do so by providing a lower bound for $V_{opt}(n)$ and an upper bound for $V_h(n)$. Therefore, we have that $\lim_{n\to\infty} V_h(n)/V_{opt}(n) \to 0$, which implies our result.

Introducing the class of networks: Consider the following class of networks indexed by n. The n^{th} network is a collection of n network components (in particular, of components $i = 1, \ldots, n$) where the i^{th} component $G(\mathcal{S}_i \cup \mathcal{B}_i, E_i)$ consists of one seller type and one buyer type (see the network below). We set $c_i = 0$ for all $i \in \mathcal{S}$. For each $i \in \{1, \ldots, n\}$, we define the distributions as

$$\begin{aligned} F_{s_i}(v) &:= \min\left\{\frac{v}{\bar{v}_{s_i}}, 1\right\}, \qquad \text{for any } v \geq 0 \text{ with } \bar{v}_{s_i} = 4^n, \\ F_{b_i}(v) &:= \min\left\{\frac{v}{\bar{v}_{b_i}}, 1\right\}, \qquad \text{for any } v \geq 0 \text{ with } \bar{v}_{b_i} = 4^i. \end{aligned} \tag{1.169}$$

and the population sizes to be

$$s_i := \bar{v}_{s_i} 8^{-i} \qquad b_i := \bar{v}_{b_i} 16^{-i}. \tag{1.170}$$



Simplifying the expressions for the equilibrium revenues. Next, we use the special structure of our networks to simplify the expressions for the revenues generated at equilibrium for any feasible commission-subscription pair $(\boldsymbol{\gamma}, \boldsymbol{\mu}) \in \Gamma \times \mathcal{U}$ where $\gamma_i^s < 1$ for all $i \in \mathcal{S}$. By Proposition 1, there exists a price vector $(p_i)_{i=1}^n$ with type-i supply given by $q_i^s = s_i \min\{\frac{1}{\bar{v}_{s_i}}[(1-\gamma_i^s)p_i - \mu_i^s]^+, 1\}$ (condition (1.2a)), and type-i demand equal to $q_i^b = b_i[1 - \min\{\frac{1}{\bar{v}_{b_i}}[(1+\gamma_i^b)p_i + \mu_i^b], 1\}]$ (as per (1.2b)), and flow conservation $q_i^s = x_{ii} = q_i^b$ (as per (1.2c)).

We can leverage the special structure of our instances to simplify these expressions by considering the following two properties of the competitive equilibrium:

1. assuming $(1-\gamma_i^s)p_i - \mu_i^s \geq 0$ for all $i \in \{1, \ldots, n\}$ is without loss of generality. Suppose there exists $i \in \{1, \ldots, n\}$ such that $(1-\gamma_i^s)p_i - \mu_i^s < 0$, then we have $q_i^s = 0$ and thus $q_i^b = 0$ and $(1+\gamma_i^b)p_i + \mu_i^b \geq \bar{v}_{b_i}$. Given that $\gamma_i^s < 1$, we can consider an alternative equilibrium by increasing p_i to the level where $(1-\gamma_i^s)p_i - \mu_i^s = 0$. This does not

change the equilibrium supply as $q_i^s = 0$ and thus, it is without loss of generality to consider $(1-\gamma_i^s)p_i - \mu_i^s \geq 0$ for all $i \in \{1,\dots,n\}$;

2. assuming $(1-\gamma_i^s)p_i - \mu_i^s \leq \bar{v}_{s_i}$ for all $i \in \{1,\dots,n\}$ is without loss of optimality. Suppose towards contradiction that there exits i such that $(1-\gamma_i^s)p_i - \mu_i^s > \bar{v}_{s_i}$. On the supply side, the assumption implies that $x_{ii} = q_i^s = s_i > 0$. However, on the demand side, given that $\frac{1+\gamma_i^b}{1-\gamma_i^s} \geq 1$ and $\mu_i^s, \mu_i^b \geq 0$ (by feasibility of the commissions-subscriptions) and $4^n = \bar{v}_{s_i} \geq \bar{v}_{b_i} = 4^i$, the assumption implies that $(1+\gamma_i^b)p_i + \mu_i^b \geq (1+\gamma_i^b)\frac{\bar{v}_{s_i}+\mu_i^s}{1-\gamma_i^s} + \mu_i^b \geq \bar{v}_{b_i}$, which further implies $x_{ii} = q_i^b = 0$. As a result, we achieve a contradiction from $0 < x_{ii} = 0$. Thus, we have $(1-\gamma_i^s)p_i - \mu_i^s \leq \bar{v}_{s_i}$.

Leveraging these two simplifications, we end up with competitive equilibria where, for $i \in \{1,\dots,n\}$, we have

$$s_i \frac{1}{\bar{v}_{s_i}} \left[(1-\gamma_i^s)p_i - \mu_i^s\right] = x_{ii} = b_i - b_i \min\left\{\frac{1}{\bar{v}_{b_i}}\left[(1+\gamma_i^b)p_i + \mu_i^b\right], 1\right\}. \tag{1.171}$$

Reorganizing the expression in (1.171) we can obtain the following expressions for (p_i, x_{ii}) for $i \in \{1,\dots,n\}$

$$\begin{aligned} p_i &= \begin{cases} \frac{b_i - (b_i/\bar{v}_{b_i})\mu_i^b + (s_i/\bar{v}_{s_i})\mu_i^s}{(s_i/\bar{v}_{s_i})(1-\gamma_i^s) + (b_i/\bar{v}_{b_i})(1+\gamma_i^b)}, & \text{if } \frac{1+\gamma_i^b}{1-\gamma_i^s}\mu_i^s + \mu_i^b \leq \bar{v}_{b_i} \\ \frac{\mu_i^s}{1-\gamma_i^s}, & \text{if } \frac{1+\gamma_i^b}{1-\gamma_i^s}\mu_i^s + \mu_i^b \geq \bar{v}_{b_i} \end{cases} \\ x_{ii} &= \begin{cases} (s_i/\bar{v}_{s_i})\frac{(1-\gamma_i^s)b_i - (b_i/\bar{v}_{b_i})[(1-\gamma_i^s)\mu_i^b + (1+\gamma_i^b)\mu_i^s]}{(s_i/\bar{v}_{s_i})(1-\gamma_i^s) + (b_i/\bar{v}_{b_i})(1+\gamma_i^b)} & \text{if } \frac{1+\gamma_i^b}{1-\gamma_i^s}\mu_i^s + \mu_i^b \leq \bar{v}_{b_i} \\ 0 & \text{if } \frac{1+\gamma_i^b}{1-\gamma_i^s}\mu_i^s + \mu_i^b \geq \bar{v}_{b_i} \end{cases} \end{aligned} \tag{1.172}$$

To ease notation, define:

$$r_i := \frac{1+\gamma_i^b}{1-\gamma_i^s} - 1, \tag{1.173a}$$

$$\xi_i := (r_i + 1)\mu_i^s + \mu_i^b. \tag{1.173b}$$

Let V_i be the revenue generated by the i^{th} network component $G(\mathcal{S}_i \cup \mathcal{B}_i, E_i)$ in a competitive equilibrium. We can show that the expression for V_i satisfies

$$
\begin{aligned}
V_i \; &= \Big[(\gamma_i^s + \gamma_i^b)p_i + (\mu_i^s + \mu_i^b)\Big]x_{ii} \\
&\overset{(a)}{=} \begin{cases} \Big[[1+\gamma_i^b - (1-\gamma_i^s)]\frac{b_i - (b_i/\bar{v}_{b_i})\mu_i^b + (s_i/\bar{v}_{s_i})\mu_i^s}{(s_i/\bar{v}_{s_i})(1-\gamma_i^s) + (b_i/\bar{v}_{b_i})(1+\gamma_i^b)} + (\mu_i^s + \mu_i^b)\Big] & \\ \cdot (s_i/\bar{v}_{s_i})\frac{(1-\gamma_i^s)b_i - (b_i/\bar{v}_{b_i})[(1-\gamma_i^s)\mu_i^b + (1+\gamma_i^b)\mu_i^s]}{(s_i/\bar{v}_{s_i})(1-\gamma_i^s) + (b_i/\bar{v}_{b_i})(1+\gamma_i^b)} & \text{if } \frac{1+\gamma_i^b}{1-\gamma_i^s}\mu_i^s + \mu_i^b \le \bar{v}_{b_i} \\ & \\ 0 & \text{if } \frac{1+\gamma_i^b}{1-\gamma_i^s}\mu_i^s + \mu_i^b \ge \bar{v}_{b_i} \end{cases} \\
&\overset{(b)}{=} \begin{cases} \frac{(s_i/\bar{v}_{s_i})(b_i/\bar{v}_{b_i})}{[(s_i/\bar{v}_{s_i}) + (r_i+1)(b_i/\bar{v}_{b_i})]^2} \cdot (\bar{v}_i^b - \xi_i)[r_i b_i + \xi_i(s_i/\bar{v}_{s_i} + b_i/\bar{v}_{b_i})] & \text{if } \xi_i \le \bar{v}_{b_i} \\ & \\ 0 & \text{if } \xi_i \ge \bar{v}_{b_i} \end{cases} \\
&\overset{(c)}{=} \frac{(s_i/\bar{v}_{s_i})(b_i/\bar{v}_{b_i})}{[(s_i/\bar{v}_{s_i}) + (r_i+1)(b_i/\bar{v}_{b_i})]^2} \cdot (\bar{v}_i^b - \xi_i)^+[r_i b_i + \xi_i(s_i/\bar{v}_{s_i} + b_i/\bar{v}_{b_i})] \\
&\overset{(d)}{=} \frac{8^{-i}16^{-i}}{[8^{-i} + (1+r_i)16^{-i}]^2}(4^i - \xi_i)^+[4^{-i}r_i + (8^{-i} + 16^{-i})\xi_i] \\
&= \frac{8^{-i}}{[2^i + (1+r_i)]^2}(4^i - \xi_i)^+[4^i r_i + (2^i+1)\xi_i], \qquad (1.174)
\end{aligned}
$$

where step (a) follows from replacing with the expressions of p_i and x_{ii} in (1.172). Step (b) follows from reorganizing of the expression in step (a) using the changes of variables $r_i = \frac{1+\gamma_i^b}{1-\gamma_i^s} - 1$ and $\xi_i = (r_i+1)\mu_i^s + \mu_i^b$. Step (c) follows from writing the expression in step (b) compactly as $(\bar{v}_i^b - \xi_i)^+ = \max\{\bar{v}_i^b - \xi_i, 0\}$. Step (d) follows from plugging in the values of $\bar{v}_{s_i} = 4^n$, $\bar{v}_{b_i} = 4^i$, $s_i/\bar{v}_{s_i} = 8^{-i}$ and $b_i/\bar{v}_{b_i} = 16^{-i}$.

Abusing some notation, we can express the equilibrium revenue in network component $G(\mathcal{S}_i \cup \mathcal{B}_i, E_i)$ as a function of (r_i, ξ_i) as:

$$V_i(r_i, \xi_i) \; = \; \frac{8^{-i}}{[2^i + (1+r_i)]^2}(4^i - \xi_i)^+[4^i r_i + (2^i+1)\xi_i]. \qquad (1.175)$$

Lower bound on $V_{opt}(n)$. We now derive a lower bound for the the optimal revenue under the heterogeneous commission-subscription pair. Consider a feasible commission-subscription pair $(\boldsymbol{\gamma}, \boldsymbol{\mu})$ where $\gamma_i^s = \gamma_i^b = 0$, $\mu_i^s = 0$, and $\mu_i^b = 2^{2i-1}$ for each $i \in \{1, \ldots, n\}$. Based on (1.173), this is equivalent to setting $r_i = 0$ and $\xi_i = \frac{4^i}{2}$ for all $i \in \{1, \ldots, n\}$. We further deduce that

$$\begin{aligned} V_{opt}(n) &\overset{(e)}{\geq} \sum_{i=1}^{n} V_i(r_i, \xi_i) \\ &= \sum_{i=1}^{n} \frac{8^{-i}}{[2^i + (1 + r_i)]^2} (4^i - \xi_i)^+ [4^i r_i + (2^i + 1)\xi_i] \\ &= \frac{1}{4} \sum_{i=1}^{n} \frac{2^i}{1 + 2^i} \overset{(f)}{\geq} \frac{1}{8} n, \end{aligned} \tag{1.176}$$

where in step (e), for any $i \in \mathcal{S}$, since the constructed $(\boldsymbol{\gamma}, \boldsymbol{\mu})$ satisfies that $\gamma_i^s = 0 < 1$, we leverage the simplified revenue expression in (1.175). Step (e) holds because the constructed $(\boldsymbol{r}, \boldsymbol{\xi})$ is feasible but not necessarily optimal. Step (f) follows from $\frac{2^i}{1+2^i} \geq \frac{1}{2}$ for all $i \in \{1, \ldots, n\}$.

Upper bound on $V_h(n)$. In this case, we can only use one vector of commission-subscription pair $(\gamma^s, \gamma^b, \mu^s, \mu^b)$, which corresponds to homogeneous (r, ξ). If $\gamma^s = 1$, then we have $(1 - \gamma^s)p_i - \mu^s \leq 0$, which implies that $q_i^s = 0$ for all $i \in \mathcal{S}$ in any induced competitive equilibrium and the platform's revenue is zero. Consider any feasible homogeneous commission-subscription pair $(\gamma^s, \gamma^b, \mu^s, \mu^b)$ where $\gamma^s < 1$. Using the revenue expression in (1.174), we derive the following upper bound for the optimal revenue

$$V_h(n) \overset{(g)}{\leq} \max_{r,\xi \geq 0} \sum_{i=1}^{n} \frac{8^{-i}}{[2^i + (1 + r)]^2} (4^i - \xi)^+ [4^i r + (2^i + 1)\xi], \tag{1.177}$$

where in inequality (g), when $\gamma^s < 1$, given the change of variables $r = \frac{1+\gamma^b}{1-\gamma^s} - 1$ and $\xi = (r+1)\mu^s + \mu^b$, we see that $\gamma^s, \gamma^b, \mu^s, \mu^b \geq 0$ would imply $r, \xi \geq 0$. Thus, the expression in (1.177) is an upper bound for the platform's revenue induced by homogeneous commissions-subscription pair $(\gamma^s, \gamma^b, \mu^s, \mu^b)$ with $\gamma^s < 1$. Moreover, it is easy to see that the upper bound expression (1.177) is at least 0, which implies that it is also an upper bound for any feasible homogeneous commissions-subscription pair $(\gamma^s, \gamma^b, \mu^s, \mu^b)$ with $\gamma^s = 1$. Thus, the upper bound of $V_h(n)$ in step (g) holds.

We denote (r, ξ) as the optimal solution for the upper bound optimization problem in (1.177). Moreover, if $\xi \geq 4^n$, then the upper bound expression (1.177) is 0. Thus, it is without loss of optimality to say that there exists $s_\xi \in \{0, \dots, n\}$ and $t_\xi \in [-1, 3 \cdot 4^{s_\xi})$ such that $\xi = 4^{s_\xi} + t_\xi$. For any $\xi \geq 0$, by considering the first order optimality condition for the expression in (1.177) in component $i \in \{1, \dots, n\}$, we deduce that it is decreasing in r for $r \geq (1 + 2^i) - 2\xi\frac{2^i+1}{4^i}$. Thus, the upper bound expression (1.177) is decreasing in r for $r \geq 2^{n+1}$. Thus, it is without loss of optimality to consider $s_r \in \{0, \dots, n\}$ and $t_r \in [-1, 2^{s_r})$ such that $r = 2^{s_r} + t_r$. Following the expression transformation $r = 2^{s_r} + t_r$ and $\xi = 4^{s_\xi} + t_\xi$, we can further derive that

$$\begin{aligned}
V_h(n) &\overset{(h)}{\leq} \sum_{i=1}^{n} \frac{8^{-i}}{(2^i + 1 + r)^2} (4^i - \xi)^+ [4^i r + (2^i + 1)\xi] \\
&= \sum_{i=1}^{n} \frac{8^{-i}}{(2^i + 1 + r)^2} \Big[- (2^i + 1)\xi^2 + 4^i(2^i + 1 - r)\xi + 16^i r \Big]^+ \\
&\overset{(i)}{\leq} \sum_{i=1}^{n} \frac{8^{-i}}{(2^i + 1 + r)^2} 16^i r + \sum_{i=1}^{n} \frac{8^{-i}}{(2^i + 1 + r)^2} \Big[- (2^i + 1)\xi^2 + 4^i(2^i + 1 - r)\xi \Big]^+ \\
&\overset{(j)}{\leq} \sum_{i=1}^{n} \frac{2^i r}{(2^i + r)^2} + \sum_{i=1}^{n} \frac{8^{-i}}{(2^i + 1)^2} \Big[- (2^i + 1)\xi^2 + 4^i(2^i + 1)\xi \Big]^+ \\
&\overset{(k)}{=} \sum_{i=1}^{n} \frac{1}{\frac{2^{s_r + t_r}}{2^i} + \frac{2^i}{2^{s_r + t_r}} + 2} + \sum_{i=1}^{n} \frac{8^{-i}}{2^i + 1} \xi \Big[4^i - \xi \Big]^+ \\
&\overset{(l)}{\leq} \sum_{i=1}^{n} \frac{2}{2^{s_r - i} + 2^{i - s_r} + 4 - \frac{1}{2^i}} + \sum_{i=s_\xi}^{n} \frac{8^{-i}}{2^i + 1} 4^i 4^{s_\xi + 1} \\
&\leq 2\Big[\sum_{i=1}^{s_r} \frac{1}{2^{s_r - i}} + \sum_{i=s_r}^{n} \frac{1}{2^{i - s_r}} \Big] + \sum_{i=s_\xi}^{n} \frac{1}{4^{i - s_\xi - 1}} \\
&\overset{(m)}{\leq} 6 + \frac{16}{3}, \qquad (1.178)
\end{aligned}$$

where step (h) follows from implementing the upper bound of $V_h(n)$ in (1.177). In step (i), we implement the following upper bound $\Big[-(2^i+1)\xi^2+4^i(2^i+1-r)\xi+16^ir\Big]^+\leq \Big[-(2^i+1)\xi^2+4^i(2^i+1-r)\xi\Big]^+ +16^ir$. In step (j), we implement $2^i+1+r\geq 2^i+r$ in the first expression, and we also implement $2^i+1+r\geq 2^i+1$ and $2^i+1-r\leq 2^i+1$ in the second expression. Step (k) follows from plugging in the expression $r=2^{s_r}+t_r$. In step (l), we first implement inequalities $\frac{2^{s_r}+t_r}{2^i}\geq\frac{1}{2}\cdot 2^{s_r-i}-\frac{1}{2^i}$ and $\frac{2^i}{2^{s_r}+t_r}\geq\frac{1}{2}\cdot 2^{i-s_r}$ in the first expression. In the second expression, we have $4^i-\xi\leq 0$ for all $i\leq s_\xi-1$, we have $-\xi^2+4^i\xi\leq 4^i\xi\leq 4^i4^{s_\xi+1}$. As a result of this derivation, we end up with the upper bound expression in step (l). The inequality in step (m) follows from $\sum_{i=1}^{s_r}\frac{1}{2^{s_r-i}}\leq 2$, $\sum_{i=s_r+1}^{n}\frac{1}{2^{i-s_r}}\leq 1$, and $\sum_{i=s_\xi}^{n}\frac{1}{4^{i-s_\xi-1}}\leq\frac{16}{3}$.

Upper bound on $V_h(n)/V_{opt}(n)$. We now combine the upper bound on $V_h(n)$ with the lower bound on $V_{opt}(n)$ to obtain that

$$\frac{V_h(n)}{V_{opt}(n)}\leq\frac{8}{n}\left(6+\frac{16}{3}\right)\leq\frac{272}{n}$$

This allows us to derive that $\frac{V_h(n)}{V_{opt}(n)}\to 0$ as $n\to\infty$, which implies our result. □

Proof of Proposition 6. Proof of Claim (i): By Proposition 1, for any $(\mathbf{0},\boldsymbol{\mu})\in\Gamma\times\mathcal{U}$, the induced equilibrium supply-demand vector $(\boldsymbol{q}^s,\boldsymbol{q}^b)$ is unique. Thus, we can define function $q_i^s:\mathcal{U}\to\mathbb{R}^+$ for all $i\in\mathcal{S}$ and $q_j^b:\mathcal{U}\to\mathbb{R}^+$ for all $j\in\mathcal{B}$ such that $(\boldsymbol{p},\boldsymbol{x},\boldsymbol{q}^s(\boldsymbol{\mu}),\boldsymbol{q}^b(\boldsymbol{\mu}))\in\mathcal{X}(\mathbf{0},\boldsymbol{\mu})$. We consider the following lemma on the parametrized equilibrium supply-demand vectors, the proof of which can be found in Section (1.15) of (40).

Lemma 7. *For any $\boldsymbol{\mu}\in\mathcal{U}$, we have*

(i) *for all $j_0\in\mathcal{B}$, function $q_i^s(\boldsymbol{\mu})$ is weakly decreasing in $\mu_{j_0}^b$ for all $i\in\mathcal{S}$;*

(ii) *for all $i_0\in\mathcal{S}$, function $q_j^b(\boldsymbol{\mu})$ is weakly decreasing in $\mu_{i_0}^s$ for all $j\in\mathcal{B}$.*

Let $\bar{\boldsymbol{\mu}} = (\bar{\boldsymbol{\mu}}^s, \bar{\boldsymbol{\mu}}^b)$ denote the optimal subscription vector obtained from Theorem 1. Then

$$\begin{aligned}
\max\{V_s, V_b\} \;&\geq\; \max\left\{\sum_{i\in\mathcal{S}} \mu_i^s q_i^s(\bar{\boldsymbol{\mu}}^s, \mathbf{0}), \sum_{j\in\mathcal{B}} \mu_j^b q_j^b(\mathbf{0}, \bar{\boldsymbol{\mu}}^b)\right\} \\
&\geq \frac{1}{2}\left[\sum_{i\in\mathcal{S}} \bar{\mu}_i^s q_i^s(\bar{\boldsymbol{\mu}}^s, \mathbf{0}) + \sum_{j\in\mathcal{B}} \bar{\mu}_j^b q_j^b(\mathbf{0}, \bar{\boldsymbol{\mu}}^b)\right] \\
&\overset{(a)}{\geq} \frac{1}{2}\left[\sum_{i\in\mathcal{S}} \bar{\mu}_i^s q_i^s(\bar{\boldsymbol{\mu}}^s, \bar{\boldsymbol{\mu}}^b) + \sum_{j\in\mathcal{B}} \mu_j^b q_j^b(\bar{\boldsymbol{\mu}}^s, \bar{\boldsymbol{\mu}}^b)\right] \;\overset{(b)}{=}\; \frac{1}{2}V_{opt}.
\end{aligned} \tag{1.179}$$

where inequality (a) follows directly from Lemma 7, which proves that $q_i^s(\bar{\boldsymbol{\mu}}^s, \mathbf{0}) \geq q_i^s(\bar{\boldsymbol{\mu}}^s, \bar{\boldsymbol{\mu}}^b)$ and $q_j^b(\mathbf{0}, \bar{\boldsymbol{\mu}}^b) \geq q_j^b(\bar{\boldsymbol{\mu}}^s, \bar{\boldsymbol{\mu}}^b)$. Equality (b) follows from the optimality of subscrition vector $\bar{\boldsymbol{\mu}}$.

Proof of Claim (ii). By Lemma 4(ii), the revenue optimization problem (1.7) can be formulated as an instance of the framework problem (1.38) where $g_j(q) = F_b^{-1}(1 - \frac{q}{b_j})q$, $h_i(q) = F_{s_i}^{-1}(\frac{q}{s_i})q$, and $w_{ij} = c_i$. Moreover, Assumptions (1.12.1-1) - (1.12.1-3) and Assumptions (1.12.1-7) - (1.12.1-8) hold in this instance. Let $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{q}}^s, \bar{\boldsymbol{q}}^b)$ be an optimal solution to problem (1.38). By Proposition 12(vi), there exists a unique dual optimal solution, which we denote by $(\boldsymbol{\theta}^b, \boldsymbol{\theta}^s, \boldsymbol{\eta}^b, \boldsymbol{\eta}^s, \boldsymbol{\pi})$ such that the primal/dual optimal pair satisfies (1.40a) - (1.40h). Note that by Theorem 1, the optimal objective value of this problem is V_{opt}.

We prove the claim in the following two steps. In step (1), we show that $V_s = V_{opt}$ when we have $F_{b_j}(v) = F_b(v)$ for all $j \in \mathcal{B}$. In step (2), we show that $V_b = V_{opt}$ when we have $F_{s_i}(v) = F_s(v)$ and $c_i = 0$ for all $i \in \mathcal{S}$.

Step 1: $V_s = V_{opt}$ when $F_{b_j}(v) = F_b(v)$ for all $j \in \mathcal{B}$: To establish the claim, we first show that there exists $\boldsymbol{\gamma}^s \in \Gamma^s$ and $\boldsymbol{\mu}^s \in \mathcal{U}^s$ such that for some $\boldsymbol{p} \geq \mathbf{0}$, we have $(\boldsymbol{p}, \bar{\boldsymbol{x}}, \bar{\boldsymbol{q}}^s, \bar{\boldsymbol{q}}^b) \in \mathcal{X}(\boldsymbol{\gamma}^s, \mathbf{0}, \boldsymbol{\mu}^s, \mathbf{0})$. Then, we establish that the revenue of the platform induced at this equilibrium is given by V_{opt}.

Since $F_{b_j}(v) = F_b(v)$ for all $j \in \mathcal{B}$, we have $g_j(q_j^b) = b_j g(\frac{q}{b_j})$ with $g(r) = F_b^{-1}(1 - r)r$.

Thus, Assumption (1.12.1-4) holds. By Proposition 12(vii), for any $i \in \mathcal{S}$, we have

$$\frac{\bar{q}_j^b}{b_j} = \frac{\bar{q}_{j'}^b}{b_{j'}}, \quad \text{for all } j \text{ and } j' : \bar{x}_{ij} > 0, \bar{x}_{ij'} > 0. \tag{1.180}$$

We claim that

$$F_b^{-1}\left(1 - \frac{\bar{q}_j^b}{b_j}\right) \;\geq\; F_{s_i}^{-1}\left(\frac{\bar{q}_i^s}{s_i}\right) + c_i, \quad \forall (i,j) : \bar{x}_{ij} > 0. \tag{1.181}$$

For some $i \in \mathcal{S}$ and $j \in \mathcal{B}$ such that $\bar{x}_{ij} > 0$, by Proposition 12(v), we have $\bar{q}_j^b < b_j$, which further implies that $\eta_j^b = 0$ (by (1.40d)) and $g_j'(\bar{q}_j^b) = \theta_j^b$ (by (1.40a)). Moreover, by condition (1.40e), we have $\eta_i^s \geq 0$, which in turn implies $\theta_i^s \geq h_i'(\bar{q}_i^s)$ (by (1.40b)). Lastly, since $\bar{x}_{ij} > 0$, we have $\pi_{ij} = 0$ by (1.40f) and $\theta_j^b = \theta_i^s + c_i$ by (1.40c). Recall that we have $g_j(q) = F_b^{-1}(1 - \frac{q}{b_j})q$ and $h_i(q) = F_{s_i}^{-1}(\frac{q}{s_i})q$. Proposition 12(iii) implies that $g_j'(\bar{q}_j^b)$ and $h_i'(\bar{q}_i^s)$ is finite. Recall that $g_j'(q)$ is continuous in $(0, b_j)$ (Assumption (1.12.1-1)), and $g_j'(0) = \lim_{q \downarrow 0} g_j'(q)$, $g_j'(b_j) = \lim_{q \uparrow b_j} g_j'(q)$ (by definition). Thus, there exists a sequence $\{q_k^1 \in (0, b_j) : k = 1, 2, \dots\}$ such that $\lim_{k\to\infty} q_k^1 = \bar{q}_j^b$ and $\lim_{k\to\infty} g_j'(q_k^1) = g_j'(\bar{q}_j^b)$. Similarly, recall that $h_i'(q)$ is continuous in $(0, s_i)$ (Assumption (1.12.1-2)), and $h_i'(0) = \lim_{q\downarrow 0} h_i'(q)$, $h_i'(s_i) = \lim_{q \uparrow s_i} h_i'(q)$ (by definition). Thus, there exists a sequence $\{q_k^2 \in (0, s_i) : k = 1, 2, \dots\}$ such that $\lim_{k\to\infty} q_k^2 = \bar{q}_i^s$ and $\lim_{k\to\infty} h_i'(q_k^2) = h_i'(\bar{q}_i^s)$. Using Assumption 1, the fact that F_b and F_{s_i} are continuously differentiable and strictly increasing respectively in $(0, \bar{v}_{b_j})$ and $(0, \bar{v}_{s_i})$, and the inverse function theorem it follows that F_b^{-1} and $F_{s_i}^{-1}$ are continuously differentiable and strictly increasing in $(0, 1)$. Using these observations, it follows that:

$$\begin{aligned} g_j'(\bar{q}_j^b) \;&=\; \lim_{k\to\infty} g_j'(q_k^1) \;=\; \lim_{k\to\infty} \frac{q_k^1}{b_j}\big[F_b^{-1}\big]'\left(1 - \frac{q_k^1}{b_j}\right) + F_b^{-1}\left(1 - \frac{q_k^1}{b_j}\right) \;\leq\; F_b^{-1}\left(1 - \frac{\bar{q}_j^b}{b_j}\right) \\ h_i'(\bar{q}_i^s) \;&=\; , \lim_{k\to\infty} h_i'(q_k^2) \;=\; \lim_{k\to\infty} \frac{q_k^2}{s_i}\big[F_{s_i}^{-1}\big]'\left(\frac{q_k^2}{s_i}\right) + F_{s_i}^{-1}\left(\frac{q_k^2}{s_i}\right) \;\geq\; F_{s_i}^{-1}\left(\frac{\bar{q}_i^s}{s_i}\right). \end{aligned} \tag{1.182}$$

These observations imply that $F_b^{-1}(1 - \frac{\bar{q}_j^b}{b_j}) - F_s^{-1}(\frac{\bar{q}_i^s}{s_i}) \geq g_j'(\bar{q}_j^b) - h_i'(\bar{q}_i^s) \geq \theta_j^b - \theta_i^s = c_i$. Since, we focused on any (i, j) such that $\bar{x}_{ij} > 0$, (1.181) follows.

Next, we construct $(\boldsymbol{p}, \boldsymbol{\gamma}^s, \boldsymbol{\mu}^s)$ such that $(\boldsymbol{p}, \bar{\boldsymbol{x}}, \bar{\boldsymbol{q}}^s, \bar{\boldsymbol{q}}^b) \in \mathcal{X}(\boldsymbol{\gamma}^s, \boldsymbol{0}, \boldsymbol{\mu}^s, \boldsymbol{0})$ then verify the feasibility of $(\boldsymbol{\gamma}^s, \boldsymbol{\mu}^s)$. We set $\boldsymbol{\gamma}^s = \boldsymbol{0}$, which clearly satisfies $\boldsymbol{\gamma}^s \in \Gamma^s$. To construct $\boldsymbol{\mu}^s$, for each $i \in \mathcal{S}$, we consider two cases: (a) $\bar{q}_i^s > 0$: we pick any $j : \bar{x}_{ij} > 0$ and set $p_i = F_b^{-1}(1 - \frac{\bar{q}_j^b}{b_j}) - c_i$ and $\mu_i^s = F_b^{-1}(1 - \frac{\bar{q}_j^b}{b_j}) - F_{s_i}^{-1}(\frac{\bar{q}_i^s}{s_i}) - c_i$. It can be seen from (1.180) that it does not matter which $j : \bar{x}_{ij} > 0$ is chosen. By (1.181), we have that $\mu_i^s \geq 0$ and $p_i \geq 0$; (b) $\bar{q}_i^s = 0$: we pick any $j \in \arg\max_{j':(i,j') \in E}\{F_b^{-1}(1 - \frac{\bar{q}_{j'}^b}{b_{j'}}) - c_i\}$ and let $p_i = \mu_i^s = \max\{F_b^{-1}(1 - \frac{\bar{q}_j^b}{b_j}) - c_i, 0\}$. In this case as well, we have $p_i, \mu_i^s \geq 0$ by construction. Thus, we conclude that our construction satisfies that $\boldsymbol{\mu}^s \in \mathcal{U}^s$.

We claim that $(\boldsymbol{p}, \bar{\boldsymbol{x}}, \bar{\boldsymbol{q}}^s, \bar{\boldsymbol{q}}^b) \in \mathcal{X}(\boldsymbol{\gamma}^s, \boldsymbol{0}, \boldsymbol{\mu}^s, \boldsymbol{0})$. To establish this claim, we check the equilibrium expressions (1.2a)-(1.2d) for the constructed tuple:

(1) (1.2a): by construction of $\boldsymbol{p}$ and $(\boldsymbol{\gamma}^s, \boldsymbol{\mu}^s)$, we obtain that $p_i - \mu_i^s = F_s^{-1}(\frac{\bar{q}_i^s}{s_i})$, which implies that

$$s_i F_{s_i}\Big((1 - \gamma_i^s)p_i - \mu_i^s\Big) \;=\; s_i F_{s_i}\left(F_s^{-1}\left(\frac{\bar{q}_i^s}{s_i}\right)\right) \;=\; \bar{q}_i^s. \tag{1.183}$$

(2) (1.2b): for any $j \in \mathcal{B}$, if $\bar{x}_{i_1 j} > 0$ and $\bar{x}_{i_2 j} > 0$, then the constructions of p_{i_1} and p_{i_2} imply that $p_{i_1} + c_{i_1} = F_b^{-1}(1 - \frac{\bar{q}_j^b}{b_j}) = p_{i_2} + c_{i_2}$. Suppose instead that $\bar{x}_{i_1 j} > 0$ and for some $(i_2, j) \in E$, we have $\bar{x}_{i_2 j} = 0$. From the construction of p_{i_1}, we have $p_{i_1} + c_{i_1} = F_b^{-1}(1 - \frac{\bar{q}_j^b}{b_j})$. If $\bar{q}_{i_2}^s > 0$, by the feasibility of $(\boldsymbol{x}, \boldsymbol{q}^s, \boldsymbol{q}^b)$ in the instance of the framework problem that corresponds to (1.7), for some j_2 we have $\bar{x}_{i_2 j_2} > 0$. Proposition 12(vii) implies that $\frac{\bar{q}_j^b}{b_j} \geq \frac{\bar{q}_{j_2}^b}{b_{j_2}}$. Since we have $p_{i_2} + c_{i_2} = F_b^{-1}(1 - \frac{\bar{q}_{j_2}^b}{b_{j_2}})$, by construction of p_{i_2}, and F_b^{-1} is strictly increasing in $(0, 1)$, we conclude that $p_{i_1} + c_{i_1} = F_b^{-1}(1 - \frac{\bar{q}_j^b}{b_j}) \leq F_b^{-1}(1 - \frac{\bar{q}_{j_2}^b}{b_{j_2}}) = p_{i_2} + c_{i_2}$. Similarly, if $\bar{q}_{i_2} = 0$, then by the construction of p_{i_2}, we have $p_{i_2} + c_{i_2} = \max_{j':(i_2,j') \in E} F_b^{-1}(1 - \frac{\bar{q}_{j'}^b}{b_{j'}})$. Since $(i_2, j) \in E$, we have that $p_{i_1} + c_{i_1} = F_b^{-1}(1 - \frac{\bar{q}_j^b}{b_j}) \leq \max_{j':(i_2,j') \in E} F_b^{-1}(1 - \frac{\bar{q}_{j'}^b}{b_{j'}}) = p_{i_2} + c_{i_2}$.

Combining these observations, we conclude that

$$\text{if } \bar{x}_{ij} > 0, \text{ then } p_i + c_i \ = \min_{i':(i',j)\in E}\{p_{i'} + c_{i'}\} = F_b^{-1}\left(1 - \frac{\bar{q}_j^b}{b_j}\right). \tag{1.184}$$

Next, we verify the equilibrium condition (1.2b). First note that for j such that $\bar{q}_j^b > 0$, (1.184) implies that

$$b_j\left[1 - F_b\left(\min_{i:(i,j)\in E}\{p_i + c_i\}\right)\right] = b_j\left[1 - F_b\left(F_b^{-1}\Big(1 - \frac{\bar{q}_j^b}{b_j}\Big)\right)\right] = \bar{q}_j^b. \tag{1.185}$$

Consider instead j such that $\bar{q}_j^b = 0$. If there exists $i_0 : (i_0, j) \in E$ with $\bar{q}_{i_0}^s > 0$, then we can find $j_0 : \bar{x}_{i_0 j_0} > 0$. Note that since $\bar{q}_j^b = 0$, we have $\bar{x}_{i_0 j} = 0$, and Proposition 12(vii) implies that $\frac{\bar{q}_j^b}{b_j} \geq \frac{\bar{q}_{j_0}^b}{b_{j_0}}$. The strict increasingness of F_b^{-1} in $(0, 1)$ implies that $F_b^{-1}(1 - \frac{\bar{q}_j^b}{b_j}) \leq F_b^{-1}(1 - \frac{\bar{q}_{j_0}^b}{b_{j_0}})$. Moreover, by $\bar{x}_{i_0 j_0} > 0$ and (1.184), it follows that $p_{i_0} + c_{i_0} = \min_{i:(i,j_0)\in E}\{p_i + c_i\} = F_b^{-1}(1 - \frac{\bar{q}_{j_0}^b}{b_{j_0}})$. These observations imply that

$$\begin{aligned} 0 \ &\leq b_j\left[1 - F_b\left(\min_{i:(i,j)\in E}\{p_i + c_i\}\right)\right] \\ &= b_j\left[1 - F_b\left(F_b^{-1}\Big(1 - \frac{\bar{q}_{j_0}^b}{b_{j_0}}\Big)\right)\right] \ \leq \ b_j\left[1 - F_b\left(F_b^{-1}\Big(1 - \frac{\bar{q}_j^b}{b_j}\Big)\right)\right] \ = \ \bar{q}_j^b \ = \ 0. \end{aligned} \tag{1.186}$$

If for all $i_0 : (i_0, j) \in E$, we have $\bar{q}_{i_0}^s = 0$, then the construction of $\boldsymbol{p}$ implies that $p_{i_0} + c_{i_0} = \max_{j':(i_0,j')\in E} F_b^{-1}(1 - \frac{\bar{q}_{j'}^b}{b_{j'}}) \ \geq \ F_b^{-1}(1 - \frac{\bar{q}_j^b}{b_j})$ for any i_0 such that $(i_0, j) \in E$. Thus, we conclude that

$$0 \ \leq b_j\left[1 - F_b\left(\min_{i_0:(i_0,j)\in E}\{p_{i_0} + c_{i_0}\}\right)\right] \ \leq \ b_j\left[1 - F_b\left(F_b^{-1}\Big(1 - \frac{\bar{q}_j^b}{b_j}\Big)\right)\right] \ = \ \bar{q}_j^b \ = \ 0. \tag{1.187}$$

Using (1.185), (1.186) and (1.187), it follows that the equilibrium condition (1.2b) holds;

(3) (1.2c) follows directly from constraints (1.38b) and (1.38c) in the instance of the

framework problem (1.38) that corresponds to (1.7);

(4) (1.2d): for any $(i,j) \in E$, by (1.38f), we have $x_{ij} \geq 0$. This observation, together with (1.184) implies (1.2d).

Summarizing, (1.2a) - (1.2d) hold for the constructed tuple, thereby implying that $(\boldsymbol{p}, \bar{\boldsymbol{x}}, \bar{\boldsymbol{q}}^s, \bar{\boldsymbol{q}}^b) \in \mathcal{X}(\boldsymbol{\gamma}^s, \mathbf{0}, \boldsymbol{\mu}^s, \mathbf{0})$. The fact that the revenue induced at this equilibrium is equal to V_{opt} can be established by following the same steps as in the proof of Theorem 1. In particular, for all $i \in \mathcal{S}$, given that $(1-\gamma_i^s)p_i - \mu_i^s = F_{s_i}^{-1}(\frac{\bar{q}_i^s}{s_i})$ by construction, the inequality in step (b) of (1.138) holds with equality (in the proof of Theorem 1). This in turn implies that step (e) of (1.139) also holds with equality. These observations imply that $V(\boldsymbol{\gamma}^s, \mathbf{0}, \boldsymbol{\mu}^s, \mathbf{0}) = h(\bar{\boldsymbol{q}}^s, \bar{\boldsymbol{q}}^b)$. Since the latter term is the optimal objective value of (1.7), which is equal to V_{opt} by Theorem 1, we conclude that $V(\boldsymbol{\gamma}^s, \mathbf{0}, \boldsymbol{\gamma}^b, \mathbf{0}) = V_{opt}$.

Step 2: $V_b = V_{opt}$ when $F_{s_i}(v) = F_s(v)$ and $c_i = 0$ for all $i \in \mathcal{S}$: To establish the claim, we show that there exists $\boldsymbol{\gamma}^b \in \Gamma^b$ and $\boldsymbol{\mu}^b \in \mathcal{U}^b$ such that for some $\boldsymbol{p} \geq \mathbf{0}$, we have $(\boldsymbol{p}, \bar{\boldsymbol{x}}, \bar{\boldsymbol{q}}^s, \bar{\boldsymbol{q}}^b) \in \mathcal{X}(\mathbf{0}, \boldsymbol{\gamma}^b, \mathbf{0}, \boldsymbol{\mu}^b)$. Then we show that the optimal revenue induced at this competitive equilibrium is given by by V_{opt}.

When $F_{s_i}(v) = F_s(v)$ for all $i \in \mathcal{S}$, we have $h_i(q_i^s) = s_i h\left(\frac{q}{s_i}\right)$ where $h(r) = F_s^{-1}(r)r$. With $c_i = 0$ for all $i \in \mathcal{S}$, we establish that Assumption (1.12.1-5) holds. From Proposition 12(viii), we have

$$\frac{\bar{q}_i^s}{s_i} = \frac{\bar{q}_{i'}^s}{s_{i'}}, \quad \text{for all } i \text{ and } i' : \bar{x}_{ij} > 0, \bar{x}_{i'j} > 0. \tag{1.188}$$

Given that $c_i = 0$ for all $i \in \mathcal{S}$, repeating the same arguments in (1.181) yields

$$F_{b_j}^{-1}\left(1 - \frac{\bar{q}_j^b}{b_j}\right) \;\geq\; F_s^{-1}\left(\frac{\bar{q}_i^s}{s_i}\right), \quad \forall (i,j) : \bar{x}_{ij} > 0. \tag{1.189}$$

To proceed, we construct $(\boldsymbol{p}, \boldsymbol{\gamma}^b, \boldsymbol{\mu}^b)$ such that $(\boldsymbol{p}, \bar{\boldsymbol{x}}, \bar{\boldsymbol{q}}^s, \bar{\boldsymbol{q}}^b) \in \mathcal{X}(\mathbf{0}, \boldsymbol{\gamma}^b, \mathbf{0}, \boldsymbol{\mu}^b)$, and verify the feasibility of $(\boldsymbol{\gamma}^b, \boldsymbol{\mu}^b)$. We first set $p_i = F_s^{-1}(\frac{\bar{q}_i^s}{s_i})$ for all $i \in \mathcal{S}$, which clearly satisfies

$p_i \geq 0$. We then let $\boldsymbol{\gamma}^b = \mathbf{0}$, which readily guarantees $\boldsymbol{\gamma}^b \in \Gamma^b$. To construct $\boldsymbol{\mu}^b$, we consider the following two cases: (a) for all j such that $\bar{q}_j^b > 0$, we pick any $i : \bar{x}_{ij} > 0$ and let $\mu_j^b = F_{b_j}^{-1}(1 - \frac{\bar{q}_j^b}{b_j}) - F_s^{-1}(\frac{\bar{q}_i^s}{s_i})$. From condition (1.188), it can be seen that it does not matter which $i : \bar{x}_{ij} > 0$ is picked. By condition (1.189), it follows that $\mu_j^b \geq 0$; (b) for all j such that $\bar{q}_j^b = 0$, we pick $i \in \arg\min_{i':(i',j)\in E}\{p_{i'}\}$ such that $\mu_j^b = \max\{F_{b_j}^{-1}(1 - \frac{\bar{q}_j^b}{b_j}) - F_s^{-1}(\frac{\bar{q}_i^s}{s_i}), 0\}$, which again satisfies $\mu_j^b \geq 0$. Thus, we conclude that our construction satisfies $\boldsymbol{\mu}^b \in \mathcal{U}^b$.

We claim that $(\boldsymbol{p}, \boldsymbol{\gamma}^b, \boldsymbol{\mu}^b)$ satisfies $(\boldsymbol{p}, \bar{\boldsymbol{x}}, \bar{\boldsymbol{q}}^s, \bar{\boldsymbol{q}}^b) \in \mathcal{X}(\mathbf{0}, \boldsymbol{\gamma}^b, \mathbf{0}, \boldsymbol{\mu}^b)$. To establish this claim, we verify the equilibrium expressions (1.2a)-(1.2d):

(1) (1.2a): for any $i \in \mathcal{S}$, given the construction of p_i that satisfies $p_i = F_s^{-1}(\frac{\bar{q}_i^s}{s_i})$, we obtain that

$$s_i F_s\big(p_i\big) = s_i F_s\left(F_s^{-1}\left(\frac{\bar{q}_i^s}{s_i}\right)\right) = \bar{q}_i^s; \tag{1.190}$$

(2) (1.2b): for all $j \in \mathcal{B}$, if there exists i_1, i_2 such that $\bar{x}_{i_1 j} > 0$ and $\bar{x}_{i_2 j} = 0$, then $\frac{\bar{q}_{i_1}^s}{s_{i_1}} \leq \frac{\bar{q}_{i_2}^s}{s_{i_2}}$ (Proposition 12(viii)). By the strict increasingness of $F_s^{-1}(\cdot)$ in $(0,1)$ and by the construction of p_{i_1}, p_{i_2}, we establish that $p_{i_1} = F_s^{-1}(\frac{\bar{q}_{i_1}^s}{s_{i_1}}) \leq F_s^{-1}(\frac{\bar{q}_{i_2}^s}{s_{i_2}}) = p_{i_2}$. Suppose instead that $\bar{x}_{i_1 j} > 0$ and $\bar{x}_{i_2 j} > 0$, by Proposition 12(viii), we have $\frac{\bar{q}_{i_1}^s}{s_{i_1}} = \frac{\bar{q}_{i_2}^s}{s_{i_2}}$, which implies that $p_{i_1} = F_s^{-1}(\frac{\bar{q}_{i_1}^s}{s_{i_1}}) = F_s^{-1}(\frac{\bar{q}_{i_2}^s}{s_{i_2}}) = p_{i_2}$. These observations imply that if $\bar{x}_{ij} > 0$, then $p_i = \min_{i':(i',j)\in E}\{p_{i'}\}$. Moreover, if $\bar{x}_{ij} > 0$, we can deduce from the construction of p_i and μ_j^b that $p_i + \mu_j^b = F_s^{-1}(\frac{\bar{q}_i^s}{s_i}) + \mu_j^b = F_{b_j}^{-1}(1 - \frac{\bar{q}_j^b}{b_j})$. Combining these observations, we conclude that

$$\text{if } \bar{x}_{ij} > 0, \text{ then } p_i + \mu_j^b \ = \min_{i':(i',j)\in E}\{p_{i'}\} + \mu_j^b \ = \ F_{b_j}^{-1}\left(1 - \frac{\bar{q}_j^b}{b_j}\right). \tag{1.191}$$

Note that for all $j \in \mathcal{B}$ such that $\bar{q}_j^b > 0$, (1.191) implies that

$$b_j\left[1 - F_{b_j}\left(\min_{i:(i,j)\in E}\{p_i\} + \mu_j^b\right)\right] = b_j\left[1 - F_{b_j}\left(F_{b_j}^{-1}\Big(1 - \frac{\bar{q}_j^b}{b_j}\Big)\right)\right] = \bar{q}_j^b. \tag{1.192}$$

Consider instead j such that $\bar{q}^b_j = 0$. Since $\mu^b_j = \max\{F^{-1}_{b_j}(1 - \frac{\bar{q}^b_j}{b_j}) - F^{-1}_s(\frac{\bar{q}^s_i}{s_i}), 0\}$ by construction, it follows that $p_i + \mu^b_j = F^{-1}_s(\frac{\bar{q}^s_i}{s_i}) + \mu^b_j \geq F^{-1}_{b_j}(1 - \frac{\bar{q}^b_j}{b_j})$. Using the weakly increasingness of $F_{b_j}(v)$ in $v \in \mathbb{R}$ (by definition of F_{b_j}), we deduce that

$$0 \leq b_j \left[1 - F_{b_j}\left(\min_{i:(i,j)\in E}\{p_i\} + \mu^b_j\right)\right] \leq b_j \left[1 - F_{b_j}\left(F^{-1}_{b_j}\Big(1 - \frac{\bar{q}^b_j}{b_j}\Big)\right)\right] = \bar{q}^b_j = 0. \tag{1.193}$$

From (1.192) and (1.193), we conclude that the equilibrium condition (1.2b) holds;

(3) (1.2c) follows directly from constraints (1.38b) and (1.38c) in the instance of the framework problem (1.38) associated with problem (1.7);

(4) (1.2d): for any $(i,j) \in E$, $x_{ij} \geq 0$ follows from (1.38f). Together with (1.191), this implies that (1.2d) holds.

Summarizing the constructed tuple satisfies (1.2a) - (1.2d), we conclude that $(\boldsymbol{p}, \bar{\boldsymbol{x}}, \bar{\boldsymbol{q}}^s, \bar{\boldsymbol{q}}^b) \in \mathcal{X}(\mathbf{0}, \boldsymbol{\gamma}^b, \mathbf{0}, \boldsymbol{\mu}^b)$. We use the same steps as in the proof of Theorem 1 to establish that the induced revenue at this competitive equilibrium is V_{opt}. In particular, for all $i \in \mathcal{S}$, since $(1 - \gamma^s_i)p_i - \mu^s_i = F^{-1}_{s_i}(\frac{\bar{q}^s_i}{s_i})$ by construction, we establish that the inequality in step (b) of (1.138) (in the proof of Theorem 1) is tight. This also implies that step (e) of (1.139) (in the proof of Theorem 1) holds. Thus, we conclude that $V(\mathbf{0}, \boldsymbol{\gamma}^b, \mathbf{0}, \boldsymbol{\mu}^b) = V_{opt}$. □

Proof of Lemma 7. Without loss of generality, we can assume that network $G(\mathcal{S} \cup \mathcal{B}, E)$ is connected. (If it is not connected, we can prove the claims in each disjoint network component.) Let $\boldsymbol{\mu}^b_{-j}$ be the subvector of $\boldsymbol{\mu}^b$ obtained by excluding component μ^b_j, and define $\boldsymbol{\mu}^s_{-i}$ analogously. By Proposition 1, for any given $\boldsymbol{\mu} \in \mathcal{U}$, the equilibrium supply-demand vector $(\boldsymbol{q}^s, \boldsymbol{q}^b)$ associated to fees $(\mathbf{0}, \boldsymbol{\mu})$ is unique. In what follows, we fix $\boldsymbol{\gamma} = 0$ and, abusing some notation, we denote the supply-demand vector by its parameterized values $(\boldsymbol{q}^s(\boldsymbol{\mu}), \boldsymbol{q}^b(\boldsymbol{\mu}))$ in a competitive equilibrium $(\boldsymbol{p}, \boldsymbol{x}, \boldsymbol{q}^s(\boldsymbol{\mu}), \boldsymbol{q}^b(\boldsymbol{\mu})) \in \mathcal{X}(\mathbf{0}, \boldsymbol{\mu})$.

Given $\boldsymbol{\gamma} = \mathbf{0}$, by Lemma 4(i), the equilibrium problem (1.16) can be formulated as an instance of the framework problem (1.38) where Assumptions (1.12.1-1) - (1.12.1-3) hold. Given $\boldsymbol{\gamma} = \mathbf{0}$, we abuse some notation by letting $g_j(q, \mu_j^b) = \int_0^q [F_{b_j}^{-1}(1 - \frac{x}{b_j}) - \mu_j^b]dx$ and $h_i(q, \mu_i^s) = \int_0^q [F_{s_i}^{-1}(\frac{x}{s_i}) + \mu_i^s]dx$ in this instance of the framework problem (1.38). Moreover, we have $w_{ij} = c_i$ for all $(i, j) \in E$. We let $(\boldsymbol{\theta}^b, \boldsymbol{\theta}^s, \boldsymbol{\eta}^b, \boldsymbol{\eta}^s, \boldsymbol{\pi})$ be the dual optimal solution vector that satisfies the conditions in (1.40), where the dual variables correspond to constraints (1.38b) - (1.38f), respectively. By Proposition 12(vi), $(\boldsymbol{\theta}^b, \boldsymbol{\theta}^s, \boldsymbol{\eta}^b, \boldsymbol{\eta}^s, \boldsymbol{\pi})$ is unique. We denote the parameterized dual optimal solution as $(\boldsymbol{\theta}^b(\boldsymbol{\mu}), \boldsymbol{\theta}^s(\boldsymbol{\mu}), \boldsymbol{\eta}^b(\boldsymbol{\mu}), \boldsymbol{\eta}^s(\boldsymbol{\mu}), \boldsymbol{\pi}(\boldsymbol{\mu}))$.

We divide the proof arguments in the following steps.

Step 1: continuity of $\boldsymbol{q}^s(\boldsymbol{\mu})$ and $\boldsymbol{q}^b(\boldsymbol{\mu})$ in $\boldsymbol{\mu} \in \mathcal{U}$. We leverage the Maximum theorem (see page 116 of (49)) to establish that $q_i^s(\boldsymbol{\mu})$ and $q_j^b(\boldsymbol{\mu})$ are continuous in $\boldsymbol{\mu} \in \mathcal{U}$ for all $i \in \mathcal{S}$ and $j \in \mathcal{B}$. Using the notation in their framework, we let $X = \mathcal{U} = \{\boldsymbol{\mu} : \mu_i^s \geq 0, \forall i \in \mathcal{S}, \mu_j^b \geq 0, \forall j \in \mathcal{B}\}$ and $Y = \{(\boldsymbol{x}, \boldsymbol{q}^s, \boldsymbol{q}^b) : q_i^s \in [0, s_i], q_j^b \in [0, b_j], x_{ij} \in [0, \max\{s_i, b_j\}]\}$. By Assumption 1, we have $\int_0^{b_j} F_{b_j}^{-1}(1 - \frac{x}{b_j})dx \in (0, \infty)$ for all $j \in \mathcal{B}$ and $\int_0^{s_i} F_{s_i}^{-1}(\frac{x}{s_i}) \in (0, \infty)$ for all $i \in \mathcal{S}$. Since Assumption (1.12.1-1) - (1.12.1-2) hold, we obtain that $g_j(q, \mu_j^b)$ is continuous in $q \in [0, b_j]$ for all $j \in \mathcal{B}$ and $h_i(q, \mu_i^s)$ is continuous in $q \in [0, s_i]$ for all $i \in \mathcal{S}$. Thus, the objective function $\sum_{j \in \mathcal{B}} g_j(\bar{q}_j^b, \mu_j^b) - \sum_{i \in \mathcal{S}} h_i(\bar{q}_i^s, \mu_i^s) - \sum_{(i,j) \in E} c_i \bar{x}_{ij}$ are continuous in $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{q}}^s, \bar{\boldsymbol{q}}^b) \in Y$ for all $\boldsymbol{\mu} \in X$. Let $\Gamma(\boldsymbol{\mu}) = \{(\bar{\boldsymbol{x}}, \bar{\boldsymbol{q}}^s, \bar{\boldsymbol{q}}^b) :$ constraints (1.38b) - (1.38f) hold.$\}$, which is independent from $\boldsymbol{\mu}$ and thus, both upper and lower hemicontinuous at $\boldsymbol{\mu} \in \mathcal{U}$. By the Maximum theorem (page 116 of (49)), the correspondence of the optimal solutions $\Phi(\boldsymbol{\mu}) = \{((\bar{\boldsymbol{x}}, \bar{\boldsymbol{q}}^s, \bar{\boldsymbol{q}}^b)) : (\bar{\boldsymbol{x}}, \bar{\boldsymbol{q}}^s, \bar{\boldsymbol{q}}^b)$ is optimal in problem (1.38) given $\boldsymbol{\mu}\}$ is upper hemicontinuous at $\boldsymbol{\mu} \in X$. By definition of upper hemicontinuity, for any $\bar{\boldsymbol{\mu}}_k, \bar{\boldsymbol{\mu}} \in X$ such that $\bar{\boldsymbol{\mu}}_k \to \bar{\boldsymbol{\mu}}$ and $(\bar{\boldsymbol{x}}_k, \bar{\boldsymbol{q}}_k^s, \bar{\boldsymbol{q}}_k^b) \in \Phi(\bar{\boldsymbol{\mu}}_k)$ such that $(\bar{\boldsymbol{x}}_k, \bar{\boldsymbol{q}}_k^s, \bar{\boldsymbol{q}}_k^b) \to (\bar{\boldsymbol{x}}, \bar{\boldsymbol{q}}^s, \bar{\boldsymbol{q}}^b)$, we have $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{q}}^s, \bar{\boldsymbol{q}}^b) \in \Phi(\bar{\boldsymbol{\mu}})$. Note that since $\bar{\boldsymbol{q}}_k^s = \boldsymbol{q}^s(\bar{\boldsymbol{\mu}}_k)$ and $\bar{\boldsymbol{q}}^s = \boldsymbol{q}^s(\bar{\boldsymbol{\mu}})$, we obtain that $\boldsymbol{q}^s(\boldsymbol{\mu})$ is continuous in $\boldsymbol{\mu} \in \mathcal{U}$. Similarly, since $\bar{\boldsymbol{q}}_k^b = \boldsymbol{q}^b(\bar{\boldsymbol{\mu}}_k)$ and $\bar{\boldsymbol{q}}^b = \boldsymbol{q}^b(\bar{\boldsymbol{\mu}})$, we conclude that $\boldsymbol{q}^b(\boldsymbol{\mu})$ is continuous in $\boldsymbol{\mu} \in \mathcal{U}$.

Step 2: trading components under $\boldsymbol{\mu} \in \mathcal{U}$. For any $\boldsymbol{\mu} \in \mathcal{U}$, from the conditions in Proposition 12(iii), we cluster sellers and buyers $(\mathcal{S}, \mathcal{B})$ into disjoint components $\{(\mathcal{S}_k, \mathcal{B}_k)\}_{k=1}^l$

where (1) $\mathcal{B}_k \subset \mathcal{B}$ is such that $\theta^b_j(\boldsymbol{\mu}) = \theta^b_{j'}(\boldsymbol{\mu})$ for all $j, j' \in \mathcal{B}_k$, and (2) $\mathcal{S}_k \subset \mathcal{S}$ is such that $\pi_{ij}(\boldsymbol{\mu}) = 0$ for all $(i,j) \in E$ with $i \in \mathcal{S}_k$ and $j \in \mathcal{B}_k$. By construction, we can match $\mathcal{S}_k$ with $\mathcal{B}_k$ such that $\theta^s_i(\boldsymbol{\mu}) + c_i = \theta^b_j(\boldsymbol{\mu})$ for $i \in \mathcal{S}_k$ and $j \in \mathcal{B}_k$ (condition (1.40c)). Note that the degenerate cases are possible: (1) $\mathcal{S}_k \neq \emptyset$ and $\mathcal{B}_k = \emptyset$, which corresponds to $q^s_i(\boldsymbol{\mu}) = 0$ for all $i \in \mathcal{S}_k$; (2) $\mathcal{S}_k = \emptyset$ and $\mathcal{B}_k \neq \emptyset$, which corresponds to $q^b_j(\boldsymbol{\mu}) = 0$ for all $j \in \mathcal{B}_k$.

To ease notation, for any $k \in \{1, \ldots, l\}$, we let $t_k := \theta^b_j(\boldsymbol{\mu})$ for $j \in \mathcal{B}_k$. Without loss of generality, we sort the components $(\mathcal{S}_k, \mathcal{B}_k)$ by increasing order of $\{t_k\}_{k=1}^l$ i.e., $t_0 \leq t_1 < t_2 \cdots < t_l < t_{l+1}$ (for completion, let $t_0 = 0$ and $t_{l+1} = \infty$). We show that if $t_1 = 0$, this corresponds to the case where $q^b_j(\boldsymbol{\mu}) = 0$ for $j \in \mathcal{B}_1$. Suppose towards contradiction that there exits $j \in \mathcal{B}_1$ such that $q^b_j(\boldsymbol{\mu}) > 0$. Then, we can pick i with $(i,j) \in E$ and $q^s_i(\boldsymbol{\mu}) > 0$, which implies that $\pi_{ij}(\boldsymbol{\mu}) = 0$ (condition (1.40f) and $\theta^b_j(\boldsymbol{\mu}) = \theta^s_i(\boldsymbol{\mu})$ (condition (1.40c)). By condition (1.40b), as $q^s_i(\boldsymbol{\mu}) > 0$, we obtain that $\theta^s_i(\boldsymbol{\mu}) = F^{-1}_{s_i}(\frac{q^s_i(\boldsymbol{\mu}))}{s_i}) + \mu^s_i + \eta^s_i > 0$, which further implies that $\theta^b_1(\boldsymbol{\mu}) > 0$, thereby leading to a contradiction to $\theta^b_j(\boldsymbol{\mu}) = t_1 = 0$. Thus,

$$\text{if } t_1 = 0, \text{ then } q^b_j(\boldsymbol{\mu}) = 0, \quad \forall j \in \mathcal{B}_1. \tag{1.194}$$

It is also worth noting that

$$\text{there exists no edge } (i,j) \in E \text{ where } i \in \mathcal{S}_{k_1}, j \in \mathcal{B}_{k_2}, \text{and } k_1 < k_2. \tag{1.195}$$

To prove condition (1.195), suppose towards contradiction that there exists $(i,j) \in E$ where $i \in \mathcal{S}_{k_1}$, $j \in \mathcal{B}_{k_2}$, and $k_1 < k_2$. Given $\theta^s_i(\boldsymbol{\mu}) + c_i - \pi_{ij}(\boldsymbol{\mu}) = \theta^b_j(\boldsymbol{\mu})$ (condition (1.40c)) and $\pi_{ij}(\boldsymbol{\mu}) \geq 0$ (condition (1.40f)), we establish that $\theta^s_i(\boldsymbol{\mu}) + c_i \geq \theta^b_j(\boldsymbol{\mu})$. Moreover, given $t_{k_1} = \theta^s_i(\boldsymbol{\mu}) + c_i$, $t_{k_2} = \theta^b_j(\boldsymbol{\mu})$ and $t_{k_1} < t_{k_2}$, we have $\theta^s_i(\boldsymbol{\mu}) + c_i < \theta^b_j(\boldsymbol{\mu})$, thereby leading to a contradiction. Thus, condition (1.195) holds. Moreover, we also deduce that

$$x_{ij} = 0, \quad \forall (i,j) \in E \text{ with } i \in \mathcal{S}_{k_1},\ j \in \mathcal{B}_{k_2},\ k_1 > k_2. \tag{1.196}$$

To prove this claim, suppose towards contradiction that there exists $(i, j) \in E$ with $x_{ij} > 0$ where $i \in \mathcal{S}_{k_1}$, $j \in \mathcal{B}_{k_2}$ and $k_1 > k_2$. By condition (1.40f), we have $\pi_{ij}(\boldsymbol{\mu}) = 0$, thereby leading to a contradiction with $k_1 = k_2$ (by the construction of clusters $\{(\mathcal{S}_k, \mathcal{B}_k)\}_{k=1}^{l}$).

By Proposition 1, we can consider the equilibrium price $p_i = \theta_i^s(\boldsymbol{\mu}) = t_k - c_i$ for all $i \in \mathcal{S}_k$ and $k \in \{1, 2, \dots, l\}$. By (1.195) and (1.196), buyers in $\mathcal{B}_k$ only trade with sellers in $\mathcal{S}_k$ and for all $k \in \{1, 2, \dots, l\}$. Moreover, if type-j buyers trade with type-i sellers where $i \in \mathcal{S}_k$ and $j \in \mathcal{B}_k$, they trade at a price of $p_i = t_k - c_i$.

Step 3: proof of claim (i). For all $j_0 \in \mathcal{B}$, we apply a sensitivity analysis to establish that $q_i^s(\boldsymbol{\mu})$ is weakly decreasing in $\mu_{j_0}^b$ for all $i \in \mathcal{S}$. Fix $j_0 \in \mathcal{B}$. We find the index $k_0 \in \{1, \dots, l\}$ such that $j_0 \in \mathcal{B}_{k_0}$. To prove the claim, from the continuity of $\boldsymbol{q}^s(\boldsymbol{\mu})$ in $\boldsymbol{\mu} \in \mathcal{U}$, it is sufficient to establish that there exists $\delta_0 > 0$ such that, for any $\delta \in [0, \delta_0]$, any competitive equilibrium $(\tilde{\boldsymbol{p}}, \tilde{\boldsymbol{x}}, \boldsymbol{q}^s(\tilde{\boldsymbol{\mu}}), \boldsymbol{q}^b(\tilde{\boldsymbol{\mu}})) \in \mathcal{X}(\mathbf{0}, \tilde{\boldsymbol{\mu}})$ induced by the alterantive subscription profile $\tilde{\boldsymbol{\mu}} = (\boldsymbol{\mu}^s, \mu_{j_0}^b + \delta, \boldsymbol{\mu}_{-j_0}^b)$ satisfies that $q_i^s(\tilde{\boldsymbol{\mu}}) \leq q_i^s(\boldsymbol{\mu})$ for all $i \in \mathcal{S}$.

We prove the claim by discussing the following two cases:

Case-(i-1): $q_{j_0}^b(\boldsymbol{\mu}) > 0$. Pick any i with $x_{ij_0} > 0$ such that $q_{i_0}^s(\boldsymbol{\mu}) > 0$ and thus $\pi_{ij_0}(\boldsymbol{\mu}) = 0$ (by condition (1.40f)). Moreover, we can find the matching cluster $\mathcal{S}_{k_0}$ such that $i \in \mathcal{S}_{k_0}$ (by construction). Since $q_{j_0}^b(\boldsymbol{\mu}) > 0$, we have $t_{k_0} > 0$ (by condition (1.194)). In the subnetwork $G(\mathcal{S}_{k_0} \cup \mathcal{B}_{k_0}, E_{k_0})$ where $E_{k_0} \subset E$ is the maximal edge set induced by $(\mathcal{S}_{k_0}, \mathcal{B}_{k_0})$, we take the maximal connected component $G(\mathcal{S}'_{k_0} \cup \mathcal{B}'_{k_0}, E'_{k_0})$ such that $j_0 \in \mathcal{B}'_{k_0}$. The goal is to show that after we increase $\mu_{j_0}^b$ by a small amount, it would only impact the trades in the network component $G(\mathcal{S}'_{k_0} \cup \mathcal{B}'_{k_0}, E'_{k_0})$.

Recalling that the buyers in $\mathcal{B}'_{k_0}$ only trade with the sellers in $\mathcal{S}'_{k_0}$ at a price of $t_{k_0} - c_i$ for all $i \in \mathcal{S}'_{k_0}$, we have $\sum_{j \in \mathcal{B}'_{k_0}} q_j^b(\boldsymbol{\mu}) - \sum_{i \in \mathcal{S}'_{k_0}} q_i^s(\boldsymbol{\mu}) = 0$. Expressing $q_i^s(\boldsymbol{\mu})$ and $q_j^b(\boldsymbol{\mu})$ using the equilibrium conditions in (1.2a) and (1.2b), we have

$$\sum_{j \in \mathcal{B}'_{k_0}} b_j [1 - F_{b_j}(t_{k_0} + \mu_j^b)] - \sum_{i \in \mathcal{S}'_{k_0}} s_i F_{s_i}(t_{k_0} - c_i - \mu_i^s) \;=\; 0. \tag{1.197}$$

By the continuity and the weakly increasingness of $F_{b_j}(v)$ and $F_{s_i}(v)$ in $v \in \mathbb{R}$ (by definition), we obtain that function $\sum_{j\in\mathcal{B}'_{k_0}} b_j(1 - F_{b_j}(t + \mu^b_j)) - \sum_{i\in\mathcal{S}'_{k_0}} s_i F_{s_i}(t - c_i - \mu^s_i)$ is continuous and weakly decreasing in $t \in \mathbb{R}$. Moreover, as $q^b_{j_0}(\boldsymbol{\mu}) > 0$ (the assumption in this case) and $q^b_{j_0}(\boldsymbol{\mu}) < b_{j_0}$ (Proposition 12(v)), it follows that $t_{k_0} + \mu^b_{j_0} \in (0, \bar{v}_{b_{j_0}})$. As $F_{b_j}(v)$ is stricly increasing in $v \in (0, \bar{v}_{b_j})$ (Assumption 1), we obtain that $\sum_{j\in\mathcal{B}'_{k_0}} b_j(1 - F_{b_j}(t + \mu^b_j)) - \sum_{i\in\mathcal{S}'_{k_0}} s_i F_{s_i}(t - c_i - \mu^s_i)$ is strictly decreasing in the neighborhood of $t = t_{k_0}$. From (1.197) and $t_{k_0-1} < t_{k_0}$ (the ordering of t_k), it follows that

$$\sum_{j\in\mathcal{B}'_{k_0}} b_j[1 - F_{b_j}(t_{k_0-1} + \mu^b_j)] - \sum_{i\in\mathcal{S}'_{k_0}} s_i F_{s_i}(t_{k_0-1} - c_i - \mu^s_i) \ > \ 0. \tag{1.198}$$

Fix $\delta_0 > 0$ small enough such that, if we further pick any $\delta \in [0, \delta_0]$ and set $\tilde{\boldsymbol{\mu}} = (\boldsymbol{\mu}^s, \mu^b_{j_0} + \delta, \boldsymbol{\mu}^b_{-j_0})$, we obtain

$$\begin{aligned}
&\sum_{j\in\mathcal{B}'_{k_0}} b_j[1 - F_{b_j}(t_{k_0} + \tilde{\mu}^b_j)] - \sum_{i\in\mathcal{S}'_{k_0}} s_i F_{s_i}(t_{k_0} - c_i - \tilde{\mu}^s_i) \ \le \ 0, \\
&\sum_{j\in\mathcal{B}'_{k_0}} b_j[1 - F_{b_j}(t_{k_0-1} + \tilde{\mu}^b_j)] - \sum_{i\in\mathcal{S}'_{k_0}} s_i F_{s_i}(t_{k_0-1} - c_i - \tilde{\mu}^s_i) \ > \ 0.
\end{aligned} \tag{1.199}$$

For any $\delta \in [0, \delta_0]$, by the continuity of function $\sum_{j\in\mathcal{B}'_{k_0}} b_j(1 - F_{b_j}(t + \tilde{\mu}^b_j)) - \sum_{i\in\mathcal{S}'_{k_0}} s_i F_{s_i}(t - c_i - \tilde{\mu}^s_i)$ in $t \in \mathbb{R}$, there exists $\tilde{t}_{k_0} \in (t_{k_0-1}, t_{k_0}]$ such that

$$\sum_{j\in\mathcal{B}'_{k_0}} b_j[1 - F_{b_j}(\tilde{t}_{k_0} + \tilde{\mu}^b_j)] - \sum_{i\in\mathcal{S}'_{k_0}} s_i F_{s_i}(\tilde{t}_{k_0} - c_i - \tilde{\mu}^s_i) = 0. \tag{1.200}$$

To move from $(\boldsymbol{p}, \boldsymbol{x}, \boldsymbol{q}^s(\boldsymbol{\mu}), \boldsymbol{q}^b(\boldsymbol{\mu})) \in \mathcal{X}(\boldsymbol{0}, \boldsymbol{\mu})$ to $(\tilde{\boldsymbol{p}}, \tilde{\boldsymbol{x}}, \boldsymbol{q}^s(\tilde{\boldsymbol{\mu}}), \boldsymbol{q}^b(\tilde{\boldsymbol{\mu}})) \in \mathcal{X}(\boldsymbol{0}, \tilde{\boldsymbol{\mu}})$, we consider the following adjustment:

(1-1) adjust the equilibrium price $\tilde{\boldsymbol{p}}$ such that $\tilde{p}_i = \tilde{t}_{k_0} - c_i$ for all $i \in \mathcal{S}'_{k_0}$ and $\tilde{p}_i = p_i$ for all $i \notin \mathcal{S}'_{k_0}$. Given that $\tilde{t}_{k_0} \in (t_{k_0-1}, t_{k_0}]$, the price adjustment ensures that the buyers in components $\mathcal{B}_k$ would still only trade with the sellers in components $\mathcal{S}_k$ for $k \neq k_0$.

Moreover, since the buyers in components $\mathcal{B}_{k_0}/\mathcal{B}'_{k_0}$ are not connected with $\mathcal{S}'_{k_0}$, they would only trade with the sellers in $\mathcal{S}_{k_0}/\mathcal{S}'_{k_0}$. Finally, the buyers in components $\mathcal{B}'_{k_0}$ would only trade with the sellers in $\mathcal{S}'_{k_0}$. By condition (1.195) and condition (1.196), this ensures that incentive compatibility constraint in the equilibrium expression (1.2d) is preserved;

(1-2) adjust the equilibrium supply to $q^s_i(\tilde{\boldsymbol{\mu}})$ such that $q^s_i(\tilde{\boldsymbol{\mu}}) = s_i F_{s_i}(\tilde{t}_{k_0} - c_i - \tilde{\mu}^s_i)$ for all $i \in \mathcal{S}'_{k_0}$ and $q^s_i(\tilde{\boldsymbol{\mu}}) = q^s_i(\boldsymbol{\mu})$ for all $i \notin \mathcal{S}'_{k_0}$. This preserves the equilibrium expression (1.2a). It is also worth noting that since $\tilde{t}_{k_0} \in (t_{k_0-1}, t_{k_0}]$ and $\tilde{\boldsymbol{\mu}}^s = \boldsymbol{\mu}^s$, we have $q^s_i(\tilde{\boldsymbol{\mu}}) \leq q^s_i(\boldsymbol{\mu})$ for all $i \in \mathcal{S}'_{k_0}$ by the weak increasingness of $F_{s_i}(v)$ for $v \in \mathbb{R}$ (by definition);

(1-3) adjust the demand to $q^b_j(\tilde{\boldsymbol{\mu}})$ such that $q^b_j(\tilde{\boldsymbol{\mu}}) = b_j[1 - F_{b_j}(\tilde{t}_{k_0} + \tilde{\mu}^b_j)]$ for $j \in \mathcal{B}'_{k_0}$ and $q^b_j(\tilde{\boldsymbol{\mu}}) = q^b_j(\boldsymbol{\mu})$ for all $j \notin \mathcal{B}'_{k_0}$. Since $\tilde{t}_{k_0} \in (t_{k_0-1}, t_{k_0}]$, this preserves the equilibrium expression (1.2b). Moreover, from $\tilde{\boldsymbol{\mu}}^b_{-j_0} = \boldsymbol{\mu}^b_{-j_0}$, we have $q^b_j(\tilde{\boldsymbol{\mu}}) \geq q^b_j(\boldsymbol{\mu})$ for all $j \in \mathcal{B}'_{k_0}/\{j_0\}$ by the weak increasingness of $F_{b_j}(v)$ for $v \in \mathbb{R}$ (by definition). Moreover, by the supply-demand balance condition in (1.200), we have $q^b_{j_0}(\tilde{\boldsymbol{\mu}}) \leq q^b_{j_0}(\boldsymbol{\mu})$;

(1-4) adjust the flow to $\tilde{\boldsymbol{x}}$ by sending a flow of $q^b_{j_0}(\boldsymbol{\mu}) - q^b_{j_0}(\tilde{\boldsymbol{\mu}})$ from j_0 to satisfy the supply decrease in $i \in \mathcal{S}'_{k_0}$ and the demand increase in $j \in \mathcal{B}'_{k_0}/\{j_0\}$. By the connectedness of $G(\mathcal{S}'_{k_0} \cup \mathcal{B}'_{k_0}, E'_{k_0})$ and supply-demand balance equations (1.197) and (1.200), this adjustment is feasible and the equilibrium expression (1.2c) and the non-negativity of the flow in the equilibrium expression (1.2d) are preserved.

Summarizing, we now end up with another equilibrium $(\tilde{\boldsymbol{p}}, \tilde{\boldsymbol{x}}, \boldsymbol{q}^s(\tilde{\boldsymbol{\mu}}), \boldsymbol{q}^b(\tilde{\boldsymbol{\mu}})) \in \mathcal{X}(\boldsymbol{0}, \tilde{\boldsymbol{\mu}})$ where $q^s_i(\tilde{\boldsymbol{\mu}}) \leq q^s_i(\boldsymbol{\mu})$ for all $i \in \mathcal{S}$;

Case-(i-2): $q^b_{j_0}(\boldsymbol{\mu}) = 0$. Then, increasing $\mu^b_{j_0}$ to $\tilde{\mu}^b_{j_0} = \mu^b_{j_0} + \delta$ for any $\delta > 0$ still returns $q^b_{j_0}(\tilde{\boldsymbol{\mu}}) = 0$ (by the equilibrium expression (1.2b) given the weak increasingness of $F_{b_{j_0}}(v)$ in $v \in \mathbb{R}$). The supply demand vector in the new competitive equilibrium does not change, so we have $q^s_i(\tilde{\boldsymbol{\mu}}) = q^s_i(\boldsymbol{\mu})$ for all $i \in \mathcal{S}$.

In summary of case-(1) and case-(2), we conclude that claim (i) holds.

Step 4: proof of claim (ii). We follow a similar sensitivity arguments as in claim (i). For all $i_0 \in \mathcal{S}$, we show that $q^b_j(\boldsymbol{\mu})$ is weakly decreasing in $\mu^s_{i_0}$ for all $j \in \mathcal{B}$. We first fix any $i_0 \in \mathcal{S}$ and find the component index $k_0 \in \{1, \ldots, l\}$ such that $i_0 \in \mathcal{S}_{k_0}$. Given the continuity of $\boldsymbol{q}^b(\boldsymbol{\mu})$ in $\boldsymbol{\mu} \in \mathcal{U}$, it is sufficient to establish that there exists $\delta_0 > 0$ such that for any $\delta \in [0, \delta_0]$, a competitive equilibrium $(\tilde{\boldsymbol{p}}, \tilde{\boldsymbol{x}}, \boldsymbol{q}^s(\tilde{\boldsymbol{\mu}}), \boldsymbol{q}^b(\tilde{\boldsymbol{\mu}})) \in \mathcal{X}(\mathbf{0}, \tilde{\boldsymbol{\mu}})$ induced by the alternative subscription profile $\tilde{\boldsymbol{\mu}} = (\mu^s_{i_0} + \delta, \boldsymbol{\mu}^s_{-i_0}, \boldsymbol{\mu}^b)$ satisfies $q^b_j(\tilde{\boldsymbol{\mu}}) \leq q^b_j(\boldsymbol{\mu})$ for all $j \in \mathcal{B}$. We prove the claim by discussing the following two cases:

Case - (ii-1): $q^s_{i_0}(\boldsymbol{\mu}) \in (0, s_i)$. Pick any j with $x_{i_0 j} > 0$ such that $q^b_{j_0}(\boldsymbol{\mu}) > 0$ and $\pi_{i_0 j}(\boldsymbol{\mu}) = 0$ (by condition (1.40f)). Moreover, we can find the matching cluster $\mathcal{B}_{k_0}$ such that $j \in \mathcal{B}_{k_0}$. In the subnetwork $G(\mathcal{S}_{k_0} \cup \mathcal{B}_{k_0}, E_{k_0})$ where $E_{k_0} \subset E$ is the maximal edge set induced by node set $(\mathcal{S}_{k_0}, \mathcal{B}_{k_0})$, we take the maximal connected component $G(\mathcal{S}'_{k_0} \cup \mathcal{B}'_{k_0}, E'_{k_0})$ such that $i_0 \in \mathcal{S}'_{k_0}$. The goal is to show that when we increase $\mu^s_{i_0}$ by a small amount, it would only impact the trades in network component $G(\mathcal{S}'_{k_0} \cup \mathcal{B}'_{k_0}, E'_{k_0})$.

Recall that the buyers in $\mathcal{B}'_{k_0}$ only trade with the sellers in $\mathcal{S}'_{k_0}$ at a price $p_i = t_{k_0} - c_i$ for all $i \in \mathcal{S}'_{k_0}$. It follows that $\sum_{j \in \mathcal{B}'_{k_0}} q^b_j(\boldsymbol{\mu}) - \sum_{i \in \mathcal{S}'_{k_0}} q^s_i(\boldsymbol{\mu}) = 0$. Expressing $q^s_i(\boldsymbol{\mu})$ and $q^b_j(\boldsymbol{\mu})$ by the equilibrium expressions (1.2a) and (1.2b), we have

$$\sum_{j \in \mathcal{B}'_{k_0}} b_j[1 - F_{b_j}(t_{k_0} + \mu^b_j)] - \sum_{i \in \mathcal{S}'_{k_0}} s_i F_{s_i}(t_{k_0} - c_i - \mu^s_i) \ = \ 0. \tag{1.201}$$

By the continuity and the weak increasingness of $F_{b_j}(v)$ and $F_{s_i}(v)$ in $v \in \mathbb{R} \cup \{-\infty, \infty\}$ (by definition), we obtain that function $\sum_{j \in \mathcal{B}'_{k_0}} b_j(1 - F_{b_j}(t + \mu^b_j)) - \sum_{i \in \mathcal{S}'_{k_0}} s_i F_{s_i}(t - c_i - \mu^s_i)$ is continuous and weakly decreasing in $t \in \mathbb{R} \cup \{-\infty, \infty\}$. Moreover, since $q^s_{i_0}(\boldsymbol{\mu}) \in (0, s_i)$, we have $t_{k_0} - c_i - \mu^s_{i_0} \in (0, \bar{v}_{s_{i_0}})$. By the strict increasingness of $F_{s_i}(v)$ in $v \in (0, \bar{v}_{s_i})$ (Assumption 1), this implies that $\sum_{j \in \mathcal{B}'_{k_0}} b_j(1 - F_{b_j}(t + \mu^b_j)) - \sum_{i \in \mathcal{S}'_{k_0}} s_i F_{s_i}(t - c_i - \mu^s_i)$ is strictly decreasing in the neighborhood of $t = t_{k_0}$. Given condition (1.201) and $t_{k_0} < t_{k_0+1}$ (by the ordering of t_k), this implies that

$$\sum_{j \in \mathcal{B}'_{k_0}} b_j[1 - F_{b_j}(t_{k_0+1} + \mu^b_j)] - \sum_{i \in \mathcal{S}'_{k_0}} s_i F_{s_i}(t_{k_0+1} - c_i - \mu^s_i) \ < \ 0. \tag{1.202}$$

Fix $\delta_0 > 0$ small enough such that if we further pick any $\delta \in [0, \delta_0]$ and let $\tilde{\boldsymbol{\mu}} = (\mu^s_{i_0} + \delta, \boldsymbol{\mu}^s_{-i_0}, \boldsymbol{\mu}^b)$, we obtain

$$\begin{aligned} \sum_{j\in\mathcal{B}'_{k_0}} b_j[1 - F_{b_j}(t_{k_0} + \tilde{\mu}^b_j)] - \sum_{i\in\mathcal{S}'_{k_0}} s_iF_{s_i}(t_{k_0} - c_i - \tilde{\mu}^s_i) &\geq 0, \\ \sum_{j\in\mathcal{B}'_{k_0}} b_j[1 - F_{b_j}(t_{k_0+1} + \tilde{\mu}^b_j)] - \sum_{i\in\mathcal{S}'_{k_0}} s_iF_{s_i}(t_{k_0+1} - c_i - \tilde{\mu}^s_i) &< 0. \end{aligned} \tag{1.203}$$

For any $\delta \in [0, \delta_0]$, by the continuity of function $\sum_{j\in\mathcal{B}'_{k_0}} b_j(1-F_{b_j}(t+\tilde{\mu}^b_j))-\sum_{i\in\mathcal{S}'_{k_0}} s_iF_{s_i}(t-c_i - \tilde{\mu}^s_i)$ in t, there exists $\tilde{t}_{k_0} \in [t_{k_0}, t_{k_0+1})$ such that

$$\sum_{j\in\mathcal{B}'_{k_0}} b_j[1 - F_{b_j}(\tilde{t}_{k_0} + \tilde{\mu}^b_j)] - \sum_{i\in\mathcal{S}'_{k_0}} s_iF_{s_i}(\tilde{t}_{k_0} - c_i - \tilde{\mu}^s_i) = 0. \tag{1.204}$$

To move from $(\boldsymbol{p}, \boldsymbol{x}, \boldsymbol{q}^s(\boldsymbol{\mu}), \boldsymbol{q}^b(\boldsymbol{\mu})) \in \mathcal{X}(\mathbf{0}, \boldsymbol{\mu})$ to $(\tilde{\boldsymbol{p}}, \tilde{\boldsymbol{x}}, \boldsymbol{q}^s(\tilde{\boldsymbol{\mu}}), \boldsymbol{q}^b(\tilde{\boldsymbol{\mu}})) \in \mathcal{X}(\mathbf{0}, \tilde{\boldsymbol{\mu}})$, we consider the following adjustment:

(2-1) adjust the equilibrium price $\tilde{\boldsymbol{p}}$ such that $\tilde{p}_i = \tilde{t}_{k_0} - c_i$ for all $i \in \mathcal{S}'_{k_0}$ and $\tilde{p}_i = p_i$ for all $i \notin \mathcal{S}'_{k_0}$. Given that $\tilde{t}_{k_0} \in [t_{k_0}, t_{k_0+1})$, the price adjustment ensures that buyers in components $\mathcal{B}_k$ would still only trade with sellers in components $\mathcal{S}_k$ for $k \neq k_0$. Moreover, since buyers in components $\mathcal{B}_{k_0}/\mathcal{B}'_{k_0}$ are not connected with $\mathcal{S}'_{k_0}$, they would only trade with sellers in $\mathcal{S}_{k_0}/\mathcal{S}'_{k_0}$. Finally, buyers in components $\mathcal{B}'_{k_0}$ would only trade with $\mathcal{S}'_{k_0}$. By condition (1.195) and condition (1.196), this ensures that incentive compatibility constraint in the equilibrium expression (1.2d) is preserved;

(2-2) adjust the equilibrium demand to $q^b_j(\tilde{\boldsymbol{\mu}})$ such that $q^b_j(\tilde{\boldsymbol{\mu}}) = b_j[1 - F_{b_j}(\tilde{t}_{k_0} + \tilde{\mu}^b_j)]$ for $j \in \mathcal{B}'_{k_0}$ and $q^b_j(\tilde{\boldsymbol{\mu}}) = q^b_j(\boldsymbol{\mu})$ for all $j \notin \mathcal{B}'_{k_0}$. Since $\tilde{t}_{k_0} \in [t_{k_0}, t_{k_0+1})$, this preserves the equilibrium expression (1.2b). Moreover, since $\tilde{\boldsymbol{\mu}}^b = \boldsymbol{\mu}^b$, we have $q^b_j(\tilde{\boldsymbol{\mu}}) \leq q^b_j(\boldsymbol{\mu})$ for all $j \in \mathcal{B}'_{k_0}$ by the weak increasingness of $F_{b_j}(v)$ for $v \in \mathbb{R}$ (by definition);

(2-3) adjust the equilibrium supply to $q^s_i(\tilde{\boldsymbol{\mu}})$ such that $q^s_i(\tilde{\boldsymbol{\mu}}) = s_iF_{s_i}(\tilde{t}_{k_0} - c_i - \tilde{\mu}^s_i)$ for all $i \in \mathcal{S}'_{k_0}$ and $q^s_i(\tilde{\boldsymbol{\mu}}) = q^s_i(\boldsymbol{\mu})$ for all $i \notin \mathcal{S}'_{k_0}$. This preserves the equilibrium expression

(1.2a). Since $\tilde{t}_{k_0} \in [t_{k_0}, t_{k_0+1})$ and $\tilde{\boldsymbol{\mu}}^s_{-i_0} = \boldsymbol{\mu}^s_{-i_0}$, we obtain that $q^s_i(\tilde{\boldsymbol{\mu}}) \geq q^s_i(\boldsymbol{\mu})$ for all $i \in \mathcal{S}'_{k_0}/\{i_0\}$ by the weak increasingness of $F_{s_i}(v)$ for $v \in \mathbb{R}$ (by definition). Moreover, by the supply-demand balance condition in (1.200), we have $q^s_{i_0}(\tilde{\boldsymbol{\mu}}) \leq q^s_{i_0}(\boldsymbol{\mu})$;

(2-4) adjust the flow to $\tilde{\boldsymbol{x}}$ by sending a flow of $q^s_{i_0}(\boldsymbol{\mu}) - q^s_{i_0}(\tilde{\boldsymbol{\mu}})$ from i_0 to satisfy the supply increase in $i \in \mathcal{S}'_{k_0}/\{i_0\}$ and the demand decrease in $j \in \mathcal{B}'_{k_0}$. By the connectedness of $G(\mathcal{S}'_{k_0} \cup \mathcal{B}'_{k_0}, E'_{k_0})$ and the supply-demand balance equations (1.201) and (1.204), the equilibrium condition (1.2c) and the non-negativity of the flow in the equilibrium condition (1.2d) are preserved.

Case - (ii-2): $q^s_{i_0}(\boldsymbol{\mu}) = 0$. Given the equilibrium price $\boldsymbol{p}$, increasing $\mu^s_{i_0}$ to $\tilde{\mu}^s_i = \mu^s_i + \delta$ for any $\delta > 0$ still results in $q^s_{i_0}(\tilde{\boldsymbol{\mu}}) = 0$ (by the equilibrium expression (1.2a) and the weak increasingness of $F_{s_{i_0}}(v)$ in $v \in \mathbb{R}$). Thus, we have $q^b_j(\tilde{\boldsymbol{\mu}}) = q^b_j(\boldsymbol{\mu})$ for all $j \in \mathcal{B}$;

Case - (ii-3): $q^s_{i_0}(\boldsymbol{\mu}) = s_{i_0}$. We further discuss the following two subcases. If $p_{i_0} - \mu^s_{i_0} = \tilde{F}^{-1}_{s_{i_0}}(\frac{q^s_{i_0}(\boldsymbol{\mu})}{s_{i_0}})$, then we repeat the same arguments as in case - (ii-1) to establish that $q^b_j(\tilde{\boldsymbol{\mu}}) \leq q^b_j(\boldsymbol{\mu})$ for all $j \in \mathcal{B}$. If $p_{i_0} - \mu^s_{i_0} > \tilde{F}^{-1}_{s_{i_0}}(\frac{q^s_{i_0}(\boldsymbol{\mu})}{s_{i_0}})$, given the equilibrium price $\boldsymbol{p}$, increasing $\mu^s_{i_0}$ by any $\delta \in (0, p_{i_0} - \mu^s_{i_0} - \tilde{F}^{-1}_{s_{i_0}}(\frac{q^s_{i_0}(\boldsymbol{\mu})}{s_{i_0}}))$ still returns $q^s_{i_0}(\tilde{\boldsymbol{\mu}}) = s_{i_0}$. Thus, we have $q^b_j(\tilde{\boldsymbol{\mu}}) = q^b_j(\boldsymbol{\mu})$ for all $j \in \mathcal{B}$.

In summary, we end up with another equilibrium $(\tilde{\boldsymbol{p}}, \tilde{\boldsymbol{x}}, \boldsymbol{q}^s(\tilde{\boldsymbol{\mu}}), \boldsymbol{q}^b(\tilde{\boldsymbol{\mu}})) \in \mathcal{X}(\boldsymbol{0}, \tilde{\boldsymbol{\mu}})$ where $q^b_j(\tilde{\boldsymbol{\mu}}) \leq q^b_j(\boldsymbol{\mu})$ for all $j \in \mathcal{B}$. □

1.16 Proofs of Results in Section 1.7

Auxiliary Results

We start by providing an alternative expression for the welfare at an equilibrium.

Lemma 8. *For any* $(\boldsymbol{\gamma}, \boldsymbol{\mu}) \in \Gamma \times \mathcal{U}$ *and any* $(\boldsymbol{p}, \boldsymbol{x}, \boldsymbol{q}^s, \boldsymbol{q}^b) \in \mathcal{X}(\boldsymbol{\gamma}, \boldsymbol{\mu})$*, the induced welfare*

$W(\boldsymbol{\gamma}, \boldsymbol{\mu})$ *satisfies*

$$W(\boldsymbol{\gamma}, \boldsymbol{\mu}) \;=\; \sum_{j\in\mathcal{B}} \int_0^{q_j^b} F_{b_j}^{-1}\Big(1 - \frac{q}{b_j}\Big) dq - \sum_{i\in\mathcal{S}} \int_0^{q_i^s} F_{s_i}^{-1}\Big(\frac{q}{s_i}\Big) dq - \sum_{i\in\mathcal{S}} c_i q_i^s. \qquad (1.205)$$

Proof of Lemma 8. By Assumption 1, F_{b_j} and F_{s_i} are continuously differentiable and strictly increasing in $(0, \bar{v}_{b_j})$ and $(0, \bar{v}_{s_i})$, which implies that $F_{b_j}^{-1}$ and $F_{s_i}^{-1}$ are continuously differentiable and strictly increasing in $(0, 1)$ (by the inverse function theorem). We define the differentiable functions $\hat{g}_j : [0, b_j] \to \mathbb{R}$ and $\hat{h}_i : [0, s_i] \to \mathbb{R}$ such that:

$$\hat{g}_j(q) = b_j \int_{F_{b_j}^{-1}(1-q/b_j)}^{\bar{v}_{b_j}} v dF_{b_j}(v), \quad \forall j \in \mathcal{B}, \qquad (1.206a)$$

$$\hat{h}_i(q) = s_i \int_0^{F_{s_i}^{-1}(q/s_i)} v dF_{s_i}(v), \quad \forall i \in \mathcal{S}. \qquad (1.206b)$$

It follows from Assumption 1 that $\hat{g}_j$ is continuous and bounded function in $[0, b_j]$ and $\hat{h}_i$ is a continuous and bounded function in $[0, s_i]$. By the definition of $\hat{g}_i$ and the fact that $F_{b_j}^{-1}(1) = \bar{v}_{b_j}$, we obtain that $\hat{g}(0) = b_j \int_{F_{b_j}^{-1}(1)}^{\bar{v}_{b_j}} v dF_{b_j}(v) = 0$. Similarly, it also follows that $\hat{h}_i(0) = 0$. Based on (1.206), we obtain:

$$\begin{aligned} \hat{g}_j'(q) \;&=\; \frac{d}{dq}\left[b_j \int_{F_{b_j}^{-1}(1-q/b_j)}^{\bar{v}_{b_j}} v dF_{b_j}(v) \right] \\ &\overset{(a)}{=} F_{b_j}^{-1}\left(1 - \frac{q}{b_j}\right) F_{b_j}'\left(F_{b_j}^{-1}\left(1 - \frac{q}{b_j}\right)\right) \left[F_{b_j}^{-1}\right]'\left(1 - \frac{q}{b_j}\right) \\ &\overset{(b)}{=} F_{b_j}^{-1}\left(1 - \frac{q}{b_j}\right), \end{aligned} \qquad (1.207)$$

for all $q \in (0, b_j)$ and $j \in \mathcal{B}$. Similarly,

$$\begin{aligned}
\hat{h}_i'(q) &= \frac{d}{dq}\left[s_i \int_0^{F_{s_i}^{-1}(q/s_i)} v dF_{s_i}(v) \right] \\
&\overset{(c)}{=} F_{s_i}^{-1}\left(\frac{q}{s_i}\right) F_{s_i}'\left(F_{s_i}^{-1}\left(\frac{q}{s_i}\right)\right), \left[F_{s_i}^{-1}\right]'\left(\frac{q}{s_i}\right) \\
&\overset{(d)}{=} F_{s_i}^{-1}\left(\frac{q}{s_i}\right), \qquad (1.208)
\end{aligned}$$

for all $q \in (0, s_i)$ and $i \in \mathcal{S}$. Here step (a) and step (c) follow from Leibniz's rule, and step (b) and step (d) follow from the inverse function theorem. Since, $\hat{g}_j$, $\hat{h}_i$ are continuous, $\hat{g}_j(0) = 0$, $\hat{h}_i(0) = 0$, and $F_{b_j}^{-1}$ and $F_{s_i}^{-1}$ are continuously differentiable in $(0, 1)$, using the fundamental theorem of calculus, we obtain that

$$\hat{g}_j(q) = \int_0^q F_{b_j}^{-1}\left(1 - \frac{x}{b_j}\right) dx, \quad \forall j \in \mathcal{B}, \qquad (1.209a)$$

$$\hat{h}_i(q) = \int_0^q F_{s_i}^{-1}\left(\frac{x}{s_i}\right) dx, \quad \forall i \in \mathcal{S}. \qquad (1.209b)$$

For any $(\boldsymbol{p}, \boldsymbol{x}, \boldsymbol{q}^s, \boldsymbol{q}^b) \in \mathcal{X}(\boldsymbol{\gamma}, \boldsymbol{\mu})$ and any $(\boldsymbol{\gamma}, \boldsymbol{\mu}) \in \Gamma \times \mathcal{U}$, using the fact that $v_{b_j}^m = F_{b_j}^{-1}(1 - \frac{q_j^b}{b_j})$ and $v_{s_i}^m = F_{s_i}^{-1}(\frac{q_i^s}{s_i})$, we conclude that the induced welfare satisfies

$$\begin{aligned}
W(\boldsymbol{\gamma}, \boldsymbol{\mu}) &\overset{(e)}{=} \sum_{j \in \mathcal{B}} b_j \int_{v_{b_j}^m}^{\bar{v}_{b_j}} v dF_{b_j}(v) - \sum_{i \in \mathcal{S}} s_i \int_0^{v_{s_i}^m} v dF_{s_i}(v) - \sum_{(i,j) \in E} c_i x_{ij} \\
&\overset{(f)}{=} \sum_{j \in \mathcal{B}} \int_0^{q_j^b} F_{b_j}^{-1}\left(1 - \frac{x}{b_j}\right) dx - \sum_{i \in \mathcal{S}} \int_0^{q_i^s} F_{s_i}^{-1}\left(\frac{x}{s_i}\right) dx - \sum_{(i,j) \in E} c_i x_{ij} \\
&\overset{(g)}{=} \sum_{j \in \mathcal{B}} \int_0^{q_j^b} F_{b_j}^{-1}\left(1 - \frac{x}{b_j}\right) dx - \sum_{i \in \mathcal{S}} \int_0^{q_i^s} F_{s_i}^{-1}\left(\frac{x}{s_i}\right) dx - \sum_{i \in \mathcal{S}} c_i q_i^s. \qquad (1.210)
\end{aligned}$$

Here step (e) follows from (1.15), step (f) follows from the definition of $\{v_{b_j}^m, v_{s_i}^m\}$, (1.206) and (1.209). Step (g) follows since $\sum_{j:(i,j) \in E} x_{ij} = q_i^s$ by the equilibrium condition (1.2c). Thus, the claim holds. $\square$

Using this alternative expression for welfare, the welfare maximization problem can be stated as follows:

$$W_{opt} = \max_{(\boldsymbol{p},\boldsymbol{x},\boldsymbol{q}^s,\boldsymbol{q}^b,\boldsymbol{\gamma},\boldsymbol{\mu})} \sum_{j\in\mathcal{B}} \int_0^{q_j^b} F_{b_j}^{-1}\left(1-\frac{q}{b_j}\right)dq - \sum_{i\in\mathcal{S}} \int_0^{q_i^s} F_{s_i}^{-1}\left(\frac{q}{s_i}\right)dq - \sum_{i\in\mathcal{S}} c_i q_i^s \tag{1.211a}$$

$$\text{s.t.} \quad (\boldsymbol{p},\boldsymbol{x},\boldsymbol{q}^s,\boldsymbol{q}^b) \in \mathcal{X}(\boldsymbol{\gamma},\boldsymbol{\mu}), \tag{1.211b}$$

$$(\boldsymbol{\gamma},\boldsymbol{\mu}) \in \Gamma \times \mathcal{U}. \tag{1.211c}$$

Following a similar approach to the revenue maximization problem, as opposed to solving problem (1.211) directly, we consider the following (convex) optimization problem:

$$\max_{\boldsymbol{x},\boldsymbol{q}^s,\boldsymbol{q}^b} \quad \sum_{j\in\mathcal{B}} \int_0^{q_j^b} F_{b_j}^{-1}\left(1-\frac{x}{b_j}\right)dx - \sum_{i\in\mathcal{S}} \int_0^{q_i^s} F_{s_i}^{-1}\left(\frac{x}{s_i}\right)dx - \sum_{(i,j)\in E} c_i x_{ij} \tag{1.212a}$$

$$\text{s.t.} \quad \sum_{j:(i,j)\in E} x_{ij} = q_i^s \quad \forall i \in \mathcal{S}, \tag{1.212b}$$

$$\sum_{i:(i,j)\in E} x_{ij} = q_j^b \quad \forall j \in \mathcal{B}, \tag{1.212c}$$

$$q_i^s \leq s_i \quad \forall i \in \mathcal{S}, \tag{1.212d}$$

$$q_j^b \leq b_j \quad \forall j \in \mathcal{B}, \tag{1.212e}$$

$$x_{ij} \geq 0, \quad \forall (i,j) \in E. \tag{1.212f}$$

We proceed by establishing that the optimal objective values of problem (1.211) and problem (1.212) coincide. Moreover, the maximum welfare is obtained for $(\boldsymbol{\gamma},\boldsymbol{\mu}) = \mathbf{0}$, and the welfare-maximizing equilibrium can be characterized through the solution of (1.212).

Lemma 9. *The optimal solution of problem* (1.212) *yields the trade amounts at a welfare-maximizing equilibrium i.e., the optimal solution* $(\bar{\boldsymbol{x}},\bar{\boldsymbol{q}}^s,\bar{\boldsymbol{q}}^b)$ *to problem* (1.212) *is such that for some price vector* $\boldsymbol{p}$*, we have* $(\boldsymbol{p},\bar{\boldsymbol{x}},\bar{\boldsymbol{q}}^s,\bar{\boldsymbol{q}}^b) \in \mathcal{X}(\mathbf{0},\mathbf{0})$*. Moreover, the maximum welfare is achieved when* $(\boldsymbol{\gamma},\boldsymbol{\mu}) = \mathbf{0}$ *i.e.,* $W_{opt} = W(\mathbf{0},\mathbf{0}) \geq W(\boldsymbol{\gamma},\boldsymbol{\mu})$ *for all* $(\boldsymbol{\gamma},\boldsymbol{\mu}) \in \Gamma \times \mathcal{U}$*.*

Proof of Lemma 9. Recall that problem (1.212) is an instance of the equilibrium problem (1.16) where $(\boldsymbol{\gamma}, \boldsymbol{\mu}) = \mathbf{0}$. Let $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{q}}^s, \bar{\boldsymbol{q}}^b)$ be an optimal solution to problem (1.212) and W_0 be the corresponding optimal objective value. By Proposition 9, $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{q}}^s, \bar{\boldsymbol{q}}^b)$ is supported in a competitive equilibrium for $(\boldsymbol{\gamma}, \boldsymbol{\mu}) = \mathbf{0}$ i.e., there exists $\boldsymbol{p}$ such that $(\boldsymbol{p}, \bar{\boldsymbol{x}}, \bar{\boldsymbol{q}}^s, \bar{\boldsymbol{q}}^b) \in \mathcal{X}(\mathbf{0}, \mathbf{0})$.

On the other hand, by Lemma 8, the corresponding objective value W_0 of problem (1.212) is such that $W(\mathbf{0}, \mathbf{0}) = W_0$. Note that eliminating constraint (1.211c) and relaxing constraint (1.211b) (the equilibrium conditions (1.2a)-(1.2d)) to (1.212b) - (1.212f) yields (1.212). This observation implies that problem (1.212) is a relaxation of problem (1.211). Hence, it follows that

$$W_{opt} \;\leq\; W_0 \;=\; W(\mathbf{0}, \mathbf{0}). \tag{1.213}$$

Since by (1.211), W_{opt} is the largest welfare obtained under any commissions-subscription pair, it also follows that $W_{opt} \geq W(\mathbf{0}, \mathbf{0})$. Thus, we conclude that $W(\mathbf{0}, \mathbf{0}) = W_0 = W_{opt} \geq W(\boldsymbol{\gamma}, \boldsymbol{\mu})$ for any $(\boldsymbol{\gamma}, \boldsymbol{\mu}) \in \Gamma \times \mathcal{U}$, and the claim follows. □

1.16.1 Proof of Welfare Results

Proof of Proposition 7. The proof follows directly from Lemma 9. □

Proof of Proposition 8. Proof of claim (i). To prove the claim, we consider a class of problems indexed by $n \in \{1, 2 \ldots\}$. Let problem instance n be a network with one seller type and one buyer type where the population vector satisfies $(s_n, b_n) = (n, 2n)$ and the disutility parameter is $c = 0$. For this problem, we let $\bar{v}_b^n = 2$, $\bar{v}_s^n = 1$, and define buyer/seller value distributions $F_b^n : [0, \bar{v}_b] \to [0,1]$ and $F_s^n : [0, \bar{v}_s] \to [0,1]$ such that

$$F_b^n(v) \;=\; \begin{cases} \frac{n}{2}v, & \text{for } 0 \le v \le \frac{1}{n} \\ 1 - \frac{1}{2nv}, & \text{for } \frac{1}{n} \le v \le 1 \\ 1 - \frac{1}{n} + \frac{1}{2n}v, & \text{for } 1 \le v \le 2 \\ 1, & \text{for } v \ge 2 \end{cases} \tag{1.214a}$$

$$F_s^n(v) \;=\; \begin{cases} nv, & \text{for } 0 \le v \le \frac{1}{n} \\ 1, & \text{for } v \ge \frac{1}{n}. \end{cases} \tag{1.214b}$$

It can be readily checked that, the construction in (1.214) satisfies Assumption 1 and Assumption 2.

Since $(s_n, b_n) = (n, 2n)$, it follows that

$$[F_b^n]^{-1}\Big(1 - \frac{q}{b_n}\Big) - [F_s^n]^{-1}\Big(\frac{q}{s_n}\Big) \;=\; \begin{cases} 2 - q - \frac{1}{n^2}q, & \text{for } 0 \le q \le 1, \\ \frac{1}{q} - \frac{1}{n^2}q, & \text{for } 1 \le q \le n. \end{cases} \tag{1.215}$$

It can be readily checked that $[F_b^n]^{-1}(1 - \frac{q}{b_n}) - [F_s^n]^{-1}(\frac{q}{s_n})$ is strictly decreasing, continuously differentiable in $q \in (0, n)$, and continuous at $q = 0$ and $q = n$. Using this observation, and the fact that in problem instance n, at most n units of buyers and sellers trade, it follows from Lemma 9 that the maximum welfare in this problm instance is given by:

$$W_{opt}^n \;=\; \max_{q \in [0,n]} \int_0^q [F_b^n]^{-1}\Big(1 - \frac{x}{b_n}\Big) - [F_s^n]^{-1}\Big(\frac{x}{s_n}\Big)dx. \tag{1.216}$$

We denote by $\tilde{q}$ the optimal solution to problem (1.216). By applying the first order optimality condition to (1.216), we obtain $[F_b^n]^{-1}\big(1 - \frac{\tilde{q}}{b_n}\big) = [F_s^n]^{-1}\big(\frac{\tilde{q}}{s_n}\big)$, which by (1.215) implies that $\tilde{q} = n$. Thus, the maximum welfare in problem instance n can be expressed as

$$W_{opt}^n \;=\; \int_0^{\tilde{q}} [F_b^n]^{-1}\Big(1 - \frac{x}{b_n}\Big) - [F_s^n]^{-1}\Big(\frac{x}{s_n}\Big)dx \;=\; \ln n + 1. \tag{1.217}$$

In comparison, from the expression (1.215), we deduce that the revenue optimization problem (1.7) can be expressed as

$$V^n_{opt} \;=\; \max_{q\in[0,n]} \;\Big[[F^n_b]^{-1}\Big(1-\frac{q}{b_n}\Big)-[F^n_s]^{-1}\Big(\frac{q}{s_n}\Big)\Big]\,q. \tag{1.218}$$

We let $\bar{q}$ be the optimal solution to problem (1.218). Given the expression (1.215), by applying the first order optimality condition in problem (1.218), we obtain that $2-2\bar{q}-\frac{2}{n^2}\bar{q}=0$, which implies that $\bar{q}=\frac{n^2}{n^2+1}$. By Theorem 1, there exists $(\boldsymbol{\gamma}^n,\boldsymbol{\mu}^n)$ that supports $\bar{q}$ in a competitive equilibrium. Based on the expression (1.215), by Lemma 9, the induced welfare satisfies

$$\begin{aligned} W^n(\boldsymbol{\gamma}^n,\boldsymbol{\mu}^n) \;&=\; \int_0^{\bar{q}} F_b^{-1}\Big(1-\frac{x}{b_n}\Big)-F_s^{-1}\Big(\frac{x}{s_n}\Big)dx \;=\; \Big(2q-\frac{q^2}{2}-\frac{q^2}{2n^2}\Big)\Big|_0^{\frac{n^2}{n^2+1}} \\ &\le\; \Big(2q-\frac{q^2}{2}-\frac{q^2}{2n^2}\Big)\Big|_0^1 \;=\; \frac{3}{2}-\frac{1}{2n^2}\;\le\;\frac{3}{2}. \end{aligned} \tag{1.219}$$

As $n\to\infty$, we conclude that

$$\frac{W^n(\boldsymbol{\gamma}',\boldsymbol{\mu}')}{W^n_{opt}}\to 0. \tag{1.220}$$

Hence, the claim follows.

Proof of claim (ii). Given $F_{s_i}(v)=\frac{v}{\bar{v}_{s_i}}$ for all $v\in[0,\bar{v}_{s_i}]$ and $F_{b_j}(v)=\frac{v}{\bar{v}_{b_j}}$ for all $v\in[0,\bar{v}_{b_j}]$, it follows that $F_{b_j}^{-1}(u)=\bar{v}_{b_j}u$ and $F_{s_i}^{-1}(u)=\bar{v}_{s_i}u$ for $u\in[0,1]$. We start by providing alterantive representations of the revenue optimization problem (1.7) and the welfare maximization problem (1.212).

Let $\boldsymbol{A}^b$ be a $|\mathcal{B}|\times|E|$ binary matrix where for any $j\in\mathcal{B}$, $A^b_{je}=1$ if $e=(i,j)\in E$ and $A^b_{je}=0$ otherwise. Note that the constraint $\sum_{i:(i,j)\in E}x_{ij}=q^b_j$ can be compactly expressed as

$$\boldsymbol{A}^b\boldsymbol{x} = \boldsymbol{q}^b, \tag{1.221}$$

where $\boldsymbol{x}$ and $\boldsymbol{q}^b$ are column vectors of length $|E|$ and $|\mathcal{B}|$ respectively.

Similarly, we let $\boldsymbol{A}^s$ be a $|\mathcal{S}| \times |E|$ binary matrix such that for all $i \in \mathcal{S}$, $A^s_{ie} = 1$ if $e = (i,j) \in E$ and $A^s_{ie} = 0$ otherwise, and represent the constraint $\sum_{j:(i,j)\in E} x_{ij} = q^s_i$ compactly as

$$\boldsymbol{A}^s\boldsymbol{x} = \boldsymbol{q}^s. \tag{1.222}$$

Let $\boldsymbol{s} = (s)_{i\in\mathcal{S}}$, $\boldsymbol{b} = (b_j)_{j\in\mathcal{B}}$, $\bar{\boldsymbol{v}}^s = (\bar{v}_{s_i})_{i\in\mathcal{S}}$ and $\bar{\boldsymbol{v}}^b = (\bar{v}_{b_j})_{j\in\mathcal{B}}$ denote the vectors whose entries correspond to the primitives of our model. Using this shorthand notation, we define the following diagonal matrices

$$\boldsymbol{\Lambda}^s = \operatorname{diag}(\boldsymbol{s}),\ \boldsymbol{\Lambda}^b = \operatorname{diag}(\boldsymbol{b}),\ \boldsymbol{\Lambda}^{v^s} = \operatorname{diag}(\bar{\boldsymbol{v}}^s),\ \boldsymbol{\Lambda}^{v^b} = \operatorname{diag}(\bar{\boldsymbol{v}}^b), \tag{1.223}$$

where $\operatorname{diag}(\boldsymbol{z})$ denotes the diagonal matrix, whose diagonal entries consist of the entries of $\boldsymbol{z}$ for any given vector $\boldsymbol{z}$. Based on (1.221),(1.222) and (1.223), constraints $q^s_i \le s_i$ for all $i \in \mathcal{S}$ and $q^b_j \le b_j$ for all $j \in \mathcal{B}$ can be compactly expressed as

$$(\boldsymbol{\Lambda}^s)^{-1}\boldsymbol{A}^s\boldsymbol{x} \le \boldsymbol{1}, \quad (\boldsymbol{\Lambda}^b)^{-1}\boldsymbol{A}^b\boldsymbol{x} \le \boldsymbol{1}. \tag{1.224}$$

Let vector $\tilde{\boldsymbol{c}} \in \mathbb{R}^{|E|}_+$ be such that $\tilde{c}_e = c_i$ if $e = (i,j) \in E$. We define $(\boldsymbol{r}, \boldsymbol{H}, \boldsymbol{A})$ such that

$$\begin{aligned} &\boldsymbol{r} = (\boldsymbol{A}^b)^\intercal\bar{\boldsymbol{v}}^b - \tilde{\boldsymbol{c}}, \quad \boldsymbol{A} = \begin{pmatrix} (\boldsymbol{\Lambda}^b)^{-1}\boldsymbol{A}^b \\ (\boldsymbol{\Lambda}^s)^{-1}\boldsymbol{A}^s \end{pmatrix}, \\ &\boldsymbol{H} = (\boldsymbol{A}^b)^\intercal\boldsymbol{\Lambda}^{v^b}(\boldsymbol{\Lambda}^b)^{-1}\boldsymbol{A}^b + (\boldsymbol{A}^s)^\intercal\boldsymbol{\Lambda}^{v^s}(\boldsymbol{\Lambda}^s)^{-1}\boldsymbol{A}^s. \end{aligned} \tag{1.225}$$

By construction, matrix $\boldsymbol{H}$ is symmetric. Since $\boldsymbol{\Lambda}^{v^b}(\boldsymbol{\Lambda}^b)^{-1}$ and $\boldsymbol{\Lambda}^{v^s}(\boldsymbol{\Lambda}^s)^{-1}$ are diagonal matrices with positive entries, it follows that $\boldsymbol{\Lambda}^{v^b}(\boldsymbol{\Lambda}^b)^{-1} \succ 0$ and $\boldsymbol{\Lambda}^{v^s}(\boldsymbol{\Lambda}^s)^{-1} \succ 0$. Thus, we conclude that, for all $\boldsymbol{z} \in \mathbb{R}^{|E|}$,

$$\boldsymbol{z}^\mathsf{T}\boldsymbol{H}\boldsymbol{z} \;=\; (\boldsymbol{A}^b\boldsymbol{z})^\mathsf{T}\boldsymbol{\Lambda}^{v^b}(\boldsymbol{\Lambda}^b)^{-1}(\boldsymbol{A}^b\boldsymbol{z}) + (\boldsymbol{A}^s\boldsymbol{z})^\mathsf{T}\boldsymbol{\Lambda}^{v^s}(\boldsymbol{\Lambda}^s)^{-1}(\boldsymbol{A}^s\boldsymbol{z}) \;\geq\; 0, \tag{1.226}$$

which implies that matrix $\boldsymbol{H}$ is positive semidefinite. Since $\boldsymbol{\Lambda}^{v^b}(\boldsymbol{\Lambda}^b)^{-1} \succ 0$ and $\boldsymbol{\Lambda}^{v^s}(\boldsymbol{\Lambda}^s)^{-1} \succ 0$, for all $\boldsymbol{z} \in \mathbb{R}^{|E|}$, it follows that

$$\text{if } \boldsymbol{A}^b\boldsymbol{z} \neq \boldsymbol{0} \text{ or } \boldsymbol{A}^s\boldsymbol{z} \neq \boldsymbol{0}, \text{ then } \boldsymbol{z}^\mathsf{T}\boldsymbol{H}\boldsymbol{z} > 0. \tag{1.227}$$

Using this notation, and the fact that the value distributions are uniform, the objective function (1.7a) of the revenue optimization problem (1.7) can be expressed as follows:

$$\begin{aligned}
&\sum_{j\in\mathcal{B}} F_{b_j}^{-1}\Big(1-\frac{q_j^b}{b_j}\Big)q_j^b - \sum_{i\in\mathcal{S}} F_{s_i}^{-1}\Big(\frac{q_i^s}{s_i}\Big)q_i^s - \sum_{(i,j)\in E} c_i x_{ij}\\
&\overset{(a)}{=} \;\sum_{j\in\mathcal{B}} \bar{v}_{b_j}\Big(1-\frac{q_j^b}{b_j}\Big)q_j^b - \sum_{i\in\mathcal{S}} \bar{v}_{s_i}\Big(\frac{q_i^s}{s_i}\Big)q_i^s - \sum_{(i,j)\in E} c_i x_{ij}\\
&\overset{(b)}{=} \;\Big[\bar{\boldsymbol{v}}^b - [\boldsymbol{\Lambda}^{v^b}(\boldsymbol{\Lambda}^b)^{-1}]^\mathsf{T}\boldsymbol{q}^b\Big]^\mathsf{T}\boldsymbol{q}^b - (\boldsymbol{q}^s)^\mathsf{T}[\boldsymbol{\Lambda}^{v^s}(\boldsymbol{\Lambda}^s)^{-1}]\boldsymbol{q}^s - \tilde{\boldsymbol{c}}^\mathsf{T}\boldsymbol{x}\\
&\overset{(c)}{=} \;\Big[(\bar{\boldsymbol{v}}^b)^\mathsf{T}\boldsymbol{A}^b - \tilde{\boldsymbol{c}}^\mathsf{T}\Big]\boldsymbol{x} - \boldsymbol{x}^\mathsf{T}\Big[(\boldsymbol{A}^b)^\mathsf{T}\boldsymbol{\Lambda}^{v^b}(\boldsymbol{\Lambda}^b)^{-1}\boldsymbol{A}^b + (\boldsymbol{A}^s)^\mathsf{T}\boldsymbol{\Lambda}^{v^s}(\boldsymbol{\Lambda}^s)^{-1}\boldsymbol{A}^s\Big]\boldsymbol{x}\\
&\overset{(d)}{=} \;\boldsymbol{x}^\mathsf{T}(\boldsymbol{r} - \boldsymbol{H}\boldsymbol{x}),
\end{aligned} \tag{1.228}$$

where step (a) follows from $F_{b_j}^{-1}(u) = \bar{v}_{b_j}u$ and $F_{s_i}^{-1}(u) = \bar{v}_{s_i}u$. Step (b) uses the matrix notations in (1.223). Step (c) follows from (1.221) and (1.222). Step (d) follows from the notations in (1.225).

Consider the following convex quadratic program

$$
\begin{aligned}
V_{opt} \;=\; \max_{\boldsymbol{x}} \quad & \boldsymbol{x}^{\mathsf{T}}(\boldsymbol{r}-\boldsymbol{H}\boldsymbol{x}) && \text{(1.229a)}\\
\text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} \le \mathbf{1}, && \text{(1.229b)}\\
& \boldsymbol{x} \ge \mathbf{0}. && \text{(1.229c)}
\end{aligned}
$$

It can be seen from (1.224) and (1.228) that problem (1.229) is equivalent to problem (1.7). In particular, (1.229) is obtained from (1.7) by eliminating the equality constraints, and given as an optimal solution $\bar{\boldsymbol{x}}$ to problem (1.229), by setting $\bar{\boldsymbol{q}}^b = \boldsymbol{A}^b\bar{\boldsymbol{x}}$ and $\bar{\boldsymbol{q}}^s = \boldsymbol{A}^s\bar{\boldsymbol{x}}$, an optimal solution $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{q}}^s, \bar{\boldsymbol{q}}^b)$ to problem (1.7) can be obtained. Moreover, they share the same optimal objective value.

Similarly, in the welfare maximization problem (1.212), the objective function (1.212a) can be compactly expressed as

$$
\begin{aligned}
&\sum_{j\in\mathcal{B}}\int_0^{\bar{q}_j^b} F_{b_j}^{-1}\Big(1-\frac{q}{b_j}\Big)dq - \sum_{i\in\mathcal{S}}\int_0^{\bar{q}_i^s} F_{s_i}^{-1}\Big(\frac{q}{s_i}\Big)dq - \sum_{(i,j)\in E} c_i\bar{x}_{ij}\\
&\overset{(e)}{=} \sum_{j\in\mathcal{B}} \bar{v}_{b_j}\Big(1-\frac{\bar{q}_j^b}{2b_j}\Big)\bar{q}_j^b - \sum_{i\in\mathcal{S}} \bar{v}_{s_i}\Big(\frac{\bar{q}_i^s}{2s_i}\Big)\bar{q}_i^s - \sum_{(i,j)\in E} c_i\bar{x}_{ij}\\
&\overset{(f)}{=} \Big[\bar{\boldsymbol{v}}^b - \frac{1}{2}[\boldsymbol{\Lambda}^{v^b}(\boldsymbol{\Lambda}^b)^{-1}]^{\mathsf{T}}\bar{\boldsymbol{q}}^b\Big]^{\mathsf{T}}\bar{\boldsymbol{q}}^b - \frac{1}{2}(\bar{\boldsymbol{q}}^s)^{\mathsf{T}}[\boldsymbol{\Lambda}^{v^s}(\boldsymbol{\Lambda}^s)^{-1}]\bar{\boldsymbol{q}}^s - \tilde{\boldsymbol{c}}^{\mathsf{T}}\bar{\boldsymbol{x}}\\
&\overset{(g)}{=} ((\bar{\boldsymbol{v}}^b)^{\mathsf{T}}\boldsymbol{A}^b - \tilde{\boldsymbol{c}}^{\mathsf{T}})\bar{\boldsymbol{x}} - \frac{1}{2}\bar{\boldsymbol{x}}^{\mathsf{T}}\Big[(\boldsymbol{A}^b)^{\mathsf{T}}\boldsymbol{\Lambda}^{v^b}(\boldsymbol{\Lambda}^b)^{-1}\boldsymbol{A}^b + (\boldsymbol{A}^s)^{\mathsf{T}}\boldsymbol{\Lambda}^{v^s}(\boldsymbol{\Lambda}^s)^{-1}\boldsymbol{A}^s\Big]\bar{\boldsymbol{x}}\\
&\overset{(h)}{=} \bar{\boldsymbol{x}}^{\mathsf{T}}(\boldsymbol{r} - \frac{1}{2}\boldsymbol{H}\bar{\boldsymbol{x}}), \qquad \text{(1.230)}
\end{aligned}
$$

where step (e) follows from $F_{b_j}^{-1}(u) = \bar{v}_{b_j}u$ and $F_{s_i}^{-1}(u) = \bar{v}_{s_i}u$. Step (f) uses the matrix notations in (1.223). Step (g) follows from (1.221) and (1.222). Step (h) follows from the notations in (1.225).

Consider the following convex quadratic program

$$W_{opt} = \max_{\boldsymbol{x}} \quad \boldsymbol{x}^{\mathsf{T}}(\boldsymbol{r} - \frac{1}{2}\boldsymbol{H}\boldsymbol{x}) \tag{1.231a}$$

$$\text{s.t.} \quad \boldsymbol{A}\boldsymbol{x} \leq \mathbf{1}, \tag{1.231b}$$

$$\boldsymbol{x} \geq \mathbf{0}. \tag{1.231c}$$

Using (1.224) and (1.230), the same arguments as before (see (1.229)) yield that the welfare maximization problem (1.212) is equivalent to problem (1.231); the solution of one can be used to construct a solution for the other, and they share the same optimal objective value.

Let $\bar{\boldsymbol{x}}$ be any optimal solution to problem (1.229) and $\tilde{\boldsymbol{x}}$ be any optimal solution to problem (1.231). Note that if $\boldsymbol{r} \in \mathbb{R}_{-}^{|E|}$, then given that all entries in matrix $\boldsymbol{H}$ are nonnegative (by the expression of $\boldsymbol{H}$ in (1.225)), the gradient of the objective function (1.229) satisfies $\boldsymbol{r} - 2\boldsymbol{H}\boldsymbol{x} \in \mathbb{R}_{-}^{|E|}$ for all feasible $\boldsymbol{x}$ in problem (1.229). Similarly, the gradient of the objective function (1.231a) satisfies $\boldsymbol{r} - \boldsymbol{H}\boldsymbol{x} \in \mathbb{R}_{-}^{|E|}$ for all feasible $\boldsymbol{x}$ in problem (1.231). These observations imply that $\bar{\boldsymbol{x}} = \tilde{\boldsymbol{x}} = \mathbf{0}$. By the welfare expression (1.205) in Lemma 8, both the maximum welfare and the welfare induced in the revenue-optimal implementation are zero. Hence this case is ruled out under the assumptions of our proposition.

Without loss of generality, assume that vector $\boldsymbol{r}$ has at least one positive component. Let $\mathcal{L}_v$ be the Lagrangian of problem (1.229) which satisfies

$$\mathcal{L}_v(\boldsymbol{x}, \boldsymbol{\pi}, \boldsymbol{\theta}) = \boldsymbol{x}^{\mathsf{T}}(\boldsymbol{r} - \boldsymbol{H}\boldsymbol{x}) - \boldsymbol{\pi}^{\mathsf{T}}(\boldsymbol{A}\boldsymbol{x} - \mathbf{1}) + \boldsymbol{\theta}^{\mathsf{T}}\boldsymbol{x}. \tag{1.232}$$

Let $\bar{\boldsymbol{x}}$ be a primal optimal solution to problem (1.229) and $(\bar{\boldsymbol{\pi}}, \bar{\boldsymbol{\theta}})$ be corresponding optimal dual multipliers for constraints (1.229b) - (1.229c). Given the Lagrangian (1.232), the optimal primal-dual pair $\bar{\boldsymbol{x}}$ and $(\bar{\boldsymbol{\pi}}, \bar{\boldsymbol{\theta}})$ satisfy the following KKT optimality conditions:

$$\boldsymbol{r} - 2\boldsymbol{H}\bar{\boldsymbol{x}} - \boldsymbol{A}^{\mathsf{T}}\bar{\boldsymbol{\pi}} + \bar{\boldsymbol{\theta}} \ = \ \boldsymbol{0}, \tag{1.233a}$$

$$\bar{\boldsymbol{\pi}} \ \geq \ \boldsymbol{0} \quad \perp \quad \boldsymbol{A}\bar{\boldsymbol{x}} - \boldsymbol{1} \ \leq \ \boldsymbol{0}, \tag{1.233b}$$

$$\bar{\boldsymbol{\theta}} \ \geq \ \boldsymbol{0} \quad \perp \quad \bar{\boldsymbol{x}} \ \geq \ \boldsymbol{0}. \tag{1.233c}$$

When vector $\boldsymbol{r}$ has at least one positive component, we claim that $\bar{\boldsymbol{x}} \neq \boldsymbol{0}$. Let $e \in E$ be such that $r_e > 0$. Suppose towards contradiction that $\bar{\boldsymbol{x}} = \boldsymbol{0}$. By condition (1.233b), we have $\bar{\boldsymbol{\pi}} = \boldsymbol{0}$. Since $\bar{\boldsymbol{\theta}} \geq \boldsymbol{0}$ (condition (1.233c)), it follows that $r_e + \bar{\theta}_e > 0$, thereby leading to a contradiction with condition (1.233a). Thus, we conclude that $\bar{\boldsymbol{x}} \neq \boldsymbol{0}$. Since the entries of $\boldsymbol{A}^b$ and $\boldsymbol{A}^s$ are nonnegative, $\boldsymbol{x} \neq \boldsymbol{0}$, and $\boldsymbol{x} \geq \boldsymbol{0}$, it follows that $\boldsymbol{A}^b\bar{\boldsymbol{x}} \neq \boldsymbol{0}$ and $\boldsymbol{A}^s\bar{\boldsymbol{x}} \neq \boldsymbol{0}$. Hence, by condition (1.227), we have

$$\bar{\boldsymbol{x}}^{\mathsf{T}}\boldsymbol{H}\bar{\boldsymbol{x}} \ > \ 0. \tag{1.234}$$

Set $\bar{\boldsymbol{q}}^s = \boldsymbol{A}^s\bar{\boldsymbol{x}}$ and $\bar{\boldsymbol{q}}^b = \boldsymbol{A}^b\bar{\boldsymbol{x}}$. By Theorem 1, there exists $(\boldsymbol{\gamma}', \boldsymbol{\mu}')$ that supports $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{q}}^s, \bar{\boldsymbol{q}}^b)$ in a competitive equilibrium. By Lemma 8 and equation (1.230), the welfare induced by revenue-maximum implementation can be compactly expressed as

$$W(\boldsymbol{\gamma}', \boldsymbol{\mu}') = \bar{\boldsymbol{x}}^{\mathsf{T}}(\boldsymbol{r} - \frac{1}{2}\boldsymbol{H}\bar{\boldsymbol{x}}). \tag{1.235}$$

Given the dual optimal solution $\bar{\boldsymbol{\pi}}$ from problem (1.229), consider the following Lagrangian relaxation problem for the welfare maximization problem (1.231) :

$$\tilde{W}_{opt} \ = \ \max_{\boldsymbol{x}} \quad \boldsymbol{x}^{\mathsf{T}}(\boldsymbol{r} - \frac{1}{2}\boldsymbol{H}\boldsymbol{x}) - \bar{\boldsymbol{\pi}}^{\mathsf{T}}(\boldsymbol{A}\boldsymbol{x} - \boldsymbol{1}) \tag{1.236a}$$

$$\text{s.t.} \quad \boldsymbol{x} \geq \boldsymbol{0}. \tag{1.236b}$$

Since (1.236) is a relaxation to problem (1.231c), it follows that

$$W_{opt} \leq \tilde{W}_{opt}. \tag{1.237}$$

Let $\tilde{\mathcal{L}}_w$ be the corresponding Lagrangian to problem (1.236), given by

$$\tilde{\mathcal{L}}_w(\boldsymbol{x}, \boldsymbol{\theta}) = \boldsymbol{x}^\intercal(\boldsymbol{r} - \frac{1}{2}\boldsymbol{H}\boldsymbol{x}) - \bar{\boldsymbol{\pi}}^\intercal(\boldsymbol{A}\boldsymbol{x} - \mathbf{1}) + \boldsymbol{\theta}^\intercal\boldsymbol{x}. \tag{1.238}$$

We claim that $(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{\theta}})$ such that $\tilde{\boldsymbol{x}} = 2\bar{\boldsymbol{x}}$ and $\tilde{\boldsymbol{\theta}} = \bar{\boldsymbol{\theta}}$ is an optimal solution to problem (1.236). To establish this, it suffices to use the Lagrangian (1.238), and verify the KKT conditions:

$$\boldsymbol{r} - \boldsymbol{H}\tilde{\boldsymbol{x}} - \boldsymbol{A}^\intercal\bar{\boldsymbol{\pi}} + \tilde{\boldsymbol{\theta}} \overset{(i)}{=} \mathbf{0}, \tag{1.239a}$$

$$\tilde{\boldsymbol{\theta}} \geq \mathbf{0} \quad \overset{(j)}{\perp} \quad \tilde{\boldsymbol{x}} \geq \mathbf{0}. \tag{1.239b}$$

Note that since $(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{\theta}}) = (2\bar{\boldsymbol{x}}, \bar{\boldsymbol{\theta}})$, it follows that $\boldsymbol{r} - \boldsymbol{H}\tilde{\boldsymbol{x}} - \boldsymbol{A}^\intercal\bar{\boldsymbol{\pi}} + \tilde{\boldsymbol{\theta}} = \boldsymbol{r} - 2\boldsymbol{H}\bar{\boldsymbol{x}} - \boldsymbol{A}^\intercal\bar{\boldsymbol{\pi}} + \bar{\boldsymbol{\theta}} = \mathbf{0}$ (condition (1.233a)), and similarly $\tilde{\boldsymbol{\theta}} \geq \mathbf{0} \perp \tilde{\boldsymbol{x}} \geq \mathbf{0}$ (condition (1.233c)). Thus, both (i) and (j) hold and $(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{\theta}})$ is an optimal solution to problem (1.236).

Using these observations, we conclude that

$$\begin{aligned}
\frac{W(\boldsymbol{\gamma}', \boldsymbol{\mu}')}{W_{opt}} \overset{(k)}{\geq} \frac{W(\boldsymbol{\gamma}', \boldsymbol{\mu}')}{\tilde{W}_{opt}} &\overset{(l)}{=} \frac{\bar{\boldsymbol{x}}^\intercal(\boldsymbol{r} - \frac{1}{2}\boldsymbol{H}\bar{\boldsymbol{x}})}{\tilde{\boldsymbol{x}}^\intercal(\boldsymbol{r} - \frac{1}{2}\boldsymbol{H}\tilde{\boldsymbol{x}}) - \bar{\boldsymbol{\pi}}^\intercal(\boldsymbol{A}\tilde{\boldsymbol{x}} - \mathbf{1})} \\
&\overset{(m)}{=} \frac{\bar{\boldsymbol{x}}^\intercal(\boldsymbol{r} - \frac{1}{2}\boldsymbol{H}\bar{\boldsymbol{x}}) - \bar{\boldsymbol{\pi}}^\intercal(\boldsymbol{A}\bar{\boldsymbol{x}} - \mathbf{1})}{\tilde{\boldsymbol{x}}^\intercal(\boldsymbol{r} - \frac{1}{2}\boldsymbol{H}\tilde{\boldsymbol{x}}) - \bar{\boldsymbol{\pi}}^\intercal(\boldsymbol{A}\tilde{\boldsymbol{x}} - \mathbf{1})} \\
&\overset{(n)}{=} \frac{\bar{\boldsymbol{x}}^\intercal(\boldsymbol{r} - 2\boldsymbol{H}\bar{\boldsymbol{x}} - \boldsymbol{A}^\intercal\bar{\boldsymbol{\pi}}) + \bar{\boldsymbol{x}}^\intercal\bar{\boldsymbol{\theta}} + \frac{3}{2}\bar{\boldsymbol{x}}^\intercal\boldsymbol{H}\bar{\boldsymbol{x}} + \bar{\boldsymbol{\pi}}^\intercal\mathbf{1}}{\tilde{\boldsymbol{x}}^\intercal(\boldsymbol{r} - \boldsymbol{H}\tilde{\boldsymbol{x}} - \boldsymbol{A}^\intercal\bar{\boldsymbol{\pi}}) + \tilde{\boldsymbol{x}}^\intercal\tilde{\boldsymbol{\theta}} + \frac{1}{2}\tilde{\boldsymbol{x}}^\intercal\boldsymbol{H}\tilde{\boldsymbol{x}} + \bar{\boldsymbol{\pi}}^\intercal\mathbf{1}} \\
&\overset{(o)}{=} \frac{\bar{\boldsymbol{x}}^\intercal(\boldsymbol{r} - 2\boldsymbol{H}\bar{\boldsymbol{x}} - \boldsymbol{A}^\intercal\bar{\boldsymbol{\pi}} + \bar{\boldsymbol{\theta}}) + \frac{3}{2}\bar{\boldsymbol{x}}^\intercal\boldsymbol{H}\bar{\boldsymbol{x}} + \bar{\boldsymbol{\pi}}^\intercal\mathbf{1}}{\tilde{\boldsymbol{x}}^\intercal(\boldsymbol{r} - \boldsymbol{H}\tilde{\boldsymbol{x}} - \boldsymbol{A}^\intercal\bar{\boldsymbol{\pi}} + \tilde{\boldsymbol{\theta}}) + \frac{1}{2}\tilde{\boldsymbol{x}}^\intercal\boldsymbol{H}\tilde{\boldsymbol{x}} + \bar{\boldsymbol{\pi}}^\intercal\mathbf{1}} \\
&\overset{(p)}{=} \frac{\frac{3}{2}\bar{\boldsymbol{x}}^\intercal\boldsymbol{H}\bar{\boldsymbol{x}} + \bar{\boldsymbol{\pi}}^\intercal\mathbf{1}}{\frac{4}{2}\bar{\boldsymbol{x}}^\intercal\boldsymbol{H}\bar{\boldsymbol{x}} + \bar{\boldsymbol{\pi}}^\intercal\mathbf{1}} \overset{(q)}{\geq} \frac{\frac{3}{2}\bar{\boldsymbol{x}}^\intercal\boldsymbol{H}\bar{\boldsymbol{x}}}{\frac{4}{2}\bar{\boldsymbol{x}}^\intercal\boldsymbol{H}\bar{\boldsymbol{x}}} \geq \frac{3}{4},
\end{aligned} \tag{1.240}$$

where step (k) follows from (1.237). Step (l) follows directly from (1.235) and the optimality of $\tilde{\boldsymbol{x}}$ in (1.236), and step (m) follows from (1.233b). In step (n), we add $\bar{\boldsymbol{x}}^\intercal\bar{\boldsymbol{\theta}}$ (where $\bar{\boldsymbol{x}}^\intercal\bar{\boldsymbol{\theta}} = 0$ by (1.233c)) to the numerator and $\tilde{\boldsymbol{x}}^\intercal\tilde{\boldsymbol{\theta}}$ (where $\tilde{\boldsymbol{x}}^\intercal\tilde{\boldsymbol{\theta}} = 0$ by (1.239b)) to the denominator, and rearrange the terms. Step (o) follows by collecting common terms. Step (p) follows by noting that $\boldsymbol{r} - 2\boldsymbol{H}\bar{\boldsymbol{x}} - \boldsymbol{A}^\intercal\bar{\boldsymbol{\pi}} + \bar{\boldsymbol{\theta}} = \boldsymbol{0}$ (condition (1.233a)), $\boldsymbol{r} - \boldsymbol{H}\tilde{\boldsymbol{x}} - \boldsymbol{A}^\intercal\bar{\boldsymbol{\pi}} + \tilde{\boldsymbol{\theta}} = \boldsymbol{0}$ (condition (1.239a)) and $\tilde{\boldsymbol{x}} = 2\bar{\boldsymbol{x}}$ (by construction). Step (q) uses the fact that $\bar{\boldsymbol{\pi}}^\intercal\boldsymbol{1} \geq 0$ (condition (1.233b)) and $\bar{\boldsymbol{x}}^\intercal\boldsymbol{H}\bar{\boldsymbol{x}} > 0$ (condition (1.234)). Thus, as claimed, we have

$$\frac{W(\boldsymbol{\gamma}', \boldsymbol{\mu}')}{W_{opt}} \geq \frac{3}{4}. \tag{1.241}$$

To show the tightness of (1.241), consider a network with one seller type and one buyer type. Let the value distributions be $F_s(v) = F_b(v) = \min\{1, v\}$ for $v \geq 0$ and the population profile be $s = b = 1$. In this example, welfare maximization problem (1.231) can be expressed as

$$\max_{0 \leq x \leq 1} x - \frac{1}{2}x^2 - \frac{1}{2}x^2. \tag{1.242}$$

By taking the first order optimality condition, it can be seen that the optimal solution $\tilde{x}$ of (1.242) is given by $\tilde{x} = \frac{1}{2}$ and $W_{opt} = \frac{1}{2} - \frac{1}{8} - \frac{1}{8} = \frac{1}{4}$.

Similarly, the revenue optimization problem (1.229) can be expressed as

$$\max_{0 \leq x \leq 1} x - x^2 - x^2. \tag{1.243}$$

By the first order optimality condition, it follows that the optimal solution $\bar{x}$ to (1.243) is given by $\bar{x} = \frac{1}{4}$. By (1.235), we obtain $W(\boldsymbol{\gamma}', \boldsymbol{\mu}') = \frac{1}{4} - \frac{1}{32} - \frac{1}{32} = \frac{3}{16}$. Thus, we conclude that $\frac{W(\boldsymbol{\gamma}', \boldsymbol{\mu}')}{W_{opt}} = \frac{3}{4}$, and the lower bound in (1.241) is tight.

Proof of claim (iii). Recall that the maximum welfare W_{opt} is achieved when $(\boldsymbol{\gamma}, \boldsymbol{\mu}) = \boldsymbol{0}$ (by Lemma 9). Let $(\tilde{\boldsymbol{p}}, \tilde{\boldsymbol{x}}, \tilde{\boldsymbol{q}}^s, \tilde{\boldsymbol{q}}^b) \in \mathcal{X}(\boldsymbol{0}, \boldsymbol{0})$ be any welfare-maximum equilibrium, obtained from the equilibrium problem (1.16) with $(\boldsymbol{\gamma}, \boldsymbol{\mu}) = \boldsymbol{0}$. Using Assumptions 1 - 3, we next obtain an expression for the maximum welfare.

Define $\tilde{g}(r) = \int_0^r F_b^{-1}(1-x)dx$ and $\tilde{h}(r) = \int_0^r F_s^{-1}(x)dx$. We let $\tilde{f}(\cdot)$ be defined as in (1.17) i.e., $\tilde{f}(t) = \max_{r\in[0,\min\{1,t\}]} \tilde{g}(r) - t\tilde{h}(\frac{r}{t})$ for $t > 0$. Assumptions 1 - 3 together with Proposition 11(i) imply that Assumptions (1.91) - (1.93) hold for $\tilde{g}(\cdot)$ and $\tilde{h}(\cdot)$. Moreover, since $(\boldsymbol{\gamma}, \boldsymbol{\mu}) = 0$ and $F_b^{-1}(1) > 0$, it follows that Assumption (1.94) also holds (Proposition 11(i)). We let $\tilde{f}(0) = \lim_{t\downarrow 0} \tilde{f}(t)$ where the limit exists by Lemma 2(ii). Let $\tilde{\rho}(\cdot)$ be defined as in (1.18) i.e., $\tilde{\rho}(t) = \arg\max_{r\in[0,\min\{1,t\}]} \tilde{g}(r) - t\tilde{h}(\frac{r}{t})$ for $t > 0$. Noting that $\tilde{\rho}(t) \in [0, \min\{1, t\}]$, we set $\tilde{\rho}(0) = \lim_{t\downarrow 0} \tilde{\rho}(t) = 0$.

Let $\boldsymbol{y}^\star$ be any optimal solution to problem (1.19) with $f(\cdot) = \tilde{f}(\cdot)$, and let $\tilde{r}_j = \tilde{\rho}(\frac{y_j^\star}{b_j})$ be the optimal solution to problem (1.18) for $t = \frac{y_j^\star}{b_j}$, $g(\cdot) = \tilde{g}(\cdot)$ and $h(\cdot) = \tilde{h}(\cdot)$. Note that by Lemma 3, vector $\boldsymbol{y}^\star > \boldsymbol{0}$ is unique, and it is the lexicographically optimal base of the polymatroid $\mathcal{P} = \{\boldsymbol{y} \geq \boldsymbol{0} : \sum_{j\in B} y_j \leq \sum_{i\in N_E(B)} s_i,\ \forall B \subset \mathcal{B}\}$ with respect to weight vector $\boldsymbol{b}$. Note that Lemma 2(i) implies that for $t > 0$, we have $\tilde{\rho}(t) \in (0, 1)$. Since $y_j^\star > 0$ and $\tilde{r}_j = \tilde{\rho}(\frac{y_j^\star}{b_j})$, this observation implies that

$$\tilde{r}_j \in (0, 1). \tag{1.244}$$

Moreover, recalling the definitions of $\{\tilde{g}(\cdot), \tilde{h}(\cdot)\}$ and the optimality condition (1.109) (in step 2 of the proof of Lemma 2), we obtain that

$$\tilde{r}_j \;=\; \max\left\{ r \in [0,1] : F_b^{-1}(1-r) - F_s^{-1}\left(\frac{r}{y_j^\star/b_j}\right) \;\geq\; 0,\ r \;\leq\; \frac{y_j^\star}{b_j} \right\}. \tag{1.245}$$

Using this observation, we obtain

$$\begin{aligned} W_{opt} \;=\; W(\boldsymbol{0},\boldsymbol{0}) &\overset{(a)}{=} \sum_{j\in\mathcal{B}} \int_0^{\tilde{q}_j^b} F_b^{-1}\left(1 - \frac{x}{b_j}\right) dx - \sum_{i\in\mathcal{S}} \int_0^{\tilde{q}_i^s} F_s^{-1}\left(\frac{x}{s_i}\right) dx \\ &\overset{(b)}{=} \sum_{j\in\mathcal{B}} b_j \tilde{f}\left(\frac{y_j^\star}{b_j}\right) \\ &\overset{(c)}{=} \sum_{j\in\mathcal{B}} b_j \left[\int_0^{\tilde{r}_i} F_b^{-1}\big(1 - x\big) - F_s^{-1}\left(\frac{x}{y_j^\star/b_j}\right) dx \right]. \end{aligned} \tag{1.246}$$

Here, step (a) uses the fact that $c_i = 0$ for all $i \in \mathcal{S}$ (Assumption 3), $(\tilde{\boldsymbol{p}}, \tilde{\boldsymbol{x}}, \tilde{\boldsymbol{q}}^s, \tilde{\boldsymbol{q}}^b) \in \mathcal{X}(\mathbf{0}, \mathbf{0})$, and Lemma 8. Recalling that $(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{q}}^s, \tilde{\boldsymbol{q}}^b)$ is an optimal solution to the equilibrium problem (1.16) for $(\boldsymbol{\gamma}, \boldsymbol{\mu}) = \mathbf{0}$ and $\boldsymbol{y}^\star$ is the optimal solution to problem (1.19) for $f(\cdot) = \tilde{f}(\cdot)$, Proposition 11(iii) implies that these instances of problem (1.16) and problem (1.19) share the same optimal objective values. Thus, step (b) holds. Since for all $j \in \mathcal{B}$, $\tilde{r}_j = \tilde{\rho}(\frac{y_j^\star}{b_j})$ is optimal in problem (1.17) with $t = \frac{y_j^\star}{b_j}$, $g(\cdot) = \tilde{g}(\cdot)$ and $h(\cdot) = \tilde{h}(\cdot)$, we obtain that $\tilde{f}(\frac{y_j^\star}{b_j}) = \tilde{g}(\tilde{r}_j) - \frac{y_j^\star}{b_j}\tilde{h}(\frac{\tilde{r}_j}{y_j^\star/b_j})$. By definition of $\tilde{g}(\cdot)$ and $\tilde{h}(\cdot)$, it follows that $\tilde{g}(\tilde{r}_j) = \int_0^{\tilde{r}_i} F_b^{-1}\Big(1 - x\Big)dx$ and $\frac{y_j^\star}{b_j}\tilde{h}(\frac{\tilde{r}_j}{y_j^\star/b_j}) = \frac{y_j^\star}{b_j}\int_0^{\frac{\tilde{r}_j}{y_j^\star/b_j}} F_s^{-1}(x)dx = \int_0^{\tilde{r}_j} F_s^{-1}(\frac{x}{y_j^\star/b_j})dx$ (by a change of the variables). Thus, step (c) follows.

Next, we consider the revenue optimization problem (1.7). Let $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{q}}^s, \bar{\boldsymbol{q}}^b)$ be an optimal solution to problem (1.7). Define $\bar{g}(r) = F_b^{-1}(1-r)r$ and $\bar{h}(r) = F_s^{-1}(r)r$. Let $\bar{f}(\cdot)$ be defined as in (1.17) i.e., $\bar{f}(t) = \max_{r \in [0, \min\{1,t\}]} \bar{g}(r) - t\bar{h}(\frac{r}{t})$ for $t > 0$ given functions $\bar{g}(\cdot)$, $\bar{h}(\cdot)$. Using Assumptions 1 - 3, and applying Proposition 10(i), we conclude that Assumptions (1.91) - (1.94) hold for $g(\cdot) = \bar{g}(\cdot)$ and $h(\cdot) = \bar{h}(\cdot)$. We let $\bar{f}(0) = \lim_{t \downarrow 0} \bar{f}(t)$, where as before, the limit is well-defined by Lemma 2(ii). Let $\bar{\rho}(\cdot)$ be defined as in (1.18) i.e., $\bar{\rho}(t) = \arg\max_{r \in [0, \min\{1,t\}]} \bar{g}(r) - t\bar{h}(\frac{r}{t})$ for $t > 0$. Noting that $\bar{\rho}(t) \in [0, \min\{1, t\}]$, we set $\bar{\rho}(0) = \lim_{t \downarrow 0} \bar{\rho}(t) = 0$.

Let $\bar{\boldsymbol{y}}$ be the optimal solution to problem (1.19) with $f(\cdot) = \bar{f}(\cdot)$. Note that $\bar{\boldsymbol{y}} > \mathbf{0}$ is the lexicographically optimal base for polymatroid $\mathcal{P} = \{\boldsymbol{y} \geq \mathbf{0} : \sum_{j \in B} y_j \leq \sum_{i \in N_E(B)} s_i,\ \forall B \subset \mathcal{B}\}$ with respect to weight vector $\boldsymbol{b}$ (by Lemma 3). Since $\boldsymbol{y}^\star$ is also the lexicographically optimal base, by the uniqueness of the lexicographically optimal base (Theorem 3.1 of (42)), we have $\bar{\boldsymbol{y}} = \boldsymbol{y}^\star$. For all $j \in \mathcal{B}$, let $\bar{r}_j = \bar{\rho}(\frac{y_j^\star}{b_j})$ be the optimal solution to problem (1.18) with $t = \frac{y_j^\star}{b_j}$.

For all $j \in \mathcal{B}$, using the same arguments as in (1.244), we obtain that

$$\bar{r}_j \in (0, 1). \tag{1.247}$$

Moreover, recalling the definitions of $\{\bar{g}(\cdot), \bar{h}(\cdot)\}$ and the optimality condition (1.109) (in step 2 of Lemma 2), we obtain that

$$\begin{aligned}\bar{r}_j \;=\; \max\Bigg\{ & r \in [0,1] : F_b^{-1}(1-r) - F_s^{-1}\left(\frac{r}{y_j^\star/b_j}\right) - r[F_b^{-1}]'(1-r) \\ & - \frac{r}{y_j^\star/b_j}\big[F_s^{-1}\big]'\left(\frac{r}{y_j^\star/b_j}\right) \;\geq\; 0,\; r \;\leq\; \frac{y_j^\star}{b_j}\Bigg\}. \end{aligned} \tag{1.248}$$

By Theorem 1, there exists $(\boldsymbol{\gamma}', \boldsymbol{\mu}')$ that supports $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{q}}^s, \bar{\boldsymbol{q}}^b)$ in a competitive equilibrium. We claim that $\bar{q}_i^s > 0$ for all $i \in \mathcal{S}$. By Lemma 4(ii), the revenue optimization problem (1.7) can be formulated as an instance of the framework problem (1.38) with functions $\{g_j(\cdot), h_i(\cdot)\}$ that satisfy Assumptions (1.12.1-1) -(1.12.1-3) and Assumption (1.12.1-8). Moreover, these functions are given by $g_j(q) = F_{b_j}^{-1}(1 - \frac{q}{b_j})q$, $h_i(q) = F_s^{-1}(\frac{q}{s_i})q$ and $w_{ij} = 0$. By using this observation, Assumption 3 and Assumption (1.12.1-6) also hold. Using this observation, Proposition 12(ix) readily implies that $\bar{q}_i^s > 0$ for all $i \in \mathcal{S}$. Since $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{q}}^s, \bar{\boldsymbol{q}}^b)$ is supported in a competitive equilibrium, this implies that for any $i \in \mathcal{S}$, there exists $j \in \mathcal{B}$ such that $\bar{x}_{ij} > 0$. These observations, (1.247), Proposition 10(ii), and the definition of $\bar{\boldsymbol{y}}$ and $\bar{\boldsymbol{r}}$ imply that $\bar{q}_j^b = \bar{r}_j b_j > 0$ and $\bar{q}_i^s = \frac{\bar{r}_j b_j}{\bar{y}_j} = \frac{\bar{r}_j b_j}{y_j^\star}$ for any j such that $\bar{x}_{ij} > 0$. Hence, we conclude the following:

$$\begin{aligned}
W(\boldsymbol{\gamma}', \boldsymbol{\mu}') &\overset{(d)}{=} \sum_{j \in \mathcal{B}} \int_0^{\bar{q}_j^b} F_b^{-1}\left(1 - \frac{x}{b_j}\right) dx - \sum_{i \in \mathcal{S}} \int_0^{\bar{q}_i^s} F_s^{-1}\left(\frac{x}{s_i}\right) dx \\
&\overset{(e)}{=} \sum_{j \in \mathcal{B}} b_j \int_0^{\bar{r}_i} F_b^{-1}\Big(1 - x\Big) dx - \sum_{i \in \mathcal{S}} \left[\sum_{j : \bar{x}_{ij} > 0} \frac{\bar{x}_{ij}}{\bar{q}_i^s}\right] s_i \int_0^{\frac{\bar{q}_i^s}{s_i}} F_s^{-1}(x)\, dx \\
&\overset{(f)}{=} \sum_{j \in \mathcal{B}} b_j \int_0^{\bar{r}_i} F_b^{-1}\Big(1 - x\Big) dx - \sum_{i \in \mathcal{S}} \sum_{j : \bar{x}_{ij} > 0} \bar{x}_{ij} \frac{y_j^\star}{\bar{r}_j b_j} \int_0^{\frac{\bar{r}_j b_j}{y_j^\star}} F_s^{-1}(x)\, dx \\
&\overset{(g)}{=} \sum_{j \in \mathcal{B}} b_j \int_0^{\bar{r}_i} F_b^{-1}\Big(1 - x\Big) dx - \sum_{j \in \mathcal{B}} \sum_{i : \bar{x}_{ij} > 0} \bar{x}_{ij} \frac{y_j^\star}{\bar{q}_j^b} \frac{b_j}{y_j^\star} \int_0^{\bar{r}_j} F_s^{-1}\left(\frac{x}{y_j^\star / b_j}\right) dx \\
&\overset{(h)}{=} \sum_{j \in \mathcal{B}} b_j \int_0^{\bar{r}_i} F_b^{-1}\Big(1 - x\Big) dx - \sum_{j \in \mathcal{B}} b_j \int_0^{\bar{r}_j} F_s^{-1}\left(\frac{x}{y_j^\star / b_j}\right) dx \\
&= \sum_{j \in \mathcal{B}} b_j \left[\int_0^{\tilde{r}_i} F_b^{-1}\Big(1 - x\Big) - F_s^{-1}\left(\frac{x}{y_j^\star / b_j}\right) dx\right]. \qquad (1.249)
\end{aligned}$$

Here step (d) follows by using Lemma 8 and noting that $(\boldsymbol{\gamma}', \boldsymbol{\mu}')$ supports $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{q}}^s, \bar{\boldsymbol{q}}^b)$ in a competitive equilibrium and $c_i = 0$ for all $i \in \mathcal{S}$ (Assumption 3). In step (e), we apply a change of variables in the integrals and use the fact that $\sum_{j : \bar{x}_{ij} > 0} \bar{x}_{ij} = \bar{q}_i^s$ since $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{q}}^s, \bar{\boldsymbol{q}}^b)$ constitutes a trading outcome in a competitive equilibrium. Step (f) follows by using the fact that $\bar{q}_j^b = \bar{r}_j b_j > 0$ and $\bar{q}_i^s = \frac{\bar{r}_j b_j}{y_j^\star}$ for any $j \in \mathcal{B}$ and $i \in \mathcal{S}$ such that $\bar{x}_{ij} > 0$. In step (g), we use the same observation, and apply another change of variables in the second integral. Step (h) follows once again using $\sum_{i : \bar{x}_{ij} > 0} \bar{x}_{ij} = \bar{q}_j^b$.

In what follows, we use the shorthand notation

$$G_j(r) \;=\; F_b^{-1}\Big(1 - r\Big) - F_s^{-1}\left(\frac{r}{y_j^\star / b_j}\right), \quad \text{for } r \in \left[0, \min\left\{1, \frac{y_j^\star}{b_j}\right\}\right]. \qquad (1.250)$$

Note that using this notation, (1.245) and (1.248) can be expressed as follows:

$$\tilde{r}_j \;=\; \max\left\{r \in [0,1] : G_j(r) \;\geq\; 0,\; r \;\leq\; \frac{y_j^\star}{b_j}\right\}, \tag{1.251a}$$

$$\bar{r}_j \;=\; \max\left\{r \in [0,1] : G_j(r) + rG_j'(r) \;\geq\; 0,\; r \;\leq\; \frac{y_j^\star}{b_j}\right\}. \tag{1.251b}$$

We claim that $0 < \bar{r}_j \leq \tilde{r}_j$ for all $j \in \mathcal{B}$. Since $F_b(v)$ and $F_s(v)$ are nonatomic and strictly increasing in $v \in (0, \bar{v}_b)$ and $v \in (0, \bar{v}_s)$ respectively (Assumption 1), we have that $F_b^{-1}(r)$ and $F_s^{-1}(r)$ strictly increase in $r \in [0,1]$. Moreover, since the aforementioned functions are also continuously differentiable respectively in $v \in (0, \bar{v}_b)$ and $v \in (0, \bar{v}_s)$ (Assumption 1), it follows that $F_b^{-1}(r)$ and $F_s^{-1}(r)$ are continuously differentiable in $r \in (0,1)$ (by the inverse function theorem). It can be readily checked that if $\lim_{r\downarrow 0}[F_b^{-1}]'(r)$, $\lim_{r\downarrow 0}[F_s^{-1}]'(r)$ have finite values, they correspond to the left derivatives $F_b^{-1}(r)$ and $F_s^{-1}(r)$ at $r = 0$. Similarly, if $\lim_{r\uparrow 1}[F_b^{-1}]'(r)$, and $\lim_{r\uparrow 1}[F_s^{-1}]'(r)$ have finite values, they correspond to the right derivatives of $F_b^{-1}(r)$ and $F_s^{-1}(r)$ at $r = 1$. Using these observations and the definition of G_j, we conclude that

$$\begin{aligned} &G_j'(r) < 0 \text{ for all } r \in \left(0, \min\left\{1, \frac{y_j^\star}{b_j}\right\}\right), \\ &G_j(r) \text{ is strictly decreasing in } r \in \left[0, \min\left\{1, \frac{y_j^\star}{b_j}\right\}\right]. \end{aligned} \tag{1.252}$$

For all $j \in \mathcal{B}$, recall that $\bar{r}_j \in (0,1)$ (by condition (1.247)) and $\tilde{r}_j \in (0,1)$ (by condition (1.244)). Moreover, by (1.252), (and the definition of the derivatives of F_b^{-1} and F_s^{-1} at the end points of their domain), we obtain that $-\bar{r}_j G_j'(\bar{r}_j) > 0$. Using these observations, together with (1.247), (1.251b) and (1.251a) readily imply that

$$0 \;<\; \bar{r}_j \;\leq\; \tilde{r}_j, \tag{1.253}$$

for all $j \in \mathcal{B}$. To proceed, fix an arbitrary $j \in \mathcal{B}$ and consider the following two cases:

(1) if $\bar{r}_j = \frac{y_j^\star}{b_j}$, by (1.251a) and (1.253), we have $\tilde{r}_j = \frac{y_j^\star}{b_j}$. Thus, it follows that

$$\frac{b_j \int_0^{\bar{r}_j} G_j(x)dx}{b_j \int_0^{\tilde{r}_i} G_j(x)dx} \;=\; 1; \tag{1.254}$$

(2) if $\bar{r}_j < \frac{y_j^\star}{b_j}$, by (1.247), we have that $\bar{r}_j < \min\{1, \frac{y_j^\star}{b_j}\}$. Using this observation, we obtain

$$-\bar{r}_j G_j'(\bar{r}_j) \overset{(i)}{=} G_j(\bar{r}_j) \overset{(j)}{\geq} G_j(\bar{r}_j) - G_j(\tilde{r}_j) \overset{(k)}{\geq} -(\tilde{r}_j - \bar{r}_j)G_j'(\bar{r}_j). \tag{1.255}$$

Here step (i) follows from (1.251b) using the continuous differentiability of $G_j(r)$ (for $r \in (0, \min\{1, \frac{y_j^\star}{b_j}\})$) and the fact that $\bar{r}_j \in (0, \min\{1, \frac{y_j^\star}{b_j}\})$. Step (j) follows since $G_j(\tilde{r}_j) \geq 0$ by (1.251a). Since $F_b(v)$ and $-F_s(v)$ are convex respectively in $v \in [0, \bar{v}_b]$ and $v \in [0, \bar{v}_s]$ (by the assumption of the proposition), it follows that $F_b^{-1}(r)$ and $-F_s^{-1}(r)$ are concave, and hence the function $G_j(r)$ is concave in $r \in [0, \min\{1, \frac{y_j^\star}{b_j}\}]$. Step (k) readily follows from the concavity of function $G_j(r)$.

Recall that by (1.253), we have $\bar{r}_j > 0$ and hence $\bar{r}_j \in (0, \min\{1, \frac{y_j^\star}{b_j}\})$. Thus, (1.252) implies that $-G_j'(\bar{r}_j) > 0$, which together with (1.255) yields that $\bar{r}_j \geq \tilde{r}_j - \bar{r}_j$. Rearranging the terms, we obtain:

$$\bar{r}_j \;\geq\; \frac{1}{2}\tilde{r}_j \;>\; 0. \tag{1.256}$$

Since $0 < \bar{r}_j \le \tilde{r}_j$ (by (1.253)), we deduce that

$$\begin{aligned}
\int_{\bar{r}_j}^{\tilde{r}_i} G_j(x)dx &\overset{(l)}{\le} \int_{\bar{r}_j}^{\tilde{r}_i} G_j(\bar{r}_j) + G_j'(\bar{r}_j)(x - \bar{r}_j)dx \\
&\overset{(m)}{\le} G_j(\bar{r}_j)(\tilde{r}_j - \bar{r}_j) + \frac{1}{2}G_j'(\bar{r}_j)(\tilde{r}_j + \bar{r}_j)(\tilde{r}_j - \bar{r}_j) - \bar{r}_j G_j'(\bar{r}_j)(\tilde{r}_j - \bar{r}_j) \\
&\overset{(n)}{\le} G_j(\bar{r}_j)(\tilde{r}_j - \bar{r}_j) + \frac{3}{2}\bar{r}_j G_j'(\bar{r}_j)(\tilde{r}_j - \bar{r}_j) - \bar{r}_j G_j'(\bar{r}_j)(\tilde{r}_j - \bar{r}_j) \\
&\overset{(o)}{\le} \frac{1}{2}G_j(\bar{r}_j)(\tilde{r}_j - \bar{r}_j),
\end{aligned} \tag{1.257}$$

where step (l) follows from the concavity of $G_j(r)$ in $r \in [0, \min\{1, \frac{y_j^\star}{b_j}\}]$. Step (m) follows from integrating the expression in step (l). In step (n), we use condition (1.256) to obtain that $\bar{r}_j + \tilde{r}_j \le 3\bar{r}_j$, which we use to bound the second term. Step (o) follows from $G_j(\bar{r}_j) = -\bar{r}_j G_j'(\bar{r}_j)$ by (1.255). Based on (1.256), we have

$$\begin{aligned}
0 &\overset{(p)}{<} \bar{r}_j G_j(\bar{r}_j) \overset{(q)}{<} \int_0^{\bar{r}_j} G_j(x)dx \overset{(r)}{\le} \int_0^{\tilde{r}_j} G_j(x)dx, \\
0 &\overset{(s)}{\le} \int_{\bar{r}_j}^{\tilde{r}_i} G_j(x)dx \overset{(t)}{\le} \frac{1}{2}G_j(\bar{r}_j)(\tilde{r}_j - \bar{r}_j) \overset{(u)}{\le} \frac{1}{2}G_j(\bar{r}_j)\bar{r}_j.
\end{aligned} \tag{1.258}$$

Here step (p) follows since $\bar{r}_j \in (0, \min\{1, \frac{y_j^\star}{b_j}\})$, and $G_j(\bar{r}_j) = -\bar{r}_j G_j'(\bar{r}_j) > 0$ by (1.252) and (1.255). Recall that Assumption (1.91) and (1.92) hold for $\tilde{g}(r) = \int_0^r F_b^{-1}(1-x)dx$ and $\tilde{h}(r) = \int_0^r F_s^{-1}(x)dx$, hence, these functions are continuous at $r = 0$. Thus, it follows that $\int_0^r G_j(x)dx$ is continuous at $r = 0$. By the continuity and the strict decreasingness of $G_j(r)$ in $r \in (0, \bar{r}_j)$ (condition (1.252)), step (q) follows. Both step (r) and step (s) follow since $G_j(r) \ge 0$ for $r \in [0, \tilde{r}_j]$ (by condition (1.251a), (1.252) and (1.253)). Step (t) follows from condition (1.257). Step (u) follows from (1.256).

Note that (1.258) implies that

$$\frac{b_j \int_0^{\bar{r}_j} G_j(x)dx}{b_j \int_0^{\tilde{r}_i} G_j(x)dx} = \frac{\int_0^{\bar{r}_j} G_j(x)dx}{\int_0^{\bar{r}_j} G_j(x)dx + \int_{\bar{r}_j}^{\tilde{r}_i} G_j(x)dx} \geq \frac{G_j(\bar{r}_j)\bar{r}_j}{G_j(\bar{r}_j)\bar{r}_j + \frac{1}{2}G_j(\bar{r}_j)\bar{r}_j} = \frac{1}{1+\frac{1}{2}} = \frac{2}{3}. \tag{1.259}$$

Using (1.246), (1.249), (1.250), as well as (1.254) and (1.259), we conclude that

$$W_r = \frac{W(\boldsymbol{\gamma}', \boldsymbol{\mu}')}{W_{opt}} = \frac{\sum_{j\in\mathcal{B}} b_j \int_0^{\bar{r}_j} G_j(x)dx}{\sum_{j\in\mathcal{B}} b_j \int_0^{\tilde{r}_i} G_j(x)dx} \geq \frac{2}{3}, \tag{1.260}$$

as claimed.

To show the tightness of the lower bound, consider a sequence of problem instances indexed by $n \in \{2, 3, \dots\}$. In the problem instance n, we assume that there is only one seller type and one buyer type, and the population vector satisfies $(s_n, b_n) = (2n, \frac{4n}{2n-1})$. We define the inverses of the value distribution functions as follows

$$[F_b^n]^{-1}(u) = \begin{cases} 2u, & \text{for } 0 \leq u \leq \frac{2n+1}{4n}, \\ 1 + \frac{2}{2n-1}(1-u) + \frac{1}{n}\log\left(1 + n\left(1 - (1-u)\frac{4n}{2n-1}\right)\right), & \text{for } \frac{2n+1}{4n} \leq u \leq 1, \end{cases}$$

$$[F_s^n]^{-1}(u) = u, \quad \text{for } u \in [0,1]. \tag{1.261}$$

It can be readily verified that $[F_b^n]^{-1}$ and $[F_s^n]^{-1}$ satisfy $\int_0^1 [F_b^n]^{-1}(x)dx < \infty$ and $\int_0^1 [F_s^n]^{-1}(x)dx < \infty$. Moreover, $[F_b^n]^{-1}$ and $[F_s^n]^{-1}$ are continuously differentiable and strictly increasing in $(0,1)$. Using the inverse function theorem, we conclude that Assumption 1 holds. Similarly, it can be readily verified that Assumptions 2 and 3 are satisfied by these distributions, and F_b^n is convex and F_s^n is concave in its domain. Hence, the conditions in part (iii) of the proposition hold. For any $n \geq 2$, we have

$$[F_b^n]^{-1}\Big(1 - \frac{q}{b_n}\Big) - [F_s^n]^{-1}\Big(\frac{q}{s_n}\Big) \;=\; \begin{cases} 1 + \frac{1}{n}\log\left(1 + n(1-q)\right), & \text{for } 0 \leq q \leq 1, \\ 2 - q, & \text{for } 1 \leq q \leq \frac{4n}{2n-1}. \end{cases} \tag{1.262}$$

Let $\tilde{q}$ be the welfare-maximizing equilibrium trade quantity, which using (1.246) can be given as $\tilde{q} = \arg\max_{0 \leq q \leq \frac{4n}{2n-1}} \{\int_0^q [F_b^n]^{-1}(1 - \frac{x}{b_n}) - [F_s^n]^{-1}(\frac{x}{s_n}) dx\}$. Using the first order conditions, we obtain that $\tilde{q} = 2$. Similarly, we let $\bar{q}$ be the revenue-maximizing equilibrium trade quantity, which using Theorem 1 can be given as $\bar{q} = \arg\max_{0 \leq q \leq \frac{4n}{2n-1}} \{[F_b^n]^{-1}(1 - \frac{q}{b_n})q - [F_s^n]^{-1}(\frac{q}{s_n})q\}$. By taking the first order condition, we have $\bar{q} = 1$. Denote the welfare induced in the aforementioned outcomes respectively as W_{opt}^n and $W^n(\boldsymbol{\gamma}', \boldsymbol{\mu}')$. Using these observations and Lemma 8, we can express the ratio of the induced welfares as follows:

$$W_r^n \;=\; \frac{W^n(\boldsymbol{\gamma}', \boldsymbol{\mu}')}{W_{opt}^n} \;=\; \frac{\int_0^{\bar{q}} [F_b^n]^{-1}\Big(1 - \frac{q}{b_n}\Big) - [F_s^n]^{-1}\Big(\frac{q}{s_n}\Big) dq}{\int_0^{\tilde{q}} [F_b^n]^{-1}\Big(1 - \frac{q}{b_n}\Big) - [F_s^n]^{-1}\Big(\frac{q}{s_n}\Big) dq} \;=\; \frac{1 + \frac{(n+1)\log(n+1) - n}{n^2}}{\frac{3}{2} + \frac{(n+1)\log(n+1) - n}{n^2}}. \tag{1.263}$$

Note that $\lim_{n \to \infty} W_r^n = \frac{2}{3}$. Thus, the lower bound is tight, and the claim follows. □

1.17 Neighborhood Map

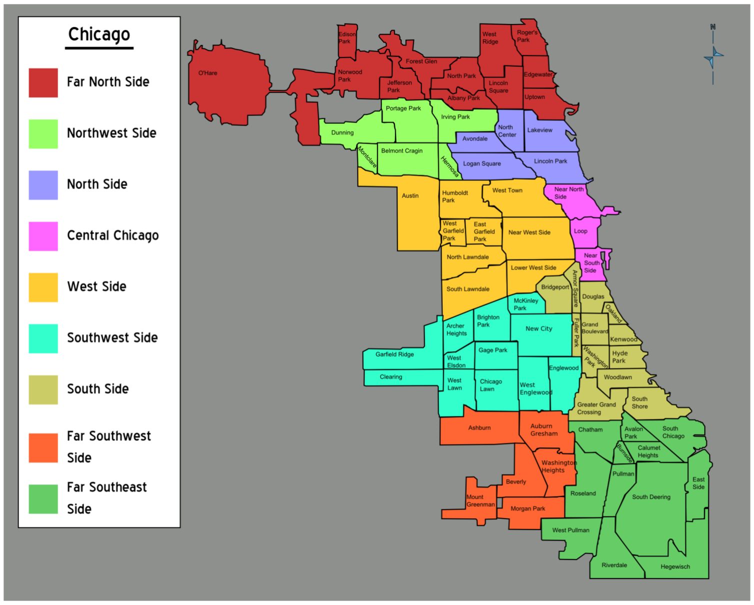


Figure 1.4: Chicago Community Map

CHAPTER 2

MARKDOWN POLICIES FOR DEMAND LEARNING WITH FORWARD-LOOKING CUSTOMERS

2.1 Introduction

2.1.1 Background and Overview

Markdowns enable a seller to earn higher profits by segmenting the market over time. Starting with a list price and carefully choosing the amount and timing of price reductions, the seller can effectively price discriminate among customers and capture a substantial portion of consumer surplus. But, the implementation of markdowns involves challenges that can potentially offset the benefits. First of all, markdown management requires a basic understanding of how prices affect demand, which is typically formulated as a *demand model* that gives the expected demand as a function of price. When selling a new product, the seller does not necessarily have perfect information on the relationship between price and demand, and needs to choose markdowns while facing *demand model uncertainty.* Secondly, another key factor in markdown management is *forward-looking customer behavior*. Some patient and savvy customers take markdowns into account while timing their purchasing decisions and choose to wait for lower prices. Such forward-looking customer behavior decreases the seller's ability to price discriminate, thereby significantly influencing the demand model.

This paper studies the design of dynamic markdown policies in the presence of demand model uncertainty and forward-looking customers. The particular markdown pricing problem we consider has three key features: (a) there is a seller who dynamically marks down the price of a product over a multiperiod time horizon; (b) the demand for the product is generated by a heterogeneous mixture of forward-looking and myopic customers with different patience levels; and (c) the seller initially knows neither the customers' valuations nor the breakdown of customers by patience level, but can learn them based on sequential demand observations.

Regarding problem feature (b), the customers' forward-looking behavior depends on their patience levels as well as how they trade off buying now versus later. Patient customers can look further into the future and strategically time their purchases—accordingly, we hereafter use the terms "patient" and "forward-looking" interchangeably. Due to problem feature (c), this problem entails a tradeoff between *learning* a demand model to increase future profits versus *earning* immediate profits. To investigate this tradeoff, we consider a Bayesian setting where the seller has a prior belief distribution on a number of demand models, and we measure performance by the seller's expected cumulative *profit loss* relative to a clairvoyant who knows the underlying demand model with certainty. In an asymptotic regime where the time horizon and market size are scaled up proportionally, we characterize how the seller's profit loss depends on the problem scale under various different settings.

We study two forms of forward-looking customer behavior: (i) an *exogenous* form where the customers' predictions of future prices depend only on the price history, and (ii) an *endogenous* form where the customers' predictions depend on the seller's pricing policy as well as the price history. Both forms of forward-looking customer behavior create intertemporal dependencies that complicate the classical exploration-exploitation tradeoff: due to the price predictions of forward-looking customers, the seller's cost of exploration spills over into subsequent exploitation periods. In fact, when forward-looking customer behavior is introduced into the problem formulation, we find that no admissible policy can achieve good performance in general—this is fundamentally different from the traditional findings based only on myopic customers. A key factor in improving the seller's performance is based on limiting the exploration cost that extends to later periods, and we design policies that achieve this goal. In the case of exogenous forward-looking customer behavior, we analyze two commonly used approaches, namely *forced exploration* and *passive learning*, and characterize their performance. In the case of endogenous forward-looking customer behavior, customers are strategic and fully anticipate the expected prices induced by the seller's policy, and we provide an asymptotic optimality analysis for this case as well. Perhaps surprisingly, we show that in this case, it is easier for the seller to limit the impact of early exploration on the customers' future purchasing decisions. Finally, we also study how forward-looking customers might potentially benefit the firm from a learning perspective by delaying the demand and creating further exploitation opportunities for the seller.

2.1.2 Practical Motivation

Markdown management and demand model uncertainty are observed in many sectors. For example, firms selling fashion or high-technology products typically introduce their new products with a list price and then gradually mark down the price to target customers who are willing to pay less than the list price. A major challenge in the practical implementation of markdowns is that the demand for new fashion and high-technology products entails significant uncertainty in advance of a selling season, because firms often have only partial information on the customer preferences and potential market size for a new product. Thus, a firm facing the aforementioned uncertainty runs the risk of marking down its product's price too early. As reported by McKinsey & Company, charging a low price for a new product not only leaves potential revenues on the table but also reduces the product's value in the market, thereby making it virtually impossible to raise prices to higher levels (52). In addition, increasing prices causes further concerns from the seller's perspective. First, deviating from markdown pricing creates an incentive for customers to take advantage of resale opportunities. To curb such resale incentives, it is to the seller's interest to use a markdown policy. Second, a price increase can potentially result in media backlash and customer complaints, which deter the seller from raising the price. Due to these challenges, judicious management of markdowns has become a focus of attention in practice, making markdown optimization a key feature of business analytics software and services (53, 54, 55)see, e.g.,. Motivated by these markdown management applications, we focus on a theoretical problem formulation with the hope of gaining high-level insights on markdown management in practice.

2.1.3 Main Contributions and Qualitative Insights

Characterizing the impact of forward-looking customers on learning performance. Our work identifies two effects of forward-looking customers on the tradeoff between learning and earning in markdown pricing.

First, forward-looking customers introduce an intertemporal dependency through which the seller's early exploration efforts cause deviations from the best possible price path in hindsight—this subsequently influences later purchasing decisions of forward-looking customers. We establish that this intertemporal dependency leads to a fundamental difference from the antecedent studies on dynamic pricing and demand learning that consider only myopic customers (56, 57, 58, 59)see, e.g.,. In particular, we show that when the customers' forward-looking behavior is exogenous and their memory includes the entire price history, the seller could incur a large profit loss that is linear in the problem size (see Theorem 6(ii)). This result highlights that in the worst case, forward-looking customers could severely hurt a seller who faces demand model uncertainty.

Second, despite the aforementioned negative result obtained in the worst case, we also discover a potential benefit of forward-looking customers from a learning perspective. To see this, we note that a seller facing customers with different patience levels has an interesting dilemma in pricing. When customers are myopic, it is easier for the seller to construct a well-performing pricing policy. This is due to the straightforward purchasing behavior of myopic customers. But, the impatient nature of myopic customers also poses a challenge to the seller: any pricing error due to demand model uncertainty has an irreversible cost when customers are myopic and impatient. By contrast, forward-looking customers can strategically delay their purchases. Thus, although forward-looking customer behavior might have a negative impact on the seller's profit performance under demand model uncertainty, it can also potentially provide "cheaper" exploration opportunities to the seller due to delayed demand. Our work compares the magnitude of these effects, showing that the impact of customer impatience on the profit loss due to demand model uncertainty could be substantially higher than the impact of forward-looking customers. To that end, we characterize the best achievable profit performance benchmarks when customers are extremely forward-looking and when they are myopic, and we prove that these benchmarks stand in stark contrast to previous results. As discussed in §2.4, under certain conditions, it is possible to design markdown policies that achieve asymptotically vanishing profit loss when customers are extremely forward-looking (Theorem 10), whereas any policy has to incur a profit loss that grows logarithmically in the problem size when customers are myopic (Corollary 4).

Based on these findings, we identify that the seller's profit performance under demand model uncertainty depends crucially on how forward-looking the customers are. To the best of our knowledge, this comparison has not been addressed in the related literature.

Achieving asymptotic optimality under exogenous and endogenous forward-looking customer behaviors. In view of the intertemporal dependency caused by forward-looking customers, there are three crucial factors to consider in reducing the profit loss due to demand model uncertainty: (i) collecting information about the demand model, (ii) preserving the market size in early periods, and (iii) limiting the impact of the seller's exploration on forward-looking customers.

In the case of exogenous forward-looking behavior, the customers use a prediction function to infer the future price path, and make optimal purchasing decisions based on their price predictions. In light of Theorem 6(ii), we observe that the intertemporal dependency induced by forward-looking customer behavior could cause a large profit loss for the seller. Despite this negative result, we identify conditions under which a forced exploration policy achieves asymptotic optimality. In Theorem 7, we establish that when customers have bounded memory, the profit loss due to demand model uncertainty grows logarithmically in the problem scale, which matches the best achievable growth rate of the profit loss in Theorem 6(i). In Theorem 8, we establish that when the customers have unbounded memory but have a prediction function satisfies a variant of Lipschitz continuity, the profit loss can grow sublinearly in the problem size. Regarding this analysis, we reiterate that due to the intertemporal dependency caused by forward-looking customers, the seller's cost of exploration is extended to later periods. As a major technical contribution, our analysis of the customer memory and the properties of the forward-looking behavior in Theorems 7 and 8 sheds light on the importance of limiting the impact of the seller's exploration on forward-looking customers.

In the case of endogenous forward-looking behavior, the customers' prediction of future prices coincide with the expected price path induced by the seller's policy. In this case, the best achievable performance benchmark is based on a subgame perfect equilibrium as in (60). Interestingly, in this case, it is easier for the seller to implement a strategy that limits the impact of early exploration efforts on forward-looking customers. To that end, the seller can announce and commit to a forced exploration policy. Due to the customers' ability to fully predict the policy-induced price path in this case, sufficient learning about the demand model in the early periods allows the seller to implement the remaining price path in the subgame perfect equilibrium, and thus, the customers who arrive later in the system would not deviate from their optimal purchasing decisions. To formalize this intuition, we prove in Theorem 9 that, under endogenous forward-looking customer behavior, the profit loss due to demand model uncertainty grows logarithmically in the problem size, which matches the best achievable growth rate of the profit loss in this case (see Proposition 15).

It is worth noting that our paper provides a novel treatment of modeling of forward-looking customer behavior. Since there has been no standard answer to determine forward-looking customer behavior in the presence of model uncertainty, we believe that our paper sheds light on the seller's performance outcomes under previously unexplored modeling choices.

Organization of the paper. This subsection ends with a review of related literature. In §2.3, we present our problem formulation, and in §2.3.1, we construct our policies and study their performance. subsection 2.4 presents our further analysis of the impact of forward-looking customers. Finally, §3.6 provides a summary of the paper. All proofs are in appendices.

2.2 Related Literature

Our work is related to two streams of research: (i) markdown management and forward-looking customer behavior, and (ii) dynamic pricing with demand model uncertainty and learning.

Markdown management and forward-looking customer behavior. In the economics literature, an early related study is by (61), who conjectured that customers' forward-looking behavior could eliminate a monopolistic seller's market power, forcing the seller to price at marginal cost. This notion, widely known as the Coase conjecture, is subsequently studied in detail by (62) ((62), (63)) and (64). In the context of markdown management, (65) demonstrated how markdowns help a monopolistic seller to capture a larger profit via intertemporal pricing, whereas (60) considered the case of forward-looking customers, using a game-theoretic approach to characterize how forward-looking customers influence a seller's markdown policy and profits. More recently, (66) empirically investigated optimal dynamic pricing strategies in the presence of forward-looking customers. (67) studied intertemporal pricing with forward-looking customers, proving the optimality of markdown policies when high-value customers are impatient or when low-value customers are patient. (68) showed that the profit loss caused by ignoring forward-looking customer behavior is substantial, and from a seller's perspective, price commitment in the form of pre-announced markdowns can be more advantageous than a contingent pricing scheme. Following this work, (69), (70), and (71) studied in detail the optimal structure of pre-announced markdown policies in the presence of forward-looking customers. (72) formulated a dynamic pricing setting where customers can strategically signal their willingness to pay, and studied the optimal pricing policy in this setting. (73) and (74) analyzed the value and effectiveness of a seller's quick response capability when dealing with forward-looking customers. (75) analyzed price commitment policies in a general setting, and identified a cyclic pricing structure under the optimal policy. On the other hand, (76) studied contingent pricing policies characterized by subgame-perfect equilibria, showing that ignoring forward-looking customer behavior decreases a seller's profits significantly in this context. (77) considered the case of product variety and proved that ignoring forward-looking customer behavior results in significantly suboptimal pricing and product portfolio choices. (78) analyzed a contingent price markdown mechanism in a setting with forward-looking customers, and showed that it can outperform the fixed-price mechanism and the pre-announced discount mechanism. (79) studied a structural model and empirically showed that retailers can benefit from implementing randomized markdowns when customers can monitor prices. As in many of the studies mentioned above, the markdown management problem we study entails forward-looking customer behavior; i.e., in our setting, there exist customers who compare current and future prices to make purchase decisions. But, unlike the aforementioned studies, our problem features *demand model uncertainty* and *dynamic learning* faced by a seller in the presence of a mixture of customers with heterogeneous patience levels. Our analysis quantifies the impacts of myopic and forward-looking customer behaviors on the profit loss due to demand model uncertainty, shedding light on how customer patience (or lack thereof) affects the performance of a dynamic pricing-and-learning policy (see §2.1.3 for further details).

Dynamic pricing with demand model uncertainty and learning. The tradeoff between learning and earning has been studied extensively in the dynamic pricing literature. In this context, (80) considered a partially observable Markov decision process framework for dynamic pricing with demand learning and analyzed heuristic pricing policies that approximate the optimal policy structure. (56) studied a stochastic control framework for dynamic pricing under demand model uncertainty, characterizing the optimal policy structure. (81) formulated a general dynamic pricing framework in which a seller uses an aggregating algorithm to predict the behavior of its customers, showing that the performance of the proposed algorithm is robust to distributional assumptions on demand. (57) studied a dynamic pricing problem with demand model uncertainty and designed well-performing heuristic policies based on dynamic programming. (58) showed that, in the context of dynamic pricing with demand learning, a myopic policy can suffer from incomplete learning, which results in poor profit performance; they also developed variants of the myopic policy that are asymptotically optimal. Furthermore, (82), (83), and (59) studied dynamic pricing problems with parametric demand model uncertainty and designed asymptotically optimal policies that balance the learn-and-earn tradeoff. (84) considered a Brownian stochastic control problem with dynamic learning and a single price-change decision, and characterized the optimal policy structure for this problem. More recently, (85) considered the case of finitely many price changes in dynamic pricing with demand learning, characterizing the best achievable profit performance and constructing asymptotically optimal policies in their setting. In the broader context of operations management, (86) analyzed a planning problem where a seller learns about market potential via advance sales, and showed how advance sales information influence capacity decisions, whereas (87) studied how uncertainty about quality costs influences the design of vertically differentiated products. Our work investigates the interplay between dynamic markdown management, demand learning, and forward-looking customer behavior. First of all, to provide practical guidelines for markdown management, we characterize how forward-looking customer behavior affects the design of well-performing markdown policies under demand model uncertainty. Secondly, our work contrasts the effects of myopic and forward-looking customer behaviors on the policy performance in a dynamic pricing-and-learning setting (see §2.1.3 for a detailed explanation of our contributions).

2.3 Problem Formulation

Basic model elements. Consider a firm, called the *seller*, that sells a product over a discrete time horizon of T periods. The product's marginal cost is $c \geq 0$, and the seller can dynamically adjust the product's price over the time horizon. There is a population of potential customers who are heterogeneous in (a) their valuations (i.e., how much they are willing to pay for the product), (b) their arrival times to the market, and (c) their patience levels (i.e., how many periods they are willing to stay in the market).

Regarding (a), the customers' valuations follow a market density function $\lambda : [0, \infty) \to [0, \bar{\lambda}]$, where $\bar{\lambda} > 0$.[1] Customers discount their future utilities with a discount factor $\delta \in (0, 1)$; therefore, if a customer with valuation v purchases the product κ periods from the current period at price p, then s/he derives a utility of $u(v, p, \kappa) = \delta^{\kappa}(v - p)$. Regarding (b), customers arrive gradually to the market over T periods. A customer with arrival period $\tau \in \{1, 2, \ldots, T\}$ joins the market in the beginning of period τ and starts considering her/his purchasing options thereafter. Regarding (c), the seller faces customers with varying patience levels. We represent a customer's decision horizon by a variable called *patience* $w \in \{0, 1, \ldots, T - 1\}$, which denotes the number of periods the customer is willing to stay in the market (not including the arrival period). The mass of customers arriving in period τ with patience w is $\alpha_{\tau w}$, where $\alpha_{\tau w} \in [0, 1]$ for all τ and w, and $\sum_{\tau=1}^{T} \sum_{w=0}^{T-1} \alpha_{\tau w} = 1$. Thus, the heterogeneity in (b) and (c) is expressed as a matrix $\boldsymbol{\alpha}$ whose $(\tau, w)^{\text{th}}$ entry is $\alpha_{\tau w}$.

We formulate forward-looking customer behavior by considering customers who evaluate their purchasing decisions within a specific time window. This formulation of patience follows (70) and (75), who describe such time windows as the patience levels of customers. Looking forward within their patience windows, customers make decisions by comparing their utilities of purchasing in one of the periods in their patience level (or leaving the market without a purchase). The tradeoff between making an immediate purchase and a potential future purchase is also influenced by a discount factor δ, which is a standard way to model forward-looking behavior; see, e.g., (60), (73), (77). In the antecedent work, a homogeneous discount factor is usually considered to be induced by a competitive financial market, and the heterogeneous patience levels correspond to idiosyncratic outside options of customers. By introducing both the patience level and discount factor in our model, we obtain a fairly general formulation of forward-looking customer behavior.

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[1] For simplicity, the value of $\int_{v \geq 0} \lambda(v) dv$ can be normalized to 1, but we do not make this assumption in our analysis.

The seller cannot directly observe the heterogeneity in (a)-(c) and thus faces an uncertainty about the demand model. We formulate this uncertainty as a prior belief distribution on N hypotheses: the seller initially believes that, for $i = 1, 2, \ldots, N$,

$$\Big(\lambda(\cdot), \boldsymbol{\alpha}, \delta\Big) = \Big(\lambda^i(\cdot), \boldsymbol{\alpha}^i, \delta^i\Big) \ \text{ with probability } b^i,$$

where $\sum_{i=1}^N b^i = 1$, and $b^i \in [0,1]$ for $i = 1, 2, \ldots, N$. We let $H_i := \Big\{\Big(\lambda(\cdot), \boldsymbol{\alpha}, \delta\Big) = \Big(\lambda^i(\cdot), \boldsymbol{\alpha}^i, \delta^i\Big)\Big\}$ and refer to H_i as *demand hypothesis* i. Thus, b^i is interpreted as the seller's prior belief that H_i is correct. For brevity, we also let $\boldsymbol{\theta} = (\boldsymbol{\theta}^1, \ldots, \boldsymbol{\theta}^N)$, where $\boldsymbol{\theta}^i := \Big(\lambda^i(\cdot), \boldsymbol{\alpha}^i, \delta^i\Big)$ for $i = 1, 2, \ldots, N$. We refer to $\boldsymbol{\theta}$ as the *problem parameter vector* and suppose that $\boldsymbol{\theta}$ resides in a compact set Θ.

Price skimming and customer decisions. To implement intertemporal price differentiation, the seller sequentially marks down the price of the product over T periods. That is, letting p_t be the product's price in period t, we have $p_1 \geq p_2 \geq \cdots \geq p_T$. A customer arriving in period τ with patience w, hereafter called a *type-*(τ, w) *customer*, stays in the market at most until the end of period $\tau + w$ or T, whichever is sooner. We denote by $\mathcal{T}_{\tau w t} = \Big\{t + 1, \ldots, \min\{\tau + w, T\}\Big\}$ the set of remaining future periods for type-(τ, w) customers as of period $t \geq \tau$. In any period $t \geq \tau$, a type-(τ, w) customer with valuation v would compare the utility of an immediate purchase with the discounted utilities for making a purchase in one of the periods in $\mathcal{T}_{\tau w t}$. Neither the seller nor the customers know future prices with certainty. Customers form a sequence of price predictions for future periods and act according to those predictions. We denote by $\hat{p}_s$ the customers' price prediction for period $s \in \mathcal{T}_{\tau w t}$. Based on the price predictions $\{\hat{p}_s, s \in \mathcal{T}_{\tau w t}\}$, a type-$(\tau, w)$ customer's decision in period t is as follows:

$$\text{customer decision} = \begin{cases} \text{purchase} & \text{if } u(v, p_t, 0) \geq \max\Big\{0, \max_{s \in \mathcal{T}_{\tau w t}}\{u(v, \hat{p}_s, s-\tau)\}\Big\}, \\ \text{wait} & \text{if } \max_{s \in \mathcal{T}_{\tau w t}}\{u(v, \hat{p}_s, s-\tau)\} \geq \max\{0, u(v, p_t, 0)\}, \\ \text{abandon} & \text{if } \max\Big\{u(v, p_t, 0), \max_{s \in \mathcal{T}_{\tau w t}}\{u(v, \hat{p}_s, s-\tau)\}\Big\} \leq 0. \end{cases} \tag{2.1}$$

To make an immediate purchase, a customer must obtain the highest utility from purchasing in the current period (compared to purchasing in the future or abandoning the market). If the customer finds it more attractive to purchase in any of the future periods within her/his patience level, then s/he waits in the system. There is also a portion of customers who derive nonpositive utility from buying in any period within their patience level. These customers choose the outside option of no purchase and abandon the market immediately. Note that if a customer does not look forward (or equivalently, has a patience level of $w = 0$) then s/he either makes a purchase or abandons within her/his arrival period, depending on whether s/he derives nonnegative utility for purchasing at the price charged in that period. Such a myopic customer would never choose to wait.[2]

Forward-looking customers predict future prices as follows. Let $\boldsymbol{p}^{(t)} = (p_1, \ldots, p_t)$ be the vectorized history of prices through period t. In period t, customers use prediction functions $\{\rho_t^\kappa : \kappa = 1, 2, \ldots\}$ such that $\rho_t^\kappa : [\underline{p}, \bar{p}]^t \to [\underline{p}, \bar{p}]$ is a continuous function that maps $\boldsymbol{p}^{(t)}$ to a price prediction $\hat{p}_{t+\kappa} = \rho_t^\kappa\big(\boldsymbol{p}^{(t)}\big)$ for period $t + \kappa$, where $\underline{p} = c$ and $\bar{p} \in (c, \infty)$. Based on this, we denote the *customer prediction behavior* by $\rho = \{\rho_t^\kappa : t, \kappa = 1, 2, \ldots\}$, which resides in a set $\mathfrak{P}$ that satisfies a markdown condition on prices, i.e., $\hat{p}_{t+\kappa} \geq \hat{p}_{t+\kappa+1}$ for all $t \in \{1, \ldots, T\}$ and $\kappa \in \{1, 2, \ldots\}$.[3]

We study both exogenous and endogenous prediction behaviors in this paper. In the case

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[2] Since the market density is finite, the marginal customers indifferent between two distinct decisions have measure zero. Thus, the decisions of these marginal customers do not impact the aggregate demand.

[3] For completeness, we let $\rho_t^0(\boldsymbol{p}^{(t)}) = p_t$ for all $\boldsymbol{p}^{(t)} \in [\underline{p}, \bar{p}]^t$, which is consistent with the fact that the customers see the price p_t before making purchasing decisions in period t.

of exogenous predictions, the customers do not necessarily know the seller's pricing policy. Thus, their prediction behavior ρ depends on the price history but not on the seller's policy. This model of prediction behavior is along a stream of theoretical and empirical research literature in economics, marketing, and operations research. In some notable empirical studies, price processes are typically assumed to be exogenous, which evolve adaptively based on historical price observations; see, e.g., (88), (89), (66), (90). On the theory side, (81) focused on limited strategic customer behavior, assuming a Markovian price process that customers can anticipate. Moreover, (91) studied a framework of minimizing regret in hindsight: in this framework, the decision maker chooses an action that only depends on the history and achieves asymptotic optimality. For each period t, we have a general prediction function $\rho_t^{\kappa}(\cdot)$, which includes all of these assumptions as special cases.

As an illustrative example, consider the setting below where, in every period, customers infer a linear price trend induced by the price history $\boldsymbol{p}^{(t)}$, making decisions based on their most recent price predictions.

Example 5. ***(linear price prediction)*** *In period* $t = 1, 2, \ldots, T-1$, *the customer prediction behavior* $\rho = \{\rho_t^{\kappa} : t, \kappa = 1, 2, \ldots\}$ *satisfies* $\rho_t^{\kappa}\left(\boldsymbol{p}^{(t)}\right) = p_t - \beta_t\left(\boldsymbol{p}^{(t)}\right)\kappa$ *for* $\boldsymbol{p}^{(t)} \in [\underline{p}, \bar{p}]^t$ *and* $\kappa \in \{1, 2, \ldots, T-t\}$, *where* $\beta_t\left(\boldsymbol{p}^{(t)}\right) = \arg\min_{\beta \in \mathbb{R}} \left\{ \sum_{s=1}^{t} [p_s - p_t - \beta(t-s)]^2 \right\}$.

Note that Example 5 is an illustrative instance of the general prediction behavior described above. In general, our formulation also covers nonlinear prediction functions.

In the case of endogenous predictions, motivated by the notion of the rational expectation equilibrium (92, 93)see, the customers know the seller's pricing policy, and for any period t, their predicted price path based on ρ coincides with the expected price path induced by the seller's policy. This assumption follows the existing literature on strategic customer behavior in deterministic settings (60, 67, 68, 75)see, in which the customers are assumed to know the seller's pricing policy and can perfectly foresee the seller's price path as well as the behavior of all other customers. (Unlike the existing literature, it is also possible to consider extensions of this model such that customers form sequences of prediction distributions that coincide with the distributions of future prices induced by the seller's policy. For a brief discussion on this extension, see §3.6.)

Demand observations. Given demand hypothesis i, price history $\boldsymbol{p}^{(t)}$, and prediction behavior ρ, we denote by $\mathcal{V}^i_{\tau w t}\left(\boldsymbol{p}^{(t)}, \rho\right) \subseteq [0, \infty)$ the set of valuations of type-(τ, w) customers remaining at the beginning of period $t+1$. This set of customers is inherited from the type-(τ, w) customers who were in the market at the beginning of period t and chose to wait during period t. Formally, for $t \in \{\tau, \ldots, \min\{\tau + w, T\}\}$, said set corresponds to the customers with valuations $v \in \mathcal{V}^i_{\tau w(t-1)}(\boldsymbol{p}^{(t-1)}, \rho)$ such that $u^i(v, \hat{p}_s, s-\tau) \geq \max\{0, u^i(v, p_t, 0)\}$ for some $s \in \mathcal{T}_{\tau w t}$, where $u^i(v, p, \kappa) := (\delta^i)^\kappa (v - p)$ denotes the customers' utility function under H_i. Thus, we can express $\mathcal{V}^i_{\tau w t}\left(\boldsymbol{p}^{(t)}, \rho\right)$ as follows:

$$\mathcal{V}^i_{\tau w t}\left(\boldsymbol{p}^{(t)}, \rho\right) = \begin{cases} \left\{ v \in \mathcal{V}^i_{\tau w(t-1)}\left(\boldsymbol{p}^{(t-1)}, \rho\right) : \max\limits_{s \in \mathcal{T}_{\tau w t}} \{u^i(v, \hat{p}_s, s-\tau)\} \geq \max\{0, u^i(v, p_t, 0)\} \right\} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } \tau \leq t \leq \tau + w, \\ [0, \infty) \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \text{if } t = \tau - 1, \\ \emptyset \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{otherwise,} \end{cases}$$

for $i \in \{1, \ldots, N\}$, $\tau \in \{1, \ldots, T\}$, $w \in \{0, \ldots, T-1\}$, and $t \in \{1, \ldots, T\}$. Let $\mathcal{U}^i_{\tau w t}\left(\boldsymbol{p}^{(t)}, \rho\right) \subseteq [0, \infty)$ be the set of valuations of type-(τ, w) customers who are willing to make a purchase in period t. This valuation set corresponds to the type-(τ, w) customers whose utility function satisfies $u^i(v, p_t, 0) \geq 0$ and $u^i(v, p_t, 0) \geq u^i(v, \hat{p}_s, s-\tau)$ for all $s \in \mathcal{T}_{\tau w t}$. Therefore,

$$\mathcal{U}^i_{\tau w t}(\boldsymbol{p}^{(t)}, \rho) = \left\{ v \in \mathcal{V}^i_{\tau w(t-1)}(\boldsymbol{p}^{(t-1)}, \rho) : u^i(v, p_t, 0) \geq \max\{0, \max_{s \in \mathcal{T}_{\tau w t}} u^i(v, \hat{p}_s, s-\tau)\} \right\} \tag{2.2}$$

for $i \in \{1, \ldots, N\}$, $\tau \in \{1, \ldots, T\}$, $w \in \{0, \ldots, T-1\}$, and $t \in \{1, \ldots, T\}$. Consequently, given the price history $\boldsymbol{p}^{(t)}$ and customer prediction behavior ρ, the expected demand in period t under H_i is

$$d_t^i\left(\boldsymbol{p}^{(t)}, \rho\right) = \sum_{\tau=1}^{t} \sum_{w=0}^{T-1} \alpha_{\tau w}^i \int_{\mathcal{U}_{\tau wt}^i(\boldsymbol{p}^{(t)},\rho)} \lambda^i(v)dv \quad \text{for } t = 1, \ldots, T. \tag{2.3}$$

Figure 2.1 illustrates customer choices and expected demand quantities in a simple two-period setting.

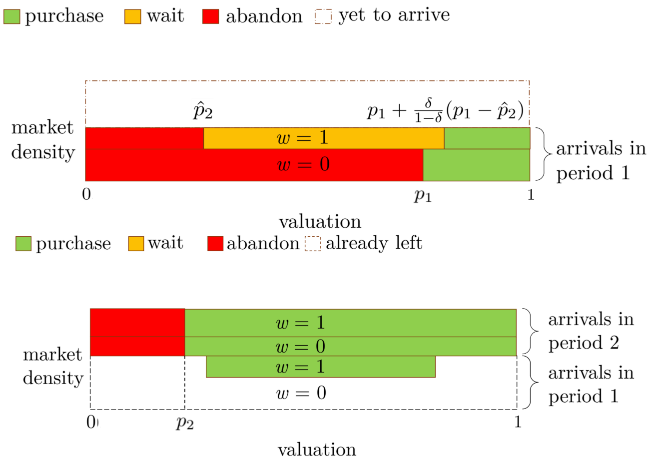

Figure 2.1: The time horizon is $T = 2$, and the customer valuations follow a uniform distribution; i.e., $v \sim U[0, 1]$. The left panel illustrates the expected demand in period 1. Impatient customers ($w = 0$) choose to purchase or abandon the market based on whether their valuation $v \geq p_1$ or not, whereas patient customers ($w = 1$) purchase if $v \geq p_1 + \frac{\delta}{1-\delta}(p_1 - \hat{p}_2)$, wait if $\hat{p}_2 \leq v \leq p_1 + \frac{\delta}{1-\delta}(p_1 - \hat{p}_2)$, or abandon the market if $v \leq \hat{p}_2$. The right panel illustrates the expected demand in period 2, where all of the remaining customers and the new customers make a purchase if their valuation $v \geq p_2$.

The seller's demand realizations are subject to unobservable temporal shocks on the market size, which are unexplained by the demand hypotheses. If the underlying hypothesis is H_i then the demand realization in period t, denoted by D_t, is a random variable with mean $d_t^i\left(\boldsymbol{p}^{(t)}, \rho\right)$ and support $[\underline{d}, \bar{d}]$, where $-\infty < \underline{d} < \bar{d} < \infty$. Under H_i, letting $\varepsilon_t = D_t - d_t^i\left(\boldsymbol{p}^{(t)}, \rho\right)$ denote the demand shock in period t, we can decompose D_t as follows:

$$D_t = d_t^i\left(\boldsymbol{p}^{(t)}, \rho\right) + \varepsilon_t \quad \text{for } t = 1, \ldots, T. \tag{2.4}$$

To accommodate a general class of demand realizations, we suppose that $\{\varepsilon_t,\, t = 1, 2, \ldots\}$

follows a martingale difference sequence adapted to the filtration $\mathcal{F}_t = \sigma(\boldsymbol{D}^{(t)}, \boldsymbol{p}^{(t+1)})$, where $\boldsymbol{D}^{(t)} = (D_1, \ldots, D_t)$. This means that the temporal demand shock ε_t is allowed to depend on the entire history of demand realizations and price decisions. Given $d_t^i\big(\boldsymbol{p}^{(t)}, \rho\big)$ and $\mathcal{F}_{t-1}$, we denote the conditional probability density function of ε_t under H_i by $\mathcal{L}_t^i(\cdot, d_t^i(\boldsymbol{p}^{(t)}, \rho)|\mathcal{F}_{t-1})$ and suppose that $\mathcal{L}_t^i(x, d_t^i(\boldsymbol{p}^{(t)}, \rho)|\mathcal{F}_{t-1}) \in [\underline{f}_d, \bar{f}_d]$ for all $x \in [\underline{d}, \bar{d}]$, where $\bar{f}_d > \underline{f}_d > 0$.

Admissible policies, posterior beliefs, and induced probabilities. We denote the set of all feasible belief vectors by $\mathcal{B} = \{\boldsymbol{b} = (b^1, \ldots, b^N) \in [0,1]^N, \sum_{i=1}^N b^i = 1\}$ and define an *admissible policy* as a sequence of non-anticipating functions $\pi = \{\pi_t,\, t = 1, 2, \ldots\}$ such that π_t is a measurable mapping from $\mathcal{B} \times [\underline{p}, \bar{p}]^{t-1}$ to $[\underline{p}, \bar{p}]$ satisfying $\pi_t\big(\boldsymbol{b}, \boldsymbol{p}^{(t-1)}\big) \leq p_{t-1}$ for all $\boldsymbol{b} \in \mathcal{B}$ and $\boldsymbol{p}^{(t-1)} \in [\underline{p}, \bar{p}]^{t-1}$. Moreover, we denote by Π the set of all admissible policies for the seller. For a policy $\pi \in \Pi$, we define the sequence of prices $\{p_t \in [\underline{p}, \bar{p}],\, t = 1, 2, \ldots\}$ and posterior beliefs $\{\boldsymbol{b}_t \in \mathcal{B},\, t = 1, 2, \ldots\}$ with the following iterative relations. In the beginning of every period t, the seller selects $p_t = \pi_t\big(\boldsymbol{b}_t, \boldsymbol{p}^{(t-1)}\big)$. After that, the seller observes the demand realization D_t and computes $\boldsymbol{b}_{t+1} = (b_{t+1}^1, \ldots, b_{t+1}^N)$ via Bayes' rule:

$$b_{t+1}^i = \nu_t^i(\boldsymbol{b}_t, \boldsymbol{D}^{(t)}, \boldsymbol{p}^{(t)}, \rho) := \frac{b_t^i\, \mathcal{L}_t^i(D_t, d_t^i(\boldsymbol{p}^{(t)}, \rho)|\mathcal{F}_{t-1})}{\sum_{i'=1}^N b_t^{i'}\, \mathcal{L}_t^{i'}(D_t, d_t^{i'}(\boldsymbol{p}^{(t)}, \rho)|\mathcal{F}_{t-1})} \quad \text{for } i = 1, \ldots, N. \tag{2.5}$$

Thus, every policy $\pi \in \Pi$ induces a probability measure on the sample space of posterior beliefs. To see this, let

$$\psi_t(\boldsymbol{b}_t, \boldsymbol{p}^{(t)}, \boldsymbol{y}, \rho) = \int_{\left\{\boldsymbol{\xi} \in \mathbb{R}^t \,:\, \nu_t^i(\boldsymbol{b}_t, \boldsymbol{\xi}, \boldsymbol{p}^{(t)}, \rho) = y_i, i = 1, \ldots, N\right\}} \Big[\sum_{i=1}^N b_t^i \mathcal{L}_t^i(\xi_t, d_t^i(\boldsymbol{p}^{(t)}, \rho)|\mathcal{F}_{t-1})\Big]\, d\xi_t, \tag{2.6}$$

for all $\boldsymbol{y} \in \mathcal{B}$. Given $\boldsymbol{b}_1 = \boldsymbol{b} \in \mathcal{B}$, define $\mathbb{P}_{\pi\rho}^{\boldsymbol{b}}\{\cdot\}$ as follows: $\mathbb{P}_{\pi\rho}^{\boldsymbol{b}}\{\boldsymbol{b}_1 = \boldsymbol{b}\} = 1$, and

$$\mathbb{P}_{\pi\rho}^{\boldsymbol{b}}\{\boldsymbol{b}_{t+1} \in d\boldsymbol{y} \,|\, \boldsymbol{b}_1, \ldots, \boldsymbol{b}_t, p_1, \ldots, p_t\} = \psi_t(\boldsymbol{b}_t, \boldsymbol{p}^{(t)}, \boldsymbol{y}, \rho)\, d\boldsymbol{y} \tag{2.7}$$

for all $t = 1, 2, \ldots, T$, and $\boldsymbol{y} \in \mathcal{B}.R_t^i\big(\boldsymbol{p}^{(t)}, \rho\big) = (p_t - c) d_t^i\big(\boldsymbol{p}^{(t)}, \rho\big)$ **Performance met-**

ric. We measure performance by the seller's expected T-period profit loss relative to a clairvoyant who knows the underlying demand model with certainty. To define this metric for an exogenous customer prediction behavior ρ, we denote by $R_t^i\big(\boldsymbol{p}^{(t)}, \rho\big) = (p_t - c)d_t^i\big(\boldsymbol{p}^{(t)}, \rho\big)$ the seller's expected profit in period t under H_i and ρ. Furthermore, we let $V^i = \max_{\pi^i \in \Pi} \big\{ \sum_{t=1}^{T} R_t^i(\boldsymbol{p}^{(t)}, \rho)\big\}$ be the seller's optimal full-information T-period profit under H_i. Thus, given the seller's prior belief $\boldsymbol{b}_1 = \boldsymbol{b}$, the T-period profit loss due to demand model uncertainty under policy π, customer prediction behavior ρ, and demand hypothesis i is

$$\Delta_{\pi\rho}^{i} = V^i - \mathbb{E}_{\pi\rho}^{\boldsymbol{b}} \Bigg\{ \sum_{t=1}^{T} R_t^i(\boldsymbol{p}^{(t)}, \rho) \Bigg\}, \tag{2.8}$$

where $\mathbb{E}_{\pi\rho}^{\boldsymbol{b}}\{\cdot\}$ is the expectation operator associated with $\mathbb{P}_{\pi\rho}^{\boldsymbol{b}}\{\cdot\}$. Consequently, if the customer prediction behavior ρ is exogenous, the seller's expected T-period profit loss due to demand model uncertainty under policy π is

$$\Delta_{\pi\rho}^{\boldsymbol{b}} = \sum_{i=1}^{N} b^i \Delta_{\pi\rho}^{i}. \tag{2.9}$$

On the other hand, if the customer prediction behavior ρ is endogenous, the customers know the seller's policy π, and their price predictions coincide with the expected price path induced by π. Formally, we consider a pair (ρ, π), consisting of a customer prediction behavior ρ and a pricing policy π, that satisfies the following condition.

Assumption 4. (endogenous forward-looking customer behavior) *For any* $t \in \{1, \ldots, T\}$, $\boldsymbol{b}_t \in \mathcal{B}$, *and* $\boldsymbol{p}^{(t-1)} \in [\underline{p}, \bar{p}]^{t-1}$ *with* $p_1 \geq p_2 \geq \cdots \geq p_t$, *the pair* (ρ, π), *consisting of customer prediction behavior* $\rho \in \mathfrak{P}$ *and pricing policy* $\pi \in \Pi$, *induces a price path* $(p_{t+1}, \ldots, p_T)$ *that satisfies* $\rho_t^{\kappa}(\boldsymbol{p}^{(t)}) = \mathbb{E}_{\pi\rho}^{\boldsymbol{b}}\big\{p_{t+\kappa} | \boldsymbol{p}^{(t)}, \boldsymbol{b}_t\big\}$ *for all* $\kappa = 1, \ldots, T - t$.

The above condition follows the modeling assumptions in the existing literature on strategic customer behavior, where customers form a sequence of predicted prices based on the knowledge of the seller's pricing policy; see, e.g., (60), (67), (68), (75).

To measure performance, we again use the seller's expected T-period profit loss relative to a clairvoyant who knows the underlying demand model with certainty. To define the clairvoyant benchmark in this case, consider a pair (ρ^i, π^i) that satisfies the following two properties under H_i: (i) $\pi^i \in \arg\max_{\pi \in \Pi} \left\{ \sum_{t=1}^{T} R_t^i(\boldsymbol{p}^{(t)}, \rho^i) \right\}$, and (ii) (ρ^i, π^i) satisfies Condition 4. That is, the seller's policy π^i is optimal under H_i and ρ^i, and based on the prediction behavior ρ^i, the customers' purchasing decisions defined in (2.1) are also optimal under H_i and π^i. Thus, (ρ^i, π^i) constitutes a subgame perfect equilibrium under H_i.[4] Without loss of generality, we choose the pair (ρ^i, π^i) with an induced equilibrium price path $(p_1^i, \ldots, p_T^i)$ under H_i such that $V_{\text{eq}}^i = \sum_{t=1}^{T} R_t^i(\boldsymbol{p}^{i(t)}, \rho^i)$ is the seller's optimal full-information profit in equilibrium. Hence, given the seller's prior belief $\boldsymbol{b}_1 = \boldsymbol{b}$, the T-period profit loss under policy π, customer prediction behavior ρ satisfying Condition 4, and demand hypothesis i is

$$\tilde{\Delta}_{\pi\rho}^i = V_{\text{eq}}^i - \mathbb{E}_{\pi\rho}^{\boldsymbol{b}} \left\{ \sum_{t=1}^{T} R_t^i(\boldsymbol{p}^{(t)}, \rho) \right\}. \tag{2.10}$$

As a result, if the customer prediction behavior ρ is endogenous as described in Condition 4, the seller's expected T-period profit loss due to demand model uncertainty is

$$\tilde{\Delta}_{\pi\rho}^{\boldsymbol{b}} = \sum_{i=1}^{N} b^i \tilde{\Delta}_{\pi\rho}^i. \tag{2.11}$$

Asymptotic regime. Because the problem described above defies exact solution, we study an asymptotic regime in which the seller's time horizon and market size grow proportionally large. Consider a sequence of problems, indexed by $k = 1, 2, \ldots$, such that in the k^{th} problem the number of periods and the customer population are scaled up by k; that is, we replace T with $T_k = kT$ and $\boldsymbol{\alpha}^i$ with $k\boldsymbol{\alpha}^i$ for all $i \in \{1, \ldots, N\}$.[5] This scaling can also be viewed as

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[4] We prove the existence of (ρ^i, π^i) in Proposition 18 of Appendix 2.8.

[5] In the k^{th} problem of this asymptotic regime, $k\alpha_{\tau w}^i$ corresponds to the mass of type-(τ, w) customers

increasing the frequency of pricing decisions in a fixed sales horizon. In this interpretation, T represents the length of a sales season, and kT corresponds to the maximum number of prices changes that the seller can implement. To ensure that the mass of customers arriving between two pricing decisions does not become negligibly small in this asymptotic regime, we assume that $\sum_{w=0}^{T_k-1} k\alpha_{\tau w}^i \in [\underline{\alpha}, \overline{\alpha}]$ for all $\tau \in \{1, \ldots, T_k\}$ and $i \in \{1, \ldots, N\}$, where $0 < \underline{\alpha} < \overline{\alpha} < \infty$. (Without this assumption, it is possible to have virtually no customer arrivals in the vast majority of periods, in which case the arrivals would be concentrated on a few periods, yielding a regime where the number of pricing decisions effectively does not grow.) In the case of exogenous forward-looking customer behavior, after adjusting the definitions in (2.8)-(2.9) based on the above scaling, we let $\Delta_{\pi\rho}^{\boldsymbol{b}}(k)$ be the seller's expected profit loss in the k^{th} problem. Similarly, in the case of endogenous forward-looking customer behavior, upon adjusting the definitions (2.10)-(2.11) with the same scaling, we let $\tilde{\Delta}_{\pi\rho}^{\boldsymbol{b}}(k)$ be the seller's expected profit loss in the k^{th} problem. With slight abuse of notation, we also express the dependence of these metrics on the problem parameter vector $\boldsymbol{\theta}$ by writing $\Delta_{\pi\rho}^{\boldsymbol{b}}(k, \boldsymbol{\theta})$ and $\tilde{\Delta}_{\pi\rho}^{\boldsymbol{b}}(k, \boldsymbol{\theta})$ instead of $\Delta_{\pi\rho}^{\boldsymbol{b}}(k)$ and $\tilde{\Delta}_{\pi\rho}^{\boldsymbol{b}}(k)$, respectively. In the subsequent sections, we characterize the optimal growth rate of $\tilde{\Delta}_{\pi\rho}^{\boldsymbol{b}}(k)$ in terms of k.

2.3.1 Learning and Earning with Forward-looking Customers

Structure of Customer Valuation Sets

As a first step of our analysis, we study the structure of the valuation sets belonging to customers remaining in the system and to those purchasing in each period.

Lemma 10. (interval structure of valuation sets) *Let* $i \in \{1, \ldots, N\}$ *and* $k \in \{1, 2, \ldots\}$. *Set* $\underline{v}_{\tau w(\tau-1)}^i = 0$ *and* $\overline{v}_{\tau w(\tau-1)}^i = \infty$ *for all* $\tau \in \{1, \ldots, T_k\}$ *and* $w \in \{0, \ldots, T_k - 1\}$. *Then,*

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under H_i, and the total mass of customers under H_i equals $\sum_{\tau=1}^{T_k} \sum_{w=0}^{T_k-1} k\alpha_{\tau w}^i = k$ for all $i \in \{1, \ldots, N\}$.

for all $t \in \{\tau\} \cup \mathcal{T}_{\tau w \tau} = \Big\{\tau, \ldots, \min\{\tau + w, T_k\}\Big\}$,

$$\text{(i)} \qquad \mathcal{V}^i_{\tau wt}\Big(\boldsymbol{p}^{(t)}, \rho\Big) = \Big[\underline{v}^i_{\tau wt}(\boldsymbol{p}^{(t)}, \rho), \overline{v}^i_{\tau wt}(\boldsymbol{p}^{(t)}, \rho)\Big],$$

$$\text{(ii)} \qquad \mathcal{U}^i_{\tau wt}\Big(\boldsymbol{p}^{(t)}, \rho\Big) = \Big[\underline{u}^i_{\tau wt}(\boldsymbol{p}^{(t)}, \rho), \overline{u}^i_{\tau wt}(\boldsymbol{p}^{(t)}, \rho)\Big],$$

where $\underline{v}^i_{\tau wt}(\boldsymbol{p}^{(t)}, \rho) = \max\Big\{\underline{v}^i_{\tau w(t-1)}(\boldsymbol{p}^{(t-1)}, \rho),\, \hat{p}_{\min\{\tau+w, T_k\}}\Big\}$,

$\overline{v}^i_{\tau wt}(\boldsymbol{p}^{(t)}, \rho) = \min\Big\{\overline{v}^i_{\tau w(t-1)}(\boldsymbol{p}^{(t-1)}, \rho),\, \max_{s \in \mathcal{T}_{\tau wt}} \Big\{p_t + \frac{(\delta^i)^{s-t}}{1-(\delta^i)^{s-t}}(p_t - \hat{p}_s)\Big\}\Big\}$,

$\underline{u}^i_{\tau wt}(\boldsymbol{p}^{(t)}, \rho) = \overline{v}^i_{\tau wt}(\boldsymbol{p}^{(t)}, \rho)$,

$\overline{u}^i_{\tau wt}(\boldsymbol{p}^{(t)}, \rho) = \overline{v}^i_{\tau w(t-1)}(\boldsymbol{p}^{(t-1)}, \rho)$.

Lemma 10 characterizes the structure of $\mathcal{V}^i_{\tau wt}\Big(\boldsymbol{p}^{(t)}, \rho\Big)$ and $\mathcal{U}^i_{\tau wt}\Big(\boldsymbol{p}^{(t)}, \rho\Big)$, namely the valuation sets of customers who choose waiting and those who make a purchase, respectively. Based on the definitions of these sets in (2.2), the above lemma proves that both sets are intervals in $[0, \infty)$. This implies that the customers with valuations above an upper threshold purchase immediately, whereas those with valuations below a lower threshold abandon the market immediately. The customers whose valuations are between these lower and upper thresholds neither purchase in the current period nor abandon immediately, because they find it more beneficial to wait for the next period. Figure 2.2 illustrates these intervals in the numerical example used in Figure 2.1.

2.3.2 Characterizing the Best Achievable Performance

In this subsection, we derive lower bounds on the smallest possible growth rate of the profit loss due to demand model uncertainty in terms of the problem scale k. We obtain these bounds for both exogenous and endogenous forms of forward-looking customer behavior.

The following result presents the best achievable profit performance in the case of exogenous customer predictions.

Theorem 6. (lower bound on profit loss under exogenous customer predictions) *Let* $\boldsymbol{b} \in \mathcal{B}$ *such that* $b^i < 1$ *for all* $i \in \{1, \ldots, N\}$. *Then,*

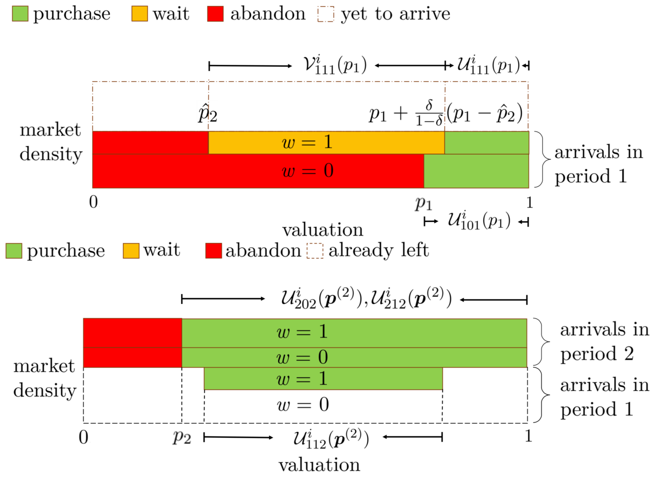

Figure 2.2: The left panel shows $\mathcal{U}^i_{101}(p_1, \rho)$ and $\mathcal{U}^i_{111}(p_1, \rho)$ (namely, the sets of type-(1,0) and type-(1,1) customers who make a purchase in period 1) as well as $\mathcal{V}^i_{111}(p_1, \rho)$ (i.e., the set of type-(1,1) customers who decide to wait in period 1); all other customers arriving in period 1 abandon the market immediately. The right panel displays $\mathcal{U}^i_{202}(\boldsymbol{p}^{(2)}, \rho)$, $\mathcal{U}^i_{212}(\boldsymbol{p}^{(2)}, \rho)$, and $\mathcal{U}^i_{112}(\boldsymbol{p}^{(2)}, \rho)$ (namely, the sets of type-(2,0), type-(2,1), and type-(1,1) customers who make a purchase in period 2). Since period 2 is the last period, no customers choose to wait in this period.

(i) *there exists a constant* $c_1 > 0$ *such that, under any policy* $\pi \in \Pi$ *and any exogenous customer prediction behavior* $\rho \in \mathfrak{P}$*,*

$$\sup_{\boldsymbol{\theta} \in \Theta} \{\Delta^{\boldsymbol{b}}_{\pi\rho}(k, \boldsymbol{\theta})\} \geq c_1 \log k$$

for $k = 1, 2, \ldots,$ *and*

(ii) *there exist an exogenous customer prediction behavior* $\rho \in \mathfrak{P}$ *and a constant* $c_2 > 0$ *such that, under any policy* $\pi \in \Pi$*,*

$$\sup_{\boldsymbol{\theta} \in \Theta} \{\Delta^{\boldsymbol{b}}_{\pi\rho}(k, \boldsymbol{\theta})\} \geq c_2 k$$

for $k = 1, 2, \ldots$

Theorem 6 provides two lower bounds on the profit loss: (i) a lower bound of order $\log k$, which holds for any exogenous $\rho \in \mathfrak{P}$, and (ii) a lower bound of order k, which holds for a special class of exogenous prediction behaviors $\rho \in \mathfrak{P}$.

Theorem 6(i) states that, for all exogenous customer prediction behaviors and all admissible pricing policies, the profit loss grows at least logarithmically in the problem scale k. The intuition behind this result is as follows: in the early periods, the seller has limited information on the underlying demand model and is likely to make a pricing error. Due to such pricing errors, customers with low patience levels abandon the market. Thus, missing the opportunity to offer better prices to these customers results in a profit loss. Theorem 6(i) provides a benchmark for good profit performance in our asymptotic setting: we say that a policy is *asymptotically optimal* if the growth rate of its profit loss in k matches the logarithmic growth rate of the lower bound in Theorem 6(i).

In contrast to Theorem 6(i), Theorem 6(ii) shows that there is an exogenous customer prediction behavior such that the profit loss under any admissible pricing policy grows linearly in the problem scale k, which is the highest possible growth rate of the profit loss. This finding establishes that the negative impact of forward-looking customers is excessively large in the worst case. The problem instance used in the derivation of Theorem 6(ii) is one where the customers have perfect memory, i.e., they use the entire price history when they make predictions under ρ. In such problem instances, early markdowns have a lasting impact on the customers' forward-looking behavior. Even if the seller can gather sufficient demand information later on, perfect customer memory poses the risk of incurring large profit losses due to early pricing errors. Theorem 6(ii) formalizes this intuition.

The distinction between parts (i) and (ii) of Theorem 6 sheds a new light on the impact of forward-looking customers on dynamic pricing with demand learning. In the absence of forward-looking customers, (82) and (83) have constructed pricing policies with cumulative losses growing logarithmically in the problem scale in their settings. Theorem 6(i) shows that our setting is at least as complex as these settings in the literature because one cannot achieve better performance in our setting. On top of this, Theorem 6(ii) establishes that the introduction of forward-looking customers makes the learning-and-earning problem fundamentally more complex. There exist customer prediction behaviors that make the profit loss of all policies increase at the largest possible rate—this is not possible in the previously studied settings with no forward-looking customers.

To complement Theorem 6, the next result characterizes the best achievable profit performance in the case of endogenous customer predictions.

Proposition 15. (lower bound on profit loss under endogenous customer predictions) *Let* $\boldsymbol{b} \in \mathcal{B}$ *such that* $b^i < 1$ *for all* $i \in \{1, \ldots, N\}$*. Then, there exists* $c_3 > 0$ *such that, under any policy* $\pi \in \Pi$ *and any endogenous customer prediction behavior* $\rho \in \mathfrak{P}$ *with* (ρ, π) *satisfying Condition 4,*

$$\sup_{\boldsymbol{\theta} \in \Theta} \{\tilde{\Delta}^{\boldsymbol{b}}_{\pi\rho}(k, \boldsymbol{\theta})\} \geq c_3 \log k$$

for $k = 1, 2, \ldots$

The intuition for Proposition 15 is similar to that for Theorem 6(i): early pricing errors curtail potential profits from impatient customers, resulting in a logarithmically growing profit loss relative to the case of full information.

In light of Theorem 6(ii) and Proposition 15, it is worth noting that the smallest growth rate of the profit loss can be significantly higher in the case of exogenous customer predictions. This is because if ρ is exogenous, the seller cannot directly influence the intertemporal dependency induced by ρ. Therefore, the consequences of using a suboptimal price path in early periods could spill over into later periods. However, if ρ is endogenous, the seller's pricing policy π influences ρ as customers perfectly predict the expected price path of the policy π. This allows the seller to have further control over customer predictions and the intertemporal dependency they induce. Thus, from a learning perspective, endogenous customer predictions gives the seller an advantage to possibly improve performance. In the subsequent sections, we investigate whether this advantage can indeed yield better learning performance. To that end, we design and analyze policies that achieve the asymptotically optimal performance benchmarks in Theorem 6 and Proposition 15.

2.3.3 Asymptotically Optimal Policies for Exogenous Forward-looking Behavior

In markdown management with demand model uncertainty, the early pricing decisions should be not only informative but also market-preserving. If the seller selects a low price in the early periods, this would reduce the product's value in the market. As a consequence, high-valuation customers would purchase the product at a low price, and a large portion of the seller's earning opportunities would be wasted. Thus, the success of a dynamic learning policy relies on the existence of at least one market-preserving and informative price path, which is formally described as follows.

Assumption 5. (informative market preservation under exogenous customer predictions) *Let $k \in \{1, 2, \ldots\}$, and for all $i \in \{1, \ldots, N\}$, let $p^i = (p_1^i, p_2^i, \ldots, p_{T_k}^i)$ be the optimal full-information price path under H_i. Suppose that there is a price path $\mathfrak{p} = (\mathfrak{p}_1, \ldots, \mathfrak{p}_{T_k})$ such that, given $t \in \{1, \ldots, T_k\}$ and $\rho \in \mathfrak{P}$, we have:* (i) *$\mathfrak{p}_t \geq \max_{i \in \{1,\ldots,N\}}\{p_t^i\}$, and* (ii) *there exists $\epsilon_d > 0$ such that $|d_t^i(\mathfrak{p}^{(t)}, \rho) - d_t^{i'}(\mathfrak{p}^{(t)}, \rho)| \geq \epsilon_d$ for all $i, i' \in \{1, \ldots, N\}$ satisfying $i \neq i'$.*

Under Condition 5, the seller can differentiate between the distinct demand hypotheses without dropping the price too early. This condition is essential to achieve a reasonable profit performance: we prove in Appendix 2.9 that, without this condition, the profit loss of *any* admissible policy can grow linearly in the problem size k; see Proposition 19(i). We emphasize that this is the worst possible growth rate of the profit loss, meaning that no pricing policy can perform well if there exists no price path that is informative and market-preserving. Thus, throughout subsection 2.3.3, we focus on settings where Condition 5 holds.

2.3.4 Optimal Full-information Policy

The asymptotically optimal policies we design later in this subsubsection are based on the optimal policy in case of full information. To express this policy, we recall that $R_t^i(\boldsymbol{p}^{(t)}, \rho) =$

$(p_t - c)d_t^i(\boldsymbol{p}^{(t)}, \rho)$ is the expected profit in period t under H_i. Hence, given the problem scale $k \in \{1, 2, \dots\}$, the seller's markdown pricing problem under H_i is given by

$$V^i = \max_{\pi \in \Pi} \left\{ \sum_{t=1}^{T_k} R_t^i\Big(\boldsymbol{p}^{(t)}, \rho\Big) \right\}. \tag{2.12}$$

Formulating (2.12) as a dynamic program, we note that the Bellman equation that characterizes the optimal full-information policy is

$$U_t^i\Big(\boldsymbol{p}^{(t-1)}\Big) = \max_{p_t \in [\underline{p}, \bar{p}]} \Big\{ R_t^i\Big(\boldsymbol{p}^{(t)}, \rho\Big) + U_{t+1}^i\Big(\boldsymbol{p}^{(t)}\Big) : p_t \leq p_{t-1} \Big\} \ \text{ for } t = 1, 2, \dots, T_k, \tag{2.13}$$

subject to the boundary condition $U_{T_k+1}^i\Big(\boldsymbol{p}^{(T_k)}\Big) = 0$. Here, $U_t^i\Big(\boldsymbol{p}^{(t-1)}\Big)$ is the maximum expected total profit in periods $\{t, t+1, \dots, T_k\}$ when the price history is $\boldsymbol{p}^{(t-1)}$ at the beginning of period t, and the seller knows that the underlying demand hypothesis is H_i. Letting $p^i = (p_1^i, p_2^i, \dots, p_{T_k}^i)$ be the optimal full-information price path obtained by solving (2.13), we note that there is a sequence of pricing functions $\varphi^i = \{\varphi_t^i : t = 1, 2, \dots, T_k\}$ with $\varphi_t^i : [\underline{p}, \bar{p}]^{t-1} \to [\underline{p}, \bar{p}]$ satisfying $p_t^i = \varphi_t^i(\boldsymbol{p}^{(t-1)})$.

Based on the interval structure of customer valuation sets and the optimal policy in case of full information, we now construct two policies that are motivated by commonly used approaches to dynamic learning problems, namely a *passive learning* policy based on the certainty-equivalence principle and a *forced exploration* policy that dedicates a certain number of periods to learning.

Passive Learning: the Certainty-Equivalence Policy For all $\boldsymbol{b} \in \mathcal{B}$, define $\lambda^{\boldsymbol{b}}(\cdot) = \sum_{i=1}^N b^i \lambda^i(\cdot)$, $\boldsymbol{\alpha}^{\boldsymbol{b}} = \sum_{i=1}^N b^i \boldsymbol{\alpha}^i$, and $\delta^{\boldsymbol{b}} = \sum_{i=1}^N b^i \delta^i$. Based on the seller's belief $\boldsymbol{b}_t$ in period t, the certainty-equivalence policy, abbreviated CE, operates as if the problem parameter vector $\boldsymbol{\theta} = \Big(\lambda(\cdot), \boldsymbol{\alpha}, \delta\Big)$ were equal to $\boldsymbol{\theta}^{\boldsymbol{b}_t} := \Big(\lambda^{\boldsymbol{b}_t}(\cdot), \boldsymbol{\alpha}^{\boldsymbol{b}_t}, \delta^{\boldsymbol{b}_t}\Big)$. In period $t \in \{1, 2, \dots, T_k\}$, given the seller's belief $\boldsymbol{b}_t$ and the price history $\boldsymbol{p}^{(t-1)}$, CE chooses $p_t = \varphi_t^{\boldsymbol{b}_t}(\boldsymbol{p}^{(t-1)})$, where $\varphi_t^{\boldsymbol{b}}(\cdot)$ is the optimal full-information pricing function in period t under $H_{\boldsymbol{b}} := \{(\lambda, \boldsymbol{\alpha}, \delta) = (\lambda^{\boldsymbol{b}}, \boldsymbol{\alpha}^{\boldsymbol{b}}, \delta^{\boldsymbol{b}})\}$

for all $\boldsymbol{b} \in \mathcal{B}$. (For this construction, the readers are reminded that $\varphi_t^{\boldsymbol{b}}(\cdot)$ is the pricing function in the optimal full-information solution.) Although CE is a forward-looking policy that accounts for the dynamic evolution of the market size, it is myopic with regard to demand learning: under CE, the seller's belief is updated via Bayes' rule (3.28), but while choosing prices, CE ignores demand learning entirely and chooses prices as if beliefs will not be updated in the future. Our next result characterizes the performance of CE.

Proposition 16. (profit loss of CE) *Let* $\pi = \mathrm{CE}$, *and* $\boldsymbol{b} \in \mathcal{B}$ *such that* $b^i < 1$ *for all* $i \in \{1, \ldots, N\}$. *Then, there exists a constant* $c_4 > 0$ *such that, under any exogenous customer prediction behavior* $\rho \in \mathfrak{P}$,

$$\sup_{\boldsymbol{\theta} \in \Theta} \{\Delta_{\pi\rho}^{\boldsymbol{b}}(k, \boldsymbol{\theta})\} \geq c_4 k$$

for $k = 1, 2, \ldots$

Proposition 16 shows that CE performs very poorly in markdown management with demand learning, incurring a profit loss that grows linearly in the problem scale k.

The antecedent literature contains theoretical results that establish the poor performance of certainty-equivalence policies in dynamic pricing problems that involve demand learning; see, e.g., (58), (59), (94). The reason why CE performs poorly in our setting is different. The aforementioned studies showed that certainty-equivalence pricing policies suffer from a phenomenon called "incomplete learning," which makes the decision-maker's beliefs (or estimates) converge to incorrect values with positive probability. However, in our setting, we observe that the poor performance of CE is caused by the markdown structure: even if the seller gathers sufficient information about the demand model and completely avoids the incomplete learning trap, it still needs to implement a markdown policy under demand model uncertainty. Because CE is myopic in terms of demand learning, this policy causes the seller to drop the price too early with positive probability. Consequently, the seller's failure to preserve market size results in extensive profit loss.

Forced Exploration: the Learn-And-Then-Earn Policy

Let $n \in \{1, 2, \ldots, T_k\}$. The learn-and-then-earn policy with parameter n, denoted by LATE(n), charges the market-preserving experimental price path $(\mathfrak{p}_1, \ldots, \mathfrak{p}_n)$ in Condition 5 in the first n periods, whereas in the remaining periods it chooses prices assuming that the most likely hypothesis in period $n+1$ is correct. To be precise, LATE(n) chooses $p_t = \mathfrak{p}_t$ for $t \in \{1, 2, \ldots, n\}$. In the beginning of period $n+1$, LATE(n) infers the underlying hypothesis by computing $\hat{\imath}_n = \arg\max_{i \in \{1,\ldots,N\}} \{b^i_{n+1}\}$. Then, given that $\boldsymbol{p}^{(n)} = \boldsymbol{\mathfrak{p}}^{(n)}$, LATE($n$) constructs a price path $\tilde{p} = (\tilde{p}_{n+1}, \tilde{p}_{n+2}, \ldots, \tilde{p}_{T_k})$ that maximizes $\sum_{t=n+1}^{T_k} R_t^{\hat{\imath}_n}(\boldsymbol{p}^{(t)}, \rho)$, namely the expected full-information profit in periods $\{n+1, \ldots, T_k\}$ under $H_{\hat{\imath}_n}$. In the remainder of the time horizon, LATE(n) chooses $p_t = \tilde{p}_t$ for $t \in \{n+1, n+2, \ldots, T_k\}$. This means that LATE($n$) divides the time horizon into two phases: the learning-and-market-preservation phase in periods $\{1, \ldots, n\}$, and the earning phase in periods $\{n+1, \ldots, T_k\}$. In the learning-and-market-preservation phase, LATE(n) preserves the market size by charging a sequence of high experimental prices, and as shown below, this can yield a sufficiently fast learning rate. In the earning phase, LATE(n) employs the optimal full-information price path based on the inferred demand hypothesis, $H_{\hat{\imath}_n}$.

To study the performance of LATE(n), we first quantify its rate of learning. In the following result, we characterize the convergence rate of beliefs for policies that ensure a minimum positive difference of expected demand between the true hypothesis and other hypotheses.

Lemma 11. (belief concentration under exogenous customer predictions) *Let $\boldsymbol{b} \in \mathcal{B}$ such that $b^i < 1$ for all $i \in \{1, \ldots, N\}$. Then, there exist constants $\zeta > 0$ and $\eta > 0$ such that, under any exogenous customer prediction behavior $\rho \in \mathfrak{P}$,*

$$\mathbb{E}^{\boldsymbol{b}}_{\pi\rho}\{1 - b^i_{t+1} \,|\, H_i\} \leq \zeta e^{-\eta t}$$

for all $t \in \{1, 2, \dots\}$, $i \in \{1, \dots, N\}$, *and* $\pi \in \Pi$ *satisfying* $\min_{i' \neq i} \left\{ |d_s^i(\boldsymbol{p}^{(s)}) - d_s^{i'}(\boldsymbol{p}^{(s)})| \right\} \geq \epsilon_d$ *almost surely for all* $s \in \{1, \dots, t\}$, *where* ϵ_d *is as given in Condition 5.*

Lemma 11 presents an upper bound on the expected distance between the seller's belief and the underlying hypothesis. When the underlying demand hypothesis is H_i, this distance converges to zero exponentially under policies that guarantee a minimum positive difference between the expected demand under H_i and those under any other alternative $H_{i'}$. We recall that, under Condition 5, efficient learning is possible, and hence, the seller can employ LATE(n) to simultaneously preserve the market price and learn about the demand model, thereby achieving the belief concentration rate in Lemma 11. (As explained earlier, if Condition 5 does not necessarily hold, then the seller runs the risk of incurring a profit loss growing linearly in k under *any* given policy, making efficient learning impossible regardless of the policy exercised.)

Although LATE(n) can attain rapid concentration of beliefs as characterized in Lemma 11, the asymptotic optimality of LATE(n) remains to be shown. To that end, we observe that the customer prediction behavior ρ plays a vital role in the seller's profit loss. From the seller's perspective, there are two main consequences of choosing suboptimal prices for learning purposes. On one hand, a suboptimal price directly affects the seller's immediate profit. On the other hand, suboptimal prices also have an indirect effect on the customers' future purchasing behavior, which can lead to further profit loss for the seller in the subsequent periods. To study these effects, we consider two cases: one where the customer memory is *bounded*, and another where it is *unbounded.*

Forced exploration with bounded customer memory. In practice, customers often have limited access to the past historical prices and may not be able to keep track of all historical prices. Motivated by this observation, we consider a class of customer prediction behaviors ρ that are based on bounded customer memory. In this case, we suppose that customers have access to the latest τ_p prices charged by the seller, where τ_p is a positive integer independent of the problem scale k. Formally, this corresponds to a subset of customer prediction behaviors in which only the most recent τ_p prices matter. This subset of customer prediction behaviors can be expressed as

$$\mathfrak{P}_1 = \left\{ \begin{array}{r} \rho \in \mathfrak{P} : \text{ if } p_s = \bar{p}_s \text{ for } s = \max\{1, t - \tau_p\}, \ldots, t, \\ \text{then } \rho_t^\kappa(\boldsymbol{p}^{(t)}) = \rho_t^\kappa(\bar{\boldsymbol{p}}^{(t)}) \text{ for } \kappa = 0, 1, \ldots \end{array} \right\} \subset \mathfrak{P}. \tag{2.14}$$

The prediction behaviors in (2.14) exhibit bounded memory because prices older than τ_p periods do not affect the customers' predictions of future prices. The following result provides a performance guarantee for LATE(n) when the customer memory is bounded as in (2.14).

Theorem 7. (profit loss of LATE under exogenous customer predictions and bounded customer memory) *Let* $\pi = \text{LATE}(n)$ *with* $n = \lceil \frac{1}{\eta} \log k \rceil$ *and* η *as given in Lemma 11, and* $\boldsymbol{b} \in \mathcal{B}$ *such that* $b^i < 1$ *for all* $i \in \{1, \ldots, N\}$*. Then, there exists a constant* $c_5 > 0$ *such that, under any exogenous customer prediction behavior* $\rho \in \mathfrak{P}_1$*,*

$$\sup_{\boldsymbol{\theta} \in \Theta} \{\Delta_{\pi\rho}^{\boldsymbol{b}}(k, \boldsymbol{\theta})\} \leq c_3 \log k$$

for $k = 2, 3, \ldots$

Theorem 7 shows that the profit loss of our learn-and-then-earn policy grows logarithmically in the problem scale k for any exogenous customer prediction behavior with bounded customer memory. This matches the order of the lower bound on the profit loss in Theorem 6(i), implying that LATE(n) is asymptotically optimal in this case, as long as the policy parameter n is selected carefully. The key insight here is that LATE(n) allows the seller to achieve asymptotic optimality by simultaneously learning about the demand model, preserving the market size, and controlling the forward-looking customer behavior.

Forced exploration with unbounded customer memory. We now relax the assumption of bounded customer memory and consider the case when customers have access to all past prices. As explained earlier, the main challenge in this case is that a suboptimal price charged early in the time horizon could have a significant impact on the subsequent purchasing behavior of the customers. Recall that when customers have unbounded memory, Theorem 6(ii) establishes that the worst-case profit loss under any given policy grows linearly in the problem scale k, which is the highest possible growth rate of the profit loss. The reason behind this result is that, due to demand model uncertainty, the seller's markdowns in the early periods could be suboptimal, and the impact of this suboptimality on the later purchasing behavior of forward-looking customers is so large that it causes an extreme profit loss. In such a setting, no policy can achieve a profit loss growing sublinearly in k.

While unbounded customer memory can cause a large profit loss in general, we show that, under mild regularity conditions on the customer prediction behavior, our learn-and-then-earn policy achieves a sublinear profit loss, regardless of customer memory size. For this purpose, consider the following subset of customer prediction behaviors: let

$$\mathfrak{P}_2(a,r) = \left\{ \begin{array}{c} \rho \in \mathfrak{P} \ : \rho_t^\kappa \in \mathcal{C} \text{ and } \left| \frac{\partial}{\partial p_s} \rho_t^\kappa(\boldsymbol{p}^{(t)}) \right| \le a\kappa \left(\frac{s}{t} \right)^{1+r} \\ \text{for } t, \kappa = 1, 2, \ldots, \text{ and } s = 1, \ldots, t \end{array} \right\} \subset \mathfrak{P}, \tag{2.15}$$

for $a > 0$ and $r > 0$, where $\mathcal{C}$ denotes the set of continuously differentiable functions. The set $\mathfrak{P}_2(a,r)$ captures the following properties for customer prediction behavior ρ: (i) range of future price predictions grows with the prediction horizon κ (as the partial derivatives of $\rho_t^\kappa(\cdot)$ are allowed to grow linearly in κ); (ii) the impact of a single historical price point on future price predictions is limited, and more recent prices have a stronger impact on future price predictions; and (iii) if two historical price paths $\boldsymbol{p}^{(t)}$ and $\tilde{\boldsymbol{p}}^{(t)}$ are significantly different (e.g., $|p_s^i - \tilde{p}_s^i|$ exceeds a positive constant for $s = 1, \ldots, t$), then the difference in the future price predictions based on $\boldsymbol{p}^{(t)}$ and $\tilde{\boldsymbol{p}}^{(t)}$ is allowed to grow linearly in terms of t, which corresponds to a wide range of predictions. A simple example satisfying these regularity conditions is the linear prediction function in Example 5.

The next result presents a performance guarantee for LATE(n) under the above regularity conditions.

Theorem 8. (profit loss of LATE under exogenous customer predictions and unbounded customer memory) *Let* $a > 0$*,* $r > 0$*,* $\pi = \mathrm{LATE}(n)$ *with* $n = \lceil \frac{1}{\eta} \log k \rceil$ *and* η *as given in Lemma 11, and* $\boldsymbol{b} \in \mathcal{B}$ *such that* $b^i < 1$ *for all* $i \in \{1, \ldots, N\}$*. Then, under any exogenous customer prediction behavior* $\rho \in \mathfrak{P}_2(a, r)$*,*

$$\frac{1}{k} \sup_{\boldsymbol{\theta} \in \Theta} \{\Delta^{\boldsymbol{b}}_{\pi\rho}(k, \boldsymbol{\theta})\} \to 0$$

as $k \to \infty$*.*

Theorem 8 states that, if the customer prediction behavior is exogenous and satisfies the regularity conditions in $\mathfrak{P}_2(a, r)$, then the seller's profit loss under LATE(n) is sublinear in the problem size k. This stands in contrast to the linearly growing profit loss in Theorem 6(ii). Even though the worst-case impact of exogenous forward-looking customer behavior is extremely detrimental to the seller, our learn-and-then-earn policy can substantially mitigate this negative impact under mild regularity conditions.

2.3.5 Asymptotically Optimal Policies for Endogenous Forward-looking Behavior

As in the preceding subsection, achieving asymptotic optimality in the case of endogenous customer predictions depends on the existence of at least one market-preserving and informative price path, which we express in the following variant of Condition 5.

Assumption 6. (informative market preservation under endogenous customer predictions) *Let* $k \in \{1, 2, \ldots\}$*. Suppose that there is a non-increasing price path* $\mathfrak{p} = (\mathfrak{p}_1, \ldots, \mathfrak{p}_{T_k})$ *such that, given any set of subgame perfect equilibria* $\{(\rho^i, \pi^i) : i = 1, \ldots, N\}$ *with* (ρ^i, π^i) *inducing a price path* $p^i = (p^i_1, p^i_2, \ldots, p^i_{T_k})$ *under* H_i*, any* $t \in \{1, \ldots, T_k\}$*, and any pair* (ρ, π) *inducing the price path* $(\mathfrak{p}_1, \ldots, \mathfrak{p}_t)$ *while satisfying Condition 4, we have:* (i) $\mathfrak{p}_t \geq \max_{i \in \{1, \ldots, N\}} \{p^i_t\}$*, and* (ii) *there exists* $\epsilon_d > 0$ *such that* $|d^i_t(\mathfrak{p}^{(t)}, \rho) - d^{i'}_t(\mathfrak{p}^{(t)}, \rho)| \geq \epsilon_d$ *for all* $i, i' \in \{1, \ldots, N\}$ *satisfying* $i \neq i'$*.*

The preceding condition describes a market-preserving price path that enables the seller to distinguish between different demand hypotheses under endogenous customer predictions. Like its earlier counterpart, this condition is key in achieving asymptotic optimality. To establish its importance, we show that without Condition 6, the profit loss of *any* admissible policy can grow linearly in the problem size k (see Proposition 19(ii) in Appendix 2.9). Thus, in this subsubsection, we focus on settings in which Condition 6 holds.

To construct a version of LATE(n) for the case of endogenous customer predictions, we let the seller announce and commit to the following policy: given $n \in \{1, 2, \ldots, T_k\}$, and subgame perfect equilibrium price path $(p_1^i, \ldots, p_{T_k}^i)$ for each H_i, suppose that in the first n periods, the seller uses the market-preserving price path $(\mathfrak{p}_1, \ldots, \mathfrak{p}_n)$ in Condition 6, whereas in the remaining periods, the seller chooses the subgame perfect equilibrium price path $(p_{n+1}^{\hat{\imath}_n}, \ldots, p_{T_k}^{\hat{\imath}_n})$ assuming that the most likely hypothesis is $H_{\hat{\imath}_n}$ in period $n+1$ is correct. Based on the definition of the profit loss in (2.10)-(2.11), we have the following performance guarantee for this version of LATE(n).

Theorem 9. (profit loss of LATE under endogenous customer predictions) *Let* $\pi = \text{LATE}(n)$ *with* $n = \lceil \frac{1}{\eta} \log k \rceil$ *with some* $\eta > 0$, *and* $\boldsymbol{b} \in \mathcal{B}$ *such that* $b^i < 1$ *for all* $i \in \{1, \ldots, N\}$. *Then, there exists a constant* $c_6 > 0$ *such that, under any endogenous customer prediction behavior* $\rho \in \mathfrak{P}$ *with* (ρ, π) *satisfying Condition 4,*

$$\sup_{\boldsymbol{\theta} \in \Theta} \{\tilde{\Delta}_{\pi\rho}^{\boldsymbol{b}}(k, \boldsymbol{\theta})\} \leq c_6 \log k$$

for $k = 2, 3, \ldots$

Comparing Theorem 9 with Proposition 15, we deduce that the above version of our learn-and-then-earn policy is asymptotically optimal under endogenous forward-looking customer behavior.

We emphasize that our performance guarantee in the case of endogenous customer predictions is stronger than the best achievable performance in the case of exogenous customer predictions, where the profit loss can grow linearly in the problem scale k (Theorem 6(ii)). Given the customers' knowledge of the seller's policy in this case, this result might seem surprising at first glance. But looking from a learning perspective, the situation is different. As alluded to earlier, endogenous customer predictions are easier to control as the intertemporal dependency induced by forward-looking customers is more limited. By announcing and committing to a policy, the seller takes advantage of the fact that the customers can foresee the expected price path induced by the seller's policy. In particular, in the periods following the initial learning-and-market-preservation phase, the seller is able to avoid incurring large losses by using the subgame perfect equilibrium price path. This results in zero profit loss from the customers who arrive after the learning-and-market-preservation phase, helping the seller achieve a tighter performance benchmark.

2.4 Do Forward-looking Customers Always Hurt Learning Performance?

The preceding subsection demonstrates the potential negative impact of forward-looking customers on learning performance as well as how this impact can be mitigated in different settings. Based on these findings, a natural follow-up question is whether forward-looking customers are always detrimental to the seller's learning performance. This subsection investigates this question.

We show below that forward-looking customer behavior does not necessarily hurt the performance of a learning policy. To that end, we establish that the seller's learning efforts may become more "costly" under myopic customer behavior. When customers are myopic, the seller might benefit from their simple purchasing behavior. But, myopic customers are also impatient, meaning that they could abandon the market because of an error in early markdown decisions. Under demand model uncertainty, such markdown errors are inevitable early in the time horizon. This results in irreversible profit losses because the seller cannot offer another price to the myopic customers who left the market. When customers are forward-looking, the seller suffers from the negative impact of the customers' forward-looking behavior on profits. However, forward-looking customers are nevertheless patient. Therefore, the seller has multiple opportunities to earn profits from forward-looking customers and can exploit their patience while learning about the demand model. Consequently, errors in early markdown decisions are less costly when customers are patient and forward-looking.

In what follows, we formalize this intuition in our setting. To provide a problem instance where forward-looking customer behavior is beneficial for a dynamic learning policy, we use the case where the customers' valuations are uniformly distributed. We note that uniform valuation distributions are closely connected to linear demand functions: for instance, in our setting, a uniform valuation distribution induces a piecewise linear demand function in each period, as long as the customers' prediction function $\rho_t^k(\cdot)$ is also linear. For purposes of exposition and analytical tractability, such assumptions on the demand model have been widely used in the antecedent literature; see e.g., (60), (59), and (83). With slight abuse of notation, we let $\lambda^i > 0$ denote the uniform market density under demand hypothesis i. That is, under H_i, the market density of valuation v is λ^i if $v \in [0, \bar{v}^i]$ and zero otherwise, where $\bar{v}^i > 0$. Accordingly, we let $\Theta' \subseteq \Theta$ denote the set of problem parameter vectors $\boldsymbol{\theta}$ for which the customers' valuations are uniformly distributed as indicated above.

Myopic customers. To define the problem instances where all customers are myopic, we let $\Theta_{\text{myopic}} = \{\boldsymbol{\theta} \in \Theta' : \alpha_{\tau w}^i = 0 \text{ for all } i, \tau, w \text{ satisfying } w \neq 0\}$. In this case, we have the following.

Corollary 4. (profit loss with myopic customers) *Let $\boldsymbol{b} \in \mathcal{B}$ such that $b^i < 1$ for all $i \in \{1, \dots, N\}$. Then, there exists a constant $c_7 > 0$ such that, under any customer prediction behavior $\rho \in \mathfrak{P}$ and any policy $\pi \in \Pi$,*

$$\sup_{\boldsymbol{\theta} \in \Theta_{\text{myopic}}} \{\Delta_{\pi\rho}^{\boldsymbol{b}}(k, \boldsymbol{\theta})\} \geq c_5 \log k$$

for $k = 1, 2, \dots$

Corollary 4 follows from the arguments developed in the proof of Theorem 6(i). This result states that if all customers are myopic, the seller's profit loss must grow at least logarithmically in the problem size k.

Extremely forward-looking customers. We now construct another case where customers have the highest patience level ($w = T_k - 1$). To express the problem parameter vectors for this case, we define $\Theta_{\text{forward-looking}} = \{\boldsymbol{\theta} \in \Theta' : \alpha^i_{\tau w} = 0$ for all i, τ, w satisfying $w \neq T_k - 1\}$. For brevity, we suppress in the sequel the dependence of $\alpha^i_{\tau w}$ on w, and write α^i_τ instead of $\alpha^i_{\tau w}$ (noting that $w = T_k - 1$ for all customers). With this simplification, the nonzero entries of the arrival matrix $\boldsymbol{\alpha}^i$ are given by $(\alpha^1_1, \ldots, \alpha^i_{T_k})$. We also define a cumulative arrival matrix $\boldsymbol{A}$ whose $(i,\tau)^{\text{th}}$ entry is $A^i_\tau = \sum_{s=1}^{\tau} \alpha^i_s$ for $i \in \{1, \ldots, N\}$ and $\tau \in \{1, \ldots, T_k\}$.

To represent extremely forward-looking customer behavior, we also consider a particular set of exogenous customer prediction behaviors. We describe this set by constructing an auxiliary scenario that serves as a hypothetical benchmark for the optimal full-information prices. In this auxiliary scenario, we suppose that all customers are myopic in their purchasing decisions but never abandon the market, which corresponds to the case when the discount factor is not too large. Given the problem parameters λ^i, $\bar{v}^i$, $\boldsymbol{\alpha}^i$ under H_i, the optimal price path in the auxiliary scenario is given by the following deterministic Bellman equation:

$$\bar{U}^i_t(p_{t-1}) = \max_{p_t \in [\underline{p}, \bar{p}]} \left\{ R^i_t(p_{t-1}, p_t) + \bar{U}^i_{t+1}(p_t) : p_t \leq p_{t-1} \right\} \tag{2.16}$$

for $t = 1, 2, \ldots, T_k$, where: $p_0 = \bar{p}$, $\bar{U}^i_{T_k+1}(p_{T_k}) = 0$, $R^i_t(p_{t-1}, p_t) = (p_t - c) d^i_t(p_{t-1}, p_t)$ and $d^i_t(p_{t-1}, p_t) = \lambda^i k \alpha^i_t (\bar{v}^i - p_t) + \lambda^i k A^i_{t-1}(p_{t-1} - p_t)$ for all $t = 1, \ldots, T_k$. The auxiliary scenario above corresponds to a hypothetical case of myopic and patient customers; we emphasize that we use this scenario as a benchmark to ultimately study the case of forward-looking and patient customers. Let $(\bar{p}^i_1, \ldots, \bar{p}^i_{T_k})$ be the price path that satisfies the auxiliary Bellman equation (2.16) under H_i. Recalling that the optimal full-information price path under H_i, namely $(p_1^i, \ldots, p_{T_k}^i)$, is given by (2.13), we define a subset of exogenous customer prediction behaviors as follows:

$$\mathfrak{P}_3 = \left\{ \begin{array}{r} \rho \in \mathfrak{P}_2(a,r) : \max_{t=1,\ldots,T_k} \{\rho_t^{T_k - t}(\boldsymbol{p}^{(t)})\} \leq p_{T_k}^i, \text{ and} \\ \max_{kt_1 \leq t \leq kt_2} \{p_t^i - \bar{p}_t^i\} \geq -\sigma_k \text{ for } i \in \{1, \ldots, N\} \end{array} \right\} \subset \mathfrak{P}, \qquad (2.17)$$

where $a > 0$, $r > 0$, t_1 and t_2 satisfy $0 < t_1 \leq t_2 < T$, and $\sigma_k \geq 0$. In (2.17), the first condition corresponds to extremely forward-looking customer behavior in the sense that customers' price predictions do not exceed the optimal full-information price at the end of the time horizon. Under this condition, customers expect sufficiently low prices at the end of the time horizon, and accordingly, they are inclined to stay in the market to take advantage of marked-down prices. As a result, the first condition in (2.17) effectively limits impatient customer abandonments. The second condition in (2.17) captures the impact of the forward-looking customer behavior on the optimal full-information price path. We know that forward-looking customers tend to delay their purchases as they might benefit from waiting for a future purchase. Hence, the seller's optimal strategy is to use a flatter price path to limit the impact of the customers' forward-looking behavior. By contrast, in the auxiliary scenario, customers make purchases myopically, and thus, the seller uses steeper price skimming to collect more profit. Based on this observation, the second condition in (2.17) states that the optimal full-information price path (which uses flatter price reductions) would not be significantly dominated by the auxiliary price path (which uses steeper price reductions). Formally, this condition states that there is at least one period t in the window $[kt_1, kt_2]$ such that $\bar{p}_t^i$ does not exceed p_t^i by more than a tolerance parameter $\sigma_k \geq 0$. This condition is typically satisfied in settings where the forward-looking customer behavior is significant. This is because, in such settings, the optimal full-information price path becomes flatter whereas the auxiliary price path remains steeply decreasing (since it is unaffected by the forward-looking customer behavior). Figure 2.3 illustrates how this condition is satisfied in a numerical example.

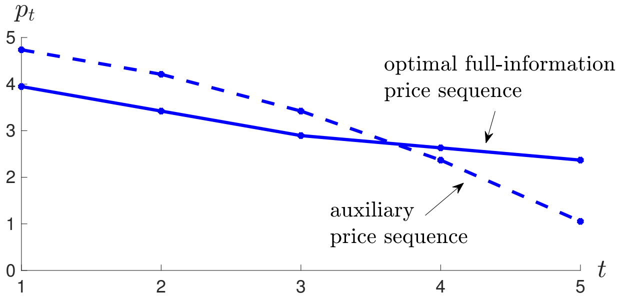

Figure 2.3: The figure displays the optimal full-information price path $(p_1, \ldots, p_5)$ that satisfies (2.13), and the auxiliary price path $(\bar{p}_1, \ldots, \bar{p}_5)$ that satisfies (2.16). The time horizon is 5 periods, and the marginal cost of production is $c = 0$. The market density for customer valuation v equals 1 if $v \in [0, 5]$, and 0 otherwise. The discount factor is $\delta = 0.5$, the fraction of customers arriving in each period is 0.2, and the customer prediction behavior is as in Example 6. The first condition in (2.17) is satisfied because $\rho_t^T(\boldsymbol{p}^{(t)}) = 0$, and the second condition in (2.17) is satisfied because $p_4 \geq \bar{p}_4$.

Our next result shows that forward-looking customer behavior could have a possibly unexpected effect on the seller's profit loss due to demand model uncertainty.

Theorem 10. (profit loss with extremely forward-looking customers) *Let* π *be* LATE(n) *with* $n = \lceil \frac{2}{\eta} \log k \rceil$ *and* η *as given in Lemma 11, and* $\boldsymbol{b} \in \mathcal{B}$ *such that* $b^i < 1$ *for all* $i \in \{1, \ldots, N\}$. *Suppose that the customer prediction behavior* $\rho \in \mathfrak{P}_3$ *with* $\sigma_k \leq \vartheta k e^{-\sqrt{k}}$, $\vartheta > 0$, $0 < t_1 < t_2 < T$, $a > 0$, *and* $r > 1$. *Then,*

$$\sup_{\boldsymbol{\theta} \in \Theta_{\text{forward-looking}}} \{\Delta_{\pi\rho}^{\boldsymbol{b}}(k, \boldsymbol{\theta})\} \to 0$$

as $k \to \infty$.

Theorem 10 stands in stark contrast to Corollary 4. When all customers are myopic and impatient, the seller incurs a profit loss of order $\log k$ because it does not have sufficient opportunities to learn about the underlying demand model before too many customers abandon the market. However, when customers are extremely forward-looking and patient, the seller might benefit significantly from the opportunities to learn about the demand model in the early sales periods while preserving the market size. This result suggests that, perhaps surprisingly, it is possible for the LATE(n) policy to achieve an asymptotically vanishing profit loss when customers are forward-looking.

As explained earlier, the conditions in $\mathfrak{P}_3$ correspond to the settings where the forward-looking customer behavior is significant. To illustrate such settings, we provide below a simple example.

Example 6. ***(a Markovian price prediction)*** *Let* $\underline{p} = c = 0$. *In period* $t = 1, 2, \ldots, T_k - 1$, *the customer prediction behavior* $\rho = \{\rho_t^\kappa : t, \kappa = 1, 2, \ldots\}$ *satisfies* $\rho_t^\kappa(\boldsymbol{p}^{(t)}) = \left(\frac{1-\delta^{T_k-t-\kappa}}{1-\delta^{T_k-t}}\right)p_t$ *for* $\boldsymbol{p}^{(t)} \in [\underline{p}, \bar{p}]^t$ *and* $\kappa \in \{1, 2, \ldots, T_k - t\}$, *where* $\delta = \min_{i=1,\ldots,N}\{\delta^i\}$.

In Example 6, the price predictions depend only on the current price p_t, and customers use a markdown rate based on the possible discount rates. Figure 2.4 shows an instance of this behavior.

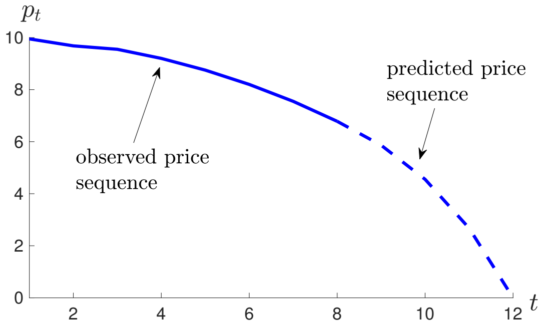

Figure 2.4: The figure displays an instance of the customer prediction behavior ρ in Example 6. The time horizon is 12 periods, and $\delta = 0.75$. In period $t = 8$, the price predictions $\{\rho_t^\kappa(\boldsymbol{p}^{(t)}),\ \kappa = 0, 1, 2, 3, 4\}$ are shown by the dashed curve, where $\rho_t^0(\boldsymbol{p}^{(t)}) = p_t$ and $\rho_t^{T_k-t}(\boldsymbol{p}^{(t)}) = 0$.

We conclude the subsection by formally confirming that the setting in Example 6 is contained in $\mathfrak{P}_3$.

Lemma 12. (Markovian price predictions in $\mathfrak{P}_3$) *There exists* $\boldsymbol{\theta} \in \Theta_{\text{forward-looking}}$ *such that the customer prediction behavior* ρ *in Example 6 is in* $\mathfrak{P}_3$ *with* $\sigma_k = \vartheta k e^{-\sqrt{k}}$ *and* $\vartheta > 0$.

2.4.1 Concluding Remarks

Summary. In this paper, we study a markdown management problem featuring demand model uncertainty and a heterogeneous mixture of forward-looking and myopic customers. Our analysis provides key insights into the impact of forward-looking customers on markdown pricing with demand learning. In the worst case, the seller may incur an extremely large profit loss due to forward-looking customers. However, if the customers' memory is bounded or their price predictions endogenously depend on the seller's pricing strategy, then it is possible to substantially improve the revenue performance and achieve asymptotic optimality. Accordingly, we design a forced exploration policy and prove that it is asymptotically optimal in these cases. Furthermore, even when customers have unbounded memory, we show that our forced exploration policy performs well under mild regularity conditions. We also demonstrate that forward-looking customer behavior can sometimes improve the performance of learning policy by delaying the demand of forward-looking and patient customers. We hope that the insights gleaned from our theoretical analysis can help guide the design of markdown policies for demand learning in practice.

More on endogenous customer predictions. One of our findings is that endogenous forward-looking customer behavior provides an advantage to learning policies: as the customers endogenously form a sequence of price predictions based on the seller's pricing policy, this actually adds a restriction to the customers' forward-looking behavior, and thus the seller has more control on the impact of forward-looking customers on later sales opportunities. It is possible to consider variants of this setting such that customers form sequences of prediction distributions that coincide with the distributions of future prices induced by the seller's pricing policy. This variant of endogenous customer predictions adds even further restriction on the customers' forward-looking behavior, and our approach is applicable this setting. In this case, the seller could still employ a forced exploration policy by first accumulating information on the demand model and then implementing the subgame perfect equilibrium price path. By the more restrictive prediction condition, the purchasing decisions of the customers who arrive in later periods will be the same as those in our current analysis, resulting in similar theoretical performance guarantees. For purposes of exposition and comparability with the extant literature, we use an endogenous prediction condition that follows the commonly used modeling assumptions in the literature on strategic customers, where customers form a sequence of predicted prices based on the knowledge of the seller's pricing policy (60, 67, 68, 75)see, for example,.

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2.5 Proofs of the Results in Section 2.3.1

For brevity in the following proofs, we suppress the dependence of the mathematical expressions on ρ, and emphasize this dependence on ρ only when necessary.

Proof of Lemma 10. We first recall that, in period t, the set of remaining future periods for type-(τ, w) customers is $\mathcal{T}_{\tau wt} = \{t+1, \dots \min\{\tau + w, T_k\}\}$. In their purchasing decisions, customers compare the discounted utilities $u^i(v, \hat{p}_s, s-t)$ for all $s \in \mathcal{T}_{\tau wt}$, where $u^i(v, \hat{p}_{t+\kappa}, \kappa) = (\delta^i)^\kappa (v - \hat{p}_{t+\kappa})$ is the utility of purchasing in period $t+\kappa$.

Note that a type-(τ, w) customer with valuation v chooses to wait in period t if and only if $\max_{s\in\mathcal{T}_{\tau wt}} \left\{(\delta^i)^{s-t}(v - \hat{p}_s)\right\} \geq v - p_t$ and $\min_{s\in\mathcal{T}_{\tau wt}} \left\{(\delta^i)^{s-t}(v - \hat{p}_s)\right\} \geq 0$. Because prices are marked down, we deduce that the condition $\max_{s\in\mathcal{T}_{\tau wt}} \left\{(\delta^i)^{s-t}(v - \hat{p}_s)\right\} \geq v - p_t$ is equivalent to $v \leq \max_{s\in\mathcal{T}_{\tau wt}} \left\{p_t + \frac{(\delta^i)^{s-t}}{1-(\delta^i)^{s-t}}(p_t - \hat{p}_s)\right\}$, and furthermore, the condition $\min_{s\in\mathcal{T}_{\tau wt}} \left\{(\delta^i)^{s-t}(v - \hat{p}_s)\right\} \geq 0$ is equivalent to $v \geq \hat{p}_{\min\{\tau+w,T_k\}}$. Because $\max_{s\in\mathcal{T}_{\tau wt}} \left\{p_t + \frac{(\delta^i)^{s-t}}{1-(\delta^i)^{s-t}}(p_t - \hat{p}_s)\right\} \geq p_t \geq \hat{p}_{\min\{\tau+w,T_k\}}$, we conclude that a type-(τ, w) customer with valuation v waits in period t if and only if $v \in \mathcal{V}^i_{\tau wt}\Big(\boldsymbol{p}^{(t)}\Big) = [\underline{v}^i_{\tau wt}(\boldsymbol{p}^{(t)}), \overline{v}^i_{\tau wt}(\boldsymbol{p}^{(t)})]$, where
$\underline{v}^i_{\tau wt}(\boldsymbol{p}^{(t)}) = \max\left\{\underline{v}^i_{\tau w(t-1)}(\boldsymbol{p}^{(t-1)}), \hat{p}_{\min\{\tau+w,T_k\}}\right\}$ and
$\overline{v}^i_{\tau wt}(\boldsymbol{p}^{(t)}) = \min\left\{\overline{v}^i_{\tau w(t-1)}(\boldsymbol{p}^{(t)}), \max_{s\in\mathcal{T}_{\tau wt}} \left\{p_t + \frac{(\delta^i)^{s-t}}{1-(\delta^i)^{s-t}}(p_t - \hat{p}_s)\right\}\right\}$.

A type-(τ, w) customer with valuation v purchases in period t if and only if $v - p_t \geq 0$ and $v - p_t \geq \max_{s\in\mathcal{T}_{\tau wt}} \left\{(\delta^i)^{s-t}(v - \hat{p}_s)\right\}$, which is equivalent to $v \geq p_t + \max_{s\in\mathcal{T}_{\tau wt}} \left\{\frac{(\delta^i)^{s-t}}{1-(\delta^i)^{s-t}}(p_t - \hat{p}_s)\right\}$. Thus, a type-$(\tau, w)$ customer with valuation v purchases in period t if and only if $v \in \mathcal{U}^i_{\tau wt}(\boldsymbol{p}^{(t)}) = [\underline{u}^i_{\tau wt}(\boldsymbol{p}^{(t)}), \overline{u}^i_{\tau wt}(\boldsymbol{p}^{(t)})]$, where
$\underline{u}^i_{\tau wt}(\boldsymbol{p}^{(t)}) = \min\left\{\overline{v}^i_{\tau w(t-1)}(\boldsymbol{p}^{(t-1)}), \max_{s\in\mathcal{T}_{\tau wt}} \left\{p_t + \frac{(\delta^i)^{s-t}}{1-(\delta^i)^{s-t}}(p_t - \hat{p}_s)\right\}\right\}$ and $\overline{u}^i_{\tau wt}(\boldsymbol{p}^{(t)}) = \overline{v}^i_{\tau w(t-1)}(\boldsymbol{p}^{(t)})$. □

Proof of Theorem 6. Let $\mathbb{P}^{\boldsymbol{b},i}_\pi\{\cdot\} = \mathbb{P}^{\boldsymbol{b}}_\pi\{\cdot \mid H_i\}$ and $\mathbb{E}^{\boldsymbol{b},i}_\pi\{\cdot\} = \mathbb{E}^{\boldsymbol{b}}_\pi\{\cdot \mid H_i\}$ for all $\boldsymbol{b} \in \mathcal{B}$, $i \in \{1, \dots, N\}$, and $\pi \in \Pi$. We prove the two claims stated in Theorem 6.

Proof of Theorem 6(i).

Step 1: Construct a problem instance $\boldsymbol{\theta} \in \Theta$. Suppose that there are two hypotheses, namely H_1 and H_2. Thus, $N = 2$. Let $\underline{p} = c = 0$ and $\bar{p} = 2$. Moreover, let $T = 1$; hence, if the problem scale is $k \in \{1, 2, \dots\}$ then the time horizon is $T_k = k$. Under the problem scale $k = 1, 2, \dots$, let the arrival matrix $\boldsymbol{\alpha}^i$ satisfy $\alpha^i_{\tau 0} = \frac{1}{k}$ for all τ, and $\alpha^i_{\tau w} = 0$ for all τ and w such that $w \geq 1$. Under H_1, let the market density function be $\lambda^1(v) = 2$ for all $v \in [0, 4]$. Under H_2, let the market density function be $\lambda^2(v) = 1$ for all $v \in [0, 3]$. By construction, for any $\boldsymbol{p}^{(t)} \in [0, 2]^t$, the expected demand in period t under H_1 and H_2 are

$$\begin{aligned} d^1_t\Big(\boldsymbol{p}^{(t)}\Big) &= k\alpha^1_{t0} \int_{v \geq p_t} \lambda^1(v)dv = 4 - 2p_t, \\ d^2_t\Big(\boldsymbol{p}^{(t)}\Big) &= k\alpha^2_{t0} \int_{v \geq p_t} \lambda^2(v)dv = 3 - p_t. \end{aligned}$$

respectively. Based on this, the optimal full-information price path under the two hypotheses are given by $p^1_t = \arg\max_{p_t \in [\underline{p}, \bar{p}]}\{p_t(4 - 2p_t)\} = 1$ and $p^2_t = \arg\max_{p_t \in [\underline{p}, \bar{p}]}\{p_t(3 - p_t)\} = \frac{3}{2}$ for all $t = 1, \dots T_k$. For simplicity of notation, we denote the price optimizers by $p^1 = 1$ and $p^2 = \frac{3}{2}$. Let $\underline{\mathfrak{d}} = \min\{4 - 2\bar{p}, 3 - \bar{p}\} = 0$ and $\bar{\mathfrak{d}} = \max\{4 - 2\underline{p}, 3 - \underline{p}\} = 4$. Suppose that the demand shocks $\{\varepsilon_t\}$ follow a sequence of i.i.d. random variables with truncated standard normal distribution. Denote by $\phi(\cdot)$ and $\Phi(\cdot)$ the standard normal density function and the corresponding cumulative distribution function, respectively. If the underlying demand hypothesis is H_i, the demand realization satisfies $D_t = d^i_t + \varepsilon_t \in [\underline{d}, \bar{d}]$. Choose $\underline{d} = -2^{k+1} + \underline{\mathfrak{d}}$ and $\bar{d} = 2^{k+1} + \bar{\mathfrak{d}}$; thus, $\Phi(\underline{d} - \underline{\mathfrak{d}}) = \Phi(-2^{k+1}) \leq 2^{-(k+1)}$, and $\Phi(\bar{d} - \bar{\mathfrak{d}}) = \Phi(2^{k+1}) = 1 - \Phi(-2^{k+1}) \geq 1 - 2^{-(k+1)}$. Consequently,

$$\Phi(\bar{d} - \bar{\mathfrak{d}}) - \Phi(\underline{d} - \underline{\mathfrak{d}}) \geq 1 - 2^{-k}. \tag{2.18}$$

Given $t = 1, \dots, T_k$, and $\xi_1, \dots, \xi_t \in [\underline{d}, \bar{d}]$, the demand density function under H_i satisfies

$$\mathbb{P}^{\boldsymbol{b}, i}_{\pi}\Big\{D_1 \in d\xi_1, \dots, D_t \in \xi_t\Big\} = \prod_{s=1}^{t} \frac{\phi\Big(\xi_s - d^i_s(\boldsymbol{p}^{(s)})\Big)}{\Phi\Big(\bar{d} - d^i_s\Big(\boldsymbol{p}^{(s)}\Big)\Big) - \Phi\Big(\underline{d} - d^i_s\Big(\boldsymbol{p}^{(s)}\Big)\Big)}$$

Step 2: Derive an expression for Kullback-Leibler divergence. Given $\pi \in \Pi$ and $\rho \in \mathfrak{P}$, let $(p_1, \ldots, p_{T_k})$ be the price path generated under π and ρ. The profit loss of π under H_1 and H_2 are

$$\begin{aligned} \Delta_\pi^1(k) &= \mathbb{E}_\pi^{\boldsymbol{b},1}\left\{ \sum_{s=1}^{T_k} p_s^1 d_s^1\Big(\boldsymbol{p}^{1(s)}\Big) - p_s d_s^1\Big(\boldsymbol{p}^{(s)}\Big) \right\} \\ &= \mathbb{E}_\pi^{\boldsymbol{b},1}\left\{ \sum_{s=1}^{T_k} p_s^1(4-2p_s^1) - p_s(4-2p_s) \right\} = \mathbb{E}_\pi^{\boldsymbol{b},1}\left\{ \sum_{s=1}^{T_k} 2(1-p_s)^2 \right\}, \qquad (2.19) \\ \Delta_\pi^2(k) &= \mathbb{E}_\pi^{\boldsymbol{b},2}\left\{ \sum_{s=1}^{T_k} p_s^2 d_s^2(\boldsymbol{p}^{2(s)}) - p_s d_s^2(\boldsymbol{p}^{(s)}) \right\} \\ &= \mathbb{E}_\pi^{\boldsymbol{b},2}\left\{ \sum_{s=1}^{T_k} p_s^2(3-p_s^2) - p_s(3-p_s) \right\} = \mathbb{E}_\pi^{\boldsymbol{b},2}\left\{ \sum_{s=1}^{T_k} (\tfrac{3}{2}-p_s)^2 \right\}, \qquad (2.20) \end{aligned}$$

respectively. Furthermore, the Kullback-Leibler divergence from $\mathbb{P}_\pi^1$ to $\mathbb{P}_\pi^2$ is given by

$$\begin{aligned} &\mathrm{KL}(\mathbb{P}_\pi^1, \mathbb{P}_\pi^2) \\ &= \mathbb{E}_\pi^{\boldsymbol{b},1}\left\{ \log\left(\prod_{s=1}^{T_k} \frac{\phi\Big(D_s - d_s^1\Big(\boldsymbol{p}^{(s)}\Big)\Big)}{\Phi(\bar{d} - d_s^1\Big(\boldsymbol{p}^{(s)}\Big)\Big) - \Phi\Big(\underline{d} - d_s^1\Big(\boldsymbol{p}^{(s)}\Big)\Big)} \Bigg/ \frac{\phi\Big(D_s - d_s^2(p_s)\Big)}{\Phi\Big(\bar{d} - d_s^2\Big(\boldsymbol{p}^{(s)}\Big)\Big) - \Phi\Big(\underline{d} - d_s^2\Big(\boldsymbol{p}^{(s)}\Big)\Big)} \right) \right\} \\ &\overset{(a)}{=} -\tfrac{1}{2}\,\mathbb{E}_\pi^{\boldsymbol{b},1}\left\{ \sum_{s=1}^{T_k} \Big[D_s - (4-2p_s)\Big]^2 - \Big[D_s - (3-p_s)\Big]^2 \right\} + \mathbb{E}_\pi^{\boldsymbol{b},1}\Big\{G_\pi(T_k)\Big\} \\ &\overset{(b)}{=} -\tfrac{1}{2}\,\mathbb{E}_\pi^{\boldsymbol{b},1}\left\{ \sum_{s=1}^{T_k} \varepsilon_s^2 - \Big[\varepsilon_s - (1-p_s)\Big]^2 \right\} + \mathbb{E}_\pi^{\boldsymbol{b},1}\Big\{G_\pi(T_k)\Big\} \\ &= \tfrac{1}{2}\,\mathbb{E}_\pi^{\boldsymbol{b},1}\left\{ \sum_{s=1}^{T_k} (1-p_s)^2 \right\} + \mathbb{E}_\pi^{\boldsymbol{b},1}\Big\{G_\pi(T_k)\Big\} \\ &\overset{(c)}{=} \tfrac{1}{4}\Delta_\pi^1(k) + \mathbb{E}_\pi^{\boldsymbol{b},1}\Big\{G_\pi(k)\Big\}, \end{aligned}$$

where: $G_\pi(T_k) = \sum_{s=1}^{T_k} -\log[\Phi(\bar{d} - d_s^1(\boldsymbol{p}^{(s)})) - \Phi(\underline{d} - d_s^1(\boldsymbol{p}^{(s)}))] + \log[\Phi(\bar{d} - d_s^2(\boldsymbol{p}^{(s)})) - \Phi(\underline{d} - d_s^2(\boldsymbol{p}^{(s)}))]$, (a) follows by elementary algebra, (b) follows because $D_s - (4-2p_s) = \varepsilon_s$ under H_1, and (c) follows from (2.19).

Step 3: Derive a lower bound on the profit loss. Letting $\eta_0 = \frac{1}{64} \in (0,1)$, we consider two cases:

Case 1: $\mathrm{KL}(\mathbb{P}_\pi^1, \mathbb{P}_\pi^2) \leq \eta_0 \log k + 1$. Let $I^i = [p^i - \frac{1}{8}, p^i + \frac{1}{8}]$, where $p^1 = 1$ and $p^2 = \frac{3}{2}$. By construction, we have $I^1 \cap I^2 = \emptyset$. Let the event $\mathcal{X}_t$ be such that

$$\mathcal{X}_t \; = \; \begin{cases} 1 & \text{if } p_t \in I^1, \\ 0 & \text{if } p_t \notin I^1. \end{cases}$$

Then,

$$\begin{aligned} \Delta_\pi^1(k) + \Delta_\pi^2(k) &= \sum_{s=1}^{T_k} \mathbb{E}_\pi^{\boldsymbol{b},1}\Big\{2(1-p_s)^2\Big\} + \mathbb{E}_\pi^{\boldsymbol{b},2}\Big\{(\tfrac{3}{2} - p_s)^2\Big\} \\ &\overset{\text{(d)}}{\geq} \eta_0 \sum_{s=1}^{T_k} \Big[\mathbb{P}_\pi^{\boldsymbol{b},1}\{p_s \notin I^1\} + \mathbb{P}_\pi^{\boldsymbol{b},2}\{p_s \notin I^2\}\Big] \\ &\overset{\text{(e)}}{\geq} \eta_0 \sum_{s=1}^{T_k} \Big[\mathbb{P}_\pi^{\boldsymbol{b},1}\{\mathcal{X}_s = 0\} + \mathbb{P}_\pi^{\boldsymbol{b},2}\{\mathcal{X}_s = 1\}\Big] \\ &\overset{\text{(f)}}{\geq} \tfrac{\eta_0}{4} \exp(-\eta_0 \log k - 1) kT = \tfrac{\eta_0 T}{4e} k^{1-\eta_0}, \end{aligned}$$

where: (d) follows because $\eta_0 = \frac{1}{64}$, which implies that $\mathbb{E}_\pi^{\boldsymbol{b},1}\Big\{2(1-p_s)^2\Big\} \geq \frac{2}{64}\mathbb{P}_\pi^{\boldsymbol{b},1}\{p_s \notin I^1\}$ and $\mathbb{E}_\pi^{\boldsymbol{b},2}\Big\{(\frac{3}{2} - p_s)^2\Big\} \geq \frac{1}{64}\mathbb{P}_\pi^{\boldsymbol{b},2}\{p_s \notin I^2\}$; (e) follows because $\{\mathcal{X}_s = 1\} = \{p_s \in I^1\} \subset \{p_s \notin I^2\}$; and (f) follows by Theorem 2.2 of (95, p. 90). Thus,

$$\Delta_\pi^{\boldsymbol{b}}(k) = b^1 \Delta_\pi^1(k) + b^2 \Delta_\pi^2(k) \geq \tfrac{\eta_0 T}{4e} \min\{b^1, b^2\} k^{1-\eta_0}.$$

Case 2: $\mathrm{KL}(\mathbb{P}^1_\pi, \mathbb{P}^2_\pi) > \eta_0 \log k + 1$. Then,

$$\begin{aligned}
\mathbb{E}^{\boldsymbol{b},1}_\pi\Big\{G^{\boldsymbol{b},1}_\pi(k)\Big\} &= \mathbb{E}^{\boldsymbol{b},1}_\pi\Bigg\{\sum_{s=1}^{T_k} -\log\Big[\Phi\Big(\bar{d} - d^1_s(\boldsymbol{p}^{(s)})\Big) - \Phi\Big(\underline{d} - d^1_s(\boldsymbol{p}^{(s)})\Big)\Big] \\
&\qquad\qquad + \log\Big[\Phi\Big(\bar{d} - d^2_s(\boldsymbol{p}^{(s)})\Big) - \Phi\Big(\underline{d} - d^2_s(\boldsymbol{p}^{(s)})\Big)\Big]\Bigg\} \\
&\overset{\text{(g)}}{\leq} \mathbb{E}^{\boldsymbol{b},1}_\pi\Bigg\{\sum_{s=1}^{T_k} -\log\Big[\Phi\Big(\bar{d} - d^1_s(\boldsymbol{p}^{(s)})\Big) - \Phi\Big(\underline{d} - d^1_s(\boldsymbol{p}^{(s)})\Big)\Big]\Bigg\} \\
&\overset{\text{(h)}}{\leq} \sum_{s=1}^{T_k} -\log\Big[\Phi\Big(\bar{d} - \bar{\mathfrak{d}}\Big) - \Phi\Big(\underline{d} - \underline{\mathfrak{d}}\Big)\Big] \\
&\overset{\text{(i)}}{\leq} \sum_{s=1}^{T_k} \frac{1}{\Phi(\bar{d} - \bar{\mathfrak{d}}) - \Phi(\underline{d} - \underline{\mathfrak{d}})} - 1 \\
&\overset{\text{(j)}}{\leq} k\Big(\tfrac{1}{1-2^{-k}} - 1\Big) \leq \tfrac{k}{2^k - 1} \leq 1,
\end{aligned}$$

where: (g) follows because $\log[\Phi(\bar{d} - d^2_s(\boldsymbol{p}^{(s)})) - \Phi(\underline{d} - d^2_s(\boldsymbol{p}^{(s)}))] < 0$; (h) follows because logarithm is an increasing function, which implies that $\Phi(\bar{d} - d^1_s(\boldsymbol{p}^{(s)})) \geq \Phi\Big(\bar{d} - \bar{\mathfrak{d}}\Big)$ and $\Phi(\underline{d} - d^1_s(\boldsymbol{p}^{(s)})) \leq \Phi(\underline{d} - \underline{\mathfrak{d}})$; (i) follows because $-\log(x) \geq \frac{1}{x} - 1$; and (j) follows from (2.18). Consequently,

$$\begin{aligned}
\Delta^{\boldsymbol{b}}_\pi(k) &= b^1\Delta^1_\pi(k) + b^2\Delta^2_\pi(k) \\
&\geq b^1\Delta^1_\pi(k) = 4b^1\Big[\mathrm{KL}(\mathbb{P}^1_\pi, \mathbb{P}^2_\pi) - \mathbb{E}^{\boldsymbol{b},1}_\pi\Big\{G^1_\pi(T_k)\Big\}\Big] \;\geq\; 4b^1\eta_0 \log k.
\end{aligned}$$

Choose $c_1 > 0$ such that $c_1 < \min\{(1-\eta_0)\frac{\eta_0}{4e}\min\{b^1, b^2\}, 4b^1\eta_0\}$. Thus, for all $k \geq 1$, we have $\frac{\eta_0 T}{4e}\min\{b^1, b^2\}k^{1-\eta_0} \geq c_1 \log k$ and $4b^1\eta_0 \log k > c_1 \log k$. We conclude the proof by noting that $\Delta^{\boldsymbol{b}}_\pi(k) = b^1\Delta^1_\pi(k) + b^2\Delta^2_\pi(k) \geq c_1 \log k$ for all $k = 1, 2, \ldots$.

Proof of Theorem 6(ii). To prove this result, we construct a problem instance. First, suppose that there are two hypotheses, H_1 and H_2. Let $T = 1$; thus, if the problem scale is $k \in \{1, 2, \ldots\}$ then the time horizon is $T_k = k$. In addition, we let $\underline{p} = c = 0$ and $\bar{p} = 5$. For each problem scale $k \in \{1, 2, \ldots\}$, let the arrival matrix $\boldsymbol{\alpha}^i$ be such that $\alpha^i_{\tau(k-1)} = \frac{1}{k}$ for all τ, and $\alpha^i_{\tau w} = 0$ for all τ and w such that $w \neq k-1$. Under H_i, the customers' discount factor is $\delta^i = \frac{1}{100k}$, and the market density is a function $\lambda^i(\cdot)$ such that $\lambda^i(v)$ equals 1 if $v \in [0, 3-i]$ and zero otherwise.

We first consider a customer prediction behavior ρ such that $\rho^\kappa_t(\boldsymbol{p}^{(t)}) = \underline{p}$ for all t and κ. Given this customer prediction behavior ρ, denote by $(p^i_1, \dots, p^i_{T_k})$ the optimal full-information price path. In this construction, since $\alpha_{\tau w} = 0$ for all $w \neq k-1$ and $\rho^\kappa_t(\boldsymbol{p}^{(t)}) = \underline{p}$, every customer waits until the end of the sales horizon if no purchase is made, and a customer with valuation v purchases in period t if and only if $v - p_t \geq (\delta^i)^{k-t} v$. Considering the dynamic program (2.13), we deduce from Lemma 10 that

$$\overline{u}^i_{\tau w t}(\boldsymbol{p}^{(t)}) = \min_{s \in \{\tau, \dots, t-1\}} \left\{ \frac{p_s}{1 - (\delta^i)^{k-s}} \right\}, \tag{2.21a}$$

$$\underline{u}^i_{\tau w t}(\boldsymbol{p}^{(t)}) = \min_{s \in \{\tau, \dots, t\}} \left\{ \frac{p_s}{1 - (\delta^i)^{k-s}} \right\}, \tag{2.21b}$$

for $\tau \in \{1, \dots, k-1\}$, $w = k - \tau$, and $t \in \{\tau, \dots, k-1\}$. Letting $(\tilde{p}^i_1, \dots, \tilde{p}^i_{T_k})$ be a feasible price path that satisfies $\tilde{p}^i_s = (3-i)\frac{3k-s}{3k}$ for all $s \in \{1, \dots, k\}$, we obtain the following for all $s \in \{1, \dots, k-2\}$:

$$\begin{aligned}\frac{\tilde{p}^i_s}{1 - (\delta^i)^{k-s}} &= \frac{(3-i)(3k-s)}{3k\Big(1 - (\delta^i)^{k-s}\Big)} \\ &\overset{\text{(a)}}{\geq} \frac{(3-i)(3k-s-1)}{3k\Big(1 - (\delta^i)^{k-(s+1)}\Big)} = \frac{\tilde{p}^i_{s+1}}{1 - (\delta^i)^{k-s-1}}.\end{aligned} \tag{2.22}$$

To prove (a), it is sufficient to show that $1 - (1-\delta^i)(3k-s)(\delta^i)^{k-s-1} - (\delta^i)^{k-s} \geq 0$. By construction, $\delta^i = \frac{1}{100k}$. Because $(1-\delta^i)(3k-s)(\delta^i)^{k-s-1} \leq 3k\delta^i \leq \frac{3}{100}$ and $(\delta^i)^{k-s} \leq \frac{1}{100k}$, we deduce that $1 - (1-\delta^i)(3k-s)(\delta^i)^{k-s-1} - (\delta^i)^{k-s} \geq 1 - \frac{3}{100} - \frac{1}{100k} > 0$, which implies that (a) holds. By (2.21a)-(2.21b) and (2.22), we have the following:

$$\overline{u}^{i}_{\tau w t}(\tilde{\boldsymbol{p}}^{i(t)}) = \frac{\tilde{p}^{i}_{t-1}}{1-(\delta^{i})^{k-t+1}}, \tag{2.23a}$$

$$\underline{u}^{i}_{\tau w t}(\tilde{\boldsymbol{p}}^{i(t)}) = \frac{\tilde{p}^{i}_{t}}{1-(\delta^{i})^{k-t}} \tag{2.23b}$$

for $\tau \in \{1,\dots,k-1\}$, $t \in \{\tau,\dots,k-1\}$ and $w = k-\tau$. By (2.3), Lemma 10, and (2.23a)-(2.23b), we deduce that there exists k_0 such that, for all $k \geq k_0$,

$$\begin{aligned}
U^{i}_{1}(k) &\overset{(b)}{\geq} \sum_{t=1}^{k-1} \tilde{p}^{i}_{t}\left[(3-i) + \frac{(t-1)\tilde{p}^{i}_{t-1}}{1-(\delta^{i})^{k-t+1}} - \frac{t\tilde{p}^{i}_{t}}{1-(\delta^{i})^{k-t}}\right] \\
&\qquad + \tilde{p}^{i}_{k}\left[(3-i) + \frac{(k-1)\tilde{p}^{i}_{k-1}}{1-\delta^{i}} - k\tilde{p}^{i}_{k}\right] \\
&\overset{(c)}{\geq} \sum_{t=1}^{k} \tilde{p}^{i}_{t}\left[(3-i) + (t-1)\tilde{p}^{i}_{t-1} - \frac{t\tilde{p}^{i}_{t}}{1-\delta^{i}}\right] \\
&\overset{(d)}{=} (3-i)^2 \sum_{t=1}^{k} \left[1 + \frac{(t-1)[3k-(t-1)]}{3k} - \frac{100kt(3k-t)}{(100k-1)3k}\right]\frac{3k-t}{3k} \\
&\overset{(e)}{\geq} \frac{(3-i)^2}{(3k)^2} \sum_{t=1}^{k} \left[(2t-1)(3k-t) - \frac{3t(3k-t)}{100}\right] \\
&= \frac{(3-i)^2}{(3k)^2} \sum_{t=1}^{k} \left[6tk - 2t^2 - (3k-t) - \frac{9tk}{100} + \frac{3t^2}{100}\right] \\
&\overset{(f)}{=} \frac{(3-i)^2}{(3k)^2}\left[\left(3-\frac{9}{200}\right)k^2(k+1) - \left(\frac{2}{3}-\frac{1}{100}\right)k(k+1)\left(k+\frac{1}{2}\right) - 3k^2 + \frac{1}{2}k(k+1)\right] \\
&= \frac{(3-i)^2}{(3k)^2}\left(\frac{1379}{600}k^3 - \frac{318}{600}k^2 + \frac{103}{600}k\right) \\
&\overset{(g)}{\geq} \frac{137(3-i)^2k^3}{60(3k)^2} = \frac{137}{540}(3-i)^2k.
\end{aligned} \tag{2.24}$$

To prove (b), we note that the constructed price path $(\tilde{p}^{i}_{1},\dots,\tilde{p}^{i}_{T_k})$ is a feasible solution but not necessarily optimal, and that the expression on the right hand side of (b) is the profit induced by the constructed feasible price path. To obtain (c), we first eliminate the coefficient $\frac{1}{1-(\delta^{i})^{k-t+1}}$ in the term $\frac{(t-1)\tilde{p}^{i}_{t-1}}{1-(\delta^{i})^{k-t+1}}$ for $t = 1,\dots,k$. We next increase the coefficient from $\frac{1}{1-(\delta^{i})^{k-t}}$ to $\frac{1}{1-\delta^{i}}$ in the term $\frac{t\tilde{p}^{i}_{t}}{1-(\delta^{i})^{k-t}}$ for all $t = 1,\dots,k-1$. Lastly, we increase the coefficient from 1 to $\frac{1}{1-\delta^i}$ in the term $k\tilde{p}_k^i$. Aggregating the coefficient changes, we end up with the expression on the right hand side of (c). Identity (d) follows directly from plugging in $\tilde{p}_t^i = (3-i)\frac{3k-t}{3k}$ and $\delta^i = \frac{1}{100k}$, and for inequality (e), we first note that $\frac{100k}{100k-1} = 1 + \frac{1}{100k-1}$ and then use the fact that $\frac{3k-t}{100k-1} \leq \frac{3}{100}$. Identity (f) follows directly from the calculation of the summation terms. For inequality (g), we first pick the smallest integer k_0 such that $(\frac{1379}{600} - \frac{137}{60})k_0 \geq \frac{318}{600}$. Then, (g) follows for all $k \geq k_0$.

By construction, the optimal full-information price paths under H_1 and H_2 are distinct; i.e., letting $\boldsymbol{p}^{i(t)} = (p_1^i, \ldots, p_t^i)$ for all i and t, we have $\boldsymbol{p}^{1(t)} \neq \boldsymbol{p}^{2(t)}$ for all $t = 1, \ldots, T_k$. To prove a lower bound on $\Delta_\pi^{\boldsymbol{b}}(k)$, we construct a prediction behavior in the following two cases:

Case 1: $\boldsymbol{p}^{(t)} = \boldsymbol{p}^{1(t)}$ or $\boldsymbol{p}^{(t)} = \boldsymbol{p}^{2(t)}$. Let $\rho_t^\kappa(\boldsymbol{p}^{(t)}) = \underline{p}$. This construction is consistent with the $\boldsymbol{p}^{i(t)}$ being the optimal full-information price path. From (2.24), we obtain the following:

$$U_1^i(k) \geq \frac{137}{540}(3-i)^2 k \quad \text{for all } k \geq k_0. \tag{2.25}$$

Case 2: $p_t \neq p_t^i$ for some $i \in \{1,2\}$ and $t \in \{1, \ldots, T_k\}$. Consider a prediction behavior ρ be such that $\rho_t^\kappa(\boldsymbol{p}^{(t)}) = p_t$. Since the predicted price satisfies $\hat{p}_s = p_t$ for all $s \in \mathcal{T}_{\tau wt}$, customers with valuation $v \geq p_t$ always make a purchase because $v - p_t \geq (\delta^i)^\kappa (v - p_t) = (\delta^i)^\kappa (v - \hat{p}_s)$ for any $s \in \mathcal{T}_{\tau wt}$. In comparison, customers with valuation $v < p_t$ leave the system immediately. Thus, we effectively end up with a system where the full-information dynamic program in (2.13) reduces to a repeated single-period problem, and the optimal price path $(\bar{p}_1^i, \ldots, \bar{p}_{T_k}^i)$ is given by $\bar{p}_t^i = \arg\max_{p \in [\underline{p}, \bar{p}]}(3-i-p)p = \frac{3-i}{2}$. In this case, the induced optimal profit satisfies

$$\underline{U}_1^i(k) = \frac{1}{4}(3-i)^2 k \quad \text{for all } k = 1, 2 \ldots. \tag{2.26}$$

Given that $U_1^i(k) > \underline{U}_1^i(k)$ for all $k \in \{1, 2, \ldots\}$ and $i \in \{1,2\}$, we first choose a positive constant $\tilde{C}_0 < \min\{b^1, b^2\} \min_{i \in \{1,2\}, k \in \{1, \ldots, k_0\}} \{U_1^i(k) - \underline{U}_1^i(k)\}$. Next, by (2.25) and (2.26), we have $U_1^i(k) - \underline{U}_1^i(k) \geq \frac{1}{270}k$ for all $k \geq k_0$. Based on this, we choose another positive constant $\tilde{C}_1 < \frac{1}{270}\min\{b^1, b^2\}$. Let $c_4 = \min\{\tilde{C}_0, \tilde{C}_1\}$. Then, for any problem scale $k \in \{1, 2 \ldots\}$ and for any policy π that generates a price path $(p_1, \ldots, p_{T_k})$, the profit loss $\Delta_\pi^{\boldsymbol{b}}(k)$ satisfies

$$\begin{aligned}
\Delta_\pi^{\boldsymbol{b}}(k) &\overset{\text{(h)}}{\geq} \mathbb{P}_\pi^{\boldsymbol{b}}\Big\{p_1 = p_1^1\Big\} b^2 \Big[U_1^2(k) - \underline{U}_1^2(k)\Big] + \mathbb{P}_\pi^{\boldsymbol{b}}\Big\{p_1 = p_1^2\Big\} b^1 \Big[U_1^1(k) - \underline{U}_1^1(k)\Big] \\
&\quad + \mathbb{P}_\pi^{b}\Big\{p_1 \neq p_1^1, p_1 \neq p_1^2\Big\} \Big[b^1(U_1^1(k) - \underline{U}_1^1(k)) + b^2(U_1^2(k) - \underline{U}_1^2(k))\Big] \\
&\overset{\text{(i)}}{\geq} c_4 k.
\end{aligned}$$

To prove (h), note that the event $\{p_1 = p_1^1\}$ implies $\{p_1 \neq p_1^2\}$, in which case the expected profit loss is at least $b^2\Big[U_1^2(k) - \underline{U}_1^2(k)\Big]$. Similarly, the event $\{p_1 = p_1^2\}$ implies $\{p_1 \neq p_1^1\}$, in which case the expected profit loss is at least $b^1\Big[U_1^1(k) - \underline{U}_1^1(k)\Big]$. Finally, the event $\{p_1 \neq p_1^1, p_1 \neq p_1^2\}$ implies that the expected profit loss is at least $b^1(U_1^1(k) - \underline{U}_1^1(k)) + b^2(U_1^2(k) - \underline{U}_1^2(k))$. Inequality (i) holds because $b^2\Big[U_1^1(k) - \underline{U}_1^1(k)\Big] \geq c_4 k$, $b^1\Big[U_1^1(k) - \underline{U}_1^1(k)\Big] \geq c_4 k$, and $b^1(U_1^1(k) - \underline{U}_1^1(k)) + b^2(U_1^2(k) - \underline{U}_1^2(k)) \geq c_4 k$ for $i \in \{1, 2\}$ and $k \in \{1, 2 \ldots\}$. $\square$

Proof of Proposition 15. The proof follows from the same problem instance and arguments used to derive Theorem6(i). $\square$

Proof of Proposition 16. Given $\pi = \text{CE}$, we let $\mathbb{P}_\pi^{\boldsymbol{b},i}\{\cdot\} = \mathbb{P}_\pi^{\boldsymbol{b}}\{\cdot \,|\, H_i\}$ and $\mathbb{E}_\pi^{\boldsymbol{b},i}\{\cdot\} = \mathbb{E}_\pi^{\boldsymbol{b}}\{\cdot \,|\, H_i\}$ for all $\boldsymbol{b} \in \mathcal{B}$ and $i \in \{1, \ldots, N\}$. To complete the proof, we first construct a problem instance $\boldsymbol{\theta} \in \Theta$. Suppose that there are two hypotheses, namely H_1 and H_2. Therefore, we have $N = 2$. Let $T = 1$, implying that if the problem scale is $k \in \{1, 2, \ldots\}$ then the time horizon is $T_k = k$. Moreover, let $\underline{p} = c = 0$ and $\bar{p} = 5$. Under the problem scale $k \in \{1, 2, \ldots\}$, let the arrival matrix $\boldsymbol{\alpha}^i$ be such that $\alpha_{\tau 0}^i = \frac{1}{k}$ for all τ, and $\alpha_{\tau w}^i = 0$ for all τ and w such that $w \geq 1$. We let the market density function under H_i be $\lambda^i(v) = \frac{1}{3-i}$ for all $v \in [0, 3-i]$. By construction, for any $\boldsymbol{p}^{(t)} \in [0, 5]^t$, the expected demand in period t

under H_i is

$$d_t^i(\boldsymbol{p}^{(t)}) = k\alpha_{\tau 0}^i \int_{v \geq p_t} \lambda^i(v)dv \; = \; \begin{cases} 3 - i - \frac{1}{3-i}p_t & \text{if } \; p_t \leq 3 - i, \\ 0 & \text{if } \; p_t \geq 3 - i. \end{cases}$$

for $i = 1, 2$. Note that, under H_i, the optimal full-information price path is given by $p_t^i = \arg\max_{p \in [\underline{p}, \bar{p}]} \left\{p(3 - i - \frac{1}{3-i}p)\right\} = \frac{(3-i)^2}{2}$ for all $t = 1, \ldots, T_k$. Letting $p^i = \frac{(3-i)^2}{2}$, we have $p^i \in [\underline{p}, \bar{p}]$ for $i = 1, 2$. We also have $p^1 > p^2$, and $p^1 \int_{v \geq p^1} \lambda^1(v)dv = 2 > \frac{1}{4} = p^2 \int_{v \geq p^2} \lambda^2(v)dv$. Expressing the prior belief $\boldsymbol{b}_1$ as $\boldsymbol{b}_1 = (b, 1 - b)$ for $b \in (0, 1)$, we observe that the pricing decision under $\pi = \text{CE}$ is given by the mapping $b \mapsto p^b$, where $p^b = \arg\max_p \left\{p\Big[1 + b - (\frac{b}{2} + \frac{1-b}{1})p\Big]\right\} = \frac{1+b}{2-b}$. Note that p^b is contained in $[p^1, p^2]$ and is increasing in b.

Let $(\boldsymbol{b}_1, \ldots, \boldsymbol{b}_{T_k})$ be the seller's belief process, where $\boldsymbol{b}_t = (b_t, 1 - b_t)$, and $(p_1, \ldots, p_{T_k})$ be the seller's price process, both of which are induced by $\pi = \text{CE}$. The price process satisfies $p_t = \pi_t(\boldsymbol{b}_t, \boldsymbol{p}^{(t-1)}) = \min\{p^{b_t}, p_{t-1}\}$. For any prior belief $\boldsymbol{b}_1 = (b, 1 - b)$ with $b \in (0, 1)$, the price induced by CE in period 1 satisfies $p_1 = \frac{1+b}{2-b} < p^1$. Since prices are marked down, we have $p_t \leq p_1 < p^1$ for all $t \geq 2$. Let $\delta_v = \frac{1}{2}(p^1 - p_1)^2 = \frac{1}{2}\Big(2 - \frac{1+b}{2-b}\Big)^2 > 0$, we further deduce that the profit loss per period under H_1 is

$$\begin{aligned} p^1 \int_{v \in [p^1, 2]} \lambda^1(v)dv - p_t \int_{v \in [p_t, 2]} \lambda^1(v)dv &= p^1(2 - \tfrac{1}{2}p^1) - p_t(2 - \tfrac{1}{2}p_t) \\ &= 2 + \tfrac{1}{2}(p_t - 2)^2 - 2 = \tfrac{1}{2}(p^1 - p_t)^2 \geq \delta_v > 0 \end{aligned}$$

for all $t = 1, \ldots, T_k$. Let $c_2 = b\delta_v > 0$. For all $k = 1, 2 \ldots$, we conclude the proof by deriving that

$$\Delta_\pi^{\boldsymbol{b}}(k) \geq b\, \mathbb{E}_\pi^{b,1}\left\{ \sum_{t=1}^{T_k} \Big[p^1\Big(2 - \tfrac{1}{2}p^1\Big) - p_t\Big(2 - \tfrac{1}{2}p_t\Big)\Big] \right\} \geq b\delta_v k = c_2 k. \qquad \square$$

Proof of Lemma 11. Let $\mathbb{P}_\pi^{\boldsymbol{b},i}\{\cdot\} = \mathbb{P}_\pi^{\boldsymbol{b}}\{\cdot \,|\, H_i\}$ and $\mathbb{E}_\pi^{\boldsymbol{b},i}\{\cdot\} = \mathbb{E}_\pi^{\boldsymbol{b}}\{\cdot \,|\, H_i\}$ for all $\boldsymbol{b} \in \mathcal{B}$,

$i \in \{1, \dots, N\}$, and $\pi \in \Pi$. We complete the proof in three steps.

Step 1: Derive a multiperiod belief updating inequality. Let $i, j \in \{1, \dots, N\}$ such that $i \neq j$, and $t \in \{1, 2, \dots\}$. For brevity, we suppress the dependence of $d_s^i(\boldsymbol{p}^{(s)})$ on $\boldsymbol{p}^{(s)}$ by letting $d_s^i = d_s^i(\boldsymbol{p}^{(s)})$ for all $s \in \{1, 2, \dots, t\}$. Thus, under H_i, we have $D_s = d_s^i + \varepsilon_s$ for all s, and

$$\begin{aligned} b_{t+1}^j &= \frac{b_1^j \prod_{s=1}^t \mathcal{L}_s^j(D_s, d_s^j | \mathcal{F}_{s-1})}{\sum_{i'=1}^N b_1^{i'} \prod_{s=1}^t \mathcal{L}_s^{i'}(D_s, d_s^{i'} | \mathcal{F}_{s-1})} \\ &= \frac{1}{1 + \sum_{i' \neq j} \frac{b_1^{i'}}{b_1^j} \exp\left(\sum_{s=1}^t \log \frac{\mathcal{L}_s^{i'}(D_s, d_s^{i'} | \mathcal{F}_{s-1})}{\mathcal{L}_s^j(D_s, d_s^j | \mathcal{F}_{s-1})} \right)} \\ &\leq \frac{1}{1 + \frac{b_1^i}{b_1^j} \exp\left(\sum_{s=1}^t \log \frac{\mathcal{L}_s^i(D_s, d_s^i | \mathcal{F}_{s-1})}{\mathcal{L}_s^j(D_s, d_s^j | \mathcal{F}_{s-1})} \right)}. \end{aligned} \tag{2.27}$$

Step 2: Construct a martingale. For $s = 1, 2, \dots$, let $X_s = \mathbb{E}_\pi^{\boldsymbol{b}, i}\left[\log \frac{\mathcal{L}_s^i(D_s, d_s^i | \mathcal{F}_{s-1})}{\mathcal{L}_s^j(D_s, d_s^j | \mathcal{F}_{s-1})} \right]$ and $Z_s = \log \frac{\mathcal{L}_s^i(D_s, d_s^i | \mathcal{F}_{s-1})}{\mathcal{L}_s^j(D_s, d_s^j | \mathcal{F}_{s-1})} - X_s$. Define an identity function $I : [\underline{d}, \bar{d}] \to [\underline{d}, \bar{d}]$ satisfying $I(d) = d$ for all $d \in [\underline{d}, \bar{d}]$, and let $u = \max\{|\underline{d}|, |\bar{d}|\}$. Set $\gamma = \frac{\epsilon_d^2}{2u^2}$, where ϵ_d is as in the statement of the lemma. Note that, for any pair of probability density functions $\mu, \nu : [\underline{d}, \bar{d}] \to \mathbb{R}^+$ satisfying $\mu(x), \nu(x) \in [\underline{f}_d, \bar{f}_d]$ for $x \in [\underline{d}, \bar{d}]$ and $|\int_{\underline{d}}^{\bar{d}} x\mu(x)dx - \int_{\underline{d}}^{\bar{d}} x\nu(x)dx| \geq \epsilon_d$, we have the following:

$$\begin{aligned} \int_{\underline{d}}^{\bar{d}} \log \frac{\mu(x)}{\nu(x)} \mu(x) dx &\overset{\text{(a)}}{\geq} \frac{1}{2} \|\mu - \nu\|_1^2 \\ &\overset{\text{(b)}}{\geq} \frac{\left\| I(\mu - \nu) \right\|_1^2}{2\|I\|_\infty^2} \\ &\overset{\text{(c)}}{=} \frac{1}{2u^2} \left| \int_{\underline{d}}^{\bar{d}} \left| x\big(\mu(x) - \nu(x)\big) \right| dx \right|^2 \\ &\overset{\text{(d)}}{\geq} \frac{1}{2u^2} \left| \int_{\underline{d}}^{\bar{d}} x\mu(x)dx - \int_{\underline{d}}^{\bar{d}} x\nu(x)dx \right|^2 \geq \frac{\epsilon_d^2}{2u^2}, \end{aligned} \tag{2.28}$$

where: $\|\mu - \nu\|_1 = \int_{\underline{d}}^{\bar{d}} |\mu(x) - \nu(x)| dx$, $\|I(\mu - \nu)\|_1 = \int_{\underline{d}}^{\bar{d}} \left| I(x)\big(\mu(x) - \nu(x)\big) \right| dx$, $\|I\|_\infty =$

$\sup_{x\in[\underline{d},\bar{d}]}|I(x)|$, (a) follows by Pinsker's inequality, (b) follows by Hölder's inequality, (c) follows because $\|I\|_\infty = u$, and (d) follows by Minkowski's inequality. Since $\mathcal{L}_s^i(\cdot, d_s^i|\mathcal{F}_{s-1})$ and $\mathcal{L}_s^j(\cdot, d_s^j|\mathcal{F}_{s-1})$ are two probability density functions satisfying the above properties of $\mu(\cdot)$ and $\nu(\cdot)$, we deduce that (3.200) holds for $\mu(\cdot) = \mathcal{L}_s^i(\cdot, d_s^i|\mathcal{F}_{s-1})$ and $\nu(\cdot) = \mathcal{L}_s^j(\cdot, d_s^j|\mathcal{F}_{s-1})$. That is, $X_s \geq \gamma = \frac{\epsilon_d^2}{2u^2}$ for all s. Moreover, Z_s satisfies the following properties for all s: (i) $|Z_s| \leq \bar{z}$, where $\bar{z} = 2\log\frac{\bar{f}_d}{\underline{f}_d} > 0$, and (ii) $\mathbb{E}_\pi^{\boldsymbol{b},i}[Z_s|\mathcal{F}_{s-1}] = \mathbb{E}_\pi^{\boldsymbol{b},i}\Big[\log\frac{\mathcal{L}_s^i(D_s,d_s^i|\mathcal{F}_{s-1})}{\mathcal{L}_s^j(D_s,d_s^j|\mathcal{F}_{s-1})} - X_s\Big|\mathcal{F}_{s-1}\Big] = \mathbb{E}_\pi^{\boldsymbol{b},i}\Big[\log\frac{\mathcal{L}_s^i(D_s,d_s^i|\mathcal{F}_{s-1})}{\mathcal{L}_s^j(D_s,d_s^j|\mathcal{F}_{s-1})}\Big|\mathcal{F}_{s-1}\Big] - \mathbb{E}_\pi^{\boldsymbol{b},i}\Big[\log\frac{\mathcal{L}_s^i(D_s,d_s^i|\mathcal{F}_{s-1})}{\mathcal{L}_s^j(D_s,d_s^j|\mathcal{F}_{s-1})}\Big|\mathcal{F}_{s-1}\Big] = 0$. As a result, $\{Z_s\}$ is a bounded martingale difference sequence adapted to $\mathcal{F}_s$.

Step 3: Derive an upper bound on the convergence rate. Let $\pi \in \Pi$ be such that, under π, we have $|d_s^i - d_s^j| \geq \epsilon_d > 0$ for all $s \in \{1, 2, \ldots, t\}$. Let $A_t = \sum_{s=1}^t X_s$ and $W_t = \sum_{s=1}^t Z_s$. By (3.199), we have

$$\mathbb{E}_\pi^{\boldsymbol{b},i}\{b_{t+1}^j\} \leq \mathbb{E}_\pi^{\boldsymbol{b},i}\left\{\frac{1}{1+\frac{b_1^i}{b_1^j}\exp(A_t+W_t)}\right\}.$$

Letting $B_t = \{W_t \leq -\frac{1}{2}A_t\}$, we deduce from the preceding inequality that

$$\begin{aligned}\mathbb{E}_\pi^{\boldsymbol{b},i}\{b_{t+1}^j\} &\leq \mathbb{E}_\pi^{\boldsymbol{b},i}\left\{\frac{1}{1+\frac{b_1^i}{b_1^j}\exp(A_t+W_t)}\mathbb{I}\{B_t\}\right\} + \mathbb{E}_\pi^{\boldsymbol{b},i}\left\{\frac{1}{1+\frac{b_1^i}{b_1^j}\exp(A_t+W_t)}\mathbb{I}\{B_t^c\}\right\}\\ &\overset{\text{(e)}}{\leq} \mathbb{P}_\pi^{\boldsymbol{b},i}\{B_t\} + \mathbb{E}_\pi^{\boldsymbol{b},i}\left\{\frac{1}{1+\frac{b_1^i}{b_1^j}\exp(A_t+W_t)}\mathbb{I}\{B_t^c\}\right\}\\ &\overset{\text{(f)}}{\leq} \mathbb{P}_\pi^{\boldsymbol{b},i}\{B_t\} + \mathbb{E}_\pi^{\boldsymbol{b},i}\left\{\frac{1}{1+\frac{b_1^i}{b_1^j}\exp(\frac{1}{2}A_t)}\mathbb{I}\{B_t^c\}\right\},\end{aligned} \tag{2.29}$$

where: $\mathbb{I}\{\cdot\}$ is the indicator function (i.e., given condition $\mathcal{A}$, $\mathbb{I}\{\mathcal{A}\} = 1$ if $\mathcal{A}$ holds, and 0 otherwise); (e) follows because $\frac{b_1^i}{b_1^j}\exp(A_t+W_t) \geq 0$, and $\mathbb{E}_\pi^{\boldsymbol{b},i}\{\mathbb{I}\{B_t\}\} = \mathbb{P}_\pi^{\boldsymbol{b},i}\{B_t\}$; and (f) follows because $A_t + W_t \ge \frac{1}{2}A_t$ on B^c_t. Under π, $A_t \ge \gamma t$. Therefore, (3.201) implies that

$$\mathbb{E}^{\boldsymbol{b},i}_{\pi}\{b^j_{t+1}\} \le \mathbb{P}^{\boldsymbol{b},i}_{\pi}\{B_t\} + \mathbb{E}^{\boldsymbol{b},i}_{\pi}\left\{\frac{1}{1+\frac{b^i_1}{b^j_1}\exp(\frac{1}{2}\gamma t)}\,\mathbb{I}\{B^c_t\}\right\} \overset{\text{(g)}}{\le} \mathbb{P}^{\boldsymbol{b},i}_{\pi}\{B_t\} + \frac{b^j}{b^i}\exp(-\tfrac{1}{2}\gamma t), \tag{2.30}$$

where (g) follows because $\mathbb{I}\{B^c_t\} \le 1$. Note that

$$\begin{aligned}\mathbb{P}^{\boldsymbol{b},i}_{\pi}\{B_t\} = \mathbb{P}^{\boldsymbol{b},i}_{\pi}\{W_t \le -\tfrac{1}{2}A_t\} &\overset{\text{(h)}}{\le} \mathbb{P}^{\boldsymbol{b},i}_{\pi}\{W_t \le -\tfrac{1}{2}\gamma t\}\\ &\overset{\text{(i)}}{\le} \mathbb{P}^{\boldsymbol{b},i}_{\pi}\{|W_t| \ge \tfrac{1}{2}\gamma t\}\\ &\overset{\text{(j)}}{\le} 2\exp(-\tfrac{\gamma^2 t}{2\bar{z}}),\end{aligned} \tag{2.31}$$

where: (h) follows because $A_t \ge \gamma t$, (i) follows because $\{W_t \ge -\frac{1}{2}\gamma t\} \subset \{|W_t| \ge \frac{1}{2}\gamma t\}$, (j) follows by the Azuma-Hoeffding inequality and the fact that $|W_s - W_{s-1}| = |Z_s| \le \bar{z}$ for all s. Combining (3.202) and (3.203), we deduce that

$$\mathbb{E}^{\boldsymbol{b},i}_{\pi}\{b^j_{t+1}\} \le 2\exp(-\tfrac{\gamma^2 t}{2\bar{z}}) + \tfrac{b^j}{b^i}\exp(-\tfrac{1}{2}\gamma t) \le \left(2 + \tfrac{b^i}{b^j}\right)\exp(-\eta t).$$

As a result, $\mathbb{E}^{\boldsymbol{b},i}_{\pi}\{1 - b^i_{t+1}\} = \sum_{j\ne i}\mathbb{E}^{\boldsymbol{b},i}_{\pi}\{b^j_{t+1}\} \le \zeta e^{-\eta t}$, where $\zeta = \max_i\left\{\sum_{j\ne i}\left(2 + \frac{b^i}{b^j}\right)\right\}$, and $\eta = \frac{1}{2}\gamma\min\{1, \frac{\gamma}{\bar{z}}\}$. □

Proof of Theorem 7. Given $\pi = \text{LATE}(n)$, we define $\mathbb{P}^{\boldsymbol{b},i}_{\pi}\{\cdot\} = \mathbb{P}^{\boldsymbol{b}}_{\pi}\{\cdot \mid H_i\}$ and $\mathbb{E}^{\boldsymbol{b},i}_{\pi}\{\cdot\} = \mathbb{E}^{\boldsymbol{b}}_{\pi}\{\cdot \mid H_i\}$ for all $\boldsymbol{b} \in \mathcal{B}$ and $i \in \{1,\dots,N\}$. Let $\boldsymbol{\theta} \in \Theta$, $M = \max_{i\in\{1,\dots,N\}}\left\{\max\{\overline{\alpha},1\}\int_{v\ge 0}\bar{p}\lambda^i(v)dv\right\}$, and $(p_1,\dots,p_{T_k})$ be the price path generated under π. Denote by $g^i_{\pi}(\tau)$ the profit to be collected from the customers arriving in period τ under H_i. Then,

$$g^i_{\pi}(\tau) = \sum_{t=\tau}^{T_k}(p_t - c)\sum_{w=0}^{T_k-1}k\alpha^i_{\tau w}\int_{v\in\mathcal{U}^i_{\tau w t}(\boldsymbol{p}^{(t)})}\lambda^i(v)dv.$$

Because $\sum_{w=0}^{T_k-1} k\alpha^i_{\tau w} \le \bar{\alpha}$, and $\sum_{t=\tau}^{T_k} \int_{v\in\mathcal{U}^i_{\tau w t}(\boldsymbol{p}^{(t)})} \lambda^i(v)dv \le \int_{v\ge 0}\lambda^i(v)dv$, we deduce that $g^i_\pi(\tau) \le M$ for all $\tau = 1,\dots,T_k$. Moreover, for every demand hypothesis $i \in \{1,\dots,N\}$ and problem scale $k \in \{1,2,\dots\}$, we have the following upper bound for the optimal full-information profit under H_i:

$$V^i \le \sum_{\tau=1}^{T_k}\sum_{w=0}^{T_k-1} k\alpha^i_{\tau w}\,\bar{p}\int_{v\ge 0}\lambda^i(v)dv \overset{\text{(a)}}{\le} Mk, \tag{2.32}$$

where (a) follows because $\sum_{\tau=1}^{T_k}\sum_{w=0}^{T_k-1}\alpha^i_{\tau w} = 1$ and $\max_{i\in\{1,\dots,N\}}\Big\{\int_{v\ge 0}\bar{p}\lambda^i(v)dv\Big\} \le M$. Let π^i denote the optimal full-information policy under H_i, and $(p^i_1,\dots,p^i_{T_k})$ be the optimal full-information price path generated under π^i. We show that there exists $c_3 > 0$ such that $\Delta^1_\pi(k) = V^1 - \mathbb{E}^{\boldsymbol{b},1}_\pi\Big\{\sum_{t=1}^{T_k} R^1_t(\boldsymbol{p}^{(t)})\Big\} \le c_3\log k$ for all k. To that end, consider an auxiliary policy $\bar{\pi}$ that generates a price path $(\bar{p}_1,\dots,\bar{p}_{T_k})$ satisfying $\bar{p}_s = \mathfrak{p}_s$ for $s \in \{1,\dots,n\}$ and $\bar{p}_s = p^{\hat{\imath}_n}_s$ where $\hat{\imath}_n = \arg\max_{i\in\{1,\dots,N\}}\{b^i_{n+1}\}$ for $s \in \{n+1,\dots,T_k\}$. Note that the policy $\pi = \mathrm{LATE}(n)$ and the auxiliary policy $\bar{\pi}$ both use the experimental price path $\mathfrak{p}^{(n)}$ in the first n periods. Consequently,

$$\mathbb{P}^{\boldsymbol{b},1}_\pi\{\hat{\imath}_n = i\} = \mathbb{P}^{\boldsymbol{b},1}_{\bar{\pi}}\{\hat{\imath}_n = i\} \;\text{ for } i \in \{1,\dots,N\}. \tag{2.33}$$

Moreover, after the initial experimental price path $\mathfrak{p}^{(n)}$, LATE(n) employs the optimal price path under H_1, whereas the auxiliary policy $\bar{\pi}$ employs a feasible (but not necessarily optimal) price path; hence,

$$\mathbb{E}^{\boldsymbol{b},1}_\pi\Bigg\{\sum_{\tau=1}^{T_k} g^1_\pi(\tau)\,\Bigg|\,\hat{\imath}_n = 1\Bigg\} \ge \mathbb{E}^{\boldsymbol{b},1}_{\bar{\pi}}\Bigg\{\sum_{\tau=1}^{T_k} g^1_{\bar{\pi}}(\tau)\,\Bigg|\,\hat{\imath}_n = 1\Bigg\}. \tag{2.34}$$

In addition, since $V^1 \le Mk$, we have

$$\mathbb{E}^{\boldsymbol{b},1}_{\bar{\pi}}\Bigg\{\sum_{\tau=1}^{T_k} g^1_{\bar{\pi}}(\tau)\,\Bigg|\,\hat{\imath}_n = i\Bigg\} - Mk \le \mathbb{E}^{\boldsymbol{b},1}_\pi\Bigg\{\sum_{\tau=1}^{T_k} g^1_\pi(\tau)\,\Bigg|\,\hat{\imath}_n = i\Bigg\} \;\text{ for all } i \ne 1. \tag{2.35}$$

As a result,

$$
\begin{aligned}
&\Delta^1_{\pi}(k)\\
&= V^1 - \sum_{i=1}^{N} \mathbb{P}^{\boldsymbol{b},1}_{\pi}\{\hat{\imath}_n = i\}\, \mathbb{E}^{\boldsymbol{b},1}_{\pi}\left\{ \sum_{\tau=1}^{T_k} g^1_{\pi}(\tau) \,\middle|\, \hat{\imath}_n = i \right\}\\
&\overset{\text{(b)}}{\leq} V^1 - \sum_{i=1}^{N} \mathbb{P}^{\boldsymbol{b},1}_{\bar{\pi}}\{\hat{\imath}_n = i\}\, \mathbb{E}^{\boldsymbol{b},1}_{\bar{\pi}}\left\{ \sum_{\tau=1}^{T_k} g^1_{\bar{\pi}}(\tau) \,\middle|\, \hat{\imath}_n = i \right\} + \mathbb{P}^{\boldsymbol{b},1}_{\bar{\pi}}\{\hat{\imath}_n \neq 1\} Mk\\
&= \Delta^1_{\bar{\pi}}(k) + Mk\, \mathbb{P}^{\boldsymbol{b},1}_{\bar{\pi}}\{\hat{\imath}_n \neq 1\}\\
&\leq \left| \mathbb{E}^{\boldsymbol{b},1}_{\bar{\pi}}\left\{ \sum_{\tau=1}^{n+\tau_p} \left[g^1_{\pi^1}(\tau) - g^1_{\bar{\pi}}(\tau) \right] \right\} \right| + \left| \mathbb{E}^{\boldsymbol{b},1}_{\bar{\pi}}\left\{ \sum_{\tau=n+\tau_p+1}^{T_k} \left[g^1_{\pi^1}(\tau) - g^1_{\bar{\pi}}(\tau) \right] \right\} \right| + Mk\, \mathbb{P}^{\boldsymbol{b},1}_{\bar{\pi}}\{\hat{\imath}_n \neq 1\},
\end{aligned}
\tag{2.36}
$$

where π^1 is the optimal full-information policy under H_1, and (b) follows from (2.33), (2.34), and (2.35). Regarding the first term on the right hand side of (2.36), we note that

$$
\left| \mathbb{E}^{\boldsymbol{b},1}_{\bar{\pi}}\left\{ \sum_{\tau=1}^{n+\tau_p} \left[g^1_{\pi^1}(\tau) - g^1_{\bar{\pi}}(\tau) \right] \right\} \right| \leq \left| \sum_{\tau=1}^{n+\tau_p} g^1_{\pi^1}(\tau) \right| + \left| \mathbb{E}^{\boldsymbol{b},1}_{\bar{\pi}}\left\{ \sum_{\tau=1}^{n+\tau_p} g^1_{\bar{\pi}}(\tau) \right\} \right| \overset{\text{(c)}}{\leq} 2M(n+\tau_p),
\tag{2.37}
$$

where (c) follows because $g^1_{\pi^1}(\tau) \leq M$ and $g^1_{\bar{\pi}}(\tau) \leq M$. Regarding the second term on the right hand side of (2.36), we have

$$
\begin{aligned}
&\left| \mathbb{E}^{\boldsymbol{b},1}_{\bar{\pi}}\left\{ \sum_{\tau=n+\tau_p+1}^{T_k} \left[g^1_{\pi^1}(\tau) - g^1_{\bar{\pi}}(\tau) \right] \right\} \right|\\
&\qquad\qquad \leq \sum_{i=1}^{N} \mathbb{P}^{\boldsymbol{b},1}_{\bar{\pi}}\{\hat{\imath}_n = i\} \left| \mathbb{E}^{\boldsymbol{b},1}_{\bar{\pi}}\left\{ \sum_{\tau=n+\tau_p+1}^{T_k} \left[g^1_{\pi^1}(\tau) - g^1_{\bar{\pi}}(\tau) \right] \,\middle|\, \hat{\imath}_n = i \right\} \right|\\
&\qquad\qquad \overset{\text{(d)}}{\leq} 2Mk\, \mathbb{P}^{\boldsymbol{b},1}_{\bar{\pi}}\{\hat{\imath}_n \neq 1\} + \left| \mathbb{E}^{\boldsymbol{b},1}_{\bar{\pi}}\left\{ \sum_{\tau=n+\tau_p+1}^{T_k} \left[g^1_{\pi^1}(\tau) - g^1_{\bar{\pi}}(\tau) \right] \,\middle|\, \hat{\imath}_n = 1 \right\} \right|,
\end{aligned}
\tag{2.38}
$$

where (d) follows because $g^1_{\pi^1}(\tau) \leq M$, $g^1_{\bar{\pi}}(\tau) \leq M$, and $\mathbb{P}^{\boldsymbol{b},1}_{\bar{\pi}}\{\hat{\imath}_n = 1\} \leq 1$. By construction of the auxiliary policy $\bar{\pi}$, if $\hat{\imath}_n = 1$, we have $\bar{p}_t = p_t^1$ for all $t \geq n+\tau_p+1$. Thus, for all customers arriving in period $\tau \geq n + \tau_p + 1$, the future price predictions satisfy $\rho_t^{\kappa}(\boldsymbol{p}^{1(t)}) = \rho_t^{\kappa}(\bar{\boldsymbol{p}}^{(t)})$ for all $t \geq \tau$ and $\kappa = 1, \ldots, T_k - t$, where $\boldsymbol{p}^{1(t)} = (p_1^1, \ldots, p_t^1)$ and $\bar{\boldsymbol{p}}^{(t)} = (\bar{p}_1, \ldots, \bar{p}_t)$. By Lemma 10(ii), this implies that, for all $\tau \geq n + \tau_p + 1$ and $t \geq \tau$, the set of valuations of type-(τ, w) customers who make a purchase in period t satisfies $\mathcal{U}_{\tau w t}^1(\boldsymbol{p}^{1(t)}) = \mathcal{U}_{\tau w t}^1(\bar{\boldsymbol{p}}^{(t)})$. As a result, we have $\mathbb{E}_{\bar{\pi}}^{\boldsymbol{b},1}\big\{ \sum_{t=n+\tau_p+1}^{T_k} [g_{\pi^1}^1(\tau) - g_{\bar{\pi}}^1(\tau)] \,|\, \hat{\imath}_n = 1 \big\} = 0$. Combining this identity with (2.38), we obtain the following:

$$\begin{aligned} \left| \mathbb{E}_{\bar{\pi}}^{\boldsymbol{b},1}\left\{ \sum_{\tau=n+\tau_p+1}^{T_k} \Big[g_{\pi^1}^1(\tau) - g_{\bar{\pi}}^1(\tau) \Big] \right\} \right| \leq 2Mk\, \mathbb{P}_{\bar{\pi}}^{\boldsymbol{b},1}\{\hat{\imath}_n \neq 1\} &\overset{\text{(e)}}{\leq} 2Mk\, \mathbb{P}_{\bar{\pi}}^{\boldsymbol{b},1}\Big\{ 1 - b_{n+1}^1 \geq \tfrac{1}{2} \Big\} \\ &\overset{\text{(f)}}{\leq} 4Mk\, \mathbb{E}_{\bar{\pi}}^{\boldsymbol{b},1}\{1 - b_{n+1}^1\} \\ &\overset{\text{(g)}}{\leq} 4Mk\zeta e^{-\eta n}, \end{aligned} \tag{2.39}$$

where (e) follows from $\{\hat{\imath}_n \neq 1\} \subset \{b_{n+1}^1 \leq \frac{1}{2}\}$, (f) follows by Markov's inequality, and (g) follows by Lemma 11. Regarding the third term on the right hand side of (2.36), the arguments used to derive (2.39) imply that

$$Mk\, \mathbb{P}_{\bar{\pi}}^{\boldsymbol{b},1}\{\hat{\imath}_n \neq 1\} \leq Mk\, \mathbb{P}_{\bar{\pi}}^{\boldsymbol{b},1}\Big\{ 1 - b_{n+1}^1 \geq \tfrac{1}{2} \Big\} \leq 2Mk\zeta e^{-\eta n}. \tag{2.40}$$

Finally, given that $n = \lceil \frac{1}{\eta} \log k \rceil$, choose $\tilde{C}_1 > 0$ such that $2M\Big(1 + \frac{1}{\eta}\log k\Big) + 2M\tau_p + 6\zeta M \leq \tilde{C}_1 \log k$ for all $k > 1$. Combining (2.36), (2.37), (2.39), (2.40), we deduce that, for all $k = 2, 3, \ldots,$

$$\begin{aligned} \Delta_{\pi}^1(k) &\leq 2M(n + \tau_p) + 4Mk\zeta e^{-\eta n} + 2Mk\zeta e^{-\eta n} \\ &\overset{\text{(h)}}{\leq} 2M(1 + \tfrac{1}{\eta}) \log k + 2M\tau_p + 6\zeta M \;\leq\; \tilde{C}_1 \log k, \end{aligned} \tag{2.41}$$

where (h) follows because $n = \lceil \frac{1}{\eta} \log k \rceil$. Repeating the arguments used to derive (2.41) for all demand hypotheses $i \neq 1$, we deduce that there exists $\tilde{C}_i > 0$ such that $\Delta_{\pi}^i(k) \leq \tilde{C}_i \log k$.

Letting $c_3 = \max_{i \in \{1,\dots,N\}} \{\tilde{C}_i\}$, we conclude that $\Delta^{\boldsymbol{b}}_\pi(k) = \sum_{i=1}^N b^i \Delta^i_\pi(k) \le c_3 \log k$ for all $k = 2, 3, \dots$ $\square$

Proof of Theorem 8. Given $\pi = \text{LATE}(n)$, let $\mathbb{P}^{\boldsymbol{b},i}_\pi\{\cdot\} = \mathbb{P}^{\boldsymbol{b}}_\pi\{\cdot \mid H_i\}$, $\mathbb{E}^{\boldsymbol{b},i}_\pi\{\cdot\} = \mathbb{E}^{\boldsymbol{b}}_\pi\{\cdot \mid H_i\}$ for all $\boldsymbol{b} \in \mathcal{B}$ and $i \in \{1, \dots, N\}$. In addition, let $(p_1, \dots, p_{T_k})$ be the price path generated under π, and $g^i_\pi(\tau) = \sum_{t=\tau}^{T_k} (p_t - c) \sum_{w=0}^{T_k - 1} k\alpha^i_{\tau w} \int_{v \in \mathcal{U}^i_{\tau w t}(\boldsymbol{p}^{(t)})} \lambda^i(v) dv$ be the profit to be collected from customers arriving in period τ under H_i. Moreover, let $\boldsymbol{\theta} \in \Theta$, and $M = \max_{i \in \{1,\dots,N\}} \left\{ \max\{\overline{\alpha}, 1\} \int_{v \ge 0} \bar{p} \lambda^i(v) dv \right\}$. Note that, by the arguments used to derive (2.32), we have $V^i \le Mk$ for each demand hypothesis $i \in \{1, \dots, N\}$ and problem scale $k \in \{1, 2, \dots\}$.

We first prove that $\frac{1}{k} \Delta^1_\pi(k) \to 0$. To prove said claim, let $\bar{\pi}$ be an auxiliary policy such that its price path $(\bar{p}_1, \dots, \bar{p}_{T_k})$ satisfies the following conditions: $\bar{p}_s = \mathfrak{p}_s$ for $s \in \{1, \dots, n\}$, and $\bar{p}_s = p^{\hat{\imath}_n}_s$ where $\hat{\imath}_n = \arg\max_{i \in \{1,\dots,N\}} \{b^i_{n+1}\}$ for $s \in \{n+1, \dots, T_k\}$. By the arguments used to derive (2.36), we deduce the following for any $n_2 > n$:

$$\begin{aligned} \Delta^1_\pi(k) &\le \Delta^1_{\bar{\pi}}(k) + Mk\, \mathbb{P}^{\boldsymbol{b},1}_{\bar{\pi}}\{\hat{\imath}_n \ne 1\} \\ &\le \left| \mathbb{E}^{\boldsymbol{b},1}_{\bar{\pi}} \left\{ \sum_{\tau=1}^{n_2} g^1_{\pi^1}(\tau) - g^1_{\bar{\pi}}(\tau) \right\} \right| + \left| \mathbb{E}^{\boldsymbol{b},1}_{\bar{\pi}} \left\{ \sum_{\tau=n_2+1}^{T_k} g^1_{\pi^1}(\tau) - g^1_{\bar{\pi}}(\tau) \right\} \right| + Mk\, \mathbb{P}^{\boldsymbol{b},1}_{\bar{\pi}}\{\hat{\imath}_n \ne 1\}, \end{aligned} \tag{2.42}$$

where π^1 is the optimal full-information policy under H_1, which generates the price path $(p^1_1, \dots, p^1_{T_k})$. Regarding the first term on the right hand side of (2.42), we note that

$$\left| \mathbb{E}^{\boldsymbol{b},1}_{\bar{\pi}} \left\{ \sum_{\tau=1}^{n_2} g^1_{\pi^1}(\tau) - g^1_{\bar{\pi}}(\tau) \right\} \right| \le \left| \sum_{\tau=1}^{n_2} g^1_{\pi^1}(\tau) \right| + \left| \mathbb{E}^{\boldsymbol{b},1}_{\bar{\pi}} \left\{ \sum_{\tau=1}^{n_2} g^1_{\bar{\pi}}(\tau) \right\} \right| \overset{\text{(a)}}{\le} 2Mn_2, \tag{2.43}$$

where (a) follows because $g^1_\pi(\tau) \le M$ and $g^1_{\bar{\pi}}(\tau) \le M$. Regarding the second term on the right hand side of (2.42), we have

$$\begin{aligned}
&\left|\mathbb{E}_{\bar\pi}^{\boldsymbol{b},1}\left\{\sum_{\tau=n_2+1}^{T_k} g_{\pi^1}^1(\tau) - g_{\bar\pi}^1(\tau)\right\}\right| \\
&\quad \le \sum_{i=1}^{N} \mathbb{P}_{\bar\pi}^{\boldsymbol{b},1}\{\hat\imath_n = i\} \left|\mathbb{E}_{\bar\pi}^{\boldsymbol{b},1}\left\{\sum_{\tau=n_2+1}^{T_k} g_{\pi^1}^1(\tau) - \sum_{\tau=n_2+1}^{T_k} g_{\bar\pi}^1(\tau) \,\middle|\, \hat\imath_n = i\right\}\right| \\
&\quad \overset{\text{(b)}}{\le} 2Mk\,\mathbb{P}_{\bar\pi}^{\boldsymbol{b},1}\{\hat\imath_n \neq 1\} + \left|\mathbb{E}_{\bar\pi}^{\boldsymbol{b},1}\left\{\sum_{\tau=n_2+1}^{T_k} g_{\pi^1}^1(\tau) - \sum_{\tau=n_2+1}^{T_k} g_{\bar\pi}^1(\tau) \,\middle|\, \hat\imath_n = 1\right\}\right|, \qquad (2.44)
\end{aligned}$$

where (b) follows because $g_{\pi}^1(\tau) \le M$, $g_{\bar\pi}^1(\tau) \le M$, and $\mathbb{P}_{\bar\pi}^{\boldsymbol{b},1}\{\hat\imath_n = 1\} \le 1$. By construction of the auxiliary policy $\bar\pi$, if $\hat\imath_n = 1$, then for all customers arriving in period $\tau \ge n_2 + 1$, the profit loss $g_{\pi^1}^1(\tau) - g_{\bar\pi}^1(\tau)$ is caused by two components:

(1) We denote by $L_w(\tau)$ the profit loss caused by customers who arrive in period τ and wait for a longer time under policy $\bar\pi$ than under policy π^1. For any $t \ge \tau \ge n_2 + 1$, we have

$$\overline{u}_{\tau w t}^1(\boldsymbol{p}^{(t)}) = \min_{l\in\{\tau,\dots t\}}\left\{p_l + \max_{s\in\mathcal{T}_{\tau w l}}\left\{\frac{(\delta^1)^{s-l}}{1-(\delta^1)^{s-l}}\Big(p_l - \rho_l^{s-l}\big(\boldsymbol{p}^{(l)}\big)\Big)\right\}\right\}, \qquad (2.45)$$

which is piecewise Lipschitz-continuous in $\boldsymbol{p}^{(t)} \in [\underline{p}, \bar p]^t$. Note that $\bar p_s = p_s^1$ for all $s \in \{n+1, \dots, \tau\}$ under $\bar\pi$. Moreover, because $\rho \in \mathfrak{P}_2(a, r)$, we have $\left|\frac{\partial}{\partial p_s}\rho_l^\kappa(\boldsymbol{p}^{(l)})\right| \le a\left(\frac{s}{l}\right)^{1+r}$ for $s \le n$, $l \ge \tau \ge n_2 + 1$, and $\kappa \in \{1, \dots, T_k - 1\}$. Thus,

$$\left|\frac{\partial}{\partial p_s}\rho_l^\kappa(\boldsymbol{p}^{(l)})\right| \le a n^{1+r} n_2^{-(1+r)} \quad \text{for } s \le n \text{ and } l \ge n_2 + 1. \qquad (2.46)$$

We deduce from (2.45) and (2.46) that there exists a constant $\bar a \ge \max_{\kappa\in\{1,\dots,T_k\}}\left\{\frac{(\delta^1)^\kappa}{1-(\delta^1)^\kappa}\kappa a\right\}$

such that

$$\left|\overline{u}^1_{\tau wt}(\bar{\boldsymbol{p}}^{(t)}) - \overline{u}^1_{\tau wt}(\boldsymbol{p}^{1(t)})\right| \le \bar{a} n^{1+r} n_2^{-(1+r)} \sum_{s=1}^{t} \left|\bar{p}_s - p^1_s\right| \quad \text{for } w \in \{0,1,\dots,T_k-1\}. \tag{2.47}$$

Thus,

$$\begin{aligned}
L_w(\tau) &\le \sum_{w=0}^{T_k-1} k\alpha^1_{\tau w} \left| \sum_{t=\tau}^{T_k} (p^1_t - c) \int_{\overline{u}^1_{\tau wt}(\bar{\boldsymbol{p}}^{(t)})}^{\overline{u}^1_{\tau wt}(\boldsymbol{p}^{1(t)})} \lambda^i(v)dv \right| \\
&\overset{(c)}{\le} \sum_{w=0}^{T_k-1} k\alpha^1_{\tau w} \sum_{t=\tau}^{T_k} \bar{p}\bar{\lambda} \left|\overline{u}^1_{\tau wt}(\bar{\boldsymbol{p}}^{(t)}) - \overline{u}^1_{\tau wt}(\boldsymbol{p}^{1(t)})\right| \\
&\overset{(d)}{\le} \left(\sum_{w=0}^{T_k-1} k\alpha^1_{\tau w}\bar{p}\bar{\lambda} \right) \max_{w\in\{0,1,\dots,T_k-1\}} \left\{ \sum_{t=\tau}^{T_k} \left|\overline{u}^1_{\tau wt}(\bar{\boldsymbol{p}}^{(t)}) - \overline{u}^1_{\tau wt}(\boldsymbol{p}^{1(t)})\right| \right\} \\
&\overset{(e)}{\le} \overline{\alpha}\bar{p}\bar{\lambda} \max_{w\in\{0,1,\dots,T_k-1\}} \left\{ \sum_{t=\tau}^{T_k} \bar{a} n^{1+r} n_2^{-(1+r)} \sum_{s=1}^{n} \left|\bar{p}_s - p^1_s\right| \right\} \\
&\overset{(f)}{\le} \overline{\alpha}\bar{p}\bar{\lambda} \left(kT\bar{a} n^{1+r} n_2^{-(1+r)} \sum_{s=1}^{n} \left|p^1_s - \bar{p}_s\right| \right) \overset{(g)}{\le} 2\overline{\alpha}\bar{p}^2\bar{\lambda}\bar{a}Tn^{2+r}n_2^{-(1+r)}k,
\end{aligned} \tag{2.48}$$

where (c) follows because $p^1_t - c \le \bar{p}$ and $\lambda^i(v) \le \bar{\lambda}$, (d) follows from Hölder's inequality, (e) holds from (2.47) and the fact that $\sum_{w=0}^{T_k-1} k\alpha^1_{\tau w} \le \overline{\alpha}$, (f) follows because $T_k - t \le kT$, and (g) follows because $|\bar{p}_s - p^1_s| = \bar{p}_s - p^1_s \le \bar{p}$.

(2) We denote by $L_a(\tau)$ the profit loss caused by customers who abandon under policy $\bar{\pi}$ but stay under policy π^1. For customers arriving in period $\tau \ge n_2 + 1$, we let $\bar{t}_{\tau w} = \min\{\tau + w, T_k\}$ and $\underline{v}_{\tau w}(\boldsymbol{p}^{(\bar{t}_{\tau w})}) = \max_{s\in\{\tau\}\cup\mathcal{T}_{\tau w\tau}}\{\underline{v}^1_{\tau ws}(\boldsymbol{p}^{(s)})\}$. By construction, $\underline{v}_{\tau w}(\boldsymbol{p}^{(\bar{t}_{\tau w})})$ captures the marginal value of customers who are indifferent between abandoning and staying in the system. By Lemma 10(i), we have

$$\underline{v}_{\tau w}(\boldsymbol{p}^{(\bar{t}_{\tau w})}) = \max_{l\in\{\tau,\dots,\bar{t}_{\tau w}\}} \left\{ \rho_l^{\bar{t}_{\tau w}-l}(\boldsymbol{p}^{(l)}) \right\}. \tag{2.49}$$

Furthermore, because $\rho \in \mathfrak{P}_2(a,r)$, we obtain the following for $s \le n$ and $l \ge n_2 + 1$:

$\left|\frac{\partial \rho_l^{\bar{t}_{\tau w}-l}(\boldsymbol{p}^{(l)})}{\partial p_s}\right| \le a(\bar{t}_{\tau w}-l)s^{1+r}l^{-(1+r)} \le aT_k n^{1+r} n_2^{-(1+r)}$. Thus,

$$\rho_l^{\bar{t}_{\tau w}-l}(\bar{\boldsymbol{p}}^{(l)}) - \rho_l^{\bar{t}_{\tau w}-l}(\boldsymbol{p}^{1(l)}) \le \sum_{s=1}^{n} akTn^{1+r}n_2^{-(1+r)}\left|\bar{p}_s - p_s^1\right| \quad \text{for } l \ge n_2+1. \tag{2.50}$$

Based on the above, we further deduce that

$$\begin{aligned}
L_a(\tau) &\overset{\text{(h)}}{\le} \sum_{w=0}^{T_k-1} k\alpha_{\tau w}^1 \bar{p}\left[\int_{\underline{v}_{w\tau}(\boldsymbol{p}^{1(\bar{t}_{\tau w})})}^{\underline{v}_{w\tau}(\bar{\boldsymbol{p}}^{(\bar{t}_{\tau w})})} \lambda^1(v)dv\right]^+ \\
&\overset{\text{(i)}}{\le} \sum_{w=0}^{T_k-1} k\alpha_{\tau w}^1 \bar{p}\bar{\lambda}\left[\underline{v}_{w\tau}(\bar{\boldsymbol{p}}^{(\bar{t}_{\tau w})}) - \underline{v}_{w\tau}(\boldsymbol{p}^{1(\bar{t}_{\tau w})})\right]^+ \\
&\overset{\text{(j)}}{\le} \sum_{w=0}^{T_k-1} k\alpha_{\tau w}^1 \bar{p}\bar{\lambda}\left[\max_{l\in\{\tau,\ldots,\bar{t}_{\tau w}\}}\left\{\rho_l^{\bar{t}_{\tau w}-l}(\bar{\boldsymbol{p}}^{(l)})\right\} - \max_{l\in\{\tau,\ldots,\bar{t}_{\tau w}\}}\left\{\rho_l^{\bar{t}_{\tau w}-l}(\boldsymbol{p}^{1(l)})\right\}\right]^+ \\
&\overset{\text{(k)}}{\le} \sum_{w=0}^{T_k-1} k\alpha_{\tau w}^1 \bar{p}\bar{\lambda}\left[\max_{l\in\{\tau,\ldots,\bar{t}_{\tau w}\}}\left\{\rho_l^{\bar{t}_{\tau w}-l}(\bar{\boldsymbol{p}}^{(l)}) - \rho_l^{\bar{t}_{\tau w}-l}(\boldsymbol{p}^{1(l)})\right\}\right]^+ \\
&\overset{\text{(l)}}{\le} \overline{\alpha}\bar{p}\bar{\lambda}akTn^{1+r}n_2^{-(1+r)}\sum_{s=1}^{n}|p_s^1 - \bar{p}_s| \\
&\le 2\overline{\alpha}\bar{p}^2\bar{\lambda}aTn^{2+r}n_2^{-(1+r)}k,
\end{aligned} \tag{2.51}$$

where (h) follows from a natural upper bound on the profit reduction due to the abandonment under the suboptimal policy $\bar{\pi}$, relative to the optimal π^1; (i) follows because $\lambda^1(v) \le \bar{\lambda}$ for $v \ge 0$; (j) follows from (2.49); (k) follows because $\rho_l^{\bar{t}_{\tau w}-l}(\boldsymbol{p}^{1(l)}) \le \max_{l\in\{\tau,\ldots,\bar{t}_{\tau w}\}}\left\{\rho_l^{\bar{t}_{\tau w}-l}(\boldsymbol{p}^{1(l)})\right\}$ and the mapping $x \mapsto [x]^+ = \max\{x, 0\}$ is increasing; and (l) follows from (2.50) and the fact that $\sum_{w=0}^{T_k-1} k\alpha_{\tau w}^1 \le \overline{\alpha}$.

Let $\tilde{C}_1 = 4\overline{\alpha}\bar{p}^2\bar{\lambda}\max\{\bar{a}, a\}T$. By (2.48) and (2.51), we have

$$g_{\pi^1}^1(\tau) - g_{\bar{\pi}}^1(\tau) \le L_w(\tau) + L_a(\tau) \le \tilde{C}_1 n^{2+r} n_2^{-(1+r)} k. \tag{2.52}$$

Combining (2.44) and (2.52), we further deduce that

$$\left|\mathbb{E}_{\bar{\pi}}^{\boldsymbol{b},1}\left\{\sum_{\tau=n_2+1}^{T_k} g^1_{\pi^1}(\tau)-g^1_{\bar{\pi}}(\tau)\right\}\right| \le 2Mk\,\mathbb{P}_{\bar{\pi}}^{\boldsymbol{b},1}\{\hat{\imath}_n\neq 1\}+\tilde{C}_1Tn^{2+r}n_2^{-(1+r)}k^2. \tag{2.53}$$

Regarding the third term on the right hand side of (2.42) (which is also the first term on the right hand side of (2.53)), we note that

$$\mathbb{P}_{\bar{\pi}}^{\boldsymbol{b},1}\{\hat{\imath}_n\neq 1\}Mk \overset{\text{(m)}}{\le} \mathbb{P}_{\bar{\pi}}^{\boldsymbol{b},1}\left\{1-b^1_{n+1}\ge\frac{1}{2}\right\}Mk \overset{\text{(n)}}{\le} 2\mathbb{E}_{\bar{\pi}}^{\boldsymbol{b},1}\{1-b^1_{n+1}\} \overset{\text{(o)}}{\le} 2\zeta e^{-\eta n}Mk, \tag{2.54}$$

where (m) follows because $\{\hat{\imath}_n\neq 1\}\subset\left\{1-b^1_{n+1}\ge\frac{1}{2}\right\}$, (n) follows from Markov's inequality, and (o) follows from Lemma 11. Now, by (2.42), (2.43), (2.53), and (2.54), we obtain the following:

$$\frac{1}{k}\Delta^1_{\pi}(k)\le\frac{1}{k}\Big[2Mn_2+\tilde{C}_1Tn^{2+r}n_2^{-(1+r)}k^2+6\zeta e^{-\eta n}Mk\Big]. \tag{2.55}$$

Recall that $n=\lceil\frac{1}{\eta}\log k\rceil$ and consider two cases:
Case 1: $r\ge 1$. Let $n_2=(1+\frac{1}{\eta})k^{\frac{1+\epsilon}{2}}$ for some $\epsilon\in(0,\frac{1}{2})$. As $k\to\infty$, we have

$$\frac{2Mn_2}{k}\le 2M\left(1+\frac{1}{\eta}\right)k^{-\frac{1-\epsilon}{2}}\to 0, \tag{2.56a}$$

$$\frac{\tilde{C}_1Tn^{2+r}n_2^{-(1+r)}k^2}{k}\le\tilde{C}_1T\left(1+\frac{1}{\eta}\log k\right)^{2+r}\left(1+\frac{1}{\eta}\right)^{-(1+r)}k^{1-\frac{1+\epsilon}{2}(1+r)}\to 0, \tag{2.56b}$$

$$\frac{6\zeta e^{-\eta n}Mk}{k}\le 6\zeta Mk^{-1}\to 0; \tag{2.56c}$$

Case 2: $0 < r < 1$. Let $n_2 = (1 + \frac{1}{\eta})k^{1-\frac{r}{2}}$. As $k \to \infty$, we have

$$\frac{2Mn_2}{k} \leq 2M\left(1 + \frac{1}{\eta}\right)k^{-\frac{r}{2}} \to 0, \tag{2.57a}$$

$$\frac{\tilde{C}_1 T n^{2+r} n_2^{-(1+r)} k^2}{k} \leq \tilde{C}_1 T \left(1 + \frac{1}{\eta}\log k\right)^{2+r} \left(1 + \frac{1}{\eta}\right)^{-(1+r)} k^{-\frac{1}{2}r(1-r)} \to 0, \tag{2.57b}$$

$$\frac{6\zeta e^{-\eta n} Mk}{k} \leq 6\zeta M k^{-1} \to 0. \tag{2.57c}$$

Combining (2.55) with (2.56a)-(2.57c), we deduce that

$$\frac{1}{k}\Delta_{\pi}^{1}(k) \to 0. \tag{2.58}$$

Finally, repeating the above arguments used to derive (2.58) for the demand hypotheses $i \neq 1$, we conclude that

$$\frac{1}{k}\Delta_{\pi}^{\boldsymbol{b}}(k) \to 0. \qquad \square$$

Proof of Theorem 9. Given π be the version of LATE(n) in §2.3.5, and $\rho \in \mathfrak{P}$ such that the pair (ρ, π) satisfies Condition 4, we define $\mathbb{P}_{\pi\rho}^{\boldsymbol{b},i}\{\cdot\} = \mathbb{P}_{\pi\rho}^{\boldsymbol{b}}\{\cdot \mid H_i\}$ and $\mathbb{E}_{\pi\rho}^{\boldsymbol{b},i}\{\cdot\} = \mathbb{E}_{\pi\rho}^{\boldsymbol{b},i}\{\cdot \mid H_i\}$ for all $\boldsymbol{b} \in \mathcal{B}$ and $i \in \{1, \ldots, N\}$. Let $\boldsymbol{\theta} \in \Theta$, $M = \max_{i\in\{1,\ldots,N\}} \left\{ \max\{\overline{\alpha}, 1\} \int_{v\geq 0} \bar{p}\lambda^i(v)dv \right\}$, and $(p_1, \ldots, p_{T_k})$ be the price path generated under π. Denote by $g_{\pi\rho}^{i}(\tau)$ the profit to be collected from the customers arriving in period τ under H_i. Then,

$$g_{\pi\rho}^{i}(\tau) = \sum_{t=\tau}^{T_k} (p_t - c) \sum_{w=0}^{T_k - 1} k\alpha_{\tau w}^{i} \int_{v\in\mathcal{U}_{\tau w t}^{i}(\boldsymbol{p}^{(t)},\rho)} \lambda^i(v)dv.$$

Because $\sum_{w=0}^{T_k-1} k\alpha_{\tau w}^{i} \leq \overline{\alpha}$, and $\sum_{t=\tau}^{T_k} \int_{v\in\mathcal{U}_{\tau w t}^{i}(\boldsymbol{p}^{(t)},\rho)} \lambda^i(v)dv \leq \int_{v\geq 0} \lambda^i(v)dv$, we deduce that $g_{\pi\rho}^{i}(\tau) \leq M$ for all $\tau = 1, \ldots, T_k$. Moreover, for every demand hypothesis $i \in \{1, \ldots, N\}$ and problem scale $k \in \{1, 2, \ldots\}$, we have the following upper bound for the optimal full-information profit under H_i:

$$V_{\text{eq}}^i \leq \sum_{\tau=1}^{T_k} \sum_{w=0}^{T_k-1} k\alpha_{\tau w}^i \, \bar{p} \int_{v\geq 0} \lambda^i(v)dv \overset{\text{(a)}}{\leq} Mk, \tag{2.59}$$

where (a) follows because $\sum_{\tau=1}^{T_k} \sum_{w=0}^{T_k-1} \alpha_{\tau w}^i = 1$ and $\max_{i\in\{1,\ldots,N\}} \Big\{ \int_{v\geq 0} \bar{p}\lambda^i(v)dv \Big\} \leq M$. Let (ρ^i, π^i) be the subgame perfect equilibrium that induces the optimal full-information profit V_{eq}^i for the seller under H_i, and $(p_1^i, \ldots, p_{T_k}^i)$ be the corresponding equilibrium price path induced by the seller's policy π^i. To complete the proof, we first show that there exists $c_6 > 0$ such that $\Delta_\pi^1(k) = V^1 - \mathbb{E}_\pi^{\boldsymbol{b},1}\Big\{ \sum_{t=1}^{T_k} R_t^1(\boldsymbol{p}^{(t)}) \Big\} \leq c_6 \log k$ for all $k \in \{2, 3, \ldots\}$. Note that

$$\begin{aligned} &V_{\text{eq}}^1 - \sum_{i=1}^{N} \mathbb{P}_{\pi\rho}^{\boldsymbol{b},1}\{\hat{\imath}_n = i\}\, \mathbb{E}_{\pi\rho}^{\boldsymbol{b},1}\bigg\{ \sum_{\tau=1}^{T_k} g_{\pi\rho}^1(\tau) \,\bigg|\, \hat{\imath}_n = i \bigg\} \\ &\qquad \leq \bigg| \mathbb{E}_{\pi\rho}^{\boldsymbol{b},1}\bigg\{ \sum_{\tau=1}^{n} \Big[g_{\pi^1\rho^1}^1(\tau) - g_{\pi\rho}^1(\tau) \Big] \bigg\} \bigg| + \bigg| \mathbb{E}_{\pi\rho}^{\boldsymbol{b},1}\bigg\{ \sum_{\tau=n+1}^{T_k} \Big[g_{\pi^1\rho^1}^1(\tau) - g_{\pi\rho}^1(\tau) \Big] \bigg\} \bigg|, \end{aligned} \tag{2.60}$$

where (ρ^1, π^1) is the aforementioned pair of customers' prediction behavior and the seller's policy, which constitutes the subgame perfect equilibrium under H_1. Regarding the first term on the right hand side of (2.60), we note that

$$\bigg| \mathbb{E}_{\pi\rho}^{\boldsymbol{b},1}\bigg\{ \sum_{\tau=1}^{n} \Big[g_{\pi^1\rho^1}^1(\tau) - g_{\pi\rho}^1(\tau) \Big] \bigg\} \bigg| \leq \bigg| \sum_{\tau=1}^{n} g_{\pi^1\rho^1}^1(\tau) \bigg| + \bigg| \mathbb{E}_{\pi\rho}^{\boldsymbol{b},1}\bigg\{ \sum_{\tau=1}^{n} g_{\pi\rho}^1(\tau) \bigg\} \bigg| \overset{\text{(b)}}{\leq} 2Mn, \tag{2.61}$$

where (b) follows because $g_{\pi^1\rho^1}^1(\tau) \leq M$ and $g_{\pi\rho}^1(\tau) \leq M$. Regarding the second term on the right hand side of (2.60), we have

$$\begin{aligned}
&\left|\mathbb{E}^{\boldsymbol{b},1}_{\pi\rho}\left\{\sum_{\tau=n+1}^{T_k}\Big[g^1_{\pi^1\rho^1}(\tau)-g^1_{\pi\rho}(\tau)\Big]\right\}\right| \\
&\qquad\qquad\le \sum_{i=1}^{N}\mathbb{P}^{\boldsymbol{b},1}_{\pi\rho}\{\hat{\imath}_n=i\}\left|\mathbb{E}^{\boldsymbol{b},1}_{\pi\rho}\left\{\sum_{\tau=n+1}^{T_k}\Big[g^1_{\pi^1\rho^1}(\tau)-g^1_{\pi\rho}(\tau)\Big]\,\middle|\,\hat{\imath}_n=i\right\}\right| \\
&\qquad\qquad\overset{\text{(c)}}{\le} 2Mk\,\mathbb{P}^{\boldsymbol{b},1}_{\pi\rho}\{\hat{\imath}_n\neq 1\}+\left|\mathbb{E}^{\boldsymbol{b},1}_{\pi\rho}\left\{\sum_{\tau=n+1}^{T_k}\Big[g^1_{\pi^1\rho^1}(\tau)-g^1_{\pi\rho}(\tau)\Big]\,\middle|\,\hat{\imath}_n=1\right\}\right|, \qquad (2.62)
\end{aligned}$$

where (c) follows because $g^1_{\pi^1\rho^1}(\tau)\le M$, $g^1_{\pi\rho}(\tau)\le M$, and $\mathbb{P}^{\boldsymbol{b},1}_{\pi\rho}\{\hat{\imath}_n=1\}\le 1$. By construction of the version of LATE(n) in §2.3.5, if $\hat{\imath}_n=1$, we have $p_t=p^1_t$ for all $t\ge n+1$, and by Condition 4, the future price predictions satisfy $\rho^{1,\kappa}_t(\boldsymbol{p}^{1(t)})=\rho^{\kappa}_t(\bar{\boldsymbol{p}}^{(t)})=p^1_{t+\kappa}$ for all $t\ge\tau$ and $\kappa=1,\ldots,T_k-t$, where $\boldsymbol{p}^{1(t)}=(p^1_1,\ldots,p^1_t)$ and $\boldsymbol{p}^{(t)}=(p_1,\ldots,p_t)$. By Lemma 10(ii), this implies that, for all $\tau\ge n+1$ and $t\ge\tau$, the set of valuations of type-(τ,w) customers who make a purchase in period t satisfies $\mathcal{U}^1_{\tau wt}(\boldsymbol{p}^{1(t)},\rho^1)=\mathcal{U}^1_{\tau wt}(\boldsymbol{p}^{(t)},\rho)$. As a result, we have $\mathbb{E}^{\boldsymbol{b},1}_{\pi\rho}\big\{\sum_{t=n+1}^{T_k}[g^1_{\pi^1\rho^1}(\tau)-g^1_{\pi\rho}(\tau)]\,|\,\hat{\imath}_n=1\big\}=0$. Combining this identity with (2.62), we obtain the following:

$$\begin{aligned}
\left|\mathbb{E}^{\boldsymbol{b},1}_{\pi\rho}\left\{\sum_{\tau=n+1}^{T_k}\Big[g^1_{\pi^1\rho^1}(\tau)-g^1_{\pi\rho}(\tau)\Big]\right\}\right| \le 2Mk\,\mathbb{P}^{\boldsymbol{b},1}_{\pi\rho}\{\hat{\imath}_n\neq 1\} &\overset{\text{(d)}}{\le} 2Mk\,\mathbb{P}^{\boldsymbol{b},1}_{\pi\rho}\Big\{1-b^1_{n+1}\ge\tfrac{1}{2}\Big\} \\
&\overset{\text{(e)}}{\le} 4Mk\,\mathbb{E}^{\boldsymbol{b},1}_{\pi\rho}\{1-b^1_{n+1}\} \\
&\overset{\text{(f)}}{\le} 4Mk\zeta e^{-\eta n}, \qquad (2.63)
\end{aligned}$$

where (d) follows from $\{\hat{\imath}_n\neq 1\}\subset\{b^1_{n+1}\le\frac{1}{2}\}$; (e) follows from Markov's inequality; (f) follows from a straightforward modification of Lemma 11 stating that, given Condition 6, $\mathbb{E}^{\boldsymbol{b}}_{\pi\rho}\{1-b^i_{n+1}|H_i\}\le\zeta e^{-\eta n}$ with $\eta>0$ and $\zeta>0$ as in Lemma 11. Given that $n=\lceil\frac{1}{\eta}\log k\rceil$, choose $\tilde{C}_1>0$ such that $2M\Big(1+\frac{1}{\eta}\log k\Big)+6\zeta M\le\tilde{C}_1\log k$ for all $k>1$. Combining (2.60), (2.61), and (2.63), we deduce that, for all $k = 2, 3, \ldots,$

$$\tilde{\Delta}^1_{\pi\rho}(k) \leq 2Mn + 4Mk\zeta e^{-\eta n} \overset{\text{(g)}}{\leq} 2M(1 + \tfrac{1}{\eta}) \log k + 4\zeta M \ \leq \ \tilde{C}_1 \log k, \tag{2.64}$$

where (g) follows because $n = \lceil \frac{1}{\eta} \log k \rceil$. Repeating the arguments used to derive (2.64) for all demand hypotheses $i \neq 1$, we deduce that there exists $\tilde{C}_i > 0$ such that $\Delta^i_{\pi\rho}(k) \leq \tilde{C}_i \log k$. Letting $c_6 = \max_{i \in \{1,\ldots,N\}} \{\tilde{C}_i\}$, we conclude that $\tilde{\Delta}^{\boldsymbol{b}}_{\pi\rho}(k) = \sum_{i=1}^N b^i \tilde{\Delta}^i_{\pi\rho}(k) \leq c_6 \log k$ for all $k = 2, 3, \ldots$ □

2.6 Proofs of the Results in Section 2.4

As in the preceding section, we suppress the dependence of the mathematical expressions on ρ for exposition purposes.

Proof of Corollary 4. The problem instance constructed in the proof of Theorem 6 uses a setting where all customers are myopic, and the proof follows by exactly the same arguments. □

Proof of Theorem 10. Given $\pi = \text{LATE}(n)$, we define $\mathbb{P}^{\boldsymbol{b},i}_\pi\{\cdot\} = \mathbb{P}^{\boldsymbol{b}}_\pi\{\cdot \,|\, H_i\}$, $\mathbb{E}^{\boldsymbol{b},i}_\pi\{\cdot\} = \mathbb{E}^{\boldsymbol{b}}_\pi\{\cdot \,|\, H_i\}$ for all $\boldsymbol{b} \in \mathcal{B}$ and $i \in \{1, \ldots, N\}$. Denote by $(\tilde{p}_1, \ldots, \tilde{p}_{T_k})$ the price path generated under π. Let $\bar{v} = \max\{\bar{v}^1, \ldots, \bar{v}^N\}$, and assume without loss of generality that the maximum valuations under different hypotheses ordered in the following way: $\bar{v}^1 \leq \bar{v}^2 \leq \cdots \leq \bar{v}^N$. For simplicity of notation, we suppress the index w in all variables (e.g., we write α^i_τ instead of $\alpha^i_{\tau w}$) and also let $A^i_\tau = \sum_{s=1}^\tau \alpha^i_s$ for all $\tau \in \{1, \ldots, T_k\}$ and $i \in \{1, \ldots, N\}$. Note that, letting $M = \max_{i \in \{1,\ldots,N\}} \left\{ \max\{\overline{\alpha}, 1\} \bar{p} \lambda^i \bar{v}^i \right\}$ and repeating the arguments used to derive (2.32), we have $V^i \leq Mk$ for $i \in \{1, \ldots, N\}$.

To prove that $\Delta^{\boldsymbol{b}}_\pi(k) \to 0$ as $k \to \infty$, we show that $\Delta^i_\pi(k) \to 0$ as $k \to \infty$ for all $i \in \{1, \ldots, N\}$. We complete the remainder of the proof in three steps.

Step 1: Derive a condition on $\bar{p}$. For all $i \in \{1, \ldots, N\}$, let π^i be the optimal full-information policy under H_i that induces the price path $(p^i_1, \dots, p^i_{T_k})$. We first state the following auxiliary proposition, the proof of which is deferred to Appendix 2.7.

Proposition 17. *For every prediction behavior* $\rho \in \mathfrak{P}_3$*, there exist* $k_1 > 0$*,* $c_v > 0$*,* $\delta_p > 0$ *such that the optimal full-information price path* $(p^i_1, \dots, p^i_{T_k})$ *satisfies* $\min\{\bar{v}^i, \bar{p}\} - p^i_t \leq kc_v e^{-\delta_p\sqrt{k}}$ *for all* $i \in \{1, \dots, N\}$*,* $t \in \{1, \dots, \lceil kt_1 \rceil\}$*, and* $k \geq k_1$*.*

Recalling that $\bar{v}^1 \leq \bar{v}^2 \leq \cdots \leq \bar{v}^N$, we show that $\bar{p} < \bar{v}^2$. Assume towards a contradiction that $\bar{p} \geq \bar{v}^2$. Then, by Proposition 17, we have $\min\{\bar{v}^i, \bar{p}\} - p^i_1 \leq kc_v e^{-\delta_p\sqrt{k}}$. Consequently, for any $p \in [\max_{i\in\{1,\dots,N\}}\{p^i_1\}, \bar{p}]$, the expected demand difference between H_1 and H_2 in period 1 satisfies $|d^1_1(p) - d^2_1(p)| \leq |k\alpha^1_1(\bar{v}^1 - p)^+| + |k\alpha^2_1(\bar{v}^2 - p)^+| \leq 2k\overline{\alpha}c_v e^{-\delta_p\sqrt{k}}$, which converges to zero as $k \to \infty$. This contradicts with Condition 5, and thus, we have $\bar{p} < \bar{v}^2$.

Step 2: Prove that $\Delta^i_\pi(k) \to 0$ as $k \to \infty$ for $i \in \{2, \dots, N\}$. Comparing the price paths induced by $\pi = \text{LATE}(n)$ and the optimal full-information policy π^i, we deduce that

$$\begin{aligned}
\Delta^i_\pi(k) &\leq \sum_{t=1}^{T_k} \left| \mathbb{E}^{\boldsymbol{b},i}_\pi \Big\{ R^i_t\big(\boldsymbol{p}^{i(t)}\big) - R^i_t\big(\tilde{\boldsymbol{p}}^{(t)}\big) \Big\} \right| \\
&\leq \left| \sum_{t=1}^{n} R^i_t\big(\boldsymbol{p}^{i(t)}\big) - R^i_t\big(\boldsymbol{\mathfrak{p}}^{(t)}\big) \right| + \mathbb{P}^{\boldsymbol{b},i}_\pi\{\hat{\imath}_n = i\} \left| \mathbb{E}^{\boldsymbol{b},i}_\pi \Big\{ U^i_{n+1}\big(\boldsymbol{p}^{i(n)}\big) - U^i_{n+1}\big(\boldsymbol{\mathfrak{p}}^{(n)}\big) \Big\} \right| \\
&\quad + \mathbb{P}^{\boldsymbol{b},i}_\pi\{\hat{\imath}_n \neq i\} \left| \mathbb{E}^{\boldsymbol{b},i}_\pi \left\{ \sum_{t=n+1}^{T_k} R^i_t\big(\boldsymbol{p}^{i(t)}\big) - R^i_t\big(\tilde{\boldsymbol{p}}^{(t)}\big) \right\} \right| \\
&\overset{(a)}{\leq} \left| \sum_{t=1}^{n} R^i_t\big(\boldsymbol{p}^{i(t)}\big) - R^i_t\big(\boldsymbol{\mathfrak{p}}^{(t)}\big) \right| + \left| U^i_{n+1}\big(\boldsymbol{p}^{i(n)}\big) - U^i_{n+1}\big(\boldsymbol{\mathfrak{p}}^{(n)}\big) \right| + 2Mk\,\mathbb{P}^{\boldsymbol{b},i}_\pi\{\hat{\imath}_n \neq i\},
\end{aligned} \tag{2.65}$$

where (a) follows because $\mathbb{P}^{\boldsymbol{b},i}_\pi\{\hat{\imath}_n = i\} \leq 1$ and $V^i \leq Mk$ for $i \in \{1, \dots, N\}$.

Regarding the first term on the right hand side of (2.65), we note that because $\rho \in \mathfrak{P}_3$, both $\underline{u}^i_{\tau t}(\boldsymbol{p}^{(t)})$ and $\overline{u}^i_{\tau t}(\boldsymbol{p}^{(t)})$ are piecewise Lipschitz-continuous functions. Thus, given $k\alpha^i_\tau \leq \overline{\alpha}$ and $\lambda^i \leq \bar{\lambda}$, there exists a constant $\bar{a} = \max_{i\in\{1,\dots,N\}} \Big\{ 1 + \max_{s\in\{1,\dots,T_k\}} \Big\{ \frac{(\delta^i)^s}{1-(\delta^i)^s}(1 +$

$as)\Big\}\Big\}$ such that

$$\sum_{t=1}^{n}\sum_{\tau=1}^{t}\left|k\alpha^i_\tau\lambda^i\Big(\overline{u}^i_{\tau t}(\boldsymbol{p}^{i(t)}) - \overline{u}^i_{\tau t}(\mathfrak{p}^{(t)})\Big)\right| \le n^2\overline{\alpha}\bar{\lambda}\bar{a}\Big\|\mathfrak{p}^{(n)} - \boldsymbol{p}^{i(n)}\Big\|_1, \tag{2.66a}$$

$$\sum_{t=1}^{n}\sum_{\tau=1}^{t}\left|k\alpha^i_\tau\lambda^i\Big(\underline{u}^i_{\tau t}(\boldsymbol{p}^{i(t)}) - \underline{u}^i_{\tau t}(\mathfrak{p}^{(t)})\Big)\right| \le n^2\overline{\alpha}\bar{\lambda}\bar{a}\Big\|\mathfrak{p}^{(n)} - \boldsymbol{p}^{i(n)}\Big\|_1. \tag{2.66b}$$

As a result,

$$\begin{aligned}
&\left|\sum_{t=1}^{n} R^i_t\Big(\boldsymbol{p}^{i(t)}\Big) - \sum_{t=1}^{n} R^i_t\Big(\mathfrak{p}^{(t)}\Big)\right| \\
&\overset{\text{(b)}}{=} \left|\sum_{t=1}^{n}(p^i_t - c)\sum_{\tau=1}^{t} k\alpha^i_\tau\lambda^i\Big(\overline{u}^i_{\tau t}(\boldsymbol{p}^{i(t)}) - \underline{u}^i_{\tau t}(\boldsymbol{p}^{i(t)})\Big) - \sum_{t=1}^{n}(\mathfrak{p}_t - c)\sum_{\tau=1}^{t} k\alpha^i_\tau\lambda^i\Big(\overline{u}^i_{\tau t}(\mathfrak{p}^{(t)}) - \underline{u}^i_{\tau t}(\mathfrak{p}^{(t)})\Big)\right| \\
&\overset{\text{(c)}}{\le} \left|\sum_{t=1}^{n}(p^i_t - c)\sum_{\tau=1}^{t} k\alpha^i_\tau\lambda^i\Big(\overline{u}^i_{\tau t}(\boldsymbol{p}^{i(t)}) - \underline{u}^i_{\tau t}(\boldsymbol{p}^{i(t)})\Big) - \sum_{t=1}^{n}(\mathfrak{p}_t - c)\sum_{\tau=1}^{t} k\alpha^i_\tau\lambda^i\Big(\overline{u}^i_{\tau t}(\boldsymbol{p}^{i(t)}) - \underline{u}^i_{\tau t}(\boldsymbol{p}^{i(t)})\Big)\right| \\
&+ \left|\sum_{t=1}^{n}(\mathfrak{p}_t - c)\sum_{\tau=1}^{t} k\alpha^i_\tau\lambda^i\Big(\overline{u}^i_{\tau t}(\boldsymbol{p}^{i(t)}) - \underline{u}^i_{\tau t}(\boldsymbol{p}^{i(t)})\Big) - \sum_{t=1}^{n}(\mathfrak{p}_t - c)\sum_{\tau=1}^{t} k\alpha^i_\tau\lambda^i\Big(\overline{u}^i_{\tau t}(\mathfrak{p}^{(t)}) - \underline{u}^i_{\tau t}(\mathfrak{p}^{(t)})\Big)\right| \\
&\overset{\text{(d)}}{\le} \Big\|\boldsymbol{p}^{i(n)} - \mathfrak{p}^{(n)}\Big\|_1 \sum_{\tau=1}^{n}\left|k\alpha^i_\tau\lambda^i\sum_{t=\tau}^{n}\Big(\overline{u}^i_{\tau t}(\boldsymbol{p}^{i(t)}) - \underline{u}^i_{\tau t}(\boldsymbol{p}^{i(t)})\Big)\right| \\
&\qquad + \bar{p}\sum_{t=1}^{n}\sum_{\tau=1}^{t} k\alpha^i_\tau\lambda^i\Big|\overline{u}^i_{\tau t}(\boldsymbol{p}^{i(t)}) - \overline{u}^i_{\tau t}(\mathfrak{p}^{(t)})\Big| + \bar{p}\sum_{t=1}^{n}\sum_{\tau=1}^{t} k\alpha^i_\tau\lambda^i\Big|\underline{u}^i_{\tau t}(\boldsymbol{p}^{i(t)}) - \underline{u}^i_{\tau t}(\mathfrak{p}^{(t)})\Big| \\
&\overset{\text{(e)}}{\le} \Big\|\boldsymbol{p}^{i(n)} - \mathfrak{p}^{(n)}\Big\|_1\Big(\overline{\alpha}\bar{\lambda}\bar{v}n + \bar{p}\overline{\alpha}\bar{\lambda}\bar{a}n^2 + \bar{p}\overline{\alpha}\bar{\lambda}\bar{a}n^2\Big) \overset{\text{(f)}}{\le} \tilde{C}_0 n^2\Big\|\mathfrak{p}^{(n)} - \boldsymbol{p}^{i(n)}\Big\|_1,
\end{aligned} \tag{2.67}$$

where (b) follows because $R^i_t\Big(\boldsymbol{p}^{(t)}\Big) = (p_t - c)d^i_t\Big(\boldsymbol{p}^{(t)}\Big)$ with $d^i_t(\boldsymbol{p}^{(t)}) = \sum_{\tau=1}^{t} k\alpha^i_\tau\lambda^i\Big(\overline{u}^i_{\tau t}(\boldsymbol{p}^{(t)}) - \underline{u}^i_{\tau t}(\boldsymbol{p}^{(t)})\Big)$ given by (2.3); (c) follows by the triangle inequality; (d) follows by the Cauchy-Schwarz inequality, the fact that the norm $\|\cdot\|_\ell$ is decreasing in ℓ (from which we deduce that $\|\boldsymbol{x}\boldsymbol{y}\|_1 \le \|\boldsymbol{x}\|_2\|\boldsymbol{y}\|_2 \le \|\boldsymbol{x}\|_1\|\boldsymbol{y}\|_1$), and the fact that $\mathfrak{p}_t - c \le \bar{p}$; (e) follows from (2.66) and the fact that $k\alpha^i_\tau \le \overline{\alpha}$, $\lambda^i \le \bar{\lambda}$, $\sum_{t=\tau}^{n}\Big[\overline{u}^i_{\tau t}(\boldsymbol{p}^{i(t)}) - \underline{u}^i_{\tau t}(\boldsymbol{p}^{i(t)})\Big] \le \bar{v}$; and (f) follows from letting $\tilde{C}_0 = \overline{\alpha}\bar{\lambda}\bar{v} + 2\bar{p}\overline{\alpha}\bar{\lambda}\bar{a}$.

To analyze the second term on the right hand side of (2.65), we introduce the following notation: for a continuous function $f(\cdot)$, denote by $\partial f(x)$ the subdifferential (superdifferential) of $f(\cdot)$ at x if $f(x)$ is locally convex (concave) at x. With slight abuse of notation, we let $\left|\frac{\partial}{\partial x} f(x)\right|_\star = \sup\{\partial f(x), -\partial f(x)\}$. Noting that that the value function of the optimal full-information policy is given by

$$U_t^i(\boldsymbol{p}^{(t-1)}) = \max_{p_t \le p_{t-1}} \left\{ (p_t - c) \sum_{\tau=1}^{t} k\alpha_\tau^i \lambda^i \Big[\overline{u}_{\tau t}^i(\boldsymbol{p}^{(t)}) - \underline{u}_{\tau t}^i(\boldsymbol{p}^{(t)}) \Big] + U_{t+1}^i(\boldsymbol{p}^{(t)}) \right\}, \tag{2.68}$$

for $t \in \{1, \ldots, T_k\}$, we first derive the following:

$$\left| \frac{\partial}{\partial p_s} (p_t - c) \sum_{\tau=1}^{t} k\alpha_\tau^i \lambda^i \Big(\overline{u}_{\tau t}^i(\boldsymbol{p}^{(t)}) - \underline{u}_{\tau t}^i(\boldsymbol{p}^{(t)}) \Big) \right|_\star \overset{\text{(g)}}{\le} 2\bar{p}\overline{\alpha}\bar{\lambda}\bar{a}Tk \quad \text{for } s = 1, \ldots, t-1, \tag{2.69}$$

where (g) follows because $p_t - c \le \bar{p}$, $k\alpha_\tau^i \le \overline{\alpha}$, $\lambda^i \le \bar{\lambda}$, $\left| \frac{\partial}{\partial p_s} \Big(\overline{u}_{\tau t}^i(\boldsymbol{p}^{(t)}) - \underline{u}_{\tau t}^i(\boldsymbol{p}^{(t)}) \Big) \right|_\star \le 2\bar{a}$, and $t \le kT$. Denote by $\mathcal{P}_t$ the set of optimal solutions to (2.68) and by Γ_t the set of induced dual optimal solutions corresponding to constraint $p_t \le p_{t-1}$. Then, by the Karush-Kuhn-Tucker condition of optimality, we have the following for all $p_t \in \mathcal{P}_t$ and $\gamma_t \in \Gamma_t$:

$$\gamma_t \le \left| \frac{\partial}{\partial p_t} (p_t - c) \sum_{\tau=1}^{t} k\alpha_\tau^i \lambda^i \Big(\overline{u}_{\tau t}^i(\boldsymbol{p}^{(t)}) - \underline{u}_{\tau t}^i(\boldsymbol{p}^{(t)}) \Big) \right|_\star + \left| \frac{\partial}{\partial p_t} U_{t+1}^i(\boldsymbol{p}^{(t)}) \right|_\star . \tag{2.70}$$

Next, we deduce that

$$\begin{aligned}
&\left|\frac{\partial U_t^i(\boldsymbol{p}^{(t-1)})}{\partial p_s}\right|_\star \\
&\overset{\text{(h)}}{\le} \sup_{p_t\in\mathcal{P}_t,\gamma_t\in\Gamma_t}\left\{\left|\frac{\partial}{\partial p_s}(p_t-c)\sum_{\tau=1}^{t}k\alpha_\tau^i\lambda^i\Big(\overline{u}_{\tau t}^i(\boldsymbol{p}^{(t)})-\underline{u}_{\tau t}^i(\boldsymbol{p}^{(t)})\Big)\right|_\star+\left|\frac{\partial U_{t+1}^i(\boldsymbol{p}^{(t)}))}{\partial p_s}\right|_\star\right\} \\
&\overset{\text{(i)}}{\le} \sup_{p_t\in\mathcal{P}_t}\left\{2\bar{p}\bar{\alpha}\bar{\lambda}\bar{a}Tk+\left|\frac{\partial U_{t+1}^i(\boldsymbol{p}^{(t)})}{\partial p_s}\right|_\star\right\}\overset{\text{(j)}}{\le} 2\bar{p}\bar{\alpha}\bar{\lambda}\bar{a}T^2k^2 \quad \text{for } s=1,\dots,t-2;
\end{aligned}\tag{2.71a}$$

$$\begin{aligned}
&\left|\frac{\partial U_t^i(\boldsymbol{p}^{(t-1)})}{\partial p_{t-1}}\right|_\star \\
&\overset{\text{(k)}}{\le} \sup_{p_t\in\mathcal{P}_t,\gamma_t\in\Gamma_t}\left\{\left|\frac{\partial}{\partial p_{t-1}}(p_t-c)\sum_{\tau=1}^{t}k\alpha_\tau^i\lambda^i\Big(\overline{u}_{\tau t}^i(\boldsymbol{p}^{(t)})-\underline{u}_{\tau t}^i(\boldsymbol{p}^{(t)})\Big)\right|_\star+\left|\frac{\partial U_{t+1}^i(\boldsymbol{p}^{(t)})}{\partial p_{t-1}}\right|_\star+|\gamma_t|\right\} \\
&\overset{\text{(l)}}{\le} \sup_{p_t\in\mathcal{P}_t}\left\{\left|\frac{\partial}{\partial p_{t-1}}(p_t-c)\sum_{\tau=1}^{t}k\alpha_\tau^i\lambda^i\Big(\overline{u}_{\tau t}^i(\boldsymbol{p}^{(t)})-\underline{u}_{\tau t}^i(\boldsymbol{p}^{(t)})\Big)\right|_\star+\left|\frac{\partial U_{t+1}^i(\boldsymbol{p}^{(t)})}{\partial p_{t-1}}\right|_\star\right. \\
&\qquad\left.+\left|\frac{\partial}{\partial p_t}(p_t-c)\sum_{\tau=1}^{t}k\alpha_\tau^i\lambda^i\Big(\overline{u}_{\tau t}^i(\boldsymbol{p}^{(t)})-\underline{u}_{\tau t}^i(\boldsymbol{p}^{(t)})\Big)\right|_\star+\left|\frac{\partial}{\partial p_t}U_{t+1}^i(\boldsymbol{p}^{(t)})\right|_\star\right\} \\
&\overset{\text{(m)}}{\le} \sup_{p_t\in\mathcal{P}_t}\left\{4\bar{p}\bar{\alpha}\bar{\lambda}\bar{a}T^2k^2+4\bar{p}\bar{\alpha}\bar{\lambda}\bar{a}T^2k^2+\left|\frac{\partial U_{t+1}^i(\boldsymbol{p}^{(t)})}{\partial p_t}\right|_\star\right\}\overset{\text{(n)}}{\le} 8\bar{p}\bar{\alpha}\bar{\lambda}\bar{a}T^3k^3.
\end{aligned}\tag{2.71b}$$

To prove (h) and (k), we use Proposition 6 of (96). For this purpose, note that the objective function in (2.68) is a piecewise Lipschitz-continuous, and the constraint functions (i.e., the left hand sides of the constraints $p_t - p_{t-1} \le 0$ and $\underline{p} - p_t \le 0$) are continuously differentiable in (p_t, p_{t-1}). Viewing $(p_1, \dots, p_{t-1})$ as the problem parameter, we also note that the feasible set of actions is uniformly compact. Furthermore, constraint $p_t \in [\underline{p}, p_{t-1}]$ satisfies Slater's condition, which implies the Mangasarian-Fromovitz constraint qualification (MFCQ). Based on these, we deduce from Proposition 6 in (96) that (h) and (k) hold. On top of this, (i) follows from (2.69), (j) and (n) follow from recursively applying the upper bound for at most kT times, (l) follows from (2.70), and (m) follows from (2.69) and (2.71a). Letting

$\tilde{C}_1 = 10\bar{p}\bar{\alpha}\bar{\lambda}\bar{a}T^3$, we deduce from (2.71) that

$$\left|U^i_{n+1}\Big(\boldsymbol{p}^{i(n)}\Big) - U^i_{n+1}\Big(\mathfrak{p}^{(n)}\Big)\right| \leq \tilde{C}_1 k^3 \Big\|\mathfrak{p}^{(n)} - \boldsymbol{p}^{i(n)}\Big\|_1, \tag{2.72}$$

which is an upper bound on the second term on the right hand side of (2.65).

Regarding the third term of (2.65), we deduce that

$$\begin{aligned} 2Mk\mathbb{P}^{\boldsymbol{b},i}_{\pi}\{\hat{\imath}_n \neq i\} &\overset{\text{(o)}}{\leq} 2Mk\,\mathbb{P}^{\boldsymbol{b},i}_{\pi}\left\{1 - b^i_{n+1} \geq \frac{1}{2}\right\} \\ &\overset{\text{(p)}}{\leq} 4Mk\,\mathbb{E}^{\boldsymbol{b},i}_{\pi}\{1 - b^i_{n+1}\} \\ &\overset{\text{(q)}}{\leq} 4\zeta e^{-\eta n} Mk, \end{aligned} \tag{2.73}$$

where (o) follows because $\{\hat{\imath}_n \neq i\} \subset \{b^i_{n+1} \leq \frac{1}{2}\}$, (p) follows from Markov's inequality, and (q) follows from Lemma 11.

Given that $n = \lceil \frac{2}{\eta} \log k \rceil$, let k_0 be the smallest positive integer such that $\lceil \frac{2}{\eta} \log k \rceil \leq tk$ for all $k \geq k_0$. For all $k \geq \min\{k_0, k_1\}$, we have

$$\begin{aligned} \Delta^i_{\pi}(k) &\overset{\text{(r)}}{\leq} (\tilde{C}_0 n^2 + \tilde{C}_1 k^3)\Big\|\boldsymbol{p}^{i(n)} - \mathfrak{p}^{(n)}\Big\|_1 + 4\zeta e^{-\eta n} Mk \\ &\overset{\text{(s)}}{=} (\tilde{C}_0 n^2 + \tilde{C}_1 k^3) \sum_{s=1}^{n} \left|\bar{p} - p^i_s\right| + 4\zeta e^{-\eta n} Mk \\ &\overset{\text{(t)}}{\leq} \left[\tilde{C}_0\left(1 + \frac{2}{\eta}\log k\right)^2 + \tilde{C}_1 k^3\right]\left(1 + \frac{2}{\eta}\log k\right) k c_v e^{-\delta_p \sqrt{k}} + \frac{4\zeta M}{k}, \end{aligned}$$

where (r) follows from (2.66), (2.72), and (2.73); (s) follows because $\mathfrak{p}_t - p^i_t \leq \bar{p} - p^i_t$; and (t) follows from Proposition 17 and the fact that $k \geq \max\{k_0, k_1\}$. Consequently, as $k \to \infty$,

$$\Delta^i_{\pi}(k) \to 0 \;\text{ for } i \in \{2, \ldots, N\}. \tag{2.74}$$

Step 3: Prove that $\Delta^1_{\pi}(k) \to 0$ as $k \to \infty$. Note that, if $\bar{v}^1 = \bar{v}^2$ or $\bar{p} < \bar{v}^1$, then the argument to prove that $\Delta^1_{\pi}(k) \to 0$ as $k \to \infty$ is exactly the same as the argument used for proving Step 2. Now, suppose that $\bar{v}^1 \leq \bar{p} < \bar{v}^2$. Recalling that $n = \lceil \frac{2}{\eta} \log k \rceil$, consider an auxiliary policy $\bar{\pi}$ that generates the price path $(\bar{p}_1, \ldots, \bar{p}_{T_k})$ where $\bar{p}_s = \mathfrak{p}_s$ for $s \in \{1, \ldots, n\}$ and $\bar{p}_s = p_s^{\hat{\imath}_n}$ with $\hat{\imath}_n = \arg\max_{i \in \{1,\ldots,N\}} \{b_{n+1}^i\}$ for $s \in \{n+1, \ldots, T_k\}$. Given that $\rho \in \mathfrak{P}_3$, we let $n_2 = \lceil t_1 k \rceil$ and recall that k_0 is the smallest integer such that $t_1 k \geq 1 + \frac{2}{\eta} \log k$ for all $k \geq k_0$. Because $\bar{\pi}$ and π both employ the price path $\mathfrak{p}^{(n)}$ in the first n periods, we deduce that

$$\mathbb{P}_{\pi}^{\boldsymbol{b},1}\{\hat{\imath}_n = i\} = \mathbb{P}_{\bar{\pi}}^{\boldsymbol{b},1}\{\hat{\imath}_n = i\} \quad \text{for } i \in \{1, \ldots, N\}. \tag{2.75}$$

If $\pi = \text{LATE}(n)$ correctly identifies that the underlying demand hypothesis is H_1, then upon charging the price path $\mathfrak{p}^{(n)}$ in the first n periods, π uses the optimal solution under H_1. However, $\bar{\pi}$ uses a feasible (but not necessarily optimal) solution in this case. Thus,

$$\mathbb{E}_{\pi}^{\boldsymbol{b},1}\left\{ \sum_{t=1}^{T_k} R_t^1\left(\tilde{\boldsymbol{p}}^{(t)}\right) \middle| \hat{\imath}_n = 1 \right\} \geq \mathbb{E}_{\bar{\pi}}^{\boldsymbol{b},1}\left\{ \sum_{t=1}^{T_k} R_t^1\left(\bar{\boldsymbol{p}}^{(t)}\right) \middle| \hat{\imath}_n = 1 \right\}. \tag{2.76}$$

If π does not correctly identify the underlying demand hypothesis, then we note that

$$\mathbb{E}_{\pi}^{\boldsymbol{b},1}\left\{ \sum_{t=1}^{T_k} R_t^1\left(\tilde{\boldsymbol{p}}^{(t)}\right) \middle| \hat{\imath}_n \neq 1 \right\} \geq \mathbb{E}_{\bar{\pi}}^{\boldsymbol{b},1}\left\{ \sum_{t=1}^{T_k} R_t^1\left(\bar{\boldsymbol{p}}^{(t)}\right) \middle| \hat{\imath}_n \neq 1 \right\} - Mk. \tag{2.77}$$

because $V^i \le Mk$. Consequently, we deduce that

$$
\begin{aligned}
\Delta^1_\pi(k) &= V^1 - \sum_{i=1}^{N} \mathbb{P}^{\boldsymbol{b},1}_{\pi}\{\hat{\imath}_n = i\}\, \mathbb{E}^{\boldsymbol{b},1}_{\pi}\left\{ \sum_{t=1}^{T_k} R^1_t\Big(\tilde{\boldsymbol{p}}^{(t)}\Big) \,\middle|\, \hat{\imath}_n = i \right\} \\
&\overset{\text{(u)}}{\le} V^1 - \sum_{i=1}^{N} \mathbb{P}^{\boldsymbol{b},1}_{\bar{\pi}}\{\hat{\imath}_n = i\}\, \mathbb{E}^{\boldsymbol{b},1}_{\bar{\pi}}\left\{ \sum_{t=1}^{T_k} R^1_t\Big(\bar{\boldsymbol{p}}^{(t)}\Big) \,\middle|\, \hat{\imath}_n = i \right\} + Mk\, \mathbb{P}^{\boldsymbol{b},1}_{\bar{\pi}}\{\hat{\imath}_n \ne 1\} \\
&\overset{\text{(v)}}{\le} \mathbb{P}^{\boldsymbol{b},1}_{\bar{\pi}}\{\hat{\imath}_n = 1\} \left| \mathbb{E}^{\boldsymbol{b},1}_{\bar{\pi}}\left\{ \sum_{t=1}^{n_2} R^1_t\Big(\boldsymbol{p}^{1(t)}\Big) - \sum_{t=1}^{n_2} R^1_t\Big(\bar{\boldsymbol{p}}^{(t)}\Big) \,\middle|\, \hat{\imath}_n = 1 \right\} \right| \\
&\quad + \mathbb{P}^{\boldsymbol{b},1}_{\bar{\pi}}\{\hat{\imath}_n = 1\} \left| \mathbb{E}^{\boldsymbol{b},1}_{\bar{\pi}}\left\{ \sum_{t=n_2+1}^{T_k} R^1_t\Big(\boldsymbol{p}^{1(t)}\Big) - \sum_{t=1}^{n_2} R^1_t\Big(\bar{\boldsymbol{p}}^{(t)}\Big) \,\middle|\, \hat{\imath}_n = 1 \right\} \right| \\
&\quad + \mathbb{P}^{\boldsymbol{b},1}_{\bar{\pi}}\{\hat{\imath}_n \ne 1\} \left| \mathbb{E}^{\boldsymbol{b},1}_{\bar{\pi}}\left\{ \sum_{t=1}^{T_k} R^1_t\Big(\boldsymbol{p}^{1(t)}\Big) - R^1_t\Big(\bar{\boldsymbol{p}}^{(t)}\Big) \,\middle|\, \hat{\imath}_n \ne 1 \right\} \right| + Mk\, \mathbb{P}^{\boldsymbol{b},1}_{\bar{\pi}}\{\hat{\imath}_n \ne 1\} \\
&\overset{\text{(w)}}{\le} \sum_{t=1}^{n_2} \left| R^1_t\Big(\boldsymbol{p}^{1(t)}\Big) - R^1_t\Big(\bar{\boldsymbol{p}}^{1(t)}\Big) \right| + \sum_{t=n_2+1}^{T_k} \left| R^1_t\Big(\boldsymbol{p}^{1(t)}\Big) - R^1_t\Big(\bar{\boldsymbol{p}}^{1(t)}\Big) \right| \\
&\quad + 2Mk\, \mathbb{P}^{\boldsymbol{b},1}_{\bar{\pi}}\{\hat{\imath}_n \ne 1\} + Mk\, \mathbb{P}^{\boldsymbol{b},1}_{\bar{\pi}}\{\hat{\imath}_n \ne 1\} \\
&= \sum_{t=1}^{n_2} \left| R^1_t\Big(\boldsymbol{p}^{1(t)}\Big) - R^1_t\Big(\bar{\boldsymbol{p}}^{1(t)}\Big) \right| + \sum_{t=n_2+1}^{T_k} \left| R^1_t\Big(\boldsymbol{p}^{1(t)}\Big) - R^1_t\Big(\bar{\boldsymbol{p}}^{1(t)}\Big) \right| + 3Mk\, \mathbb{P}^{\boldsymbol{b},1}_{\bar{\pi}}\{\hat{\imath}_{n_2} \ne 1\},
\end{aligned}
\tag{2.78}
$$

where (u) follows from (2.75), (2.76), and (2.77); (v) follows because $V^1 = \sum_{t=1}^{T_k} R^1_t(\boldsymbol{p}^{1(t)})$; and (w) follows because $\mathbb{P}^{\boldsymbol{b},1}_{\bar{\pi}}\{\hat{\imath}_n = 1\} \le 1$ and $V^i \le Mk$.

To analyze the first term on the right hand side of (2.78), we let k_2 be the smallest integer such that $kc_v e^{-\delta_p\sqrt{k}} \le \bar{p} - \bar{v}^1$ for all $k \ge k_2$. Thus, for $k \ge \max\{k_0, k_1, k_2\}$ and $s \in \{1, \ldots, n_2\}$, we have $\mathfrak{p}_s \ge p^2_s \ge \bar{p} - kc_v e^{-\delta_p\sqrt{k}} > \bar{v}^1$. This implies that $\overline{u}^1_{\tau s}(\mathfrak{p}^{(s)}) = \bar{v}^1$ and $\underline{u}^1_{\tau s}(\mathfrak{p}^{(s)}) = \bar{v}^1$ for all $\tau \in \{1, \ldots, n_2\}$ and $s \in \{\tau, \ldots, n_2\}$, which further implies that

$$
R^1_s\Big(\bar{\boldsymbol{p}}^{1(s)}\Big) = R^1_s\Big(\boldsymbol{\mathfrak{p}}^{(s)}\Big) = R^1_s\Big(\bar{\boldsymbol{v}}^{1(s)}\Big) = 0 \ \text{ for } s \in \{1, \ldots, n_2\}.
$$

Thus, by the arguments used to derive (2.67), we deduce that there exists $\tilde{C}_0 > 0$ such that

$$\sum_{t=1}^{n_2}\Big|R_t^1\big(\boldsymbol{p}^{1(t)}\big) - R_t^1\big(\bar{\boldsymbol{p}}^{1(t)}\big)\Big| = \sum_{t=1}^{n_2}\Big|R_t^1\big(\boldsymbol{p}^{1(t)}\big) - R_t^1\big(\bar{\boldsymbol{v}}^{1(t)}\big)\Big| \le \tilde{C}_0 n_2^2 \Big\|\boldsymbol{p}^{1(n)} - \bar{\boldsymbol{v}}^{1(n)}\Big\|_1. \tag{2.79}$$

Regarding the second term on the right hand side of (2.78), we deduce from Lemma 10 that the following holds for all $\tau \in \{1, \ldots, T_k\}$ and $t \in \{\tau, \ldots, T_k\}$:

$$\overline{u}_{\tau t}^1(\boldsymbol{p}^{(t)}) = \min_{l \in \{\tau, \ldots, t\}} \left\{ p_l + \max_{s \in \mathcal{T}_{\tau l}} \left\{ \frac{(\delta^i)^{s-l}}{1 - (\delta^i)^{s-l}} \Big(p_s - \rho_s^{s-l}(\boldsymbol{p}^s) \Big) \right\} \right\} \text{ with } \mathcal{T}_{\tau l} = \{l+1, \ldots, T_k\}. \tag{2.80}$$

Let

$$\tilde{u}_{\tau t}^1(\boldsymbol{p}^{(t)}) = \min_{l \in \{\max\{n_2, \tau\}, \ldots, t\}} \left\{ p_l + \max_{s \in \mathcal{T}_{\tau l}} \left\{ \frac{(\delta^i)^{s-l}}{1 - (\delta^i)^{s-l}} \Big(p_l - \rho_s^{s-l}(\boldsymbol{p}^s) \Big) \right\} \right\} \text{ with } \mathcal{T}_{\tau l} = \{l+1, \ldots, T_k\}. \tag{2.81}$$

By (2.80) and (2.81), $\overline{u}_{\tau t}^1(\boldsymbol{p}^{(t)}) \le \tilde{u}_{\tau t}^1(\boldsymbol{p}^{(t)})$. Moreover, if $\tilde{u}_{\tau t}^1(\boldsymbol{p}^{1(t)}) > \overline{u}_{\tau t}^1(\boldsymbol{p}^{1(t)})$, this implies that

$$p_{n_2}^1 \le \min_{l \in \{\tau, \ldots, \max\{n_2, \tau\}\}} \left\{ p_l + \max_{s \in \mathcal{T}_{\tau l}} \left\{ \frac{(\delta^i)^{s-l}}{1 - (\delta^i)^{s-l}} \Big(p_s - \rho_s^{s-l}(\boldsymbol{p}^s) \Big) \right\} \right\} = \overline{u}_{\tau t}^1(\boldsymbol{p}^{1(t)}) < \tilde{u}_{\tau t}^1(\boldsymbol{p}^{1(t)}) \le \bar{v}^1.$$

Thus,

$$0 \le \tilde{u}_{\tau t}^1(\boldsymbol{p}^{1(t)}) - \overline{u}_{\tau t}^1(\boldsymbol{p}^{1(t)}) \le \bar{v}^1 - p_{n_2}^1. \tag{2.82}$$

In addition, under $\bar{\pi}$, we have

$$\bar{p}_s^1 = p_s^1 \text{ for } s \in \{n+1, \ldots, T_k\}.$$

Because $\rho \in \mathfrak{P}_3$, we have $\rho \in \mathfrak{P}_2(a,r)$, from which we deduce that $\left|\frac{\partial}{\partial p_s}\rho_l^\kappa(\boldsymbol{p}^l)\right| \le a\left(\frac{s}{l}\right)^{1+r}$ for all $s \le n$, $l \ge \tau \ge n_2+1$, and $\kappa \in \{1,\ldots,T_k-1\}$. Thus,

$$\left|\frac{\partial}{\partial p_s}\rho_l^\kappa(\boldsymbol{p}^l)\right| \le a n^{1+r} n_2^{-(1+r)} \quad \text{for } s \le n \text{ and } l \ge n_2+1. \tag{2.83}$$

By (2.57) and (2.83), there exists a constant $\bar{a} \ge 1 + \max_{\kappa \in \{1,\ldots,T_k\}} \left\{\frac{(\delta^1)^\kappa}{1-(\delta^1)^\kappa}(1+\kappa a)\right\}$ such that

$$\left[\tilde{u}_{\tau t}^1(\bar{\boldsymbol{p}}^{1(t)}) - \tilde{u}_{\tau t}^1(\boldsymbol{p}^{1(t)})\right]^+ \le \bar{a} n^{1+r} n_2^{-(1+r)} \sum_{s=1}^{n} \left|p_s^1 - \bar{p}_s\right|. \tag{2.84}$$

Therefore, we deduce that

$$\begin{aligned}
\sum_{t=n_2+1}^{T_k} \left|R_t^1\left(\boldsymbol{p}^{1(t)}\right) - R_t^1\left(\bar{\boldsymbol{p}}^{1(t)}\right)\right| &\overset{\text{(x)}}{\le} \sum_{\tau=1}^{T_k} \alpha_\tau^1 \sum_{t=\max\{n_2+1,\tau\}}^{T_k} \bar{p}k\bar{\lambda}\left[\bar{u}_{\tau t}^1(\bar{\boldsymbol{p}}^{1(t)}) - \bar{u}_{\tau t}^1(\boldsymbol{p}^{1(t)})\right]^+ \\
&\overset{\text{(y)}}{\le} \sum_{\tau=1}^{T_k} \alpha_\tau^1 \sum_{t=\max\{n_2+1,\tau\}}^{T_k} \bar{p}k\bar{\lambda}\left[\tilde{u}_{\tau t}^1(\bar{\boldsymbol{p}}^{1(t)}) - \tilde{u}_{\tau t}^1(\boldsymbol{p}^{1(t)})\right]^+ \\
&\qquad + \sum_{\tau=1}^{T_k} \alpha_\tau^1 \sum_{t=\max\{n_2+1,\tau\}}^{T_k} \bar{p}k\bar{\lambda}\left[\tilde{u}_{\tau t}^1(\boldsymbol{p}^{1(t)}) - \bar{u}_{\tau t}^1(\boldsymbol{p}^{1(t)})\right]^+ \\
&\overset{\text{(z)}}{\le} \sum_{\tau=1}^{T_k} \alpha_\tau^1 \sum_{t=\max\{n_2+1,\tau\}}^{T_k} k\bar{p}\bar{\lambda} \sum_{s=1}^{n} \bar{a} n^{1+r} n_2^{-(1+r)} \left|\bar{p}_s^1 - p_s^1\right| \\
&\qquad + \sum_{\tau=1}^{T_k} \alpha_\tau^1 \sum_{t=\max\{n_2+1,\tau\}}^{T_k} k\bar{p}\bar{\lambda}|\bar{v}^1 - p_{n_2}^1| \\
&\overset{\text{(aa)}}{\le} \bar{p}\bar{\lambda}Tk^2\left(\bar{a} n^{1+r} n_2^{-(1+r)} \sum_{s=1}^{n} |p_s^1 - \bar{p}_s|\right) + \bar{p}\bar{\lambda}Tk^2\left|\bar{v}^1 - p_{n_2}^1\right| \\
&\overset{\text{(bb)}}{\le} 2\bar{p}^2\bar{\lambda}\bar{a} n^{2+r} n_2^{-(1+r)} Tk^2 + \bar{p}\bar{\lambda}Tk^2\left|\bar{v}^1 - p_{n_2}^1\right| \\
&\overset{\text{(cc)}}{=} \tilde{C}_1 n^{2+r} n_2^{-(1+r)} k^2 + \tilde{C}_2 k^2 \left|\bar{v}^1 - p_{n_2}^1\right|,
\end{aligned} \tag{2.85}$$

where (x) follows from a natural upper bound on the profit loss caused by the customers who delay their purchase under $\bar{\pi}$, (y) follows from adding and subtracting the term $\tilde{u}_{\tau t}^1(\boldsymbol{p}^{1(t)})$

and using the fact that the mapping $x \mapsto [x]^+ = \max\{x, 0\}$ is non-decreasing, (z) follows from (2.82) and (2.84), (aa) follows because $\sum_{\tau=1}^{T_k} \alpha^1_\tau \le 1$ and $T_k - \max\{n_2 + 1, \tau\} \le T_k$, (bb) follows because $|p^1_s - \bar{p}_s| \le 2\bar{p}$, and (cc) follows by setting $\tilde{C}_1 = 2\bar{p}^2\bar{\lambda}\bar{a}T$ and $\tilde{C}_2 = \bar{p}\bar{\lambda}T$.

Regarding the third term on the right hand side of (2.78), we apply the argument used to derive (2.73) to obtain the following:

$$Mk\,\mathbb{P}^{\boldsymbol{b},i}_{\pi}\{\hat{\imath}_{n_2} \neq 1\} \le 2Mk\,\zeta e^{-\eta n}. \tag{2.86}$$

Given that $n = \lceil \frac{2}{\eta} \log k \rceil$ and $n_2 = \lceil t_1 k \rceil$, we deduce the following for all $k \ge \max\{k_0, k_1, k_2\}$:

$$\begin{aligned}
\Delta^1_\pi(k) &\overset{\text{(dd)}}{\le} \tilde{C}_0 n_2^2 \left\| \boldsymbol{p}^{1(n_2)} - \bar{\boldsymbol{v}}^1 \right\|_1 + \tilde{C}_1 k^2 n^{2+r} n_2^{-(1+r)} + \tilde{C}_2 k^2 |\bar{v}^1 - p^1_{n_2}| + 6Mk\,\zeta e^{-\eta n} \\
&\overset{\text{(ee)}}{\le} \tilde{C}_0 (1 + t_1 k)^3 k c_v e^{-\delta_p \sqrt{k}} + \tilde{C}_1 \left(1 + \frac{2}{\eta} \log k\right)^{2+r} (t_1 k)^{-(1+r)} k^2 \\
&\quad + \tilde{C}_2 k^3 c_v e^{-\delta_p \sqrt{k}} + \frac{6\zeta M}{k},
\end{aligned} \tag{2.87}$$

where (dd) follows from (2.78), (2.79), (2.85) and (2.86); and (ee) follows by noting that $n = \lceil \frac{2}{\eta} \log k \rceil$ and $n_2 = \lceil t_1 k \rceil$, and invoking Proposition 17, which implies that $\bar{v}_1 - p^1_\ell \le k c_v e^{-\delta_p \sqrt{k}}$ for all $\ell \in \{1, \ldots, n_2\}$. Because $r > 1$, each term on the right hand side of (2.87) converges to 0 as $k \to \infty$. Thus,

$$\Delta^1_\pi(k) \to 0 \tag{2.88}$$

as $k \to \infty$. From (2.74) and (2.88), we conclude that $\Delta^{\boldsymbol{b}}_\pi(k) = \sum_{i=1}^N b^i \Delta^i_\pi(k) \to 0$ as $k \to \infty$. □

Proof of Lemma 12. As before, for simplicity of notation, we suppress the index w in all variables (e.g., we write α^i_τ instead of $\alpha^i_{\tau w}$) and also let $A^i_\tau = \sum_{s=1}^{\tau} \alpha^i_s$ for all $\tau \in \{1, \ldots, T_k\}$ and $i \in \{1, \ldots, N\}$. Consider the following problem instance $\boldsymbol{\theta}$: let $\alpha^i_\tau = \frac{1}{T_k}$ such that $A^i_\tau = \frac{\tau}{T_k}$ for all $\tau = 1, \ldots, T_k$. Letting $\underline{\alpha} = \frac{1}{2T}$, we also have $k\alpha_\tau \ge \underline{\alpha}$ for all $\tau = 1, \ldots, T_k$.

Choose t_1, t_2 such that $0 < t_1 < t_2 < T$. To show $\rho \in \mathfrak{P}_3$, we verify the conditions characterizing $\mathfrak{P}_3$.

(1) Recall that $\rho_t^{\kappa}(\boldsymbol{p}^{(t)}) = \Big(\frac{1-\delta^{T_k-t-\kappa}}{1-\delta^{T_k-t}}\Big)p_t$ for $\boldsymbol{p}^{(t)} \in [\underline{p}, \bar{p}]^t$ and $\kappa \in \{1, 2, \dots, T_k - t\}$. To prove that $\rho \in \mathfrak{P}_2(a, r)$ for some $a > 0$ and $r > 1$, choose $a = 1$ and $r = 2$ such that $\Big|\frac{\partial}{\partial p_s}\rho_t^{\kappa}(\boldsymbol{p}^{(t)})\Big| = 0 \le a\kappa\Big(\frac{s}{t}\Big)^{1+r}$ for $s = 1, \dots, t-1$, and $\Big|\frac{\partial}{\partial p_t}\rho_t^{\kappa}(\boldsymbol{p}^{(t)})\Big| \le \frac{1-\delta^{T_k-t-\kappa}}{1-\delta^{T_k-t}} \le 1 \le a\kappa\Big(\frac{t}{t}\Big)^{1+r}$. Thus,

$$\rho \in \mathfrak{P}_2(a, r). \tag{2.89}$$

(2) Because $\rho_t^{T_k-t}(\boldsymbol{p}^{(t)}) = \Big(\frac{1-\delta^0}{1-\delta^{T_k-t}}\Big)p_t = 0 \le \min_{i\in\{1,\dots,N\}}\{p_t^i\}$, we deduce that

$$\rho_t^{T_k-t}(\boldsymbol{p}^{(t)}) \le \min_{i\in\{1,\dots,N\}}\{p_{T_k}^i\}. \tag{2.90}$$

(3) We know that $d_t^i(\boldsymbol{p}^{(t)}) = \sum_{\tau=1}^{t} k\alpha_\tau^i \lambda^i\Big(\overline{u}_{\tau t}^i(\boldsymbol{p}^{(t)}) - \underline{u}_{\tau t}^i(\boldsymbol{p}^{(t)})\Big)$ for all $t \ge \tau$. By Lemma 10, we have

$$\underline{u}_{\tau(\tau-1)}^i(\boldsymbol{p}^{(t)}) = \overline{u}_{\tau\tau}^i(\boldsymbol{p}^{(t)}) = \bar{v}^i, \tag{2.91a}$$

$$\begin{aligned} \overline{u}_{\tau(t+1)}^i(\boldsymbol{p}^{(t)}) &= \underline{u}_{\tau t}^i(\boldsymbol{p}^{(t)}) \\ &= \min\left\{\overline{u}_{\tau t}^i(\boldsymbol{p}^{(t)}), p_t + \max_{\kappa\in\{1,\dots,T_k-t\}}\left\{\frac{(\delta^i)^\kappa}{1-(\delta^i)^\kappa}\Big(p_t - \rho_t^\kappa\big(\boldsymbol{p}^{(t)}\big)\Big)\right\}\right\} \\ &\qquad \text{for } t \in \{\tau, \dots, T_k - 1\}, \end{aligned} \tag{2.91b}$$

$$\underline{u}_{\tau T_k}^i(\boldsymbol{p}^{(T_k)}) = p_{T_k}. \tag{2.91c}$$

Note that, for all $t = 1, \dots, T_k - 1$,

$$\begin{aligned}
p_t + \frac{(\delta^i)^{T_k-t}}{1-(\delta^i)^{T_k-t}}\Big(p_t - \rho_t^{T_k-t}\big(\boldsymbol{p}^{(t)}\big)\Big) &\le p_t + \max_{\kappa\in\{1,\dots,T_k-t\}} \left\{\frac{(\delta^i)^\kappa}{1-(\delta^i)^\kappa}\Big(p_t - \rho_t^\kappa\big(\boldsymbol{p}^{(t)}\big)\Big)\right\} \\
&\overset{\text{(a)}}{=} \max_{\kappa\in\{1,\dots,T_k-t\}} \left\{p_t + \frac{(\delta^i)^\kappa \delta^{T_k-t-\kappa}(1-\delta^\kappa)}{\Big(1-(\delta^i)^\kappa\Big)\Big(1-\delta^{T_k-t}\Big)} p_t\right\} \\
&\overset{\text{(b)}}{\le} \max_{\kappa\in\{1,\dots,T_k-t\}} \left\{p_t + \frac{(\delta^i)^\kappa (\delta^i)^{T_k-t-\kappa}(1-(\delta^i)^\kappa)}{\Big(1-(\delta^i)^\kappa\Big)\Big(1-(\delta^i)^{T_k-t}\Big)} p_t\right\} \\
&= p_t + \frac{(\delta^i)^{T_k-t}}{1-(\delta^i)^{T_k-t}}\, p_t \\
&\overset{\text{(c)}}{=} p_t + \frac{(\delta^i)^{T_k-t}}{1-(\delta^i)^{T_k-t}}\Big(p_t - \rho_t^{T_k-t}\big(\boldsymbol{p}^{(t)}\big)\Big),
\end{aligned} \tag{2.92}$$

where (a) and (c) follow because $\rho_t^\kappa(\boldsymbol{p}^{(t)}) = \left(\frac{1-\delta^{T_k-t-\kappa}}{1-\delta^{T_k-t}}\right)p_t$, and (b) follows because $\delta \le \delta^i$ and the mapping $x \mapsto \frac{x^{T_k-t-\kappa}(1-x^\kappa)}{1-x^{T_k-t}}$ is monotone increasing in $(0,1)$. For simplicity of notation, let

$$\xi_t^i = \frac{1}{1-(\delta^i)^{T_k-t}} \quad \text{for } t = 1, \dots, T_k - 1, \tag{2.93a}$$

$$\xi_{T_k}^i = 1. \tag{2.93b}$$

From (2.91a)-(2.91c), (2.92) and (2.93a)-(2.93b), we obtain the following:

$$\overline{u}_{\tau\tau}^i(\boldsymbol{p}^{(\tau)}) = \underline{u}_{\tau(\tau-1)}^i(\boldsymbol{p}^{(\tau-1)}) = \bar{v}^i, \tag{2.94a}$$

$$\overline{u}_{\tau(t+1)}^i(\boldsymbol{p}^{(t+1)}) = \underline{u}_{\tau t}^i(\boldsymbol{p}^{(t)}) = \min\{\bar{v}^i, \xi_\tau^i p_\tau, \dots, \xi_t^i p_t\} \quad \text{for } t \in \{\tau, \dots, T_k - 1\}, \tag{2.94b}$$

$$\underline{u}_{\tau T_k}^i(\boldsymbol{p}^{(T_k)}) = p_{T_k}. \tag{2.94c}$$

Now, for any $i \in \{1, \dots, N\}$, denote by $\mathcal{C} = \{\boldsymbol{p} \in [0, \bar{v}^i]^{T_k} : p_1 \ge \dots p_{T_k}\}$ the set of feasible price paths. In addition, let

$$\bar{V}(\boldsymbol{p}) = \sum_{t=1}^{T_k} p_t \Big[kA_{t-1}^i \lambda^i p_{t-1} + k\alpha_t^i \lambda^i \bar{v}^i - kA_t^i \lambda^i p_t \Big], \tag{2.95a}$$

$$V(\boldsymbol{p}, \delta^i) = \sum_{t=1}^{T_k} p_t \sum_{\tau=1}^{t} k\alpha_\tau^i \lambda^i \Big[\overline{u}_{\tau t}^i(\boldsymbol{p}^{(t)}, \delta^i) - \underline{u}_{\tau t}^i(\boldsymbol{p}^{(t)}, \delta^i) \Big]. \tag{2.95b}$$

Denote by $(\bar{p}_1^i, \ldots, \bar{p}_{T_k}^i) = \arg\max_{\boldsymbol{p}\in\mathcal{C}} \Big\{ \bar{V}(\boldsymbol{p}) \Big\}$ be the price path that maximizes (2.95a). Similarly, for any $i \in \{1, \ldots, N\}$, we let $(p_1^i, \ldots, p_{T_k}^i) \in \arg\max_{\boldsymbol{p}\in\mathcal{C}} \Big\{ V(\boldsymbol{p}, \delta^i) \Big\}$. We deduce from (2.94a)-(2.94c) that $\lim_{\delta^i \to 0} \Big\{ \underline{u}_{\tau t}^i(\boldsymbol{p}^{(t)}, \delta^i) \Big\} = p_t$, $\lim_{\delta^i \to 0} \Big\{ \overline{u}_{\tau t}^i(\boldsymbol{p}^{(t)}, \delta^i) \Big\} = p_{t-1}$, and

$$\lim_{\delta^i \to 0} \Big\{ V(\boldsymbol{p}, \delta^i) \Big\} = V(\boldsymbol{p}, 0) = \bar{V}(\boldsymbol{p}) \quad \text{for } \boldsymbol{p} \in \mathcal{C}. \tag{2.96}$$

Next, we show that $V(\boldsymbol{p}, \delta^i)$ is Lipschitz-continuous in $\boldsymbol{p}$ with a Lipschitz constant of $5k^2 T \bar{v}^i$ for all $\delta^i \in [0, 0.5]$. By (2.94a)-(2.94c), we have the following for all $\delta^i \in [0, 0.5]$, $\tau \in \{1, \ldots, T_k\}$, and $t \in \{\tau, \ldots T_k\}$:

$$\Big| \overline{u}_{\tau t}^i(\boldsymbol{p}^{(t)}, \delta^i) - \underline{u}_{\tau t}^i(\boldsymbol{p}^{(t)}, \delta^i) \Big| \leq 2\bar{v}^i \tag{2.97a}$$

$$\Big| \overline{u}_{\tau t}^i(\boldsymbol{p}^{1(t)}, \delta^i) - \overline{u}_{\tau t}^i(\boldsymbol{p}^{2(t)}, \delta^i) \Big| \leq 2 \sum_{l=1}^{t} \Big| p_l^1 - p_l^2 \Big| \tag{2.97b}$$

$$\Big| \underline{u}_{\tau t}^i(\boldsymbol{p}^{1(t)}, \delta^i) - \underline{u}_{\tau t}^i(\boldsymbol{p}^{2(t)}, \delta^i) \Big| \leq 2 \sum_{l=1}^{t} \Big| p_l^1 - p_l^2 \Big| \tag{2.97c}$$

Consequetly, for all $\boldsymbol{p}^1, \boldsymbol{p}^2 \in \mathcal{C}$, and $\delta^i \in [0, 0.5]$,

$$\begin{aligned}
&\Big|V(\boldsymbol{p}^1,\delta^i) - V(\boldsymbol{p}^2,\delta^i)\Big| \\
&\overset{\text{(d)}}{\leq} \Bigg| \sum_{t=1}^{T_k} p_t^1 \sum_{\tau=1}^{t} k\alpha_\tau^i \lambda^i \Big[\overline{u}_{\tau t}^i(\boldsymbol{p}^{1(t)},\delta^i) - \underline{u}_{\tau t}^i(\boldsymbol{p}^{1(t)},\delta^i)\Big] \\
&\quad - \sum_{t=1}^{T_k} p_t^2 \sum_{\tau=1}^{t} k\alpha_\tau^i \lambda^i \Big[\overline{u}_{\tau t}^i(\boldsymbol{p}^{2(t)},\delta^i) - \underline{u}_{\tau t}^i(\boldsymbol{p}^{2(t)},\delta^i)\Big]\Bigg| \\
&\overset{\text{(e)}}{\leq} \Bigg| \sum_{t=1}^{T_k} p_t^1 \sum_{\tau=1}^{t} k\alpha_\tau^i \lambda^i \Big[\overline{u}_{\tau t}^i(\boldsymbol{p}^{1(t)},\delta^i) - \overline{u}_{\tau t}^i(\boldsymbol{p}^{2(t)},\delta^i)\Big] + \Big[\underline{u}_{\tau t}^i(\boldsymbol{p}^{2(t)},\delta^i) - \underline{u}_{\tau t}^i(\boldsymbol{p}^{1(t)},\delta^i)\Big]\Bigg| \\
&\quad + \Bigg| \sum_{t=1}^{T_k} (p_t^1 - p_t^2) \sum_{\tau=1}^{t} k\alpha_\tau^i \lambda^i \Big[\overline{u}_{\tau t}^i(\boldsymbol{p}^{2(t)},\delta^i) - \underline{u}_{\tau t}^i(\boldsymbol{p}^{2(t)},\delta^i)\Big]\Bigg| \\
&\overset{\text{(f)}}{\leq} 5kT\bar{v}^i \sum_{t=1}^{T_k} \Big|p_t^1 - p_t^2\Big|,
\end{aligned} \tag{2.98}$$

where (d) follows by (2.95a)-(2.95b), (e) follows by the triangle inequality, and (f) follows by (2.97a)-(2.97c). From (2.96) and (2.97a)-(2.97c), we conclude that there exists a non-increasing sequence $\{\delta_m^i,\, m = 1, 2, \ldots\}$ with $\lim_{m\to\infty}\{\delta_m^i\} = 0$ such that $V(\boldsymbol{p}, \delta_m^i)$ converges uniformly to $\bar{V}(\boldsymbol{p})$. Letting $\boldsymbol{p}^i(\delta_m^i) \in \arg\max_{\boldsymbol{p}\in\mathcal{C}} \Big\{V(\boldsymbol{p}, \delta_m^i)\Big\}$, we have

$$\lim_{m\to\infty} \Big\{\boldsymbol{p}^i(\delta_m^i)\Big\} = \bar{\boldsymbol{p}}^i, \tag{2.99a}$$

$$\lim_{m\to\infty} \Big\{\xi_t^i(\delta_m^i)\Big\} = 1 \quad \text{for } t \in \{1, \ldots, T_k\}. \tag{2.99b}$$

With $\sigma_k = \vartheta k e^{-\sqrt{k}}$, we deduce from (2.99a)-(2.99b) that there exists $M \in \mathbb{N}$ such that for all $m \geq M$,

$$p_t^i - \bar{p}_t^i \geq -\sigma_k \quad \text{for } i \in \{1, \ldots, N\} \text{ and } t \in \{1, \ldots, T_k\}. \tag{2.100}$$

As a result, the prediction behavior ρ is in $\mathfrak{P}_3$. $\square$

2.7 Proofs of Auxiliary Results

Proof of Proposition 17. Denote by $(\bar{p}^i_1, \ldots, \bar{p}^i_{T_k})$ the price path characterized by the Bellman equation (2.16). For $\rho \in \mathfrak{P}_3$, there exists $t_0 \in [kt_1, kt_2]$ such that $p^i_{t_0} - \bar{p}^i_{t_0} \geq -\vartheta k e^{-\sqrt{k}}$. Thus,

$$\min\{\bar{v}^i, \bar{p}\} - p^i_{t_0} \leq \min\{\bar{v}^i, \bar{p}\} - \bar{p}^i_{t_0} + \vartheta k e^{-\sqrt{k}}. \tag{2.101}$$

We complete the remainder of the proof in the following two steps.

Step 1: Derive an upper bound on $\min\{\bar{v}^i, \bar{p}\} - \bar{p}^i_\ell$ for $\ell = 1, \ldots, T_k$. Let $\beta^i_\ell = \frac{1}{\lambda^i A^i_\ell}$. For the optimal solution to the Bellman equation (2.16), we state and prove the following lemma.

Lemma 13. (optimal policy in the auxiliary scenario) *For all $t = 1, 2, \ldots, T_k$ define a mapping $\varphi^i_t : [\underline{p}, \bar{p}] \to [\underline{p}, \bar{p}]$ satisfying*

$$\varphi^i_t(p) \ := \ \min\left\{\bar{v}^i - \beta^i_t\,[s^i_{t-1}(p) + q^i_t(p)],\ p\right\} \ \text{for all } p \in [\underline{p}, \bar{p}], \tag{2.102}$$

where $q^i_t(p) = \frac{1}{2}[(1 + y^i_{t+1})\bar{v}^i - (\beta^i_t - 2z^i_{t+1})s^i_{t-1}(p)]/(\beta^i_t - z^i_{t+1})$ for all $p \in [\underline{p}, \bar{p}]$ and $t = 1, 2, \ldots, T_k$, $\beta^i_\ell = \frac{1}{\lambda^i A^i_\ell}$, and $\{y^i_t\}$ and $\{z^i_t\}$ are constructed via the following backward iterations: $y^i_t = y^i_{t+1} - \frac{1}{2}[(1 + y^i_{t+1})(\beta^i_t - 2z^i_{t+1})]/(\beta^i_t - z^i_{t+1})$ and $z^i_t = \frac{1}{4}(\beta^i_t)^2/(\beta^i_t - z^i_{t+1})$ for $t = 1, 2, \ldots, T_k$, with $y^i_{T_k+1} = z^i_{T_k+1} = 0$. Then, $\bar{p}^i_t = \varphi^i_t(\bar{p}^i_{t-1})$ for all $t = 1, 2, \ldots, T_k$, where $\bar{p}^i_0 = \min\{\bar{v}^i, \bar{p}\}$.

Proof of Lemma 13. For simplicity of notation, we suppress the scale parameter k in this proof. We first prove that there exists $l \in \{1, 2, \ldots, T\}$ such that $\bar{p}^i_1 = \cdots = \bar{p}^i_l > \bar{p}^i_{l+1} > \cdots > \bar{p}^i_T$. Assume towards a contradiction that there exists $s \in \{2, 3, \ldots, T-1\}$ such that $\bar{p}^i_{s-1} > \bar{p}^i_s = \bar{p}^i_{s+1}$. Let $\varepsilon = \frac{1}{2}(\bar{p}^i_{s-1} - \bar{p}^i_s) > 0$, and consider the price path $\tilde{p} = (\tilde{p}_1, \tilde{p}_2, \ldots, \tilde{p}_T)$ satisfying $\tilde{p}_s = \bar{p}^i_s + \varepsilon$ and $\tilde{p}_t = \bar{p}^i_t$ for all $t \in \{1, 2, \ldots, T\} \setminus \{s\}$. By elementary algebra, we deduce that

$$
\begin{aligned}
&\sum_{t=1}^{T} R_t^i(\tilde{p}_{t-1}, \tilde{p}_t) - \sum_{t=1}^{T} R_t^i(\bar{p}_{t-1}^i, \bar{p}_t^i) \\
&= R_s^i(\tilde{p}_{s-1}, \tilde{p}_s) + R_{s+1}^i(\tilde{p}_s, \tilde{p}_{s+1}) - R_s^i(\bar{p}_{s-1}^i, \bar{p}_s^i) - R_{s+1}^i(\bar{p}_s^i, \bar{p}_{s+1}^i) \\
&\overset{\text{(a)}}{=} R_s^i(\bar{p}_{s-1}^i, \tilde{p}_s) + R_{s+1}^i(\tilde{p}_s, \bar{p}_{s+1}^i) - R_s^i(\bar{p}_{s-1}^i, \bar{p}_s^i) - R_{s+1}^i(\bar{p}_s^i, \bar{p}_{s+1}^i) \\
&\overset{\text{(b)}}{=} (\tilde{p}_s - c)d_s^i(\bar{p}_{s-1}^i, \tilde{p}_s) + (\bar{p}_{s+1}^i - c)d_{s+1}^i(\tilde{p}_s, \bar{p}_{s+1}^i) - (\bar{p}_s^i - c)d_s^i(\bar{p}_{s-1}^i, \bar{p}_s^i) \\
&- (\bar{p}_{s+1}^i - c)d_{s+1}^i(\bar{p}_s^i, \bar{p}_{s+1}^i)
\end{aligned}
$$

where (a) follows because $\tilde{p}_t = \bar{p}_t^i$ for all $t \in \{1, 2, \ldots, T\} \setminus \{s\}$, and (b) follows because $R_t^i(p_{t-1}, p_t) = (p_t - c)d_t^i(p_{t-1}, p_t)$ for all t. Therefore,

$$
\begin{aligned}
&\sum_{t=1}^{T} R_t^i(\tilde{p}_{t-1}, \tilde{p}_t) - \sum_{t=1}^{T} R_t^i(\bar{p}_{t-1}^i, \bar{p}_t^i) \\
&= (\tilde{p}_s - \bar{p}_s^i)d_s^i(p_{s-1}^i, \tilde{p}_s) + (\bar{p}_s^i - c)\Big[d_s^i(\bar{p}_{s-1}^i, \tilde{p}_s) - d_s^i(\bar{p}_{s-1}^i, \bar{p}_s^i)\Big] \\
&+ (\bar{p}_{s+1}^i - c)\Big[d_{s+1}^i(\tilde{p}_s, \bar{p}_{s+1}^i) - d_{s+1}^i(\bar{p}_s^i, \bar{p}_{s+1}^i)\Big] \\
&\overset{\text{(c)}}{=} \varepsilon d_s^i(\bar{p}_{s-1}^i, \tilde{p}_s) + (\bar{p}_s^i - c)\Big[d_s^i(\bar{p}_{s-1}^i, \tilde{p}_s) + d_{s+1}^i(\tilde{p}_s, \bar{p}_{s+1}^i) - d_s^i(\bar{p}_{s-1}^i, \bar{p}_s^i) - d_{s+1}^i(\bar{p}_s^i, \bar{p}_{s+1}^i)\Big] \\
&\overset{\text{(d)}}{=} \varepsilon d_s^i(\bar{p}_{s-1}^i, \tilde{p}_s) = \varepsilon\Big[\lambda^i \beta_{s-1}^i(\bar{p}_{s-1}^i - \tilde{p}_s) + \lambda^i \alpha_s^i(\bar{v}^i - \bar{p}_s^i)\Big] > 0,
\end{aligned}
$$

where (c) follows because $\tilde{p}_s = \bar{p}_s^i + \varepsilon$ and $\bar{p}_s^i = \bar{p}_{s+1}^i$, and (d) follows because $d^i(\bar{p}_{s-1}^i, \tilde{p}_s) + d^i(\tilde{p}_s, \bar{p}_{s+1}^i) = d^i(\bar{p}_{s-1}^i, \bar{p}_s^i) + d^i(\bar{p}_s^i, \bar{p}_{s+1}^i)$. Thus, a price path $\bar{p}^i$ cannot be optimal if there exists $s \in \{2, 3, \ldots, T-1\}$ such that $\bar{p}_{s-1}^i > \bar{p}_s^i = \bar{p}_{s+1}^i$. Consequently, there exists $l \in \{1, 2, \ldots, T\}$ such that $\bar{p}_1^i = \cdots = \bar{p}_l^i > \bar{p}_{l+1}^i > \cdots > \bar{p}_T^i$.

We now prove the statement of the lemma by considering two cases for period t.

Case 1: $t > l$. We know that $\bar{p}_{t-1}^i > \bar{p}_t^i > \cdots > \bar{p}_T^i$, which we use to derive a formula for p_t^i. Note that there is a one-to-one correspondence between p_{t-1} and $s_{t-1}^i(p_{t-1}) = \lambda A_{t-1}^i(\bar{v}^i - p_{t-1})$. Thus, we can employ $s_{t-1}^i(p_{t-1})$ as the state descriptor of the dynamic program characterized by (2.16). More formally, we let $h^i_t(s) = \bar{v}^i - s/(\lambda A^i_t)$, and define $\tilde{U}^i_t(s) = U^i_t\Big(h^i_{t-1}(s)\Big)$ for all $s \in \mathbb{R}$. Therefore, $\tilde{U}^i_t\Big(s^i_{t-1}(p_{t-1})\Big) = U^i_t(p_{t-1})$. With slight abuse of notation, we suppress the dependence of $s^i_{t-1}(p_{t-1})$ and $q^i_t(p_{t-1})$ on p_{t-1}, letting $s^i_{t-1} = s^i_{t-1}(p_{t-1})$ and $q^i_t = q^i_t(p_{t-1})$ for all $t = 1, 2, \ldots, T$. To complete the proof for the case where $t > l$, we use the following lemma, the proof of which is deferred to the end of this section.

Lemma 14. (optimal prices and value function in the auxiliary scenario) *For all $t > l$,*

$$\begin{aligned}
\bar{p}^i_t &= \bar{v}^i - \beta^i_t\,(s^i_{t-1} + q^i_t), && (2.103\text{a})\\
\tilde{U}^i_t(s^i_{t-1}) &= x^i_t(\bar{v}^i - c)^2 + y^i_t(\bar{v}^i - c)s^i_{t-1} + z^i_t(s^i_{t-1})^2, && (2.103\text{b})\\
q^i_t &= \tfrac{1}{2}[(1 + y^i_{t+1})(\bar{v}^i - c) - (\beta^i_t - 2z^i_{t+1})s^i_{t-1}]/(\beta^i_t - z^i_{t+1}), && (2.103\text{c})
\end{aligned}$$

where:

$$\begin{aligned}
x^i_t &= x^i_{t+1} + \tfrac{1}{4}(1 + y^i_{t+1})^2/(\beta^i_t - z^i_{t+1}), && (2.104\text{a})\\
y^i_t &= y^i_{t+1} - \tfrac{1}{2}[(1 + y^i_{t+1})(\beta^i_t - 2z^i_{t+1})]/(\beta^i_t - z^i_{t+1}), && (2.104\text{b})\\
z^i_t &= \tfrac{1}{4}(\beta^i_t)^2/(\beta^i_t - z^i_{t+1}), && (2.104\text{c})
\end{aligned}$$

with $x^i_{T+1} = y^i_{T+1} = z^i_{T+1} = 0$.

By (2.103a) in Lemma 14, we deduce that the optimal solution in period $t > l$ is $\varphi^i_t(p_{t-1}) = \bar{v}^i - \beta^i_t\,[s^i_{t-1}(p_{t-1}) + q^i_t(p_{t-1})]$.

Case 2: $t \le l$. We know that the optimal solution in period $t \le l$ is $\varphi^i_t(p_{t-1}) = p_{t-1}$.

Combining our findings in Cases 1 and 2, we conclude that the optimal solution in period $t \in \{1, 2, \ldots, T\}$ is $\varphi^i_t(p_{t-1}) = \min\Big\{\bar{v}^i - \beta^i_t\,[s^i_{t-1}(p_{t-1}) + q^i_t(p_{t-1})],\, p_{t-1}\Big\}$. □

Having proven Lemma 13, we now complete the remainder of the proof of Proposition 17. In what follows, we prove by induction that $\min\{\bar{v}^i, \bar{p}\} - \bar{p}^i_\ell \le \ell(\bar{v}^1 - c)(1 - Gk^{-1/2})^{kT-\ell}$

for all $\ell \in \{1, \ldots, T_k\}$ and sufficiently large values of k (which are specified below). For the base step, we apply Lemma 13 and obtain that

$$\bar{p}_1^i = \psi_1^i(\bar{p}_0^i) = \min\left\{\bar{p}_0^i - \frac{q_1^i(\bar{p}_0^i)}{k\lambda^i A_1^i}, \bar{p}_0^i\right\} = \bar{p}_0^i - \frac{q_1^i(\bar{p}_0^i)}{k\lambda^i A_1^i}. \tag{2.105}$$

In the preceding identity, $q_1^i(\bar{p})$ is given by

$$q_1^i(\bar{p}_0^i) \overset{(a)}{=} \frac{(1+y_2^i)(\bar{v}^i - c)}{2\left(\frac{1}{k\lambda^i A_1^i} - z_2^i\right)} \overset{(b)}{=} 2k^2(\lambda^i)2(A_1^i)^2 z_1^i(1+y_2^i)(\bar{v}^i - c), \tag{2.106}$$

where (a) and (b) follow by (2.103c) and (2.104c), respectively, in Lemma 14. Let $X_t^i = 2k\lambda^i A_t^i z_t^i$. Then, by (2.106), $q_1^i(\bar{p})$ satisfies

$$q_1^i(\bar{p}_0^i) = k\lambda^i A_1^i X_1^i(1+y_2^i)(\bar{v}^i - c) \overset{(c)}{\leq} k\lambda^i A_1^i(1+y_1^i)(\bar{v}^i - c), \tag{2.107}$$

where (c) follows by (2.104b) in Lemma 14, which implies that $X_1^i(1+y_2^i) = 1 + y_1^i$. Now, consider the following lemma, the proof of which is deferred to the end of this section.

Lemma 15. (coefficients of auxiliary value function) *Given $i \in \{1, \ldots, N\}$, the sequences $\{x_t^i\}$, $\{y_t^i\}$, and $\{z_t^i\}$ satisfy the following:*

(i) *Let $k \in \{1, 2, \ldots\}$. Then, $z_{t+1}^i < z_t^i$, and $z_t^i < 1/(2k\lambda^i A_t^i)$ for $t = 1, \ldots, T_k$. Furthermore, $z_t^i \leq M_z$ for $t = 1, \ldots, T_k$, where M_z is a positive constant that is independent of k.*

(ii) *There exist positive constants $\mathcal{K}$ and G such that, if $k \geq \mathcal{K}$, then $(1+y_t^i)/(1+y_{t+1}^i) \leq 1 - Gk^{-1/2}$, and $1 + y_t^i \leq (1 - Gk^{-1/2})^{kT-t}$ for $t = 1, \ldots, T_k$.*

By Lemma 15(ii), we deduce that, if $k \geq \mathcal{K}$, then $q_1^i(\bar{p}_0^i) \leq k\lambda^i A_1^i(\bar{v}^i - c)(1 - Gk^{-1/2})^{kT-1}$. Combining this inequality with (2.105), we further deduce that, if $k \geq \mathcal{K}$, then we have the following:

$$\bar{v}^i - \bar{p}^i_1 = \frac{q^i_1(\bar{p}^i_0)}{k\lambda^i A^i_1} \leq (\bar{v}^i - c)(1 - Gk^{-1/2})^{kT-1}. \tag{2.108}$$

For the induction step, assume that $\min\{\bar{v}^i, \bar{p}\} - \bar{p}^i_{\ell-1} \leq (\ell-1)(\bar{v}^i - c)(1 - Gk^{-1/2})^{kT-\ell+1}$, where $\ell \geq 2$. Note that, if $\bar{p}^i_\ell = \bar{p}^i_{\ell-1}$, then

$$\begin{aligned} \min\{\bar{v}^i, \bar{p}\} - \bar{p}^i_\ell &= \min\{\bar{v}^i, \bar{p}\} - \bar{p}^i_{\ell-1} \\ &\leq (\ell-1)(\bar{v}^i - c)(1 - Gk^{-1/2})^{kT-\ell+1} \leq \ell(\bar{v}^i - c)(1 - Gk^{-1/2})^{kT-\ell}. \end{aligned} \tag{2.109}$$

On the other hand, if $\bar{p}^i_\ell < \bar{p}^i_{\ell-1}$, then we have

$$\begin{aligned} \min\{\bar{v}^i, \bar{p}\} - \bar{p}^i_\ell = \min\{\bar{v}^i, \bar{p}\} - \varphi^i_\ell(\bar{p}^i_{\ell-1}) &\overset{\text{(d)}}{=} \frac{q^i_\ell(\bar{p}^i_{\ell-1}) + s^i_{\ell-1}(\bar{p}^i_{\ell-1})}{k\lambda^i A^i_\ell} \\ &\overset{\text{(e)}}{=} \frac{(1 + y^i_{\ell+1})(\bar{v}^i - c) + \frac{s^i_{\ell-1}(\bar{p}^i_{\ell-1})}{k\lambda^i A^i_\ell}}{2k\lambda^i A^i_\ell\left(\frac{1}{k\lambda^i A^i_\ell} - z^i_{\ell+1}\right)} \\ &\overset{\text{(f)}}{=} 2k\lambda^i A^i_\ell z^i_\ell\left[(1 + y^i_{\ell+1})(\bar{v}^i - c) + \frac{s^i_{\ell-1}(\bar{p}^i_{\ell-1})}{k\lambda^i A^i_\ell}\right] \\ &\overset{\text{(g)}}{=} X^i_\ell(1 + y^i_{\ell+1})(\bar{v}^i - c) + \frac{X^i_\ell s^i_{\ell-1}(\bar{p}^i_{\ell-1})}{k\lambda^i A^i_\ell}, \end{aligned} \tag{2.110}$$

where (d) follows by Lemma 13, and the fact that $\bar{p}^i_\ell < \bar{p}^i_{\ell-1}$; (e) and (f) follow by (2.103c) and (2.104c), respectively, in Lemma 14; and (g) follows because $X^i_\ell = 2k\lambda^i A^i_\ell z^i_\ell$. Note that, by the induction hypothesis, $s^i_{\ell-1}(\bar{p}^i_{\ell-1}) = k\lambda^i A^i_{\ell-1}(\bar{v}^i - \bar{p}^i_{\ell-1}) \leq k\lambda^i A^i_{\ell-1}(\ell-1)(\bar{v}^i - c)(1 -$

$Gk^{-1/2})^{kT-\ell+1}$. Therefore, (2.110) implies that, if $k \geq \mathcal{K}$, then

$$\begin{aligned}
\min\{\bar{v}^i, \bar{p}\} - \bar{p}^i_\ell &\overset{\text{(h)}}{\leq} (\bar{v}^i - c)(1 - Gk^{-1/2})^{kT-\ell} + \frac{X^i_\ell s^i_{\ell-1}(\bar{p}^i_{\ell-1})}{k\lambda^i A^i_\ell} \\
&\overset{\text{(i)}}{\leq} (\bar{v}^i - c)(1 - Gk^{-1/2})^{kT-\ell} + (\ell-1)(\bar{v}^i - c)\frac{A^i_{\ell-1}}{A^i_\ell}(1 - Gk^{-1/2})^{kT-\ell+1} \\
&\leq \ell(\bar{v}^i - c)(1 - Gk^{-1/2})^{kT-\ell}, \qquad (2.111)
\end{aligned}$$

where (h) follows by Lemma 15(ii) and the fact that $X^i_\ell = (1 + y^i_\ell)/(1 + y^i_{\ell+1})$; and (i) follows because $s^i_{\ell-1}(\bar{p}^i_{\ell-1}) \leq k\lambda^i A^i_{\ell-1}(\ell-1)(\bar{v}^i - c)(1 - Gk^{-1/2})^{kT-\ell+1}$ under the induction hypothesis, and $X^i_\ell \leq 1$ by Lemma 15(i). This implies that, if $k \geq \mathcal{K}$, then $\min\{\bar{v}^i, \bar{p}\} - \bar{p}^i_\ell \leq \ell(\bar{v}^i - c)(1 - Gk^{-1/2})^{kT-\ell}$ for all $\ell = 1, \ldots, T_k$ and all $i \in \{1, \ldots, N\}$.

Step 2: Derive the desired upper bound on $\min\{\bar{v}^i, \bar{p}\} - p^i_t$. Given $t_0 \in [kt_1, kt_2]$, we have the following for all $t \leq t_0$ and $k \geq k_1 = \max\{\mathcal{K}, G^2\}$:

$$\begin{aligned}
\min\{\bar{v}^i, \bar{p}\} - p^i_t &\overset{\text{(j)}}{\leq} \min\{\bar{v}^i, \bar{p}\} - p^i_{t_0} \\
&\overset{\text{(k)}}{\leq} \min\{\bar{v}^i, \bar{p}\} - \bar{p}^i_{t_0} + \vartheta k e^{-\sqrt{k}} \\
&\overset{\text{(l)}}{\leq} t_0(\bar{v}^i - c)(1 - Gk^{-1/2})^{kT-t_0} + \vartheta k e^{-\sqrt{k}} \\
&\overset{\text{(m)}}{\leq} kt_2(\bar{v}^i - c)(1 - Gk^{-1/2})^{(T-t_2)k} + \vartheta k e^{-\sqrt{k}} \\
&\overset{\text{(n)}}{\leq} kt_2(\bar{v}^i - c)e^{-\frac{1}{2}G(T-t_2)\sqrt{k}} + \vartheta k e^{-\sqrt{k}} \\
&\overset{\text{(o)}}{\leq} kc_v e^{-\delta_p\sqrt{k}},
\end{aligned}$$

where (j) follows because $p^i_t \geq p^i_{t_0}$ for $t \leq t_0$; (k) follows from (2.101); (l) follows from (2.111); (m) follows because $t_0 \in [kt_1, kt_2]$, which implies that $t_0 \leq kt_2$ and $kT - t_0 \geq (T - t_2)k$; (n) follows because $(1 - Gk^{-1/2})^{\sqrt{k}} \leq e^{-G}$ for $k \geq G^2$; and (o) follows by choosing $\delta_p \in \left(0, \min\left\{1, \frac{1}{2}G(T - t_2)\right\}\right)$ such that $e^{-\frac{1}{2}G(T-t_2)\sqrt{k}} \leq e^{-\delta_p\sqrt{k}}$ and $e^{-\sqrt{k}} \leq e^{-\delta_p\sqrt{k}}$. To complete the proof, we choose $c_v > 0$ such that $c_v \geq t_2(\max_{i\in\{1,\ldots,N\}}\{\bar{v}^i\} - c) + \vartheta$ for all

$k = 1, 2, \ldots$ $\square$

Proof of Lemma 14. We prove the lemma by backward induction. For the base step, note that $A_T^i = 1$, and

$$\tilde{U}_T^i(s_{T-1}^i) = \max_{p_T}\left\{(p_T - c)\, d_T^i\Big(h_{T-1}^i(s_{T-1}^i), p_T\Big)\right\} = \max_{p_T}\left\{(p_T - c)\left(\frac{\bar{v}^i - p_T}{\beta_T^i} - s_{T-1}^i\right)\right\}$$
$$\overset{\text{(a)}}{=} \max_{q_T}\left\{[\bar{v}^i - c - \beta_T^i(s_{T-1}^i + q_T)]q_T\right\}, \tag{2.112}$$

where (a) follows by the change of variable $q_T = (\bar{v}^i - p_T)/\beta_T^i - s_{T-1}^i$. The maximizer of the optimization problem on the right hand side of (2.112) is $q_T^i = (\bar{v}^i - c - \beta_T^i s_{T-1}^i)/(2\beta_T^i)$, implying that the optimal price for period T is $\bar{p}_T^i = \bar{v}^i - \beta_T^i(s_{T-1}^i + q_T^i)$. Therefore, the optimal value function in period T is $\tilde{U}_T^i(s_{T-1}^i) = x_T^i(\bar{v}^i - c)^2 + y_T^i(\bar{v}^i - c)s_{T-1}^i + z_T^i(s_{T-1}^i)^2$, where: $x_T^i = 1/(4\beta_T^i)$, $y_T^i = -1/2$, and $z_T^i = 4\beta_T^i$. For the induction step, assume that

$$\bar{p}_k^i = \bar{v}^i - \beta_k^i\,(s_{k-1}^i + q_k^i),$$
$$\tilde{U}_k^i(s_{k-1}^i) = x_k^i(\bar{v}^i - c)^2 + y_k^i(\bar{v}^i - c)s_{k-1}^i + z_k^i(s_{k-1}^i)^2,$$

for all $k \in \{t+1, t+2, \ldots, T\}$. Consequently, we have the following for period t:

$$\tilde{U}_t^i(s_{t-1}^i) = \max_{p_t}\left\{(p_t - c)d_t^i\Big(h_{t-1}^i(s_{t-1}^i), p_t\Big) + \tilde{U}_{t+1}^i(s_t^i)\right\}$$
$$= \max_{p_t}\left\{(p_t - c)\left(\frac{\bar{v}^i - p_t}{\beta_t^i} - s_{t-1}^i\right) + \tilde{U}_{t+1}^i(s_t^i)\right\}$$
$$\overset{\text{(b)}}{=} \max_{q_t}\left\{[\bar{v}^i - c - \beta_t^i(s_{t-1}^i - q_t)]q_t + \tilde{U}_{t+1}^i(s_t^i)\right\},$$

where (b) follows by the change of variable $q_t = (\bar{v}^i - p_t)/\beta_t^i - s_{t-1}^i$. Thus,

$$\begin{aligned}
\tilde{U}_t^i(s_{t-1}^i) &\overset{\text{(c)}}{=} \max_{q_t} \Big\{ - \beta_t^i q_t^2 + (\bar{v}^i - c - \beta_t^i s_{t-1}^i) q_t \\
&\qquad\qquad + x_{t+1}^i (\bar{v}^i - c)^2 + y_{t+1}^i (\bar{v}^i - c)(s_{t-1}^i + q_t) + z_{t+1}^i (s_{t-1}^i + q_t)^2 \Big\} \\
&= \max_{q_t} \Big\{ (-\beta_t^i + z_{t+1}^i) q_t^2 + \big[(1 + y_{t+1}^i)(\bar{v}^i - c) - (\beta_t^i - 2 z_{t+1}^i) s_{t-1}^i \big] q_t \\
&\qquad\qquad + x_{t+1}^i (\bar{v}^i - c)^2 + y_{t+1}^i (\bar{v}^i - c) s_{t-1}^i + z_{t+1}^i (s_{t-1}^i)^2 \Big\}, \qquad (2.113)
\end{aligned}$$

where (c) follows by the induction hypothesis and the fact that $q_t = (\bar{v}^i - p_t)/\beta_t^i - s_{t-1}^i = s_t^i - s_{t-1}^i$. By Lemma 15(i), we have $z_{t+1}^i \le 1/(2\lambda A_{t+1}^i) < 1/(2\lambda A_t^i) < 1/(\lambda A_t^i) = \beta_t^i$, which implies that $-\beta_t^i + z_{t+1}^i < 0$. Hence, the maximization problem on the right hand side of (2.113) has a concave objective function, and the associated maximizer is

$$q_t^i = \frac{(1 + y_{t+1}^i)(\bar{v}^i - c) - (\beta_t^i - 2 z_{t+1}^i) s_{t-1}^i}{2(\beta_t^i - z_{t+1}^i)}.$$

This implies that $\bar{p}_t^i = \bar{v}^i - \beta_t^i (s_{t-1}^i + q_t^i)$. As a conclusion, the optimal value function in period t is

$$\begin{aligned}
\tilde{U}_t^i(s_{t-1}^i) &= \left(x_{t+1}^i + \frac{(1 + y_{t+1}^i)^2}{4(\beta_t^i - z_{t+1}^i)} \right) (\bar{v}^i - c)^2 + \left(y_{t+1}^i - \frac{(1 + y_{t+1}^i)(\beta_t^i - 2 z_{t+1}^i)}{2(\beta_t^i - z_{t+1}^i)} \right) (\bar{v}^i - c) s_{t-1}^i \\
&\qquad + \left(z_{t+1}^i + \frac{(\beta_t^i - 2 z_{t+1}^i)^2}{4(\beta_t^i - z_{t+1}^i)} \right) (s_{t-1}^i)^2 \\
&\overset{\text{(d)}}{=} x_t^i (\bar{v}^i - c)^2 + y_t^i (\bar{v}^i - c) s_{t-1}^i + z_t^i (s_{t-1}^i)^2,
\end{aligned}$$

where (d) follows by (2.104a), (2.104b), and (2.104c). □

Proof of Lemma 15. To simplify notation, we suppress the superscripts in $\{x_t^i\}$, $\{y_t^i\}$, and $\{z_t^i\}$.

We prove (i) by induction. For the base step, we first note that $z_{T_k+1} = 0$, and that $z_{T_k} = 1/(4k\lambda) < 1/(2k\lambda) = 1/(2k\lambda A_{T_k})$. This implies that $z_{T_k+1} < z_{T_k}$, and $z_{T_k} < 1/(2k\lambda A_{T_k})$.

For the induction step, suppose that (i) holds for period $t+1$. Then,

$$z_t = \frac{1}{4k\lambda A_t - (2k\lambda A_t)^2 z_{t+1}} < \frac{1}{4k\lambda A_t - \frac{2k\lambda A_t^2}{A_{t+1}}} < \frac{1}{4k\lambda A_t - 2k\lambda A_t} = \frac{1}{2k\lambda A_t}.$$

Furthermore, $z_t = 1/(4k\lambda A_t - 4k^2\lambda^2 A_t^2 z_{t+1}) < z_{t+1}$, because $-4k^2\lambda^2 A_t^2 z_{t+1}^2 + 4k\lambda A_t z_{t+1} - 1 = -(1 - 2k\lambda A_t z_{t+1})^2 < 0$. This proves that $z_{t+1} < z_t$, and $z_t < 1/(2k\lambda A_t)$ for $t = 1, \ldots, T_k$. Now, since $A_t \geq \underline{\alpha}/k$, we further have $z_t < 1/(2k\lambda A_t) \leq M_z$, where $M_z = 1/(2\underline{\alpha}\lambda)$.

To prove (ii), we note that (2.104b) and (2.104c) in Lemma 14 implies that

$$\frac{1+y_t}{1+y_{t+1}} = \frac{1}{2k\lambda A_t(\frac{1}{k\lambda A_t} - z_{t+1})} = 2k\lambda A_t z_t.$$

Letting $X_t = 2k\lambda A_t z_t$, the preceding identity becomes $1 + y_t = X_t(1 + y_{t+1})$. Moreover, letting $r_t = A_t/A_{t+1} < 1$, we deduce from (2.104c) in Lemma 14 that

$$X_t = 2k\lambda A_t z_t = \frac{1}{2 - 2k\lambda A_t z_{t+1}} = \frac{1}{2 - r_t X_{t+1}}$$

for all $t = 1, \ldots, T_k - 1$. Note that $X_t > 0$, and $X_{T_k} = 2k\lambda A_{T_k} z_{T_k} = \frac{1}{2}$. Because $\underline{\alpha} \leq k\alpha_t$ for all t, we have $r_t \leq (k - \underline{\alpha})/k < 1$ for all $t \in \{1, \ldots, T_k\}$. Let $\gamma_k = (k - \underline{\alpha})/k$, and define a sequence $\{Y_t\}$ such that $Y_{T_k} = \frac{1}{2}$, and

$$Y_{t-1} = \frac{1}{2 - \gamma_k Y_t} \qquad \text{for all } t \leq T_k.$$

Now, we find $G > 0$ and $\mathcal{K} > 0$ such that, if $k \geq \mathcal{K}$, then $\max_{t \in \{1, \ldots, T_k\}}\{X_t\} \leq 1 - Gk^{-1/2}$. To that end, we first show by induction that, for all $t \leq T_k$, (a) $Y_{t-1} \geq Y_t$, (b) $Y_t < 1$, and (c) $Y_t \geq X_t$. For the base step, note that, because $\gamma_k > 0$ and $Y_{T_k} > 0$, we have $Y_{T_k - 1} \geq \frac{1}{2} = Y_{T_k}$. Moreover, $Y_{T_k} = \frac{1}{2} < 1$, and $Y_{T_k} = X_{T_k} = \frac{1}{2}$. Thus, (a-c) hold for $t = T_k$. For the induction step, assume that (a-c) hold for period $t+1$. Then, $Y_{t-1} =$

$\frac{1}{2-\gamma_k Y_t} \geq \frac{1}{2-\gamma_k Y_{t+1}} = Y_t$. In addition, $Y_t = \frac{1}{2-\gamma_k Y_{t+1}} < \frac{1}{2-\gamma_k} < 1$. Finally, since $\gamma_k \geq r_t$ and $Y_{t+1} \geq X_{t+1}$, we have $Y_t = \frac{1}{2-\gamma_k Y_{t+1}} \geq \frac{1}{2-r_t X_{t+1}} = X_t$. This proves that (a-c) hold for all $t \leq T_k$. Because the mapping $y \mapsto \frac{1}{2-\gamma_k y}$ is continuous on $[\frac{1}{2}, 1]$, we deduce from (a) and (b) that there exists $Y \in \mathbb{R}$ such that $\lim_{t \to -\infty}\{Y_t\} = Y = \frac{1}{2-\gamma_k Y}$. Since $Y_t < 1$ for all $t \leq T_k$, this implies that

$$\lim_{t \to -\infty}\{Y_t\} = Y = \frac{1-\sqrt{1-\gamma_k}}{\gamma_k} = \frac{1}{1+\sqrt{1-\gamma_k}} = \frac{1}{1+\sqrt{\underline{\alpha}/k}}.$$

Consequently, letting $\mathcal{K} = \underline{\alpha}$, we deduce the following if $k \geq \mathcal{K}$:

$$Y = \frac{1}{1+\sqrt{\underline{\alpha}/k}} = 1 - \frac{\sqrt{\underline{\alpha}}}{\sqrt{k}+\sqrt{\underline{\alpha}}} \leq 1 - \frac{\sqrt{\underline{\alpha}}}{2\sqrt{k}}. \tag{2.114}$$

By (iii), $Y_t \geq X_t$ for all $t \leq T_k$; hence, (2.114) implies that, if $k \geq \mathcal{K}$, then

$$\max_{t \in \{1,\ldots,T_k\}}\{X_t\} \leq \max_{t \in \{1,\ldots,T_k\}}\{Y_t\} \leq Y \leq 1 - Gk^{-1/2},$$

where $G = \frac{1}{2}\sqrt{\underline{\alpha}}$. As a result, $(1+y_t)/(1+y_{t+1}) = X_t \leq 1 - Gk^{-1/2}$, and $1 + y_t = X_t(1+y_{t+1}) \leq (1-Gk^{-1/2})^{kT-t}(1+y_{T_k+1}) = (1-Gk^{-1/2})^{kT-t}$ for all $t \in \{1, \ldots, T_k\}$. $\square$

2.8 On Rational Expectation Equilibria

In this section, we discuss the existence of rational expectation equilibria in our problem formulation, which includes the subgame perfect equilibria under H_i as special cases. Formally, a rational expectation equilibrium in the context of our setting is defined as follows. (As before, we suppress the dependence of the mathematical expressions on ρ for purposes of exposition.)

Definition 4. (rational expectation equilibrium) *A rational expectation equilibrium is a pair (ρ, π) consisting of a prediction behavior ρ and a pricing policy π such that for any $t \in \{1, \ldots, T_k\}$, $\boldsymbol{b}_t \in \mathcal{B}$, and $\boldsymbol{p}^{(t-1)} \in [\underline{p}, \bar{p}]^{t-1}$ with $p_1 \geq p_2 \cdots \geq p_{t-1}$, the pair (ρ, π)*

induces a price path $(p_t, \dots, p_{T_k})$ *satisfying the following three properties:*

(i) *optimality of the customers' purchasing decisions* (2.1) *under the prediction behavior* ρ*,*

(ii) *optimality of the seller's pricing policy* π*; i.e.,* $\pi \in \arg\max_{\tilde{\pi} \in \Pi} \mathbb{E}_{\tilde{\pi}}^{\boldsymbol{b}}\Big\{ \sum_{t=1}^{T_k} R_t\Big(\boldsymbol{b}_t, \boldsymbol{p}^{(t)}\Big)\Big\}$,

(iii) *consistency of predictions:* $\mathbb{E}_{\pi}^{\boldsymbol{b}}\Big\{\rho_{\ell}^{\kappa}(\boldsymbol{p}^{(\ell)}) | \boldsymbol{p}^{(t-1)}, \boldsymbol{b}_t\Big\} = \mathbb{E}_{\pi}^{\boldsymbol{b}}\Big\{p_{\ell+\kappa} | \boldsymbol{p}^{(t-1)}, \boldsymbol{b}_t\Big\}$ *for* $\ell = t, \dots, T_k - 1$ *and* $\kappa = 1, \dots, T_k - \ell$*.*

In the following result, we prove the existence of rational expectation equilibria in our setting.

Proposition 18. (existence of equilibrium) *There exists a rational expectation equilibrium in our setting.*

Proposition 18 establishes that our general formulation accommodates a rational expectation equilibrium where the customers' prediction behavior ρ and the seller's pricing policy π depend on each other. This is based on the assumption that every customer knows with certainty the seller's pricing policy as well as the other customers' prediction behavior. While our formulation subsumes this equilibrium as a special case, our goal is to shed light not only on this case but also on a broader set of scenarios motivated by practical applications. Therefore, for certain parts of our analysis, we relax the assumption on the customers' knowledge of the seller's policy and other customers' predictions and consider a more general formulation.

Proof of Proposition 18. To describe the optimal policy under demand model uncertainty, we first let $R_t\Big(\boldsymbol{b}_t, \boldsymbol{p}^{(t)}\Big) = (p_t - c) \sum_{i=1}^{N} b_t^i d_t^i\Big(\boldsymbol{p}^{(t)}\Big)$ be the seller's expected profit in period t. Given the problem scale $k \in \{1, 2, \dots\}$, the aforementioned optimal policy solves the following markdown pricing problem:

$$V^{\boldsymbol{b}} = \max_{\pi \in \Pi} \mathbb{E}_{\pi}^{\boldsymbol{b}} \Bigg\{ \sum_{t=1}^{T_k} R_t\Big(\boldsymbol{b}_t, \boldsymbol{p}^{(t)}\Big) \Bigg\}. \tag{2.115}$$

We can formulate (2.115) as a dynamic program in which the seller's belief $\boldsymbol{b}_t$ and price history $\boldsymbol{p}^{(t-1)}$ serve as state descriptors. Accordingly, the Bellman equation characterizing the optimal policy can be expressed as

$$U_t\Big(\boldsymbol{b}_t, \boldsymbol{p}^{(t-1)}\Big) = \max_{p_t \in [\underline{p}, \bar{p}]} \left\{ R_t(\boldsymbol{b}_t, \boldsymbol{p}^{(t)}) + \int_{\boldsymbol{y} \in \mathcal{B}} U_{t+1}(\boldsymbol{y}, p_t)\, \psi(\boldsymbol{b}_t, \boldsymbol{p}^{(t)}, \boldsymbol{y})\, d\boldsymbol{y} \, : \, p_t \leq p_{t-1} \right\} \tag{2.116}$$

for $t = 1, 2, \ldots, T_k$, subject to the boundary condition $U_{T_k+1}\Big(\boldsymbol{b}_{T_k+1}, \boldsymbol{p}^{(T_k)}\Big) = 0$. In the Bellman equation (2.116), $U_t\Big(\boldsymbol{b}_t, \boldsymbol{p}^{(t-1)}\Big)$ is the maximum expected total profit in periods $\{t, t+1, \ldots, T_k\}$ given that the seller's belief and price history are $\boldsymbol{b}_t$ and $\boldsymbol{p}^{(t-1)}$, respectively, at the beginning of period t.

For any prediction behavior ρ, we first show the existence of an optimal policy π. For any $t \in \{1, \ldots, T_k\}$, given state variable $(\boldsymbol{p}^{(t-1)}, \boldsymbol{b}_t)$ and prediction behavior ρ, this is equivalent to show that there exists an optimal solution to the Bellman equation (2.116).

We prove this claim by induction. For $\ell = T_k$, given that $p_1 \geq p_2 \geq \cdots \geq p_{\ell-1}$ and $\boldsymbol{b}_{\ell+1} \in \mathcal{B}$, we have $U_{\ell+1}(\boldsymbol{b}_{\ell+1}, \boldsymbol{p}^{(\ell)}) = 0$; hence, the function $U_{\ell+1}(\boldsymbol{b}_{\ell+1}, \boldsymbol{p}^{(\ell)})$ is continuous. This implies that, for $\ell = T_k$, the objective value function in (2.116) is continuous in p_ℓ. Given that $p_\ell \in [\underline{p}, p_{\ell-1}]$, we deduce from Weierstrass's extreme value theorem that there exists an optimal solution to Bellman equation (2.116) for $\ell = T_k$. By Berge's maximum theorem, the value function $U_\ell(\boldsymbol{b}_\ell, \boldsymbol{p}^{(\ell-1)})$ is continuous at $(\boldsymbol{b}_\ell, \boldsymbol{p}^{(\ell-1)})$. Inductively, we replicate the proof arguments to show that there exists an optimal solution to Bellman equation (2.116) for all $\ell \in \{1, \ldots, T_k\}$. Moreover, the optimal policy $\pi_\ell(\boldsymbol{b}_\ell, \boldsymbol{p}^{(\ell-1)})$ is upper hemicontinuous for each $\ell \in \{1, \ldots, T_k\}$.

To complete the proof of the proposition, it is sufficient to show that given $\ell \in \{1, \ldots, T_k\}$, $\boldsymbol{b}_\ell \in \mathcal{B}$, and $\boldsymbol{p}^{(\ell-1)} \in [\underline{p}, \bar{p}]^{\ell-1}$ with $p_1 \geq p_2 \geq \cdots \geq p_{\ell-1}$, there exists a prediction behavior ρ and an optimal policy π such that condition (iii) in Definition 4 is satisfied. For each $\ell = 1, \ldots, T_k$, define a convex compact set $\mathcal{P}_\ell = \{(y_\ell, \ldots, y_{T_k}) \in [\underline{p}, p_{\ell-1}]^{T_k-\ell+1} : y_\ell \geq \cdots \geq$

$y_{T_k}\}$. We consider the class of constant prediction functions ρ where $\rho_t^\kappa\Big(\boldsymbol{p}^{(\ell-1)}, x_\ell, \ldots, x_t\Big) = y_{t+\kappa}$ for all decreasing vectors $(x_\ell, \ldots, x_t) \in [\underline{p}, p_{\ell-1}]^{t-\ell+1}$ with $t \in \{\ell, \ldots, T_k - 1\}$, $\kappa \in \{1, \ldots, T_k - \ell\}$, and $\boldsymbol{y}_t = (y_t, \ldots, y_{T_k}) \in \mathcal{P}_t$. Under this class of constant prediction behavior, we define the correspondence $\Phi : \mathcal{P}_t \to 2^{\mathcal{P}_t}$ that maps $\boldsymbol{y}_t \in \mathcal{P}_t$ to the set of expected price paths induced by the optimal policy π. To abuse some notation, we parameterize the optimal solution $\pi_t(\boldsymbol{b}_t, \boldsymbol{p}^{(t-1)}, \boldsymbol{y}_t)$ of (2.116) in terms of vector $\boldsymbol{y}_t$. Formally, we can express $\Phi(\cdot)$ as

$$\begin{aligned}\Phi(\boldsymbol{y}_\ell) = &\Big\{(x_\ell, \ldots, x_{T_k}) : x_t = \mathbb{E}^{\boldsymbol{b}}_{\pi\rho}\big\{p_t \,|\, \boldsymbol{b}_\ell, \boldsymbol{p}^{(\ell-1)}\big\}, \\ &\qquad \text{where } p_t = \pi_t(\boldsymbol{b}_t, \boldsymbol{p}^{(t-1)}, \boldsymbol{y}_t),\ \text{for } t = \ell, \ldots, T_k\Big\}. \end{aligned} \tag{2.117}$$

Since $\pi_t(\boldsymbol{b}_t, \boldsymbol{p}^{(t-1)}, \boldsymbol{y}_t)$ is upper hemicontinuous for all $t \in \{\ell, \ldots, T_k\}$, it readily follows that $\Phi(\boldsymbol{y}_t)$ is upper hemicontinuous. Since $\mathcal{P}_t$ is a compact set, it follows that $\Phi(\boldsymbol{y}_t)$ is closed. By the closed graph theorem, the correspondence $\Phi(\boldsymbol{y}_t)$ is a closed graph. Thus, we deduce from Kakutani's fixed point theorem that there exists $\boldsymbol{y}_t \in \mathcal{P}_t$ such that $\boldsymbol{y}_t \in \Phi(\boldsymbol{y}_t)$.

Since the arguments can be applied to all $t \in \{1, \ldots, T_k\}$, this completes the proof that a rational expectation equilibrium exists. □

2.9 Discussions about Conditions 5 and 6

In a markdown setting, the existence of a market-preserving and demand discriminative price path in the early periods of sales horizons is crucial. If Condition 5 or Condition 6 is not satisfied, the seller would potentially run into a situation where either there is no demand-discriminative price or all demand-discriminative prices are too low. Thus, the seller would either have insufficient information on the demand model or be unable to preserve the market size in the early periods. Both of these outcomes would result in extensive profit loss. In the next result, we quantify the amount of profit loss due to demand model uncertainty in this scenario.

Proposition 19. (no market-preserving and informative price)

(i) *Suppose that Condition 5 does not necessarily hold. Let* $\pi \in \Pi$, *and* $\boldsymbol{b} \in \mathcal{B}$ *such that* $b^i < 1$ *for all* $i \in \{1, \ldots, N\}$. *Then, there exists a constant* $c_8 > 0$ *such that, under any exogenous customer prediction behavior* $\rho \in \mathfrak{P}$,

$$\sup_{\boldsymbol{\theta} \in \Theta} \{\Delta^{\boldsymbol{b}}_{\pi\rho}(k, \boldsymbol{\theta})\} \geq c_8 k \tag{2.118}$$

for $k = 1, 2, \ldots$

(ii) *Now, suppose that Condition 6 does not necessarily hold. Let* $\pi \in \Pi$, *and* $\boldsymbol{b} \in \mathcal{B}$ *such that* $b^i < 1$ *for all* $i \in \{1, \ldots, N\}$. *Then, there exists* $c_9 > 0$ *such that, under any endogenous customer prediction behavior* $\rho \in \mathfrak{P}$ *with* (ρ, π) *satisfying Condition 4,*

$$\sup_{\boldsymbol{\theta} \in \Theta} \{\tilde{\Delta}^{\boldsymbol{b}}_{\pi\rho}(k, \boldsymbol{\theta})\} \geq c_9 k \tag{2.119}$$

for $k = 1, 2, \ldots$

To derive Proposition 19, we quantify the impact of the profit reduction caused by either insufficient demand information or marking down the price too early. This proposition shows that, in the absence of Condition 5 or Condition 6, the profit loss due to demand model uncertainty grows linearly in problem scale k, which is the highest possible growth rate of the profit loss. Thus, this result shows why one needs Conditions 5 and 6 to achieve good profit performance.

Proof of Proposition 19. As in preceding proofs, we suppress the dependence of the mathematical expressions on ρ for brevity.

Proof of (i). We first construct a problem instance $\boldsymbol{\theta} \in \Theta$. Let $k \in \{1, 2, \ldots\}$, and suppose that there are two hypotheses, namely H_1 and H_2. Thus, $N = 2$. Set $c = 0$, $\underline{p} = 0$, $\bar{p} = 5$. Let $T = 1$ such that $T_k = k$. Furthermore, let the arrival matrix $\boldsymbol{\alpha}^i$ be such that $\alpha^i_{\tau 0} = \frac{1}{T_k}$ and $\alpha^i_{\tau w} = 0$ with $w \neq 0$ for $\tau = 1, \ldots, T_k$, and suppose that the possible market density functions are as follows: $\lambda^1(v) = \frac{1}{2}$ for $v \in [0, 4]$ and $\lambda^2(v) = \frac{1}{2}$ for $v \in [\frac{7}{2}, 4]$. For simplicity of notation, we suppress the problem parameter vector $\boldsymbol{\theta}$ in $\Delta_\pi^{\boldsymbol{b}}(k,\boldsymbol{\theta})$ and write $\Delta_\pi^{\boldsymbol{b}}(k)$ instead.

In the above construction, regardless of customers' prediction function $\rho \in \mathfrak{P}$, customers come to the system, decide whether or not to pruchase, and then they leave the system. Consequently, for any $\boldsymbol{p}^{(t)} \in [0,2]^t$, we deduce that the expected demand in period t under H_1 and H_2 are

$$d_t^1(\boldsymbol{p}^{(t)}) = \int_{p_t \le v \le 4} \lambda^1(v)dv = 2 - \tfrac{1}{2}p_t \quad \text{for } p_t \in [0,4],$$
$$d_t^2(\boldsymbol{p}^{(t)}) = \int_{p_t \le v \le 4} \lambda^2(v)dv = \begin{cases} 2 - \tfrac{1}{2}p_t & \text{for } p_t \in [\tfrac{7}{2},4], \\ \tfrac{1}{4} & \text{for } p_t \in [0,\tfrac{7}{2}), \end{cases}$$

respectively. The profit-maximizing prices under H_1 and H_2 are $p^1 = \arg\max_{p\in[0,4]} \big\{p(2 - \tfrac{1}{2}p)\big\} = 2$ and $p^2 = \arg\max_{p\in[\frac{7}{2},4]} \big\{p(2 - \tfrac{1}{2}p)\big\} = \tfrac{7}{2}$, respectively. Under these profit-maximizing prices, we have $d_t^1(\boldsymbol{p}^{1(t)}) = 1$ and $d_t^2(\boldsymbol{p}^{2(t)}) = \tfrac{1}{4}$ for all $t = 1,\ldots,T_k$. In this construction, we also have $p^1 > p^2$ and $p^1 d_t^1(\boldsymbol{p}^{1(t)}) = 2 > \tfrac{7}{8} = p^2 d_t^2(\boldsymbol{p}^{2(t)})$.

For any given policy π, let $(\boldsymbol{b}_1,\ldots,\boldsymbol{b}_{T_k})$ and $(\hat{p}_1,\ldots,\hat{p}_{T_k})$ be the belief and price processes induced by policy π, where $\boldsymbol{b}_t = (b_t, 1-b_t)$ for all $t = 1,\ldots,T_k$. Moreover, given $\pi \in \Pi$, we define $\mathbb{E}_\pi^{\boldsymbol{b},i}\{\cdot\} = \mathbb{E}_\pi^{\boldsymbol{b}}\{\cdot \mid H_i\}$ for all $\boldsymbol{b} \in \mathcal{B}$ and $i \in \{1,\ldots,N\}$. Consider the following two cases:

Case 1: $p_1 < 3$. By the markdown condition, we have $p_1 \ge p_t$ for all $t = 1,\ldots,T_k$. Then,

$$\Delta_\pi^{\boldsymbol{b}}(k) \ge (1-b)\mathbb{E}_\pi^{b,2}\Bigg\{\sum_{t=1}^{T_k} p^2 d_t^2(\boldsymbol{p}^{2(t)}) - p_t d_t^2(\boldsymbol{p}^{(t)})\Bigg\} = (1-b)\sum_{\tau=1}^{T_k}\left(\frac{7}{2}\cdot\frac{1}{4} - 3\cdot\frac{1}{4}\right) = \frac{1}{8}(1-b)k.$$

Case 2: $p_1 \ge 3$. In this case,

$$\Delta_\pi^{\boldsymbol{b}}(k) \ge b\,\mathbb{E}_\pi^{b,1}\Bigg\{\sum_{\tau=1}^{T_k} p^1 d_t^1(\boldsymbol{p}^{1(t)}) - p_t d_t^1(\boldsymbol{p}^{(t)})\Bigg\} \ge b\sum_{\tau=1}^{T_k}\left(2\cdot 1 - 3\cdot\frac{1}{2}\right) = \frac{1}{2}bk.$$

Letting $c_8 = \min\{\tfrac{1}{2}b, \tfrac{1}{8}(1-b)\} > 0$, we have $\Delta_\pi^{\boldsymbol{b}}(k) \ge c_8 k$ for all $k = 1,2,\ldots$.

Proof of (ii). Since the problem instance in the proof of (i) consists of customers with $w = 0$, the same problem instance can be used to establish (ii). □

CHAPTER 3

TO INTERFERE OR NOT TO INTERFERE: INFORMATION REVELATION AND PRICE-SETTING INCENTIVES IN A MULTIAGENT LEARNING ENVIRONMENT

3.1 Introduction

3.1.1 Background and overview

Marketplace platforms enable a multitude of sellers provide their products or services to customers. For example, in the accommodation industry, travelers use Airbnb to book listings offered by individual hosts; in the freelancing industry, workers advertise their skills on Upwork and arriving customers request their services. A notable feature of such platforms is that in most cases, they do not control transaction prices—instead, sellers post prices for their products or services, and arriving customers decide whether to purchase these products or services. Platforms derive revenue by extracting commissions on marketplace transactions. Fixed commission rates are stable, simple, and most importantly, credible to the participants. As a result, under such commission contracts, how sellers pick their prices has a significant impact on platforms' revenues.

In this paper, we consider how a marketplace platform maximizes its revenue in the context of *demand model uncertainty*. The platform and its sellers do not necessarily have perfect information about the relationship between prices and demand, and they need to learn this relationship from accumulating transaction data. Thus, the sellers typically have a hard time determining prices to maximize their revenues, resulting in losses for both themselves and the platform. The sellers might potentially be better off relinquishing control to the platform, because the platform has access to much more information. As large technology firms, platforms are able to collect rich contextual information such as customers' browsing and consumption history, personal information, etc. The access to this information allows platforms to estimate demand models more accurately. However, it is difficult for the platform to make credible price recommendations to the sellers, because the sellers would not necessarily believe that the platform is acting on the sellers' best interest. Thus, the platform must design incentives to motivate the sellers to choose their prices.

We consider two types of levers to accommodate the price-setting challenges under information asymmetry and demand model uncertainty. The first one is broadly related to information revelation: a platform can actively disclose some market information to the participating sellers to help them determine prices. For instance, Airbnb shares its estimated demand function with the hosts (see 97), and Upwork frequently provides reports about the popularity trend of different task categories to its freelancers (see 98). In contrast, Amazon does not share any demand information with the sellers (see 99). The second tool that a platform often uses is to offer price-setting incentives to its sellers. As an example, Airbnb currently offers hosts monetary rewards to join its premium program, resulting in hosts increasing their listing prices (see 100).

To study how to best use the aforementioned levers, we formulate a multi-period setting with a finite population of sellers. In each period, the arriving customers are characterized by a random and high-dimensional feature vector, and the sellers determine their own prices in the marketplace. Each seller's sales are generated by a *demand model*, which jointly depends on the sellers' prices and the customers' features. A key characterization of our model is that the sellers and the platform face different levels of demand model uncertainties at the beginning. The platform knows about the demand models' functional forms and is able to observe the customers' features, with the only uncertainty being the true demand model parameters; in comparison, the sellers are not able to observe the customers' features. The platform designs a stopping rule on the timing of information revelation to the sellers, which could be never or at a finite time. Once the platform's market information is disclosed, both the sellers and the platform share equal amount of information on the demand learning process. Apart from the information revelation decision, the platform can offer reward contracts to the sellers as price-setting incentives throughout the time horizon. It is worth noting that designing reward contracts before information revelation is challenging due to the platform's limited knowledge about the sellers' decision process. In this setting, we focus on the platform's revenue after paying the rewards to the sellers (i.e., the operating margin). We study the performance of different policies based on the cumulative regret over T periods i.e., the gap between the platform's T-period expected revenue under demand model uncertainty and that in a clairvoyant case where the platform and the sellers have full information about the demand model. Our analysis of this setting investigates the value of information revelation in a demand learning environment.

3.1.2 Main contributions and qualitative insights

Policy analysis. Our work identifies a dilemma that a platform faces in demand-learning environment. On one hand, the platform may seek to withhold information from sellers to exploit its informational advantage, while on the other hand, the platform may wish to share information to better manage the sellers' price-setting incentives. Thus, the value and timing of the platform's information revelation are ex-ante unclear. We consider three types of learning policies: (1) a do-nothing (DN) policy that reveals no information and offers no price-setting incentives, (2) a reveal-and-incentivize (RI) policy that reveals all information and offers price-setting incentives, and (3) a strategic-reveal-and-incentivize (SRI) policy that judiciously chooses the time at which information is revealed and offers price-setting incentives. The following table summarizes the regret results we derive for each policy.

Table 3.1: Summary of Results

policy	"good cases"	"bad cases"
do-nothing (DN)	regret $\approx -T$	regret $\approx T$
reveal-and-incentivize (RI)	regret $\approx \sqrt{T \log(T)}$	regret $\approx \sqrt{T \log(T)}$
strategic-reveal-and-incentivize (SRI)	regret $\approx -T$	regret $\approx \sqrt{T \log(T)}$

The poor performance of the do-nothing policy in the context of social optimality has been studied in the antecedent literature. Setting the performance benchmark as the platform's full-information optimal revenue (as in Proposition 20), one might think that the same intuition holds in our setting. Our analysis sheds a new light on this point: the preservation of information asymmetry in the demand learning process might create an opportunity for the platform to reduce its regret to a negative and linearly decreasing order of magnitude—see Theorem 11(i). But without the platform's intervention, the sellers might also update their prices in a way that causes a linearly growing regret for the platform—see Theorem 11(ii). Because of its drastically different learning outcomes, the do-nothing policy is a risky strategy for the platform. We characterize when the regret of the do-nothing policy turns out to be negative (the "good cases" in Table 3.1) versus positive (the "bad cases" in Table 3.1)—see Proposition 21.

To avoid the aforementioned risks, the platform can give up its informational advantage at the outset in exchange for higher accuracy in managing the sellers' price-setting incentives (see the reveal-and-incentivize policy in Algorithm 1). We establish that under this policy, the platform is able to achieve a regret that grows sublinearly in terms of the time horizon (see Theorem 12). This result highlights the benefit of information revelation in protecting the platform from triggering large revenue loss due to demand model uncertainty. However, the same result also suggests that this policy can be too conservative in trading informational advantages for easier management of the sellers' incentives.

Combining the merits of the two policies above, we include the strategic timing of information revelation in designing the price-setting incentives (see the strategic-reveal-and-incentivize policy in Algorithm 2). The key observation in designing this type of policy is that it is possible to incentivize the sellers to set a price profile that facilitates the platform's demand learning. After accumulating sufficient information on the demand models, the platform can make a clever decision on the timing of information disclosure. We show that this class of policy achieves a negative and linearly decreasing regret in the "good cases" (see Theorem 13(i)), and is also robust against the "bad cases", with a regret that grows sublinearly in the time horizon (see Theorem 13(ii)).

Theoretical contributions. We make three theoretical contributions to the literature on dynamic pricing with demand learning. First, our paper characterizes a dynamic learning problem in a platform setting. A distinguishing feature that differentiates our work from earlier studies is that the platform does not directly control the prices. In particular, few papers have analyzed how information asymmetry creates both opportunities and challenges for a central planner's learning process. Second, noting that earlier related studies focus on generalized linear demand models (see 101, 83, 102, 103), we extend the analysis of demand learning to systems of general demand models. Identifying the key properties of the empirical Fisher information matrix induced by estimation (see Assumption 13), we establish an estimation convergence result in a fairly general demand framework (see Proposition 22). We also show that some of the commonly used generalized linear demand models are special cases of our setting (see Proposition 24). Third, we design dynamic learning policies that do not use only the data generated in exploration periods. While restricting attention to the samples in exploration periods provides analytical tractability, our paper studies a more general setting without this restriction, allowing for non-i.i.d. samples.

3.2 Literature Review

Our work is broadly related to three streams of literature: (i) managing service platforms, (ii) dynamic pricing with demand learning, and (iii) multiagent learning and incentivizing explorations.

Managing service platforms. Our paper is related to the burgeoning literature on service platforms. In contrast to the central price-setting environment (see 5, 104, 105, 106, 107), an important class of the papers in this field focus on indirect control levers. For example, (108), (109), (110), and (111) focus mainly on how a platform should match the sellers and the buyers to improve the system's performance. In particular, (108) shows that focusing only on operational efficiency can hurt the platform. (109) and (111) identify that by limiting the agents' access in the market, the platform can reduce the search cost and induce higher market efficiency. (110) establishes that a forward-looking linear programming-based matching policy outperforms myopic matching policies, and is asymptotically optimal in large market regimes. Besides matching, (112) shows that to maximize profit given admission and repositioning controls, the platform sometimes needs to reject customers to induce repositioning to high-demand markets. More recently, (27) and (113) characterize how a platform can leverage commissions and subscriptions in the revenue maximization problem. Information control is another tool that the platform can apply; see for example, (114), (115), (116), (117). By contrast, our paper characterizes a setting where a platform needs to jointly decide on the timing of information revelation and the offering of price-setting incentives to induce matching under the challenge of demand model uncertainty. By addressing the platform's dilemma between preserving its informational advantage and managing the complexities in price-setting incentives, our work quantifies the asymptotic revenue loss due to demand model uncertainty.

Dynamic pricing and learning. Our paper is also related to the literature on dynamic pricing with demand model uncertainty; see, e.g., (82), (101), (83), (103) and (102). These papers assume that there is an underlying demand model whose parameters are unknown to a central planner, and formulate online learning problems based on commonly used estimation procedures such as ordinary least squares, maximum likelihood estimation, and maximum quasi-likelihood estimation. There are also papers using Bayesian learning frameworks: the central planner repeatedly updates its belief on a finite number of hypotheses on the demand model; see for example, (80), (56), (57), (58). A common feature for both streams of literature is that the central planner is assumed to have complete control over the price-setting process. To address the exploration-exploitation trade-off, the central planner carefully controls price variations for the purpose of learning the demand model while simultaneously limiting the profit loss caused by demand model uncertainty. By contrast, our paper features a setting in which the platform has to rely on decentralized sellers to determine their prices. Moreover, the information asymmetry between the platform and the sellers creates an interesting trade-off between exploiting informational advantages and managing price-setting incentives under demand model uncertainty. In a recent study, (87) analyze a firm's dynamic selling mechanism design problem under cost uncertainty. In their paper, the firm has perfect knowledge about its customers' incentive compatibility conditions. By contrast, our paper considers a setting where the sellers' incentives depend on the platform's information revelation decisions.

Multiagent learning and incentivizing exploration. Our work has a connection with the literature on multiagent learning and incentivizing exploration. (118, 119), (120), and (121) investigate information disclosure as a means of incentivizing exploration, and they develop incentive-compatible algorithms for a social planner to achieve asymptotically optimal regret. In comparison, (122) and (123) consider the optimal payment schemes that incentivize customers to explore. In these two papers, the noisy reward information becomes public to all following customers. While these papers all focus on social efficiency, our paper studies the revenue management problem of a central planner (namely, the platform). We show that the platform can leverage its informational advantage to benefit itself in a demand learning environment.

In the literature on multiagent learning, there are many studies that extend the reinforcement learning algorithms to decentralized settings. These include the extension of mirror descent algorithms (see, e.g., 124, 125, 126, 127, 128, 129, 130) and Q-learning (see, e.g., 131, 132, 133, 134, 135, 136). In comparison, the economics and finance literatures explore the frameworks of fictitious play (see, e.g., 137, 138, 139, 140, 141) and Bayesian equilibrium (see, e.g., 142, 143, 144, 145) from both theoretical and empirical perspectives. While almost all of these papers focus on investigating the multiagent learning environment itself, our work focuses on how a platform can leverage its informational advantage to guide the sellers' behavior in a multiagent learning setting.

3.3 Model Formulation

Basic model elements. We consider a marketplace platform that connects n sellers with customers who arrive over a time horizon of T periods. In each period $t \in \{1, 2, \ldots, T\}$, the platform observes a feature vector $\boldsymbol{\xi}_t$ that characterizes the customers arriving in that period. The feature vectors, $\{\boldsymbol{\xi}_t : t = 1, 2, \ldots, T\}$, are independent and identically distributed random vectors, taking values in a compact set $\Xi \subset \mathbb{R}^{\kappa_{\boldsymbol{\xi}}}$, where $\kappa_{\boldsymbol{\xi}} \in \mathbb{N}_{++}$.[1] In practice, the features may include the customers' browsing records, consumption history, demographics, or abstract metrics developed by the platform. In period t, each seller $i \in \{1, 2, \ldots, n\}$ chooses a price $p_{it} \in [l, u]$, where $0 < l < u < \infty$. We denote by $\boldsymbol{p}_t = (p_{1t}, \ldots, p_{nt}) \in [l, u]^n$ the price profile of all sellers in period t.

The expected demand for seller i in period t depends on the price profile $\boldsymbol{p}_t$, the feature vector $\boldsymbol{\xi}_t$, and a parameter vector $\boldsymbol{\theta} \in \Theta$, where $\Theta \subset \mathbb{R}^{\kappa_\theta}$ is a convex and compact set with $\kappa_\theta \in \mathbb{N}_{++}$. Let $Q_i : \mathbb{R}^n \times \mathbb{R}^{\kappa_{\boldsymbol{\xi}}} \times \mathbb{R}^{\kappa_\theta} \to \mathbb{R}^n$ be the demand function such that, given $(\boldsymbol{p}_t, \boldsymbol{\xi}_t, \boldsymbol{\theta}) \in [l, u]^n \times \Xi \times \Theta$, seller i's expected demand in period t is $Q_i(\boldsymbol{p}_t, \boldsymbol{\xi}_t, \boldsymbol{\theta})$. We also write $\boldsymbol{Q} = (Q_1, \ldots, Q_n)$. The demand function $\boldsymbol{Q}$ incorporates many widely used demand models in practice. In the example below, we present three such instances.

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[1] Here and later, we let $\mathbb{N}_{++}$ and $\mathbb{R}^d$ denote the the set of positive natural numbers and the d-dimensional Euclidean space, respectively, where $d \in \mathbb{N}_{++}$. We also denote by $\mathbb{N}_+$ the set of non-negative natural numbers.

Example 7. *Let* $\Theta = [\underline{\theta}, \overline{\theta}]^{\kappa_\theta}$ *and* $\Xi = [\underline{\xi}, \overline{\xi}]^{\kappa_{\boldsymbol{\xi}}}$ *with* $\overline{\theta} > \underline{\theta} > 0$ *and* $\overline{\xi} > \underline{\xi} > 0$*. For any* $(\boldsymbol{p}, \boldsymbol{\xi}, \boldsymbol{\theta}) \in [l, u]^n \times [\underline{\xi}, \overline{\xi}]^{\kappa_{\boldsymbol{\xi}}} \times [\underline{\theta}, \overline{\theta}]^{\kappa_\theta}$*, consider the following:*

> *(i)* ***(linear demand)*** $\boldsymbol{Q}(\boldsymbol{p}, \boldsymbol{\xi}, \boldsymbol{\theta}) = \boldsymbol{\theta}_0 + \boldsymbol{\theta}_1\boldsymbol{\xi} - \boldsymbol{\theta}_2\boldsymbol{p}$*, where* $\boldsymbol{\theta}_0 \in \mathbb{R}^n$*,* $\boldsymbol{\theta}_1 \in \mathbb{R}^{n\times n}$*, and* $\boldsymbol{\theta}_2 \in \mathbb{R}^{n\times n}$*.*[2]
>
> *(ii)* ***(multinomial logit demand)*** *for all* $i \in \{1, \ldots, n\}$*,* $Q_i(\boldsymbol{p}, \boldsymbol{\xi}, \boldsymbol{\theta}) = \exp(\theta_{0i} + \theta_{1i}\xi_i - \theta_{2i}p_i)/[1 + \sum_{j=1}^{n} \exp(\theta_{0j} + \theta_{1j}\xi_j - \theta_{2j}p_j)]$*, where* $\theta_{0i}, \theta_{1i}, \theta_{2i} \in \mathbb{R}$*.*[3]
>
> *(iii)* ***(bilinear demand)*** $\boldsymbol{Q}(\boldsymbol{p}, \boldsymbol{\xi}, \boldsymbol{\theta}) = \boldsymbol{\theta}_0 + \boldsymbol{\theta}_1\boldsymbol{\xi}\boldsymbol{p} - \boldsymbol{\theta}_2\boldsymbol{p}$*, where* $\boldsymbol{\theta}_0 \in \mathbb{R}^n$*,* $\boldsymbol{\theta}_1 \in \mathbb{R}^{n\times n}$*, and* $\boldsymbol{\theta}_2 \in \mathbb{R}^{n\times n}$*.*[4]

We suppose that the demand function $\boldsymbol{Q}$ satisfies the following continuity conditions. (These conditions hold in several special cases of our general formulation—see Proposition 23.)

Assumption 7. (smoothness of the demand function) *For any* $(\boldsymbol{p}, \boldsymbol{\xi}, \boldsymbol{\theta}) \in [l, u]^n \times \Xi \times \Theta$*, the function* $\boldsymbol{Q}$ *satisfies* $\boldsymbol{Q}(\boldsymbol{p}, \boldsymbol{\xi}, \boldsymbol{\theta}) \in \mathbb{R}^n_{++}$ *and is twice continuously differentiable. Moreover, there exists some* $L_q > 0$ *such that for any* $i \in \{1, \ldots, n\}$*,* Q_i *has a* L_q*-Lipschitz continuous gradient.*

We assume that the demand realizations in period t are subject to an ex-ante unobservable temporal demand shock $\boldsymbol{\varepsilon}_t \in \mathbb{R}^n$. The process $\{\boldsymbol{\varepsilon}_t : t = 1, 2 \ldots\}$ is a martingale difference sequence adapted to the filtration $\{\mathcal{F}_t : t = 0, 1, \ldots\}$, where $\mathcal{F}_0 = \sigma(\boldsymbol{\xi}_1)$ and $\mathcal{F}_t = \sigma(\boldsymbol{\xi}_1, \boldsymbol{\varepsilon}_1, \ldots, \boldsymbol{\xi}_t, \boldsymbol{\varepsilon}_t, \boldsymbol{\xi}_{t+1})$ for all $t \in \{1, 2, \ldots\}$. We denote by $\mathbb{P}_{\boldsymbol{\xi},\boldsymbol{\varepsilon}}\{\cdot\}$ the probability measure governing the distribution of $\{\boldsymbol{\xi}_t\}$ and $\{\boldsymbol{\varepsilon}_t\}$, and let $\mathbb{E}_{\boldsymbol{\xi},\boldsymbol{\varepsilon}}\{\cdot\}$ be the expectation operator associated with $\mathbb{P}_{\boldsymbol{\xi},\boldsymbol{\varepsilon}}\{\cdot\}$. Note that the demand shock $\boldsymbol{\varepsilon}_t$ can depend on the entire

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[2] In Example 7(i), $\kappa_\theta = n + 2n^2$ and $\kappa_{\boldsymbol{\xi}} = n$, with a slight abuse of notation regarding the matrix space $\mathbb{R}^{n\times(1+2n)}$, which is isomorphic to the vector space $\mathbb{R}^{n+2n^2}$. In general, $\kappa_{\boldsymbol{\xi}}$ can take other values in $\mathbb{N}_{++}$.

[3] In Example 7(ii), $\kappa_\theta = 3n$ and $\kappa_{\boldsymbol{\xi}} = n$.

[4] In Example 7(iii), $\kappa_\theta = n(1 + 2n)$ and $\kappa_{\boldsymbol{\xi}} = n^2$ with $\boldsymbol{\xi} \in \mathbb{R}^{n\times n}$, where the matrix space $\mathbb{R}^{n\times n}$ is isomorphic to the vector space $\mathbb{R}^{n^2}$.

history of demand and feature realizations. Because $\{\boldsymbol{\varepsilon}_t\}$ is a martingale difference sequence adapted to $\{\mathcal{F}_t\}$, $\mathbb{E}_{\boldsymbol{\xi},\boldsymbol{\varepsilon}}\{\boldsymbol{\varepsilon}_t \,\big|\, \mathcal{F}_{t-1}\} = \mathbf{0}$ for $t \in \{1, 2, \ldots\}$. Moreover, we suppose that there exists $\bar{\sigma} \in (0, \infty)$ such that $\mathbb{E}_{\boldsymbol{\xi},\boldsymbol{\varepsilon}}\{\|\boldsymbol{\varepsilon}_t\|_2^2 \,\big|\, \mathcal{F}_{t-1}\} \leq \bar{\sigma}^2$ for $t \in \{1, 2, \ldots\}$. Given the demand shock $\boldsymbol{\varepsilon}_t$ in period t, the realized demand profile in period t is

$$\boldsymbol{D}_t \;=\; \boldsymbol{Q}(\boldsymbol{p}_t, \boldsymbol{\xi}_t, \boldsymbol{\theta}) + \boldsymbol{\varepsilon}_t \quad \text{for } t = 1, 2, \ldots, T. \tag{3.1}$$

Platform's learning process. Let $\tilde{\boldsymbol{\theta}}$ denote a generic value of the demand parameter vector. The platform knows the functional form of $\boldsymbol{Q}(\cdot, \tilde{\boldsymbol{\theta}})$ given $\tilde{\boldsymbol{\theta}} \in \Theta$, but does not know the true demand parameter vector $\boldsymbol{\theta}$. The platform observes the customer features at the beginning of each period before any decision is made. For all $t \in \{1, \ldots, T\}$, denote by $\mathcal{H}_t = (\boldsymbol{p}_1, \boldsymbol{\xi}_1, \boldsymbol{D}_1, \ldots, \boldsymbol{p}_{t-1}, \boldsymbol{\xi}_{t-1}, \boldsymbol{D}_{t-1}, \boldsymbol{\xi}_t)$ the history of the platform's observations up to the beginning of period t. Based on these observations, the platform uses maximum likelihood estimation to infer $\boldsymbol{\theta}$. To be precise, given the log-likelihood function $\ell_t(\tilde{\boldsymbol{\theta}}|\mathcal{H}_t) = \sum_{s=1}^{t-1} \log \mathbb{P}_{\boldsymbol{\xi},\boldsymbol{\varepsilon}}\{\boldsymbol{D}_s - \boldsymbol{Q}(\boldsymbol{p}_s, \boldsymbol{\xi}_s, \tilde{\boldsymbol{\theta}}) \in d\boldsymbol{\varepsilon}_s \,\big|\, \mathcal{F}_{s-1}\}$ at the beginning of period t, the platform solves the following problem to estimate $\boldsymbol{\theta}$:

$$\tilde{\boldsymbol{\theta}}_t \;\in\; \mathcal{P}_\Theta\left\{\arg\max_{\tilde{\boldsymbol{\theta}}} \left\{\ell_t\Big(\tilde{\boldsymbol{\theta}}\Big|\mathcal{H}_t\Big)\right\}\right\} \qquad \text{for } t = 1, 2, \ldots, T, \tag{3.2}$$

where $\mathcal{P}_\Theta$ denotes the projection mapping from $\mathbb{R}^{\kappa_\theta}$ onto Θ.[5] We suppose that the log-likelihood function satisfies the following condition. We show in Proposition 24 that this condition holds in several different special cases of our general formulation.

Assumption 8. (smoothness and concavity of the likelihood function) *For any $T \geq 2$, $t \in \{1, \ldots, T\}$, and $\{(\boldsymbol{p}_s, \boldsymbol{\xi}_s, \boldsymbol{D}_s) \in [l, u]^m \times \Xi \times \mathbb{R}^n : s = 1, \ldots t\}$, the difference $\ell_t(\tilde{\boldsymbol{\theta}}|\mathcal{H}_t) - \ell_{t-1}(\tilde{\boldsymbol{\theta}}|\mathcal{H}_{t-1})$ is twice continuously differentiable and weakly concave in $\tilde{\boldsymbol{\theta}} \in \Theta$.*

Maximum likelihood estimation uses the functional form of the demand model $\boldsymbol{Q}(\cdot, \tilde{\boldsymbol{\theta}})$

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[5] If the estimation problem admits multiple solutions, then without loss of generality the platform picks any of them. As explained below, our policy admits a unique estimator with high probability.

given a parameter vector $\tilde{\boldsymbol{\theta}} \in \Theta$. In practice, the platform is typically a large technology firm that has the expertise of formulating a reasonable demand model and would thus know the demand model's functional form. This kind of estimation procedures that use the functional form of a demand model have been widely used in the literature on dynamic pricing and learning—see, e.g., (82) and (83). Such estimation procedures are also consistent with the anecdotal observation that, as the platform collects more market information, its understanding of the relationship between prices and demand improves substantially (relative to the individual sellers that have more limited information). Maximum likelihood estimation is also based on the likelihood function. If such a likelihood function is not available, it is possible to use maximum quasi-likelihood estimation instead—see, e.g., (103).

Platform's admissible policies and revenue function. Because the platform does not directly set the transaction prices, it must use other market tools to influence the sellers' price-setting process. The levers we consider are (i) payments to the sellers, and (ii) disclosing information.

The first lever is a monetary incentive in the price-setting process. While the sellers determine their own prices, the platform can offer payments to the sellers such that it would be incentive compatible for the sellers to follow the platform's recommended prices. To model such transfer payment schemes, we define $\boldsymbol{\mathcal{W}} = \{\mathcal{W}_{it} : i \in \{1, \ldots, n\}, t \in \{1, \ldots, T\}\}$ where function $\mathcal{W}_{it} : [l, u] \to \mathbb{R}_+$ is such that $\mathcal{W}_{it}(p)$ denotes the platform's reward to seller i in period t if the seller sets price p. We require $\mathcal{W}_{it}(p) \geq 0$ for any $p \in [l, u]$ in this paper; i.e., the platform does not penalize the sellers in the marketplace. A penalty can discourage the sellers from participating in trading and hurt the reputation of the platform in the long run.

The second lever for the platform relates to information disclosure. At the beginning of the time horizon, the platform has an informational advantage over the sellers because it observes the customer feature process $\{\boldsymbol{\xi}_t\}$. To model information revelation, we define a stopping time $\tau \in \mathbb{N}_{++} \cup \{\infty\}$ that characterizes the period in which the platform decides to reveal its information to the sellers. Formally, we let τ be such that $\{\tau \leq t\} \in \mathcal{F}_t$ for all $t \in \{1, \ldots, T\}$. If $\tau < \infty$, the platform gives up its informational advantage at the beginning of period τ, and both the platform and the sellers have access to the same market information as time moves forward. As a result of this information equivalence, the estimated demand model $\boldsymbol{Q}(\cdot, \tilde{\boldsymbol{\theta}}_t)$ given the parameter estimate $\tilde{\boldsymbol{\theta}}_t$ becomes public knowledge in the marketplace. This is consistent with practice, where it is common for platforms to share their estimated demand models with the sellers (see 97).

Based on the above, we define an admissible policy for the platform as the pair of the stopping rule on information revelation timing and the monetary rewards to the sellers; i.e., $\pi = (\tau, \boldsymbol{\mathcal{W}})$. Consistent with the popular practice, we assume that each seller retains $\gamma \in (0, 1)$ fraction of the revenue from the sales, while the platform receives $1 - \gamma$. Given policy π, the sellers choose prices $\boldsymbol{p}_t$ in period t, resulting in total revenue $\sum_{i=1}^{n} p_{it} Q_i(\boldsymbol{p}_t, \boldsymbol{\xi}_t, \boldsymbol{\theta})$, so that the platform's collected amount from the commissions is

$$R(\boldsymbol{p}_t, \boldsymbol{\xi}_t, \boldsymbol{\theta}) \ := \ \sum_{i=1}^{n} (1 - \gamma) p_{it} Q_i(\boldsymbol{p}_t, \boldsymbol{\xi}_t, \boldsymbol{\theta}). \tag{3.3}$$

The value in (3.3) must be reduced to account for the monetary reward offered to the sellers, leading to the platform's total expected revenue (i.e., the operating margin) in period t being

$$\mathcal{R}(\boldsymbol{p}_t, \boldsymbol{\xi}_t, \boldsymbol{\theta}) \ := \ R(\boldsymbol{p}_t, \boldsymbol{\xi}_t, \boldsymbol{\theta}) - \sum_{i=1}^{n} \mathcal{W}_{it}(p_{it}). \tag{3.4}$$

Sellers' pricing decisions. In each period, the sellers select their own prices. For any $(\boldsymbol{p}_t, \boldsymbol{\xi}_t, \boldsymbol{\theta}) \in [l, u]^n \times \Xi \times \Theta$, when transactions happen in period t, the payment to seller i is

$$R_{s_i}(\boldsymbol{p}_t, \boldsymbol{\xi}_t, \boldsymbol{\theta}) \ := \ \gamma p_{it} Q_i(\boldsymbol{p}_t, \boldsymbol{\xi}_t, \boldsymbol{\theta}). \tag{3.5}$$

Under the platform's offered reward contract $\mathcal{W}_{it}$, seller i's revenue in period t is

$$\mathcal{R}_{s_i}(\boldsymbol{p}_t, \boldsymbol{\xi}_t, \boldsymbol{\theta}) \;:=\; R_{s_i}(\boldsymbol{p}_t, \boldsymbol{\xi}_t, \boldsymbol{\theta}) + \mathcal{W}_{it}(p_{it}). \tag{3.6}$$

We assume that the sellers are myopic and revenue-maximizing in each period. This assumption reflects that the decentralized sellers are more risk-averse and focus more on short-term benefits than the platform as a large-scale technology firm with longer a planning horizon. Since the platform holds informational advantages over the sellers at the beginning of the time horizon, depending on the platform's information revelation decision τ, there are two possible information regimes for the sellers:

Pre-disclosure regime: $t < \tau$. In this regime, the platform has not yet revealed any information, so the sellers cannot observe the customer features in period t. With no access to the market information, each seller $i \in \{1, \ldots, n\}$ approximates its unknown revenue function $R_{s_i}(p, \boldsymbol{p}_{-it}, \boldsymbol{\xi}_t, \boldsymbol{\theta})$ in period t by some function $\tilde{R}_{it}(p)$, and then solves the following approximate revenue optimization problem:

$$p_{it} \;:=\; \operatorname*{arg\,max}_{p \in [l,u]} \left\{ \tilde{R}_{it}(p) + \mathcal{W}_{it}(p) \right\}. \tag{3.7}$$

In problem (3.7), we make the following assumption on the approximate revenue function $\tilde{R}_{it}$.

Assumption 9. (smoothness of the approximate revenue function) *For any $i \in \{1, \ldots, n\}$ and $t \in \{1, \ldots, T\}$, the approximate revenue function $\tilde{R}_{it}(p)$ is continuously differentiable in $p \in [l, u]$, with $|\tilde{R}_{it}(p)| \le M_R^0$ and $|\frac{d}{dp}\tilde{R}_{it}(p)| \le M_R^1$ for some $M_R^0, M_R^1 > 0$.*

Before the platform discloses the information, it does not know how the sellers update their prices; that is, it does not know what exactly $\tilde{R}_{it}$ is in (3.7). The platform only knows that $\tilde{R}_{it}$ satisfies Assumption 9, and we show that this assumption holds in several commonly studied multiagent learning frameworks in literature; see Proposition 25.

Post-disclosure regime: $t \geq \tau$. After the platform discloses the market information to the sellers, both the sellers and the platform share the same amount of information, and the estimated demand model becomes more transparent to the sellers. Based on this, the sellers start using the maximum likelihood estimate $\tilde{\boldsymbol{\theta}}_t$ in (3.2). That is, each seller $i \in \{1, \ldots, n\}$ replaces $\boldsymbol{\theta}$ with $\tilde{\boldsymbol{\theta}}_t$ in the revenue function (3.6), and maximizes its expected revenue knowing that its competitors apply the same strategy. Formally, given $(\boldsymbol{\xi}_t, \tilde{\boldsymbol{\theta}}_t) \in \Xi \times \Theta$, the induced profile $\boldsymbol{p}_t$ in period t satisfies the following for all $i \in \{1, \ldots, n\}$:

$$p_{it} \ := \ \operatorname*{arg\,max}_{p \in [l,u]} \left\{ \gamma p Q_i(p, \boldsymbol{p}_{-it}, \boldsymbol{\xi}_t, \tilde{\boldsymbol{\theta}}_t) + \mathcal{W}_{it}(p) \right\}. \tag{3.8}$$

The induced price profile $\boldsymbol{p}_t$ in (3.8) is a Nash equilibrium given the estimate $\tilde{\boldsymbol{\theta}}_t$, the customers' features $\boldsymbol{\xi}_t$, and the platform's reward contracts $\boldsymbol{\mathcal{W}}_t$. Here, the sellers view the estimate $\tilde{\boldsymbol{\theta}}_t$ as trustworthy because they have access to the same amount of market information as the platform. Revealing the market information to the sellers and increasing the transparency in the marketplace, the platform gains the benefit of more accurately managing the price profile through the reward contracts.

Performance metric. Given the demand parameter vector $\boldsymbol{\theta}$, the platform's admissible policy π, and the induced price process $\{\boldsymbol{p}_t : t = 1, 2 \ldots\}$, we let $\mathbb{P}^\pi_{\boldsymbol{\theta}}\{\cdot\}$ be a probability measure such that

$$\mathbb{P}^\pi_{\boldsymbol{\theta}} \{\boldsymbol{D}_1 \in d\boldsymbol{x}_1, \ldots, \boldsymbol{D}_T \in d\boldsymbol{x}_T\} \ := \ \prod_{t=1}^{T} \mathbb{P}_{\boldsymbol{\xi},\boldsymbol{\varepsilon}} \Big\{ \boldsymbol{Q}(\boldsymbol{p}_t, \boldsymbol{\xi}_t, \boldsymbol{\theta}) + \boldsymbol{\varepsilon}_t \in d\boldsymbol{x}_t \,\Big|\, \mathcal{F}_{t-1} \Big\}, \tag{3.9}$$

where $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_T \in \mathbb{R}^n$. Letting $\mathbb{E}^\pi_{\boldsymbol{\theta}}\{\cdot\}$ be the expectation operator associated with $\mathbb{P}^\pi_{\boldsymbol{\theta}}\{\cdot\}$ and using the platform's revenue function in (3.4), we express the platform's expected revenue under policy π as

$$V^\pi_{\boldsymbol{\theta}}(T) \ := \ \mathbb{E}^\pi_{\boldsymbol{\theta}} \left\{ \sum_{t=1}^{T} \mathcal{R}(\boldsymbol{p}_t, \boldsymbol{\xi}_t, \boldsymbol{\theta}) \right\}. \tag{3.10}$$

In the case of full information, the demand model $\boldsymbol{Q}$, the feature process $\{\boldsymbol{\xi}_t : t = 1, 2, \dots\}$, and the true demand parameter vector $\boldsymbol{\theta}$ are all public knowledge to the platform and the sellers. Given $(\boldsymbol{\xi}_t, \boldsymbol{\theta}) \in \Xi \times \Theta$, we define the platform's full-information optimal revenue in period t as

$$\mathcal{R}^\star(\boldsymbol{\xi}_t, \boldsymbol{\theta}) \;:=\; \max_{\boldsymbol{p}_t^\star, \boldsymbol{\mathcal{W}}_t^\star(\cdot)} \sum_{i=1}^{n} (1-\gamma) p_{it}^\star Q_i(\boldsymbol{p}_t^\star, \boldsymbol{\xi}_t, \boldsymbol{\theta}) - \sum_{i=1}^{n} \mathcal{W}_{it}^\star(p_{it}^\star), \tag{3.11a}$$

$$\text{s.t.} \quad p_{it}^\star \in \arg\max_{p \in [l,u]} \left\{ \gamma p Q_i(p, \boldsymbol{p}_{-it}^\star, \boldsymbol{\xi}_t, \boldsymbol{\theta}) + \mathcal{W}_{it}^\star(p) \right\} \quad \text{for } i \in \{1, \dots, n\}, \tag{3.11b}$$

$$\mathcal{W}_{it}^\star(p) \geq 0 \quad \text{for } p \in [l,u],\; i \in \{1, \dots, n\}. \tag{3.11c}$$

In problem (3.11), the platform offers a reward contract $\mathcal{W}_{it}$ to each seller i in period t so that seller i finds it incentive compatible to follow the recommended price $p_{it}^\star$; i.e., constraint (3.11b) is satisfied.

The platform's expected T-period regret under policy π uses the full-information optimal revenue in (3.11) as a performance benchmark for the expected revenue in (3.10), and is given by

$$\Delta_{\boldsymbol{\theta}}^\pi(T) \;:=\; T\,\mathbb{E}_{\boldsymbol{\xi},\boldsymbol{\varepsilon}}\left\{\mathcal{R}^\star(\boldsymbol{\xi}, \boldsymbol{\theta})\right\} - V_{\boldsymbol{\theta}}^\pi(T). \tag{3.12}$$

Because the feature process $\{\boldsymbol{\xi}_t : t = 1, 2, \dots\}$ is identically and independently distributed, the expectation in the benchmark term—i.e., first term on the right-hand side of (3.12)—can be taken over the common distribution of the feature vectors $\boldsymbol{\xi}$. Our definition of the benchmark is essentially the second best solution in a mechanism design problem: the platform can only implement a price profile that is incentive compatible for the sellers in the marketplace. To evaluate the performance of the platform's policy, we consider an asymptotic setting in which we increase the time horizon T and quantify the growth of $\Delta_{\boldsymbol{\theta}}^\pi(T)$.

3.4 Policy Design and Performance Analysis for the Platform

At the beginning of the time horizon, the platform has an informational advantage over the sellers in the marketplace, causing a dilemma in its revenue management. On one hand, the platform can exploit its informational advantage: with limited market information, the sellers can mistakenly pick prices that further benefit the platform relative to the case of full information. On the other hand, because the platform has limited knowledge on how the sellers update their prices before the information disclosure, offering the price-setting incentives accurately is much more difficult. Revealing the market information to the sellers introduces more transparency in the marketplace, allowing the platform better to manage the reward contracts. Because of this trade-off, it is ex-ante unclear whether the platform should give up its informational advantage in exchange for more accurate management of the price-setting incentives.

3.4.1 Full-information Optimal Policy

To analyze our performance benchmark, we first focus on the platform's full-information optimal policy in problem (3.11). Observe that problem (3.11) requires a solution including a contract profile $\boldsymbol{\mathcal{W}}_t$ and an incentive-compatible price profile $\boldsymbol{p}_t$. The platform's goal is to implement the best incentive-compatible price profile to achieve revenue optimality. For any price profile that the platform intends to implement, the challenge is that the sellers would not necessarily be incentivized to follow that price profile. Thus, we consider a natural class of reward contracts that are incentive-compatible for the sellers: compensating the sellers for what they would have gained from deviating from the recommended price profile. The following result presents the key condition characterizing this class of reward contracts.

Proposition 20. (optimal reward contracts) *Under Assumption 7, given* $t \in \{1, \ldots, T\}$

and $(\boldsymbol{\xi}_t, \boldsymbol{\theta}) \in \Xi \times \Theta$, $(\boldsymbol{p}_t^{\boldsymbol{\theta}}, \boldsymbol{\mathcal{W}}_t^{\boldsymbol{\theta}})$ *is an optimal solution to problem* (3.11) *if and only if*

$$\mathcal{W}_{it}^{\boldsymbol{\theta}}(p_{it}^{\boldsymbol{\theta}}) \;=\; \gamma \left[\max_{p \in [l,u]} \left\{ pQ_i(p, \boldsymbol{p}_{-it}^{\boldsymbol{\theta}}, \boldsymbol{\xi}_t, \boldsymbol{\theta}) \right\} - p_{it}^{\boldsymbol{\theta}} Q_i(\boldsymbol{p}_t^{\boldsymbol{\theta}}, \boldsymbol{\xi}_t, \boldsymbol{\theta}) \right] \quad \textit{for } i \in \{1, \ldots, n\}. \tag{3.13}$$

Proposition 20 suggests that any optimal reward contract needs to compensate each seller for the opportunity cost of following the price recommendation. In the expression of $\mathcal{W}_{it}^{\boldsymbol{\theta}}(p_{it}^{\boldsymbol{\theta}})$ in (3.13), $\gamma \max_{p \in [l,u]} \left\{ pQ_i(p, \boldsymbol{p}_{-it}^{\boldsymbol{\theta}}, \boldsymbol{\xi}_t, \boldsymbol{\theta}) \right\}$ captures the maximum possible revenue that seller i can extract from the transaction. In comparison, $\gamma p_{it}^{\boldsymbol{\theta}} Q_i(\boldsymbol{p}_t^{\boldsymbol{\theta}}, \boldsymbol{\xi}_t, \boldsymbol{\theta})$ captures the revenue that seller i collects under the platform's recommended price profile $\boldsymbol{p}_t^{\boldsymbol{\theta}}$. The difference between the two corresponds to the opportunity cost for seller i.

Using the optimality condition in Proposition 20 for problem (3.11), we define an optimal price mapping $\psi^\star : \Xi \times \Theta \to [l, u]^n$ such that for any $(\tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{\theta}}) \in \Xi \times \Theta$,

$$\psi^\star(\tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{\theta}}) \;\in\; \operatorname*{arg\,max}_{\tilde{\boldsymbol{p}} \in [l,u]^n} \sum_{i=1}^{n} \left[\tilde{p}_i Q_i(\tilde{\boldsymbol{p}}, \tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{\theta}}) - \gamma \max_{p_i \in [l,u]} \tilde{p}_i Q_i(p_i, \tilde{\boldsymbol{p}}_{-i}, \tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{\theta}}) \right]. \tag{3.14}$$

We impose the following condition on the optimal price mapping ψ.

Assumption 10. (Lipschitz stability) *There exists some* $L_\psi > 0$ *such that* $\psi^\star(\tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{\theta}})$ *is a* L_ψ*-Lipschitz function of* $(\boldsymbol{\xi}, \boldsymbol{\theta}) \in \Xi \times \Theta$.

Assumption 10 specifies that the problem in (3.14) admits a unique solution, and moreover, the optimal solution is Lipschitz in terms of its parameters. This assumption strengthens the continuity of the solution mapping $\psi^\star$ by Berge's maximum theorem, and is a commonly used assumption in the literature (see, e.g., 82, 83, 103). We also establish in Proposition 23 that there are several special cases of our formulation that satisfy this condition.

Based on the structural insight of the optimality condition (3.13) in Proposition 20, we note that given $(\boldsymbol{\xi}_t, \boldsymbol{\theta}) \in \Xi \times \Theta$, the revenue-optimal price profile for the platform is $\boldsymbol{p}_t^{\boldsymbol{\theta}} = \psi^\star(\boldsymbol{\xi}_t, \boldsymbol{\theta})$. Because the optimality condition (3.13) needs to be satisfied only at the optimal price profile, there are many ways to design optimal reward contracts. In the following result, we propose a class of piecewise quadratic reward contracts that satisfy (3.13).

Lemma 16. *Let* $L_w > 0$ *be such that for all* $(\tilde{\boldsymbol{p}}, \tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{\theta}}) \in [l,u]^n \times \Xi \times \Theta$ *and* $i \in \{1, \ldots, n\}$, $L_w > |\frac{\partial^2}{\partial \tilde{p}_i^2} R_{s_i}(\tilde{\boldsymbol{p}}, \tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{\theta}})|$*, where* R_{s_i} *is as in* (3.5)*. Let* $\boldsymbol{\mathcal{W}}^{\tilde{\boldsymbol{p}}, \tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{\theta}}}$ *be such that for* $i \in \{1, \ldots, n\}$, $\mathcal{W}_i^{\tilde{\boldsymbol{p}}, \tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{\theta}}}(x) = \max\{\tilde{\mathcal{W}}_i^{\tilde{\boldsymbol{p}}, \tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{\theta}}}(x), 0\}$*, where* $\tilde{\mathcal{W}}_i^{\tilde{\boldsymbol{p}}, \tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{\theta}}}(x) = w_{0i} + w_{1i}x + w_{2i}x^2$ *for* $x \in [l,u]$ *and* (w_{0i}, w_{1i}, w_{2i}) *satisfies*

$$
\begin{aligned}
w_{0i} + w_{1i}\tilde{p}_i + w_{2i}\tilde{p}_i^2 &= \gamma \left[\max_{p \in [l,u]} \{p Q_i(p, \tilde{\boldsymbol{p}}_{-i}, \tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{\theta}})\} - \tilde{p}_i Q_i(\tilde{\boldsymbol{p}}, \tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{\theta}}) \right], && \text{(3.15a)} \\
w_{1i} + 2w_{2i}\tilde{p}_i &= -\frac{\partial}{\partial \tilde{p}_i} \gamma \tilde{p}_i Q_i(\tilde{\boldsymbol{p}}, \tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{\theta}}), && \text{(3.15b)} \\
2w_{2i} &= -L_w. && \text{(3.15c)}
\end{aligned}
$$

In period t*, given* $(\boldsymbol{\xi}_t, \boldsymbol{\theta}) \in \Xi \times \Theta$*, let* $\boldsymbol{p}_t^{\boldsymbol{\theta}} = \psi^\star(\boldsymbol{\xi}_t, \boldsymbol{\theta})$*. Then,* $\boldsymbol{\mathcal{W}}^{\boldsymbol{p}_t^{\boldsymbol{\theta}}, \boldsymbol{\xi}_t, \boldsymbol{\theta}}$ *is an optimal reward contract for problem* (3.11)*.*

The class of optimal reward contracts in Lemma 16 is uniquely determined by solving the system of linear equations in (3.15). In particular, (3.15a) guarantees that the optimality condition (3.13) holds.

Based on the definition of the price mapping $\psi^\star$ in (3.14), the optimal contract $\boldsymbol{\mathcal{W}}_t^{\boldsymbol{\theta}}(\cdot)$ in (3.13), and the revenue function $\mathcal{R}(\cdot)$ in (3.4), the platform's full-information optimal revenue in period t is

$$
\mathcal{R}^\star(\boldsymbol{\xi}_t, \boldsymbol{\theta}) = \mathcal{R}(\psi^\star(\boldsymbol{\xi}_t, \boldsymbol{\theta}), \boldsymbol{\xi}_t, \boldsymbol{\theta}). \tag{3.16}
$$

3.4.2 The Do-Nothing Policy

Let us now analyze the performance of arguably the simplest policy for the platform, one in which the platform does nothing. Under this policy, the platform does not provide any price-setting incentives and does not reveal any information. We call this policy the do-nothing (DN) policy and denote it by π_0. Formally, this policy chooses $(\tau, \boldsymbol{\mathcal{W}})$ such that $\tau = \infty$

and $\boldsymbol{\mathcal{W}}_s = \mathbf{0}$ for $s = 1, \ldots, T$. Under π_0, the sellers choose their prices according to (3.7), which captures many multiagent learning frameworks. To study this policy, we first focus on a natural reference point characterized by a Nash equilibrium.

Definition 5. ***(DN Nash Equilibrium)*** *Given $\boldsymbol{\theta} \in \Theta$, a DN Nash equilibrium is a price profile $\bar{\boldsymbol{p}}^{\boldsymbol{\theta}} \in [l, u]^n$ such that for any $i \in \{1, \ldots, n\}$,*

$$\bar{p}_i^{\boldsymbol{\theta}} \;=\; \underset{p_i \in [l,u]}{\arg\max} \left\{ \gamma p_i \mathbb{E}_{\boldsymbol{\xi},\boldsymbol{\varepsilon}} \left\{ Q_i(p_i, \bar{\boldsymbol{p}}_{-i}^{\boldsymbol{\theta}}, \boldsymbol{\xi}, \boldsymbol{\theta}) \right\} \right\}. \tag{3.17}$$

In a DN Nash equilibrium, each seller maximizes its ex-ante expected revenue before feature realizations and does not want to deviate from it given the competitors' price profile. The expectation $\mathbb{E}_{\boldsymbol{\xi},\boldsymbol{\varepsilon}}\{\cdot\}$ in (3.17) is taken over the common distribution of the customer feature vectors $\boldsymbol{\xi}$. We make the following set of assumptions on the DN Nash equilibria and the price process induced by the DN policy of the platform.

Assumption 11. (existence and uniqueness of the DN Nash equilibrium) *For $i \in \{1, \ldots, n\}$ and $\boldsymbol{p}_{-i} \in [l, u]^{n-1}$, seller i's expected revenue $\gamma p_i \mathbb{E}_{\boldsymbol{\xi},\boldsymbol{\varepsilon}}\Big\{ Q_i(p_i, \boldsymbol{p}_{-i}, \boldsymbol{\xi}, \boldsymbol{\theta}) \Big\}$ is log-concave in $p_i \in [l, u]$. Moreover, letting $\bar{U}_i(\boldsymbol{p}) = \frac{\partial}{\partial p_i} \log \Big(\gamma p_i \mathbb{E}_{\boldsymbol{\xi},\boldsymbol{\varepsilon}} \Big\{ Q_i(p_i, \boldsymbol{p}_{-i}, \boldsymbol{\xi}, \boldsymbol{\theta}) \Big\} \Big)$ for $i \in \{1, \ldots, n\}$, we have the following: $\sum_{i=1}^{n} (\bar{U}_i(\boldsymbol{p}) - \bar{U}_i(\tilde{\boldsymbol{p}}))(p_i - \tilde{p}_i) < 0$ for $\boldsymbol{p}, \tilde{\boldsymbol{p}} \in [l, u]^n$ satisfying $\boldsymbol{p} \neq \tilde{\boldsymbol{p}}$.*

Assumption 12. (convergence to the DN Nash equilibrium) *If there is a unique DN Nash equilibrium $\bar{\boldsymbol{p}}^{\boldsymbol{\theta}} \in \operatorname{int}([l, u]^n)$, then the induced price process $\{\boldsymbol{p}_t : t = 1, 2, \ldots\}$ under the platform's DN policy is such that $\boldsymbol{p}_t \to \bar{\boldsymbol{p}}^{\boldsymbol{\theta}}$ almost surely as $t \to \infty$.*

The conditions in Assumption 11 reflect the N-person concave game framework of (146). From Theorems 1 and 2 of (146), we immediately deduce the existence and the uniqueness of the DN Nash equilibrium. The main purpose of this assumption is to simplify our analysis of the DN Nash equilibrium by focusing on a single reference point under information asymmetry. We establish in Proposition 23 that this assumption holds for several special cases of our formulation. To provide further structure on how the sellers choose their prices with limited market information (i.e., how they solve problem (3.7)), we impose Assumption 12 that specifies the convergence of the seller-induced price process $\{\boldsymbol{p}_t : t = 1, 2, \dots\}$. This convergence property has both theoretical and practical foundations. Following the antecedent literature (e.g., 137, 147, 129), we show in Proposition 25 that the convergence property in Assumption 12 holds in several frameworks such as fictitious play, multiagent mirror descent, and Bayesian equilibrium under mild conditions. Some empirical studies (e.g., 139, 147) also describe such adaptive behavior in practice.

From the platform's perspective, the DN Nash equilibrium provides a potential opportunity to extract more revenues relative to the case of full information. The platform's expected revenue in the DN Nash equilibrium is $\mathbb{E}_{\boldsymbol{\xi},\boldsymbol{\varepsilon}}\{\mathcal{R}(\bar{\boldsymbol{p}}^{\boldsymbol{\theta}}, \boldsymbol{\xi}, \boldsymbol{\theta})\}$, where the platform's revenue function $\mathcal{R}$ is as in (3.4) and the DN Nash equilibrium $\bar{\boldsymbol{p}}^{\boldsymbol{\theta}}$ is as in Definition 5. In contrast, when the demand model and the feature information are public knowledge in the marketplace, the platform's expected full-information optimal revenue is $\mathbb{E}_{\boldsymbol{\xi},\boldsymbol{\varepsilon}}\{\mathcal{R}^\star(\boldsymbol{\xi}, \boldsymbol{\theta})\}$, where $\mathcal{R}^\star$ is as in (3.16). Comparing the platform's revenues in these two information states, we characterize the scenarios for the underlying demand parameter vector where the platform can benefit from the DN Nash equilibrium:

$$\tilde{\Theta} \;=\; \Big\{\tilde{\boldsymbol{\theta}} \in \Theta \;:\; \mathbb{E}_{\boldsymbol{\xi},\boldsymbol{\varepsilon}}\Big\{\mathcal{R}^\star(\boldsymbol{\xi}, \tilde{\boldsymbol{\theta}})\Big\} < \mathbb{E}_{\boldsymbol{\xi},\boldsymbol{\varepsilon}}\Big\{\mathcal{R}(\bar{\boldsymbol{p}}^{\tilde{\boldsymbol{\theta}}}, \boldsymbol{\xi}, \tilde{\boldsymbol{\theta}})\Big\}\Big\}. \tag{3.18}$$

The set $\tilde{\Theta}$ consists of all the scenarios in which the platform would strictly prefer the DN Nash equilibrium to the case of full information. Accordingly, we hereafter refer to $\tilde{\Theta}$ as the *beneficial set*. We observe that $\tilde{\Theta}$ plays an important role in characterizing the performance of the DN policy.

Theorem 11. (performance of the DN policy) *Let π_0 be the DN policy of the platform. Then, under Assumptions 7, 11, and 12, we have the following:*

(i) If $\boldsymbol{\theta} \in \operatorname{int}(\tilde{\Theta})$, then there exist $r_0, C_0 > 0$ such that $\Delta_{\boldsymbol{\theta}}^{\pi_0}(T) \le r_0 - C_0 T$ for all

$T \in \{1, 2 \dots\}$.

(ii) If $\boldsymbol{\theta} \in \operatorname{int}(\tilde{\Theta}^c)$*, then there exist* $r_1, C_1 > 0$ *such that* $\Delta_{\boldsymbol{\theta}}^{\pi_0}(T) \geq -r_1 + C_1 T$ *for all* $T \in \{1, 2 \dots\}$.[6]

Theorem 11 establishes two outcomes under the DN policy of the platform. In Theorem 11(ii), we identify that this policy can incur a large revenue loss due to demand model uncertainty. When the demand parameter vector is in the complement of the beneficial set, i.e., $\boldsymbol{\theta} \in \operatorname{int}(\tilde{\Theta}^c)$, the platform favors the case of full information over the DN Nash equilibrium. If the platform does not intervene, the induced price process would move towards the static price profile that is less preferred by the platform, causing a large revenue loss due to demand model uncertainty. This insight is consistent with the earlier studies characterizing the poor performance of a central planner's no-intervention policy for selfish agents (see, e.g., (148) and (118)). However, Theorem 11(i) also identifies a potential learning benefit of the DN policy. When the demand parameter vector $\boldsymbol{\theta}$ is inside the beneficial set $\tilde{\Theta}$, i.e., $\boldsymbol{\theta} \in \operatorname{int}(\tilde{\Theta})$, the platform prefers the DN Nash equilibrium to the case of full information. In other words, if the platform uses the DN policy, this induces the sellers to choose prices in a way that limits the platform's regret. Because the sellers try to maximize their ex-ante expected revenue before feature realizations, the platform's cost for preserving the sellers' incentive compatibility is significantly reduced, resulting in a low revenue loss due to demand model uncertainty. This insight stands in stark contrast to the common intuition about the DN policy.

To illustrate the intuition behind Theorem 11, we present a problem instance.

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[6] The analysis of the boundary case where $\boldsymbol{\theta} \in \partial\Theta$ is deferred to Appendix 3.7. We show that in this boundary case, the growth rate of regret depends on the convergence speed of the price process. Furthermore, we provide some numerical evidence in Figure 3.2 that the boundary case can be negligible.

Example 8. *Consider the following instance of Example 7(iii):*

$$\boldsymbol{Q}(\boldsymbol{p},\boldsymbol{\xi},\boldsymbol{\theta}) \;=\; \begin{pmatrix} 10 \\ 10 \end{pmatrix} - \left[\begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix} + \begin{pmatrix} 1 & \theta_{12} \\ \theta_{21} & 1 \end{pmatrix}\begin{pmatrix} \xi_{11} & \xi_{12} \\ \xi_{21} & \xi_{22} \end{pmatrix}\right]\begin{pmatrix} p_1 \\ p_2 \end{pmatrix}, \tag{3.19}$$

where $\boldsymbol{\theta} = (\theta_{12}, \theta_{21})$, $\boldsymbol{\xi} = \left(\begin{smallmatrix} \xi_{11} & \xi_{12} \\ \xi_{21} & \xi_{22} \end{smallmatrix}\right)$, $\xi_{12} = 2X_1 - 1$, $\xi_{21} = 2X_2 - 1$, $X_1, X_2 \sim \mathit{Beta}\left(\frac{1}{10}, \frac{1}{10}\right)$, *and* $\xi_{11} = \xi_{22} = -\frac{1}{2}(\xi_{12} + \xi_{21})$. *Moreover, we let* $\gamma = 0.9$ *and* $\boldsymbol{\varepsilon}_t \sim \mathcal{N}(0, I_n)$ *for all* $t \in \{1, 2 \ldots\}$.

In the setting of Example 8, suppose that the sellers apply an online mirror descent algorithm (see (3.25) for details). As shown in Figure 3.1, the DN policy can perform extremely well or extremely poorly (Figure 3.1) in different special cases of Example 8.

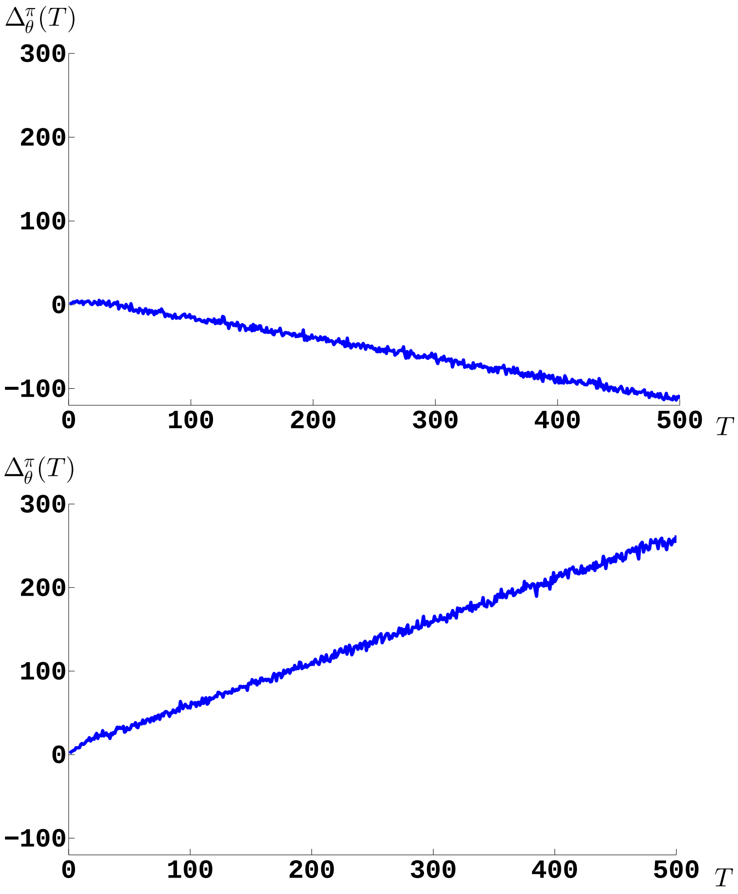

Figure 3.1: The graphs show the platform's T-period regret under the do-nothing policy in two special cases of Example 8. In panel (a), $\boldsymbol{\theta} = (0, 0)$, whereas in panel (b), $\boldsymbol{\theta} = (1, 1)$. The sellers uses the online mirror descent algorithm specified in (3.25) with $\eta_t = \frac{5}{t}$, where $\Theta = [-0.5, 1.5]^2$, $\Xi = [-1, 1]^2$, $l = 1$, and $u = 3$. The displayed regret values are computed by averaging the realized regret over 5000 sample paths.

Existence of the beneficial set. Note that one difference between our paper and the extant literature such as (148) and (122) is that the sellers' incentives can be influenced by the platform's information revelation decision. The drastically different performance outcomes for the DN policy suggest that from a learning perspective, the potential benefit of the policy comes with a nontrivial risk. Thus, it is crucial to have a better understanding about the beneficial set. To that end, there are two key issues that need to be addressed: the existence of the beneficial set, and the estimation of its location. We study the existence of the beneficial set in the remainder of this section, and defer the analysis of its estimation to Section 3.4.4.

To provide some insight into when the beneficial set exists, we study the stochasticity of the feature process. Towards this direction, we first develop the following condition under which the beneficial set $\tilde{\Theta}$ is empty.

Lemma 17. (condition for the nonexistence of the beneficial set) *Under Assumptions 7 and 11, if there exists some constant vector* $\bar{\boldsymbol{\xi}} \in \Xi$ *such that* $\boldsymbol{\xi}_t = \bar{\boldsymbol{\xi}}$ *for any* $t \in \{1, 2 \ldots\}$, *then* $\tilde{\Theta} = \emptyset$.

For any demand parameter vector $\tilde{\boldsymbol{\theta}} \in \Theta$, when the feature vector $\boldsymbol{\xi}_t$ is a constant $\bar{\boldsymbol{\xi}}$, the platform can implement the DN-Nash equilibrium by not offering any reward profile to the sellers (i.e., setting $\boldsymbol{\mathcal{W}}_t = \boldsymbol{0}$), which is a feasible reward contract in problem (3.11). Moreover, the DN-Nash equilibrium $\bar{\boldsymbol{p}}^{\tilde{\boldsymbol{\theta}}}$ satisfies the incentive compatibility constraints (3.11b), suggesting that the price profile $\bar{\boldsymbol{p}}^{\tilde{\boldsymbol{\theta}}}$ is feasible in (3.11). Thus, from the definition of the beneficial set $\tilde{\Theta}$ in (3.18), the platform's full-information optimal revenue $\mathcal{R}^{\star}(\bar{\boldsymbol{\xi}}, \tilde{\boldsymbol{\theta}})$ always dominates the DN-Nash equilibrium revenue $\mathcal{R}(\bar{\boldsymbol{p}}^{\tilde{\boldsymbol{\theta}}}, \bar{\boldsymbol{\xi}}, \tilde{\boldsymbol{\theta}})$.

In contrast, when the feature process $\{\boldsymbol{\xi}_t : t = 1, 2, \ldots\}$ is stochastic, the beneficial set $\tilde{\Theta}$ may become non-empty. To provide some analytical insight, we focus on a class of problem instances from Example 7(iii), which allows us to find the closed-form solutions. Given matrix $\boldsymbol{\theta}_1$, we define $\boldsymbol{J}_{\boldsymbol{\theta}_1} = \boldsymbol{\theta}_1 + \boldsymbol{\Lambda}(\boldsymbol{\theta}_1)$ and $\boldsymbol{\Lambda}(\boldsymbol{\theta}_1) = diag(\boldsymbol{\theta}_1)$. Furthermore, given $\boldsymbol{\theta}_1, \boldsymbol{\theta}_2$, we also let $\boldsymbol{\theta}_{\tilde{\boldsymbol{\xi}}} = \boldsymbol{\theta}_1 + \boldsymbol{\theta}_2 \tilde{\boldsymbol{\xi}}$, $\boldsymbol{\Lambda}_{\tilde{\boldsymbol{\xi}}} = diag(\boldsymbol{\theta}_{\tilde{\boldsymbol{\xi}}})$, $\boldsymbol{G}_{\tilde{\boldsymbol{\xi}}} = \frac{2-\gamma}{4}(\boldsymbol{\theta}_{\tilde{\boldsymbol{\xi}}} + \boldsymbol{\theta}_{\tilde{\boldsymbol{\xi}}}^{\intercal}) + \frac{\gamma}{4}(\boldsymbol{\theta}_{\tilde{\boldsymbol{\xi}}}^{\intercal} \boldsymbol{\Lambda}_{\tilde{\boldsymbol{\xi}}}^{-1} \boldsymbol{\theta}_{\tilde{\boldsymbol{\xi}}} + \boldsymbol{\Lambda}_{\tilde{\boldsymbol{\xi}}}^{-1})$, and $\boldsymbol{H}_{\tilde{\boldsymbol{\xi}}} = [(1 - \frac{\gamma}{2})\boldsymbol{I} + \frac{\gamma}{4}(\boldsymbol{\theta}_{\tilde{\boldsymbol{\xi}}}^{\intercal} \boldsymbol{\Lambda}_{\tilde{\boldsymbol{\xi}}}^{-1} + \boldsymbol{\Lambda}_{\tilde{\boldsymbol{\xi}}}^{-1} \boldsymbol{\theta}_{\tilde{\boldsymbol{\xi}}})]$ for any $\tilde{\boldsymbol{\xi}} \in \Xi$. From the simplicity of notations, we can summarize the expressions of the expected full-information revenue $\mathbb{E}_{\boldsymbol{\xi},\boldsymbol{\varepsilon}}\Big\{\mathcal{R}^{\star}(\boldsymbol{\xi}, \boldsymbol{\theta})\Big\}$ and the DN Nash equilibrium revenue $\mathbb{E}_{\boldsymbol{\xi},\boldsymbol{\varepsilon}}\Big\{\mathcal{R}(\bar{\boldsymbol{p}}^{\boldsymbol{\theta}}, \boldsymbol{\xi}, \boldsymbol{\theta})\Big\}$ in the following result.

Lemma 18. *In Example 7(iii), let* $\boldsymbol{\theta} = (\boldsymbol{\theta}_0, \boldsymbol{\theta}_1, \boldsymbol{\theta}_2)$ *and* $\boldsymbol{\xi}$ *be the random feature such that* $\mathbb{E}_{\boldsymbol{\xi},\boldsymbol{\varepsilon}}\{\boldsymbol{\xi}\} = \boldsymbol{0}$. [7]

(i) Assuming that the diagonal entries of $\boldsymbol{\theta}_1$ *are positive and* $\boldsymbol{J}_{\boldsymbol{\theta}_1}^{-1} \boldsymbol{\theta}_0 \in \mathrm{int}([l, u]^n)$, *the*

---

[7] In the following two results, we impose assumptions on the concavity of the objective functions and on the interior property of the price solution to simplify the analytical solutions.

platform's expected revenue from the DN Nash equilibrium $\bar{\boldsymbol{p}}^{\boldsymbol{\theta}}$ *satisfies*

$$\mathbb{E}_{\boldsymbol{\xi},\boldsymbol{\varepsilon}}\Big\{\mathcal{R}(\bar{\boldsymbol{p}}^{\boldsymbol{\theta}},\boldsymbol{\xi},\boldsymbol{\theta})\Big\} \;=\; (1-\gamma)\boldsymbol{\theta}_0^\intercal\left[\frac{1}{2}[\boldsymbol{J}_{\boldsymbol{\theta}_1}^{-1}]^\intercal + \frac{1}{2}[\boldsymbol{J}_{\boldsymbol{\theta}_1}^{-1}] - [\boldsymbol{J}_{\boldsymbol{\theta}_1}^{-1}]^\intercal\boldsymbol{\theta}_1[\boldsymbol{J}_{\boldsymbol{\theta}_1}^{-1}]\right]\boldsymbol{\theta}_0. \tag{3.20}$$

(ii) Assuming that for any $\tilde{\boldsymbol{p}} \in [l,u]^n$ *and* $\tilde{\boldsymbol{\xi}} \in \Xi$, *the diagonal entries for* $\boldsymbol{\theta}_{\tilde{\boldsymbol{\xi}}}$ *are positive,* $\boldsymbol{G}_{\tilde{\boldsymbol{\xi}}} \succ 0$, $\frac{1}{2}\boldsymbol{G}_{\tilde{\boldsymbol{\xi}}}^{-1}\boldsymbol{H}_{\tilde{\boldsymbol{\xi}}}\boldsymbol{\theta}_0 \in \operatorname{int}([l,u]^n)$ *and* $\frac{1}{2}\boldsymbol{\Lambda}_{\tilde{\boldsymbol{\xi}}}^{-1}\boldsymbol{\theta}_0 - \frac{1}{2}(\boldsymbol{\theta}_{\tilde{\boldsymbol{\xi}}} - \boldsymbol{\Lambda}_{\tilde{\boldsymbol{\xi}}})\tilde{\boldsymbol{p}} \in \operatorname{int}([l,u]^n)$, *the platform's expected full-information optimal revenue satisfies*

$$\mathbb{E}_{\boldsymbol{\xi},\boldsymbol{\varepsilon}}\Big\{\mathcal{R}^\star(\boldsymbol{\xi},\boldsymbol{\theta})\Big\} = \boldsymbol{\theta}_0^\intercal\mathbb{E}_{\boldsymbol{\xi},\boldsymbol{\varepsilon}}\left\{\frac{1}{4}\boldsymbol{H}_{\boldsymbol{\xi}}\boldsymbol{G}_{\boldsymbol{\xi}}^{-1}\boldsymbol{H}_{\boldsymbol{\xi}} - \frac{\gamma}{4}\boldsymbol{\Lambda}_{\boldsymbol{\xi}}^{-1}\right\}\boldsymbol{\theta}_0, \tag{3.21}$$

Given the closed-form expressions in Lemma 18, we observe that whether the beneficial set $\tilde{\Theta}$ is non-empty depends crucially on the eigendecomposition structure of matrix $\boldsymbol{X}(\boldsymbol{\theta}_1,\boldsymbol{\theta}_2)$ below

$$\boldsymbol{X}(\boldsymbol{\theta}_1,\boldsymbol{\theta}_2) \;=\mathbb{E}_{\boldsymbol{\xi},\boldsymbol{\varepsilon}}\left\{\frac{1}{4}\boldsymbol{H}_{\boldsymbol{\xi}}\boldsymbol{G}_{\boldsymbol{\xi}}^{-1}\boldsymbol{H}_{\boldsymbol{\xi}} - \frac{\gamma}{4}\boldsymbol{\Lambda}_{\boldsymbol{\xi}}^{-1}\right] - (1-\gamma)\left[\frac{1}{2}[\boldsymbol{J}_{\boldsymbol{\theta}_1}^{-1}]^\intercal + \frac{1}{2}[\boldsymbol{J}_{\boldsymbol{\theta}_1}^{-1}] - [\boldsymbol{J}_{\boldsymbol{\theta}_1}^{-1}]^\intercal\boldsymbol{\theta}_1[\boldsymbol{J}_{\boldsymbol{\theta}_1}^{-1}]\right\}. \tag{3.22}$$

From the expression of $\boldsymbol{X}(\boldsymbol{\theta}_1,\boldsymbol{\theta}_2)$ in (3.22), we observe that the inequality $\mathbb{E}_{\boldsymbol{\xi},\boldsymbol{\varepsilon}}\Big\{\mathcal{R}^\star(\boldsymbol{\xi},\boldsymbol{\theta})\Big\} < \mathbb{E}_{\boldsymbol{\xi},\boldsymbol{\varepsilon}}\Big\{\mathcal{R}(\bar{\boldsymbol{p}}^{\boldsymbol{\theta}},\boldsymbol{\xi},\boldsymbol{\theta})\Big\}$ is equivalent to $\boldsymbol{\theta}_0^\intercal\boldsymbol{X}(\boldsymbol{\theta}_1,\boldsymbol{\theta}_2)\boldsymbol{\theta}_0 < 0$. Towards this direction, we summarize our observation in the following result.

Proposition 21. ***(Sufficient and necessary condition for existence of the beneficial set)*** *In the problem instances of Example 7(iii) from Lemma 18, we let* $\boldsymbol{\lambda}(\boldsymbol{\theta}_1,\boldsymbol{\theta}_2)$ *be the vector of eigenvalues for matrix* $\boldsymbol{X}(\boldsymbol{\theta}_1,\boldsymbol{\theta}_2)$, *and let* $\boldsymbol{U}(\boldsymbol{\theta}_1,\boldsymbol{\theta}_2)$ *be the corresponding eigenvector matrix. We also let* $\mathcal{V}(\boldsymbol{\theta}_1,\boldsymbol{\theta}_2) = \{\boldsymbol{y}\circ\boldsymbol{y} : \boldsymbol{y} = \boldsymbol{U}(\boldsymbol{\theta}_1,\boldsymbol{\theta}_2)^\intercal\boldsymbol{\theta}_0,\ (\boldsymbol{\theta}_0,\boldsymbol{\theta}_1,\boldsymbol{\theta}_2) \in \Theta\}$[8] *and* $\mathcal{V}^\star(\boldsymbol{\theta}_1,\boldsymbol{\theta}_2)$ *be its corresponding dual cone. Then, we have that* $\tilde{\Theta} \neq \emptyset$ *if and only if there exists* $(\boldsymbol{\theta}_0,\boldsymbol{\theta}_1,\boldsymbol{\theta}_2) \in \Theta$ *such that* $\boldsymbol{\lambda}(\boldsymbol{\theta}_1,\boldsymbol{\theta}_2) \notin \mathcal{V}^\star(\boldsymbol{\theta}_1,\boldsymbol{\theta}_2)$.

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[8] In the definition of $\mathcal{V}(\boldsymbol{\theta}_1,\boldsymbol{\theta}_2)$, "∘" is the Hadamard product.

In the following figure, we numerically demonstrate the shape of the beneficial set $\tilde{\Theta}$ from Example 8. With $\boldsymbol{\theta} = (\theta_{12}, \theta_{21})$, by we let $\delta(\boldsymbol{\theta}) = \mathbb{E}_{\boldsymbol{\xi},\boldsymbol{\varepsilon}}\{\mathcal{R}^\star(\boldsymbol{\xi},\boldsymbol{\theta})\} - \mathbb{E}_{\boldsymbol{\xi},\boldsymbol{\varepsilon}}\{\mathcal{R}(\bar{\boldsymbol{p}}^{\boldsymbol{\theta}},\boldsymbol{\xi},\boldsymbol{\theta})\}$ be the difference of the expected platform's full-information optimal revenue and the expected revenue from the DN Nash equilibrium. We then plot $\delta(\boldsymbol{\theta})$ in Figure 3.2 and furthermore, provide some numerical insight into the structure of the beneficial set $\tilde{\Theta}$ in Figure 3.2.

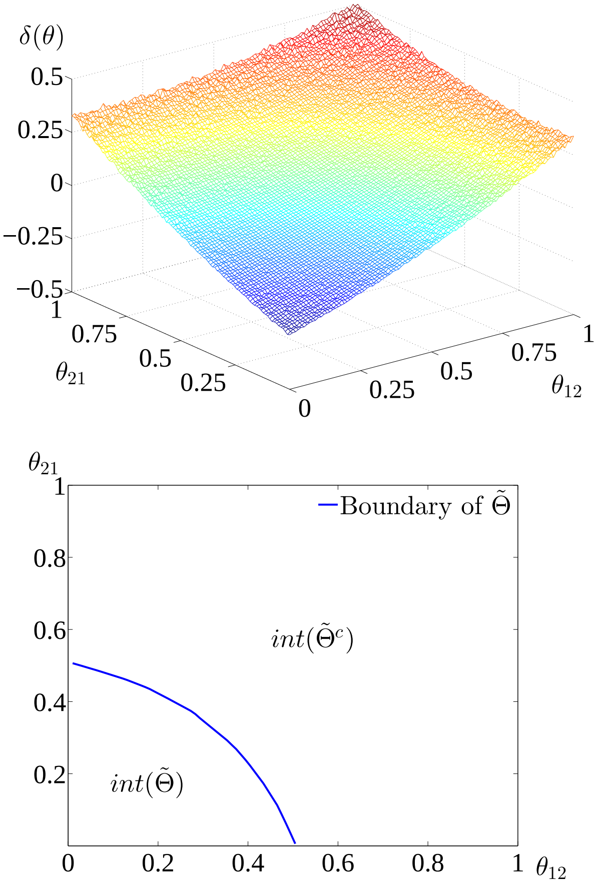

Figure 3.2: Consider the problem instance specified in Example 8.

3.4.3 Reveal-And-Incentivize Policy

In this section, we consider a class of policies in which the platform reveals its private information to the sellers at the beginning of the time horizon. The main purpose of this particular type of information revelation decision is to maximize the transparency in the marketplace and improve the platform's ability to manage the sellers' price-setting incentives. Once the platform gives up its informational advantages, both the sellers and the platform have the same amount of market information, and the only uncertainty in the marketplace is in terms of the demand parameter vector. Given the sellers' incentive compatibility conditions specified in problem (3.8), the platform is able to more accurately design the price-setting incentives in the remaining planning season. As a result, the key challenge becomes the classical exploration-exploitation trade-off in the demand learning process.

Under the RI policy, the platform reveals the market information in the first period i.e., $\tau = 1$. Moreover, it focuses on the following two types of reward contracts: (i) the exploration reward contracts $\boldsymbol{\mathcal{W}}_t^0$ and (ii) the exploitation reward contracts $\boldsymbol{\mathcal{W}}_t^\star$. The purpose of the exploration reward contracts $\boldsymbol{\mathcal{W}}_t^0$ is to incentivize the sellers to pick a price profile that is informative in demand learning. To construct the exploration reward contracts in period t, the platform first randomly picks a price profile $\boldsymbol{p}_t^0$ that satisfies $p_{it}^0 = l + (u - l)X_{it}$ where $X_{it} \sim Bernoulli(\frac{1}{2})$ is i.i.d across seller i in period t (a random extreme point of $[l, u]^n$). We consider the class of reward contracts specified in Lemma 16: given $(\boldsymbol{p}_t^0, \boldsymbol{\xi}_t, \tilde{\boldsymbol{\theta}}_t)$, we let $\boldsymbol{\mathcal{W}}_t^0 = \boldsymbol{\mathcal{W}}^{\boldsymbol{p}_t^0, \boldsymbol{\xi}_t, \tilde{\boldsymbol{\theta}}_t}$. The key idea is that it is incentive compatible for the sellers to follow the price profile $\boldsymbol{p}_t^0$ under the contracts $\boldsymbol{\mathcal{W}}_t^0$. In contrast to the exploration reward contracts, the purpose of the exploitation reward contracts $\boldsymbol{\mathcal{W}}_t^\star$ is to incentivize the sellers to follow the certainty-equivalent price profile $\boldsymbol{p}_t^\star = \psi^\star(\boldsymbol{\xi}_t, \tilde{\boldsymbol{\theta}}_t)$ given $(\boldsymbol{\xi}_t, \tilde{\boldsymbol{\theta}}_t) \in \Xi \times \Theta$. We again consider the class of the reward contracts in Lemma 16 and let $\boldsymbol{\mathcal{W}}_t^\star = \boldsymbol{\mathcal{W}}^{\boldsymbol{p}_t^\star, \boldsymbol{\xi}_t, \tilde{\boldsymbol{\theta}}_t}$. Summarizing the designs of the reward contracts above, we proceed to formally define the platform's RI policy.

Algorithm 1 reveal-and-incentive policy

```
Initiate $t = 0$ and $\tau = 1$
$t \leftarrow t + 1$
repeat
    if $t \leq \lceil \sqrt{T \log(T)} \rceil$ then
        set $\boldsymbol{\mathcal{W}}_t = \boldsymbol{\mathcal{W}}_t^0$
    else
        set $\boldsymbol{\mathcal{W}}_t = \boldsymbol{\mathcal{W}}_t^\star$
    end if
    update the maximum likelihood estimator $\tilde{\boldsymbol{\theta}}_{t+1} \in \mathcal{P}_\Theta\Big\{ \arg\max_{\tilde{\boldsymbol{\theta}}} \big\{ \ell_t\big(\tilde{\boldsymbol{\theta}} \big| \mathcal{H}_t\big)\big\}\Big\}$;
until $t = T$
```

In summary of the RI policy, after the platform reveals the market information at the beginning of the planning season, the platform devotes the first $\lceil \sqrt{T \log(T)} \rceil$ periods to implement the exploration reward contracts and the remaining periods to implement the exploitation contracts.

Before evaluating the performance of the platform's RI policy, we study the convergence of the estimator process. Given that we have a general demand model $\boldsymbol{Q}$ with a general noise process $\{\boldsymbol{\varepsilon}_t\}$ in this paper, we need to impose some further structures. We summarize the technical conditions below and verify in Proposition 24 later in this section that some widely-applied problem instances of Example 7(i) - (iii), all satisfy these conditions. Given the twice continuous differentiability of $\ell_t(\cdot)$ in Assumption 8, we can further denote by $\mathcal{I}_t(\tilde{\boldsymbol{\theta}}|\mathcal{H}_t)$ the induced empirical Fisher information matrix: for any $i, j \in \{1, \ldots, \kappa_{\boldsymbol{\theta}}\}$, the $(i,j)^{th}$ component of $\mathcal{I}_t(\tilde{\boldsymbol{\theta}}|\mathcal{H}_t)$ is given by $-\frac{\partial^2}{\partial\theta_i \partial\theta_j} \ell_t(\tilde{\boldsymbol{\theta}}|\mathcal{H}_t)$. For the rest of the paper, we consider the following conditions on $\mathcal{I}_t(\tilde{\boldsymbol{\theta}}|\mathcal{H}_t)$.

Assumption 13. ***(Information accumulation)*** *There exists $\bar{\lambda}_I, \kappa_I > 0$ such that for any $T \geq 2$, $t \in \{1, \ldots, T\}$, and $\mathcal{H}_t$ induced by $\{(\boldsymbol{p}_s, \boldsymbol{\xi}_s, \boldsymbol{D}_s) : s = 1, \ldots t\}$, we have* *(i)* $\lambda_{max}\Big\{\mathcal{I}_t(\tilde{\boldsymbol{\theta}}|\mathcal{H}_t) - \mathcal{I}_{t-1}(\tilde{\boldsymbol{\theta}}|\mathcal{H}_{t-1})\Big\} \ \le\ \bar{\lambda}_I$ *for any* $\tilde{\boldsymbol{\theta}} \in \Theta$*;*

(ii) $\frac{1}{\kappa_I}\mathcal{I}_t(\tilde{\boldsymbol{\theta}}_2|\mathcal{H}_t) \preceq \mathcal{I}_t(\tilde{\boldsymbol{\theta}}_1|\mathcal{H}_t) \preceq \kappa_I \mathcal{I}_t(\tilde{\boldsymbol{\theta}}_2|\mathcal{H}_t)$ *for any* $\tilde{\boldsymbol{\theta}}_1, \tilde{\boldsymbol{\theta}}_2 \in \Theta$*.*

Moreover, there exists $\lambda_0, k_0 > 0$ *such that given the true demand parameter vector* $\boldsymbol{\theta} \in \Theta$*, if policy* π *induces a price path* $\{\boldsymbol{p}_s : s = 1, 2, \ldots, t\}$ *such that for some* $\mathcal{T}_t \subset \{1, \ldots, t\}$ *with* $|\mathcal{T}_t| = \tau(t) \in \mathbb{N}_{++}$*, the price vector* $\{\boldsymbol{p}_s : s \in \mathcal{T}_t\}$ *are i.i.d with* $Cov(\boldsymbol{p}_s) \succ 0$ *for* $s \in \mathcal{T}_t$*, then*

$$\mathbb{P}^{\pi}_{\boldsymbol{\theta}}\Big\{\lambda_{min}\left(\mathcal{I}_t\left(\boldsymbol{\theta}|\mathcal{H}_t\right)\right) \le \lambda_0 \tau(t)\Big\} \ \le\ \frac{k_0}{\tau(t)}. \tag{3.23}$$

There are several conditions that specified above. Assumption 13(i) characterizes the uniform boundedness of the eigenvalues for the per-period empirical Fisher information matrix. Assumption 13(ii) defines a variant of the self-concordant condition from convex optimization, which has been established in the class of generalized linear demand models in the dynamic learning literature (e.g., see 101, 83, 103, 102). Condition (3.23) connects the growth rate of the information matrix's minimum eigenvalue with the number of i.i.d samples. Under Assumption 13, we can characterize the convergence of the demand parameter vector's estimator $\tilde{\boldsymbol{\theta}}_t$ under the RI policy.

Proposition 22. *(**Convergence of the estimator**) Under Assumptions 8 and 13, let* π_1 *be the RI policy. There exists* $K_1 > 0$ *such that for any* $T \ge 2$ *and* $t \in \{2, \ldots, T\}$*, we have* $\tau(t) = \min\{t, \lceil\sqrt{T\log(T)}\rceil\}$ *and moreover,*

$$\mathbb{E}^{\pi_1}_{\boldsymbol{\theta}}\Big\{\|\tilde{\boldsymbol{\theta}}_t - \boldsymbol{\theta}\|_2^2\Big\} \ \le\ K_1 \frac{\log(t)}{\tau(t)}. \tag{3.24}$$

Proposition 22 establishes the convergence rate of the estimator $\tilde{\boldsymbol{\theta}}_t$. We highlight three aspects of this convergence result. First, given the generalized linear demand models in the demand learning literature (e.g., see 101, 83, 103, 102), our paper makes an important extension to the generalized demand model. By showing that Assumption 13 holds for Example 7(i)-(iii) in Proposition 24 of Section 3.5.2, we establish that our framework problem covers these generalized linear demand models as special cases. Second, it is worth pointing out that the analysis in (82) and (103) is based on the critical assumption that the estimator is evaluated only with the i.i.d random samples from the exploration periods. Since $\{\|\tilde{\boldsymbol{\theta}}_t - \boldsymbol{\theta}\|_2^2 : t = 1, 2, \dots\}$ is not necessarily monotone if we include non-i.i.d samples in the online estimation procedure, it is ex-ante unclear whether the convergence arguments still hold. By establishing the consistency of the estimator, our work largely relaxes this assumption by including the entire up-to-date sample history in the online procedure. Third, note that this particular relaxation also better fits our model setting: from the implementation perspective, it is much easier for the platform to share the demand models that the sellers can verify simply with the entire samples.

Based on the observation in Proposition 22 and our design of the RI policy specified in Algorithm 1, we establish the performance for the RI policy in the following result.

Theorem 12. ***(Performance of the RI policy)*** *Under Assumptions 7, 8, 10, and 13, let* π_1 *be the platform's RI policy. There exist* $r_2, C_2 > 0$ *such that* $\Delta_{\boldsymbol{\theta}}^{\pi_1}(T) \leq r_2 + C_2\sqrt{T \log(T)}$ *for all* $T \in \{1, 2, \dots\}$.

The key idea in Theorem 12 is to balance the regret induced in the exploration and exploitation periods. By setting the length of the exploration periods to be $\lceil\sqrt{T\log(T)}\rceil$, we ensure that the regret induced in the exploration periods is bounded above by $\mathcal{O}(\sqrt{T\log(T)})$. In the exploitation periods, given the estimator convergence property in Proposition 22, we can establish that the per-period regret caused by estimation error is bounded by $\mathcal{O}(\sqrt{\frac{\log(T)}{T}})$. As a result, the regret from the exploitation periods can be upper bounded $\mathcal{O}(\sqrt{T\log(T)})$. Note that this argument allows us to improve the asymptotic regret bound from $\mathcal{O}(\sqrt{T}\log(T))$ in the past literature (e.g., see 101, 83, 103). To numerically illustrate the performance of the RI policy, we again explore the problem instance of Example 8 (as in Figure 3.1).

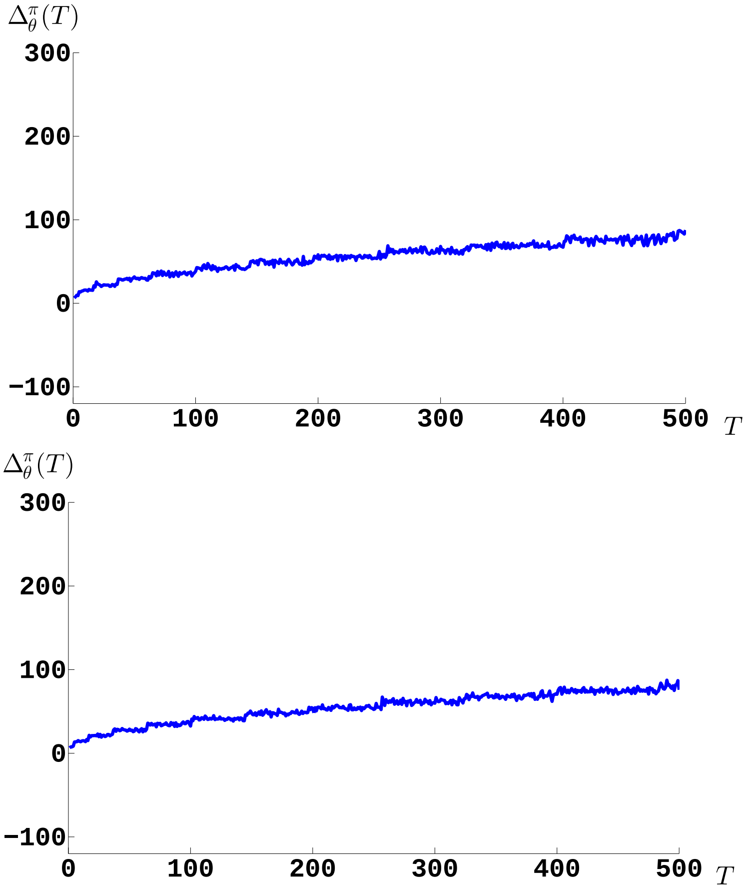

Figure 3.3: Consider the two problem instances of Example 8 in Figure 3.1.

From the learning perspective, we observe by comparing the regret plots in Figure 3.1 and Figure 3.3 that the RI policy is very conservative. The plot in Figure 3.3 suggests that when $\boldsymbol{\theta} \in \mathrm{int}(\tilde{\Theta}^c)$, the platform effectively avoids large regret by providing more accurate price-setting incentives under the challenge of the exploration-exploitation trade-off. In comparison, when $\boldsymbol{\theta} \in \mathrm{int}(\tilde{\Theta})$, it is perhaps equally interesting to observe from Figure 3.3 that the RI policy limits the platform's opportunity to exploit its informational advantages over the sellers. The key takeaway is that while the conservativeness of the RI protects the platform from incurring large regret, it simultaneously eliminates the opportunity for the platform to benefit from its informational advantages.

3.4.4 Strategic-Reveal-And-Incentivize Policy

Observe that either the DN policy of the RI policy covers only one of the two ends in terms of information revelation, which is either too risky or too conservative. In this section, we consider the strategic-reveal-and-incentivize(SRI) policy which more carefully manages the timing of information revelation. We want to highlight that there are two hurdles to overcome in the designing of the SRI policy. The first one is about the statistical evaluation of whether $\boldsymbol{\theta} \in \operatorname{int}(\tilde{\Theta})$ in the demand learning process. Since the platform's preference for the DN Nash equilibrium depends on whether $\boldsymbol{\theta} \in \operatorname{int}(\tilde{\Theta})$, challenges are induced by the fact that the true demand parameter vector $\boldsymbol{\theta}$ are unknown to the platform throughout the planning season. Thus, it is crucial for the platform to avoid any false positive error that causes failure to intervene. The second challenge is induced by the fact that the platform has limited knowledge about the sellers' pricing decision process before it reveals the market information. As a result, while the platform holds information advantages, it is impossible for the platform to accurately offer price-setting incentives for the purpose of price coordination.

To address these two challenges, we design the SRI policy such that it includes three types of reward contracts and an statistical test on $\boldsymbol{\theta} \in \operatorname{int}(\tilde{\Theta})$. The three types of reward contracts include: (i) the exploration reward contracts $\boldsymbol{\mathcal{W}}_t^0$, (ii) the protective exploitation reward contracts $\boldsymbol{\mathcal{W}}_t^\star$, and (iii) the do-nothing contracts $\boldsymbol{\mathcal{W}}_t^{DN}$. In the exploration contracts $\boldsymbol{\mathcal{W}}_t^0$, given the platform's limited knowledge about the sellers' approximate revenue functions $\tilde{R}_{it}$ (i.e., the uniform bounds M_R^0 and M_R^1 for function values and its derivatives in Assumption 9), we let $\mathcal{W}_{it}^0(p) = M_R^0 + M_R^1 u + Z_{it} M_R^1 p$ where $Z_{it} = 2X_{it} - 1$ and $X_{it} \sim \mathit{Bernoulli}(\frac{1}{2})$ for any $i \in \{1, \ldots, n\}$ and any $p \in [l, u]$. The main goal of exploration reward contracts $\boldsymbol{\mathcal{W}}_t^0$ is to incentivize the sellers to pick a price profile $\boldsymbol{p}_t \in \{l, u\}^n$ that is i.i.d. In comparison, to design the protective exploitation reward contracts $\boldsymbol{\mathcal{W}}_t^\star$, we leverage the construction in Lemma 16 again and let $\boldsymbol{\mathcal{W}}_t^\star = \boldsymbol{\mathcal{W}}^{\boldsymbol{p}_t^\star, \boldsymbol{\xi}_t, \tilde{\boldsymbol{\theta}}_t}$ such that given $(\boldsymbol{\xi}_t, \tilde{\boldsymbol{\theta}}_t) \in \Xi \times \Theta$, it is incentive-compatible for the sellers to pick the certainty-equivalence price profile $\boldsymbol{p}_t^\star = \psi^\star(\boldsymbol{\xi}_t, \tilde{\boldsymbol{\theta}}_t)$. The do-nothing reward contracts are defined as $\boldsymbol{\mathcal{W}}_t^{DN} = \mathbf{0}$. Besides the reward contracts, the

SRI policy includes a statistical testing procedure on estimating whether $\boldsymbol{\theta} \in \text{int}(\tilde{\Theta})$. From the definition of the beneficial set $\tilde{\Theta}$ in (3.18), given the demand parameter vector's estimator $\tilde{\boldsymbol{\theta}}_t$ in period t, we let the statistical test $\frac{1}{t}\sum_{s=1}^{t}\left[\mathcal{R}^{\star}(\boldsymbol{\xi}_s, \tilde{\boldsymbol{\theta}}_t) - \mathcal{R}(\bar{\boldsymbol{p}}^{\tilde{\boldsymbol{\theta}}_t}, \boldsymbol{\xi}_s, \tilde{\boldsymbol{\theta}}_t)\right] < 0$ to evaluate whether $\boldsymbol{\theta} \in \text{int}(\tilde{\Theta})$. It is worth highlighting that the SRI policy relies on the following two observations: (1) despite the fact that the platform has limited knowledge about the sellers' revenue maximization problems before its information disclosure, it is still possible to offer price-setting incentives to achieve the exploration purpose; (2) there is an efficient statistical testing procedure to evaluate the location of the true demand parameter vector relative to the beneficial set.

Summarizing all of the components above, we formally define the SRI policy below.

Algorithm 2 strategic-reveal-and-incentivize policy

```
Initiate $t = 0$
repeat
    $t \leftarrow t + 1$
    set $\boldsymbol{\mathcal{W}}_t = \boldsymbol{\mathcal{W}}_t^0$
until $t = \lceil \sqrt{T \log(T)} \rceil$
Update the estimator $\tilde{\boldsymbol{\theta}}_{t+1} \in \mathcal{P}_\Theta \Big\{ \arg\max_{\tilde{\boldsymbol{\theta}}} \big\{ \ell_t \big( \tilde{\boldsymbol{\theta}} \big| \mathcal{H}_t \big) \big\} \Big\}$
if $\frac{1}{t} \sum_{s=1}^{t} \Big[ \mathcal{R}^\star(\boldsymbol{\xi}_s, \tilde{\boldsymbol{\theta}}_t) - \mathcal{R}(\bar{\boldsymbol{p}}^{\tilde{\boldsymbol{\theta}}_t}, \boldsymbol{\xi}_s, \tilde{\boldsymbol{\theta}}_t) \Big] < 0$ then
    set $\tau = \infty$
else
    set $\tau = \lceil \sqrt{T \log(T)} \rceil + 1$
end if
repeat
    $t \leftarrow t + 1$
    if $t \geq \tau$, then $\boldsymbol{\mathcal{W}}_t = \boldsymbol{\mathcal{W}}_t^\star$
    if $t \leq \tau$, then $\boldsymbol{\mathcal{W}}_t = \boldsymbol{\mathcal{W}}_t^{DN}$
    Update the estimator $\tilde{\boldsymbol{\theta}}_{t+1} = \max_{\tilde{\boldsymbol{\theta}} \in \Theta} \ell_t(\tilde{\boldsymbol{\theta}} | \mathcal{H}_t)$;
until $t = T$
```

Observe that the SRI policy has an exploration stage and an exploitation stage. In the first $\lceil \sqrt{T \log(T)} \rceil$ periods, the main purpose for the platform is to collect informative price samples and learn about the demand parameter vector. Although the platform cannot accurately manage the price-setting incentives due to its limited knowledge about the sellers' decision-making process, it is still able to incentivize the sellers to pick informative prices by offering the exploration reward contracts $\boldsymbol{\mathcal{W}}_t^0$. At the end of the exploration period (or the beginning of period $t = \lceil \sqrt{T \log(T)} \rceil + 1$), the platform evaluates whether $\boldsymbol{\theta} \in \text{int}(\tilde{\Theta})$. Note that in the statistical test $\frac{1}{t} \sum_{s=1}^{t} \Big[\mathcal{R}^\star(\boldsymbol{\xi}_s, \tilde{\boldsymbol{\theta}}_t) - \mathcal{R}(\bar{\boldsymbol{p}}^{\tilde{\boldsymbol{\theta}}_t}, \boldsymbol{\xi}_s, \tilde{\boldsymbol{\theta}}_t) \Big] < 0$, it is sufficient for the platform to rely only the the feature history $\{\boldsymbol{\xi}_s : s = 1, \ldots t\}$ and the update-to-date estimator $\tilde{\boldsymbol{\theta}}_t$. Given the estimator $\tilde{\boldsymbol{\theta}}_t$, we can also estimate the DN Nash equilibrium price profile by $\bar{\boldsymbol{p}}^{\tilde{\boldsymbol{\theta}}_t}$ and the platform's corresponding long-run expected revenue by $\frac{1}{t}\sum_{s=1}^{t}\mathcal{R}(\bar{\boldsymbol{p}}^{\tilde{\boldsymbol{\theta}}_t}, \boldsymbol{\xi}_s, \tilde{\boldsymbol{\theta}}_t)$. In comparison, the platform's expected full-information revenue can be estimated by $\frac{1}{t}\sum_{s=1}^{t}\mathcal{R}^{\star}(\boldsymbol{\xi}_s, \tilde{\boldsymbol{\theta}}_t)$. If the testing outcome is $\frac{1}{t}\sum_{s=1}^{t}\Big[\mathcal{R}^{\star}(\boldsymbol{\xi}_s, \tilde{\boldsymbol{\theta}}_t) - \mathcal{R}(\bar{\boldsymbol{p}}^{\tilde{\boldsymbol{\theta}}_t}, \boldsymbol{\xi}_s, \tilde{\boldsymbol{\theta}}_t)\Big] < 0$, which suggests that $\tilde{\boldsymbol{\theta}} \in \operatorname{int}(\tilde{\Theta})$, then the platform would continue to hide the market information from the sellers (i.e., $\tau = \infty$) and move on with the do-nothing reward contracts $\boldsymbol{\mathcal{W}}_t^{DN}$. Alternatively, the platform would reveal the market information to the sellers (i.e., $\tau = \lceil\sqrt{T\log(T)}\rceil + 1$) and exploitation reward contracts $\boldsymbol{\mathcal{W}}_t^{\star}$.

The SRI policy allows the platform to be strategic about whether to reveal its market information, which should combine the merits of the aforementioned policies. Towards this direction, we summarize the performance of the SRI policy from the learning perspective in the following result.

Theorem 13. ***(Performance of the SRI policy)*** *Under Assumptions 7 - 13, let π_2 be the SRI policy.*

(i) If $\boldsymbol{\theta} \in \operatorname{int}(\tilde{\Theta})$, then there exist $r_3, C_3 > 0$ such that $\Delta_{\boldsymbol{\theta}}^{\pi_2}(T) \leq r_3 - C_3 T$ for all $T \in \{1, 2, \dots\}$.

(ii) If $\boldsymbol{\theta} \in \operatorname{int}(\tilde{\Theta}^c)$, then there exist $r_4, C_4 > 0$ such that $\Delta_{\boldsymbol{\theta}}^{\pi_2}(T) \leq r_4 + C_4\sqrt{T\log(T)}$ for all $T \in \{1, 2 \dots\}$.

By comparing Theorem 13 with Theorem 11 in Section 3.4.2 and Theorem 12 in Section 3.4.3, we immediately observe that the SRI policy addresses the disadvantages in both the DN policy and the RI policy. In the "good" cases (i.e., $\boldsymbol{\theta} \in \operatorname{int}(\tilde{\Theta})$), the SRI policy allows the platform to exploit its informational advantages with high confidence and achieve a total regret of $\mathcal{O}(-T)$. In contrast, when the platform faces the "bad" cases (i.e., $\boldsymbol{\theta} \in \operatorname{int}(\tilde{\Theta}^c)$), the SRI policy protects the platform from failing to intervene, and the strategy of active coordination of the market incurs a regret of $\mathcal{O}(\sqrt{T\log(T)})$. These properties demonstrate that by gathering enough sufficient knowledge before making a strategic decision on information revelation, the platform improves the robustness of performance via the SRI policy.

To illustrate this observation from a numerical perspective, we use the aforementioned problem instance to generate the regret plot for the SRI policy.

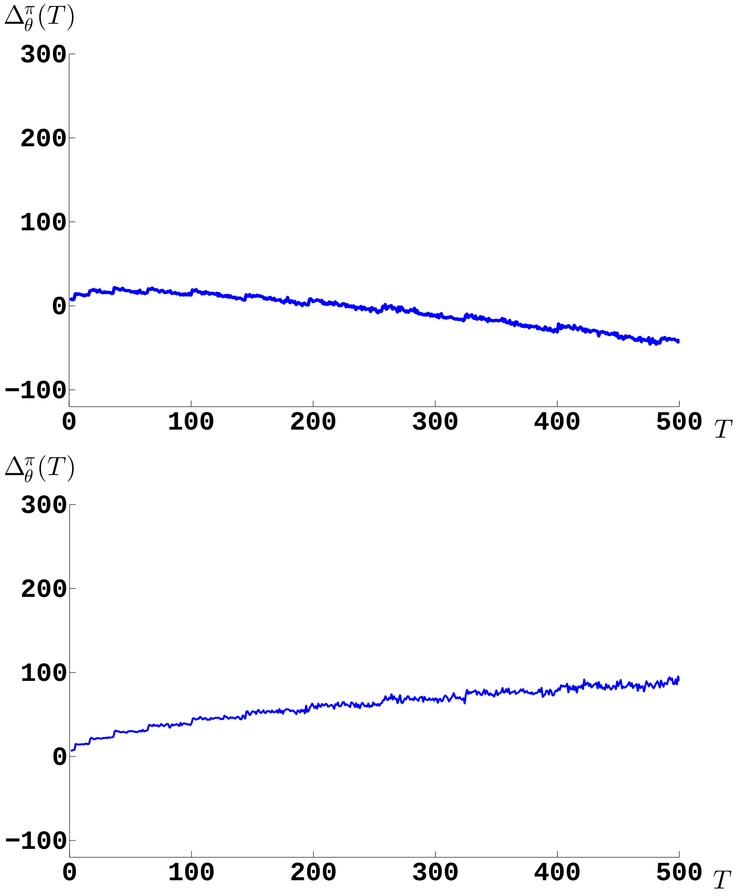

Figure 3.4: Consider the same problem instances of Example 8 as in Figure 3.1.

The numerical performance in Figure 3.4 suggests that the SRI policy allows the platform to exploit the informational advantages in the case of $\boldsymbol{\theta} \in \text{int}(\tilde{\Theta})$, which outperforms the regret of RI policy (as shown in Figure 3.3) and delivers a comparable outcome to that from the DN policy (see Figure 3.1). In the case of $\boldsymbol{\theta} \in \text{int}(\tilde{\Theta}^c)$, Figure 3.4 shows that the SRI policy helps the platform avoid the large revenue loss due to demand model uncertainty as demonstrated in the DN policy (see Figure 3.1). Meanwhile, like the RI policy (see Figure 3.3), the SRI policy achieves a regret of sublinear order (see Figure 3.4). The numerical example summarizes the success of the SRI policy against both "good" and "bad" scenarios.

3.5 Connecting the Examples and the Assumptions

Since the results that we have established in this paper depend on Assumptions 7 - 13, it is important to understand whether these assumptions can apply to problem instances such as the ones in Example 7(i) - (iii). In this section, we establish the connection between the problem instances and the assumptions of the paper.

3.5.1 Assumptions related to the Demand Model

We first focus on Assumptions 7, 10, and 11, which characterize properties of the demand model $\boldsymbol{Q}$. In the following result, we establish that these assumptions admit a nontrivial set of problem instances.

Proposition 23. *There exists non-empty subset of problem instances of Example 7(i) - (iii) in which Assumptions 7, 10, and 11 hold.*

We would like to point out that the continuity conditions in Assumption 7 and the concavity conditions in Assumption 11 can cover a very broad class of problem instances beyond Example 7(i) - (iii). In comparison, the main purpose of Lipschitz stability condition in Assumption 10 is to facilitate the analysis of the technical results. Without this assumption, we believe that the regret upper bound in Theorem 12 and Theorem 13(ii) could change from $\mathcal{O}(\sqrt{T\log(T)})$ to $\mathcal{O}(T^{\frac{2}{3}}\log^{\frac{1}{4}}(T))$, which is still sublinear in terms of the number of periods T.

3.5.2 Assumptions related to the Platform's Learning Process

Next, we consider the Assumptions 8 and 13 regarding the platform's learning process. We use the following result to establish that these two assumptions in Example 7(i) - (iii).

Proposition 24. *Consider problem instances of Example 7(i) - (iii) with i.i.d feature process* $\{\boldsymbol{\xi}_t : t = 1, 2 \ldots\}$ *satisfying* $Cov(\boldsymbol{\xi}_t) \succ 0$ *for all* $t \in \{1, 2 \ldots\}$.

(i) Under the linear demand model in Example 7(i), if the demand noise process $\{\boldsymbol{\varepsilon}_t : t = 1, 2, \ldots\}$ *follow a multivariate normal distribution i.e.,* $\boldsymbol{\varepsilon}_t \sim \mathcal{N}(\mathbf{0}, \Sigma_t)$ *where the eigenvalues of the covariance matrix* Σ_t *are uniformly bounded by* $[\underline{\lambda}_\epsilon, \overline{\lambda}_\epsilon]$ *for some* $\overline{\lambda}_\epsilon > \underline{\lambda}_\epsilon > 0$*, then Assumption 8 and Assumption 13 hold.*

(ii) Under the multinomial logit demand model in Example 7(ii), Assumption 8 and Assumption 13 hold.

(iii) Under the variation class of linear demand model in Example 7(iii) with the same demand noise process $\{\boldsymbol{\varepsilon}_t : t = 1, 2, \ldots\}$ *as in part (i), Assumption 8 and Assumption 13 hold.*

3.5.3 Assumptions related to the Sellers' Pricing Decisions.

In this section, we bridge the gap between the sellers' approximate revenue optimization problems in (3.7) and some widely-applied multiagent learning frameworks in the literature.

Multiagent reinforcement learning. One class of multiagent learning models extends the reinforcement learning algorithms into the decentralized setting. For example, (131), (132), (133) (134),(135), and (136) applied Q-Learning algorithms into the settings of cooperative and non-cooperative games. More recently, based on (149)), the extensions to the multiagent mirror descent algorithms have also been investigated by (124), (125), (126), (127), (128), (129) and (130). Using the multiagent mirror descent framework as an example in our setting, we can fit the approximate revenue optimization problem (3.7) into this framework by defining

$$\tilde{R}_{i(t+1)}(p) \;=\; \eta_t \tilde{g}_{it} p - M_{h_i}(p, p_{it}). \tag{3.25}$$

In (3.25), function $M_{h_i}(p, \tilde{p})$ is the Bregman divergence which can be expressed as $M_{h_i}(x, y) = h_i(x) - h_i(y) - h_i'(y)(x - y)$ for some continuously differentiable l_h-strongly convex function $h_i : \mathbb{R} \to \mathbb{R}$ for some $l_h > 0$. It can be easily observed that when $h_i(x) = \frac{1}{2}x^2$ and

$\mathcal{W}_{it}(p) = 0$ for any $p \in [l, u]$, then solving problem (3.7) in the scenario with an interior optimal solution is the same as the online stochastic gradient descent algorithm i.e., $p_{i(t+1)} = p_{it} + \eta_t \tilde{g}_{it}$. Random value $\tilde{g}_{it}$ captures seller i's perceived direction of updating her price to improve her revenue in the next iteration. We assume that $|\tilde{g}_{it}| < \bar{\sigma}_g$ for some $\bar{\sigma}_g > 0$ and $\mathbb{E}_{\boldsymbol{\xi},\boldsymbol{\varepsilon}}\{\tilde{g}_{it}|\mathcal{F}^0_{t-1}\} = \frac{\frac{\partial}{\partial p_{it}}\mathbb{E}_{\boldsymbol{\xi},\boldsymbol{\varepsilon}}\{R_{s_i}(\boldsymbol{p}_t,\boldsymbol{\xi}_t,\boldsymbol{\theta})|\mathcal{F}^0_{t-1}\}}{\mathbb{E}_{\boldsymbol{\xi},\boldsymbol{\varepsilon}}\{R_{s_i}(\boldsymbol{p}_t,\boldsymbol{\xi}_t,\boldsymbol{\theta})|\mathcal{F}^0_{t-1}\}}$ where $\mathcal{F}^0_t = \sigma(\boldsymbol{\varepsilon}_1 \ldots, \boldsymbol{\varepsilon}_t)$. In this implementation, $\mathbb{E}_{\boldsymbol{\xi},\boldsymbol{\varepsilon}}\{\tilde{g}_{it}|\mathcal{F}^0_{t-1}\}$ can treated as the normalized derivative for seller i's expected revenue function (3.5) in the next period. Since function (3.25) is one-dimensional, it is sufficient to use $\tilde{g}_{it}$ to capture the perceived direction of revenue improvement. It is worth noting that the multiagent mirror descent algorithm framework makes the assumption that the perceived direction of price update is unbiased under filtration $\mathcal{F}^0_t$. Relaxation of this assumption is allowed as long as the biased estimations of the gradient directions diminishes. The exogenous step length η_t captures the sellers' willingness to update their prices in each period.

Fictitious play. The approximate revenue optimization problem in (3.7) can also be implemented in the fictitious play framework: each agent chooses a strategy given the empirical distribution of the competitors' strategies (e.g., see 137, 138, 139, 140, 141, 150, 151, 152, 153). For example, we can implement the most classical setting in (137): for any seller i in period t, let $\tilde{R}_{it}(p)$ be such that the maximizer $p_{it} = \arg\max_{p\in[l,u]} \tilde{R}_{it}(p)$ satisfies

$$p_{it} = \underset{p\in\mathcal{S}_i}{\arg\max} \ \tilde{R}_i(p, \tilde{\boldsymbol{b}}_{it}), \tag{3.26}$$

where in (3.26), $\mathcal{S}_i \subset [l, u]$ is a finite subset. Vector $\tilde{\boldsymbol{b}}_{it}$ is considered as seller i's observed empirical distribution over the competitors' price profile i.e., $\tilde{\boldsymbol{b}}_{it} = (\tilde{b}_{it}(\boldsymbol{p}_{-i}))_{\boldsymbol{p}_{-i}\in\mathcal{S}_{-i}}$ where we let $\tilde{b}_{it}(\tilde{\boldsymbol{p}}_{-i}) = \frac{\eta^t_i(\tilde{\boldsymbol{p}}_{-i})}{\sum_{\tilde{\boldsymbol{p}}_{-i}\in\mathcal{S}_{-i}} \eta^t_i(\tilde{\boldsymbol{p}}_{-i})}$ and $\eta^t_i(\boldsymbol{p}_{-i})$ be the total number of observations for price profile $\boldsymbol{p}_{-i} \in \mathcal{S}_{-i}$ in period t. Function $\tilde{R}_i(p, \tilde{\boldsymbol{b}}_{it})$ can be perceived as seller i's expected payoff induced by belief vector $\tilde{\boldsymbol{b}}_{it}$ i.e., $\tilde{R}_i(p, \tilde{\boldsymbol{b}}_{it}) = \sum_{\tilde{\boldsymbol{p}}_{-i}\in\mathcal{S}_{-i}} b_{it}(\tilde{\boldsymbol{p}}_{-i})\mathbb{E}_{\boldsymbol{\xi},\boldsymbol{\varepsilon}}\{R_{s_i}(p, \tilde{\boldsymbol{p}}_{-i}, \boldsymbol{\xi}_t, \boldsymbol{\theta})\}$. Similar as before, the fictitious play framework assumes that the bias mainly comes from the

the price profile induced by the competitors.

Bayesian equilibrium. Besides the two learning frameworks above, we can further consider Bayesian equilibrium models to capture the sellers' pricing decisions in (3.7), which has a more rigorous economic foundation. Based on the seminal paper by (143) for example, we implement an extension of the dynamic pricing paper (58) into a multiagent learning setting: we assume that there is a finite number of possibilities of the demand models $\{Q_i^{(k)}(\boldsymbol{p}, \boldsymbol{\theta}^{(k)}) : k = 0, \ldots, K\}$ for seller i given $K \in \mathbb{N}_+$ and $\boldsymbol{p} \in [l, u]^n$. We assume that $Q_i^{(k)}(\boldsymbol{p}, \boldsymbol{\theta}^{(k)})$ is twice continuously differentiable in $\boldsymbol{p} \in [l, u]^n$ for any $k \in \{0, \ldots, K\}$. Every possible demand model from a seller's perspective is formed at the beginning. Since the sellers are not able to observe customers features $\{\boldsymbol{\xi}_t : t = 1, 2 \ldots\}$, we assume that $\boldsymbol{\theta}^{(0)} = \boldsymbol{\theta}$ and $Q_i^{(0)}(\boldsymbol{p}, \boldsymbol{\theta}^{(0)}) = \mathbb{E}_{\boldsymbol{\xi},\boldsymbol{\varepsilon}}\{\boldsymbol{Q}(\boldsymbol{p}, \boldsymbol{\xi}, \boldsymbol{\theta})]$ for any $\boldsymbol{p} \in [l, u]^n$ without loss of generality. Let $\boldsymbol{b}_{it} = (b_{it}^{(k)})_{k=0}^K \in [0, 1]^{K+1}$ be seller i' belief distribution over the possibility of the true demand models $\boldsymbol{Q}$ in period t where $\sum_{k=0}^K b_{it}^{(k)} = 1$. For the approximate revenue function $\tilde{R}_{it}(p)$, we can define

$$\tilde{R}_{it}(p) \;=\; \gamma p \sum_{k=0}^{K} b_{it}^{(k)} Q_i^{(k)}(p, \boldsymbol{p}_{-it}, \boldsymbol{\theta}^{(k)}), \tag{3.27}$$

where $\boldsymbol{p}_{-it}$ is the competitor's induced price profile for any $i \in \{1, \ldots, n\}$ and $k \in \{0, \ldots, K\}$. Seller i observes demand realization $D_{it} = Q_i^{(0)}(\boldsymbol{p}_t, \boldsymbol{\theta}^{(0)}) + \nu_{it}$ where the demand noise $\nu_{it} = Q_i(\boldsymbol{p}_t, \boldsymbol{\xi}_t, \boldsymbol{\theta}) - Q_i^{(0)}(\boldsymbol{p}, \boldsymbol{\theta}^{(0)}) + \varepsilon_{it}$ is induced by both the customers features and the exogenous demand shock. Given filtration $\mathcal{F}_t^0 = \sigma(\boldsymbol{\varepsilon}_1, \ldots, \boldsymbol{\varepsilon}_t)$, we can make a stronger assumption that D_{it} is uniformly bounded for simplicity of analysis. We also assume that seller i has full knowledge about the likelihood function $\mathcal{L}_{it}^{(k)}\{D_{it}, Q_i^{(k)}(\boldsymbol{p}_t, \boldsymbol{\theta}^{(k)}) | \mathcal{F}_{t-1}^0\} = \mathbb{P}_{\boldsymbol{\xi},\boldsymbol{\varepsilon}}\{D_{it} - Q_i^{(k)}(\boldsymbol{p}_t, \boldsymbol{\xi}_t, \boldsymbol{\theta}^{(k)}) \in d\varepsilon_{it} \;\Big|\; \mathcal{F}_{t-1}^0\} \in [\underline{f}, \overline{f}]$ for some $\overline{f} > \underline{f} > 0$, and moreover, updates her belief via the Bayes' rule

$$b_{i(t+1)}^{k} \;=\; \frac{b_{it}^{(k)}\,\mathcal{L}_{it}^{(k)}\Big\{D_{it}, Q_i^{(k)}(\boldsymbol{p}_t, \boldsymbol{\theta}^{(k)}) | \mathcal{F}_{t-1}^{0}\Big\}}{\sum_{k'=0}^{K} b_{it}^{(k')}\,\mathcal{L}_{it}^{(k')}\Big\{D_{it}, Q_i^{(k')}(\boldsymbol{p}_t, \boldsymbol{\theta}^{(k')}) | \mathcal{F}_{t-1}^{0}\Big\}}. \tag{3.28}$$

Establishing the assumptions in multiagent learning frameworks. In all three frameworks above, we can focus on settings where Assumption 11 holds such that we have a unique DN Nash equilibrium. Moreover, we can assume that the DN Nash equilibrium satisfies $\bar{\boldsymbol{p}}^{\boldsymbol{\theta}} \in \mathrm{int}([l,u]^n)$. In the following result, we show that smoothness property in Assumption 9 and the convergence property in Assumption 12 are broadly satisfied in the aforementioned multiagent learning frameworks.

Proposition 25. *(i) Under the multiagent mirror descent framework in* (3.25)*, Assumption 9 holds. Moreover, if* $\sum_{s=1}^{\infty} \eta_s = \infty$ *and* $\sum_{s=1}^{\infty} \eta_s^2 < \infty$*, then Assumption 12 also holds.*

(ii) Under the fictitious play framework in (3.26)*, Assumption 9 holds if* $\tilde{R}_{it}$ *with a maximizer satisfying* (3.26) *belongs to a compact subspace of* $C^1([l,u])$*. Moreover, if the game in Definition 5 is a finite weighted potential game and* $\bar{\boldsymbol{p}}^{\boldsymbol{\theta}} \in \mathcal{S}$*, then Assumption 12 holds.*[9]

(iii) Under the Bayesian equilibrium framework in (3.27)*, Assumption 9 holds. If there exists* $\delta_d > 0$ *such that* $|Q_i^{(k)}(\boldsymbol{p}, \boldsymbol{\theta}^{(k)}) - Q_i^{(k')}(\boldsymbol{p}, \boldsymbol{\theta}^{(k')})| \geq \delta_d$ *for any* $\boldsymbol{p} \in [l,u]^n$*, any* $k, k' \in \{0, \ldots, K\}$ *with* $k \neq k'$*, and any* $i \in \{1, \ldots, n\}$*, then Assumption 12 holds.*

3.6 Conclusion

This paper investigates how a platform can leverage information disclosure and payments to maximize its revenue under the challenge of demand model uncertainty and information

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[9] The definition of the ordinal potential game is defined in (154).

asymmetry. Our analysis sheds light on the value of information revelation from the perspective of demand learning and revenue management. While we establish that the DN policy can perform poorly in some settings, we establish the opportunities for the platform to exploit the information asymmetry in a demand learning environment. At the other extreme, if the platform is willing to blindly trade its informational advantages for more accurate management of the price-setting incentives, it is able to avoid large revenue loss due to demand model uncertainty. However, such a policy is shown to be too conservative sometimes. We develop a policy on the timing of information revelation and the offering of price-setting incentives such that the platform can exploit the informational advantages and simultaneously avoid large revenue loss in general. We hope that this insight can help guide the effort of price coordination under demand uncertainty in practice.

3.7 The Boundary Scenario $\boldsymbol{\theta} \in \partial\tilde{\Theta}$

Recall that in Theorem 11 of Section 3.4.2 and Theorem 13 of Section 3.4.4, we have established the regret for the case of $\boldsymbol{\theta} \in \mathrm{int}(\tilde{\Theta})$ and $\boldsymbol{\theta} \in \mathrm{int}(\tilde{\Theta}^c)$. In this section, we consider the special case that remains to be discussed. As the special case, the demand parameter vector $\boldsymbol{\theta}$ is located on the boundary of the beneficial set $\tilde{\Theta}$ i.e., $\boldsymbol{\theta} \in \partial\tilde{\Theta}$ where

$$\partial\tilde{\Theta} \;=\; \{\tilde{\boldsymbol{\theta}} : \mathbb{E}_{\boldsymbol{\xi},\boldsymbol{\varepsilon}}\{\mathcal{R}^{\star}(\boldsymbol{\xi},\tilde{\boldsymbol{\theta}})\} \;=\; \mathbb{E}_{\boldsymbol{\xi},\boldsymbol{\varepsilon}}\{\mathcal{R}(\bar{\boldsymbol{p}}^{\tilde{\boldsymbol{\theta}}},\boldsymbol{\xi},\tilde{\boldsymbol{\theta}})\}\}. \tag{3.29}$$

In the boundary case $\boldsymbol{\theta} \in \partial\tilde{\Theta}$, we explore the connection between the growth of regret and the induced price process $\{\boldsymbol{p}_s : s = 1, 2, \ldots, T\}$. If we take the multiagent mirror descent in 3.25 as an example, we observe that the regret depends on the step length parameter η_t from (3.25). We summarize our observation in the following result.

Theorem 14. *(**Performance of policies for the boundary case**) Let π_0 be the platform's DN policy and π_2 be the SRI policy. Abusing some notation, let $\eta_0 = \infty$ be a dummy parameter such that $\frac{1}{\eta_0} = 0$. Suppose that Assumptions 7 - 13 hold and that $\boldsymbol{\theta} \in \partial\tilde{\Theta}$.*

(i) There exists $C_5 > 0$ *such that* $\Delta_{\boldsymbol{\theta}}^{\pi_0}(T) \le C_5\Big[\sum_{t=1}^{T} \max\Big\{\frac{1}{\eta_t} - \frac{1}{\eta_{t-1}}, 0\Big\} + \sum_{t=1}^{T} \eta_t\Big]$ *for any* $T \in \{1, 2, \dots\}$.

(ii) There exists $C_6, C_7 > 0$ *such that* $\Delta_{\boldsymbol{\theta}}^{\pi_2}(T) \le C_6\Big[\sum_{t=1}^{T} \max\Big\{\frac{1}{\eta_t} - \frac{1}{\eta_{t-1}}, 0\Big\} + \sum_{t=1}^{T} \eta_t\Big] + C_7\sqrt{T\log(T)}$ *for any* $T \in \{1, 2 \dots\}$.

The proof of Theorem 14 is delayed to the Appendix Section 3.9. In this result, we show that when the demand parameter vector $\boldsymbol{\theta}$ is on the boundary of the beneficial set $\tilde{\Theta}$, then the induced regret depends on the convergence speed of the price process $\{\boldsymbol{p}_s : s = 1, 2, \dots\}$ i.e., the step length process $\{\eta_t : t = 1, 2, \dots\}$. It is worth noticing that if for some $\epsilon > 0$, we take step length $\eta_t = t^{-(\frac{1}{2}+\epsilon)}$ for any $i \in \{1, \dots, n\}$ and $t \in \{1, \dots, T\}$ (which satisfies the condition in Proposition 25(i)), then we recover the classical result about the sublinear growth of regret from Theorem 14, i.e., $\Delta_{\boldsymbol{\theta}}^{\pi_0}(T), \Delta_{\boldsymbol{\theta}}^{\pi_2}(T) \sim \mathcal{O}(T^{\frac{1}{2}+\epsilon})$.

3.8 Proof of Results in Section 3.4

Proof of Proposition 20. Fix any $t \in \{1, \dots, T\}$, we prove the claim by establishing the sufficiency and the necessity.

Before proceeding, we let $(\bar{\boldsymbol{p}}_t, \bar{\boldsymbol{\mathcal{W}}}_t)$ be any optimal solution to problem (3.11) and $\boldsymbol{\mathcal{W}}_t^{\boldsymbol{\theta}}$ satisfy the expression in (3.13). For any $i \in \{1, \dots, n\}$ and $t \in \{1, \dots, T\}$, we obtain that

$$\begin{aligned}
\gamma\bar{p}_{it}Q_i(\bar{\boldsymbol{p}}_t, \boldsymbol{\xi}_t, \boldsymbol{\theta}) + \bar{\mathcal{W}}_{it}(\bar{p}_{it}) &\overset{(a)}{=} \max_{p_{it}\in[l,u]} \Big\{\gamma p_{it}Q_i(p_{it}, \bar{\boldsymbol{p}}_{-it}, \boldsymbol{\xi}_t, \boldsymbol{\theta}) + \bar{\mathcal{W}}_{it}(p_{it})\Big\} \\
&\overset{(b)}{\ge} \max_{p_{it}\in[l,u]} \Big\{\gamma p_{it}Q_i(p_{it}, \bar{\boldsymbol{p}}_{-it}, \boldsymbol{\xi}_t, \boldsymbol{\theta})\Big\} \\
&\overset{(c)}{=} \gamma\bar{p}_{it}Q_i(\bar{\boldsymbol{p}}_t, \boldsymbol{\xi}_t, \boldsymbol{\theta}) + \mathcal{W}_{it}^{\boldsymbol{\theta}}(\bar{p}_{it}),
\end{aligned} \tag{3.30}$$

where step (a) follows directly from constraint (3.11b) in problem (3.11). Step (b) follows from the fact that $\bar{\mathcal{W}}_{it}(p_{it}) \ge 0$ for all $i \in \{1, \dots, n\}$ and $p_{it} \in [l, u]$. In step (c), we can leverage the expression of $\mathcal{W}_{it}^{\boldsymbol{\theta}}(\bar{p}_{it})$ in (3.13) to obtain the equation given that $\bar{\boldsymbol{p}}_t$ is an optimal price profile to problem (3.11). From (3.30),we obtain that

$$\bar{\mathcal{W}}_{it}(\bar{p}_{it}) \;\geq\; \mathcal{W}^{\boldsymbol{\theta}}_{it}(\bar{p}_{it}), \quad \forall i \in \{1, \ldots, n\}. \tag{3.31}$$

Next, to establish $\mathcal{W}^{\boldsymbol{\theta}}_{it}(\bar{p}_{it}) = \bar{\mathcal{W}}_{it}(\bar{p}_{it})$, we proceed to show that

$$\begin{aligned}
\mathcal{R}^{\star}(\boldsymbol{\xi}_t, \boldsymbol{\theta}) &\overset{(d)}{=} \sum_{i=1}^{n} (1-\gamma)\bar{p}_{it} Q_i(\bar{\boldsymbol{p}}_t, \boldsymbol{\xi}_t, \boldsymbol{\theta}) - \bar{\mathcal{W}}_{it}(\bar{p}_{it}) \\
&= \sum_{i=1}^{n} \bar{p}_{it} Q_i(\bar{\boldsymbol{p}}_t, \boldsymbol{\xi}_t, \boldsymbol{\theta}) - \left[\sum_{i=1}^{n} \gamma \bar{p}_{it} Q_i(\bar{\boldsymbol{p}}_t, \boldsymbol{\xi}_t, \boldsymbol{\theta}) + \bar{\mathcal{W}}_{it}(\bar{p}_{it})\right] \\
&= \sum_{i=1}^{n} \bar{p}_{it} Q_i(\bar{\boldsymbol{p}}_t, \boldsymbol{\xi}_t, \boldsymbol{\theta}) - \left[\sum_{i=1}^{n} \max_{p_{it} \in [l,u]} \left\{\gamma p_{it} Q_i(p_{it}, \bar{\boldsymbol{p}}_{-it}, \boldsymbol{\xi}_t, \boldsymbol{\theta}) + \bar{\mathcal{W}}_{it}(p_{it})\right\}\right] \\
&\overset{(e)}{\leq} \sum_{i=1}^{n} \bar{p}_{it} Q_i(\bar{\boldsymbol{p}}_t, \boldsymbol{\xi}_t, \boldsymbol{\theta}) - \left[\sum_{i=1}^{n} \max_{p_{it} \in [l,u]} \left\{\gamma p_{it} Q_i(p_{it}, \bar{\boldsymbol{p}}_{-it}, \boldsymbol{\xi}_t, \boldsymbol{\theta})\right\}\right] \\
&= \sum_{i=1}^{n} \bar{p}_{it} Q_i(\bar{\boldsymbol{p}}_t, \boldsymbol{\xi}_t, \boldsymbol{\theta}) - \left[\sum_{i=1}^{n} \gamma \bar{p}_{it} Q_i(\bar{\boldsymbol{p}}_t, \boldsymbol{\xi}_t, \boldsymbol{\theta}) + \mathcal{W}^{\boldsymbol{\theta}}_{it}(\bar{p}_{it})\right] \\
&= \sum_{i=1}^{n} (1-\gamma)\bar{p}_{it} Q_i(\bar{\boldsymbol{p}}_t, \boldsymbol{\xi}_t, \boldsymbol{\theta}) - \mathcal{W}^{\boldsymbol{\theta}}_{it}(\bar{p}_{it}) \;\overset{(f)}{\leq}\; \mathcal{R}^{\star}(\boldsymbol{\xi}_t, \boldsymbol{\theta}),
\end{aligned} \tag{3.32}$$

where step (d) follows directly from the optimality of $(\bar{\boldsymbol{p}}_t, \bar{\boldsymbol{\mathcal{W}}}_t)$ in problem (3.11) given the expression of the platform's revenue function in (3.4). Step (e) follows from constraint (3.11b) in problem (3.11). Step (e) follows from the fact that $\bar{\mathcal{W}}_{it}(p_{it}) \geq 0$ for any $i \in \{1, \ldots, n\}$ and $p_{it} \in [l, u]$. Step (f) follows from the optimality of $\mathcal{R}^{\star}(\boldsymbol{\xi}_t, \boldsymbol{\theta})$. Summarizing the observations in (3.32), we obtain that the inequality in step (e) is tight, which suggests that

$$\sum_{i=1}^{n} \bar{\mathcal{W}}_{it}(\bar{p}_{it}) \;=\; \sum_{i=1}^{n} \mathcal{W}^{\boldsymbol{\theta}}_{it}(\bar{p}_{it}). \tag{3.33}$$

Summarizing (3.31) and (3.33), we conclude that

$$\bar{\mathcal{W}}_{it}(\bar{p}_{it}) \;=\; \mathcal{W}^{\boldsymbol{\theta}}_{it}(\bar{p}_{it}), \quad \forall i \in \{1, \ldots, n\}. \tag{3.34}$$

To establish the sufficiency claim, from the equality in (3.34) and the fact that $\bar{\boldsymbol{\mathcal{W}}}_t$ is an optimal reward contract, we conclude that if $\boldsymbol{\mathcal{W}}_t^{\boldsymbol{\theta}}$ satisfies (3.13), then it is also optimal.

To establish the necessity claim, from the fact that the optimal solution $(\bar{\boldsymbol{p}}_t, \bar{\boldsymbol{\mathcal{W}}}_t)$ to problem (3.11) is selected arbitrarily, we can readily leverage the equality in (3.34) to establish that it must satisfy the expression in (3.13). $\square$

Proof of Lemma 16. Fix any $t \in \{1, \ldots, T\}$ and $(\boldsymbol{p}_t, \boldsymbol{\xi}_t, \boldsymbol{\theta}) \in [l, u]^n \times \Xi \times \Theta$ with $\boldsymbol{p}_t = \psi^\star(\boldsymbol{\xi}_t, \boldsymbol{\theta})$. From the construction of $\tilde{\boldsymbol{\mathcal{W}}}^{\tilde{\boldsymbol{p}}, \tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{\theta}}}$ in (3.15) for any $(\tilde{\boldsymbol{p}}, \tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{\theta}}) \in [l, u]^n \times \Xi \times \Theta$, we observe that $\tilde{\boldsymbol{\mathcal{W}}}^{\boldsymbol{p}_t, \boldsymbol{\xi}_t, \boldsymbol{\theta}}$ admits a unique solution of $(w_{0it}, w_{1it}, w_{2it})$ for any $i \in \{1, \ldots, n\}$. This further implies that the reward profile $\boldsymbol{\mathcal{W}}^{\boldsymbol{p}_t, \boldsymbol{\xi}_t, \boldsymbol{\theta}}$ is well-defined in $(\boldsymbol{p}_t, \boldsymbol{\xi}_t, \boldsymbol{\theta}) \in [l, u]^n \times \Xi \times \Theta$ with $\boldsymbol{p}_t = \psi^\star(\boldsymbol{\xi}_t, \boldsymbol{\theta})$. Moreover, the construction also suggests that $\mathcal{W}_i^{\boldsymbol{p}_t, \boldsymbol{\xi}_t, \boldsymbol{\theta}}(p) \geq 0$ for all $i \in \{1, \ldots, n\}$ and $p \in [l, u]$. Given that $\boldsymbol{p}_t$ is the optimal price profile in problem (3.16), we leverage condition (3.15a) to show that for any $i \in \{1, \ldots, n\}$, we have $\tilde{\mathcal{W}}_i^{\boldsymbol{p}_t, \boldsymbol{\xi}_t, \boldsymbol{\theta}}(p_{it}) > 0$, and thus, $\mathcal{W}_i^{\boldsymbol{p}_t, \boldsymbol{\xi}_t, \boldsymbol{\theta}}(p_{it}) = \tilde{\mathcal{W}}_i^{\boldsymbol{p}_t, \boldsymbol{\xi}_t, \boldsymbol{\theta}}(p_{it})$. This allows us to conclude that $\boldsymbol{\mathcal{W}}^{\boldsymbol{p}_t, \boldsymbol{\xi}_t, \boldsymbol{\theta}}(\boldsymbol{p}_t)$ satisfies the property in Proposition 20. Summarizing the observations above, we obtain that the construction of $\boldsymbol{\mathcal{W}}^{\boldsymbol{p}_t, \boldsymbol{\xi}_t, \boldsymbol{\theta}}$ is well-defined and optimal in problem (3.16). $\square$

Proof of Theorem 11. Let π be the platform's DN policy and $\{\boldsymbol{p}_s : s = 1, 2, \ldots\}$ be the price process induced by the sellers solving the approximate optimization problem (3.7) under policy π. Since $\{\boldsymbol{\xi}_s : s = 1, 2, \ldots\}$ are i.i.d, we let $\boldsymbol{\xi}$ be the generic random variable. We prove the two claims in this theorem.

Step 1: Proof of claim (i). If $\boldsymbol{\theta} \in \operatorname{int}(\tilde{\Theta})$, we have $\mathbb{E}_{\boldsymbol{\xi}, \boldsymbol{\varepsilon}}\Big\{\mathcal{R}^\star(\boldsymbol{\xi}, \boldsymbol{\theta}) - \mathcal{R}(\bar{\boldsymbol{p}}^{\boldsymbol{\theta}}, \boldsymbol{\xi}, \boldsymbol{\theta})\Big\} < 0$ (by the strict inequality in (3.18)). We let c_0 be a positive constant such that $c_0 = -\mathbb{E}_{\boldsymbol{\xi}, \boldsymbol{\varepsilon}}\Big\{\mathcal{R}^\star(\boldsymbol{\xi}, \boldsymbol{\theta}) - \mathcal{R}(\bar{\boldsymbol{p}}^{\boldsymbol{\theta}}, \boldsymbol{\xi}, \boldsymbol{\theta})\Big\}$. As a first step, we establish the following upper bound for $\Delta^{\pi}_{\boldsymbol{\theta}}(T)$:

$$\begin{aligned}
\Delta^{\pi}_{\boldsymbol{\theta}}(T) &\overset{(a)}{=} T\mathbb{E}_{\boldsymbol{\xi},\boldsymbol{\varepsilon}}\Big\{\mathcal{R}^{\star}(\boldsymbol{\xi},\boldsymbol{\theta})\Big\} - V^{\pi}_{\boldsymbol{\theta}}(T) \\
&\overset{(b)}{=} T\mathbb{E}_{\boldsymbol{\xi},\boldsymbol{\varepsilon}}\Big\{\mathcal{R}^{\star}(\boldsymbol{\xi},\boldsymbol{\theta})\Big\} - \mathbb{E}^{\pi}_{\boldsymbol{\theta}}\left\{\sum_{t=1}^{T}\mathcal{R}(\boldsymbol{p}_t,\boldsymbol{\xi}_t,\boldsymbol{\theta})\right\} \\
&\overset{(c)}{=} T\mathbb{E}_{\boldsymbol{\xi},\boldsymbol{\varepsilon}}\Big\{\mathcal{R}^{\star}(\boldsymbol{\xi},\boldsymbol{\theta}) - \mathcal{R}(\bar{\boldsymbol{p}}^{\boldsymbol{\theta}},\boldsymbol{\xi},\boldsymbol{\theta})\Big\} - \mathbb{E}^{\pi}_{\boldsymbol{\theta}}\left\{\sum_{t=1}^{T}\mathcal{R}(\boldsymbol{p}_t,\boldsymbol{\xi}_t,\boldsymbol{\theta}) - \mathcal{R}(\bar{\boldsymbol{p}}^{\boldsymbol{\theta}},\boldsymbol{\xi}_t,\boldsymbol{\theta})\right\} \\
&\overset{(d)}{\leq} -c_0T + \mathbb{E}^{\pi}_{\boldsymbol{\theta}}\left\{\sum_{t=1}^{T}\mathcal{R}(\boldsymbol{p}_t,\boldsymbol{\xi}_t,\boldsymbol{\theta}) - \mathcal{R}(\bar{\boldsymbol{p}}^{\boldsymbol{\theta}},\boldsymbol{\xi}_t,\boldsymbol{\theta})\right\}, \qquad (3.35)
\end{aligned}$$

where step (a) follows from the definition of $\Delta^{\pi}_{\boldsymbol{\theta}}(T)$ in (3.12). In step (b), since $\boldsymbol{p}_t$ is the price process induced by the sellers under the platform's DN policy π, we have $V^{\pi}_{\boldsymbol{\theta}}(T) = \mathbb{E}^{\pi}_{\boldsymbol{\theta}}\Big\{\sum_{t=1}^{T}\mathcal{R}(\boldsymbol{p}_t,\boldsymbol{\xi}_t,\boldsymbol{\theta})\Big\}$ (by the definition in (3.10)). In step (c), we add and subtract a common term $\mathbb{E}^{\pi}_{\boldsymbol{\theta}}\Big\{\sum_{t=1}^{T}\mathcal{R}(\bar{\boldsymbol{p}}^{\boldsymbol{\theta}},\boldsymbol{\xi}_t,\boldsymbol{\theta})\Big\}$. It is worth noting that since $\bar{\boldsymbol{p}}^{\boldsymbol{\theta}}$ is the unique DN Nash equilibrium (i.e., a static price profile) given $\boldsymbol{\theta}$, we obtain that $\mathbb{E}^{\pi}_{\boldsymbol{\theta}}\Big\{\sum_{t=1}^{T}\mathcal{R}(\bar{\boldsymbol{p}}^{\boldsymbol{\theta}},\boldsymbol{\xi}_t,\boldsymbol{\theta})\Big\} = \mathbb{E}_{\boldsymbol{\xi},\boldsymbol{\varepsilon}}\Big\{\sum_{t=1}^{T}\mathcal{R}(\bar{\boldsymbol{p}}^{\boldsymbol{\theta}},\boldsymbol{\xi}_t,\boldsymbol{\theta})\Big\}$. Furthermore, since $\{\boldsymbol{\xi}_s : s = 1,2,\dots\}$ are i.i.d, we can also obtain that $\mathbb{E}_{\boldsymbol{\xi},\boldsymbol{\varepsilon}}\Big\{\sum_{t=1}^{T}\mathcal{R}(\bar{\boldsymbol{p}}^{\boldsymbol{\theta}},\boldsymbol{\xi}_t,\boldsymbol{\theta})\Big\} = T\mathbb{E}_{\boldsymbol{\xi},\boldsymbol{\varepsilon}}\Big\{\mathcal{R}(\bar{\boldsymbol{p}}^{\boldsymbol{\theta}},\boldsymbol{\xi},\boldsymbol{\theta})\Big\}$. Together, we have $\mathbb{E}^{\pi}_{\boldsymbol{\theta}}\Big\{\sum_{t=1}^{T}\mathcal{R}(\bar{\boldsymbol{p}}^{\boldsymbol{\theta}},\boldsymbol{\xi}_t,\boldsymbol{\theta})\Big\} = T\mathbb{E}_{\boldsymbol{\xi},\boldsymbol{\varepsilon}}\Big\{\mathcal{R}(\bar{\boldsymbol{p}}^{\boldsymbol{\theta}},\boldsymbol{\xi},\boldsymbol{\theta})\Big\}$. Step (d) follows readily from the fact that $c_0 = -\mathbb{E}_{\boldsymbol{\xi},\boldsymbol{\varepsilon}}\Big\{\mathcal{R}^{\star}(\boldsymbol{\xi},\boldsymbol{\theta}) - \mathcal{R}(\bar{\boldsymbol{p}}^{\boldsymbol{\theta}},\boldsymbol{\xi},\boldsymbol{\theta})\Big\} > 0$.

By Assumption 12, we have $\boldsymbol{p}_t \to \bar{\boldsymbol{p}}^{\boldsymbol{\theta}}$ almost surely as $t \to \infty$. Recall that function $\mathcal{R}(\tilde{\boldsymbol{p}},\tilde{\boldsymbol{\xi}},\boldsymbol{\theta})$ is a continuously differentiable function in $(\tilde{\boldsymbol{p}},\tilde{\boldsymbol{\xi}}) \in [l,u]^n \times \Xi$ under policy π. By applying the continuous mapping theorem, for any $\boldsymbol{\xi} \in \Xi$, this implies that $\mathcal{R}(\boldsymbol{p}_t,\boldsymbol{\xi},\boldsymbol{\theta}) \to \mathcal{R}(\bar{\boldsymbol{p}}^{\boldsymbol{\theta}},\boldsymbol{\xi},\boldsymbol{\theta})$ almost surely as $t \to \infty$. Leveraging the uniform boundedness of $\mathcal{R}(\tilde{\boldsymbol{p}},\tilde{\boldsymbol{\xi}},\boldsymbol{\theta})$ in the compact set $[l,u]^n \times \Xi \times \Theta$, we can leverage the uniform integrable property to show that $\mathbb{E}^{\pi}_{\boldsymbol{\theta}}\Big\{\mathcal{R}(\boldsymbol{p}_t,\boldsymbol{\xi}_t,\boldsymbol{\theta}) - \mathcal{R}(\bar{\boldsymbol{p}}^{\boldsymbol{\theta}},\boldsymbol{\xi}_t,\boldsymbol{\theta})\Big\} \to 0$ as $t \to \infty$. Thus, there exists $t_0' > 0$ such that

$$\mathbb{E}^{\pi}_{\boldsymbol{\theta}}\Big\{\mathcal{R}(\boldsymbol{p}_t,\boldsymbol{\xi}_t,\boldsymbol{\theta}) - \mathcal{R}(\bar{\boldsymbol{p}},\boldsymbol{\xi}_t,\boldsymbol{\theta})\Big\} \;\leq\; \frac{c_0}{2}, \quad \forall t \geq t_0'. \qquad (3.36)$$

By letting $\bar{\mathcal{R}} = \max_{(\tilde{\boldsymbol{p}},\tilde{\boldsymbol{\xi}},\tilde{\boldsymbol{\theta}})\in[l,u]^n\times\Xi\times\tilde{\Theta}} \mathcal{R}(\tilde{\boldsymbol{p}},\tilde{\boldsymbol{\xi}},\tilde{\boldsymbol{\theta}})$, we pick $t_0 \geq t_0'$ to be the smallest integer such that for all $T \geq t_0$, we have $-\frac{c_0}{2}(T + t_0') + 2t_0'\bar{\mathcal{R}} < 0$. Given the positive constants t_0, t_0', we can further pick a constant $C_0 \in (0, \frac{c_0}{2})$ such that for any $T \geq t_0$,

$$-C_0 T \;\geq\; -\frac{c_0}{2}(T + t_0') + 2t_0'\bar{\mathcal{R}} \tag{3.37}$$

Summarizing the aforementioned arguments, for any $T \geq t_0$, we can establish that

$$\begin{aligned} \Delta_{\boldsymbol{\theta}}^{\pi}(T) \;&\overset{(e)}{\leq}\; -c_0 T + \frac{c_0}{2}(T - t_0') + 2t_0'\bar{\mathcal{R}} \\ &=\; -\frac{c_0}{2}(T + t_0') + 2\bar{\mathcal{R}}t_0' \;\overset{(f)}{\leq}\; -C_0 T, \end{aligned} \tag{3.38}$$

where in step (e), from step (d) of (3.35), for all $t \geq t_0'$, we have $\mathbb{E}_{\boldsymbol{\theta}}^{\pi}\Big\{\mathcal{R}(\boldsymbol{p}_t, \boldsymbol{\xi}_t, \boldsymbol{\theta}) - \mathcal{R}(\bar{\boldsymbol{p}}^{\boldsymbol{\theta}}, \boldsymbol{\xi}_t, \boldsymbol{\theta})\Big\} \leq \frac{c_0}{2}$ (by (3.36)), and for all $t \leq t_0'$, we have $\mathbb{E}_{\boldsymbol{\theta}}^{\pi}\Big\{\mathcal{R}(\boldsymbol{p}_t, \boldsymbol{\xi}_t, \boldsymbol{\theta}) - \mathcal{R}(\bar{\boldsymbol{p}}^{\boldsymbol{\theta}}, \boldsymbol{\xi}_t, \boldsymbol{\theta})\Big\} \leq 2\bar{\mathcal{R}}$ (by construction of the uniform upper bound $\bar{\mathcal{R}}$). Step (f) follows directly from (3.37).

We also pick a positive constant $r_0 \geq 2\bar{\mathcal{R}}t_0$ such that for any $T \leq t_0$,

$$\Delta_{\boldsymbol{\theta}}^{\pi}(T) \;\leq\; r_0. \tag{3.39}$$

Based on the observations in (3.38) and (3.39), given the selected positive constants r_0, C_0, we can conclude that

$$\Delta_{\boldsymbol{\theta}}^{\pi}(T) \;\leq\; r_0 - C_0 T, \quad \forall T \in \mathbb{N}. \tag{3.40}$$

This completes the proof of Claim (i).

Step 2: Proof of Claim (ii). If $\boldsymbol{\theta} \in \operatorname{int}(\tilde{\Theta}^c)$, we have $\mathbb{E}_{\boldsymbol{\xi},\boldsymbol{\varepsilon}}\Big\{\mathcal{R}^{\star}(\boldsymbol{\xi}, \boldsymbol{\theta}) - \mathcal{R}(\bar{\boldsymbol{p}}^{\boldsymbol{\theta}}, \boldsymbol{\xi}, \boldsymbol{\theta})\Big\} > 0$.

By letting $c_1 = \mathbb{E}_{\boldsymbol{\xi},\boldsymbol{\varepsilon}}\Big\{\mathcal{R}^\star(\boldsymbol{\xi},\boldsymbol{\theta}) - \mathcal{R}(\bar{\boldsymbol{p}}^{\boldsymbol{\theta}},\boldsymbol{\xi},\boldsymbol{\theta})\Big\} > 0$, we can establish that

$$\begin{aligned}
\Delta^\pi_{\boldsymbol{\theta}}(T) &\overset{(g)}{=} T\mathbb{E}_{\boldsymbol{\xi},\boldsymbol{\varepsilon}}\Big\{\mathcal{R}^\star(\boldsymbol{\xi},\boldsymbol{\theta}) - \mathcal{R}(\bar{\boldsymbol{p}}^{\boldsymbol{\theta}},\boldsymbol{\xi},\boldsymbol{\theta})\Big\} - \mathbb{E}^\pi_{\boldsymbol{\theta}}\left\{\sum_{t=1}^{T}\mathcal{R}(\boldsymbol{p}_t,\boldsymbol{\xi}_t,\boldsymbol{\theta}) - \mathcal{R}(\bar{\boldsymbol{p}}^{\boldsymbol{\theta}},\boldsymbol{\xi}_t,\boldsymbol{\theta})\right\}, \\
&\overset{(h)}{\geq} c_1 T - \mathbb{E}^\pi_{\boldsymbol{\theta}}\left\{\sum_{t=1}^{T}\mathcal{R}(\boldsymbol{p}_t,\boldsymbol{\xi}_t,\boldsymbol{\theta}) - \mathcal{R}(\bar{\boldsymbol{p}}^{\boldsymbol{\theta}},\boldsymbol{\xi}_t,\boldsymbol{\theta})\right\}, \qquad (3.41)
\end{aligned}$$

where step (g) follows from the same arguments as in step (a) - (c) of (3.35). step (h) follows from the fact that $c_1 = \mathbb{E}_{\boldsymbol{\xi},\boldsymbol{\varepsilon}}\Big\{\mathcal{R}^\star(\boldsymbol{\xi},\boldsymbol{\theta}) - \mathcal{R}(\bar{\boldsymbol{p}}^{\boldsymbol{\theta}},\boldsymbol{\xi},\boldsymbol{\theta})\Big\} > 0$.

By using the same arguments as in Claim (i), we can show that $\mathbb{E}^\pi_{\boldsymbol{\theta}}\Big[\mathcal{R}(\boldsymbol{p}_t,\boldsymbol{\xi}_t,\boldsymbol{\theta}) - \mathcal{R}(\bar{\boldsymbol{p}}^{\boldsymbol{\theta}},\boldsymbol{\xi}_t,\boldsymbol{\theta})\Big] \to 0$ as $t \to \infty$. This allows us to pick a positive integer t_1' such that

$$\mathbb{E}^\pi_{\boldsymbol{\theta}}\Big[\mathcal{R}(\boldsymbol{p}_t,\boldsymbol{\xi}_t,\boldsymbol{\theta}) - \mathcal{R}(\bar{\boldsymbol{p}}^{\boldsymbol{\theta}},\boldsymbol{\xi}_t,\boldsymbol{\theta})\Big] \;\leq\; \frac{c_1}{2}, \quad \forall t \geq t_1'. \qquad (3.42)$$

Given $\bar{\mathcal{R}} = \max_{(\tilde{\boldsymbol{p}},\tilde{\boldsymbol{\xi}},\tilde{\boldsymbol{\theta}})\in[l,u]^n\times\Xi\times\Theta}\mathcal{R}(\tilde{\boldsymbol{p}},\tilde{\boldsymbol{\xi}},\tilde{\boldsymbol{\theta}})$, we further let t_1 be the smallest integer with $t_1 \geq t_1'$ such that for all $T \geq t_1$, $\frac{c_1}{2}(T + t_1') - 2t_1'\bar{\mathcal{R}} > 0$. This allows us to pick a constant $C_1 \in (0, \frac{c_1}{2})$ such that

$$C_1 T \;\leq\; \frac{c_1}{2}(T + t_1') - 2\bar{\mathcal{R}}t_1', \quad \forall T \geq t_1. \qquad (3.43)$$

Thus, for all $T \geq t_1$, we have

$$\Delta^\pi_{\boldsymbol{\theta}}(T) \;\overset{(i)}{\geq}\; c_1 T - \frac{c_1}{2}(T - t_1') - 2\bar{\mathcal{R}}t_1' \;\overset{(j)}{\geq}\; C_1 T, \qquad (3.44)$$

where in step (i), note that for all $t \geq t_1'$, we have $\mathbb{E}^\pi_{\boldsymbol{\theta}}\mathcal{R}(\boldsymbol{p}_t,\boldsymbol{\xi}_t,\boldsymbol{\theta}) - \mathcal{R}(\bar{\boldsymbol{p}}^{\boldsymbol{\theta}},\boldsymbol{\xi}_t,\boldsymbol{\theta})\Big] \leq \frac{c_1}{2}$, and for all $t \leq t_1'$, we have $\mathbb{E}^\pi_{\boldsymbol{\theta}}\Big[\mathcal{R}(\boldsymbol{p}_t,\boldsymbol{\xi}_t,\boldsymbol{\theta}) - \mathcal{R}(\bar{\boldsymbol{p}}^{\boldsymbol{\theta}},\boldsymbol{\xi}_t,\boldsymbol{\theta})\Big] \leq 2\bar{\mathcal{R}}$. Combining with step (h) of (3.41), we obtain the inequality in step (i). Step (j) follows immediately from (3.43) .

Lastly, we can pick $r_1 \geq 2\bar{\mathcal{R}}t_1$ such that for all $T \leq t_1$,

$$\Delta^{\pi}_{\boldsymbol{\theta}}(T) \ \geq \ -r_1. \tag{3.45}$$

Summarizing (3.44) and (3.45), for the selected $r_1, C_1 > 0$, we have

$$\Delta^{\pi}_{\boldsymbol{\theta}}(T) \ \geq \ -r_1 + C_1 T, \quad \forall T \in \mathbb{N}. \tag{3.46}$$

This completes the proof of Claim (ii). ☐

Proof of Lemma 17. Given that there exists some $\bar{\boldsymbol{\xi}} \in \Xi$ such that $\boldsymbol{\xi}_t = \bar{\boldsymbol{\xi}}$ for any $t \in \{1, \ldots, T\}$, we establish $\tilde{\Theta} = \emptyset$ by contradiction. Given the definition of the beneficial set $\tilde{\Theta}$ in (3.18), suppose towards a contradiction that $\tilde{\Theta} \neq \emptyset$. Then there exists $\tilde{\boldsymbol{\theta}} \in \tilde{\Theta}$ such that

$$\begin{aligned}
\mathcal{R}^{\star}(\bar{\boldsymbol{\xi}}, \tilde{\boldsymbol{\theta}}) \ &\overset{(a)}{=} \ \max_{\boldsymbol{p} \in [l,u]^n} \ \sum_{i=1}^{n} p_i Q_i(\boldsymbol{p}, \bar{\boldsymbol{\xi}}, \tilde{\boldsymbol{\theta}}) - \max_{\tilde{p}_i \in [l,u]} \gamma \tilde{p}_i Q_i(\tilde{p}_i, \boldsymbol{p}_{-i}, \bar{\boldsymbol{\xi}}, \tilde{\boldsymbol{\theta}}) \\
&\overset{(b)}{<} \ \mathcal{R}(\bar{\boldsymbol{p}}^{\tilde{\boldsymbol{\theta}}}, \bar{\boldsymbol{\xi}}, \tilde{\boldsymbol{\theta}}) \ \overset{(c)}{=} \ \sum_{i=1}^{n} (1-\gamma) \bar{\boldsymbol{p}}_i^{\tilde{\boldsymbol{\theta}}} Q_i(\bar{\boldsymbol{p}}^{\tilde{\boldsymbol{\theta}}}, \bar{\boldsymbol{\xi}}, \tilde{\boldsymbol{\theta}}),
\end{aligned} \tag{3.47}$$

where step (a) follows directly from the expression of optimal revenue $\mathcal{R}^{\star}(\bar{\boldsymbol{\xi}}, \tilde{\boldsymbol{\theta}})$ from problem (3.11) and the property of the optimal reward contracts $\boldsymbol{\mathcal{W}}_t^{\tilde{\boldsymbol{\theta}}}$ in (3.13) of Proposition 20. In step (b), we let $\bar{\boldsymbol{p}}^{\tilde{\boldsymbol{\theta}}}$ be the DN Nash equilibrium in Definition 5 given $\tilde{\boldsymbol{\theta}} \in \tilde{\Theta}$, and the inequality here follows immediately from the assumption that $\tilde{\boldsymbol{\theta}} \in \tilde{\Theta}$. In step (c), given the DN Nash equilibrium $\bar{\boldsymbol{p}}^{\tilde{\boldsymbol{\theta}}}$ and the expression of the platform's revenue function $\mathcal{R}$ in (3.4) (where $\boldsymbol{\mathcal{W}}_t = \boldsymbol{0}$ under the DN policy π), we immediately obtain the equality in step (c).

Meanwhile, we can also show that

$$\begin{aligned}\mathcal{R}^\star(\bar{\boldsymbol{\xi}}, \tilde{\boldsymbol{\theta}}) &\overset{(d)}{=} \max_{\boldsymbol{p}\in[l,u]^n} \sum_{i=1}^{n} p_i Q_i(\boldsymbol{p}, \bar{\boldsymbol{\xi}}, \tilde{\boldsymbol{\theta}}) - \min_{\tilde{p}_i\in[l,u]} \gamma \tilde{p}_i Q_i(\tilde{p}_i, \boldsymbol{p}_{-i}, \bar{\boldsymbol{\xi}}, \tilde{\boldsymbol{\theta}}) \\ &\overset{(e)}{\geq} \max_{\boldsymbol{p}\in[l,u]^n} \sum_{i=1}^{n} p_i Q_i(\boldsymbol{p}, \bar{\boldsymbol{\xi}}, \tilde{\boldsymbol{\theta}}) - \gamma p_i Q_i(\boldsymbol{p}, \bar{\boldsymbol{\xi}}, \tilde{\boldsymbol{\theta}}) \\ &\overset{(f)}{\geq} \sum_{i=1}^{n} (1-\gamma) \bar{\boldsymbol{p}}_i^{\tilde{\boldsymbol{\theta}}} Q_i(\bar{\boldsymbol{p}}^{\tilde{\boldsymbol{\theta}}}, \bar{\boldsymbol{\xi}}, \tilde{\boldsymbol{\theta}}), \end{aligned} \tag{3.48}$$

where step (d) follows from the same argument as in step (a) of (3.47). In step (e), the inequality follows immediately from eliminating the minimization problem from the two-stage optimization problem on the left hand side. In step (f), since the DN Nash equilibrium $\bar{\boldsymbol{p}}^{\tilde{\boldsymbol{\theta}}}$ is a feasible solution in the left-hand-side maximization problem, the inequality readily follows.

Summarizing (3.47) and (3.48), we reach a contradiction, which allows us to conclude that $\tilde{\Theta} = \emptyset$. $\square$

Proof of Lemma 18. Given that $\{\boldsymbol{\xi}_s : s = 1, 2, \dots\}$ are i.i.d, we let $\boldsymbol{\xi}$ be the corresponding common random vector. Proof of Claim (i). In Example 7(iii), given $\boldsymbol{\theta} = (\boldsymbol{\theta}_0, \boldsymbol{\theta}_1, \boldsymbol{\theta}_2)$, $\boldsymbol{p} \in [l, u]^n$ and $\mathbb{E}_{\boldsymbol{\xi},\boldsymbol{\varepsilon}}\{\boldsymbol{\xi}\} = \mathbf{0}$, we observe that

$$\mathbb{E}_{\boldsymbol{\xi},\boldsymbol{\varepsilon}}\Big\{\boldsymbol{Q}(\boldsymbol{p}, \boldsymbol{\xi}, \boldsymbol{\theta})\Big\} = \boldsymbol{\theta}_0 - \boldsymbol{\theta}_1 \boldsymbol{p}. \tag{3.49}$$

By the definition of the DN Nash equilibrium (in Definition 5), the positivity of matrix $\boldsymbol{\theta}_1$'s diagonal entries (by the lemma statement), the expression $\boldsymbol{J}_{\boldsymbol{\theta}_1} = \boldsymbol{\theta}_1 + diag(\boldsymbol{\theta}_1)$, and the condition that $\boldsymbol{J}_{\boldsymbol{\theta}_1}^{-1}\boldsymbol{\theta}_0 \in \text{int}([l, u]^n)$ (by the lemma statement), we obtain the following closed-form expression for the DN Nash equilibrium:

$$\bar{\boldsymbol{p}}^{\boldsymbol{\theta}} \ = \ \boldsymbol{J}_{\boldsymbol{\theta}_1}^{-1}\boldsymbol{\theta}_0. \tag{3.50}$$

The observations above allow us to deduce that the platform's expected revenue from the DN Nash equilibrium can be expressed as

$$\begin{aligned}\mathbb{E}_{\boldsymbol{\xi},\boldsymbol{\varepsilon}}\Big\{\mathcal{R}(\bar{\boldsymbol{p}}^{\boldsymbol{\theta}},\boldsymbol{\xi},\boldsymbol{\theta})\Big\} &\overset{(a)}{=} (1-\gamma)[\bar{\boldsymbol{p}}^{\boldsymbol{\theta}}]^{\intercal}(\boldsymbol{\theta}_0-\boldsymbol{\theta}_1\bar{\boldsymbol{p}}^{\boldsymbol{\theta}})\\ &\overset{(b)}{=} (1-\gamma)\Big[\boldsymbol{J}_{\boldsymbol{\theta}_1}^{-1}\boldsymbol{\theta}_0\Big]^{\intercal}(\boldsymbol{\theta}_0-\boldsymbol{\theta}_1\boldsymbol{J}_{\boldsymbol{\theta}_1}^{-1}\boldsymbol{\theta}_0)\\ &\overset{(c)}{=} (1-\gamma)\boldsymbol{\theta}_0^{\intercal}\left[\frac{1}{2}[\boldsymbol{J}_{\boldsymbol{\theta}_1}^{-1}]^{\intercal}+\frac{1}{2}\boldsymbol{J}_{\boldsymbol{\theta}_1}^{-1}-[\boldsymbol{J}_{\boldsymbol{\theta}_1}^{-1}]^{\intercal}\boldsymbol{\theta}_1\boldsymbol{J}_{\boldsymbol{\theta}_1}^{-1}\right]\boldsymbol{\theta}_0,\end{aligned}\tag{3.51}$$

where in step (a), given the expression of $\mathbb{E}_{\boldsymbol{\xi},\boldsymbol{\varepsilon}}\Big\{\boldsymbol{Q}(\boldsymbol{p},\boldsymbol{\xi},\boldsymbol{\theta})\Big\}$ in (3.49), the platform's revenue in DN Nash equilibrium follows from the revenue function $\mathcal{R}$ in (3.4) and the fact that $\boldsymbol{\mathcal{W}}_t=\boldsymbol{0}$ in the equilibrium. In step (b), we use $\bar{\boldsymbol{p}}^{\boldsymbol{\theta}}$ in (3.50) to obtain the expression on the right hand side. Step (c) follows directly from reorganizing the expression.

Summarizing the arguments above, we conclude the proof of Claim (i).

<u>Proof of Claim (ii).</u> To establish this claim, we deduce the expression of $\mathbb{E}_{\boldsymbol{\xi},\boldsymbol{\varepsilon}}\{\mathcal{R}^{\star}(\boldsymbol{\xi},\boldsymbol{\theta})\}$. In Example 7(iii), for any $(\tilde{\boldsymbol{p}},\tilde{\boldsymbol{\xi}})\in[l,u]^n\times\Xi$, we have $\boldsymbol{Q}(\tilde{\boldsymbol{p}},\tilde{\boldsymbol{\xi}},\boldsymbol{\theta})=\boldsymbol{\theta}_0-\boldsymbol{\theta}_{\boldsymbol{\xi}}\tilde{\boldsymbol{p}}$ where $\boldsymbol{\theta}=(\boldsymbol{\theta}_0,\boldsymbol{\theta}_1,\boldsymbol{\theta}_2)$, $\boldsymbol{\theta}_{\tilde{\boldsymbol{\xi}}}=\boldsymbol{\theta}_1+\boldsymbol{\theta}_2\tilde{\boldsymbol{\xi}}$ and $\boldsymbol{\Lambda}_{\tilde{\boldsymbol{\xi}}}=diag(\boldsymbol{\theta}_{\tilde{\boldsymbol{\xi}}})$. Thus, for any $\tilde{\boldsymbol{\xi}}\in\Xi$, we can establish that

$$\begin{aligned}\mathcal{R}^{\star}(\tilde{\boldsymbol{\xi}},\boldsymbol{\theta}) &\overset{(d)}{=} \max_{\boldsymbol{p}\in[l,u]^n}\quad \sum_{i=1}^{n}p_iQ_i(\boldsymbol{p},\tilde{\boldsymbol{\xi}},\boldsymbol{\theta})-\max_{\tilde{p}_i\in[l,u]}\Big\{\gamma\tilde{p}_i\mathcal{Q}_i(\tilde{p}_i,\boldsymbol{p}_{-i},\tilde{\boldsymbol{\xi}},\boldsymbol{\theta})\Big\}\\ &\overset{(e)}{=} \max_{\boldsymbol{p}\in[l,u]^n}\quad \boldsymbol{p}^{\intercal}(\boldsymbol{\theta}_0-\boldsymbol{\theta}_{\tilde{\boldsymbol{\xi}}}\boldsymbol{p})-\frac{\gamma}{4}\Big[\boldsymbol{\theta}_0-(\boldsymbol{\theta}_{\tilde{\boldsymbol{\xi}}}-\boldsymbol{\Lambda}_{\tilde{\boldsymbol{\xi}}})\boldsymbol{p}\Big]^{\intercal}\boldsymbol{\Lambda}_{\tilde{\boldsymbol{\xi}}}^{-1}\Big[\boldsymbol{\theta}_0-(\boldsymbol{\theta}_{\tilde{\boldsymbol{\xi}}}-\boldsymbol{\Lambda}_{\tilde{\boldsymbol{\xi}}})\boldsymbol{p}\Big],\end{aligned}\tag{3.52}$$

where in step (d), we apply the expression of the optimal revenue $\mathcal{R}^{\star}$ from problem (3.11) and the property of the optimal reward contracts $\boldsymbol{\mathcal{W}}_t^{\boldsymbol{\theta}}$ in (3.13) of Proposition 20 to obtain this expression. In step (e), given that the diagonal entries of $\boldsymbol{\theta}_{\boldsymbol{\xi}}$ are positive (by the lemma statement), the inner optimization problem on the left hand side is a concave maximization problem for all $i\in\{1,\ldots,n\}$. Considering the first-order optimality condition and the assumption that $\frac{1}{2}\boldsymbol{\Lambda}_{\tilde{\boldsymbol{\xi}}}^{-1}\boldsymbol{\theta}_0-\frac{1}{2}(\boldsymbol{\theta}_{\tilde{\boldsymbol{\xi}}}-\boldsymbol{\Lambda}_{\tilde{\boldsymbol{\xi}}})\tilde{\boldsymbol{p}}\in\operatorname{int}([l,u]^n)$ for any $\tilde{\boldsymbol{p}}\in[l,u]^n$ and $\tilde{\boldsymbol{\xi}}\in\Xi$ (by the lemma statement), we can establish that the optimal solution of the inner maximization problems is located in the interior point of $[l, u]$ for each $i \in \{1, \ldots, n\}$. After some arithmetic calculation, we obtain the expression on the right hand side of step (e).

Given the notations $\boldsymbol{G}_{\tilde{\boldsymbol{\xi}}} = \frac{2-\gamma}{4}(\boldsymbol{\theta}_{\tilde{\boldsymbol{\xi}}} + \boldsymbol{\theta}_{\tilde{\boldsymbol{\xi}}}^\intercal) + \frac{\gamma}{4}(\boldsymbol{\theta}_{\tilde{\boldsymbol{\xi}}}^\intercal \boldsymbol{\Lambda}_{\tilde{\boldsymbol{\xi}}}^{-1} \boldsymbol{\theta}_{\tilde{\boldsymbol{\xi}}} + \boldsymbol{\Lambda}_{\tilde{\boldsymbol{\xi}}}^{-1})$ and $\boldsymbol{H}_{\tilde{\boldsymbol{\xi}}} = [(1 - \frac{\gamma}{2})\boldsymbol{I} + \frac{\gamma}{4}(\boldsymbol{\theta}_{\tilde{\boldsymbol{\xi}}}^\intercal \boldsymbol{\Lambda}_{\tilde{\boldsymbol{\xi}}}^{-1} + \boldsymbol{\Lambda}_{\tilde{\boldsymbol{\xi}}}^{-1} \boldsymbol{\theta}_{\tilde{\boldsymbol{\xi}}})]$, we can use (3.52) to show that $\mathcal{R}^\star(\tilde{\boldsymbol{\xi}}, \boldsymbol{\theta})$ satisfies

$$\mathcal{R}^\star(\boldsymbol{\xi}, \boldsymbol{\theta}) \;=\; \max_{\boldsymbol{p} \in [l,u]^n} \; -\boldsymbol{p}^\intercal \boldsymbol{G}_{\boldsymbol{\xi}} \boldsymbol{p} + [\boldsymbol{H}_{\boldsymbol{\xi}} \boldsymbol{\theta}_0]^\intercal \boldsymbol{p} - \frac{\gamma}{4} \boldsymbol{\theta}_0^\intercal \boldsymbol{\Lambda}_{\boldsymbol{\xi}}^{-1} \boldsymbol{\theta}_0, \tag{3.53}$$

From the assumptions $\boldsymbol{G}_{\tilde{\boldsymbol{\xi}}} \succ 0$ and $\frac{1}{2}\boldsymbol{G}_{\tilde{\boldsymbol{\xi}}}^{-1} \boldsymbol{H}_{\tilde{\boldsymbol{\xi}}} \boldsymbol{\theta}_0 \in \operatorname{int}([l, u]^n)$ (by the lemma statement), we use the first-order optimality condition to compute the closed-form optimal price profile as

$$\psi^\star(\tilde{\boldsymbol{\xi}}, \boldsymbol{\theta}) \;=\; \frac{1}{2} \boldsymbol{G}_{\tilde{\boldsymbol{\xi}}}^{-1} \boldsymbol{H}_{\tilde{\boldsymbol{\xi}}} \boldsymbol{\theta}_0. \tag{3.54}$$

Based on the expression of $\mathcal{R}^\star$ in (3.53) and the expression of $\psi^\star$ in (3.54), the induced expected full-information optimal revenue satisfies

$$\mathbb{E}_{\boldsymbol{\xi}, \boldsymbol{\varepsilon}} \left\{ \mathcal{R}^\star(\boldsymbol{\xi}, \boldsymbol{\theta}) \right\} \;=\; \boldsymbol{\theta}_0^\intercal \mathbb{E}_{\boldsymbol{\xi}, \boldsymbol{\varepsilon}} \left\{ \frac{1}{4} \boldsymbol{H}_{\boldsymbol{\xi}} \boldsymbol{G}_{\boldsymbol{\xi}}^{-1} \boldsymbol{H}_{\boldsymbol{\xi}} - \frac{\gamma}{4} \boldsymbol{\Lambda}_{\boldsymbol{\xi}}^{-1} \right\} \boldsymbol{\theta}_0. \tag{3.55}$$

With the aforementioned arguments, we conclude the proof of Claim (ii). □

Proof of Proposition 21. We first establish the sufficiency arguments. Given the symmetric matrix $\boldsymbol{X}(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2)$ in (3.22), we consider the eigendecomposition

$$\boldsymbol{X}(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2) = \boldsymbol{U}(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2) \boldsymbol{\Lambda}(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2) \boldsymbol{U}(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2)^\intercal. \tag{3.56}$$

To prove the sufficiency, we assume that there exists $(\boldsymbol{\theta}_0, \boldsymbol{\theta}_1, \boldsymbol{\theta}_2) \in \Theta$ that satisfies $\boldsymbol{\lambda}(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2) \notin \mathcal{V}^\star(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2)$. Given $\mathcal{V}(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2)$ in the proposition statement, the corresponding dual cone $\mathcal{V}^\star(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2)$ can be expressed as

$$\mathcal{V}^\star(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2) \;=\; \left\{ \boldsymbol{v} : \langle \boldsymbol{v}, \boldsymbol{z} \rangle \geq 0, \; \forall \boldsymbol{z} \in \mathcal{V}(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2) \right\}. \tag{3.57}$$

We let $\boldsymbol{y} = \boldsymbol{U}(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2)^\mathsf{T}\boldsymbol{\theta}_0$. Since $(\boldsymbol{\theta}_0, \boldsymbol{\theta}_1, \boldsymbol{\theta}_2) \in \Theta$, we can directly leverage the definition of $\mathcal{V}(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2)$ in the proposition statement to obtain that

$$\boldsymbol{y} \circ \boldsymbol{y} \ \in \ \mathcal{V}(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2). \tag{3.58}$$

This allows us to further deduce that

$$\begin{aligned} 0 \ &\overset{(a)}{>} \ \langle \boldsymbol{\lambda}(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2), \boldsymbol{y} \circ \boldsymbol{y} \rangle \ = \ \sum_{k=1}^{n} \lambda_k(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2) y_k^2 \ \overset{(b)}{=} \ \boldsymbol{y}^\mathsf{T}\boldsymbol{\Lambda}(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2)\boldsymbol{y} \\ &\overset{(c)}{=} \ \boldsymbol{\theta}_0^\mathsf{T}\boldsymbol{U}(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2)\boldsymbol{\Lambda}(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2)\boldsymbol{U}(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2)^\mathsf{T}\boldsymbol{\theta}_0 \ \overset{(d)}{=} \ \boldsymbol{\theta}_0^\mathsf{T}\boldsymbol{X}(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2)\boldsymbol{\theta}_0, \end{aligned} \tag{3.59}$$

where in step (a), based on the assumption that $\boldsymbol{\lambda}(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2) \notin \mathcal{V}^\star(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2)$, we leverage the definition of $\mathcal{V}^\star(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2)$ from (3.57) and the observation that $\boldsymbol{y} \circ \boldsymbol{y} \in \mathcal{V}(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2)$ (by (3.58)) to obtain the strict inequality. Step (b) follows from the notation of $\boldsymbol{\Lambda}(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2)$ as the diagonal eigenvalue matrix. In step (c), we implement the construction that $\boldsymbol{y} = \boldsymbol{U}(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2)^\mathsf{T}\boldsymbol{\theta}_0$. Step (d) follows immediately from the definition $\boldsymbol{X}(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2) = \boldsymbol{U}(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2)\boldsymbol{\Lambda}(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2)\boldsymbol{U}(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2)^\mathsf{T}$.

By the strictly inequality in (3.59), we have that $\tilde{\Theta} \neq \emptyset$, we establish that $\boldsymbol{\theta} \in \tilde{\Theta}$. Thus, we can conclude that $\tilde{\Theta} \neq \emptyset$.

<u>We next establish establish the necessity arguments.</u> Given that $\tilde{\Theta} \neq \emptyset$, there exists $(\boldsymbol{\theta}_0, \boldsymbol{\theta}_1, \boldsymbol{\theta}_2) \in \Theta$ such that $\boldsymbol{\theta}_0^\mathsf{T}\boldsymbol{X}(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2)\boldsymbol{\theta}_0 < 0$. To establish the necessity, we need to show that there exists $(\boldsymbol{\theta}_0, \boldsymbol{\theta}_1, \boldsymbol{\theta}_2) \in \Theta$ that satisfies $\boldsymbol{\lambda}(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2) \notin \mathcal{V}^\star(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2)$. Using the eigendecomposition $\boldsymbol{X}(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2) = \boldsymbol{U}(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2)\boldsymbol{\Lambda}(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2)\boldsymbol{U}(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2)^\mathsf{T}$, we can establish that

$$0 \ > \ \boldsymbol{\theta}_0^\mathsf{T}\boldsymbol{X}(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2)\boldsymbol{\theta}_0 \ = \ \boldsymbol{\theta}_0^\mathsf{T}\boldsymbol{U}(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2)\boldsymbol{\Lambda}(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2)\boldsymbol{U}(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2)^\mathsf{T}\boldsymbol{\theta}_0. \tag{3.60}$$

By letting $\boldsymbol{y} = \boldsymbol{U}(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2)^\mathsf{T}\boldsymbol{\theta}_0$, we immediately obtain that $\boldsymbol{y} \circ \boldsymbol{y} \in \mathcal{V}(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2)$ (by definition of $\mathcal{V}(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2)$). With $\langle \boldsymbol{\lambda}(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2), \boldsymbol{y} \circ \boldsymbol{y} \rangle = \boldsymbol{y}^\mathsf{T}\boldsymbol{\Lambda}(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2)\boldsymbol{y} < 0$, it follows that $\boldsymbol{\lambda}(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2) \notin \mathcal{V}^\star(\boldsymbol{\theta}_1, \boldsymbol{\theta}_2)$. $\square$

Proof of Proposition 22. Let π be the RI policy, and $\{\boldsymbol{p}_s : s = 1, 2, \ldots, \}$ be the price process induced by the sellers under policy π. To ease some notation, for any $\tilde{\boldsymbol{\theta}} \in \Theta$, $T \geq 2$ and $t \in \{1, \ldots, T\}$, we suppress the history information $\mathcal{H}_t$ in the expressions of $\ell_t(\tilde{\boldsymbol{\theta}}|\mathcal{H}_t)$ and $\mathcal{I}_t(\tilde{\boldsymbol{\theta}}|\mathcal{H}_t)$ to obtain the more concise expressions $\ell_t(\tilde{\boldsymbol{\theta}})$ and $\mathcal{I}_t(\tilde{\boldsymbol{\theta}})$. Define $\ell_0(\tilde{\boldsymbol{\theta}}) = 0$. For any $t \in \{1, 2 \ldots, \}$, we let $g_t(\tilde{\boldsymbol{\theta}})$ be defined as

$$g_t(\tilde{\boldsymbol{\theta}}) \;=\; \ell_t(\tilde{\boldsymbol{\theta}}) - \ell_{t-1}(\tilde{\boldsymbol{\theta}}). \tag{3.61}$$

Step 1: the unconstrained maximum likelihood estimator in period t. For any $T \geq 2$ and $t \in \{1, \ldots, T\}$, given the history information $\mathcal{H}_t$, we denote by $\tilde{\boldsymbol{\theta}}^u_{t+1}$ the unconstrained maximum likelihood estimator without projection onto Θ i.e.,

$$\tilde{\boldsymbol{\theta}}^u_{t+1} \;=\; \arg\max_{\tilde{\boldsymbol{\theta}}} \;\ell_t(\tilde{\boldsymbol{\theta}}). \tag{3.62}$$

Recalling that κ_θ is the dimension of the demand parameter vector and $\bar{\lambda}_I$ is the constant in Assumption 13(i), we pick any $\kappa_1 > \frac{1}{2}\kappa_\theta\bar{\lambda}_I$ and define event $A = \{\|\nabla \ell_t(\boldsymbol{\theta})\|_2 \leq \kappa_1 t\}$. For any $T \geq 2$, $t \in \{1, \ldots, T\}$ and $\delta \geq 0$, we let $\tau(t) = \min\{t, \lceil\sqrt{T\log(T)}\rceil\}$ and deduce that

$$\begin{aligned}
\mathbb{P}^\pi_{\boldsymbol{\theta}}\left\{\|\tilde{\boldsymbol{\theta}}^u_{t+1} - \boldsymbol{\theta}\|_2^2 > \delta\right\} \;&\leq \mathbb{P}^\pi_{\boldsymbol{\theta}}\left\{\|\tilde{\boldsymbol{\theta}}^u_{t+1} - \boldsymbol{\theta}\|_2^2 > \delta, \lambda_{min}(\mathcal{I}_t(\boldsymbol{\theta})) \leq \lambda_0\tau(t)\right\} \\
&\qquad + \mathbb{P}^\pi_{\boldsymbol{\theta}}\left\{\|\tilde{\boldsymbol{\theta}}^u_{t+1} - \boldsymbol{\theta}\|_2^2 > \delta, \lambda_{min}(\mathcal{I}_t(\boldsymbol{\theta})) \geq \lambda_0\tau(t)\right\} \\
&\leq \mathbb{P}^\pi_{\boldsymbol{\theta}}\left\{\lambda_{min}(\mathcal{I}_t(\boldsymbol{\theta})) \leq \lambda_0\tau(t)\right\} \\
&\qquad + \mathbb{P}^\pi_{\boldsymbol{\theta}}\left\{\|\tilde{\boldsymbol{\theta}}^u_{t+1} - \boldsymbol{\theta}\|_2^2 > \delta, \lambda_{min}(\mathcal{I}_t(\boldsymbol{\theta})) \geq \lambda_0\tau(t), A\right\} \\
&\qquad + \mathbb{P}^\pi_{\boldsymbol{\theta}}\left\{\|\tilde{\boldsymbol{\theta}}^u_{t+1} - \boldsymbol{\theta}\|_2^2 > \delta, \lambda_{min}(\mathcal{I}_t(\boldsymbol{\theta})) \geq \lambda_0\tau(t), A^c\right\}.
\end{aligned} \tag{3.63}$$

In the following arguments, we develop an upper bound for each term in the expression of (3.63). Step 2: upper bound for $\mathbb{P}^\pi_{\boldsymbol{\theta}}\{\lambda_{min}(\mathcal{I}_t(\boldsymbol{\theta})) \leq \lambda_0\tau(t)\}$. For simplicity of notation, we let $\mathcal{I}_0(\tilde{\boldsymbol{\theta}}) = \mathbf{0}$ and $\lambda_{min}(\mathcal{I}_0(\tilde{\boldsymbol{\theta}})) = 0$ for any $\tilde{\boldsymbol{\theta}} \in \Theta$. By construction, given the total periods T, the set of exploration periods $\mathcal{T}_t$ up to period t under the RI policy can be expressed as $\mathcal{T}_t = \{1, \ldots, \tau(t)\}$ where $\tau(t) = \min\{t, \lceil \sqrt{T\log(T)} \rceil\}$. By the inequality (3.23) in Assumption 13, we readily obtain that there exists positive constants λ_0, k_0 such that for any $T \geq 2$ and $t \in \{1, \ldots, T\}$,

$$\mathbb{P}^{\pi}_{\boldsymbol{\theta}}\Big\{\lambda_{min}(\mathcal{I}_t(\tilde{\boldsymbol{\theta}})) \leq \lambda_0 \tau(t)\Big\} \ \leq \ \frac{k_0}{\tau(t)}. \tag{3.64}$$

Step 3: upper bound for $\mathbb{P}^{\pi}_{\boldsymbol{\theta}}\{\|\tilde{\boldsymbol{\theta}}^u_{t+1} - \boldsymbol{\theta}\|_2 > \delta, \lambda_{min}(\mathcal{I}_t(\boldsymbol{\theta})) \geq \lambda_0\tau(t), A\}$. We develop an upper bound for the second term of (3.63) in the following auxiliary result.

Lemma 19. *Let π be the RI policy. There exists $K_1', K_2' > 0$ such that for any $T \geq 2$, $t \in \{1, \ldots, T\}$ and $\delta \geq 0$, with $\tau(t) = \min\{t, \lceil \sqrt{T\log(T)}\}$, we have*

$$\mathbb{P}^{\pi}_{\boldsymbol{\theta}}\Big\{\|\tilde{\boldsymbol{\theta}}^u_{t+1} - \boldsymbol{\theta}\|_2^2 > \delta, \lambda_{min}(\mathcal{I}_t(\boldsymbol{\theta})) \geq \lambda_0\tau(t), A\Big\} \ \leq \ K_1' t^{\kappa_\theta} \delta^{\frac{\kappa_\theta}{2}} \exp\left\{-K_2' \frac{\delta}{\max\{1, \sqrt{\delta}\}}\tau(t)\right\}. \tag{3.65}$$

Step 4: upper bound for $\mathbb{P}^{\pi}_{\boldsymbol{\theta}}\{\|\tilde{\boldsymbol{\theta}}^u_{t+1} - \boldsymbol{\theta}\|_2 \geq \delta, \lambda_{min}(\mathcal{I}_t(\boldsymbol{\theta})) \geq \lambda_0\tau(t), A^c\}$. Similar to the previous step, we deduce an upper bound for the third term of (3.63) in the following result

Lemma 20. *Let π be the RI policy. There exists $K_3', K_4' > 0$ such that for any $T \geq 2$, $t \in \{1, \ldots, T\}$ and $\delta \geq 0$, with $\tau(t) = \min\{t, \lceil \sqrt{T\log(T)}\}$, we have*

$$\mathbb{P}^{\pi}_{\boldsymbol{\theta}}\Big\{\|\tilde{\boldsymbol{\theta}}^u_{t+1} - \boldsymbol{\theta}\|_2^2 > \delta, \lambda_{min}(\mathcal{I}_t(\boldsymbol{\theta})) \geq \lambda_0\tau(t), A^c\Big\} \ \leq \ K_3' \exp\left\{-K_4' t\right\}. \tag{3.66}$$

Step 5: concluding the upper bound for $\mathbb{E}^{\pi}_{\boldsymbol{\theta}}\{\|\tilde{\boldsymbol{\theta}}_{t+1} - \boldsymbol{\theta}\|_2^2\}$. Given that Θ is a compact set, we let $\delta_{\boldsymbol{\theta}} = \max_{\tilde{\boldsymbol{\theta}}_1, \tilde{\boldsymbol{\theta}}_2 \in \Theta} \|\tilde{\boldsymbol{\theta}}_1 - \tilde{\boldsymbol{\theta}}_2\|_2^2$, which is a finite constant. Combining the observations from step 2 - 4, for any $T \geq 2$, $t \in \{1, 2 \ldots\}$ and $\tau(t) = \min\{t, \lceil \sqrt{T\log(T)} \rceil\}$, we establish that

$$\begin{aligned}
\mathbb{E}^{\pi}_{\boldsymbol{\theta}}\Big\{\|\tilde{\boldsymbol{\theta}}_{t+1}-\boldsymbol{\theta}\|_2^2\Big\} &= \int_0^{\infty}\mathbb{P}^{\pi}_{\boldsymbol{\theta}}\Big(\|\tilde{\boldsymbol{\theta}}_{t+1}-\boldsymbol{\theta}\|_2^2 > x\Big)dx \\
&\overset{(a)}{\leq} \int_0^{\delta_{\boldsymbol{\theta}}}\mathbb{P}^{\pi}_{\boldsymbol{\theta}}\Big(\|\tilde{\boldsymbol{\theta}}_{t+1}-\boldsymbol{\theta}\|_2^2 > x, \lambda_{min}(\mathcal{I}_t(\boldsymbol{\theta})) \leq \lambda_0\tau(t)\Big)\,dx \\
&\qquad + \int_0^{\delta_{\boldsymbol{\theta}}}\mathbb{P}^{\pi}_{\boldsymbol{\theta}}\Big(\|\tilde{\boldsymbol{\theta}}_{t+1}-\boldsymbol{\theta}\|_2^2 > x, \lambda_{min}(\mathcal{I}_t(\boldsymbol{\theta})) \geq \lambda_0\tau(t), A\Big)dx \\
&\qquad + \int_0^{\delta_{\boldsymbol{\theta}}}\mathbb{P}^{\pi}_{\boldsymbol{\theta}}\Big(\|\tilde{\boldsymbol{\theta}}_{t+1}-\boldsymbol{\theta}\|_2^2 > x, \lambda_{min}(\mathcal{I}_t(\boldsymbol{\theta})) \geq \lambda_0\tau(t), A^c\Big)dx \\
&\overset{(b)}{\leq} \int_0^{\delta_{\boldsymbol{\theta}}}\mathbb{P}^{\pi}_{\boldsymbol{\theta}}\Big(\lambda_{min}(\mathcal{I}_t(\boldsymbol{\theta})) \leq \lambda_0\tau(t)\Big)dx \\
&\qquad + \int_0^{\delta_{\boldsymbol{\theta}}}\mathbb{P}^{\pi}_{\boldsymbol{\theta}}\Big(\|\tilde{\boldsymbol{\theta}}^{u}_{t+1}-\boldsymbol{\theta}\|_2^2 > x, \lambda_{min}(\mathcal{I}_t(\boldsymbol{\theta})) \geq \lambda_0\tau(t), A\Big)\,dx \\
&\qquad + \int_0^{\delta_{\boldsymbol{\theta}}}\mathbb{P}^{\pi}_{\boldsymbol{\theta}}\Big(\|\tilde{\boldsymbol{\theta}}^{u}_{t+1}-\boldsymbol{\theta}\|_2^2 > x, \lambda_{min}(\mathcal{I}_t(\boldsymbol{\theta})) \geq \lambda_0\tau(t), A^c\Big)\,dx \\
&\overset{(c)}{\leq} \int_0^{\delta_{\boldsymbol{\theta}}}\frac{k_0}{\tau(t)}dx + \int_0^{\frac{(\kappa_\theta+1)\log(t)}{K_2'\tau(t)}\wedge 1} 1dx \\
&\quad + \int_{\frac{(\kappa_\theta+1)\log(t)}{K_2'\tau(t)}\wedge 1}^{1} K_1' t^{\kappa_\theta} x^{\frac{\kappa_\theta}{2}}\exp\Big\{-K_2' x\tau(t)\Big\}\,dx \\
&\qquad + \int_1^{\delta_{\boldsymbol{\theta}}\vee 1} K_1' t^{\kappa_\theta} x^{\frac{\kappa_\theta}{2}}\exp\Big\{-K_2'\sqrt{x}\tau(t)\Big\}\,dx + \int_0^{\delta_{\boldsymbol{\theta}}} K_3'\exp\Big\{-K_4' t\Big\}dx \\
&\overset{(d)}{\leq} \frac{\delta_{\boldsymbol{\theta}} k_0}{\tau(t)} + \frac{(\kappa_\theta+1)\log(t)}{K_2'\tau(t)} \\
&\quad + \frac{K_1'}{t} + \delta_{\boldsymbol{\theta}} K_5'\exp\Big\{-\frac{1}{4}K_2'\sqrt{t}\Big\} + \delta_{\boldsymbol{\theta}} K_3'\exp\Big\{-K_4' t\Big\} \\
&\overset{(e)}{\leq} K_1\frac{\log(t+1)}{\tau(t+1)},
\end{aligned} \tag{3.67}$$

where in in step (a), we decompose the probability $\mathbb{P}^{\pi}_{\boldsymbol{\theta}}(\|\tilde{\boldsymbol{\theta}}_{t+1}-\boldsymbol{\theta}\|_2^2 \geq x)$ into three components. Note that since $\delta_{\boldsymbol{\theta}} = \max_{\tilde{\boldsymbol{\theta}}_1,\tilde{\boldsymbol{\theta}}_2\in\Theta}\|\tilde{\boldsymbol{\theta}}_1-\tilde{\boldsymbol{\theta}}_2\|_2^2$, we have $\mathbb{P}^{\pi}_{\boldsymbol{\theta}}\{\|\tilde{\boldsymbol{\theta}}_{t+1}-\boldsymbol{\theta}\|_2^2 > x\} = 0$ for any $x > \delta_{\boldsymbol{\theta}}$. In step (b), we observe that in the first term, event $\{\|\tilde{\boldsymbol{\theta}}_{t+1}-\boldsymbol{\theta}\|_2^2 \geq x, \lambda_{min}(\mathcal{I}_t(\boldsymbol{\theta})) \leq \lambda_0\tau(t)\}$ implies $\{\lambda_{min}(\mathcal{I}_t(\boldsymbol{\theta})) \leq \lambda_0\tau(t)\}$. In the second term and the third term, given that $\tilde{\boldsymbol{\theta}}^{u}_{t+1}$ is the unconstrained maximum likelihood estimator, we observe that event $\{\|\tilde{\boldsymbol{\theta}}_{t+1}-\boldsymbol{\theta}\|_2^2 \geq x\}$ implies event $\{\|\tilde{\boldsymbol{\theta}}^{u}_{t+1}-\boldsymbol{\theta}\|_2^2 \geq x\}$. Summarizing the two observations above, we obtain the inequality in step (b). In step (c), we have three terms on the left hand side. The first term follows directly from $\mathbb{P}^{\pi}_{\boldsymbol{\theta}}(\lambda_{min}(\mathcal{I}_t(\boldsymbol{\theta})) \leq \lambda_0\tau(t)) \leq \frac{k_0}{\tau(t)}$ (by (3.64)). In the second term of step (c), we break the integral further into three components: (1) when $x \leq \frac{(\kappa_{\theta}+1)\log(t)}{K_2'\tau(t)} \wedge 1$, the integrand is upper bounded by 1; (2) when $x \in [\frac{(\kappa_{\theta}+1)\log(t)}{K_2'\tau(t)} \wedge 1, 1]$, together with (3.65) in Lemma 19, the integrand satisfies that $\frac{x}{\max\{1,\sqrt{x}\}} = x$ in the second component; and (3) when $x \geq 1$, we have $\frac{x}{\max\{1,\sqrt{x}\}} = \sqrt{x}$ in the integrand. The third term of step (c) follows readily from (3.66) of Lemma 20. In step (d), the first two terms follow from direct integration. In the third term, the integrand is upper bounded by $\frac{K_1'}{t}$. In the fourth term, given $T \geq 2$, we have $\tau(t) = \min\{t, \lceil\sqrt{T\log(T)}\rceil\} \geq \frac{1}{2}\sqrt{t} > \frac{1}{4}\sqrt{t}$. We can further pick $K_5' > 0$ such that $K_5'\exp\{-\frac{1}{4}K_2'\sqrt{t}\} \geq K_1'\delta_{\boldsymbol{\theta}}^{\frac{\kappa_{\theta}}{2}}t^{\kappa_{\theta}}\exp\{-K_2'\sqrt{t}\}$ for any $t \geq 1$. Step (e) follows readily from the fact that we can pick $K_1 > 0$ such that the inequality holds in this step for any $T \geq 2$ and $t \in \{1, \ldots, T\}$.

Summarizing the observations above, we conclude the claim of this proposition. □

Proof of Theorem 12. Let π be the platform's RI policy and $\{\boldsymbol{p}_s : s = 1, 2, \ldots\}$ be the price process induced by the sellers under policy π. Before proving the claim, we first establish the following auxiliary result.

Lemma 21. *For any* $(\tilde{\boldsymbol{p}}, \tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{\theta}}) \in [l,u]^n \times \Xi \times \Theta$*, if the mapping* $\psi^{\star}$ *in* (3.14) *satisfies Assumption 10, then under the reward contracts* $\mathcal{W}^{\tilde{\boldsymbol{p}},\tilde{\boldsymbol{\xi}},\tilde{\boldsymbol{\theta}}}$ *in Lemma 16, there exists a constant* $K_{\psi} > 0$ *such that for any* $\tilde{\boldsymbol{\theta}}, \bar{\boldsymbol{\theta}} \in \Theta$ *and* $\tilde{\boldsymbol{\xi}} \in \Xi$ *then we have*

$$\left|\mathcal{R}(\psi^{\star}(\tilde{\boldsymbol{\xi}},\tilde{\boldsymbol{\theta}}),\tilde{\boldsymbol{\xi}},\bar{\boldsymbol{\theta}}) - \mathcal{R}(\psi^{\star}(\tilde{\boldsymbol{\xi}},\bar{\boldsymbol{\theta}}),\tilde{\boldsymbol{\xi}},\bar{\boldsymbol{\theta}})\right| \;\leq\; K_{\psi}\|\tilde{\boldsymbol{\theta}} - \bar{\boldsymbol{\theta}}\|_2^2. \tag{3.68}$$

Under the RI policy π, we focus on the reward contract in Lemma 16. Given the expression of the platform's revenue function $\mathcal{R}(\tilde{\boldsymbol{p}}, \tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{\theta}})$ in (3.4), by the continuity of the demand function $\boldsymbol{Q}(\tilde{\boldsymbol{p}}, \tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{\theta}})$ and the reward contracts $\boldsymbol{\mathcal{W}}^{\tilde{\boldsymbol{p}},\tilde{\boldsymbol{\xi}},\tilde{\boldsymbol{\theta}}}$ from Lemma 16 in terms of $(\tilde{\boldsymbol{p}}, \tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{\theta}}) \in [l,u]^n \times \Theta \times \Xi$, we obtain that $\mathcal{R}(\tilde{\boldsymbol{p}}, \tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{\theta}})$ is continuous in $(\tilde{\boldsymbol{p}}, \tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{\theta}}) \in [l,u]^n \times \Theta \times \Xi$. As a result, there exists a positive constant $\bar{\mathcal{R}}$ such that $\mathcal{R}(\tilde{\boldsymbol{p}}, \tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{\theta}}) \leq \bar{\mathcal{R}}$ for any $(\tilde{\boldsymbol{p}}, \tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{\theta}}) \in [l,u]^n \times \Theta \times \Xi$.

We conclude the claim by deducing that

$$\begin{aligned}
\Delta_{\boldsymbol{\theta}}^{\pi}(T) &= \mathbb{E}_{\boldsymbol{\theta}}^{\pi}\left\{\sum_{t=1}^{T}\mathcal{R}^{\star}(\boldsymbol{\xi}_t,\boldsymbol{\theta})\right\} - V_{\boldsymbol{\theta}}^{\pi}(T)\\
&\overset{(a)}{=} \mathbb{E}_{\boldsymbol{\theta}}^{\pi}\left\{\sum_{t=1}^{T}\mathcal{R}(\psi^{\star}(\boldsymbol{\xi}_t,\boldsymbol{\theta}),\boldsymbol{\xi}_t,\boldsymbol{\theta}) - \mathcal{R}(\boldsymbol{p}_t,\boldsymbol{\xi}_t,\boldsymbol{\theta})\right\}\\
&\overset{(b)}{\leq} \mathbb{E}_{\boldsymbol{\theta}}^{\pi}\left\{\sum_{t=1}^{\lceil\sqrt{T\log(T)}\rceil}\Big|\mathcal{R}(\psi^{\star}(\boldsymbol{\xi}_t,\boldsymbol{\theta}),\boldsymbol{\xi}_t,\boldsymbol{\theta}) - \mathcal{R}(\boldsymbol{p}_t,\boldsymbol{\xi}_t,\boldsymbol{\theta})\Big|\right\}\\
&\qquad + \mathbb{E}_{\boldsymbol{\theta}}^{\pi}\left\{\sum_{t=\lceil\sqrt{T\log(T)}\rceil+1}^{T}\Big|\mathcal{R}(\psi^{\star}(\boldsymbol{\xi}_t,\boldsymbol{\theta}),\boldsymbol{\xi}_t,\boldsymbol{\theta}) - \mathcal{R}(\psi^{\star}(\boldsymbol{\xi}_t,\tilde{\boldsymbol{\theta}}_t),\boldsymbol{\xi}_t,\boldsymbol{\theta})\Big|\right\}\\
&\overset{(c)}{\leq} 2\bar{\mathcal{R}}(\sqrt{T\log(T)}+1) + K_{\psi}\sum_{t=2}^{T}\mathbb{E}_{\boldsymbol{\theta}}^{\pi}\Big\{\|\tilde{\boldsymbol{\theta}}_t-\boldsymbol{\theta}\|_2^2\Big\}\\
&\overset{(d)}{\leq} 2\bar{\mathcal{R}}(\sqrt{T\log(T)}+1) + K_{\psi}\sum_{t=2}^{T}K_1\frac{\log(t)}{\min\{\sqrt{T\log(T)},t\}}\\
&\leq 2\bar{\mathcal{R}}(\sqrt{T\log(T)}+1) + K_{\psi}\sum_{t=2}^{T}K_1\frac{\log(t)}{\sqrt{T\log(T)}} + K_{\psi}\sum_{t=1}^{T}K_1\frac{\log(t)}{t}\\
&\overset{(e)}{\leq} 2\bar{\mathcal{R}}(\sqrt{T\log(T)}+1) + K_{\psi}K_1\sqrt{T\log(T)} + K_{\psi}K_1\log(T)^2 \overset{(f)}{\leq} C_2\sqrt{T\log(T)},
\end{aligned} \tag{3.69}$$

where step (a) follows directly from the expression of $\mathcal{R}^{\star}(\boldsymbol{\xi}_t,\boldsymbol{\theta})$ in (3.16) and the definition of $V_{\boldsymbol{\theta}}^{\pi}$ in (3.10). Step (b) follows from dividing the summation expression further into two terms. In step (c), the first term follows readily from the observation that $\mathcal{R}(\psi^{\star}(\boldsymbol{\xi}_t,\boldsymbol{\theta}),\boldsymbol{\xi}_t,\boldsymbol{\theta}) - \mathcal{R}(\boldsymbol{p}_t,\boldsymbol{\xi}_t,\boldsymbol{\theta}) \leq 2\bar{\mathcal{R}}$ for all $t \in \{1,\ldots,\lceil\sqrt{T\log(T)}\rceil\}$ given the uniform upper bound $\bar{\mathcal{R}}$. The second term of step (c) follows from (3.68) in Lemma 21. Step (d) follows directly from inequality (3.24) in Proposition 22 where we have $\tau(t) = \min\{t,\lceil\sqrt{T\log(T)}\rceil\}$. In step (e), both the second term and the third term follow from the fact that $\log(t) \leq \log(T)$. In step (f), we can pick $C_2 > 4\bar{\mathcal{R}} + 4K_1K_{\psi} > 0$ such that the inequality stands in this step.

With the aforementioned arguments, we conclude the proof of the claim in this theorem.

□

Proof of Theorem 13. Let π be the platform's SRI policy and $\{\boldsymbol{p}_s : s = 1, \ldots, T\}$ be the price process induced by the sellers under policy π. To evaluate the performance of the SRI policy, we first develop a supporting result to facilitate our proof of the claim.

Lemma 22. *Let π be the SRI policy. There exists $k_2, M_2, K_2 > 0$ such that for any $T \geq 2$ and $t \in \{2, \ldots, T\}$, with $\tau(t) = \min\{t, \lceil\sqrt{T\log(T)}\rceil\}$, we have that*

$$\mathbb{E}^{\pi}_{\boldsymbol{\theta}}\Big\{\|\tilde{\boldsymbol{\theta}}_t - \boldsymbol{\theta}\|_2^2\Big\} \;\leq\; K_2 \frac{\log(t)}{\tau(t)}. \tag{3.70}$$

Moreover, the platform's information revelation timing τ under policy π satisfies that

(i) if $\boldsymbol{\theta} \in \mathrm{int}(\tilde{\Theta})$, there exists $\bar{K}_0 > 0$ such that for any $T \geq 2$,

$$\mathbb{P}^{\pi}_{\boldsymbol{\theta}}\Big\{\tau = \lceil\sqrt{T\log(T)}\rceil + 1\Big\} \;\leq\; \frac{\bar{K}_0}{\sqrt{T\log(T)}}; \tag{3.71}$$

(ii) if $\boldsymbol{\theta} \in \mathrm{int}(\tilde{\Theta}^c)$, there exists $\bar{K}_1 > 0$ such that for any $T \geq 2$,

$$\mathbb{P}^{\pi}_{\boldsymbol{\theta}}\Big\{\tau = \infty\Big\} \;\leq\; \frac{\bar{K}_1}{\sqrt{T\log(T)}}. \tag{3.72}$$

Based on the observations in Lemma 22, we prove this theorem by establishing Claim (i) and Claim (ii).

Proof of Claim (i). If $\boldsymbol{\theta} \in \mathrm{int}(\tilde{\Theta})$, we first establish that

$$\begin{aligned}
\Delta^{\pi}_{\boldsymbol{\theta}}(T) &= \mathbb{E}^{\pi}_{\boldsymbol{\theta}}\left\{\sum_{t=1}^{\lceil\sqrt{T\log(T)}\rceil} \mathcal{R}^{\star}(\boldsymbol{\xi}_t,\boldsymbol{\theta}) - \mathcal{R}(\boldsymbol{p}_t,\boldsymbol{\xi}_t,\boldsymbol{\theta})\right\} \\
&\quad + \mathbb{E}^{\pi}_{\boldsymbol{\theta}}\left\{\sum_{t=\lceil\sqrt{T\log(T)}\rceil+1}^{T} \mathcal{R}^{\star}(\boldsymbol{\xi}_t,\boldsymbol{\theta}) - \mathcal{R}(\boldsymbol{p}_t,\boldsymbol{\xi}_t,\boldsymbol{\theta})\right\} \\
&= \mathbb{E}^{\pi}_{\boldsymbol{\theta}}\left\{\sum_{t=1}^{\lceil\sqrt{T\log(T)}\rceil} \mathcal{R}^{\star}(\boldsymbol{\xi}_t,\boldsymbol{\theta}) - \mathcal{R}(\boldsymbol{p}_t,\boldsymbol{\xi}_t,\boldsymbol{\theta})\right\} \\
&\quad + \mathbb{E}^{\pi}_{\boldsymbol{\theta}}\left\{\sum_{t=\lceil\sqrt{T\log(T)}\rceil+1}^{T} \mathcal{R}^{\star}(\boldsymbol{\xi}_t,\boldsymbol{\theta}) - \mathcal{R}(\boldsymbol{p}_t,\boldsymbol{\xi}_t,\boldsymbol{\theta})\middle|\tau=\infty\right\}\mathbb{P}^{\pi}_{\boldsymbol{\theta}}\Big\{\tau=\infty\Big\} \\
&\quad + \mathbb{E}^{\pi}_{\boldsymbol{\theta}}\left\{\sum_{t=\lceil\sqrt{T\log(T)}\rceil+1}^{T} \mathcal{R}^{\star}(\boldsymbol{\xi}_t,\boldsymbol{\theta}) - \mathcal{R}(\boldsymbol{p}_t,\boldsymbol{\xi}_t,\boldsymbol{\theta})\middle|\tau=\lceil\sqrt{T\log(T)}\rceil+1\right\} \\
&\qquad \mathbb{P}^{\pi}_{\boldsymbol{\theta}}\Big\{\tau=\lceil\sqrt{T\log(T)}\rceil+1\Big\}
\end{aligned} \tag{3.73}$$

To proceed, we develop an upper bound for each of the three terms in the last expression of (3.73). In the first term of (3.73), by letting $\bar{\mathcal{R}} = \max_{(\tilde{\boldsymbol{p}},\tilde{\boldsymbol{\xi}},\tilde{\boldsymbol{\theta}})\in[l,u]^n\times\Xi\times\Theta} \mathcal{R}(\tilde{\boldsymbol{p}},\tilde{\boldsymbol{\xi}},\tilde{\boldsymbol{\theta}})$, we can establish that

$$\begin{aligned}
\mathbb{E}^{\pi}_{\boldsymbol{\theta}}\left\{\sum_{t=1}^{\lceil\sqrt{T\log(T)}\rceil} \mathcal{R}^{\star}(\boldsymbol{\xi}_t,\boldsymbol{\theta}) - \mathcal{R}(\boldsymbol{p}_t,\boldsymbol{\xi}_t,\boldsymbol{\theta})\right\} &= \mathbb{E}^{\pi}_{\boldsymbol{\theta}}\left\{\sum_{t=1}^{\lceil\sqrt{T\log(T)}\rceil} \Big|\mathcal{R}^{\star}(\boldsymbol{\xi}_t,\boldsymbol{\theta}) - \mathcal{R}(\boldsymbol{p}_t,\boldsymbol{\xi}_t,\boldsymbol{\theta})\Big|\right\} \\
&\le 2\bar{\mathcal{R}}(\sqrt{T\log(T)}+1).
\end{aligned} \tag{3.74}$$

For the second term of (3.73), given the positive constant $\bar{K}_0$ in (3.71) of Lemma 22, we first note that there exists $\tau_0' > 0$ such that for any $T \ge \tau_0'$, we have $1 - \frac{\bar{K}_0}{\sqrt{T}} \le \frac{1}{2}$. Next, based on the probability upper bound in (3.71), we deduce that

$$\mathbb{P}^{\pi}_{\boldsymbol{\theta}}\Big\{\tau=\infty\Big\} \;=\; 1-\mathbb{P}^{\pi}_{\boldsymbol{\theta}}\Big\{\tau=\lceil\sqrt{T\log(T)}\rceil+1\Big\} \;\ge\; \frac{1}{2}. \tag{3.75}$$

To proceed, we let τ_1' be the smallest positive constant such that $T-(\lceil\sqrt{T\log(T)}\rceil+1) \geq \frac{1}{2}T$ for any $T \geq \tau_1'$. Given the aforementioned positive constants τ_0', τ_1', for any $T \geq \max\{\tau_0', \tau_1'\}$, we can show that

$$
\begin{aligned}
&\mathbb{E}_{\boldsymbol{\theta}}^{\pi}\left\{\sum_{t=\lceil\sqrt{T\log(T)}\rceil+1}^{T} \mathcal{R}^{\star}(\boldsymbol{\xi}_t,\boldsymbol{\theta}) - \mathcal{R}(\boldsymbol{p}_t,\boldsymbol{\xi}_t,\boldsymbol{\theta})\middle|\tau=\infty\right\}\mathbb{P}_{\boldsymbol{\theta}}^{\pi}\Big\{\tau=\infty\Big\} \\
&\overset{(a)}{=} \mathbb{E}_{\boldsymbol{\theta}}^{\pi}\left\{\sum_{t=\lceil\sqrt{T\log(T)}\rceil+1}^{T} \mathcal{R}^{\star}(\boldsymbol{\xi}_t,\boldsymbol{\theta}) - \mathcal{R}(\bar{\boldsymbol{p}}^{\boldsymbol{\theta}},\boldsymbol{\xi}_t,\boldsymbol{\theta})\middle|\tau=\infty\right\}\mathbb{P}_{\boldsymbol{\theta}}^{\pi}\Big\{\tau=\infty\Big\} \\
&\quad + \mathbb{E}_{\boldsymbol{\theta}}^{\pi}\left\{\sum_{t=\lceil\sqrt{T\log(T)}\rceil+1}^{T} \mathcal{R}(\bar{\boldsymbol{p}}^{\boldsymbol{\theta}},\boldsymbol{\xi}_t,\boldsymbol{\theta}) - \mathcal{R}(\boldsymbol{p}_t,\boldsymbol{\xi}_t,\boldsymbol{\theta})\middle|\tau=\infty\right\}\mathbb{P}_{\boldsymbol{\theta}}^{\pi}\Big\{\tau=\infty\Big\} \\
&\overset{(b)}{=} \mathbb{E}_{\boldsymbol{\xi},\boldsymbol{\varepsilon}}\left\{\sum_{t=\lceil\sqrt{T\log(T)}\rceil+1}^{T} \mathcal{R}^{\star}(\boldsymbol{\xi}_t,\boldsymbol{\theta}) - \mathcal{R}(\bar{\boldsymbol{p}}^{\boldsymbol{\theta}},\boldsymbol{\xi}_t,\boldsymbol{\theta})\right\}\mathbb{P}_{\boldsymbol{\theta}}^{\pi}\Big\{\tau=\infty\Big\} \\
&\quad + \mathbb{E}_{\boldsymbol{\theta}}^{\pi}\left\{\sum_{t=\lceil\sqrt{T\log(T)}\rceil+1}^{T} \mathcal{R}(\bar{\boldsymbol{p}}^{\boldsymbol{\theta}},\boldsymbol{\xi}_t,\boldsymbol{\theta}) - \mathcal{R}(\boldsymbol{p}_t,\boldsymbol{\xi}_t,\boldsymbol{\theta})\middle|\tau=\infty\right\}\mathbb{P}_{\boldsymbol{\theta}}^{\pi}\Big\{\tau=\infty\Big\} \\
&\overset{(c)}{\leq} -\frac{1}{2}c_0 T\mathbb{P}_{\boldsymbol{\theta}}^{\pi}\Big\{\tau=\infty\Big\} \\
&\quad + \mathbb{E}_{\boldsymbol{\theta}}^{\pi}\left\{\sum_{t=\lceil\sqrt{T\log(T)}\rceil+1}^{T} \mathcal{R}(\bar{\boldsymbol{p}}^{\boldsymbol{\theta}},\boldsymbol{\xi}_t,\boldsymbol{\theta}) - \mathcal{R}(\boldsymbol{p}_t,\boldsymbol{\xi}_t,\boldsymbol{\theta})\middle|\tau=\infty\right\}\mathbb{P}_{\boldsymbol{\theta}}^{\pi}\Big\{\tau=\infty\Big\} \\
&\overset{(d)}{\leq} -\frac{1}{2}c_0 T\mathbb{P}_{\boldsymbol{\theta}}^{\pi}\Big\{\tau=\infty\Big\} + \frac{1}{4}c_0 T\mathbb{P}_{\boldsymbol{\theta}}^{\pi}\Big\{\tau=\infty\Big\} \overset{(e)}{\leq} -\frac{1}{8}c_0 T, \qquad (3.76)
\end{aligned}
$$

where step (a) follows from adding and subtracting the same term $\mathbb{E}_{\boldsymbol{\theta}}^{\pi}\{\sum_{\lceil\sqrt{T\log(T)}\rceil+1}^{T}\mathcal{R}(\bar{\boldsymbol{p}}^{\boldsymbol{\theta}},\boldsymbol{\xi}_t,\boldsymbol{\theta})|\tau=\infty\}\mathbb{P}_{\boldsymbol{\theta}}^{\pi}\{\tau=\infty\}$. In step (b), we replace the expectation measure from $\mathbb{E}_{\boldsymbol{\theta}}^{\pi}\{\cdot|\tau=\infty\}$ to $\mathbb{E}_{\boldsymbol{\xi},\boldsymbol{\varepsilon}}\{\cdot\}$ in the first term given the independence of features $\{\boldsymbol{\xi}_t\}$ from the event $\{\tau=\infty\}$. In step (c), from the assumption that $\boldsymbol{\theta} \in \mathrm{int}(\tilde{\Theta})$ and that $\{\boldsymbol{\xi}_t\}$ are i.i.d, there exists a positive constant c_0 such that $c_0 = \mathbb{E}_{\boldsymbol{\xi},\boldsymbol{\varepsilon}}\Big\{\mathcal{R}(\bar{\boldsymbol{p}}^{\boldsymbol{\theta}},\boldsymbol{\xi},\boldsymbol{\theta})-\mathcal{R}^{\star}(\boldsymbol{\xi},\boldsymbol{\theta})\Big\} > 0$. Moreover, since $T-(\lceil\sqrt{T\log(T)}\rceil+1) \geq \frac{1}{2}T$ for $T \geq \tau_1'$, step (c) immediately follows. In step (d), under Assumption 12, we can leverage the continuous mapping theorem to establish that $\mathbb{E}_{\boldsymbol{\theta}}^{\pi}\{\mathcal{R}(\boldsymbol{p}_t,\boldsymbol{\xi}_t,\boldsymbol{\theta})|\tau=\infty\} \to \mathbb{E}_{\boldsymbol{\theta}}^{\pi}\{\mathcal{R}(\bar{\boldsymbol{p}}^{\boldsymbol{\theta}},\boldsymbol{\xi}_t,\boldsymbol{\theta})|\tau=\infty\}$ as $t \to \infty$. This allows us to further pick a positive constant τ_2' such that $\mathbb{E}_{\boldsymbol{\theta}}^{\pi}\{\mathcal{R}(\bar{\boldsymbol{p}}^{\boldsymbol{\theta}}, \boldsymbol{\xi}_t, \boldsymbol{\theta}) - \mathcal{R}(\boldsymbol{p}_t, \boldsymbol{\xi}_t, \boldsymbol{\theta}) | \tau = \infty\} \leq \frac{1}{4}c_0$ for all $t \geq \lceil\sqrt{T\log(T)}\rceil + 1 \geq \tau_2'$. Step (e) follows readily from the inequality in (3.75).

For the third term of (3.73), we deduce that

$$\begin{aligned}
&\mathbb{E}_{\boldsymbol{\theta}}^{\pi}\left\{\sum_{t=\lceil\sqrt{T\log(T)}\rceil+1}^{T} \mathcal{R}^{\star}(\boldsymbol{\xi}_t, \boldsymbol{\theta}) - \mathcal{R}(\boldsymbol{p}_t, \boldsymbol{\xi}_t, \boldsymbol{\theta}) \middle| \tau = \lceil\sqrt{T\log(T)}\rceil + 1\right\} \mathbb{P}_{\boldsymbol{\theta}}^{\pi}\Big\{\tau = \lceil\sqrt{T\log(T)}\rceil + 1\Big\} \\
&\quad \overset{(f)}{=} \mathbb{E}_{\boldsymbol{\theta}}^{\pi}\left\{\sum_{t=\lceil\sqrt{T\log(T)}\rceil+1}^{T} \mathcal{R}(\psi^{\star}(\boldsymbol{\xi}_t, \boldsymbol{\theta}), \boldsymbol{\xi}_t, \boldsymbol{\theta}) - \mathcal{R}(\psi^{\star}(\boldsymbol{\xi}_t, \tilde{\boldsymbol{\theta}}_t), \boldsymbol{\xi}_t, \boldsymbol{\theta}) \middle| \tau = \lceil\sqrt{T\log(T)}\rceil + 1\right\} \\
&\quad \mathbb{P}_{\boldsymbol{\theta}}^{\pi}\Big\{\tau = \lceil\sqrt{T\log(T)}\rceil + 1\Big\} \\
&\quad \overset{(g)}{\leq} \sum_{t=\lceil\sqrt{T\log(T)}\rceil+1}^{T} K_{\psi}\mathbb{E}_{\boldsymbol{\theta}}^{\pi}\Big\{\|\tilde{\boldsymbol{\theta}}_t - \boldsymbol{\theta}\|_2^2 \;\Big|\; \tau = \lceil\sqrt{T\log(T)}\rceil + 1\Big\}\mathbb{P}_{\boldsymbol{\theta}}^{\pi}\Big\{\tau = \lceil\sqrt{T\log(T)}\rceil + 1\Big\} \\
&\quad \leq \sum_{t=\lceil\sqrt{T\log(T)}\rceil+1}^{T} K_{\psi}\mathbb{E}_{\boldsymbol{\theta}}^{\pi}\Big\{\|\tilde{\boldsymbol{\theta}}_t - \boldsymbol{\theta}\|_2^2\Big\} \overset{(h)}{\leq} K_{\psi}K_2 T\frac{\log(T)}{\sqrt{T\log(T)}} \leq \bar{K}_3\sqrt{T\log(T)},
\end{aligned} \tag{3.77}$$

where in step (f), conditional on $\tau = \lceil\sqrt{T\log(T)}\rceil + 1$, the first component in the expectation term follows from the expression of $\mathcal{R}^{\star}(\boldsymbol{\xi}_t, \boldsymbol{\theta})$ in (3.16), and the second component follows from the fact that $\boldsymbol{p}_t = \psi^{\star}(\boldsymbol{\xi}_t, \tilde{\boldsymbol{\theta}}_t)$ under the SRI policy π. In step (g), we replicate the same proof arguments as in Lemma 21 in the proof of Theorem 12 to establish that there exists a positive constant K_{ψ} such that $\mathcal{R}(\psi^{\star}(\boldsymbol{\xi}_t, \boldsymbol{\theta}), \boldsymbol{\xi}_t, \boldsymbol{\theta}) - \mathcal{R}(\psi^{\star}(\boldsymbol{\xi}_t, \tilde{\boldsymbol{\theta}}_t), \boldsymbol{\xi}_t, \boldsymbol{\theta}) \leq K_{\psi}\|\tilde{\boldsymbol{\theta}}_t - \boldsymbol{\theta}\|_2^2$. Step (h) follows directly from the inequality in (3.70) of Lemma 22.

Summarizing (3.74), (3.76) and (3.77), we can find $\tau_3' > 0$ and $C_3 < \frac{1}{16}c_0$ such that $2\bar{\mathcal{R}}(\sqrt{T\log(T)} + 1) - \frac{1}{8}c_0 T + \bar{K}_3\sqrt{T\log(T)} \leq -C_3 T$ for all $T \geq \tau_3'$. Thus, for any $T \geq \max\{\tau_0', \tau_1', \tau_2', \tau_3'\}$, we establish that

$$\Delta_{\boldsymbol{\theta}}^{\pi}(T) \;\leq\; -C_3 T. \tag{3.78}$$

Furthermore, there exists $r_3 > 2\bar{\mathcal{R}}\max\{\tau_0', \tau_1', \tau_2', \tau_3'\}$ such that for any $T \leq \max\{\tau_0', \tau_1', \tau_2', \tau_3'\}$,

$$\Delta_{\boldsymbol{\theta}}^{\pi}(T) \ \leq \ r_3. \tag{3.79}$$

In summary of (3.78) and (3.79), we conclude that

$$\Delta_{\boldsymbol{\theta}}^{\pi}(T) \ \leq \ r_3 - C_3 T, \quad \forall T \geq 2. \tag{3.80}$$

Proof of Claim (ii). If $\boldsymbol{\theta} \in \operatorname{int}(\tilde{\Theta}^c)$, we first replicate the arguments in (3.73) to establish the following upper bound on $\Delta_{\boldsymbol{\theta}}^{\pi}(T)$:

$$\begin{aligned}
\Delta_{\boldsymbol{\theta}}^{\pi}(T) \ &= \mathbb{E}_{\boldsymbol{\theta}}^{\pi}\left\{\sum_{t=1}^{\lceil\sqrt{T\log(T)}\rceil} \mathcal{R}^{\star}(\boldsymbol{\xi}_t, \boldsymbol{\theta}) - \mathcal{R}(\boldsymbol{p}_t, \boldsymbol{\xi}_t, \boldsymbol{\theta})\right\} \\
&\quad + \mathbb{E}_{\boldsymbol{\theta}}^{\pi}\left\{\sum_{t=\lceil\sqrt{T\log(T)}\rceil+1}^{T} \mathcal{R}^{\star}(\boldsymbol{\xi}_t, \boldsymbol{\theta}) - \mathcal{R}(\boldsymbol{p}_t, \boldsymbol{\xi}_t, \boldsymbol{\theta})\right\} \\
&= \mathbb{E}_{\boldsymbol{\theta}}^{\pi}\left\{\sum_{t=1}^{\lceil\sqrt{T\log(T)}\rceil} \mathcal{R}^{\star}(\boldsymbol{\xi}_t, \boldsymbol{\theta}) - \mathcal{R}(\boldsymbol{p}_t, \boldsymbol{\xi}_t, \boldsymbol{\theta})\right\} \\
&\quad + \mathbb{E}_{\boldsymbol{\theta}}^{\pi}\left\{\sum_{t=\lceil\sqrt{T\log(T)}\rceil+1}^{T} \mathcal{R}^{\star}(\boldsymbol{\xi}_t, \boldsymbol{\theta}) - \mathcal{R}(\boldsymbol{p}_t, \boldsymbol{\xi}_t, \boldsymbol{\theta}) \middle| \tau = \infty\right\} \mathbb{P}_{\boldsymbol{\theta}}^{\pi}\Big\{\tau = \infty\Big\} \\
&\quad + \mathbb{E}_{\boldsymbol{\theta}}^{\pi}\left\{\sum_{t=\lceil\sqrt{T\log(T)}\rceil+1}^{T} \mathcal{R}^{\star}(\boldsymbol{\xi}_t, \boldsymbol{\theta}) - \mathcal{R}(\boldsymbol{p}_t, \boldsymbol{\xi}_t, \boldsymbol{\theta}) \middle| \tau = \lceil\sqrt{T\log(T)}\rceil + 1\right\} \\
&\quad \mathbb{P}_{\boldsymbol{\theta}}^{\pi}\Big\{\tau = \lceil\sqrt{T\log(T)}\rceil + 1\Big\}.
\end{aligned} \tag{3.81}$$

To proceed, we develop upper bounds on the three terms in the last expression of (3.81). Recalling that $\bar{\mathcal{R}} = \max_{(\tilde{\boldsymbol{p}}, \tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{\theta}}) \in [l,u]^n \times \Xi \times \Theta} \mathcal{R}(\tilde{\boldsymbol{p}}, \tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{\theta}})$, we can readily deduce that the first term of (3.81) satisfies

$$\mathbb{E}^{\pi}_{\boldsymbol{\theta}}\left\{\sum_{t=1}^{\lceil\sqrt{T\log(T)}\rceil}\mathcal{R}^{\star}(\boldsymbol{\xi}_t,\boldsymbol{\theta})-\mathcal{R}(\boldsymbol{p}_t,\boldsymbol{\xi}_t,\boldsymbol{\theta})\right\} \;\le\; 2\bar{\mathcal{R}}(\sqrt{T\log(T)}+1). \tag{3.82}$$

In the second term of (3.81), we can show that

$$\begin{aligned}\mathbb{E}^{\pi}_{\boldsymbol{\theta}}\left\{\sum_{t=\lceil\sqrt{T\log(T)}\rceil+1}^{T}\mathcal{R}(\boldsymbol{p}^{\star}_t,\boldsymbol{\xi}_t,\boldsymbol{\theta})-\mathcal{R}(\boldsymbol{p}_t,\boldsymbol{\xi}_t,\boldsymbol{\theta})\Big|\tau=\infty\right\}\mathbb{P}^{\pi}_{\boldsymbol{\theta}}\Big\{\tau=\infty\Big\} &\overset{(i)}{\le} 2\bar{\mathcal{R}}T\mathbb{P}^{\pi}_{\boldsymbol{\theta}}\Big\{\tau=\infty\Big\}\\ &\overset{(j)}{\le} 2\bar{\mathcal{R}}T\frac{\bar{K}_1}{\sqrt{T\log(T)}}\\ &\le 4\bar{\mathcal{R}}\bar{K}_1\sqrt{T},\end{aligned} \tag{3.83}$$

where step (i) follows from the observation that $\mathcal{R}^{\star}(\boldsymbol{\xi}_t,\boldsymbol{\theta})-\mathcal{R}(\boldsymbol{p}_t,\boldsymbol{\xi}_t,\boldsymbol{\theta})\le 2\bar{\mathcal{R}}$. In step (j), the inequality follows directly from (3.71) of Lemma 22.

In the third term of (3.81), we obtain that

$$\begin{aligned}&\mathbb{E}^{\pi}_{\boldsymbol{\theta}}\left\{\sum_{t=\lceil\sqrt{T\log(T)}\rceil+1}^{T}\mathcal{R}^{\star}(\boldsymbol{\xi}_t,\boldsymbol{\theta})-\mathcal{R}(\boldsymbol{p}_t,\boldsymbol{\xi}_t,\boldsymbol{\theta})\Big|\tau=\lceil\sqrt{T\log(T)}\rceil+1\right\}\mathbb{P}^{\pi}_{\boldsymbol{\theta}}\Big\{\tau=\lceil\sqrt{T\log(T)}\rceil+1\Big\}\\ &\overset{(k)}{=}\ \mathbb{E}^{\pi}_{\boldsymbol{\theta}}\left\{\sum_{t=\lceil\sqrt{T\log(T)}\rceil+1}^{T}\mathcal{R}(\psi^{\star}(\boldsymbol{\xi}_t,\boldsymbol{\theta}),\boldsymbol{\xi}_t,\boldsymbol{\theta})-\mathcal{R}(\psi^{\star}(\boldsymbol{\xi}_t,\tilde{\boldsymbol{\theta}}_t),\boldsymbol{\xi}_t,\boldsymbol{\theta})\Big|\tau=\lceil\sqrt{T\log(T)}\rceil+1\right\}\\ &\quad\mathbb{P}^{\pi}_{\boldsymbol{\theta}}\Big\{\tau=\lceil\sqrt{T\log(T)}\rceil+1\Big\}\\ &\overset{(l)}{\le}\ \sum_{t=\lceil\sqrt{T\log(T)}\rceil+1}^{T}K_{\psi}\mathbb{E}^{\pi}_{\boldsymbol{\theta}}\Big\{\|\tilde{\boldsymbol{\theta}}_t-\boldsymbol{\theta}\|_2^2\ \Big|\ \tau=\lceil\sqrt{T\log(T)}\rceil+1\Big\}\mathbb{P}^{\pi}_{\boldsymbol{\theta}}\Big\{\tau=\lceil\sqrt{T\log(T)}\rceil+1\Big\}\\ &\le\ \sum_{t=\lceil\sqrt{T\log(T)}\rceil+1}^{T}K_{\psi}\mathbb{E}^{\pi}_{\boldsymbol{\theta}}\Big\{\|\tilde{\boldsymbol{\theta}}_t-\boldsymbol{\theta}\|_2^2\Big\}\\ &\overset{(m)}{\le}\ K_{\psi}K_2T\frac{\log(T)}{\lceil\sqrt{T\log(T)}\rceil}\ \le\ \bar{K}_3\sqrt{T\log(T)},\end{aligned} \tag{3.84}$$

where in step (k), conditional on $\tau = \lceil \sqrt{T\log(T)} \rceil + 1$, the first component follows readily from the expression of $\mathcal{R}^\star(\boldsymbol{\xi}_t, \boldsymbol{\theta})$ in (3.16), and the second component follows from the observation that $\boldsymbol{p}_t = \psi^\star(\boldsymbol{\xi}_t, \tilde{\boldsymbol{\theta}}_t)$ under the platform's SRI policy π. In step (l), we can again replicate the arguments and apply Lemma 21 in the proof of Theorem 12 to establish that there exists a positive constant $K_\psi > 0$ such that $\mathcal{R}(\psi^\star(\boldsymbol{\xi}_t, \boldsymbol{\theta}), \boldsymbol{\xi}_t, \boldsymbol{\theta}) - \mathcal{R}(\psi^\star(\boldsymbol{\xi}_t, \tilde{\boldsymbol{\theta}}_t), \boldsymbol{\xi}_t, \boldsymbol{\theta}) \leq K_\psi \|\tilde{\boldsymbol{\theta}}_t - \boldsymbol{\theta}\|_2^2$. Step (m) follows readily from the inequality in (3.70) of Lemma 22.

In summary of (3.82), (3.83), and (3.84), we obtain that there exists $C_4 = 4\bar{\mathcal{R}} + 4\bar{\mathcal{R}}\bar{K}_1 + 2\bar{K}_3 > 0$ such that

$$\Delta^\pi_{\boldsymbol{\theta}}(T) \;\leq\; C_4\sqrt{T\log(T)}, \quad \forall T \geq 2. \tag{3.85}$$

□

3.9 Proofs of Results in Section 3.5.

Proof of Proposition 23. To prove the claim for Example 7(i), we establish Assumptions 7 and Assumption 11 for the following problem instance:

$$\boldsymbol{Q}(\tilde{\boldsymbol{p}}, \tilde{\xi}, \tilde{\theta}) \;=\; \begin{pmatrix} 12 + \tilde{\theta} \\ 12 + \tilde{\theta} \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \tilde{\xi} \\ \tilde{\xi} \end{pmatrix} - \begin{pmatrix} 5 & -1 \\ -1 & 5 \end{pmatrix} \begin{pmatrix} \tilde{p}_1 \\ \tilde{p}_2 \end{pmatrix}, \tag{3.86}$$

where $\tilde{\xi} \in [-1, 1]$, $[\tilde{p}_1, \tilde{p}_2] \in [0, 2]^2$, $\tilde{\theta} \in [-1, 1]$, and $\gamma = 0.1$. Note that in this problem instance, we have $[l, u] = [0, 2]^2$, $\Xi = [-1, 1]$, $\Theta = [-1, 1]$, and $\xi \sim U[-1, 1]$.

Assumption 7 in Example 7(i). It can be easily checked that $\boldsymbol{Q}(\tilde{\boldsymbol{p}}, \tilde{\xi}, \tilde{\theta}) \in \mathbb{R}^2_{++}$, and that it is twice continuously differentiable for any $(\tilde{\boldsymbol{p}}, \tilde{\xi}, \tilde{\theta}) \in [0, 2]^2 \times [-1, 1] \times [-1, 1]$. Moreover, since $[0, 2]^2 \times [-1, 1] \times [-1, 1]$ is a compact set, together with the observation that gradient of $Q_i(\tilde{\boldsymbol{p}}, \tilde{\xi}, \tilde{\theta})$ is differentiable in $(\tilde{\boldsymbol{p}}, \tilde{\xi}, \tilde{\theta})$, we obtain that there exists $L_q > 0$ such that $Q_i(\tilde{\boldsymbol{p}}, \tilde{\xi}, \tilde{\theta})$ has a L_q-Lipschitz continuous gradient.

Summarizing the arguments above, we obtain that Assumption 7 holds in this instance.

Assumption 10 in Example 7(i). With some arithmetic calculation, we obtain that the optimization problem for (3.14) in this problem instance yields a unique optimal solution

$$\psi^\star(\tilde{\xi}, \tilde{\theta}) \;=\; \frac{0.9998}{10.0002}(12 + \tilde{\xi} + \tilde{\theta}) \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{3.87}$$

From the expression above, we can immediately establish that there exists $L_\psi > 0$ such that $\psi^\star(\tilde{\xi}, \tilde{\theta})$ is L_ψ-Lipschitz continuous in $(\tilde{\xi}, \tilde{\theta}) \in [-1, 1] \times [-1, 1]$.

Assumption 11 in Example 7(i). From (3.86), we note that

$$\mathbb{E}_{\boldsymbol{\xi},\boldsymbol{\varepsilon}}\Big\{\boldsymbol{Q}(\tilde{\boldsymbol{p}}, \boldsymbol{\xi}, \tilde{\boldsymbol{\theta}})\Big\} \;=\; \begin{pmatrix} 12 + \tilde{\theta} \\ 12 + \tilde{\theta} \end{pmatrix} - \begin{pmatrix} 5 & -1 \\ -1 & 5 \end{pmatrix} \begin{pmatrix} \tilde{p}_1 \\ \tilde{p}_2 \end{pmatrix}, \tag{3.88}$$

which implies that $\log \tilde{p}_i \mathbb{E}_{\boldsymbol{\xi},\boldsymbol{\varepsilon}}\{Q_i(\tilde{\boldsymbol{p}}, \boldsymbol{\xi}, \tilde{\boldsymbol{\theta}})\}$ is a concave function in $\tilde{p}_i \in [0, 2]$ for $i \in \{1, 2\}$. Moreover, we can also check that $\sum_{i=1}^{2} \log \tilde{p}_i \mathbb{E}_{\boldsymbol{\xi},\boldsymbol{\varepsilon}}\{Q_i(\tilde{\boldsymbol{p}}, \boldsymbol{\xi}, \tilde{\boldsymbol{\theta}})\}$ is strictly concave in $\tilde{\boldsymbol{p}} \in [0, 2]^2$. By Theorem 5 of (146) (with $\boldsymbol{r} = \boldsymbol{1}$), it implies that the system satisfies the diagonally strictly concave property specified in the statement of this assumption holds. Thus, Assumption 11 holds in this instance.

To prove the claim for Example 7(ii), we establish Assumptions 7 and Assumption 11 for the following problem instance:

$$\begin{aligned} Q_1(\tilde{\boldsymbol{p}}, \tilde{\xi}, \tilde{\theta}) \;&=\; \exp(\tilde{\theta} + \tilde{\xi} - \tilde{p}_1)/[1 + \sum_{j=1}^{2} \exp(\tilde{\theta} + \tilde{\xi} - \tilde{p}_j)], \\ Q_2(\tilde{\boldsymbol{p}}, \tilde{\xi}, \tilde{\theta}) \;&=\; \exp(\tilde{\theta} + \tilde{\xi} - \tilde{p}_2)/[1 + \sum_{j=1}^{2} \exp(\tilde{\theta} + \tilde{\xi} - \tilde{p}_j)]. \end{aligned} \tag{3.89}$$

where $\tilde{\xi} \in [-1, 1]$, $[\tilde{p}_1, \tilde{p}_2] \in [0, 2]^2$, $\tilde{\theta}_1 \in [1, 2]$, and $\gamma = 0.1$. In this problem instance, we have $[l, u] = [0, 2]^2$, $\Xi = [-1, 1]$, $\Theta = [1, 2]$, and $\xi \sim U[-1, 1]$.

Assumption 7 in Example 7(ii). We can easily check that $\boldsymbol{Q}(\tilde{\boldsymbol{p}}, \tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{\theta}})$ is twice continuously differentiable in $(\tilde{\boldsymbol{p}}, \tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{\theta}}) \in [0, 3]^2 \times [-1, 1]^2 \times [1, 2]^2$. From (3.89), we readily observe that

$\boldsymbol{Q}(\tilde{\boldsymbol{p}}, \tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{\theta}}) \in [0,1]^2$ for any $(\tilde{\boldsymbol{p}}, \tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{\theta}}) \in [0,3]^2 \times [-1,1]^2 \times [1,2]^2$. Given that it is a compact set, together with the observation that gradient of $Q_i(\tilde{\boldsymbol{p}}, \tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{\theta}})$ is continuous, we obtain that there exists $L_q > 0$ such that $Q_i(\tilde{\boldsymbol{p}}, \tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{\theta}})$ has a L_q-Lipschitz continuous gradient. Thus, Assumption 7 holds.

Assumption 10 in Example 7(ii). From the construction of the problem instance, we can easily verify that $\max_{p_i \in [0,2]} p_i Q_i(p_i, \tilde{p}_{-i}, \tilde{\xi}, \tilde{\theta})$ is a concave maximization problem in $p_i \in [0,2]$ for any $(\tilde{p}_{-i}, \tilde{\xi}, \tilde{\theta}) \in [0,2] \times [-1,1] \times [1,2]$ and $i \in \{1,2\}$, which allows us to deduce that its maximizer is unique. For simplicity of notation, we let this maximizer as $\bar{p}_i$. Next, we plan to leverage Corollary 4 of (155) to establish that $\max_{p_i \in [0,2]} p_i Q_i(p_i, \tilde{p}_{-i}, \tilde{\xi}, \tilde{\theta})$ is continuously differentiable in $\tilde{p}_{-i} \in [0,2]$ for any $(\tilde{\xi}, \tilde{\theta}) \in [-1,1] \times [1,2]$ and $i \in \{1,2\}$. To see this, given that $\tilde{p}_i Q_i(\tilde{p}_i, \tilde{p}_{-i}, \tilde{\xi}, \tilde{\theta})$ is continuously differentiable in $\tilde{p}_i \in [-0.01, 2.01]$ (we look at a larger compact set) and that $\frac{\partial}{\partial \tilde{p}_{-i}} \tilde{p}_i Q_i(\tilde{p}_i, \tilde{p}_{-i}, \tilde{\xi}, \tilde{\theta})$ is continuously differentiable in $[\tilde{p}_i, \tilde{p}_{-i}] \in [-0.01, 2.01]^2$ in this problem instance, together with the observation that $\bar{p}_i = \arg\max_{p_i \in [-0.01, 2.01]} p_i Q_i(p_i, \tilde{p}_{-i}, \tilde{\xi}, \tilde{\theta})$ is the unique maximizer for any $(p_{-i}, \tilde{\xi}, \tilde{\theta}) \in [0,2] \times [-1,1] \times [1,2]$, we have that $\frac{\partial}{\partial \tilde{p}_{-i}} \tilde{p}_i Q_i(\tilde{p}_i, \tilde{p}_{-i}, \tilde{\xi}, \tilde{\theta})$ is a singleton at $\tilde{p}_i = \bar{p}_i$. Thus, we can apply Corollary 4(iii) of (155) to establish that $\max_{p_i \in [0,2]} p_i Q_i(p_i, \tilde{p}_{-i}, \tilde{\xi}, \tilde{\theta})$ is continuously differentiable in $\tilde{p}_{-i} \in [0,2]$ for any $(\tilde{\xi}, \tilde{\theta}) \in [-1,1] \times [1,2]$ and $i \in \{1,2,\}$.

To proceed, we look at optimization problem (3.14) and that the objective function has an Hessian matrix in terms of $\tilde{\boldsymbol{p}} \in [0,2]^2$ whose eigenvalue is uniformly negative for any $\tilde{\xi} \times \tilde{\theta} \in [-1,1] \times [1,2]$. For simplicity of notation, we let $H(f(\tilde{\boldsymbol{p}}))$ be the Hessian matrix of any function $f(\tilde{\boldsymbol{p}})$ evaluated at $\tilde{\boldsymbol{p}}$. We also let $\lambda(H)$ be the set of eigenvalues of matrix H. With some calculation, we can show that for any $(\tilde{\boldsymbol{p}}, \tilde{\xi}, \tilde{\theta}) \in [0,2]^2 \times [-1,1] \times [1,2]$, the eigenvalue for the objective function of optimization problem (3.14) satisfies

$$\begin{aligned}
&\max\left\{\lambda\Big\{H\Big(\sum_{i=1}^{2}\tilde{p}_iQ_i(\tilde{\boldsymbol{p}},\tilde{\boldsymbol{\xi}},\tilde{\boldsymbol{\theta}})\Big)-\max_{p_i\in[l,u]}\tilde{p}_iQ_i(p_i,\tilde{\boldsymbol{p}}_{-i},\tilde{\boldsymbol{\xi}},\tilde{\boldsymbol{\theta}})\Big\}\right\}\\
&\quad\le\ \max\left\{\lambda\Big\{H\Big(\sum_{i=1}^{2}\tilde{p}_iQ_i(\tilde{\boldsymbol{p}},\tilde{\boldsymbol{\xi}},\tilde{\boldsymbol{\theta}})\Big)\Big\}\right\}-\min\left\{\lambda\Big\{H\Big(\sum_{i=1}^{2}\gamma\max_{p_i\in[l,u]}\tilde{p}_iQ_i(p_i,\tilde{\boldsymbol{p}}_{-i},\tilde{\boldsymbol{\xi}},\tilde{\boldsymbol{\theta}})\Big\}\right\}\\
&\quad\le\ -0.01+0.05\gamma\ \le\ -0.005.
\end{aligned}\tag{3.90}$$

From (3.90), we conclude that the optimizer $\psi^\star(\tilde{\xi},\tilde{\theta})$ of problem (3.14) in this instance is uniquely determined by $(\tilde{\xi},\tilde{\theta})\in[-1,1]\times[1,2]$. Moreover, since the eigenvalue of the objective function is uniformly negative, we plan to leverage Theorem 3.1 of (156) to establish the Lipschitz stability. We need to verify Assumptions 1-4 of (156). The continuity condition in Assumption 1 holds since the optimizer and the value function of optimization problem (3.14) are both continuous in $(\tilde{\xi},\tilde{\theta})\in[-1,1]\times[1,2]$ (by applying the Berge's maximum theorem and the fact that the optimizer is unique). The singleton condition in Assumption 2 has been established by the uniqueness of optimal solution to problem (3.14) for any $(\tilde{\xi},\tilde{\theta})\in[-1,1]\times[1,2]$. To check Assumption 3, given that there are no equality constraints, we let $u_i^+=1$ and $u_i^-=-1$ for $i\in\{1,2\}$. For each $i\in\{1,2\}$, we have $u_i^+\frac{\partial}{\partial p_i}(-p_i+l)+u_i^-\frac{\partial}{\partial p_i}(p_i-u)<0$. Thus, Assumption 3 holds. The strong second order condition in Assumption 4 holds directly by the uniform upper bound value in (3.90). Summarizing Assumptions 1-4 of (156), we can apply Theorem 3.1 to establish that there exists $L_\psi>0$ such that $\psi^\star(\tilde{\xi},\tilde{\theta})$ is L_ψ-Lipschitz in $(\tilde{\xi},\tilde{\theta})\in[-1,1]\times[1,2]$.

Assumption 11 in Example 7(iii). We note that

$$\mathbb{E}_{\boldsymbol{\xi},\boldsymbol{\varepsilon}}\Big\{Q_i(\tilde{\boldsymbol{p}},\boldsymbol{\xi},\tilde{\boldsymbol{\theta}})\Big\}\ =\ \mathbb{E}_{\boldsymbol{\xi},\boldsymbol{\varepsilon}}\left\{\exp(\tilde{\theta}_i+\xi_i-\tilde{p}_i)/[1+\sum_{j=1}^{2}\exp(\tilde{\theta}_j+\xi_j-\tilde{p}_j)]\right\},\quad\forall i\in\{1,2\}.\tag{3.91}$$

Since $\log Q_i(\tilde{\boldsymbol{p}},\tilde{\boldsymbol{\xi}},\tilde{\boldsymbol{\theta}})$ is jointly log-concave in $(\tilde{\boldsymbol{p}},\tilde{\boldsymbol{\xi}})$, which implies that $\log\mathbb{E}_{\boldsymbol{\xi},\boldsymbol{\varepsilon}}\{Q_i(\tilde{\boldsymbol{p}},\boldsymbol{\xi},\tilde{\boldsymbol{\theta}})\}$

is concave in $\tilde{\boldsymbol{p}} \in [1,3]^2$, which that $\log \tilde{p}_i \mathbb{E}_{\boldsymbol{\xi},\boldsymbol{\varepsilon}}\{Q_i(\tilde{\boldsymbol{p}}, \boldsymbol{\xi}, \tilde{\boldsymbol{\theta}})\}$ is a log-concave function for $i \in \{1,2\}$. Moreover, we can easily check that $\sum_{i=1}^{2} \log \tilde{p}_i \mathbb{E}_{\boldsymbol{\xi},\boldsymbol{\varepsilon}}\{Q_i(\tilde{\boldsymbol{p}}, \boldsymbol{\xi}, \tilde{\boldsymbol{\theta}})\}$ is strictly concave in $\tilde{\boldsymbol{p}} \in [1,3]^2$. Again, by applying Theorem 5 of (146) (with $\boldsymbol{r} = \mathbf{1}$), we show that the system satisfies the diagonally strictly concave property (specified in this assumption). Thus, this problem satisfies Assumption 11.

To prove the claim for Example 7(iii), we consider the following instance

$$\boldsymbol{Q}(\tilde{\boldsymbol{p}}, \tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{\theta}}) \;=\; \begin{pmatrix} 12 \\ 12 \end{pmatrix} - \begin{pmatrix} 5 & -1 \\ -1 & 5 \end{pmatrix} \begin{pmatrix} \tilde{p}_1 \\ \tilde{p}_2 \end{pmatrix} + \begin{pmatrix} \tilde{\theta} & 0 \\ 0 & \tilde{\theta} \end{pmatrix} \begin{pmatrix} \tilde{\xi} & 0 \\ 0 & \tilde{\xi} \end{pmatrix} \begin{pmatrix} \tilde{p}_1 \\ \tilde{p}_2 \end{pmatrix}, \tag{3.92}$$

where $\tilde{\xi} \in [0,1]$, $[\tilde{p}_1, \tilde{p}_2] \in [0,2]^2$, $\tilde{\theta} \in [0,1]$, and $\gamma = 0.1$. In this problem instance, we have $[l,u] = [0,2]^2$, $\Xi = [-1,1]$, $\Theta = [0,1]$, and $\xi \sim U[0,1]$.

Assumption 7 in Example 7(iii). It can be easily checked that $\boldsymbol{Q}(\tilde{\boldsymbol{p}}, \tilde{\xi}, \tilde{\theta}) \in \mathbb{R}^2_{++}$ is twice continuously differentiable for any $(\tilde{\boldsymbol{p}}, \tilde{\xi}, \tilde{\theta}) \in [0,2]^2 \times [-1,1] \times [0,1]$. Moreover, since $[0,2]^2 \times [-1,1] \times [0,1]$ is a compact set, together with the observation that gradient of $Q_i(\tilde{\boldsymbol{p}}, \tilde{\xi}, \tilde{\theta})$ is differentiable in $(\tilde{\boldsymbol{p}}, \tilde{\xi}, \tilde{\theta})$, we obtain that there exists $L_q > 0$ such that $Q_i(\tilde{\boldsymbol{p}}, \tilde{\xi}, \tilde{\theta})$ has a L_q-Lipschitz continuous gradient. Thus, Assumption 7 holds in this instance.

Assumption 10 in Example 7(i). With some arithmetic calculation, we obtain that the optimization problem for (3.14) in this problem instance yields a unique optimal solution

$$\psi^{\star}(\tilde{\xi}, \tilde{\theta}) \;=\; \frac{12\Big[1 - \frac{1}{20}(1 + \tilde{\xi}\tilde{\theta})/(5 - \tilde{\xi}\tilde{\theta})^2\Big]}{9 - 3\tilde{\xi}\tilde{\theta} + \frac{1}{20}(1 + \tilde{\xi}\tilde{\theta})^2/(5 - \tilde{\xi}\tilde{\theta})^2} \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{3.93}$$

From the expression above, we can immediately establish that there exists $L_\psi > 0$ such that $\psi^{\star}(\tilde{\xi}, \tilde{\theta})$ is L_ψ-Lipschitz continuous in $(\tilde{\xi}, \tilde{\theta}) \in [-1,1] \times [0,1]$.

Assumption 11 in Example 7(i). From (3.86), we note that

$$\mathbb{E}_{\boldsymbol{\xi},\boldsymbol{\varepsilon}}\Big\{\boldsymbol{Q}(\tilde{\boldsymbol{p}},\xi,\tilde{\theta})\Big\} \;=\; \begin{pmatrix}12\\12\end{pmatrix} - \begin{pmatrix}5 & -1\\ -1 & 5\end{pmatrix}\begin{pmatrix}\tilde{p}_1\\ \tilde{p}_2\end{pmatrix}, \tag{3.94}$$

which implies that $\log \tilde{p}_i \mathbb{E}_{\boldsymbol{\xi},\boldsymbol{\varepsilon}}\{Q_i(\tilde{\boldsymbol{p}},\xi,\tilde{\theta})\}$ is a log-concave function in $\tilde{p}_i \in [0,2]$ for $i \in \{1,2\}$. Moreover, we can also check that $\sum_{i=1}^{2} \log \tilde{p}_i \mathbb{E}_{\boldsymbol{\xi},\boldsymbol{\varepsilon}}\{Q_i(\tilde{\boldsymbol{p}},\xi,\tilde{\theta})\}$ is strictly concave in $\tilde{\boldsymbol{p}} \in [0,2]^2$. By Theorem 5 of (146) (with $\boldsymbol{r} = \boldsymbol{1}$), it implies that the system satisfies the diagonally strictly concave property specified in the statement of this assumption holds. Thus, Assumption 11 holds in this instance. □

Proof of Proposition 24. Proof of Claim (i).

To verify Assumption 8 in Claim (i), for any $\tilde{\boldsymbol{\theta}} \in \Theta$ and $t \in \mathbb{N}_{++}$, we suppress the history information $\mathcal{H}_t = \{(\boldsymbol{p}_s, \boldsymbol{\xi}_s, \boldsymbol{D}_s) : s = 1,\ldots,t\}$ in the expressions of the log-likelihood function $\ell_t(\tilde{\boldsymbol{\theta}}|\mathcal{H}_t)$ and empirical fisher information matrix $\mathcal{I}_t(\tilde{\boldsymbol{\theta}}|\mathcal{H}_t)$ for simplicity of notation.

Under Example 7(i), the log-likelihood function in period t can be expressed as

$$\begin{aligned}\ell_t(\tilde{\boldsymbol{\theta}}) \;=\;& -\frac{nt}{2}\log 2\pi - \frac{t}{2}\log det(\Sigma_t)\\ & -\sum_{s=1}^{t}\frac{1}{2}[\boldsymbol{D}_s - (\tilde{\boldsymbol{\theta}}_0 + \tilde{\boldsymbol{\theta}}_1\boldsymbol{\xi}_s - \tilde{\boldsymbol{\theta}}_2\boldsymbol{p}_s)]^{\mathsf{T}}\Sigma_s^{-1}[\boldsymbol{D}_s - (\tilde{\boldsymbol{\theta}}_0 + \tilde{\boldsymbol{\theta}}_1\boldsymbol{\xi}_s - \tilde{\boldsymbol{\theta}}_2\boldsymbol{p}_s)].\end{aligned} \tag{3.95}$$

For simplicity of notations, for $s \in \{1,2,\ldots\}$, we define $g_s(\tilde{\boldsymbol{\theta}}) = \ell_s(\tilde{\boldsymbol{\theta}}) - \ell_{s-1}(\tilde{\boldsymbol{\theta}})$ where $\ell_0(\tilde{\boldsymbol{\theta}}) = 0$, and we let $\mathcal{M}(\boldsymbol{p}_s,\boldsymbol{\xi}_s) = (1,\boldsymbol{\xi}_s,-\boldsymbol{p}_s)(1,\boldsymbol{\xi}_s,-\boldsymbol{p}_s)^{\mathsf{T}}$. After some arithmetic calculation, we deduce that the corresponding empirical Fisher information matrix can be expressed as

$$\mathcal{I}_t(\tilde{\boldsymbol{\theta}}) \;=\; \sum_{s=1}^{t}\Sigma_s^{-1}\otimes\mathcal{M}(\boldsymbol{p}_s,\boldsymbol{\xi}_s). \tag{3.96}$$

It can be readily verified from the expression (3.95) that $g_t(\tilde{\boldsymbol{\theta}})$ is twice continuously differentiable in $\tilde{\boldsymbol{\theta}} = (\tilde{\boldsymbol{\theta}}_0, \tilde{\boldsymbol{\theta}}_1, \tilde{\boldsymbol{\theta}}_2) \in \Theta$. Moreover, since $\Sigma_s \succeq 0$ and $(1, \boldsymbol{\xi}_s, -\boldsymbol{p}_s)(1, \boldsymbol{\xi}_s, -\boldsymbol{p}_s)^\intercal \succeq 0$ for all $s \in \mathbb{N}_+$, from the expression in (3.96), we can immediately verify that $\mathcal{I}_t(\tilde{\boldsymbol{\theta}}) - \mathcal{I}_{t-1}(\tilde{\boldsymbol{\theta}}) \succeq 0$. By definition, since matrix $\mathcal{I}_t(\tilde{\boldsymbol{\theta}}) - \mathcal{I}_t(\tilde{\boldsymbol{\theta}})$ is precisely the negative Hessian function of $g_t(\tilde{\boldsymbol{\theta}})$, we obtain that $g_t(\tilde{\boldsymbol{\theta}})$ is weakly concave in $\tilde{\boldsymbol{\theta}} \in \Theta$.

To verify Assumption 13 in Claim (i), we first establish Assumption13(i), given that $[l, u]^n$ and Ξ are compact sets and that $\lambda_{min}(\Sigma_s) \geq \underline{\lambda}_\epsilon$ for any $s \in \{1, 2 \ldots\}$, we observe that there exists $\bar{\lambda}_I > \frac{1}{\underline{\lambda}_\epsilon} \max_{(\boldsymbol{p},\boldsymbol{\xi}) \in [l,u]^n \times \Xi} \max_{\|\boldsymbol{x}\|_2^2 = 1} \|(1, \boldsymbol{\xi}^\intercal, \boldsymbol{p}^\intercal)\boldsymbol{x}\|_2^2$. Thus, for any $t \in \mathbb{N}_{++}$ and any $\tilde{\boldsymbol{\theta}} \in \tilde{\Theta}$,

$$
\begin{aligned}
\lambda_{max}\left(\mathcal{I}_t(\tilde{\boldsymbol{\theta}}) - \mathcal{I}_{t-1}(\tilde{\boldsymbol{\theta}})\right) &\overset{(a)}{\leq} \lambda_{max}(\Sigma_t^{-1})\lambda_{max}((1, \boldsymbol{\xi}, \boldsymbol{p})(1, \boldsymbol{\xi}, \boldsymbol{p})^\intercal) \\
&\overset{(b)}{\leq} \frac{1}{\underline{\lambda}_\epsilon} \max_{(\boldsymbol{p},\boldsymbol{\xi}) \in [l,u]^n \times \Xi} \max_{\|\boldsymbol{x}\|_2^2 = 1} \|(1, \boldsymbol{\xi}^\intercal, \boldsymbol{p}^\intercal)\boldsymbol{x}\|_2^2 \;\leq\; \bar{\lambda}_I, \qquad (3.97)
\end{aligned}
$$

where in step (a), given the expression of $\mathcal{I}_t(\tilde{\boldsymbol{\theta}})$ in (3.96), we can apply the maximum eigenvalue property of the matrix Kronecker product. Step (b) follows from the fact that $\lambda_{max}(\Sigma_t^{-1}) \geq \frac{1}{\underline{\lambda}_\epsilon}$ and that $\bar{\lambda}_I > \frac{1}{\underline{\lambda}_\epsilon} \max_{(\boldsymbol{p},\boldsymbol{\xi}) \in [l,u]^n \times \Xi} \max_{\|\boldsymbol{x}\|_2^2 = 1} \|(1, \boldsymbol{\xi}^\intercal, \boldsymbol{p}^\intercal)\boldsymbol{x}\|_2^2$.

To establish Assumption13(ii), we first observe from (3.96) that $\mathcal{I}_t(\tilde{\boldsymbol{\theta}})$ does not depend on $\tilde{\boldsymbol{\theta}}$. By letting $\kappa_I = 1$, we can immediately establish that $\frac{1}{\kappa_1}\mathcal{I}_t(\tilde{\boldsymbol{\theta}}_1) \preceq \mathcal{I}_t(\tilde{\boldsymbol{\theta}}_2) \preceq \frac{1}{\kappa_I}\mathcal{I}_t(\tilde{\boldsymbol{\theta}}_1)$ for any $\tilde{\boldsymbol{\theta}}_1, \tilde{\boldsymbol{\theta}}_2 \in \Theta$, $t \in \mathbb{N}_{++}$ and $\{(\boldsymbol{p}_s, \boldsymbol{\xi}_s) \in [l, u]^n \times \Xi : s = 1, 2 \ldots, t\}$.

To establish the inequality in (3.23), we plan to leverage the matrix Freedman inequality. Fixing $T \geq 2$ and $t \in \{1, \ldots, T\}$, we let $\{\boldsymbol{p}_s : s = 1, 2, \ldots, t\}$ be the price process induced by the sellers under policy π. Recalling that $\mathcal{T}_t$ is the set of random exploration periods with $|\mathcal{T}_t| = \tau(t) \in \mathbb{N}_+$, we let $\mathcal{T}_t = \{t_1, \ldots, t_{\tau(t)}\}$. Note that there exists a random vector $\boldsymbol{\xi}$ with $Cov(\boldsymbol{\xi}) \succ 0$ such that $\boldsymbol{\xi}_{t_\ell} = \boldsymbol{\xi}$ for all $\ell \in \{1, \ldots, \tau(t)\}$. Under policy π, there exists a random price vector $\tilde{\boldsymbol{p}}$ with $Cov(\tilde{\boldsymbol{p}}) \succ 0$ such that $\boldsymbol{p}_{t_\ell} = \boldsymbol{p}$ for all $\ell \in \{1, \ldots, \tau(t)\}$.

With $\mathcal{M}(\boldsymbol{p}_s, \boldsymbol{\xi}_s) = (1, \boldsymbol{\xi}_s, -\boldsymbol{p}_s)(1, \boldsymbol{\xi}_s, -\boldsymbol{p}_s)^\intercal$ for all $s \in \{1, 2, \ldots\}$, for any $\ell \in \{1, \ldots, \tau(t)\}$, we simplify the notation and define

$$\begin{aligned}\mathcal{K}_\ell &= \Sigma_{t_\ell}^{-1} \otimes \mathcal{M}(\boldsymbol{p}_{t_\ell}, \boldsymbol{\xi}_{t_\ell}),\\ \mathcal{J}_\ell &= \sum_{s=1}^{\ell} \mathcal{K}_s. \end{aligned} \tag{3.98}$$

To proceed, we define filtration $\tilde{\mathcal{F}}_s = \mathcal{F}_{t_s}$ for all $s \in \{1, \ldots, \ell\}$. Based on (3.98), for any $\ell \in \{1, \ldots, \tau(t)\}$, we further define

$$\begin{aligned}X_\ell &= -\mathcal{K}_\ell + \mathbb{E}_{\boldsymbol{\theta}}^{\pi}\Big\{\mathcal{K}_\ell | \tilde{\mathcal{F}}_{\ell-1}\Big\},\\ Y_\ell &= \sum_{s=1}^{\ell} X_s,\\ W_\ell &= \Big\| \sum_{s=1}^{\ell} X_s^2 \Big\|_2. \end{aligned} \tag{3.99}$$

Before we can establish the claim, we need to establish some properties for $\{X_\ell : \ell = 1, \ldots, \tau(t)\}$, $\sum_{\ell=1}^{\tau(t)} \mathbb{E}_{\boldsymbol{\theta}}^{\pi}\Big\{\mathcal{K}_\ell \Big| \tilde{\mathcal{F}}_{\ell-1}\Big\}$ and $W_{\tau(t)}$ in (3.100) - (3.103).

By construction, we have $\mathbb{E}_{\boldsymbol{\theta}}^{\pi}\Big\{X_\ell \Big| \tilde{\mathcal{F}}_{\ell-1}\Big\} = \mathbf{0}$ and $Y_\ell = \sum_{s=1}^{\ell} X_s = -\mathcal{J}_\ell + \mathbb{E}_{\boldsymbol{\theta}}^{\pi}\Big\{\mathcal{J}_\ell | \tilde{\mathcal{F}}_{\ell-1}\Big\}$. Thus, $\{Y_s : s = 1, 2, \ldots\}$ is a matrix martingale and $\{X_s : s = 1, 2, \ldots\}$ is the corresponding martingale difference sequence adapted to filtration $\{\tilde{\mathcal{F}}_s : s = 1, 2, \ldots\}$. As a result of this, there exists $\bar{\lambda}_M > 0$ such that for all $\ell \in \{1, \ldots, \tau(t)\}$,

$$\begin{aligned}\lambda_{max}\left(X_\ell\right) &\overset{(c)}{\leq} \Big|\lambda_{max}\Big(\Sigma_{t_\ell}^{-1}\Big)\Big| \Big|\lambda_{max}\Big(\mathcal{M}(\boldsymbol{p}_{t_\ell}, \boldsymbol{\xi}_{t_\ell})\Big)\Big| \\ &\qquad + \Big|\lambda_{max}\Big(\Sigma_{t_\ell}^{-1}\Big)\Big| \Big|\lambda_{max}\Big(\mathbb{E}_{\boldsymbol{\theta}}^{\pi}\Big\{\mathcal{M}(\boldsymbol{p}_{t_\ell}, \boldsymbol{\xi}_{t_\ell}) | \tilde{\mathcal{F}}_{t-1}\Big\}\Big)\Big| \overset{(d)}{\leq} \bar{\lambda}_M, \end{aligned} \tag{3.100}$$

where in step (c), the upper bound follows from applying the eigenvalue property of Kronecker product of matrices given the expressions of X_ℓ in (3.99) and $\mathcal{K}_\ell$ in (3.98). In step (d), since the eigenvalues of each component on the left hand side of the inequality are uniformly bounded, we can pick $\bar{\lambda}_M > 0$ large enough such that the inequality holds uniformly.

For simplicity of notation, we let $(\boldsymbol{p}, \boldsymbol{\xi})$ be the random vector such that $(\boldsymbol{p}_s, \boldsymbol{\xi}_s) = (\boldsymbol{p}, \boldsymbol{\xi})$ for all $s \in \mathcal{T}_t$ under policy π. Denote by $(\bar{\boldsymbol{p}}, \bar{\boldsymbol{\xi}})$ the corresponding mean value of $(\boldsymbol{p}, \boldsymbol{\xi})$. We can proceed to establish the following expression

$$\sum_{\ell=1}^{\tau(t)} \mathbb{E}_{\boldsymbol{\theta}}^{\pi}\Big\{\mathcal{K}_\ell \Big| \tilde{\mathcal{F}}_{\ell-1}\Big\} \overset{(e)}{=} \sum_{\ell=1}^{\tau(t)} \Sigma_{t_\ell}^{-1} \otimes \begin{pmatrix} 1 & \bar{\boldsymbol{\xi}}^\intercal & -\bar{\boldsymbol{p}}^\intercal \\ \bar{\boldsymbol{\xi}} & \bar{\boldsymbol{\xi}}\bar{\boldsymbol{\xi}}^\intercal & -\bar{\boldsymbol{\xi}}\bar{\boldsymbol{p}}^\intercal \\ -\bar{\boldsymbol{p}} & -\bar{\boldsymbol{p}}\bar{\boldsymbol{\xi}}^\intercal & \bar{\boldsymbol{p}}\bar{\boldsymbol{p}}^\intercal \end{pmatrix} + \sum_{\ell=1}^{\tau} \Sigma_{t_\ell}^{-1} \otimes \begin{pmatrix} 0 & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & Cov\left(\boldsymbol{\xi}\right) & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & Cov\left(\boldsymbol{p}\right) \end{pmatrix}, \tag{3.101}$$

where in step (e), for each $\ell \in \{1, \ldots, \tau(t)\}$, given the expression of $\mathcal{K}_\ell$ in (3.98), we decompose the expectation of $\mathcal{K}_\ell$ into the two components in (3.101).

Moreover, we can establish that

$$\begin{aligned} &\lambda_{min}\Big(\sum_{\ell=1}^{\tau(t)} \mathbb{E}_{\boldsymbol{\theta}}^{\pi}\Big\{\mathcal{K}_\ell \Big| \tilde{\mathcal{F}}_{\ell-1}\Big\}\Big) \\ \overset{(f)}{\geq}\ & \frac{\tau(t)}{\bar{\lambda}_\epsilon} \min_{\|\boldsymbol{y}\|_2^2=1} \Big\{(y_0 + \boldsymbol{y}_1^\intercal \bar{\boldsymbol{\xi}} + \boldsymbol{y}_2^\intercal \bar{\boldsymbol{p}})^2 + \boldsymbol{y}_1^\intercal Cov\left(\boldsymbol{\xi}\right) \boldsymbol{y}_1 + \boldsymbol{y}_2^\intercal Cov\left(\boldsymbol{p}\right) \boldsymbol{y}_2\Big\} \\ \overset{(g)}{=}\ & \frac{\tau(t)}{\bar{\lambda}_\epsilon} \Big[(\bar{y}_0 + \bar{\boldsymbol{y}}_1^\intercal \bar{\boldsymbol{\xi}} + \bar{\boldsymbol{y}}_2^\intercal \bar{\boldsymbol{p}})^2 + \bar{\boldsymbol{y}}_1^\intercal Cov\left(\boldsymbol{\xi}\right) \bar{\boldsymbol{y}}_1 + \bar{\boldsymbol{y}}_2^\intercal Cov\left(\boldsymbol{p}\right) \bar{\boldsymbol{y}}_2\Big] \\ \overset{(h)}{=}\ & \bar{\lambda}_y \tau(t), \end{aligned} \tag{3.102}$$

where in step (f), based on the expression of $\sum_{\ell=1}^{\tau(t)} \mathbb{E}_{\boldsymbol{\theta}}^{\pi}\Big\{\mathcal{K}_\ell \Big| \tilde{\mathcal{F}}_{\ell-1}\Big\}$ in (3.101) and the assumption that $\lambda_{min}(\Sigma_{t_\ell}^{-1}) \geq \frac{1}{\bar{\lambda}_\epsilon}$ for any $\ell \in \{1, \ldots, \tau(t)\}$ (by the proposition statement), we apply the Rayleigh quotient to obtain the inequality on the right hand side. In step (g), we let $\bar{\boldsymbol{y}} = \arg\min_{|\boldsymbol{y}|_2^2=1} (y_0 + \boldsymbol{y}_1^\intercal \bar{\boldsymbol{\xi}} + \boldsymbol{y}_2^\intercal \bar{\boldsymbol{p}})^2 + \boldsymbol{y}_1^\intercal Cov\left(\boldsymbol{\xi}\right) \boldsymbol{y}_1 + \boldsymbol{y}_2^\intercal Cov\left(\boldsymbol{p}\right) \boldsymbol{y}_2$. In step (h), we define the positive constant $\bar{\lambda}_y = \frac{1}{\bar{\lambda}_\epsilon}[(\bar{y}_0 + \bar{\boldsymbol{y}}_1^\intercal \bar{\boldsymbol{\xi}} + \bar{\boldsymbol{y}}_2^\intercal \bar{\boldsymbol{p}})^2 + \bar{\boldsymbol{y}}_1^\intercal Cov\left(\boldsymbol{\xi}\right) \bar{\boldsymbol{y}}_1 + \bar{\boldsymbol{y}}_2^\intercal Cov\left(\boldsymbol{p}\right) \bar{\boldsymbol{y}}_2]$.

Next, given that $[l, u]^n$ and Ξ are compact sets and that the matrix 2-norm $\|\cdot\|_2$ is equivalent to the matrix's maximum eigenvalue, there exists a constant $\bar{\lambda}_W = 4\bar{\lambda}_I^2$ such that

$$\begin{aligned} W_{\tau(t)} &= \Big\| \sum_{\ell=1}^{\tau(t)} X_\ell^2 \Big\|_2 \;\leq\; \sum_{\ell=1}^{\tau(t)} \Big\| \Big(- \mathcal{K}_\ell + \mathbb{E}_{\boldsymbol{\theta}}^{\pi}\big\{ \mathcal{K}_\ell \big| \tilde{\mathcal{F}}_{\ell-1} \big\} \Big)^2 \Big\|_2 \\ &\leq \sum_{\ell=1}^{\tau(t)} \big\| \mathcal{K}_\ell \big\|_2^2 + \big\| \mathcal{K}_\ell \big\|_2 \big\| \mathbb{E}_{\boldsymbol{\theta}}^{\pi}\big\{ \mathcal{K}_\ell \big| \tilde{\mathcal{F}}_{\ell-1} \big\} \big\|_2 + \big\| \mathbb{E}_{\boldsymbol{\theta}}^{\pi}\big\{ \mathcal{K}_\ell \big| \tilde{\mathcal{F}}_{\ell-1} \big\} \big\|_2 \big\| \mathcal{K}_\ell \big\|_2 + \big\| \mathbb{E}_{\boldsymbol{\theta}}^{\pi}\big\{ \mathcal{K}_\ell \big| \tilde{\mathcal{F}}_{\ell-1} \big\} \big\|_2^2 \\ &\overset{(i)}{\leq} \bar{\lambda}_W \tau(t), \end{aligned} \tag{3.103}$$

where step (i) follows from the fact that the eigenvalue for each component in the left hand side of the inequality is upper bounded by $\bar{\lambda}_I^2$. Given that $\bar{\lambda}_W = 4\bar{\lambda}_I^2$, step (i) readily follows.

Based on the observations that the dimension of the matrices in $\{Y_s : s = 1, 2, \dots\}$ (defined in (3.99)) is $n(n+\kappa_{\boldsymbol{\xi}}+1)$, we leverage the properties in (3.100), (3.102) and (3.103), together with the matrix Freedman inequality to obtain that for any $\delta \geq 0$ and $\sigma \geq 0$,

$$\mathbb{P}_{\boldsymbol{\theta}}^{\pi}\Big\{ \lambda_{max}\left(Y_{\tau(t)} \right) \geq \delta,\; W_{\tau(t)} \leq \sigma^2 \Big\} \;\leq\; n(n+\kappa_{\boldsymbol{\xi}}+1) \exp\left\{ \frac{-\delta^2/2}{\sigma_2^2 + \bar{\lambda}_M \delta/3} \right\}. \tag{3.104}$$

Based on (3.104), by picking $\delta = \frac{1}{2}\bar{\lambda}_y \tau(t)$, $\sigma_2^2 = \bar{\lambda}_W \tau(t)$, and $k_\lambda = \frac{\frac{1}{8}\bar{\lambda}_y^2}{\bar{\lambda}_W + \frac{1}{6}\bar{\lambda}_M \bar{\lambda}_y}$, we can show that

$$\begin{aligned}
n(n+\kappa_{\boldsymbol{\xi}}+1)\exp\left\{-k_\lambda \tau(t)\right\} &\overset{(j)}{\geq} \mathbb{P}^{\pi}_{\boldsymbol{\theta}}\left\{\lambda_{max}\left(Y_{\tau(t)}\right) \geq \frac{1}{2}\bar{\lambda}_y \tau(t),\ W_{\tau(t)} \leq \bar{\lambda}_W \tau(t)\right\} \\
&\overset{(k)}{=} \mathbb{P}^{\pi}_{\boldsymbol{\theta}}\left\{\lambda_{max}\left(Y_{\tau(t)}\right) \geq \frac{1}{2}\bar{\lambda}_y \tau(t)\right\} \\
&\overset{(l)}{\geq} \mathbb{P}^{\pi}_{\boldsymbol{\theta}}\left\{\lambda_{max}\left(-\mathcal{J}_{\tau(t)}\right) + \lambda_{min}\Big(\sum_{\ell=1}^{\tau(t)} \mathbb{E}^{\pi}_{\boldsymbol{\theta}}\Big\{\mathcal{K}_\ell \Big| \tilde{\mathcal{F}}_{\ell-1}\Big\}\Big) \geq \frac{1}{2}\bar{\lambda}_y \tau(t)\right\} \\
&\overset{(m)}{=} \mathbb{P}^{\pi}_{\boldsymbol{\theta}}\left\{-\lambda_{min}\left(\mathcal{J}_{\tau(t)}\right) + \lambda_{min}\Big(\sum_{\ell=1}^{\tau(t)} \mathbb{E}^{\pi}_{\boldsymbol{\theta}}\Big\{\mathcal{K}_\ell \Big| \tilde{\mathcal{F}}_{\ell-1}\Big\}\Big) \geq \frac{1}{2}\bar{\lambda}_y \tau(t)\right\} \\
&= \mathbb{P}^{\pi}_{\boldsymbol{\theta}}\left\{\lambda_{min}\left(\mathcal{J}_{\tau(t)}\right) \leq \lambda_{min}\Big(\sum_{s=1}^{\tau(t)} \mathbb{E}^{\pi}_{\boldsymbol{\theta}}\Big\{\mathcal{K}_\ell \Big| \tilde{\mathcal{F}}_{\ell-1}\Big\}\Big) - \frac{1}{2}\bar{\lambda}_y \tau(t)\right\} \\
&\overset{(n)}{\geq} \mathbb{P}^{\pi}_{\boldsymbol{\theta}}\left\{\lambda_{min}\left(\mathcal{J}_{\tau(t)}\right) \leq \frac{1}{2}\bar{\lambda}_y \tau(t)\right\}. \qquad (3.105)
\end{aligned}$$

where step (j) follows directly from the matrix freedman inequality in (3.104) given the selected values $\delta = \frac{1}{2}\bar{\lambda}_y \tau(t)$, $\sigma_2^2 = \bar{\lambda}_W \tau(t)$, and $k_\lambda = \frac{\frac{1}{8}\bar{\lambda}_y^2}{\bar{\lambda}_W + \frac{1}{6}\bar{\lambda}_M \bar{\lambda}_y}$. In step (k), the equality follows from the property that $W_{\tau(t)} \leq \bar{\lambda}_W \tau(t)$ almost surely (by (3.103)). In step (l), from the expression of Y_ℓ, X_ℓ (defined in (3.99)) and $\mathcal{J}_\ell$ (defined in (3.98)), we obtain that $Y_{\tau(t)} = -\mathcal{J}_{\tau(t)} + \sum_{\ell=1}^{\tau(t)} \mathbb{E}^{\pi}_{\boldsymbol{\theta}}\Big\{\mathcal{K}_\ell \Big| \tilde{\mathcal{F}}_{\ell-1}\Big\}$. The inequality in step (l) follows from the fact that $\lambda_{max}(Y_{\tau(t)}) \geq \lambda_{max}(-\mathcal{J}_{\tau(t)}) + \lambda_{min}(\sum_{\ell=1}^{\tau(t)} \mathbb{E}^{\pi}_{\boldsymbol{\theta}}\Big\{\mathcal{K}_\ell \Big| \tilde{\mathcal{F}}_{\ell-1}\Big\})$. Step (m) follows from the fact that $\lambda_{max}(-\mathcal{J}_{\tau(t)}) = -\lambda_{min}(\mathcal{J}_{\tau(t)})$. Step (n) follows directly from (3.102).

Based on the expression of $\mathcal{I}_t$ in (3.96) and $\mathcal{J}_{\tau(t)}$ in (3.98), we can readily establish that $\mathcal{I}_t \succeq \mathcal{J}_{\tau(t)}$. By picking $\lambda_0 = \frac{1}{2}\bar{\lambda}_y$ and $k_0 > 0$ such that $n(n+\kappa_{\boldsymbol{\xi}}+1)\exp\left\{-k_\lambda z\right\} \leq \frac{k_0}{z}$ for all $z \in \{1, 2, \ldots, \}$, we complete the verification of (3.23) by deducing that

$$\begin{aligned}
\mathbb{P}^{\pi}_{\boldsymbol{\theta}}\Big\{\lambda_{min}\left(\mathcal{I}_t\right) \leq \lambda_0 \tau(t)\Big\} &= \mathbb{P}^{\pi}_{\boldsymbol{\theta}}\left\{\lambda_{min}\left(\mathcal{I}_t\right) \leq \frac{1}{2}\bar{\lambda}_y \tau(t)\right\} \\
&\overset{(o)}{\leq} \mathbb{P}^{\pi}_{\boldsymbol{\theta}}\left\{\lambda_{min}\left(\mathcal{J}_{\tau(t)}\right) \leq \frac{1}{2}\bar{\lambda}_y \tau(t)\right\} \leq \frac{k_0}{\tau(t)}. \qquad (3.106)
\end{aligned}$$

where in step (o), since the symmetric matries $\mathcal{I}_t$ and $\mathcal{J}_{\tau(t)}$ satisfy $\mathcal{I}_t \succeq \mathcal{J}_{\tau(t)} \succeq 0$, we obtain that $\lambda_{min}(\mathcal{I}_t - \mathcal{J}_{\tau(t)}) \geq 0$, which further implies that $\lambda_{min}(\mathcal{I}_t) \geq \lambda_{min}(\mathcal{J}_{\tau(t)}) + \lambda_{min}(\mathcal{I}_t - \mathcal{J}_{\tau(t)}) \geq \lambda_{min}(\mathcal{J}_{\tau(t)})$.

Proof of Claim (ii).

To verify Assumption 8 in Claim (i), for simplicity of notation, we suppress the history information $\mathcal{H}_t$ in the expressions of the log-likelihood function $\ell_t(\tilde{\boldsymbol{\theta}}|\mathcal{H}_t)$ and its corresponding empirical information matrix $\mathcal{I}_t(\tilde{\boldsymbol{\theta}}|\mathcal{H}_t)$. Under Example 7(ii), the log-likelihood function can be expressed as

$$\ell_t(\tilde{\boldsymbol{\theta}}) \;=\; \sum_{s=1}^{t}\sum_{i=1}^{n} D_{is}(\tilde{\theta}_{0i} + \tilde{\theta}_{1i}\tilde{\xi}_{is} - \tilde{\theta}_{2i}p_{is}) - \sum_{s=1}^{t} \log\left(1 + \sum_{j=1}^{n}\exp\left\{(\tilde{\theta}_{0j} + \tilde{\theta}_{1j}\tilde{\xi}_{js} - \tilde{\theta}_{2j}\tilde{p}_{js})\right\}\right). \tag{3.107}$$

We define two matrices (1) $\mathcal{M}(\boldsymbol{p}_s, \boldsymbol{\xi}_s) = (1, \boldsymbol{\xi}_s, -\boldsymbol{p}_s)(1, \boldsymbol{\xi}_s, -\boldsymbol{p}_s)^\intercal$, and (2) $B(\boldsymbol{p}_s, \boldsymbol{\xi}_s, \tilde{\boldsymbol{\theta}})$ be such that $B_{ii}(\boldsymbol{p}_s, \boldsymbol{\xi}_s, \tilde{\boldsymbol{\theta}}) = Q_i(\boldsymbol{p}_s, \boldsymbol{\xi}_s, \tilde{\boldsymbol{\theta}})[1 - Q_i(\boldsymbol{p}_s, \boldsymbol{\xi}_s, \tilde{\boldsymbol{\theta}})]$ for all $i \in \{1, \ldots, n\}$ and $B_{ij}(\boldsymbol{p}_s, \boldsymbol{\xi}_s, \tilde{\boldsymbol{\theta}}) = Q_i(\boldsymbol{p}_s, \boldsymbol{\xi}_s, \tilde{\boldsymbol{\theta}})Q_j(\boldsymbol{p}_s, \boldsymbol{\xi}_s, \tilde{\boldsymbol{\theta}})$ for all $i \neq j$. With some arithmetic derivation, we obtain that the empirical Fisher information matrix $\mathcal{I}_t(\tilde{\boldsymbol{\theta}})$ associated with the log-likelihood function $\ell_t(\tilde{\boldsymbol{\theta}})$ in (3.107) satisfies

$$\mathcal{I}_t(\tilde{\boldsymbol{\theta}}) \;=\; \sum_{s=1}^{t} B(\boldsymbol{p}_s, \boldsymbol{\xi}_s, \tilde{\boldsymbol{\theta}}) \otimes \mathcal{M}(\boldsymbol{p}_s, \boldsymbol{\xi}_s). \tag{3.108}$$

Letting $g_t(\tilde{\boldsymbol{\theta}}) = \ell_t(\tilde{\boldsymbol{\theta}}) - \ell_{t-1}(\tilde{\boldsymbol{\theta}})$ for all $t \in \mathbb{N}_+$, we can immediately establish that $g_t(\tilde{\boldsymbol{\theta}})$ is a twice continuously differentiable function in $\tilde{\boldsymbol{\theta}} \in \Theta$. Moreover, it can be easily verified that for any $(\tilde{\boldsymbol{p}}, \tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{\theta}}) \in [l, u]^n \times \Xi \times \Theta$, matrix $B(\tilde{\boldsymbol{p}}, \tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{\theta}})$ is symmetric and strictly diagonal dominant with each diagonal component being strictly positive. This implies that $B(\boldsymbol{p}_t, \boldsymbol{\xi}_t, \tilde{\boldsymbol{\theta}}) \succ 0$ for all $\tilde{\boldsymbol{\theta}} \in \Theta$. Together with the observation that $\mathcal{M}(\tilde{\boldsymbol{p}}, \tilde{\boldsymbol{\xi}}) \succeq 0$ for any $(\tilde{\boldsymbol{p}}, \tilde{\boldsymbol{\xi}}) \in [l, u]^n \times \Xi$, we obtain that $\mathcal{I}_t(\tilde{\boldsymbol{\theta}}) - \mathcal{I}_{t-1}(\tilde{\boldsymbol{\theta}}) \succeq 0$, from which we conclude that $g_t(\tilde{\boldsymbol{\theta}})$ is weakly concave in $\tilde{\boldsymbol{\theta}} \in \Theta$.

To verify Assumption 13 in Claim (ii), we first show that Assumption13(i) holds. For any $\tilde{\boldsymbol{p}} \in [l,u]^n$, $\tilde{\boldsymbol{\xi}} \in \Xi$ and $\tilde{\boldsymbol{\theta}} \in \Theta$, there exists a compact subset $\mathcal{B} \subset (0,1)^{n\times n}$ such that $B(\tilde{\boldsymbol{p}}, \tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{\theta}}) \in \mathcal{B}$. Since all entries of $B(\tilde{\boldsymbol{p}}, \tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{\theta}})$ are continuous in $(\tilde{\boldsymbol{p}}, \tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{\theta}})$ for all possible $(\tilde{\boldsymbol{p}}, \tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{\theta}}) \in [l,u]^n \times \Xi \times \Theta$, there exists $\overline{\lambda}_B > \underline{\lambda}_B > 0$ such that

$$\begin{aligned}\min_{\|\boldsymbol{x}\|_2^2=1} \min_{(\tilde{\boldsymbol{p}}, \tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{\theta}}) \in [l,u]^n \times \Xi \times \Theta} \boldsymbol{x}^\mathsf{T} B(\tilde{\boldsymbol{p}}, \tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{\theta}}) \boldsymbol{x} &= \underline{\lambda}_B > 0, \\ \max_{\|\boldsymbol{x}\|_2^2=1} \max_{(\tilde{\boldsymbol{p}}, \tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{\theta}}) \in [l,u]^n \times \Xi \times \Theta} \boldsymbol{x}^\mathsf{T} B(\tilde{\boldsymbol{p}}, \tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{\theta}}) \boldsymbol{x} &= \overline{\lambda}_B > 0. \end{aligned} \tag{3.109}$$

To proceed, we can pick $\bar{\lambda}_I > 0$ such that $\bar{\lambda}_I > \frac{1}{\underline{\lambda}_B} \max_{(\tilde{\boldsymbol{p}}, \tilde{\boldsymbol{\xi}}) \in [l,u]^n \times \Xi} \max_{\|\boldsymbol{x}\|_2^2=1} \|(1, \tilde{\boldsymbol{\xi}}^\mathsf{T}, \tilde{\boldsymbol{p}}^\mathsf{T})\boldsymbol{x}\|_2^2$. As a result, for all $t \in \mathbb{N}_{++}$, we can deduce that

$$\begin{aligned}\lambda_{max}\left(\mathcal{I}_t(\tilde{\boldsymbol{\theta}}) - \mathcal{I}_{t-1}(\tilde{\boldsymbol{\theta}})\right) &\overset{(p)}{=} \lambda_{max}\left(B(\boldsymbol{p}_t, \boldsymbol{\xi}_t, \tilde{\boldsymbol{\theta}}) \otimes \mathcal{M}(\boldsymbol{p}_t, \boldsymbol{\xi}_t)\right) \\ &\leq \max_{(\tilde{\boldsymbol{p}}, \tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{\theta}}) \in [l,u]^n \times \Xi \times \Theta} \lambda_{max}\left(B(\tilde{\boldsymbol{p}}, \tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{\theta}}) \otimes \mathcal{M}(\tilde{\boldsymbol{p}}, \tilde{\boldsymbol{\xi}})\right) \\ &= \max_{(\tilde{\boldsymbol{p}}, \tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{\theta}}) \in [l,u]^n \times \Xi \times \Theta} \max_{\|\boldsymbol{x}\|_2^2=1} \boldsymbol{x}^\mathsf{T} \Big[B(\tilde{\boldsymbol{p}}, \tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{\theta}}) \otimes \mathcal{M}(\tilde{\boldsymbol{p}}, \tilde{\boldsymbol{\xi}})\Big] \boldsymbol{x} \leq \bar{\lambda}_I, \end{aligned} \tag{3.110}$$

where step (p) follows directly from (3.108). With the observations above, we establish Assumption 13(i).

To establish Assumption13(ii), we pick a positive constant $\kappa_I > 0$ such that $\underline{\lambda}_B - \frac{1}{\kappa_I}\overline{\lambda}_B > 0$ and $\overline{\lambda}_B - \kappa_I \underline{\lambda}_B < 0$. This allows us to establish that $\frac{1}{\kappa_I}\mathcal{I}_t(\tilde{\boldsymbol{\theta}}_2) \preceq \mathcal{I}_t(\tilde{\boldsymbol{\theta}}_1) \preceq \kappa_I \mathcal{I}_t(\tilde{\boldsymbol{\theta}}_2)$ for all $\tilde{\boldsymbol{\theta}}_1, \tilde{\boldsymbol{\theta}}_2 \in \Theta$.

The verification the inequality in (3.23) is similar to that of Claim (i). Given $T \geq 2$, $t \in \{1, \ldots, T\}$ and $\mathcal{T}_t = \{t_1, \ldots, t_{\tau(t)}\}$, for any $\ell \in \{1, \ldots, \tau(t)\}$, we define

$$\begin{aligned}\mathcal{K}_\ell &= B(\boldsymbol{p}_{t_\ell}, \boldsymbol{\xi}_{t_\ell}, \boldsymbol{\theta}) \otimes \mathcal{M}(\boldsymbol{p}_{t_\ell}, \boldsymbol{\xi}_{t_\ell}), \\ \mathcal{J}_\ell &= \sum_{s=1}^{\ell} \mathcal{K}_s. \end{aligned} \tag{3.111}$$

Under filtration $\tilde{\mathcal{F}}_\ell = \mathcal{F}_{t_\ell}$ for any $\ell \in \{1, \ldots, \tau(t)\}$, we further define the stochastic process

$$\begin{aligned} X_\ell &= -\mathcal{K}_\ell + \mathbb{E}^\pi_{\boldsymbol{\theta}}\Big\{\mathcal{K}_\ell | \tilde{\mathcal{F}}_{\ell-1}\Big\}, \\ Y_\ell &= \sum_{s=1}^{\ell} X_s, \\ W_\ell &= \Big\| \sum_{s=1}^{\ell} X_s^2 \Big\|_2. \end{aligned} \tag{3.112}$$

We develop the sufficient conditions on X_ℓ, $\sum_{s=1}^{\ell} \mathbb{E}^\pi_{\boldsymbol{\theta}}\Big\{\mathcal{K}_s\Big|\tilde{\mathcal{F}}_{s-1}\Big\}$, and W_ℓ such that we can apply the matrix Freedman's inequality.

By construction of X_ℓ in (3.112), we immediately verify that $\mathbb{E}^\pi_{\boldsymbol{\theta}}\Big\{X_\ell\Big|\tilde{\mathcal{F}}_{\ell-1}\Big\} = \mathbf{0}$. Moreover, given the expression of X_ℓ, Y_ℓ in (3.112) and $\mathcal{J}_\ell$ in (3.111), we also have $Y_\ell = \sum_{s=1}^{\ell} X_s = -\mathcal{J}_\ell + \mathbb{E}^\pi_{\boldsymbol{\theta}}\Big\{\mathcal{J}_\ell | \tilde{\mathcal{F}}_{\ell-1}\Big\}$. Thus, $\{Y_s : s = 1, 2, \ldots\}$ is a matrix martingale and $\{X_s : s = 1, 2, \ldots\}$ is the corresponding martingale difference sequence adapted to filtration $\{\tilde{\mathcal{F}}_s : s = 1, 2 \ldots\}$. Given that both $[l, u]^n$ and Ξ are both compact sets, there exists $\bar{\lambda}_M > 0$ such that for all $\ell \in \{1, \ldots, \tau(t)\}$,

$$\begin{aligned} \lambda_{max}\left(X_\ell\right) &\overset{(q)}{=} \lambda_{max}\left(-\mathcal{K}_\ell + \mathbb{E}^\pi_{\boldsymbol{\theta}}\Big\{\mathcal{K}_\ell | \tilde{\mathcal{F}}_{\ell-1}\Big\}\right) \\ &\overset{(r)}{\leq} \lambda_{max}\left(-\mathcal{K}_\ell\right) + \lambda_{max}\left(\mathbb{E}^\pi_{\boldsymbol{\theta}}\Big\{\mathcal{K}_\ell | \tilde{\mathcal{F}}_{\ell-1}\Big\}\right) \\ &\leq \Big|\lambda_{max}\Big(B(\boldsymbol{p}_{t_\ell}, \boldsymbol{\xi}_{t_\ell}, \boldsymbol{\theta})\Big)\Big|\Big|\lambda_{max}\Big(\mathcal{M}(\boldsymbol{p}_{t_\ell}, \boldsymbol{\xi}_{t_\ell})\Big)\Big| \\ &\qquad + \Big|\lambda_{max}\Big(B(\boldsymbol{p}_{t_\ell}, \boldsymbol{\xi}_{t_\ell}, \boldsymbol{\theta})\Big)\Big|\Big|\lambda_{max}\Big(\mathcal{M}(\boldsymbol{p}_{t_\ell}, \boldsymbol{\xi}_{t_\ell})\Big)\Big| \\ &\overset{(s)}{\leq} \bar{\lambda}_M, \end{aligned} \tag{3.113}$$

where step (q) follows from the definition of X_ℓ in (3.112). In step (r), we implement the observation that $\mathcal{K}_\ell$ is the symmetric matrix. Step (s) follows from the observations that $B(\boldsymbol{p}_{t_\ell}, \boldsymbol{\xi}_{t_\ell}, \boldsymbol{\theta})$ and $\mathcal{M}(\boldsymbol{p}_{t_\ell}, \boldsymbol{\xi}_{t_\ell})$ have uniformly bounded eigenvalues.

Since $\{(\boldsymbol{p}_s, \boldsymbol{\xi}_s) : s \in \mathcal{T}_t\}$ are i.i.d, we denote by $(\boldsymbol{p}, \boldsymbol{\xi})$ be the random vector such that

$(\boldsymbol{p}_s, \boldsymbol{\xi}_s) = (\boldsymbol{p}, \boldsymbol{\xi})$ for any $s \in \mathcal{T}_t$. This allows us to establish that

$$\sum_{\ell=1}^{\tau(t)} \mathbb{E}_{\boldsymbol{\theta}}^{\pi}\Big\{\mathcal{K}_\ell \Big| \tilde{\mathcal{F}}_{\ell-1}\Big\} \overset{(t)}{=} \sum_{\ell=1}^{\tau(t)} \mathbb{E}_{\boldsymbol{\theta}}^{\pi}\Big\{B(\boldsymbol{p}, \boldsymbol{\xi}, \boldsymbol{\theta}) \otimes \mathcal{M}(\boldsymbol{p}, \boldsymbol{\xi})\Big\}, \tag{3.114}$$

where step (t) follow from the observation that $\{(\boldsymbol{p}_s, \boldsymbol{\xi}_s) : s \in \mathcal{T}_t\}$ are i.i.d. With (3.114), we can show that

$$\begin{aligned} \lambda_{min}\left(\sum_{\ell=1}^{\tau(t)} \mathbb{E}_{\boldsymbol{\theta}}^{\pi}\Big\{\mathcal{K}_\ell \Big| \tilde{\mathcal{F}}_{\ell-1}\Big\}\right) &\overset{(u)}{\geq} \sum_{\ell=1}^{\tau(t)} \lambda_{min}\left(\mathbb{E}_{\boldsymbol{\theta}}^{\pi}\Big\{B(\boldsymbol{p}, \boldsymbol{\xi}, \boldsymbol{\theta}) \otimes \mathcal{M}(\boldsymbol{p}, \boldsymbol{\xi})\Big\}\right) \\ &\overset{(v)}{\geq} \underline{\lambda}_B \sum_{\ell=1}^{\tau(t)} \lambda_{min}\left(\mathbb{E}_{\boldsymbol{\theta}}^{\pi}\Big\{\mathcal{M}(\boldsymbol{p}, \boldsymbol{\xi})\Big\}\right) \\ &\overset{(w)}{=} \bar{\lambda}_y \tau(t). \end{aligned} \tag{3.115}$$

where step (u) follows from the the symmetry of matrix $\mathcal{K}_\ell$ for $\ell \in \{1, \ldots, \tau(t)\}$. Step (v) follows from the fact that $\lambda_{min}(B(\tilde{\boldsymbol{p}}, \tilde{\boldsymbol{\xi}}, \boldsymbol{\theta})) \geq \underline{\lambda}_B$. In step (w), based on the expression $\mathcal{M}(\tilde{\boldsymbol{p}}, \tilde{\boldsymbol{\xi}}) = (1, \tilde{\boldsymbol{\xi}}, -\tilde{\boldsymbol{p}})(1, \tilde{\boldsymbol{\xi}}, -\tilde{\boldsymbol{p}})^\intercal$, we further let $(\bar{\boldsymbol{p}}, \bar{\boldsymbol{\xi}})$ be the mean value of $(\boldsymbol{p}, \boldsymbol{\xi})$. Similar to the arguments in (3.101) - (3.102), we let $\bar{\boldsymbol{y}} = \arg\min_{|\boldsymbol{y}|_2^2=1} (y_0 + \boldsymbol{y}_1^\intercal \bar{\boldsymbol{\xi}} + \boldsymbol{y}_2^\intercal \bar{\boldsymbol{p}})^2 + \boldsymbol{y}_1^\intercal Cov\,(\boldsymbol{\xi})\, \boldsymbol{y}_1 + \boldsymbol{y}_2^\intercal Cov\,(\boldsymbol{p})\, \boldsymbol{y}_2$. Recalling that $Cov(\boldsymbol{\xi}) \succ 0$ and $Cov(\boldsymbol{p}) \succ 0$, given the definition of $\underline{\lambda}_B$ in (3.109), there exists a positive constant $\bar{\lambda}_y = \underline{\lambda}_B[(\bar{y}_0 + \bar{\boldsymbol{y}}_1^\intercal \bar{\boldsymbol{\xi}} + \bar{\boldsymbol{y}}_2^\intercal \bar{\boldsymbol{p}})^2 + \bar{\boldsymbol{y}}_1^\intercal Cov\,(\boldsymbol{\xi})\, \bar{\boldsymbol{y}}_1 + \bar{\boldsymbol{y}}_2^\intercal Cov\,(\boldsymbol{p})\, \bar{\boldsymbol{y}}_2]$. Thus, we have that $\lambda_{min}\Big(\mathbb{E}_{\boldsymbol{\theta}}^{\pi}\big[\mathcal{M}(\boldsymbol{p}, \boldsymbol{\xi})\big]\Big) \geq \min_{|\boldsymbol{y}|_2^2=1} (y_0 + \boldsymbol{y}_1^\intercal \bar{\boldsymbol{\xi}} + \boldsymbol{y}_2^\intercal \bar{\boldsymbol{p}})^2 + \boldsymbol{y}_1^\intercal Cov\,(\boldsymbol{\xi})\, \boldsymbol{y}_1 + \boldsymbol{y}_2^\intercal Cov\,(\boldsymbol{p})\, \boldsymbol{y}_2 = \bar{\lambda}_y / \underline{\lambda}_B$.

By replicating the same arguments as in (3.103), we can pick $\bar{\lambda}_W = 4\bar{\lambda}_I^2$ such that

$$W_{\tau(t)} \leq \bar{\lambda}_W \tau(t). \tag{3.116}$$

Given (3.113), (3.115) and (3.116), we apply the matrix Freedman inequality and replicate the same arguments as in (3.105) to establish that there exists $k_0, \lambda_0 > 0$ such that

$$\mathbb{P}^{\pi}_{\boldsymbol{\theta}}\Big\{\lambda_{min}\left(\mathcal{I}_t\right) \leq \lambda_0 \tau(t)\Big\} \ \leq \ \frac{k_0}{\tau(t)} \tag{3.117}$$

Summarizing the observations above, we complete the proof of Claim (ii).

Proof of Claim (iii).

To verify Assumption 8 in Claim (i), for any $\tilde{\boldsymbol{\theta}} \in \Theta$ and $t \in \mathbb{N}_{++}$, we suppress the history information $\mathcal{H}_t = \{(\boldsymbol{p}_s, \boldsymbol{\xi}_s, \boldsymbol{D}_s) : s = 1, \ldots, t\}$ in the expressions of the log-likelihood function $\ell_t(\tilde{\boldsymbol{\theta}}|\mathcal{H}_t)$ and the empirical fisher information matrix $\mathcal{I}_t(\tilde{\boldsymbol{\theta}}|\mathcal{H}_t)$ for simplicity of notation.

Under Example 7(iii), the log-likelihood function in period t can be expressed as

$$\begin{aligned}\ell_t(\tilde{\boldsymbol{\theta}}) \ &= \ -\frac{nt}{2}\log 2\pi - \frac{t}{2}\log det(\Sigma_t) \\ &\quad - \sum_{s=1}^{t}\frac{1}{2}[\boldsymbol{D}_s - (\tilde{\boldsymbol{\theta}}_0 - \tilde{\boldsymbol{\theta}}_1\boldsymbol{p}_s + \tilde{\boldsymbol{\theta}}_2\boldsymbol{\xi}_s\boldsymbol{p}_s)]^{\intercal}\Sigma_s^{-1}[\boldsymbol{D}_s - (\tilde{\boldsymbol{\theta}}_0 - \tilde{\boldsymbol{\theta}}_1\boldsymbol{p}_s + \tilde{\boldsymbol{\theta}}_2\boldsymbol{\xi}_s\boldsymbol{p}_s)].\end{aligned} \tag{3.118}$$

With $g_t(\tilde{\boldsymbol{\theta}}) = \ell_t(\tilde{\boldsymbol{\theta}}) - \ell_{t-1}(\tilde{\boldsymbol{\theta}})$, by repeating the same arguments as in Claim ((i)), we can readily verify that it is twice continuously differentiable and weakly concave in $\tilde{\boldsymbol{\theta}} = (\tilde{\boldsymbol{\theta}}_0, \tilde{\boldsymbol{\theta}}_1, \tilde{\boldsymbol{\theta}}_2) \in \Theta$.

To verify Assumption 13 in Claim (iii), it is worth noting that in Example 7(iii), we have $\boldsymbol{Q}(\tilde{\boldsymbol{p}}, \tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{\theta}}) \ = \ \tilde{\boldsymbol{\theta}}_0 - \tilde{\boldsymbol{\theta}}_1\tilde{\boldsymbol{p}} + \tilde{\boldsymbol{\theta}}_2\tilde{\boldsymbol{\xi}}\tilde{\boldsymbol{p}}$. By letting $\tilde{\boldsymbol{y}} = \tilde{\boldsymbol{\xi}}\tilde{\boldsymbol{p}}$, the demand model in Example 7(iii) can be reformulated as an instance of Example 7(i):

$$\tilde{\boldsymbol{Q}}(\tilde{\boldsymbol{p}}, \tilde{\boldsymbol{y}}, \tilde{\boldsymbol{\theta}}) \ = \ \tilde{\boldsymbol{\theta}}_0 - \tilde{\boldsymbol{\theta}}_1\tilde{\boldsymbol{p}} + \tilde{\boldsymbol{\theta}}_2\tilde{\boldsymbol{y}}. \tag{3.119}$$

Let $vec(\cdot)$ be the vector form of a matrix. Given the policy-induced price process $\{\boldsymbol{p}_s : s = 1, 2, \ldots\}$ and the set of random exploration periods $\mathcal{T}_t$, there exists a random vector $\tilde{\boldsymbol{\xi}}$ with

$Cov(vec(\boldsymbol{\xi})) \succ 0$ such that $\boldsymbol{\xi}_{t_\ell} = \boldsymbol{\xi}$ for all $\ell \in \{1, \dots, \tau(t)\}$. Moreover, under policy π, there exists a random price vector $\tilde{\boldsymbol{p}}$ such that $\boldsymbol{p}_{t_\ell} = \boldsymbol{p}$ for all $\ell \in \{1, \dots, \tau(t)\}$ with $Cov(\boldsymbol{p}) \succ 0$. We obtain that the covariance matrix of the random vector $\boldsymbol{y}$ satisfies

$$Cov(\boldsymbol{y}) \;=\; Cov(\boldsymbol{\xi}\boldsymbol{p}) \;\succ\; 0. \tag{3.120}$$

Then we can repeat exactly the same proof arguments above to establish that Assumption 13 also holds for Example 7(iii).

With the arguments above, we establish that the classes of instances in Example 7(iii) satisfy Assumption 13. $\square$

Proof of Proposition 25. Proof of Claim (i). To establish Assumption 9, by (3.25), we have $\tilde{R}_{i(t+1)}(p) \;=\; \eta_t \tilde{g}_{it} p - M_{h_i}(p, p_{it})$. From $\sum_{s=1}^{\infty} \eta_s^2 < \infty$, we obtain that there exists $\bar{\eta} > 0$ such that $\eta_s \le \bar{\eta}$ for any $s \in \{1, 2, \dots\}$. By the instance set-up below (3.25), we have $|\tilde{g}_{it}| \le \bar{\sigma}_g$ for some $\bar{\sigma}_g \ge 0$. Given that $p \in [l, u]$ and that $M_{h_i}(p, p_{it}) = h_i(p) - h_i(p_{it}) - h_i'(p_{it})(p - p_{it})$ with continuously differentiable function $h_i(\cdot)$, we immediately observe that there exists $\bar{M}_0 > 0$ such that $|M_{h_i}(p, p_{it})| \le \bar{M}_0$ for any $i \in \{1, \dots, n\}$, $p, p_{it} \in [l, u]$. Thus, we can pick $M_R^0 \ge \bar{\eta}\bar{\sigma}_g u + \bar{M}_0 > 0$ such that $|\tilde{R}_{i(t+1)}(p)| \le \bar{M}$ for any $i \in \{1, \dots, n\}$, $t \in \{1, 2, \dots\}$, and $p \in [l, u]$. Given that $\frac{d}{dp}\tilde{R}_{i(t+1)}(p) = \eta_t \tilde{g}_{it} + h_i'(p) - h_i'(p_{it})$, by the continuously differentiable of function $h_i(\cdot)$ and that $p, p_{it} \in [l, u]$, there exists $M_R^1 > 0$ such that $|\frac{d}{dp}\tilde{R}_{i(t+1)}(p)| \le M_R^1$ for any $i \in \{1, \dots, n\}$, $t \in \{1, 2, \dots\}$, and $p \in [l, u]$.

To establish Assumption 12, we divide the proof arguments in the following steps.

Step (i) -1: establish an equivalent expression for the price process in (3.25). Let π be the platform's DN policy. It is worth noting that under this policy, the reward contract satisfies $\mathcal{W}_{it}(p) = 0$ for all $p \in [l, u]$, $i \in \{1, \dots, n\}$, and $t \in \{1, 2 \dots\}$. Moreover, the expectation operator $E_{\boldsymbol{\xi},\boldsymbol{\varepsilon}}\{\cdot\}$ is equivalent to $E_{\boldsymbol{\theta}}^{\pi}\{\cdot\}$.

By definition, from (3.25), we have $\tilde{g}_{it}$ be such that

$$\mathbb{E}_{\boldsymbol{\xi},\boldsymbol{\varepsilon}}\{\tilde{g}_{it}|\mathcal{F}^0_{t-1}\} = \frac{\partial}{\partial p_{it}} \log(\mathbb{E}_{\boldsymbol{\xi},\boldsymbol{\varepsilon}}\{R_{s_i}(\boldsymbol{p}_t, \boldsymbol{\xi}_t, \boldsymbol{\theta})|\mathcal{F}^0_{t-1}\}), \tag{3.121}$$

where $R_{s_i}(\boldsymbol{p}_t, \boldsymbol{\xi}_t, \boldsymbol{\theta}) = \gamma p_{it} Q_i(\boldsymbol{p}_t, \boldsymbol{\xi}_t, \boldsymbol{\theta})$. Based on the definition of the Bregman divergence $M_{h_i}(x, y) = h_i(x) - h_i(y) - h_i'(y)(x - y)$ in (3.25), for all $i \in \{1, \dots, n\}$, the price process in (3.25) satisfies

$$\begin{aligned} p_{i(t+1)} &= \underset{p\in[l,u]}{\arg\max} \left\{\eta_t \tilde{g}_{it} p - M_{h_i}(p, p_{it})\right\} \\ &= \underset{p\in[l,u]}{\arg\max} \left\{\eta_t \tilde{g}_{it} p - h_i(p) + h_i(p_{it}) + h_i'(p_{it})(p - p_{it})\right\} \\ &= \underset{p\in[l,u]}{\arg\max} \left\{[h_i'(p_{it}) + \eta_t \tilde{g}_{it}]p - h_i(p)\right\}, \end{aligned} \tag{3.122}$$

We define the convex conjugate for $h_i(\cdot)$ and its corresponding maximizer respectively as:

$$\begin{aligned} h_i^\star(v) &= \max_{p\in[l,u]} \{vp - h_i(p)\}, \quad \forall v \in \mathbb{R} \\ Q_i(v) &= \underset{p\in[l,u]}{\arg\max} \{vp - h_i(p)\}, \quad \forall v \in \mathbb{R}. \end{aligned} \tag{3.123}$$

Based on (3.122) and (3.123), the price process $\{\boldsymbol{p}_s : s = 1, 2, \dots\}$ induced by the sellers from solving (3.25) under policy π satisfies that for any $i \in \{1, \dots, n\}$ and $t \in \{1, 2, \dots\}$,

$$\tilde{y}_{i(t+1)} = h_i'(p_{it}) + \eta_t \tilde{g}_{it}, \tag{3.124a}$$

$$p_{i(t+1)} = Q_i(\tilde{y}_{i(t+1)}). \tag{3.124b}$$

Step (i) -2: define the Fenchel-Young inequality gap. Based on (3.123), the Fenchel-Young inequality suggests that $h_i^\star(v) + h_i(p) - vp \geq 0$. We define the gap induced by the Fenchel-

Young inequality as

$$F_i(p,v) \;=\; h_i^\star(v)+h_i(p)-vp, \quad \forall p\in\mathbb{R}, v\in\mathbb{R}. \tag{3.125}$$

Recall from the discussions in (3.25) that $h_i(\cdot)$ is a continuously differentiable and l_h-strongly convex function on $[l,u]$ for all $i\in\{1,\dots,n\}$. Note that the l_h-strongly convexity of function $h_i(p)$ in $p\in\mathbb{R}$ implies that function $h_i^\star(v)$ is $\frac{1}{l_h}$-strongly smooth in $v\in\mathbb{R}$. From the definition of function $F_i(p,v)$ in (3.125), we summarize the following property for the next step.

Lemma 23. *For all $p,v\in\mathbb{R}$, and $i\in\{1,\dots,n\}$, function $F_i(p,v)$ satisfies*

$$F_i(p,v) \;\le\; F_i(p,v')+(v-v')([h_i^\star]'(v')-p)+\frac{1}{2l_h}(v-v')^2, \quad \forall v'\in\mathbb{R}. \tag{3.126}$$

In a compact form, for all $\boldsymbol{p},\boldsymbol{v}\in\mathbb{R}^n$, we define $\boldsymbol{\eta}_t=(\eta_{1t},\dots,\eta_{nt})$, $\tilde{\boldsymbol{g}}_t=(\tilde{g}_{1t},\dots,\tilde{g}_{nt})$, $h(\boldsymbol{p})=\sum_{i=1}^n h_i(p_i)$, and $Q(\boldsymbol{v})=(Q_1(v_1),\dots,Q_n(v_n))$. Moreover, we define that

$$F(\boldsymbol{p},\boldsymbol{v})=\sum_{i=1}^n F_i(p_i,v_i). \tag{3.127}$$

Step (i)-3: show that the price process $\boldsymbol{p}_t$ is recurrent in the neighborhood of $\bar{\boldsymbol{p}}^{\boldsymbol{\theta}}$ in expectation. For any $\epsilon>0$, the goal of this step is to establish

$$\sup\left\{t\in\mathbb{N}:\mathbb{E}^\pi_{\boldsymbol{\theta}}\left\{F(\bar{\boldsymbol{p}}^{\boldsymbol{\theta}},\nabla h(\boldsymbol{p}_t))\right\}<\epsilon\right\} \;=\; \infty. \tag{3.128}$$

Recall that $\tilde{g}_{it}$ is such that $\mathbb{E}_{\boldsymbol{\xi},\boldsymbol{\varepsilon}}\{\tilde{g}_{it}|\mathcal{F}^0_{t-1}\}=\frac{\partial}{\partial p_{it}}\log(\mathbb{E}_{\boldsymbol{\xi},\boldsymbol{\varepsilon}}\{R_{s_i}(\boldsymbol{p}_t,\boldsymbol{\xi}_t,\boldsymbol{\theta})|\mathcal{F}^0_{t-1}\})$ where $R_{s_i}(\boldsymbol{p}_t,\boldsymbol{\xi}_t,\boldsymbol{\theta})=\gamma p_{it}Q_i(\boldsymbol{p}_t,\boldsymbol{\xi}_t,\boldsymbol{\theta})$ and the expectation measure $\mathbb{E}_{\boldsymbol{\xi},\boldsymbol{\varepsilon}}\{\cdot\}$ is induced by $\{(\tilde{\boldsymbol{\varepsilon}}_s,\boldsymbol{\xi}_s):s=1,2,\dots\}$. Given $\mathcal{F}^0_t=\sigma(\boldsymbol{\varepsilon}_1,\dots,\boldsymbol{\varepsilon}_t)$ and the equivalence of $E_{\boldsymbol{\xi},\boldsymbol{\varepsilon}}\{\cdot\}$ and $E^\pi_{\boldsymbol{\theta}}\{\cdot\}$, for all $i\in\{1,\dots,n\}$, we define $\bar{g}_{it}=\mathbb{E}^\pi_{\boldsymbol{\theta}}\{\tilde{g}_{it}|\mathcal{F}^0_{t-1}\}$ for simplicity of notation. We establish the following auxiliary lemma to support our proof.

Lemma 24. *Let π be the platform's DN policy with seller-induced price process $\{\boldsymbol{p}_s : s =$*

$1, 2, \dots\}$ under policy π, and let $\bar{\boldsymbol{p}}^{\boldsymbol{\theta}}$ be the unique DN Nash equilibrium defined in Definition 5. For any $t \in \{1, 2, \dots, \}$, we further let $(\boldsymbol{p}_{t+1}, \tilde{\boldsymbol{y}}_{t+1})$ be the process defined in (3.124)*. For any $i \in \{1, \dots, n\}$, we have*

(i) $F_i(\bar{p}_i^{\boldsymbol{\theta}}, h_i'(p_{i(t+1)})) \le F_i(\bar{p}_i^{\boldsymbol{\theta}}, \tilde{y}_{i(t+1)})$;

(ii) $\mathbb{E}_{\boldsymbol{\theta}}^{\pi}\Big\{F_i(\bar{p}_i^{\boldsymbol{\theta}}, h_i'(p_{i(t+1)}))|\mathcal{F}_{t-1}^0\Big\} \le F_i(\bar{p}_i^{\boldsymbol{\theta}}, h_i'(p_{it})) + \eta_t \bar{g}_{it}(p_{it} - \bar{p}_i^{\boldsymbol{\theta}}) + \frac{1}{2l_h}\mathbb{E}_{\boldsymbol{\theta}}^{\pi}\left\{\eta_t^2 \tilde{g}_{it}^2|\mathcal{F}_{t-1}^0\right\}$.

Before proceeding, we also clarify the following notations

$$\begin{aligned} \bar{U}_i(\boldsymbol{p}_t) &\overset{(a)}{=} \frac{\partial}{\partial p_{it}} \log(p_{it}\mathbb{E}_{\boldsymbol{\xi},\boldsymbol{\varepsilon}}\{Q_i(\boldsymbol{p}_t, \boldsymbol{\xi}_t, \boldsymbol{\theta})\}) \\ &\overset{(b)}{=} \frac{\partial}{\partial p_{it}} \log(\gamma p_{it}\mathbb{E}_{\boldsymbol{\theta}}^{\pi}\{Q_i(\boldsymbol{p}_t, \boldsymbol{\xi}_t, \boldsymbol{\theta})|\mathcal{F}_{t-1}^0\}) \overset{(c)}{=} \bar{g}_{it}. \end{aligned} \tag{3.129}$$

where step (a) follows from the definition of $\bar{U}_i(\boldsymbol{p}_t)$ in Assumption 11. In step (b), given $\mathcal{F}_{t-1}^0$ and price vector $\boldsymbol{p}_t$, the only uncertainty is induced by $\boldsymbol{\xi}_t$. We also replace the expectation operator from $\mathbb{E}_{\boldsymbol{\xi},\boldsymbol{\varepsilon}}\{\cdot\}$ to $\mathbb{E}_{\boldsymbol{\theta}}^{\pi}\{\cdot\}$ in step (b). Step (c) follow follows directly from the definition of $\bar{g}_{it}$.

By letting $\tilde{\boldsymbol{g}}_t = (\tilde{g}_{it})_{i=1}^n$, $\bar{\boldsymbol{g}}_t = (\bar{g}_{it})_{i=1}^n$, and $\boldsymbol{\delta}_t = \tilde{\boldsymbol{g}}_t - \bar{\boldsymbol{g}}_t$, we have that

$$\mathbb{E}_{\boldsymbol{\theta}}^{\pi}\left\{\boldsymbol{\delta}_t\middle|\mathcal{F}_{t-1}^0\right\} \overset{(d)}{=} \mathbf{0} \tag{3.130a}$$

$$\mathbb{E}_{\boldsymbol{\theta}}^{\pi}\left\{\|\boldsymbol{\delta}_t\|_2^2\middle|\mathcal{F}_{t-1}^0\right\} \overset{(e)}{\le} n\bar{\sigma}_g^2 < \infty, \tag{3.130b}$$

where in step (d), the equation follows directly from the definition of $\tilde{\boldsymbol{g}}_t$ and $\bar{\boldsymbol{g}}_t$. Step (e) follow readily from the assumptions that $|\tilde{g}_{it}| \le \bar{\sigma}_g$.

To prove the claim in (3.128), we assume towards a contradiction that there exists τ_ϵ such that $\mathbb{E}_{\boldsymbol{\theta}}^{\pi}\left[F(\bar{\boldsymbol{p}}^{\boldsymbol{\theta}}, \nabla h(\boldsymbol{p}_t))\right] \ge \epsilon$ for all $t \ge \tau_\epsilon$. Then there exists $c_\epsilon > 0$ such that for all

$t \geq \tau_\epsilon$, we have

$$0 \ > \ -c_\epsilon \ \overset{(f)}{>} \ \mathbb{E}^{\pi}_{\boldsymbol{\theta}} \left\{ \sum_{i=1}^{n} (\bar{U}_i(\boldsymbol{p}_t) - \bar{U}_i(\bar{\boldsymbol{p}}^{\boldsymbol{\theta}}))(p_{it} - \bar{p}_i^{\boldsymbol{\theta}}) \right\} \ \overset{(g)}{=} \ \mathbb{E}^{\pi}_{\boldsymbol{\theta}} \left\{ \sum_{i=1}^{n} \bar{U}_i(\boldsymbol{p}_t)(p_{it} - \bar{p}_i^{\boldsymbol{\theta}}) \right\}, \tag{3.131}$$

where in step (f), from the assumption that $\mathbb{E}^{\pi}_{\boldsymbol{\theta}} \left[F(\bar{\boldsymbol{p}}^{\boldsymbol{\theta}}, \nabla h(\boldsymbol{p}_t)) \right] \geq \epsilon$, we have $\boldsymbol{p}_t \neq \bar{\boldsymbol{p}}^{\boldsymbol{\theta}}$ for any $t \geq \tau_\epsilon$, by continuity property and the condition in Assumption 11, this implies that there exists $c_\epsilon > 0$ such that $\sum_{i=1}^{n} (\bar{U}_i(\boldsymbol{p}_t) - \bar{U}_i(\bar{\boldsymbol{p}}^{\boldsymbol{\theta}}))(p_{it} - \bar{p}_i^{\boldsymbol{\theta}}) < -c_\epsilon$ for any $t \geq \tau_\epsilon$. Step (g) follows from $\bar{\boldsymbol{p}}^{\boldsymbol{\theta}}$ being in the interior of $[l, u]^n$ (by Assumption 11), which further implies that $\bar{U}_i(\bar{\boldsymbol{p}}^{\boldsymbol{\theta}}) = 0$ for all $i \in \{1, \ldots, n\}$ (by the first order optimality condition).

Thus, for all $t \geq \tau_\epsilon$, we have

$$\mathbb{E}^{\pi}_{\boldsymbol{\theta}} \left\{ \sum_{i=1}^{n} \bar{g}_{it}(p_{it} - \bar{p}_i^{\boldsymbol{\theta}}) \right\} \ \overset{(h)}{<} \ -\bar{c}_\epsilon, \tag{3.132}$$

where in step (h), given $\bar{g}_{it} = \bar{U}_i(\boldsymbol{p}_t)$ (by (3.129)) and the inequality in (3.131), the inequality naturally follows.

Based on (3.132) and the definition that $F(\boldsymbol{p}, \boldsymbol{v}) = \sum_{i=1}^{n} F_i(p_i, v_i)$ in (3.127), we can proceed to establish that

$$\begin{aligned} \mathbb{E}^{\pi}_{\boldsymbol{\theta}} \Big\{ F(\bar{\boldsymbol{p}}^{\boldsymbol{\theta}}, \nabla h(\boldsymbol{p}_{t+1})) \Big\} \ &\overset{(i)}{\leq} \mathbb{E}^{\pi}_{\boldsymbol{\theta}} \Big\{ F(\bar{\boldsymbol{p}}^{\boldsymbol{\theta}}, \nabla h(\boldsymbol{p}_t)) \Big\} + \mathbb{E}^{\pi}_{\boldsymbol{\theta}} \Big\{ \eta_t \bar{\boldsymbol{g}}_t^{\intercal} (\boldsymbol{p}_t - \bar{\boldsymbol{p}}^{\boldsymbol{\theta}}) \Big\} + \frac{1}{2 l_h} \mathbb{E}^{\pi}_{\boldsymbol{\theta}} \Big\{ \| \eta_t \tilde{\boldsymbol{g}}_t \|_2^2 \Big\}, \\ &\overset{(j)}{\leq} \mathbb{E}^{\pi}_{\boldsymbol{\theta}} \Big\{ F(\bar{\boldsymbol{p}}^{\boldsymbol{\theta}}, \nabla h(\boldsymbol{p}_t)) \Big\} - c_\epsilon \eta_t + \frac{1}{2 l_h} \mathbb{E}^{\pi}_{\boldsymbol{\theta}} \Big\{ \| \eta_t \tilde{\boldsymbol{g}}_t \|_2^2 \Big\}, \\ &\overset{(k)}{\leq} \mathbb{E}^{\pi}_{\boldsymbol{\theta}} \Big\{ F(\bar{\boldsymbol{p}}^{\boldsymbol{\theta}}, \nabla h(\boldsymbol{p}_1)) \Big\} - \left(\sum_{s=1}^{t} \eta_s \right) \left(c_\epsilon - \frac{1}{2 l_h} \mathbb{E}^{\pi}_{\boldsymbol{\theta}} \left\{ \frac{\sum_{s=1}^{t} \eta_s^2 \| \tilde{\boldsymbol{g}}_s \|_2^2}{\sum_{s=1}^{t} \eta_s} \right] \right\}, \end{aligned} \tag{3.133}$$

where step (i) follows directly from Lemma 24(ii). In step (j) follows from (3.132). Step (k), we inductively iterate through step (i) - (j), and then reorganize the expression.

Within the last-step expression in (3.133), we show that there exists a nonnegative random variable $\tilde{G}$ with $\mathbb{E}^{\pi}_{\boldsymbol{\theta}}\{\tilde{G}\} < \infty$ such that as $t \to \infty$,

$$\sum_{s=1}^{t} \eta_s^2 \|\tilde{\boldsymbol{g}}_s\|_2^2 \overset{(l)}{\to} \tilde{G} \quad a.s., \tag{3.134}$$

where in step (l), we notice that $\{\sum_{s=1}^{t} \eta_s^2 \|\tilde{\boldsymbol{g}}_s\|_2^2 : t = 1, 2, \dots\}$ is a nonnegative submartingale series, and that $\mathbb{E}^{\pi}_{\boldsymbol{\theta}}\left\{\sum_{s=1}^{t} \eta_s^2 \|\tilde{\boldsymbol{g}}_s\|_2^2\right\} = \mathbb{E}^{\pi}_{\boldsymbol{\theta}}\left\{\sum_{s=1}^{t} \eta_s^2 \|\bar{\boldsymbol{g}}_s\|_2^2\right\} + \mathbb{E}^{\pi}_{\boldsymbol{\theta}}\left\{\sum_{s=1}^{t} \eta_s^2 \|\boldsymbol{\delta}_s\|_2^2\right\} \leq \left(\sum_{s=1}^{\infty} \eta_s^2\right)\left(\bar{U} + n\bar{\sigma}_g^2\right) < \infty$ for all $t \in \mathbb{N}_+ \cup \{\infty\}$ where $\bar{U} = \sum_{i=1}^{n} \sup_{\boldsymbol{p} \in [l,u]^n} |\bar{U}_i(\boldsymbol{p})|$. By the Doob's martingale convergence theorem, the nonnegative random sequence $\sum_{s=1}^{t} \eta_s^2 \|\tilde{\boldsymbol{g}}_s\|_2^2$ converges to a nonnegative random variable $\tilde{G}$ almost surely.

This allows us to establish that as $t \to \infty$,

$$\mathbb{E}^{\pi}_{\boldsymbol{\theta}}\left\{\frac{\sum_{s=1}^{t} \eta_s^2 \tilde{g}_{is}^2}{\sum_{s=1}^{t} \eta_s}\right\} \xrightarrow{(m)} 0, \tag{3.135}$$

where step (m) follows from the observation that $\mathbb{E}^{\pi}_{\boldsymbol{\theta}}\{\tilde{G}\} < \infty$ and the assumption that $\sum_{s=1}^{t} \eta_{is} \to \infty$ as $t \to \infty$.

From (3.133) and (3.135), with $\sum_{s=1}^{t} \eta_s \to \infty$ as $t \to \infty$, we have

$$\mathbb{E}^{\pi}_{\boldsymbol{\theta}}\Big\{F(\bar{\boldsymbol{p}}^{\boldsymbol{\theta}}, \nabla h(\boldsymbol{p}_t))\Big\} \longrightarrow -\infty, \tag{3.136}$$

which is a contradiction to the observation $\mathbb{E}^{\pi}_{\boldsymbol{\theta}}\left\{F(\bar{\boldsymbol{p}}^{\boldsymbol{\theta}}, \nabla h(\boldsymbol{p}_t))\right\} \geq 0$ (by the nonnegativity of the Fenchel-Young inequality gap). Thus, the induced price process $\{\boldsymbol{p}_t\}$ is recurrent in the neighborhood of $\bar{\boldsymbol{p}}^{\boldsymbol{\theta}}$ i.e., condition (3.128) holds.

Step (i)-4: conclude the convergence of the price process. To prove the convergence on the price process $\{\boldsymbol{p}_t\}$, we prove that $\mathbb{E}^{\pi}_{\boldsymbol{\theta}}\{F(\bar{\boldsymbol{p}}^{\boldsymbol{\theta}}, \nabla h(\boldsymbol{p}_t))\} \to 0$ as $t \to 0$. This is equivalent to show that for any $\epsilon > 0$, there exists $\tau > 0$ such that for any $t \geq \tau$, we have $\mathbb{E}^{\pi}_{\boldsymbol{\theta}}\{F(\bar{\boldsymbol{p}}^{\boldsymbol{\theta}}, \nabla h(\boldsymbol{p}_t))\} \leq \epsilon$. Fix any $\epsilon > 0$, for any $t \in \{s : E^{\pi}_{\boldsymbol{\theta}}\{F(\bar{\boldsymbol{p}}^{\boldsymbol{\theta}}, \nabla h(\boldsymbol{p}_s)\} \leq \epsilon\}$, we discuss the following two possible cases:

(1) if $\mathbb{E}^{\pi}_{\boldsymbol{\theta}}\left\{F(\bar{\boldsymbol{p}}^{\boldsymbol{\theta}}, \nabla h(\boldsymbol{p}_t))\right\} \geq \frac{\epsilon}{2}$, we can repeat the same arguments as in (3.131) - (3.132)

to establish that there exists a uniform constant $c_{\frac{\epsilon}{2}} > 0$ such that

$$\mathbb{E}_{\boldsymbol{\theta}}^{\pi}\left\{\sum_{i=1}^{n} \bar{g}_{it}(p_{it} - \bar{p}_i^{\boldsymbol{\theta}})\right\} \; < \; -c_{\frac{\epsilon}{2}}. \tag{3.137}$$

(2) if $\mathbb{E}_{\boldsymbol{\theta}}^{\pi}\left\{F(\bar{\boldsymbol{p}}^{\boldsymbol{\theta}}, \nabla h(\boldsymbol{p}_t))\right\} \le \frac{\epsilon}{2}$, then from the diagonal strict concavity condition in Assumption 11, we naturally have

$$\mathbb{E}_{\boldsymbol{\theta}}^{\pi}\left\{\sum_{i=1}^{n} \bar{g}_{it}(p_{it} - \bar{p}_i^{\boldsymbol{\theta}})\right\} \; \le \; 0. \tag{3.138}$$

Since $\eta_s \to 0$ as $s \to \infty$ for all $i \in \{1, \ldots, n\}$, there exists a positive integer τ_1 such that for all $s \ge \tau_1$, we have $\eta_s \le \frac{2l_h}{n(\bar{U}^2 + \bar{\sigma}_g^2)} c_{\frac{\epsilon}{2}}$, and moreover,

$$\begin{aligned}
\frac{1}{2l_h}\mathbb{E}_{\boldsymbol{\theta}}^{\pi}\left\{\|\eta_s \tilde{\boldsymbol{g}}_s\|_2^2\right\} &\overset{(n)}{=} \frac{1}{2l_h}\mathbb{E}_{\boldsymbol{\theta}}^{\pi}\left\{\|\eta_s(\bar{\boldsymbol{g}}_s + \boldsymbol{\delta}_s)\|_2^2\right\} \\
&\overset{(o)}{=} \frac{1}{2l_h}\mathbb{E}_{\boldsymbol{\theta}}^{\pi}\left\{\|\eta_s \bar{\boldsymbol{g}}_s\|_2^2\right\} + \frac{1}{2l_h}\mathbb{E}_{\boldsymbol{\theta}}^{\pi}\{\|\eta_s \boldsymbol{\delta}_s\|_2^2\} \\
&\overset{(p)}{\le} \frac{n(\bar{U}^2 + \bar{\sigma}_g^2)}{2l_h}\eta_s^2 \\
&\overset{(q)}{\le} c_{\frac{\epsilon}{2}}\eta_s,
\end{aligned} \tag{3.139}$$

where step (n) follows from the equation $\tilde{\boldsymbol{g}}_s = \bar{\boldsymbol{g}}_s + \boldsymbol{\delta}_s$. Step (o) holds because $\mathbb{E}_{\boldsymbol{\theta}}^{\pi}\{\boldsymbol{\delta}_s\} = \mathbf{0}$ (from (3.130)). Step (p) follows from the definition that $\|\tilde{g}_{is}\|_2^2 \le \bar{U}^2$ where $\bar{U} = \sum_{i=1}^{n} \sup_{\boldsymbol{p}\in[l,u]^n} |\bar{U}_i(\boldsymbol{p})|$ and the observation that $\mathbb{E}_{\boldsymbol{\theta}}^{\pi}\{\|\boldsymbol{\delta}_s\|_2^2\} \le n\bar{\sigma}_g^2$ (from (3.130)). Step (q) holds because $\eta_s \le \frac{2l_h}{n(\bar{U}^2 + \bar{\sigma}_g^2)} c_{\frac{\epsilon}{2}}$.

Similarly, there exists positive integer τ_2 such that for any $s \ge \tau_2$, we have that $\eta_s \le$

$\sqrt{\frac{l_h\epsilon}{n(\bar{U}^2+\bar{\sigma}_g^2)}}$, and that

$$\begin{aligned}
\frac{1}{2l_h}\mathbb{E}^{\pi}_{\boldsymbol{\theta}}\left\{\|\eta_s\tilde{\boldsymbol{g}}_s\|_2^2\right\} &\overset{(r)}{=} \frac{1}{2l_h}\mathbb{E}^{\pi}_{\boldsymbol{\theta}}\left\{\|\eta_s(\bar{\boldsymbol{g}}_s+\boldsymbol{\delta}_s)\|_2^2\right\}\\
&= \frac{1}{2l_h}\mathbb{E}^{\pi}_{\boldsymbol{\theta}}\left\{\|\eta_s\bar{\boldsymbol{g}}_s\|_2^2\right\}+\frac{1}{2l_h}\mathbb{E}^{\pi}_{\boldsymbol{\theta}}\{\|\eta_s\boldsymbol{\delta}_s\|_2^2\}\\
&\overset{(t)}{\le} \frac{n(\bar{U}^2+\bar{\sigma}_g^2)}{2l_h}\eta_s^2\\
&\overset{(u)}{\le} \frac{\epsilon}{2},
\end{aligned} \tag{3.140}$$

where step (r) - (t) follows from the same arguments as in step (n) - (p) above. Step (u) follows from the fact that $\eta_s \le \sqrt{\frac{l_h\epsilon}{n(\bar{U}^2+\bar{\sigma}_g^2)}}$.

We can pick any $\tau \in \{s : E^{\pi}_{\boldsymbol{\theta}}\{F(\bar{\boldsymbol{p}}^{\boldsymbol{\theta}}, \nabla h(\boldsymbol{p}_s)\} \le \epsilon\}$ with $\tau \ge \max\{\tau_1,\tau_2\}$ (by the recurrent property established in the previous step). In case of $\mathbb{E}^{\pi}_{\boldsymbol{\theta}}\left\{F(\bar{\boldsymbol{p}}^{\boldsymbol{\theta}}, \nabla h(\boldsymbol{p}_\tau))\right\} \le \frac{\epsilon}{2}$, by summarizing Lemma 24(ii), (3.137), and (3.139), such that

$$\mathbb{E}^{\pi}_{\boldsymbol{\theta}}\Big\{F(\bar{\boldsymbol{p}}^{\boldsymbol{\theta}}, \nabla h(\boldsymbol{p}_{\tau+1}))\Big\} \;\le\; \mathbb{E}^{\pi}_{\boldsymbol{\theta}}\Big\{F(\bar{\boldsymbol{p}}^{\boldsymbol{\theta}}, \nabla h(\boldsymbol{p}_{\tau}))\Big\} - c_{\frac{\epsilon}{2}}\eta_t + c_{\frac{\epsilon}{2}}\eta_t \;\le\; \epsilon. \tag{3.141}$$

In case of $\mathbb{E}^{\pi}_{\boldsymbol{\theta}}\left\{F(\bar{\boldsymbol{p}}^{\boldsymbol{\theta}}, \nabla h(\boldsymbol{p}_t))\right\} \le \frac{\epsilon}{2}$, then by summarizing Lemma 24(ii), (3.138), and (3.140), we obtain that

$$\mathbb{E}^{\pi}_{\boldsymbol{\theta}}\Big\{F(\bar{\boldsymbol{p}}^{\boldsymbol{\theta}}, \nabla h(\boldsymbol{p}_{\tau+1}))\Big\} \;\le\; \mathbb{E}^{\pi}_{\boldsymbol{\theta}}\Big\{F(\bar{\boldsymbol{p}}^{\boldsymbol{\theta}}, \nabla h(\boldsymbol{p}_{\tau}))\Big\} + 0 + \frac{\epsilon}{2} \;\le\; \epsilon. \tag{3.142}$$

Summarizing the arguments above, we observe that for any $\epsilon > 0$, there exists a positive integer τ such that for any $t \ge \tau$, we have $\mathbb{E}^{\pi}_{\boldsymbol{\theta}}\left\{F(\bar{\boldsymbol{p}}^{\boldsymbol{\theta}}, \nabla h(\boldsymbol{p}_t))\right\} < \epsilon$. Thus, we have $\mathbb{E}^{\pi}_{\boldsymbol{\theta}}\left[F(\bar{\boldsymbol{p}}^{\boldsymbol{\theta}}, \nabla h(\boldsymbol{p}_t))\right] \le \epsilon$. Since ϵ can be arbitrarily chosen, we conclude the proof that as $t \to \infty$,

$$\mathbb{E}^{\pi}_{\boldsymbol{\theta}}\Big\{F(\bar{\boldsymbol{p}}^{\boldsymbol{\theta}}, \nabla h(\boldsymbol{p}_t))\Big\} \;\to\; 0. \tag{3.143}$$

To establish the almost sure convergence of the price process $\{\boldsymbol{p}_s : s = 1, 2, \dots\}$. Consider a random process $\{\tilde{X}_s : s = 1, 2, \dots\}$ where

$$\tilde{X}_t \ = F(\bar{\boldsymbol{p}}^{\boldsymbol{\theta}}, \nabla h(\boldsymbol{p}_{t+1})) + \sum_{s=t+1}^{\infty} \frac{n(\bar{U}^2 + \bar{\sigma}_g^2)}{2l_h} \eta_s^2. \tag{3.144}$$

By construction, we have $\tilde{X}_t \geq 0$ for all $t \in \{1, 2, \dots\}$. To establish that $\tilde{X}_t$ is a supermartingale adapted to filtration $\mathcal{F}_t^0$, we verify that

$$\begin{aligned}
\mathbb{E}_{\boldsymbol{\theta}}^{\pi}\left\{\tilde{X}_t \middle| \mathcal{F}_{t-1}^0\right\} &\overset{(v)}{=} \mathbb{E}_{\boldsymbol{\theta}}^{\pi}\left\{F(\bar{\boldsymbol{p}}^{\boldsymbol{\theta}}, \nabla h(\boldsymbol{p}_{t+1})) \middle| \mathcal{F}_{t-1}^0\right\} + \sum_{s=t+1}^{\infty} \frac{n(\bar{U}^2 + \bar{\sigma}_g^2)}{2l_h} \eta_s^2 \\
&\overset{(w)}{\leq} F(\bar{\boldsymbol{p}}^{\boldsymbol{\theta}}, \nabla h(\boldsymbol{p}_t)) + \eta_t \bar{\boldsymbol{g}}_t^{\intercal}(\boldsymbol{p}_t - \bar{\boldsymbol{p}}^{\boldsymbol{\theta}}) + \mathbb{E}_{\boldsymbol{\theta}}^{\pi}\Big\{\eta_t^2 \|\tilde{\boldsymbol{g}}_t^2\|_2^2 | \mathcal{F}_{t-1}^0\Big\} + \sum_{s=t+1}^{\infty} \frac{n(\bar{U}^2 + \bar{\sigma}_g^2)}{2l_h} \eta_s^2 \\
&\overset{(x)}{\leq} F(\bar{\boldsymbol{p}}^{\boldsymbol{\theta}}, \nabla h(\boldsymbol{p}_t)) + \frac{n(\bar{U}^2 + \bar{\sigma}_g^2)}{2l_h} \eta_t^2 + \sum_{s=t+1}^{\infty} \frac{n(\bar{U}^2 + \bar{\sigma}_g^2)}{2l_h} \eta_s^2 \\
&= F(\bar{\boldsymbol{p}}^{\boldsymbol{\theta}}, \nabla h(\boldsymbol{p}_t)) + \sum_{s=t}^{\infty} \frac{n(\bar{U}^2 + \bar{\sigma}_g^2)}{2l_h} \eta_s^2 \ = \ \tilde{X}_{t-1},
\end{aligned} \tag{3.145}$$

where step (v) follows from the definition of $\tilde{X}_t$ in (3.144). Step (w) follows directly from Lemma 24(ii). In step (x), we leverage the observation that $\bar{\boldsymbol{g}}_t^{\intercal}(\boldsymbol{p}_t - \bar{\boldsymbol{p}}^{\boldsymbol{\theta}}) \leq 0$ (by the strict diagonal concavity condition in Assumption 11) and $\frac{1}{2l_h}\|\eta_t \tilde{\boldsymbol{g}}_t\|_2^2 \leq \frac{n(\bar{U}^2 + \gamma^2 \bar{\sigma}_g^2)}{2l_h} \eta_t^2$ (the same as step (n) - (p) of (3.139)).

Based on (3.145) and the nonnegativity of $\tilde{X}_t$, we apply Doob's martingale convergence theorem to conclude that there exists random variable $\tilde{X}$ such that

$$\tilde{X}_t \to \tilde{X}, \quad a.s. \tag{3.146}$$

Moreover, we have

$$\begin{aligned}\mathbb{E}^{\pi}_{\boldsymbol{\theta}}\Big\{\tilde{X}\Big\} &= \mathbb{E}^{\pi}_{\boldsymbol{\theta}}\Big\{\lim_{t\to\infty}\tilde{X}_t\Big\} \\ &\overset{(y)}{=} \mathbb{E}^{\pi}_{\boldsymbol{\theta}}\Big\{\lim_{t\to\infty}F(\bar{\boldsymbol{p}}^{\boldsymbol{\theta}}, \nabla h(\boldsymbol{p}_{t+1}))\Big\} \\ &\overset{(z)}{\leq} \liminf_{t\to\infty}\mathbb{E}^{\pi}_{\boldsymbol{\theta}}\Big\{F(\bar{\boldsymbol{p}}^{\boldsymbol{\theta}}, \nabla h(\boldsymbol{p}_{t+1}))\Big\} \\ &\overset{(a)}{=} 0, \end{aligned} \tag{3.147}$$

where step (y) follows from the definition of $\tilde{X}_t$ in (3.144) from which we obtain $\lim_{t\to\infty}\tilde{X}_t = \lim_{t\to\infty}F(\bar{\boldsymbol{p}}^{\boldsymbol{\theta}}, \nabla h(\boldsymbol{p}_t))$. Step (z) follows from directly from Fatou's lemma. Step (a) follows directly from (3.143).

Leveraging the observations above and the nonegative of $\tilde{X}$, we obtain that $\tilde{X} = 0$ almost surely, which further suggests that $F(\bar{\boldsymbol{p}}^{\boldsymbol{\theta}}, \nabla h(\boldsymbol{p}_t)) \to 0$ almost surely as $t \to \infty$. This allows us to conclude that $\boldsymbol{p}_t \to \bar{\boldsymbol{p}}^{\boldsymbol{\theta}}$ almost surely.

Proof of Claim (ii). By the proposition statement, $\tilde{R}_{it}$ belongs to a compact subspace of $C^1([l,u])$ for any $i \in \{1,\ldots,n\}$ and $t \in \{1,\ldots,T\}$ and $T \in \{1,2\ldots\}$. This readily implies that there exists $M_R^0, M_R^1 > 0$ such that $\sup_{p\in[l,u]}|\tilde{R}_{it}(p)| \leq M_R^0$ and $\sup_{p\in[l,u]}|\frac{d}{dp}\tilde{R}_{it}(p)| \leq M_R^1$. Thus, Assumption 9 holds.

Note that the game specified in the DN Nash equilibrium in Definition 5 is a finite weighted potential game in which the unique Nash equilibrium price profile satisfies $\bar{\boldsymbol{p}}^{\boldsymbol{\theta}} \in \mathcal{S}$. By leveraging Theorem 2.4 of (154), we immediately obtain that the price dynamics $\{\boldsymbol{p}_t : t = 1,2,\ldots\}$ in (3.26) has a fictitious play property: a game satisfies the fictitious play property if every fictitious process converges in beliefs to equilibrium ((see 157)). By the fictitious play property, fictitious play dynamics in (3.26) which induces the price process $\{\boldsymbol{p}_t : t = 1,2,\ldots\}$ converges almost surely to the unique DN Nash equilibrium $\bar{\boldsymbol{p}}^{\boldsymbol{\theta}}$ in Definition 5 as $t \to \infty$.

Proof of Claim (iii). To prove the claim, we first establish the following supporting lemma on belief convergence. For simplicity of notation, we define event $H_0 = \{\mathbb{E}_{\boldsymbol{\xi},\boldsymbol{\varepsilon}}\{\boldsymbol{Q}(\cdot, \tilde{\boldsymbol{\xi}}, \boldsymbol{\theta})\} =$

$\boldsymbol{Q}^{(0)}(\cdot, \boldsymbol{\theta}^{(0)})\}$.

Lemma 25. *Suppose that there exists* $\delta_d > 0$ *such that* $|Q_i^{(k)}(\boldsymbol{p}, \boldsymbol{\theta}^{(k)}) - Q_i^{(k')}(\boldsymbol{p}, \boldsymbol{\theta}^{(k')})| \geq \delta_d$ *for any* $\boldsymbol{p} \in [l,u]^n$*, any* $k, k' \in \{0, \dots, K\}$ *with* $k \neq k'$*, and any* $i \in \{1, \dots, n\}$*. Given* $\boldsymbol{b}_{i1} \in (0,1)^{K+1}$ *with* $i \in \{1, \dots, n\}$*, there exists a constant* $\zeta, \eta > 0$ *such that for any* $i \in \{1, \dots, n\}$ *and* $t \in \{1, 2, \dots\}$*,*

$$\mathbb{E}_{\boldsymbol{\theta}}^{\pi}\left\{1 - b_{it}^{(0)} | H_0\right\} \;\leq\; \zeta \exp\{-\eta t\}. \tag{3.148}$$

From Lemma 25, for any $\epsilon > 0$, we can establish that for any $t \in \{1, 2, \dots, \}$,

$$\begin{aligned}
\mathbb{P}_{\boldsymbol{\theta}}^{\pi}\left\{\sup_{s \geq t}\{1 - b_{is}^{(0)}\} > \epsilon | H_0)\right\} &\overset{(a)}{\leq} \sum_{s=t}^{\infty} \mathbb{P}_{\boldsymbol{\theta}}^{\pi}\left\{1 - b_{is}^{(0)} > \epsilon | H_0\right\} \\
&\overset{(b)}{\leq} \frac{1}{\epsilon} \sum_{s=t}^{\infty} \mathbb{E}_{\boldsymbol{\theta}}^{\pi}\left\{1 - b_{is}^{(0)} | H_0\right\} \\
&\overset{(c)}{\leq} \frac{1}{\epsilon} \sum_{s=t}^{\infty} \zeta \exp\{-\eta s\} \\
&\leq \zeta \exp\{-\eta t\} \sum_{s=t}^{\infty} \exp\{-\eta(s-t)\},
\end{aligned} \tag{3.149}$$

where step (a) follows from implementing the probability union bound. Step (b) holds because of the Markov's inequality. Step (c) follows directly from the conclusion of Lemma 25.

From (3.149), we can establish

$$0 \;\leq\; \mathbb{P}_{\boldsymbol{\theta}}^{\pi}\left\{\lim_{t \to \infty} \sup_{s \geq t}\{1 - b_{is}^{(0)}\} > \epsilon | H_0)\right\} \;\leq\; \lim_{t \to \infty} \zeta \exp\{-\eta t\} \sum_{s=t}^{\infty} \exp\{-\eta(s-t)\} \;=\; 0, \tag{3.150}$$

which by definition, says that $b_{it}^{(0)}$ converges to 1 almost surely as $t \to \infty$ for any $i \in \{1, \dots, n\}$. This also implies that $b_{it}^{(k)}$ converges to 0 almost surely as $t \to \infty$ for any $i \in \{1, \dots, n\}$ and $k \in \{1, \dots, K\}$.

To proceed, for any $t \in \{1, 2, \dots\}$, we know that the induced price process $\boldsymbol{p}_t$ in period t satisfies that for any $i \in \{1, \dots, n\}$,

$$p_{it} \;=\; \underset{p \in [l,u]}{\arg\max}\; \gamma p \sum_{k=0}^{K} b_{it}^{(k)} Q_i^{(k)}(p, \boldsymbol{p}_{-it}, \boldsymbol{\theta}^{(k)}). \tag{3.151}$$

For simplicity of notation, we consider function $F : [0,1]^{n(K+1)} \times [l,u]^n \to \mathbb{R}^n$ in which for any $i \in \{1, \dots, n\}$, $F_i(\boldsymbol{b}, \boldsymbol{p})$ satisfies

$$F_i(\boldsymbol{b}, \boldsymbol{p}) \;=\; \frac{\partial}{\partial p_i}\left[\gamma p_i \sum_{k=0}^{K} b_{it}^{(k)} Q_i^{(k)}(\boldsymbol{p}, \boldsymbol{\theta}^{(k)})\right]. \tag{3.152}$$

Given that the DN-Nash equilibrium $\bar{\boldsymbol{p}}^{\boldsymbol{\theta}} \in \operatorname{int}([l,u]^n)$, we let $\bar{\boldsymbol{b}} = (\bar{b}_i^{(k)})_{i,k}$ be such that $\bar{b}_i^{(0)} = 1$ and $\bar{b}_i^{(k)} = 0$ for all $k \in \{1, \dots, K\}$ and $i \in \{1, \dots, n\}$. From the definition of DN Nash equilibrium in Definition 5, by applying the first order optimality condition, we obtain that

$$F(\bar{\boldsymbol{b}}, \bar{\boldsymbol{p}}^{\boldsymbol{\theta}}) = \mathbf{0}. \tag{3.153}$$

We continue to define the corresponding Jacobian of function F as $J_F(\boldsymbol{b}, \boldsymbol{p}) = [\frac{\partial}{\partial p_j} F_i(\boldsymbol{b}, \boldsymbol{p})]_{i,j \in \{1,\dots,n\}}$. From strict diagonal dominance property (Assumption 11), we know that there exists open set $\mathcal{B} \subset \mathbb{R}^{n(K+1)}$ such that the corresponding Jacobian matrix $J(\boldsymbol{b}, \boldsymbol{p})$ is invertible for any $(\boldsymbol{b}, \boldsymbol{p}) \in \mathcal{B} \times [l,u]^n$. By the implicit function theorem, there exists a differentiable function $g : \mathcal{B} \to \mathbb{R}^n$ such that for any $\boldsymbol{b} \in \mathcal{B}$,

$$F(\boldsymbol{b}, g(\boldsymbol{b})) \;=\; \mathbf{0}. \tag{3.154}$$

Given $\boldsymbol{b}_t \to \bar{\boldsymbol{b}}$ almost sure as $t \to \infty$, by the continuously mapping theorem, we establish that $g(\boldsymbol{b}_t) \to g(\bar{\boldsymbol{b}})$ almost surely as $t \to \infty$. Since $\bar{\boldsymbol{b}} \in \mathcal{B}$, $\bar{\boldsymbol{p}}^{\boldsymbol{\theta}} = g(\bar{\boldsymbol{b}})$ and $\boldsymbol{p}_t = g(\boldsymbol{b}_t)$, we conclude the proof that $\boldsymbol{p}_t \to \bar{\boldsymbol{p}}^{\boldsymbol{\theta}}$ almost surely. □

3.10 Proof of Results in Section 3.7.

Proof of Theorem 14. Proof of Claim (i). In proving this result, we borrow some notations in the proof arguments of Proposition 25(i). We let π be the platform's DN policy under which the seller induces the price process $\{\boldsymbol{p}_s : s = 1, 2, \dots\}$ through the multiagent mirror descent framework in (3.25). From Assumptions 11 and 12, we let $\bar{\boldsymbol{p}}^{\boldsymbol{\theta}} \in \operatorname{int}([l, u]^n)$ be the DN Nash equilibrium. Given filtration $\mathcal{F}_t^0 = \sigma(\boldsymbol{\varepsilon}_s : s = 1, \dots, t)$, we can show that for any $T \geq 2$,

$$\begin{aligned}
\Delta_{\boldsymbol{\theta}}^{\pi}(T) &\overset{(a)}{=} T\mathbb{E}_{\boldsymbol{\theta}}^{\pi}\left\{\mathcal{R}^{\star}(\boldsymbol{\xi}, \boldsymbol{\theta}) - \mathcal{R}(\bar{\boldsymbol{p}}^{\boldsymbol{\theta}}, \boldsymbol{\xi}, \boldsymbol{\theta})\right\} - \mathbb{E}_{\boldsymbol{\theta}}^{\pi}\left\{\sum_{t=1}^{T} \mathcal{R}(\boldsymbol{p}_t, \boldsymbol{\xi}_t, \boldsymbol{\theta}) - \mathcal{R}(\bar{\boldsymbol{p}}^{\boldsymbol{\theta}}, \boldsymbol{\xi}_t, \boldsymbol{\theta})\right\}, \\
&\overset{(b)}{=} \mathbb{E}_{\boldsymbol{\theta}}^{\pi}\left\{\sum_{t=1}^{T} \mathcal{R}(\bar{\boldsymbol{p}}^{\boldsymbol{\theta}}, \boldsymbol{\xi}_t, \boldsymbol{\theta}) - \mathcal{R}(\boldsymbol{p}_t, \boldsymbol{\xi}_t, \boldsymbol{\theta})\right\} \\
&\overset{(c)}{=} (1-\gamma)\mathbb{E}_{\boldsymbol{\theta}}^{\pi}\left\{\sum_{t=1}^{T}\sum_{i=1}^{n} \bar{p}_i^{\boldsymbol{\theta}}\mathbb{E}_{\boldsymbol{\theta}}^{\pi}\left\{Q_i(\bar{\boldsymbol{p}}^{\boldsymbol{\theta}}, \boldsymbol{\xi}_t, \boldsymbol{\theta})\middle|\mathcal{F}_{t-1}^0\right\} - p_{it}\mathbb{E}_{\boldsymbol{\theta}}^{\pi}\left\{Q_i(\boldsymbol{p}_t, \boldsymbol{\xi}_t, \boldsymbol{\theta})\middle|\mathcal{F}_{t-1}^0\right\}\right\}
\end{aligned} \tag{3.155}$$

where step (a) follows from exactly the same arguments as in step (a) - (c) of (3.35). In step (b), given $\boldsymbol{\theta} \in \partial\tilde{\Theta}$, we have $\mathbb{E}_{\boldsymbol{\theta}}^{\pi}\{\mathcal{R}^{\star}(\boldsymbol{\xi}, \boldsymbol{\theta})\} = \mathbb{E}_{\boldsymbol{\xi},\boldsymbol{\varepsilon}}\{\mathcal{R}^{\star}(\boldsymbol{\xi}, \boldsymbol{\theta})\} = \mathbb{E}_{\boldsymbol{\xi},\boldsymbol{\varepsilon}}\{\mathcal{R}(\bar{\boldsymbol{p}}^{\boldsymbol{\theta}}, \boldsymbol{\xi}, \boldsymbol{\theta})\} = \mathbb{E}_{\boldsymbol{\theta}}^{\pi}\{\mathcal{R}(\bar{\boldsymbol{p}}^{\boldsymbol{\theta}}, \boldsymbol{\xi}, \boldsymbol{\theta})\}$ (by definition of $\partial\tilde{\Theta}$ in (3.29)), which implies that step (b) holds. In step (c), under the platform's DN policy π, we have $\mathcal{W}_{it}(p) = 0$ for all $i \in \{1, \dots, n\}$ and any $p \in [l, u]$. We also observe that $\boldsymbol{p}_t$ is determined conditional on $\mathcal{F}_{t-1}^0$. Together with the platform's revenue expression in (3.3) and (3.4), we obtain the equality in this step.

Next, for any $T \geq 2$ and $t \in \{1, \dots, T\}$, given $\mathcal{F}_{t-1}^0$, we develop an upper bound for $\bar{p}_i^{\boldsymbol{\theta}}\mathbb{E}_{\boldsymbol{\theta}}^{\pi}\left\{Q_i(\bar{\boldsymbol{p}}^{\boldsymbol{\theta}}, \boldsymbol{\xi}_t, \boldsymbol{\theta})\middle|\mathcal{F}_{t-1}^0\right\} - p_{it}\mathbb{E}_{\boldsymbol{\theta}}^{\pi}\left\{Q_i(\boldsymbol{p}_t, \boldsymbol{\xi}_t, \boldsymbol{\theta})\middle|\mathcal{F}_{t-1}^0\right\}$ in the last term of (3.155). For any $i \in \{1, \dots, n\}$, we let $\bar{U}_i(\boldsymbol{p}) = \frac{\partial}{\partial p_i}\log\left(\gamma p_i \mathbb{E}_{\boldsymbol{\xi},\boldsymbol{\varepsilon}}\left\{Q_i(p_i, \boldsymbol{p}_{-i}, \tilde{\boldsymbol{\xi}}, \boldsymbol{\theta})\right\}\right)$ for any $\boldsymbol{p} \in [l, u]^n$. Recall that conditional on $\mathcal{F}_{t-1}^0$, the induced price profile $\boldsymbol{p}_t$ is given. Abusing some notation, we can let $\bar{U}_i(\boldsymbol{p}_t) = \frac{\partial}{\partial p_{it}}\log\left(\gamma p_{it}\mathbb{E}_{\boldsymbol{\theta}}^{\pi}\left\{Q_i(p_{it}, \boldsymbol{p}_{-it}, \tilde{\boldsymbol{\xi}}, \boldsymbol{\theta})\middle|\mathcal{F}_{t-1}^0\right\}\right)$. From Assumption 11, we have

$\sum_{i=1}^{n}(\bar{U}_i(\boldsymbol{p}_t)-\bar{U}_i(\bar{\boldsymbol{p}}^{\boldsymbol{\theta}}))(p_{it}-\bar{p}_i^{\boldsymbol{\theta}})<0$ given filtration $\mathcal{F}_{t-1}^{0}$. This allows us to establish that

$$0 \;>\; \sum_{i=1}^{n}(\bar{U}_i(\boldsymbol{p}_t)-\bar{U}_i(\bar{\boldsymbol{p}}^{\boldsymbol{\theta}}))(p_{it}-\bar{p}_i^{\boldsymbol{\theta}}) \;\overset{(d)}{=}\; \sum_{i=1}^{n}\bar{U}_i(\boldsymbol{p}_t)(p_{it}-\bar{p}_i^{\boldsymbol{\theta}}), \tag{3.156}$$

where step (d) follows from applying first order optimality condition to the DN Nash equilibrium in Definition 5 to obtain $\bar{U}_i(\bar{\boldsymbol{p}}^{\boldsymbol{\theta}})=0$ for all $i\in\{1,\ldots,n\}$ given that $\bar{\boldsymbol{p}}^{\boldsymbol{\theta}}\in\operatorname{int}([l,u]^n)$.

For simplicity of notations, we define $F_i(p_i,v_i) \;=\; h_i^{\star}(v_i)+h_i(p_i)-v_ip_i$ given the continuously differentiable l_h-strongly convex function $h_i(\cdot)$ and its conjugate $h_i^{\star}(\cdot)$ for any $p_i\in[l,u]$ and $v_i\in\mathbb{R}$ (as in (3.125) of Proposition 25). Leveraging Lemma 24(ii) in the proof arguments of Proposition 25), we can establish that

$$\begin{aligned}
\mathbb{E}_{\boldsymbol{\theta}}^{\pi}\Big\{F_i(\bar{p}_i^{\boldsymbol{\theta}},h_i'(p_{i(t+1)}))\Big\} &\overset{(e)}{\le} \mathbb{E}_{\boldsymbol{\theta}}^{\pi}\Big\{F_i(\bar{p}_i^{\boldsymbol{\theta}},h_i'(p_{it}))\Big\}+\mathbb{E}_{\boldsymbol{\theta}}^{\pi}\Big\{\eta_t\bar{U}_i(\boldsymbol{p}_t)(p_{it}-\bar{p}_i^{\boldsymbol{\theta}})\Big\}+\frac{1}{2l_h}n(\bar{U}^2+\bar{\sigma}_g^2)\eta_t^2\\
&\overset{(f)}{\le} \mathbb{E}_{\boldsymbol{\theta}}^{\pi}\Big\{F_i(\bar{p}_i^{\boldsymbol{\theta}},h_i'(p_{it}))\Big\}+\mathbb{E}_{\boldsymbol{\theta}}^{\pi}\Big\{\eta_t\bar{U}_i(\boldsymbol{p}_t)(p_{it}-\bar{p}_i^{\boldsymbol{\theta}})\Big\}+\bar{G}\eta_t^2,
\end{aligned} \tag{3.157}$$

where in step (e), based on Lemma 24(ii), we have $\bar{g}_{it}=U_i(\boldsymbol{p}_t)$ (from (3.129)) and $\mathbb{E}_{\boldsymbol{\theta}}^{\pi}\{\eta_t^2\tilde{g}_{it}^2\}\le n(\bar{U}^2+\bar{\sigma}_g^2)\eta_t^2$ where $\bar{U}=\max_{i\in\{1,\ldots,n\},\boldsymbol{p}\in[l,u]^n}\bar{U}_i(\boldsymbol{p})$ (from step (n) - (p) of (3.139)). In step (f), we define a positive constant $\bar{G}=\frac{1}{2l_h}n(\bar{U}^2+\bar{\sigma}_g^2)$.

Moreover, we can also establish that

$$\begin{aligned}
&\frac{(1-\gamma)\bar{p}_i^{\boldsymbol{\theta}}\mathbb{E}_{\boldsymbol{\theta}}^{\pi}\Big\{Q_i(\bar{\boldsymbol{p}}^{\boldsymbol{\theta}},\boldsymbol{\xi}_t,\boldsymbol{\theta})\Big|\mathcal{F}_{t-1}^{0}\Big\}-(1-\gamma)p_{it}\mathbb{E}_{\boldsymbol{\theta}}^{\pi}\Big\{Q_i(\boldsymbol{p}_t,\boldsymbol{\xi}_t,\boldsymbol{\theta})\Big|\mathcal{F}_{t-1}^{0}\Big\}}{(1-\gamma)\bar{p}_i^{\boldsymbol{\theta}}\mathbb{E}_{\boldsymbol{\theta}}^{\pi}\Big\{Q_i(\bar{\boldsymbol{p}}^{\boldsymbol{\theta}},\boldsymbol{\xi}_t,\boldsymbol{\theta})\Big|\mathcal{F}_{t-1}^{0}\Big\}}\\
&\qquad\overset{(g)}{=} \frac{\gamma\bar{p}_i^{\boldsymbol{\theta}}\mathbb{E}_{\boldsymbol{\theta}}^{\pi}\Big\{Q_i(\bar{\boldsymbol{p}}^{\boldsymbol{\theta}},\boldsymbol{\xi}_t,\boldsymbol{\theta})\Big|\mathcal{F}_{t-1}^{0}\Big\}-\gamma p_{it}\mathbb{E}_{\boldsymbol{\theta}}^{\pi}\Big\{Q_i(\boldsymbol{p}_t,\boldsymbol{\xi}_t,\boldsymbol{\theta})\Big|\mathcal{F}_{t-1}^{0}\Big\}}{\gamma\bar{p}_i^{\boldsymbol{\theta}}\mathbb{E}_{\boldsymbol{\theta}}^{\pi}\Big\{Q_i(\bar{\boldsymbol{p}}^{\boldsymbol{\theta}},\boldsymbol{\xi}_t,\boldsymbol{\theta})\Big|\mathcal{F}_{t-1}^{0}\Big\}}\\
&\qquad\overset{(h)}{\le} \log\gamma\bar{p}_i^{\boldsymbol{\theta}}\mathbb{E}_{\boldsymbol{\theta}}^{\pi}\Big\{Q_i(\bar{\boldsymbol{p}}^{\boldsymbol{\theta}},\boldsymbol{\xi}_t,\boldsymbol{\theta})\Big|\mathcal{F}_{t-1}^{0}\Big\}-\log\gamma p_{it}\mathbb{E}_{\boldsymbol{\theta}}^{\pi}\Big\{Q_i(\boldsymbol{p}_t,\boldsymbol{\xi}_t,\boldsymbol{\theta})\Big|\mathcal{F}_{t-1}^{0}\Big\}\\
&\qquad\overset{(i)}{\le} -\bar{U}_i(\boldsymbol{p}_t)(p_{it}-\bar{p}_i^{\boldsymbol{\theta}}).
\end{aligned} \tag{3.158}$$

where step (g) follows from the property that $\gamma\in(0,1)$. Step (h) follows from the observation that $\frac{x-y}{x} \le \log(\frac{x}{y})$ for any $x, y > 0$. In step (i), we leverage the assumption that function $\log \gamma p_i \mathbb{E}^{\pi}_{\boldsymbol{\theta}}\{Q_i(\boldsymbol{p}, \boldsymbol{\xi}_t, \boldsymbol{\theta})\big|\mathcal{F}^0_{t-1}\}$ is concave in $p_i \in [l, u]$ and the definition that $\bar{U}_i(\boldsymbol{p}_t) = \frac{\partial}{\partial p_{it}} \log(\gamma p_{it} \mathbb{E}_{\boldsymbol{\xi},\boldsymbol{\varepsilon}}\{Q_i(\boldsymbol{p}_t, \boldsymbol{\xi}_t, \boldsymbol{\theta})|\mathcal{F}^0_{t-1}\})$ (by Assumption 11).

From the arguments above, we can establish that

$$\begin{aligned}
&\mathbb{E}^{\pi}_{\boldsymbol{\theta}}\Big\{(1-\gamma)\bar{p}_i^{\boldsymbol{\theta}}\mathbb{E}^{\pi}_{\boldsymbol{\theta}}\Big\{Q_i(\bar{\boldsymbol{p}}^{\boldsymbol{\theta}}, \boldsymbol{\xi}_t, \boldsymbol{\theta})\Big|\mathcal{F}^0_{t-1}\Big\} - (1-\gamma)p_{it}\mathbb{E}^{\pi}_{\boldsymbol{\theta}}\Big\{Q_i(\boldsymbol{p}_t, \boldsymbol{\xi}_t, \boldsymbol{\theta})\Big|\mathcal{F}^0_{t-1}\Big\}\Big\} \\
&\overset{(j)}{\le} \mathbb{E}^{\pi}_{\boldsymbol{\theta}}\Big\{(1-\gamma)\bar{p}_i^{\boldsymbol{\theta}}\mathbb{E}^{\pi}_{\boldsymbol{\theta}}\Big\{Q_i(\bar{\boldsymbol{p}}^{\boldsymbol{\theta}}, \boldsymbol{\xi}_t, \boldsymbol{\theta})\Big|\mathcal{F}^0_{t-1}\Big\} \cdot \Big[-\bar{U}_i(\boldsymbol{p}_t)(p_{it} - \bar{p}_i^{\boldsymbol{\theta}})\Big]\Big\} \\
&\overset{(k)}{=} \bar{\mathcal{R}}_i\mathbb{E}^{\pi}_{\boldsymbol{\theta}}\Big\{-\bar{U}_i(\boldsymbol{p}_t)(p_{it} - \bar{p}_i^{\boldsymbol{\theta}})\Big\} \\
&\overset{(l)}{\le} \frac{\bar{\mathcal{R}}_i}{\eta_t}\mathbb{E}^{\pi}_{\boldsymbol{\theta}}\Big\{F_i(\bar{p}_i^{\boldsymbol{\theta}}, h_i'(p_{it})) - F_i(\bar{p}_i^{\boldsymbol{\theta}}, h_i'(p_{i(t+1)}))\Big\} + \bar{G}\eta_t, \qquad (3.159)
\end{aligned}$$

where step (j) follows from the observation in (3.158). In step (k), we define a positive constant $\bar{\mathcal{R}}_i = (1-\gamma)\bar{p}_i^{\boldsymbol{\theta}}\mathbb{E}_{\boldsymbol{\xi},\boldsymbol{\varepsilon}}\{Q_i(\bar{\boldsymbol{p}}^{\boldsymbol{\theta}}, \boldsymbol{\xi}, \boldsymbol{\theta})\}$. By the i.i.d property of $\{\boldsymbol{\xi}_s\}$, we also obtain that $\bar{\mathcal{R}}_i = (1-\gamma)\bar{p}_i^{\boldsymbol{\theta}}\mathbb{E}^{\pi}_{\boldsymbol{\theta}}\{Q_i(\bar{\boldsymbol{p}}^{\boldsymbol{\theta}}, \tilde{\boldsymbol{\xi}}, \boldsymbol{\theta})|\mathcal{F}^0_{t-1}\}$. In step (l), we can directly leverage the observation in (3.157) to establish the inequality.

To conclude the arguments for this claim, we show that

$$\begin{aligned}
\Delta^{\pi}_{\boldsymbol{\theta}}(T) &\le (1-\gamma)\mathbb{E}^{\pi}_{\boldsymbol{\theta}}\left\{\sum_{t=1}^{T}\sum_{i=1}^{n}\bar{p}_i^{\boldsymbol{\theta}}\mathbb{E}^{\pi}_{\boldsymbol{\theta}}\Big\{Q_i(\bar{\boldsymbol{p}}^{\boldsymbol{\theta}}, \boldsymbol{\xi}_t, \boldsymbol{\theta})\Big|\mathcal{F}^0_{t-1}\Big\} - p_{it}\mathbb{E}^{\pi}_{\boldsymbol{\theta}}\Big\{Q_i(\boldsymbol{p}_t, \boldsymbol{\xi}_t, \boldsymbol{\theta})\Big|\mathcal{F}^0_{t-1}\Big\}\right\} \\
&\overset{(m)}{\le} \mathbb{E}^{\pi}_{\boldsymbol{\theta}}\left\{\sum_{i=1}^{n}\sum_{t=1}^{T}\frac{\bar{\mathcal{R}}_i}{\eta_t}F_i(\bar{p}_i^{\boldsymbol{\theta}}, h_i'(p_{it})) - \frac{\bar{\mathcal{R}}_i}{\eta_t}F_i(\bar{p}_i^{\boldsymbol{\theta}}, h_i'(p_{i(t+1)})) + \bar{G}\eta_t\right\} \\
&\overset{(n)}{=} \mathbb{E}^{\pi}_{\boldsymbol{\theta}}\left\{\sum_{i=1}^{n}\sum_{t=1}^{T+1}\bar{\mathcal{R}}_iF_i(\bar{p}_i^{\boldsymbol{\theta}}, h_i'(p_{it}))\left(\frac{1}{\eta_t} - \frac{1}{\eta_{t-1}}\right)\right\} + \mathbb{E}^{\pi}_{\boldsymbol{\theta}}\left\{\sum_{i=1}^{n}\sum_{t=1}^{T}\bar{G}\eta_t\right\} \\
&\overset{(j)}{\le} C_5^0\sum_{t=1}^{T}\max\left\{\frac{1}{\eta_t} - \frac{1}{\eta_{t-1}}, 0\right\} + C_5^1\sum_{t=1}^{T}\eta_t, \qquad (3.160)
\end{aligned}$$

where step (m) follows readily from the inequality in (3.159). In step (n), we reorganize the terms in the summation. Note that in this step, we abuse some notation to define two dummy variables $\eta_0 = \eta_{T+1} = \infty$ such that $\frac{1}{\eta_0} = \frac{1}{\eta_{T+1}} = 0$. In step (j), we choose

$C_5^0 > \sum_{i=1}^n \bar{\mathcal{R}}_i \max_{p,\tilde{p}\in[l,u]^2} F_i(p,\tilde{p})$ and $C_5^1 > n\bar{G}$ such that step (i) readily follows.

Summarizing the aforementioned arguments, we pick $C_5 \geq \max\{C_5^0, C_5^1\}$ such that for any $T \geq 2$, we have $\Delta_{\boldsymbol{\theta}}^{\pi}(T) \leq C_5\Big[\sum_{t=1}^T \max\Big\{\frac{1}{\eta_t} - \frac{1}{\eta_{t-1}}, 0\Big\} + \sum_{t=1}^T \eta_t\Big]$.

Proof of Claim (ii). Let π be the platform's SRI policy under which the sellers induce the price process $\{\boldsymbol{p}_s : s = 1, 2, \dots\}$ by applying the multiagent mirror descent framework in (3.25). We also let $\bar{\boldsymbol{p}}^{\boldsymbol{\theta}} \in \operatorname{int}([l,u]^n)$ be the DN Nash equilibrium in Definition 5. By the i.i.d property, we let $\boldsymbol{\xi}$ be a random vector that $\boldsymbol{\xi}_t = \boldsymbol{\xi}$ for all $t \in \{1, 2 \dots, \}$. Since $\boldsymbol{\theta} \in \partial\tilde{\Theta}$, we first show that

$$\mathbb{E}_{\boldsymbol{\theta}}^{\pi}\Big\{\mathcal{R}^{\star}(\boldsymbol{\xi},\boldsymbol{\theta}) - \mathcal{R}(\bar{\boldsymbol{p}}^{\boldsymbol{\theta}},\boldsymbol{\xi},\boldsymbol{\theta})\Big\} \overset{(a)}{=} \mathbb{E}_{\boldsymbol{\xi},\boldsymbol{\varepsilon}}\Big\{\mathcal{R}^{\star}(\boldsymbol{\xi},\boldsymbol{\theta}) - \mathcal{R}(\bar{\boldsymbol{p}}^{\boldsymbol{\theta}},\boldsymbol{\xi},\boldsymbol{\theta})\Big\} = 0, \tag{3.161}$$

where in step (a), since the only uncertainty in the expression is $\tilde{\boldsymbol{\xi}}$, the expectation operator $\mathbb{E}_{\boldsymbol{\theta}}^{\pi}\{\cdot\}$ can be replaced with $\mathbb{E}_{\boldsymbol{\xi},\boldsymbol{\varepsilon}}\{\cdot\}$.

Next, we establish that

$$\begin{aligned}
\Delta_{\boldsymbol{\theta}}^{\pi}(T) \overset{(b)}{=} \; & \mathbb{E}_{\boldsymbol{\theta}}^{\pi}\left\{\sum_{t=1}^{\lceil\sqrt{T\log(T)}\rceil} \mathcal{R}^{\star}(\boldsymbol{\xi}_t,\boldsymbol{\theta}) - \mathcal{R}(\boldsymbol{p}_t,\boldsymbol{\xi}_t,\boldsymbol{\theta})\right\} \\
& + \mathbb{E}_{\boldsymbol{\theta}}^{\pi}\left\{\sum_{t=\lceil\sqrt{T\log(T)}\rceil+1}^{T} \mathcal{R}^{\star}(\boldsymbol{\xi}_t,\boldsymbol{\theta}) - \mathcal{R}(\boldsymbol{p}_t,\boldsymbol{\xi}_t,\boldsymbol{\theta})\Big|\tau = \infty\right\}\mathbb{P}_{\boldsymbol{\theta}}^{\pi}\Big\{\tau = \infty\Big\} \\
& + \mathbb{E}_{\boldsymbol{\theta}}^{\pi}\left\{\sum_{t=\lceil\sqrt{T\log(T)}\rceil+1}^{T} \mathcal{R}^{\star}(\boldsymbol{\xi}_t,\boldsymbol{\theta}) - \mathcal{R}(\boldsymbol{p}_t,\boldsymbol{\xi}_t,\boldsymbol{\theta})\Big|\tau = \lceil\sqrt{T\log(T)}\rceil + 1\right\} \\
& \quad \mathbb{P}_{\boldsymbol{\theta}}^{\pi}\Big\{\tau = \lceil\sqrt{T\log(T)}\rceil + 1\Big\},
\end{aligned} \tag{3.162}$$

where in step (b), we leverage the same arguments as in (3.73) of Theorem 13's proof arguments.

In the first term of (3.162), we define a positive constant

$\bar{\mathcal{R}} = \max_{(\tilde{\boldsymbol{p}},\tilde{\boldsymbol{\xi}},\tilde{\boldsymbol{\theta}})\in[l,u]^n\times\Xi\times\Theta} \mathcal{R}(\tilde{\boldsymbol{p}},\tilde{\boldsymbol{\xi}},\tilde{\boldsymbol{\theta}}) > 0$, which allows us to immediately establish that

$$\mathbb{E}^{\pi}_{\boldsymbol{\theta}}\left\{\sum_{t=1}^{\lceil\sqrt{T\log(T)}\rceil} \mathcal{R}^{\star}(\boldsymbol{\xi}_t,\boldsymbol{\theta}) - \mathcal{R}(\boldsymbol{p}_t,\boldsymbol{\xi}_t,\boldsymbol{\theta})\right\} \;\leq\; 2\bar{\mathcal{R}}\Big(\sqrt{T\log(T)}+1\Big). \tag{3.163}$$

In the second term of (3.162), there exits $C_6^0, C_6^1 > 0$ such that

$$\begin{aligned}
&\mathbb{E}^{\pi}_{\boldsymbol{\theta}}\left\{\sum_{t=\lceil\sqrt{T\log(T)}\rceil+1}^{T} \mathcal{R}^{\star}(\boldsymbol{\xi}_t,\boldsymbol{\theta}) - \mathcal{R}(\boldsymbol{p}_t,\boldsymbol{\xi}_t,\boldsymbol{\theta})\Big|\tau=\infty\right\}\mathbb{P}^{\pi}_{\boldsymbol{\theta}}\Big\{\tau=\infty\Big\}\\
&\overset{(c)}{\leq}\; \mathbb{E}^{\pi}_{\boldsymbol{\theta}}\left\{\sum_{t=\lceil\sqrt{T\log(T)}\rceil+1}^{T} \mathcal{R}^{\star}(\boldsymbol{\xi}_t,\boldsymbol{\theta}) - \mathcal{R}(\bar{\boldsymbol{p}}^{\boldsymbol{\theta}},\boldsymbol{\xi}_t,\boldsymbol{\theta})\Big|\tau=\infty\right\}\mathbb{P}^{\pi}_{\boldsymbol{\theta}}\Big\{\tau=\infty\Big\}\\
&\quad + \mathbb{E}^{\pi}_{\boldsymbol{\theta}}\left\{\sum_{t=\lceil\sqrt{T\log(T)}\rceil+1}^{T} \mathcal{R}(\bar{\boldsymbol{p}}^{\boldsymbol{\theta}},\boldsymbol{\xi}_t,\boldsymbol{\theta}) - \mathcal{R}(\boldsymbol{p}_t,\boldsymbol{\xi}_t,\boldsymbol{\theta})\Big|\tau=\infty\right\}\mathbb{P}^{\pi}_{\boldsymbol{\theta}}\Big\{\tau=\infty\Big\}\\
&\overset{(d)}{=}\; \mathbb{E}^{\pi}_{\boldsymbol{\theta}}\left\{\sum_{t=\lceil\sqrt{T\log(T)}\rceil+1}^{T} \mathcal{R}(\bar{\boldsymbol{p}}^{\boldsymbol{\theta}},\boldsymbol{\xi}_t,\boldsymbol{\theta}) - \mathcal{R}(\boldsymbol{p}_t,\boldsymbol{\xi}_t,\boldsymbol{\theta})\Big|\tau=\infty\right\}\mathbb{P}^{\pi}_{\boldsymbol{\theta}}\Big\{\tau=\infty\Big\}\\
&\overset{(e)}{\leq}\; C_6^0\sum_{t=1}^{T}\max\left\{\frac{1}{\eta_t}-\frac{1}{\eta_{t-1}},0\right\} + C_6^1\sum_{t=1}^{T}\eta_t
\end{aligned} \tag{3.164}$$

where in step (c), we add and subtract a common term $\mathbb{E}^{\pi}_{\boldsymbol{\theta}}\{\sum_{t=\lceil\sqrt{T\log(T)}\rceil+1}^{T}\mathcal{R}^{\star}(\boldsymbol{\xi}_t,\boldsymbol{\theta}) - \mathcal{R}(\bar{\boldsymbol{p}}^{\boldsymbol{\theta}},\boldsymbol{\xi}_t,\boldsymbol{\theta})\Big|\tau=\infty\}\mathbb{P}^{\pi}_{\boldsymbol{\theta}}\{\tau=\infty\}$ to get the right-hand-side expression. In step (d), given the i.i.d property of $\{\boldsymbol{\xi}_t\}$, we have that

$\mathbb{E}^{\pi}_{\boldsymbol{\theta}}\{\mathcal{R}^{\star}(\boldsymbol{\xi}_t,\boldsymbol{\theta})|\tau=\infty\} = \mathbb{E}^{\pi}_{\boldsymbol{\theta}}\{\mathcal{R}^{\star}(\boldsymbol{\xi}_t,\boldsymbol{\theta})\}$ and $\mathbb{E}^{\pi}_{\boldsymbol{\theta}}\{\mathcal{R}(\bar{\boldsymbol{p}}^{\boldsymbol{\theta}},\boldsymbol{\xi}_t,\boldsymbol{\theta})|\tau=\infty\} = \mathbb{E}^{\pi}_{\boldsymbol{\theta}}\{\mathcal{R}(\bar{\boldsymbol{p}}^{\boldsymbol{\theta}},\boldsymbol{\xi}_t,\boldsymbol{\theta})\}$. From the observation in (3.161), we immediately obtain that the first term in step (d) is zero. In step (e), given that $\mathbb{P}^{\pi}_{\boldsymbol{\theta}}\{\tau=\infty\}\leq 1$, since the SRI policy is the same as the DN policy conditional on $\tau=\infty$, for $t\geq\sqrt{T\log(T)}$, we can directly leveraging the conclusion from Claim (i) to establish that there exists $C_6^0, C_6^1 > 0$ such that the inequality in this step holds.

In the third term of (3.162), we show that there exists $C_7 > 0$,

$$\begin{aligned}
&\mathbb{E}^{\pi}_{\boldsymbol{\theta}}\left\{\sum_{t=\lceil\sqrt{T\log(T)}\rceil+1}^{T} \mathcal{R}^{\star}(\boldsymbol{\xi}_t,\boldsymbol{\theta}) - \mathcal{R}(\boldsymbol{p}_t,\boldsymbol{\xi}_t,\boldsymbol{\theta})\Big|\tau = \lceil\sqrt{T\log(T)}\rceil + 1\right\}\mathbb{P}^{\pi}_{\boldsymbol{\theta}}\Big\{\tau = \lceil\sqrt{T\log(T)+1}\rceil\Big\} \\
&\overset{(f)}{\leq}\ \mathbb{E}^{\pi}_{\boldsymbol{\theta}}\left\{\sum_{t=\lceil\sqrt{T\log(T)}\rceil+1}^{T} \mathcal{R}(\psi^{\star}(\boldsymbol{\xi}_t,\boldsymbol{\theta}),\boldsymbol{\xi}_t,\boldsymbol{\theta}) - \mathcal{R}(\psi^{\star}(\boldsymbol{\xi}_t,\tilde{\boldsymbol{\theta}}_t),\boldsymbol{\xi}_t,\boldsymbol{\theta})\Big|\tau = \lceil\sqrt{T\log(T)}\rceil + 1\right\} \\
&\overset{(g)}{\leq}\ C_7\sqrt{T\log(T)}. \qquad (3.165)
\end{aligned}$$

In step (f), we leverage the fact that $\mathcal{R}^{\star}(\boldsymbol{\xi}_t,\boldsymbol{\theta})$ can be implemented by the price profile $\psi^{\star}(\boldsymbol{\xi}_t,\boldsymbol{\theta})$. Moreover, conditional on $\tau = \lceil\sqrt{T\log(T)}\rceil + 1$, for any $t \geq \lceil\sqrt{T\log(T)}\rceil + 1$, the price profile $\boldsymbol{p}_t$ satisfies $\boldsymbol{p}_t = \psi^{\star}(\boldsymbol{\xi}_t,\tilde{\boldsymbol{\theta}}_t)$ (by Algorithm 2). Moreover, we have $\mathbb{P}^{\pi}_{\boldsymbol{\theta}}\{\tau = \lceil\sqrt{T\log(T)+1}\rceil\} \leq 1$. Step (g) follows from the same derivation as in (3.77) of Theorem 13 for some positive constant $K_6 > 0$. Summarizing (3.162), (3.163), (3.164) and (3.165), we conclude that there exists positive constants $C_6 = \max\{C_6^0, C_6^1\}$, C_7 such that

$$\Delta^{\pi}_{\boldsymbol{\theta}}(T)\ \leq\ C_6\Big[\sum_{t=1}^{T}\max\left\{\frac{1}{\eta_t} - \frac{1}{\eta_{t-1}}, 0\right\} + \sum_{t=1}^{T}\eta_t\Big] + C_7\sqrt{T\log(T)}. \qquad (3.166)$$

This completes the proof. □

3.11 Supporting Results

Proof of Lemma 19. To prove the claim, for any $t \in \{1, 2, \dots\}$, we divide the arguments into the following steps.

Step 1: establish an upper bound for $\|\tilde{\boldsymbol{\theta}}^u_{t+1} - \boldsymbol{\theta}\|_{\mathcal{I}_t(\boldsymbol{\theta})}$. Recall that $\ell_t(\tilde{\boldsymbol{\theta}})$ is the log-likelihood function in period t and $\tilde{\boldsymbol{\theta}}^u_{t+1}$ is the unconstrained maximum likelihood estimator. We first consider the Taylor expansion to $\ell_t(\tilde{\boldsymbol{\theta}}^u_{t+1})$ to the second order: given the true demand parameter vector $\boldsymbol{\theta}$, there exists $\bar{\boldsymbol{\theta}}_{t+1}$ such that

$$\ell_t(\tilde{\boldsymbol{\theta}}^u_{t+1}) \;= \ell_t(\boldsymbol{\theta}) + \nabla \ell_t(\boldsymbol{\theta})^\intercal(\tilde{\boldsymbol{\theta}}^u_{t+1} - \boldsymbol{\theta}) - \frac{1}{2}(\tilde{\boldsymbol{\theta}}^u_{t+1} - \boldsymbol{\theta})^\intercal \mathcal{I}_t(\bar{\boldsymbol{\theta}}_{t+1})(\tilde{\boldsymbol{\theta}}^u_{t+1} - \boldsymbol{\theta}). \tag{3.167}$$

Given the dimension of the demand parameter vector κ_θ, for any positive semidefinite matrix $A \in \mathbb{R}^{\kappa_\theta \times \kappa_\theta}$ and vector $\boldsymbol{x} \in \mathbb{R}^{\kappa_\theta}$, we define $\|\boldsymbol{x}\|_A = \|A^{\frac{1}{2}}\boldsymbol{x}\|_2$. Recall from the lemma statement that for any $T \geq 2$ and $t \in \{1, \ldots, T\}$, we let $\tau(t) = \min\{t, \sqrt{T\log(T)}\}$. Conditional on $\lambda_{min}(\mathcal{I}_t(\boldsymbol{\theta})) \geq \lambda_0 \tau(t)$, we deduce that

$$\begin{aligned}
\|\nabla \ell_t(\boldsymbol{\theta})\|_{\mathcal{I}_t^{-1}(\boldsymbol{\theta})} \|\tilde{\boldsymbol{\theta}}^u_{t+1} - \boldsymbol{\theta}\|_{\mathcal{I}_t(\boldsymbol{\theta})} &\overset{(a)}{=} \|\mathcal{I}_t^{-\frac{1}{2}}(\boldsymbol{\theta})\nabla \ell_t(\boldsymbol{\theta})\|_2 \|\mathcal{I}_t^{\frac{1}{2}}(\boldsymbol{\theta})(\tilde{\boldsymbol{\theta}}^u_{t+1} - \boldsymbol{\theta})\|_2 \\
&\overset{(b)}{\geq} \nabla \ell_t(\boldsymbol{\theta})^\intercal(\tilde{\boldsymbol{\theta}}^u_{t+1} - \boldsymbol{\theta}) \\
&\overset{(c)}{\geq} \frac{1}{2}(\tilde{\boldsymbol{\theta}}^u_{t+1} - \boldsymbol{\theta})^\intercal \mathcal{I}_t(\bar{\boldsymbol{\theta}}_{t+1})(\tilde{\boldsymbol{\theta}}^u_{t+1} - \boldsymbol{\theta}) \\
&\overset{(d)}{\geq} \frac{1}{2\kappa_I}(\tilde{\boldsymbol{\theta}}^u_{t+1} - \boldsymbol{\theta})^\intercal \mathcal{I}_t(\boldsymbol{\theta})(\tilde{\boldsymbol{\theta}}^u_{t+1} - \boldsymbol{\theta}) \overset{(e)}{=} \frac{1}{2\kappa_I}\|\tilde{\boldsymbol{\theta}}^u_{t+1} - \boldsymbol{\theta}\|^2_{\mathcal{I}_t(\boldsymbol{\theta})}
\end{aligned} \tag{3.168}$$

where step (a) and step (e) follow directly from the definition $\|\boldsymbol{x}\|_{\mathcal{I}_t(\boldsymbol{\theta})} = \|\mathcal{I}_t^{\frac{1}{2}}(\boldsymbol{\theta})\boldsymbol{x}\|_2$ and $\|\boldsymbol{x}\|_{\mathcal{I}_t^{-1}(\boldsymbol{\theta})} = \|\mathcal{I}_t^{-\frac{1}{2}}(\boldsymbol{\theta})\boldsymbol{x}\|_2$. Step (b) follows directly from Cauchy-Schwarz inequality. In step (c), by leveraging the definition that $\tilde{\boldsymbol{\theta}}^u_{t+1} = \arg\max_{\tilde{\boldsymbol{\theta}}} \ell_t(\tilde{\boldsymbol{\theta}})$, we have $\ell_t(\tilde{\boldsymbol{\theta}}^u_{t+1}) \geq \ell_t(\boldsymbol{\theta})$. Together with the equation in (3.167), step (c) readily follows. Step (d) follows from Assumption 13(ii). In summary of the observation in (3.168), we have

$$2\kappa_I \|\nabla \ell_t(\boldsymbol{\theta})\|_{\mathcal{I}_t^{-1}(\boldsymbol{\theta})} \;\geq\; \|\tilde{\boldsymbol{\theta}}^u_{t+1} - \boldsymbol{\theta}\|_{\mathcal{I}_t(\boldsymbol{\theta})}. \tag{3.169}$$

Step 2: an upper bound for $\mathbb{P}^\pi_{\boldsymbol{\theta}}\Big\{\|\nabla \ell_t(\boldsymbol{\theta})\|_{\mathcal{I}_t^{-1}(\boldsymbol{\theta})} > \delta_t, \lambda_{min}(\mathcal{I}_t(\boldsymbol{\theta})) \geq \lambda_0\tau(t), A\Big\}$. Before proceeding, we first develop some notations. For any $s \in \{1, 2 \ldots\}$ and any $\delta \geq 0$, we let $\delta_s = \frac{1}{2\kappa_I}\sqrt{\lambda_0 \tau(s)\delta}$. We further let $\{\tilde{\boldsymbol{y}}_s : s = 1, 2, \ldots\}$ be a stochastic process where $\tilde{\boldsymbol{y}}_s$

satisfies

$$\tilde{\boldsymbol{y}}_s = \frac{\delta_s}{\|\mathcal{I}_s^{-1}(\boldsymbol{\theta})\nabla\ell_s(\boldsymbol{\theta})\|_{\mathcal{I}_t(\boldsymbol{\theta})}}\mathcal{I}_s^{-1}(\boldsymbol{\theta})\nabla\ell_s(\boldsymbol{\theta}). \tag{3.170}$$

Given the definition in (3.170), for any $s \in \{1, 2, \dots\}$, if $\|\nabla\ell_s(\boldsymbol{\theta})\|_{\mathcal{I}_s^{-1}(\boldsymbol{\theta})} > \delta_s$, it immediately follows that

$$\|\tilde{\boldsymbol{y}}_s\|^2_{\mathcal{I}_s(\boldsymbol{\theta})} \overset{(e)}{=} \frac{\nabla\ell_s(\boldsymbol{\theta})^\intercal\mathcal{I}_s^{-1}(\boldsymbol{\theta})\mathcal{I}_s(\boldsymbol{\theta})\mathcal{I}_s^{-1}(\boldsymbol{\theta})\nabla\ell_s(\boldsymbol{\theta})}{\|\mathcal{I}_s^{-1}(\boldsymbol{\theta})\nabla\ell_s(\boldsymbol{\theta})\|^2_{\mathcal{I}_s(\boldsymbol{\theta})}}\delta_s^2 \overset{(f)}{=} \frac{\|\mathcal{I}_s^{-1}(\boldsymbol{\theta})\nabla\ell_s(\boldsymbol{\theta})\|^2_{\mathcal{I}_s(\boldsymbol{\theta})}}{\|\mathcal{I}_s^{-1}(\boldsymbol{\theta})\nabla\ell_s(\boldsymbol{\theta})\|^2_{\mathcal{I}_s(\boldsymbol{\theta})}}\delta_s^2 \overset{(g)}{=} \delta_s^2, \tag{3.171}$$

where step (e) and step (f) respectively follow from the definition in (3.170) and the definition of the matrix norm $\|\cdot\|_{\mathcal{I}_s(\boldsymbol{\theta})}$. Step (g) follows directly from the assumption that $\|\nabla\ell_s(\boldsymbol{\theta})\|_{\mathcal{I}_s^{-1}(\boldsymbol{\theta})} > \delta_s \geq 0$. Recalling that $\delta_s = \frac{1}{2\kappa_I}\sqrt{\lambda_0\tau(s)\delta}$, if we further have $\lambda_{min}(\mathcal{I}_s(\boldsymbol{\theta})) \geq \lambda_0\tau(s)$, together with the observation in (3.171), we would obtain that

$$\|\tilde{\boldsymbol{y}}_s\|_2^2 \;\leq\; \frac{\lambda_{min}(\mathcal{I}_s(\boldsymbol{\theta}))}{\lambda_0\tau(s)}\|\tilde{\boldsymbol{y}}_s\|_2^2 \;\leq\; \frac{1}{\lambda_0\tau(s)}\|\tilde{\boldsymbol{y}}_s\|^2_{\mathcal{I}_s(\boldsymbol{\theta})} \;=\; \frac{\delta}{4\kappa_I^2} \tag{3.172}$$

Given the positive constant $\bar{\lambda}_I$ in Assumption 13(i) and the positive constant κ_I in Assumption 13(ii), we pick ζ_1 large enough such that $\zeta_1 \geq \exp\{\frac{\bar{\lambda}_I}{2\kappa_I\zeta_1}\}$, and then set $\nu = \max\{\sqrt{\delta}, 1\}$. For any deterministic vector $\boldsymbol{y} \in \mathbb{R}^{\kappa_\theta}$, we define a sequence $\{Z_s^{\boldsymbol{y}} : s = 1, 2, \dots\}$ where $Z_s^{\boldsymbol{y}}$ satisfies

$$Z_s^{\boldsymbol{y}} \;=\; \exp\left\{\frac{1}{\zeta_1\nu}\left(\boldsymbol{y}^\intercal\nabla\ell_s(\boldsymbol{\theta}) - \frac{1}{2}\boldsymbol{y}^\intercal\mathcal{I}_s(\boldsymbol{\theta})\boldsymbol{y}\right)\right\}. \tag{3.173}$$

Based on the definition of the $\{\tilde{\boldsymbol{y}}_s : s = 1, 2\dots\}$ in (3.170) and $\{Z_s^{\boldsymbol{y}} : s = 1, 2, \dots\}$ in

(3.173), for any $T \geq 2$, $t \in \{1, \ldots, T\}$ and $\tau(t) = \min\{t, \sqrt{T \log(T)}\}$, we can establish that

$$
\begin{aligned}
&\mathbb{P}_{\boldsymbol{\theta}}^{\pi}\left\{\|\nabla \ell_t(\boldsymbol{\theta})\|_{\mathcal{I}_t^{-1}(\boldsymbol{\theta})} > \delta_t, \lambda_{min}(\mathcal{I}_t(\boldsymbol{\theta})) \geq \lambda_0 \tau(t), A\right\} \\
&\overset{(h)}{=} \mathbb{P}_{\boldsymbol{\theta}}^{\pi}\left\{\frac{\delta_t \nabla \ell_t(\boldsymbol{\theta})^\intercal \mathcal{I}_t^{-1}(\boldsymbol{\theta}) \nabla \ell_t(\boldsymbol{\theta})}{\|\mathcal{I}_t^{-1}(\boldsymbol{\theta}) \nabla \ell_t(\boldsymbol{\theta})\|_{\mathcal{I}_t(\boldsymbol{\theta})}} > \frac{\delta_t^2 \nabla \ell_t(\boldsymbol{\theta})^\intercal \mathcal{I}_t^{-1}(\boldsymbol{\theta}) \nabla \ell_t(\boldsymbol{\theta})}{\|\mathcal{I}_t^{-1}(\boldsymbol{\theta}) \nabla \ell_t(\boldsymbol{\theta})\|_{\mathcal{I}_t(\boldsymbol{\theta})}^2}, \lambda_{min}(\mathcal{I}_t(\boldsymbol{\theta})) \geq \lambda_0 \tau(t), A\right\} \\
&\overset{(i)}{=} \mathbb{P}_{\boldsymbol{\theta}}^{\pi}\Bigg\{\frac{\delta_t \nabla \ell_t(\boldsymbol{\theta})^\intercal \mathcal{I}_t^{-1}(\boldsymbol{\theta}) \nabla \ell_t(\boldsymbol{\theta})}{\|\mathcal{I}_t^{-1}(\boldsymbol{\theta}) \nabla \ell_t(\boldsymbol{\theta})\|_{\mathcal{I}_t(\boldsymbol{\theta})}} > \frac{\delta_t^2 \nabla \ell_t(\boldsymbol{\theta})^\intercal \mathcal{I}_t^{-1}(\boldsymbol{\theta}) \mathcal{I}_t(\boldsymbol{\theta}) \mathcal{I}_t^{-1}(\boldsymbol{\theta}) \nabla \ell_t(\boldsymbol{\theta})}{\|\mathcal{I}_t^{-1}(\boldsymbol{\theta}) \nabla \ell_t(\boldsymbol{\theta})\|_{\mathcal{I}_t(\boldsymbol{\theta})}^2}, \\
&\qquad\qquad \lambda_{min}(\mathcal{I}_t(\boldsymbol{\theta})) \geq \lambda_0 \tau(t), A\Bigg\} \\
&\overset{(j)}{\leq} \mathbb{P}_{\boldsymbol{\theta}}^{\pi}\Big\{\tilde{\boldsymbol{y}}_t^\intercal \nabla \ell_t(\boldsymbol{\theta}) \geq \tilde{\boldsymbol{y}}_t^\intercal \mathcal{I}_t(\boldsymbol{\theta}) \tilde{\boldsymbol{y}}_t, \lambda_{min}(\mathcal{I}_t(\boldsymbol{\theta})) \geq \lambda_0 \tau(t), A\Big\} \\
&= \mathbb{P}_{\boldsymbol{\theta}}^{\pi}\Big\{\frac{1}{\zeta_1 \nu} \tilde{\boldsymbol{y}}_t^\intercal \nabla \ell_t(\boldsymbol{\theta}) \geq \frac{1}{\zeta_1 \nu} \tilde{\boldsymbol{y}}_t^\intercal \mathcal{I}_t(\boldsymbol{\theta}) \tilde{\boldsymbol{y}}_t, \lambda_{min}(\mathcal{I}_t(\boldsymbol{\theta})) \geq \lambda_0 \tau(t), A\Big\} \\
&= \mathbb{P}_{\boldsymbol{\theta}}^{\pi}\left\{\exp\left\{\frac{1}{\zeta_1 \nu} \tilde{\boldsymbol{y}}_t^\intercal \nabla \ell_t(\boldsymbol{\theta})\right\} \geq \exp\left\{\frac{1}{\zeta_1 \nu} \tilde{\boldsymbol{y}}_t^\intercal \mathcal{I}_t(\boldsymbol{\theta}) \tilde{\boldsymbol{y}}_t\right\}, \lambda_{min}(\mathcal{I}_t(\boldsymbol{\theta})) \geq \lambda_0 \tau(t), A\right\} \\
&\overset{(k)}{=} \mathbb{P}_{\boldsymbol{\theta}}^{\pi}\Bigg\{\exp\left\{\frac{1}{\zeta_1 \nu}\left(\tilde{\boldsymbol{y}}_t^\intercal \nabla \ell_t(\boldsymbol{\theta}) - \frac{1}{2} \tilde{\boldsymbol{y}}_t^\intercal \mathcal{I}_t(\boldsymbol{\theta}) \tilde{\boldsymbol{y}}_t\right)\right\} \geq \exp\left\{\frac{1}{2\zeta_1 \nu} \tilde{\boldsymbol{y}}_t^\intercal \mathcal{I}_t(\boldsymbol{\theta}) \tilde{\boldsymbol{y}}_t\right\}, \\
&\qquad\qquad \lambda_{min}(\mathcal{I}_t(\boldsymbol{\theta})) \geq \lambda_0 \tau(t), A\Bigg\} \\
&\overset{(l)}{=} \mathbb{P}_{\boldsymbol{\theta}}^{\pi}\left\{Z_t^{\tilde{\boldsymbol{y}}_t} \geq \exp\left\{\frac{1}{2\zeta_1 \nu} \tilde{\boldsymbol{y}}_t^\intercal \mathcal{I}_t(\boldsymbol{\theta}) \tilde{\boldsymbol{y}}_t\right\}, \lambda_{min}(\mathcal{I}_t(\boldsymbol{\theta})) \geq \lambda_0 \tau(t), A\right\} \\
&\overset{(m)}{=} \mathbb{P}_{\boldsymbol{\theta}}^{\pi}\left\{Z_t^{\tilde{\boldsymbol{y}}_t} \geq \exp\left\{\frac{\delta_t^2}{2\zeta_1 \nu}\right\}, \lambda_{min}(\mathcal{I}_t(\boldsymbol{\theta})) \geq \lambda_0 \tau(t), A\right\},
\end{aligned}
\tag{3.174}
$$

where step (h) follows from the equivalence of events on both sides of the equation given that $\|\nabla \ell_t(\boldsymbol{\theta})\|_{\mathcal{I}_t^{-1}(\boldsymbol{\theta})}^2 = \|\mathcal{I}_t^{-1}(\boldsymbol{\theta}) \nabla \ell_t(\boldsymbol{\theta})\|_{\mathcal{I}_t(\boldsymbol{\theta})}^2 > 0$. In step (i), on the right hand side of the inequality of the first event, we multiply an identity matrix $\mathcal{I}_t(\boldsymbol{\theta}) \mathcal{I}_t^{-1}(\boldsymbol{\theta}) = I$ in the numerator to obtain the equivalent expression. In step (j), we implement the expression of $\tilde{\boldsymbol{y}}_t$ from (3.170) in the first inequality, and then replace the strict inequality with the weak inequality. In step (k), we divide a common term $\exp\{\frac{1}{2\zeta_1 \nu} \tilde{\boldsymbol{y}}_t^\intercal \mathcal{I}_t(\boldsymbol{\theta}) \tilde{\boldsymbol{y}}_t\}$ on both sides of the first inequality. Step (l) follows from the definition of $Z_t^{\boldsymbol{y}}$ in (3.173). In step (m), since $\|\tilde{\boldsymbol{y}}_t\|_{\mathcal{I}_t(\boldsymbol{\theta})} = \delta_t$ (by (3.171)), it follows that $\tilde{\boldsymbol{y}}_t^\intercal \mathcal{I}_t(\boldsymbol{\theta}) \tilde{\boldsymbol{y}}_t = \|\tilde{\boldsymbol{y}}_t\|_{\mathcal{I}_t(\boldsymbol{\theta})}^2 = \delta_t^2$.

To develop an upper bound for the expression $\mathbb{P}\{Z_t^{\tilde{\boldsymbol{y}}_t} \geq \exp\{\frac{\delta_t^2}{2\zeta_1\nu}\}, \lambda_{min}(\mathcal{I}_t(\boldsymbol{\theta})) \geq \lambda_0\tau(t), A\}$, we want to consider some deterministic vector $\boldsymbol{y}$ that is close to the stochastic vector $\tilde{\boldsymbol{y}}_t$, and then leverage the supermartingale property of $Z_t^{\boldsymbol{y}}$, which we would establish below.

Step 3: establish that $\{Z_s^{\boldsymbol{y}} : s = 1, 2 \dots\}$ is a supermartingale adapted to $\{\mathcal{F}_s : s = 0, 1, \dots\}$. Before proceeding, we first show that $\{Z_s^{\boldsymbol{y}} : s = 1, 2, \dots\}$ in (3.173) is a supermartingale adapted to the filtration $\{\mathcal{F}_s : s = 0, 1, \dots\}$ where $\mathcal{F}_s = \sigma(\boldsymbol{\xi}_1, \boldsymbol{\varepsilon}_1, \dots, \boldsymbol{\xi}_s, \tilde{\boldsymbol{\varepsilon}}_s, \tilde{\boldsymbol{\xi}}_{s+1})$. Fixing $\delta \geq 0$, for any deterministic vector $\boldsymbol{y}$ such that $\|\boldsymbol{y}\|_2^2 \leq \frac{\delta}{4\kappa_I^2}$, with $g_s(\boldsymbol{\theta}) = \ell_s(\boldsymbol{\theta}) - \ell_{s-1}(\boldsymbol{\theta})$

and $\mathcal{J}_s(\boldsymbol{\theta}) = \mathcal{I}_s(\boldsymbol{\theta}) - \mathcal{I}_{s-1}(\boldsymbol{\theta})$, we can establish that

$$
\begin{aligned}
&\mathbb{E}^{\pi}_{\boldsymbol{\theta}}\left[\exp\left\{\frac{1}{\zeta_1\nu}\boldsymbol{y}^\intercal\nabla g_s(\boldsymbol{\theta})\right\}\middle|\mathcal{F}_{s-1}\right] \\
\overset{(n)}{=}\ & 1 + \frac{1}{\zeta_1\nu}\boldsymbol{y}^\intercal\mathbb{E}^{\pi}_{\boldsymbol{\theta}}\left[\nabla g_s(\boldsymbol{\theta})\middle|\mathcal{F}_{s-1}\right] + \sum_{k=2}^{\infty}\frac{1}{k!}\mathbb{E}^{\pi}_{\boldsymbol{\theta}}\left[\left(\frac{1}{\zeta_1\nu}\boldsymbol{y}^\intercal\nabla g_s(\boldsymbol{\theta})\right)^k\middle|\mathcal{F}_{s-1}\right] \\
\overset{(o)}{=}\ & 1 + \sum_{k=2}^{\infty}\frac{1}{k!}\mathbb{E}^{\pi}_{\boldsymbol{\theta}}\left[\left(\frac{1}{\zeta_1\nu}\boldsymbol{y}^\intercal\nabla g_s(\boldsymbol{\theta})\right)^k\middle|\mathcal{F}_{s-1}\right] \\
\overset{(p)}{\le}\ & 1 + \sum_{k=2}^{\infty}\frac{1}{k!}\mathbb{E}^{\pi}_{\boldsymbol{\theta}}\left[\left(\frac{1}{\zeta_1\nu}\boldsymbol{y}^\intercal\nabla g_s(\boldsymbol{\theta})\right)^2\middle|\mathcal{F}_{s-1}\right]^{\frac{k}{2}} \\
\overset{(q)}{=}\ & 1 + \mathbb{E}^{\pi}_{\boldsymbol{\theta}}\left[\left(\frac{1}{\zeta_1\nu}\boldsymbol{y}^\intercal\nabla g_s(\boldsymbol{\theta})\right)^2\middle|\mathcal{F}_{s-1}\right]\cdot\sum_{k=2}^{\infty}\frac{1}{k!}\mathbb{E}^{\pi}_{\boldsymbol{\theta}}\left[\left(\frac{1}{\zeta_1\nu}\boldsymbol{y}^\intercal\nabla g_s(\boldsymbol{\theta})\right)^2\middle|\mathcal{F}_{s-1}\right]^{\frac{k}{2}-1} \\
\overset{(r)}{=}\ & 1 + \frac{1}{2\zeta_1^2\nu^2}\boldsymbol{y}^\intercal\mathbb{E}^{\pi}_{\boldsymbol{\theta}}\left[\nabla g_s(\boldsymbol{\theta})\nabla g_s(\boldsymbol{\theta})^\intercal\middle|\mathcal{F}_{s-1}\right]\boldsymbol{y} \\
&\qquad\cdot\sum_{k=2}^{\infty}\frac{2}{k!}\left(\frac{1}{\zeta_1^2\nu^2}\boldsymbol{y}^\intercal\mathbb{E}^{\pi}_{\boldsymbol{\theta}}\left[\nabla g_s(\boldsymbol{\theta})\nabla g_s(\boldsymbol{\theta})^\intercal\middle|\mathcal{F}_{s-1}\right]\boldsymbol{y}\right)^{\frac{k}{2}-1} \\
\overset{(s)}{\le}\ & 1 + \frac{1}{2\zeta_1^2\nu^2}\boldsymbol{y}^\intercal\mathbb{E}^{\pi}_{\boldsymbol{\theta}}\left[\mathcal{J}_s(\boldsymbol{\theta})\middle|\mathcal{F}_{s-1}\right]\boldsymbol{y} \\
&\qquad\cdot\sum_{k=2}^{\infty}\frac{1}{k(k-1)}\frac{2}{(k-2)!}\left(\frac{1}{\zeta_1^2\nu^2}\boldsymbol{y}^\intercal\mathbb{E}^{\pi}_{\boldsymbol{\theta}}\left[\mathcal{J}_s(\boldsymbol{\theta})\middle|\mathcal{F}_{s-1}\right]\boldsymbol{y}\right)^{\frac{k}{2}-1} \\
\overset{(t)}{\le}\ & 1 + \frac{1}{2\zeta_1^2\nu^2}\boldsymbol{y}^\intercal\mathbb{E}^{\pi}_{\boldsymbol{\theta}}\left[\mathcal{J}_s(\boldsymbol{\theta})\middle|\mathcal{F}_{s-1}\right]\boldsymbol{y}\cdot\sum_{k=2}^{\infty}\frac{1}{(k-2)!}\left(\frac{\bar{\lambda}_I\sqrt{\delta}}{2\kappa_I\zeta_1\nu}\right)^{k-2} \\
\overset{(u)}{\le}\ & 1 + \frac{1}{2\zeta_1\nu^2}\boldsymbol{y}^\intercal\mathbb{E}^{\pi}_{\boldsymbol{\theta}}\left[\mathcal{J}_s(\boldsymbol{\theta})\middle|\mathcal{F}_{s-1}\right]\boldsymbol{y} \\
\overset{(v)}{\le}\ & \exp\left\{\frac{1}{2\zeta_1\nu}\boldsymbol{y}^\intercal\mathbb{E}^{\pi}_{\boldsymbol{\theta}}\left[\mathcal{J}_s(\boldsymbol{\theta})\middle|\mathcal{F}_{s-1}\right]\boldsymbol{y}\right\},
\end{aligned}
\tag{3.175}
$$

where step (n) follows from directly applying the Taylor expansion. Step (o) follows from the observation that $\mathbb{E}^{\pi}_{\boldsymbol{\theta}}[\nabla g_s(\boldsymbol{\theta})|\mathcal{F}_{s-1}] = \mathbf{0}$. Step (p) follows from the observation that for any $k \ge 2$, we have $\left[\mathbb{E}^{\pi}_{\boldsymbol{\theta}}[(\frac{1}{\zeta_1\nu}\boldsymbol{y}^\intercal\nabla g_s(\boldsymbol{\theta}))^k|\mathcal{F}_{s-1}]\right]^{\frac{1}{k}} \le \left[\mathbb{E}^{\pi}_{\boldsymbol{\theta}}[(\frac{1}{\zeta_1\nu}\boldsymbol{y}^\intercal\nabla g_s(\boldsymbol{\theta}))^2|\mathcal{F}_{s-1}]\right]^{\frac{1}{2}}$. In step (q), we move a common expression $\mathbb{E}^{\pi}_{\boldsymbol{\theta}}[(\frac{1}{\zeta_1\nu}\boldsymbol{y}^\intercal\nabla g_s(\boldsymbol{\theta}))^2|\mathcal{F}_{s-1}]$ out of the summation term. In step (r), since $\boldsymbol{y}$ is a deterministic vector, we have $\mathbb{E}^{\pi}_{\boldsymbol{\theta}}[(\frac{1}{\zeta_1\nu}\boldsymbol{y}^\intercal\nabla g_s(\boldsymbol{\theta}))^2|\mathcal{F}_{s-1}] =$

$\frac{1}{\zeta_1^2\nu^2}\boldsymbol{y}^\intercal\mathbb{E}^\pi_{\boldsymbol{\theta}}[\nabla g_s(\boldsymbol{\theta})\nabla g_t(\boldsymbol{\theta})^\intercal|\mathcal{F}_{s-1}]\boldsymbol{y}$. In step (s), given that $g_s(\boldsymbol{\theta}) = \ell_s(\boldsymbol{\theta}) - \ell_{s-1}(\boldsymbol{\theta})$ and $\mathcal{J}_s(\boldsymbol{\theta}) = \mathcal{I}_s(\boldsymbol{\theta}) - \mathcal{I}_{s-1}(\boldsymbol{\theta})$, we leverage the observation $\mathbb{E}^\pi_{\boldsymbol{\theta}}[\nabla g_s(\boldsymbol{\theta})\nabla g_s(\boldsymbol{\theta})^\intercal|\mathcal{F}_{s-1}] = \mathbb{E}^\pi_{\boldsymbol{\theta}}[\mathcal{J}_s(\boldsymbol{\theta})|\mathcal{F}_{s-1}]$ to deduce the inequality in this step. In step (t), for $k \geq 2$, we have $\frac{1}{k(k-1)} \leq 1$. Moreover, we also have $\boldsymbol{y}^\intercal\mathbb{E}^\pi_{\boldsymbol{\theta}}[\mathcal{J}_s(\boldsymbol{\theta})|\mathcal{F}_{s-1}]\boldsymbol{y} \leq \bar{\lambda}_I\|\boldsymbol{y}\|_2^2$ (by Assumption 13(i)). Together with the fact that $\|\boldsymbol{y}\|_2^2 \leq \frac{\delta}{4\kappa_I^2}$, we can establish step (t). In step (u), since $\sqrt{\delta} \leq \max\{\sqrt{\delta}, 1\} = \nu$ (by construction), $\exp\{\frac{\bar{\lambda}_I\sqrt{\delta}}{2\kappa_I\zeta_1\nu}\} = \sum_{k=2}^\infty \frac{1}{(k-2)!}(\frac{\bar{\lambda}_I\sqrt{\delta}}{2\kappa_I\zeta_1\nu})^{k-2}$ (by the Taylor expansion), and $\zeta_1 \geq \exp\{\frac{\bar{\lambda}_I}{2\kappa_I\zeta_1}\}$ (by construction), step (u) immediately follows from the observation that $\zeta_1 \geq \sum_{k=2}^\infty \frac{1}{(k-2)!}(\frac{\bar{\lambda}_I\sqrt{\delta}}{\zeta_1\nu})^{k-2}$. Step (v) follows from $\nu = \max\{\sqrt{\delta}, 1\} \geq 1$ (by construction).

Based on the observation in (3.175), for any $\boldsymbol{y}$ that satisfies $\|\boldsymbol{y}\|_2^2 \leq \frac{\delta_0^2}{4\kappa_I^2}$, we establish that the stochastic process $\{Z_s^{\boldsymbol{y}} : s = 1, 2, \dots\}$ in (3.173) is a supermartingale adapted to $\{\mathcal{F}_s : s = 0, 1 \dots\}$:

$$\begin{aligned}
\mathbb{E}^\pi_{\boldsymbol{\theta}}\left[Z_s^{\boldsymbol{y}}|\mathcal{F}_{s-1}\right] &\overset{(w)}{=} \mathbb{E}^\pi_{\boldsymbol{\theta}}\left[\exp\left\{\frac{1}{\zeta_1\nu}\left(\boldsymbol{y}^\intercal\nabla\ell_s(\boldsymbol{\theta}) - \frac{1}{2}\boldsymbol{y}^\intercal\mathcal{I}_s(\boldsymbol{\theta})\boldsymbol{y}\right)\right\}\middle|\mathcal{F}_{s-1}\right] \\
&= \exp\left\{\frac{1}{\zeta_1\nu}\left(\boldsymbol{y}^\intercal\nabla\ell_{s-1}(\boldsymbol{\theta}) - \frac{1}{2}\boldsymbol{y}^\intercal\mathcal{I}_{s-1}(\boldsymbol{\theta})\boldsymbol{y}\right)\right\} \\
&\quad\cdot\mathbb{E}^\pi_{\boldsymbol{\theta}}\left[\exp\left\{\frac{1}{\zeta_1\nu}\left(\boldsymbol{y}^\intercal(\nabla\ell_s(\boldsymbol{\theta}) - \nabla\ell_{s-1}(\boldsymbol{\theta})) - \frac{1}{2}\boldsymbol{y}^\intercal(\mathcal{I}_s(\boldsymbol{\theta}) - \mathcal{I}_{s-1}(\boldsymbol{\theta}))\boldsymbol{y}\right)\right\}\middle|\mathcal{F}_{s-1}\right] \\
&\overset{(x)}{=} \exp\left\{\frac{1}{\zeta_1\nu}\left(\boldsymbol{y}^\intercal\nabla\ell_{s-1}(\boldsymbol{\theta}) - \frac{1}{2}\boldsymbol{y}^\intercal\mathcal{I}_{s-1}(\boldsymbol{\theta})\boldsymbol{y}\right)\right\} \\
&\qquad\cdot\mathbb{E}^\pi_{\boldsymbol{\theta}}\left[\exp\left\{\frac{1}{\zeta_1\nu}\left(\boldsymbol{y}^\intercal\nabla g_s(\boldsymbol{\theta}) - \frac{1}{2}\boldsymbol{y}^\intercal\mathcal{J}_s(\boldsymbol{\theta})\boldsymbol{y}\right)\right\}\middle|\mathcal{F}_{s-1}\right] \\
&\overset{(y)}{\leq} \exp\left\{\frac{1}{\zeta_1\nu}\left(\boldsymbol{y}^\intercal\nabla\ell_{s-1}(\boldsymbol{\theta}) - \frac{1}{2}\boldsymbol{y}^\intercal\mathcal{I}_{s-1}(\boldsymbol{\theta})\boldsymbol{y}\right)\right\} \overset{(z)}{=} Z_{s-1}^{\boldsymbol{y}},
\end{aligned} \tag{3.176}$$

where step (w) follows from the definition of $Z_s^{\boldsymbol{y}}$ in (3.173). In step (x), given $\mathcal{F}_{s-1} = \sigma(\boldsymbol{\xi}_1, \boldsymbol{\varepsilon}_1, \dots, \boldsymbol{\xi}_{s-1}, \tilde{\boldsymbol{\varepsilon}}_{s-1}, \boldsymbol{\xi}_s)$, we separate the deterministic and the stochastic components of $\nabla\ell_s(\boldsymbol{\theta})$ and $\mathcal{I}_s(\boldsymbol{\theta})$ i.e., $\nabla g_s(\boldsymbol{\theta}) = \nabla\ell_s(\boldsymbol{\theta}) - \nabla\ell_{s-1}(\boldsymbol{\theta})$ and $\mathcal{J}_s(\boldsymbol{\theta}) = \mathcal{I}_s(\boldsymbol{\theta}) - \mathcal{I}_{s-1}(\boldsymbol{\theta})$. Step (y) readily follows from (3.175). Step (z) follows immediately from the definition of $Z_{s-1}^{\boldsymbol{y}}$ in (3.173).

Step 4: develop a cover on the set $\{\boldsymbol{y} : \|\boldsymbol{y}\|_2^2 \leq \frac{\delta}{4\kappa_I^2}\}$. For any $T \geq 2$ and $t \in \{1, \ldots, T\}$, we want to cover the random vector $\tilde{\boldsymbol{y}}_t$ with some small balls with fixed centers. We first let $L(\boldsymbol{y}) = \boldsymbol{y}^\intercal \nabla \ell_t(\boldsymbol{\theta}) - \frac{1}{2}\boldsymbol{y}^\intercal \mathcal{I}_t(\boldsymbol{\theta})\boldsymbol{y}$, which is a concave function in $\boldsymbol{y} \in \mathbb{R}^{\kappa_\theta}$. Let $\tilde{\boldsymbol{y}}_t$ be defined as in (3.170). Under the event that $\{\|\nabla \ell_t(\boldsymbol{\theta})\|_{\mathcal{I}_t^{-1}(\boldsymbol{\theta})} > \delta_t, \lambda_{min}(\mathcal{I}_t(\boldsymbol{\theta})) \geq \lambda_0 \tau(t), A\}$, we have

$$
\begin{aligned}
L(\tilde{\boldsymbol{y}}_t) - L(\boldsymbol{y}) &\overset{(a)}{=} \nabla \ell_t(\boldsymbol{\theta})^\intercal (\tilde{\boldsymbol{y}}_t - \boldsymbol{y}) - \frac{1}{2}\tilde{\boldsymbol{y}}_t^\intercal \mathcal{I}_t(\boldsymbol{\theta})\tilde{\boldsymbol{y}}_t + \frac{1}{2}\boldsymbol{y}^\intercal \mathcal{I}_t(\boldsymbol{\theta})\boldsymbol{y} \\
&= \nabla \ell_t(\boldsymbol{\theta})^\intercal (\tilde{\boldsymbol{y}}_t - \boldsymbol{y}) + \frac{1}{2}(\tilde{\boldsymbol{y}}_t - \boldsymbol{y})^\intercal \mathcal{I}_t(\boldsymbol{\theta})(\tilde{\boldsymbol{y}}_t - \boldsymbol{y}) - \tilde{\boldsymbol{y}}_t^\intercal \mathcal{I}_t(\boldsymbol{\theta})(\tilde{\boldsymbol{y}}_t - \boldsymbol{y}) \\
&\overset{(b)}{=} \nabla \ell_t(\boldsymbol{\theta})^\intercal (\tilde{\boldsymbol{y}}_t - \boldsymbol{y}) + \frac{1}{2}(\tilde{\boldsymbol{y}}_t - \boldsymbol{y})^\intercal \mathcal{I}_t(\boldsymbol{\theta})(\tilde{\boldsymbol{y}}_t - \boldsymbol{y}) \\
&\qquad - \frac{\delta_t}{\|\mathcal{I}_t^{-1}(\boldsymbol{\theta})\nabla \ell_t(\boldsymbol{\theta})\|_{\mathcal{I}_t(\boldsymbol{\theta})}} \nabla \ell_t(\boldsymbol{\theta})^\intercal (\tilde{\boldsymbol{y}}_t - \boldsymbol{y}) \\
&\overset{(c)}{\leq} \max\Big\{\nabla \ell_t(\boldsymbol{\theta})^\intercal (\tilde{\boldsymbol{y}}_t - \boldsymbol{y}), 0\Big\} + \frac{1}{2}(\tilde{\boldsymbol{y}}_t - \boldsymbol{y})^\intercal \mathcal{I}_t(\boldsymbol{\theta})(\tilde{\boldsymbol{y}}_t - \boldsymbol{y}) \\
&\overset{(d)}{\leq} \|\nabla \ell_t(\boldsymbol{\theta})\|_2 \|\tilde{\boldsymbol{y}}_t - \boldsymbol{y}\|_2 + \frac{1}{2}(\tilde{\boldsymbol{y}}_t - \boldsymbol{y})^\intercal \mathcal{I}_t(\boldsymbol{\theta})(\tilde{\boldsymbol{y}}_t - \boldsymbol{y}) \\
&\overset{(e)}{\leq} \|\nabla \ell_t(\boldsymbol{\theta})\|_2 \|\tilde{\boldsymbol{y}}_t - \boldsymbol{y}\|_2 + \frac{1}{2}\bar{\lambda}_I t \|\tilde{\boldsymbol{y}}_t - \boldsymbol{y}\|_2^2, \qquad (3.177)
\end{aligned}
$$

where step (a) follows from the definition of $L(\boldsymbol{y})$. Step (b) follows directly from the expression of $\tilde{\boldsymbol{y}}_t$ in (3.170) to the third term of the expression. In step (c), given the event that $\{\|\nabla \ell_t(\boldsymbol{\theta})\|_{\mathcal{I}_t^{-1}(\boldsymbol{\theta})} > \delta_t, \lambda_{min}(\mathcal{I}_t(\boldsymbol{\theta})) \geq \lambda_0 \tau(t), A\}$, we have $\|\nabla \ell_t(\boldsymbol{\theta})\|_{\mathcal{I}_t^{-1}(\boldsymbol{\theta})} = \|\mathcal{I}_t^{-1}(\boldsymbol{\theta})\nabla \ell_t(\boldsymbol{\theta})\|_{\mathcal{I}_t(\boldsymbol{\theta})} > \delta_t$. By combining the first and the third term, we obtain step (c). Step (d) follows directly from the Jensen's inequality i.e., $\nabla \ell_t(\boldsymbol{\theta})^\intercal (\tilde{\boldsymbol{y}}_t - \boldsymbol{y}) \leq \|\nabla \ell_t(\boldsymbol{\theta})\|_2 \|\tilde{\boldsymbol{y}} - \boldsymbol{y}\|_2$. In step (e), since $\lambda_{max}(\mathcal{J}_t(\boldsymbol{\theta})) \leq \bar{\lambda}_I$ (by Assumption 13(i)), it follows that $\lambda_{max}(\mathcal{I}_t(\boldsymbol{\theta})) \leq \bar{\lambda} t$. By applying the Rayleigh quotient rule, we establish step (e).

Summarizing the observations above, given the event $A = \{\|\nabla \ell_t(\boldsymbol{\theta})\|_2 \leq \kappa_1 t\}$ with $\kappa_1 > \frac{1}{2}\kappa_\theta \bar{\lambda}_I$, there exists a positive constant $\bar{L} = \kappa_1 + \frac{1}{2}\bar{\lambda}_I$ such that

$$
L(\tilde{\boldsymbol{y}}_t) - L(\boldsymbol{y}) \leq \bar{L}, \quad \forall \|\tilde{\boldsymbol{y}}_t - \boldsymbol{y}\|_2 \leq \frac{1}{t}. \qquad (3.178)
$$

We divide the set $\mathcal{Y}_\delta = \{\boldsymbol{y} : \|\boldsymbol{y}\|_2^2 \le \frac{\delta}{4\kappa_I^2}\}$ into finitely many balls with radius $\frac{1}{t}$. To cover $\mathcal{Y}_\delta$ with these balls, we can compute that the hypercube that covers the set $\mathcal{Y}_\delta$ has a length of $\frac{\sqrt{\delta}}{\kappa_I}$. As a result, after some arithmetic calculation, we can cover set $\mathcal{Y}_\delta$ by a total of $\bar{K}(\delta,t) = \left(\frac{\sqrt{\delta}t}{\kappa_I}\right)^{\kappa_\theta}$ balls. Let $\{B(\bar{\boldsymbol{y}}_k) : k = 1, \ldots, \bar{K}(\delta,t)\}$ be the set of the balls where $\bar{\boldsymbol{y}}_k$ is the center for the k^{th} ball for any $k \in \{1, \ldots, \bar{K}(\delta,t)\}$.

Step 5: conclude the claim on the upper bound. Combining (3.174), (3.176) and (3.178), we obtain that

$$
\begin{aligned}
&\mathbb{P}^\pi_{\boldsymbol{\theta}}\left\{Z_t^{\tilde{\boldsymbol{y}}_t} \ge \exp\left\{\frac{\delta_t^2}{2\zeta_1\nu}\right\}, \lambda_{min}(\mathcal{I}_t(\boldsymbol{\theta})) \ge \lambda_0\tau(t), A\right\} \\
&\quad \overset{(f)}{\le} \sum_{k=1}^{\bar{K}(\delta,t)} \mathbb{P}^\pi_{\boldsymbol{\theta}}\left\{Z_t^{\tilde{\boldsymbol{y}}_t} \ge \exp\left\{\frac{\delta_t^2}{2\zeta_1\nu}\right\}, \lambda_{min}(\mathcal{I}_t(\boldsymbol{\theta})) \ge \lambda_0\tau(t), A, \tilde{\boldsymbol{y}}_t \in B(\bar{\boldsymbol{y}}_k)\right\} \\
&\quad \overset{(g)}{\le} \sum_{k=1}^{\bar{K}(\delta,t)} \mathbb{P}^\pi_{\boldsymbol{\theta}}\left\{Z_t^{\bar{\boldsymbol{y}}_k} \ge \exp\left\{-\frac{\bar{L}}{\zeta_1\nu}\right\}\exp\left\{\frac{\delta_t^2}{4\zeta_1\nu}\right\}, \lambda_{min}(\mathcal{I}_t(\boldsymbol{\theta})) \ge \lambda_0\tau(t), A, \tilde{\boldsymbol{y}}_t \in B(\bar{\boldsymbol{y}}_k)\right\} \\
&\quad \le \sum_{k=1}^{\bar{K}(\delta,t)} \mathbb{P}^\pi_{\boldsymbol{\theta}}\left\{Z_t^{\bar{\boldsymbol{y}}_k} \ge \exp\left\{-\frac{\bar{L}}{\zeta_1\nu}\right\}\exp\left\{\frac{\delta_t^2}{4\zeta_1\nu}\right\}\right\} \\
&\quad \overset{(h)}{\le} \sum_{k=1}^{\bar{K}(\delta,t)} \mathbb{E}^\pi_{\boldsymbol{\theta}}\left[Z_t^{\bar{\boldsymbol{y}}_k}\right]\exp\left\{\frac{\bar{L}}{\zeta_1}\right\}\exp\left\{-\frac{\delta_t^2}{4\zeta_1\nu}\right\} \\
&\quad \overset{(i)}{\le} \bar{K}(\delta,t)\exp\left\{\frac{\bar{L}}{\zeta_1}\right\}\exp\left\{-\frac{\lambda_0}{16\kappa_I^2\zeta_1}\frac{\delta}{\max\{1,\sqrt{\delta}\}}\tau(t)\right\},
\end{aligned}
\tag{3.179}
$$

where step (f) follows from the union bound given the fact that $\{\|\tilde{\boldsymbol{y}}_t\|^2_{\mathcal{I}_t(\boldsymbol{\theta})} = \delta_t^2\} \subset \{\|\tilde{\boldsymbol{y}}_t\|_2^2 \le \frac{\delta}{4\kappa_I^2}\} \subset \cup_{k=1}^{K_n}\{\tilde{\boldsymbol{y}}_y \in B(\bar{\boldsymbol{y}}_k)\}$. In step (g), for any $\tilde{\boldsymbol{y}}_t \in B(\bar{\boldsymbol{y}}_k)$, we have $\|\tilde{\boldsymbol{y}}_t - \bar{\boldsymbol{y}}_k\|_2 \le \frac{1}{t}$, which implies that $L(\bar{\boldsymbol{y}}_k) \le L(\tilde{\boldsymbol{y}}_t) + \bar{L}$, or equivalently $Z_t^{\tilde{\boldsymbol{y}}_t} \ge \exp(-\frac{\bar{L}}{\zeta_1\nu})Z_t^{\bar{\boldsymbol{y}}_k}$ given the definition of $Z_t^{\boldsymbol{y}}$ in (3.173) for any $\boldsymbol{y}$. Step (h) follows directly from the Markov's inequality and the fact that $\nu = \max\{\sqrt{\delta}, 1\} \ge 1$. In step (i), by using the tower property to the supermartingale $Z_t^{\bar{\boldsymbol{y}}_k}$, we obtain that $\mathbb{E}^\pi_{\boldsymbol{\theta}}\left\{Z_t^{\bar{\boldsymbol{y}}_k}\right\} \le 1$. Combining with the facts that $\delta_t = \frac{1}{2\kappa_I}\sqrt{\lambda_0\tau(t)\delta}$ and $\nu = \max\{1, \delta\}$, we obtain the upper bound expression in step (i).

Based on (3.174), (3.179), and $\delta_t = \frac{1}{2\kappa_I}\sqrt{\lambda_0 \tau(t)\delta}$, we establish that

$$\begin{aligned}
&\mathbb{P}^{\pi}_{\boldsymbol{\theta}}\left\{\|\nabla \ell_t(\boldsymbol{\theta})\|_{\mathcal{I}_t^{-1}(\boldsymbol{\theta})} > \frac{1}{2\kappa_I}\sqrt{\lambda_0\tau(t)\delta}, \lambda_{min}(\mathcal{I}_t(\boldsymbol{\theta})) \geq \lambda_0\tau(t), A\right\} \\
&\quad = \mathbb{P}^{\pi}_{\boldsymbol{\theta}}\left\{\|\nabla \ell_t(\boldsymbol{\theta})\|_{\mathcal{I}_t^{-1}(\boldsymbol{\theta})} > \delta_t, \lambda_{min}(\mathcal{I}_t(\boldsymbol{\theta})) \geq \lambda_0\tau(t), A\right\} \\
&\quad \leq \bar{K}(\delta, t)\exp\left\{\frac{\bar{L}}{\zeta_1}\right\}\exp\left\{-\frac{\lambda_0}{16\kappa_I^2\zeta_1}\frac{\delta}{\max\{1,\sqrt{\delta}\}}\tau(t)\right\}.
\end{aligned} \tag{3.180}$$

Summarizing all of the observations above, we conclude that

$$\begin{aligned}
&\mathbb{P}^{\pi}_{\boldsymbol{\theta}}\left\{\|\tilde{\boldsymbol{\theta}}^u_{t+1} - \boldsymbol{\theta}\|_2^2 > \delta, \lambda_{min}(\mathcal{I}_t(\boldsymbol{\theta})) \geq \lambda_0\tau(t), A\right\} \\
&\overset{(j)}{\leq} \mathbb{P}^{\pi}_{\boldsymbol{\theta}}\left\{\|\tilde{\boldsymbol{\theta}}^u_{t+1} - \boldsymbol{\theta}\|_{\mathcal{I}_t(\boldsymbol{\theta})}^2 > \lambda_0\tau(t)\delta, \lambda_{min}(\mathcal{I}_t(\boldsymbol{\theta})) \geq \lambda_0\tau(t), A\right\} \\
&\leq \mathbb{P}^{\pi}_{\boldsymbol{\theta}}\left\{\|\tilde{\boldsymbol{\theta}}^u_{t+1} - \boldsymbol{\theta}\|_{\mathcal{I}_t(\boldsymbol{\theta})} > \sqrt{\lambda_0\tau(t)\delta}, \lambda_{min}(\mathcal{I}_t(\boldsymbol{\theta})) \geq \lambda_0\tau(t), A\right\} \\
&\overset{(k)}{\leq} \mathbb{P}^{\pi}_{\boldsymbol{\theta}}\left\{\|\nabla \ell_t(\boldsymbol{\theta})\|_{\mathcal{I}_t^{-1}(\boldsymbol{\theta})} > \frac{1}{2\kappa_I}\sqrt{\lambda_0\tau(t)\delta}, \lambda_{min}(\mathcal{I}_t(\boldsymbol{\theta})) \geq \lambda_0\tau(t), A\right\} \\
&\overset{(l)}{\leq} \bar{K}(\delta, t)\exp\left\{\frac{\bar{L}}{\zeta_1}\right\}\exp\left\{-\frac{\lambda_0}{16\kappa_I^2\zeta_1}\frac{\delta}{\max\{1,\sqrt{\delta}\}}\tau(t)\right\} \\
&\overset{(m)}{\leq} K_1' t^{\kappa_\theta}\delta^{\frac{\kappa_\theta}{2}}\exp\left\{-K_2'\frac{\delta}{\max\{1,\sqrt{\delta}\}}\tau(t)\right\}
\end{aligned} \tag{3.181}$$

where step (j) follows from $\lambda_{min}(\mathcal{I}_t(\boldsymbol{\theta})) \geq \lambda_0\tau(t)$. Step (k) follows directly from the inequality in (3.169). Step (l) follows from the inequality in (3.180). In step (m), we leverage the fact that $\bar{K}(\delta, t) = \left(\frac{\delta t}{\kappa_I}\right)^{\kappa_\theta}$. Step (m) follows from picking $K_1' = \left(\frac{1}{\kappa_I}\right)^{\kappa_\theta}$ and $K_2' = \frac{\lambda_0}{16\kappa_I^2\zeta_1}$. This concludes the proof of this claim. □

Proof of Lemma 20. To prove the claim, for any $T \geq 2$ and $t \in \{1, \ldots, T\}$, in the expression $\mathbb{P}^{\pi}_{\boldsymbol{\theta}}\{\|\tilde{\boldsymbol{\theta}}_{t+1} - \boldsymbol{\theta}\|_2 > \delta, \lambda_{min}(\mathcal{I}_t(\boldsymbol{\theta})) \geq \lambda_0\tau(t), A^c\}$, we abuse some notation by letting $g'_{kt}(\boldsymbol{\theta}) = (\nabla g_t(\boldsymbol{\theta}))_k$ for $k \in \{1, \ldots, \kappa_\theta\}$ where κ_θ is the dimension of the demand parameter vector $\boldsymbol{\theta}$ and $g_t(\boldsymbol{\theta}) = \ell_t(\boldsymbol{\theta}) - \ell_{t-1}(\boldsymbol{\theta})$. Given the positive constants κ_θ and $\bar{\lambda}_I$, we pick a positive constnt ζ_2 large enough such that $\zeta_2 \geq \exp\{\frac{\sqrt{\kappa_\theta}\bar{\lambda}_I}{\zeta_2}\}$. For any $y \in \{-1, 1\}$, we deduce that

$$\begin{aligned}
\mathbb{E}^{\pi}_{\boldsymbol{\theta}}\left[\exp\left\{\frac{y}{\zeta_2}g'_{kt}(\boldsymbol{\theta})\right\}\middle|\mathcal{F}_{t-1}\right] &\overset{(a)}{=} 1+\frac{y}{\zeta_2}\mathbb{E}^{\pi}_{\boldsymbol{\theta}}\left[g'_{kt}(\boldsymbol{\theta})\middle|\mathcal{F}_{t-1}\right]+\sum_{k=2}^{\infty}\frac{1}{k!}\mathbb{E}^{\pi}_{\boldsymbol{\theta}}\left[\left(\frac{y}{\zeta_2}g'_{kt}(\boldsymbol{\theta})\right)^k\middle|\mathcal{F}_{t-1}\right]\\
&= 1+\sum_{k=2}^{\infty}\frac{1}{k!}\mathbb{E}^{\pi}_{\boldsymbol{\theta}}\left[\left(\frac{y}{\zeta_2}g'_{kt}(\boldsymbol{\theta})\right)^k\middle|\mathcal{F}_{t-1}\right]\\
&\le 1+\sum_{k=2}^{\infty}\frac{1}{k!}\mathbb{E}^{\pi}_{\boldsymbol{\theta}}\left[\left(\frac{y}{\zeta_2}g'_{kt}(\boldsymbol{\theta})\right)^2\middle|\mathcal{F}_{t-1}\right]^{\frac{k}{2}}\\
&= 1+\mathbb{E}^{\pi}_{\boldsymbol{\theta}}\left[\left(\frac{y}{\zeta_2}g'_{kt}(\boldsymbol{\theta})\right)^2\middle|\mathcal{F}_{t-1}\right]\cdot\sum_{k=2}^{\infty}\frac{1}{k!}\mathbb{E}^{\pi}_{\boldsymbol{\theta}}\left[\left(\frac{y}{\zeta_2}g'_{kt}(\boldsymbol{\theta})\right)^2\middle|\mathcal{F}_{t-1}\right]^{\frac{k}{2}-1}\\
&\overset{(b)}{=} 1+\frac{y^2}{2\zeta_2^2}\mathbb{E}^{\pi}_{\boldsymbol{\theta}}\left[(g'_{kt}(\boldsymbol{\theta}))^2\middle|\mathcal{F}_{t-1}\right]\\
&\qquad\cdot\sum_{k=2}^{\infty}\frac{2}{k!}\left(\frac{y^2}{\zeta_2^2}\mathbb{E}^{\pi}_{\boldsymbol{\theta}}\left[(g'_{kt}(\boldsymbol{\theta}))^2\middle|\mathcal{F}_{t-1}\right]\right)^{\frac{k}{2}-1}\\
&\overset{(c)}{\le} 1+\frac{\kappa_\theta\bar{\lambda}_I y^2}{2\zeta_2^2}\cdot\sum_{k=2}^{\infty}\frac{1}{(k-2)!}\left(\frac{\sqrt{\kappa_\theta\bar{\lambda}_I}}{\zeta_2}\right)^{k-2}\\
&\overset{(d)}{\le} 1+\frac{\kappa_\theta\bar{\lambda}_I y^2}{2\zeta_2} \;\le\; \exp\left\{\frac{\kappa_\theta\bar{\lambda}_I}{2\zeta_2}y^2\right\},
\end{aligned}\tag{3.182}$$

where the derivations for steps (a) - (b) follow from the same arguments as step (n) - (s) in (3.175) of Lemma 19. In step (c), given that $\mathbb{E}^{\pi}_{\boldsymbol{\theta}}[(g'_{kt}(\boldsymbol{\theta}))^2|\mathcal{F}_{t-1}]$ is the k^{th} diagonal entry of matrix $\mathbb{E}^{\pi}_{\boldsymbol{\theta}}[\mathcal{J}_t(\boldsymbol{\theta})|\mathcal{F}_{t-1}]$ where $\mathcal{J}_t(\boldsymbol{\theta})=\mathcal{I}_t(\boldsymbol{\theta})-\mathcal{I}_{t-1}(\boldsymbol{\theta})$, from the fact that the trace of matrix $\mathbb{E}^{\pi}_{\boldsymbol{\theta}}[\mathcal{J}_t(\boldsymbol{\theta})|\mathcal{F}_{t-1}]$ is the sum of its eigenvalue upper bounded by the positive constants $\kappa_\theta\bar{\lambda}_I$, we obtain that $\mathbb{E}^{\pi}_{\boldsymbol{\theta}}[(g'_{kt}(\boldsymbol{\theta}))^2|\mathcal{F}_{t-1}]\le\kappa_\theta\bar{\lambda}_I$. This observation readily implies that step (c) holds. Step (d) follows immediately from the selection of ζ_2 which satisfies $\zeta_2\ge\exp\{\frac{\sqrt{\kappa_\theta\bar{\lambda}_I}}{\zeta_2}\}$. Based on (3.182), for any $y\in\{-1,1\}$, we establish that

$$\exp\left\{\frac{1}{\zeta_2}\left(\sum_{s=1}^{t}g'_{ks}(\boldsymbol{\theta})y-\frac{\kappa_\theta\bar{\lambda}_I t}{2}y^2\right)\right\}\;\le\;1.\tag{3.183}$$

Thus, for any $T \geq 2$ and $t \in \{1, \ldots, T\}$, we conclude the claim by showing that

$$
\begin{aligned}
&\mathbb{P}^{\pi}_{\boldsymbol{\theta}} \left\{ \|\tilde{\boldsymbol{\theta}}^{u}_{t+1} - \boldsymbol{\theta}\|_2^2 > \delta, \lambda_{min}(\mathcal{I}_t(\boldsymbol{\theta})) \geq \lambda_0 \tau(t), A^c \right\} \\
&\overset{(e)}{\leq} \mathbb{P}^{\pi}_{\boldsymbol{\theta}} \left\{ \|\nabla \ell_t(\boldsymbol{\theta})\|_1 \geq \kappa_1 t \right\} \\
&\overset{(f)}{\leq} \sum_{k=1}^{\kappa_\theta} \mathbb{P}^{\pi}_{\boldsymbol{\theta}} \left\{ \Big| \sum_{s=1}^{t} g'_{kt}(\boldsymbol{\theta}) \Big| \geq \kappa_1 t \right\} \\
&\overset{(g)}{\leq} \sum_{k=1}^{\kappa_\theta} \mathbb{P}^{\pi}_{\boldsymbol{\theta}} \left\{ \sum_{s=1}^{t} g'_{kt}(\boldsymbol{\theta}) \geq \kappa_1 t \right\} + \sum_{k=1}^{\kappa_\theta} \mathbb{P}^{\pi}_{\boldsymbol{\theta}} \left\{ - \sum_{s=1}^{t} g'_{kt}(\boldsymbol{\theta}) \geq \kappa_1 t \right\} \\
&\overset{(h)}{=} \sum_{k=1}^{\kappa_\theta} \mathbb{P}^{\pi}_{\boldsymbol{\theta}} \left\{ \exp\left\{ \frac{1}{\zeta_2} \left(\sum_{s=1}^{t} g'_{kt}(\boldsymbol{\theta}) - \frac{1}{2}\kappa_\theta \bar{\lambda}_I t \right) \right\} \geq \exp\left\{ \frac{\kappa_1 t - \frac{1}{2}\kappa_\theta \bar{\lambda}_I t}{\zeta_2} \right\} \right\} \\
&\quad + \sum_{k=1}^{\kappa_\theta} \mathbb{P}^{\pi}_{\boldsymbol{\theta}} \left\{ \exp\left\{ \frac{1}{\zeta_2} \left(- \sum_{s=1}^{t} g'_{kt}(\boldsymbol{\theta}) - \frac{1}{2}\kappa_\theta \bar{\lambda}_I t \right) \right\} \geq \exp\left\{ \frac{\kappa_1 t - \frac{1}{2}\kappa_\theta \bar{\lambda}_I t}{\zeta_2} \right\} \right\} \\
&\overset{(i)}{\leq} \sum_{k=1}^{\kappa_\theta} \exp\left\{ - \frac{\kappa_1 t - \frac{1}{2}\kappa_\theta \bar{\lambda}_I t}{\zeta_2} \right\} \mathbb{E}^{\pi}_{\boldsymbol{\theta}} \left[\exp\left\{ \frac{1}{\zeta_2} \left(\sum_{s=1}^{t} g'_{kt}(\boldsymbol{\theta}) - \frac{1}{2}\kappa_\theta \bar{\lambda}_I t \right) \right\} \right] \\
&\quad + \sum_{k=1}^{\kappa_\theta} \exp\left\{ - \frac{\kappa_1 t - \frac{1}{2}\kappa_\theta \bar{\lambda}_I t}{\zeta_2} \right\} \mathbb{E}^{\pi}_{\boldsymbol{\theta}} \left[\exp\left\{ \frac{1}{\zeta_2} \left(- \sum_{s=1}^{t} g'_{kt}(\boldsymbol{\theta}) - \frac{1}{2}\kappa_\theta \bar{\lambda}_I t \right) \right\} \right] \\
&\overset{(j)}{\leq} 2\kappa_\theta \exp\left\{ - \frac{\kappa_1 t - \frac{1}{2}\kappa_\theta \bar{\lambda}_I t}{\zeta_2} \right\} \overset{(k)}{\leq} K'_3 \exp\left\{ -K'_4 t \right\},
\end{aligned}
\tag{3.184}
$$

where step (e) follows readily from the observation that event A^c implies $\{\|\nabla \ell_t(\boldsymbol{\theta})\|_2 \geq \kappa_1 t\}$ and the fact that $\|\ell_t(\boldsymbol{\theta})\|_2 \leq \|\ell_t(\boldsymbol{\theta})\|_1$. In step (f), since $\nabla \ell_t(\boldsymbol{\theta}) = \sum_{s=1}^{t} \nabla g_s(\boldsymbol{\theta})$ (by Assumption 8) and the that event $\{\| \sum_{s=1}^{t} \nabla g_s(\boldsymbol{\theta})\|_1 \geq \kappa_1 t\}$ implies event $\cup_{k=1}^{\kappa_\theta} \{| \sum_{s=1}^{t} g'_{ks}(\boldsymbol{\theta})| \geq \kappa_1 t\}$, the inequality immediately follows. Step (g) follows directly from applying the union bound. In step (h), we leverage the facts that the events on the left hand side of the equation is equivalent to the right hand side. Step (i) follows from the Markov's inequality. In step (j), we implement the inequality in (3.183) where we respectively pick $y \in \{1, -1\}$. In step (k), we pick positive constants $K'_3 = 2\kappa_\theta$ and $K'_4 = \frac{\kappa_1 - \frac{1}{2}\kappa_\theta \bar{\lambda}_I}{\zeta_2}$, and the inequality readily follows. This concludes the proof of the claim. $\square$

Proof of Lemma 21. By the lemma statement, for any $(\tilde{\boldsymbol{p}}, \tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{\theta}}) \in [l, u]^n \times \Xi \times \Theta$ and $i \in$

$\{1, \ldots, n\}$, we consider the reward contract $\mathcal{W}_i^{\tilde{\boldsymbol{p}}, \tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{\theta}}}$ in Lemma 16, which has a L_w-Lipschitz sub(sup)-derivative. Moreover, recall that $Q_i(\boldsymbol{p}, \tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{\theta}})$ has a L_q-Lipschitz continuous gradient (by Assumption 7). we immediately obtain that there exists $L_{\mathcal{R}} > 0$ such that $\mathcal{R}(\tilde{\boldsymbol{p}}, \tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{\theta}})$ has $L_{\mathcal{R}}$-Lipschitz sup-gradients. Abusing some notation, we pick one sup-gradient, denoted as $\nabla_{\tilde{\boldsymbol{p}}} \mathcal{R}(\tilde{\boldsymbol{p}}, \tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{\theta}})$ in $\tilde{\boldsymbol{p}}$ for any $(\tilde{\boldsymbol{p}}, \tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{\theta}}) \in [l, u]^n \times \Xi \times \Theta$. Given the true demand parameter vector $\boldsymbol{\theta} \in \Theta$ and the optimal price mapping $\psi^\star$ in (3.14), for any $(\tilde{\boldsymbol{p}}, \tilde{\boldsymbol{\xi}}) \in [l, u]^n \times \Xi$, we can establish that

$$\mathcal{R}(\psi^\star(\tilde{\boldsymbol{\xi}}, \boldsymbol{\theta}), \tilde{\boldsymbol{\xi}}, \boldsymbol{\theta}) \;\leq\; \mathcal{R}(\tilde{\boldsymbol{p}}, \tilde{\boldsymbol{\xi}}, \boldsymbol{\theta}) + \nabla_{\tilde{\boldsymbol{p}}} \mathcal{R}(\tilde{\boldsymbol{p}}, \tilde{\boldsymbol{\xi}}, \boldsymbol{\theta})^\intercal (\psi^\star(\tilde{\boldsymbol{\xi}}, \boldsymbol{\theta}) - \tilde{\boldsymbol{p}}) + \frac{L_{\mathcal{R}}}{2} \|\psi^\star(\tilde{\boldsymbol{\xi}}, \boldsymbol{\theta}) - \tilde{\boldsymbol{p}}\|_2^2. \tag{3.185}$$

To proceed, we can further show that

$$\begin{aligned}
& \mathcal{R}(\psi^\star(\tilde{\boldsymbol{\xi}}, \boldsymbol{\theta}), \tilde{\boldsymbol{\xi}}, \boldsymbol{\theta}) - \mathcal{R}(\tilde{\boldsymbol{p}}, \tilde{\boldsymbol{\xi}}, \boldsymbol{\theta}) \\
\overset{(a)}{\leq}\; & \nabla_{\tilde{\boldsymbol{p}}} \mathcal{R}(\tilde{\boldsymbol{p}}, \tilde{\boldsymbol{\xi}}, \boldsymbol{\theta})^\intercal (\psi^\star(\tilde{\boldsymbol{\xi}}, \boldsymbol{\theta}) - \tilde{\boldsymbol{p}}) + \frac{L_{\mathcal{R}}}{2} \|\psi^\star(\tilde{\boldsymbol{\xi}}, \boldsymbol{\theta}) - \tilde{\boldsymbol{p}}\|_2^2 \\
\overset{(b)}{\leq}\; & [\nabla_{\tilde{\boldsymbol{p}}} \mathcal{R}(\tilde{\boldsymbol{p}}, \tilde{\boldsymbol{\xi}}, \boldsymbol{\theta}) - \nabla_{\tilde{\boldsymbol{p}}} \mathcal{R}(\psi^\star(\tilde{\boldsymbol{\xi}}, \boldsymbol{\theta}), \tilde{\boldsymbol{\xi}}, \boldsymbol{\theta})]^\intercal (\psi^\star(\tilde{\boldsymbol{\xi}}, \boldsymbol{\theta}) - \tilde{\boldsymbol{p}}) + \frac{L_{\mathcal{R}}}{2} \|\psi^\star(\tilde{\boldsymbol{\xi}}, \boldsymbol{\theta}) - \tilde{\boldsymbol{p}}\|_2^2 \\
\overset{(c)}{\leq}\; & \|\nabla_{\tilde{\boldsymbol{p}}} \mathcal{R}(\tilde{\boldsymbol{p}}, \tilde{\boldsymbol{\xi}}, \boldsymbol{\theta}) - \nabla_{\tilde{\boldsymbol{p}}} \mathcal{R}(\psi^\star(\tilde{\boldsymbol{\xi}}, \boldsymbol{\theta}), \tilde{\boldsymbol{\xi}}, \boldsymbol{\theta})\|_2 \|\psi^\star(\tilde{\boldsymbol{\xi}}, \boldsymbol{\theta}) - \boldsymbol{p})\|_2 + \frac{L_{\mathcal{R}}}{2} \|\psi^\star(\tilde{\boldsymbol{\xi}}, \boldsymbol{\theta}) - \tilde{\boldsymbol{p}}\|_2^2 \\
\overset{(d)}{\leq}\; & 3 L_{\mathcal{R}} \|\psi^\star(\tilde{\boldsymbol{\xi}}, \boldsymbol{\theta}) - \tilde{\boldsymbol{p}}\|_2^2,
\end{aligned} \tag{3.186}$$

where step (a) follows directly from (3.185) and the fact that $\mathcal{R}(\psi^\star(\tilde{\boldsymbol{\xi}}, \boldsymbol{\theta}), \tilde{\boldsymbol{\xi}}, \boldsymbol{\theta}) \geq \mathcal{R}(\tilde{\boldsymbol{p}}, \tilde{\boldsymbol{\xi}}, \boldsymbol{\theta})$ for any $(\tilde{\boldsymbol{p}}, \tilde{\boldsymbol{\xi}}) \in [l, u]^n \times \Xi$. In step (b), given the optimality of $\psi^\star(\tilde{\boldsymbol{\xi}}, \boldsymbol{\theta})$ in problem (3.14), the inequality from the observation that $\nabla_{\tilde{\boldsymbol{p}}} \mathcal{R}(\psi^\star(\tilde{\boldsymbol{\xi}}, \boldsymbol{\theta}), \tilde{\boldsymbol{\xi}}, \boldsymbol{\theta})^\intercal (\tilde{\boldsymbol{p}} - \psi^\star(\tilde{\boldsymbol{\xi}}, \boldsymbol{\theta})) \leq 0$ (the optimality condition). In step (c), we leverage Cauchy-Schwarz inequality to establish the inequality. Step (d) follows the selected gradient $\nabla_{\tilde{\boldsymbol{p}}} \mathcal{R}$ is $L_{\mathcal{R}}$-Lipschitz continuous, which readily implies that $\|\nabla_{\tilde{\boldsymbol{p}}} \mathcal{R}(\psi^\star(\tilde{\boldsymbol{\xi}}, \boldsymbol{\theta}), \tilde{\boldsymbol{\xi}}, \boldsymbol{\theta}) - \nabla_{\tilde{\boldsymbol{p}}} \mathcal{R}(\tilde{\boldsymbol{p}}, \tilde{\boldsymbol{\xi}}, \boldsymbol{\theta})\|_2 \leq 2 L_{\mathcal{R}} \|\psi^\star(\tilde{\boldsymbol{\xi}}, \boldsymbol{\theta}) - \tilde{\boldsymbol{p}}\|_2$. Together with the last term, the inequality follows.

For any $\tilde{\boldsymbol{\theta}} \in \Theta$, if we pick $\tilde{\boldsymbol{p}} = \psi^\star(\tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{\theta}})$, we can leverage the observation in (3.186) and the L_ψ-Lipschitz continuity of $\psi^\star(\tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{\theta}})$ (by Assumption 10) to establish that $\|\psi^\star(\tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{\theta}}) - \psi^\star(\tilde{\boldsymbol{\xi}}, \boldsymbol{\theta})\|_2 \leq L_\psi \|\tilde{\boldsymbol{\theta}} - \boldsymbol{\theta}\|_2$. As a result, there exists a positive constant $K_\psi = 3L_{\mathcal{R}} L_\psi^2$ such that

$$\Big|\mathcal{R}(\psi^\star(\tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{\theta}}), \tilde{\boldsymbol{\xi}}, \boldsymbol{\theta}) - \mathcal{R}(\psi^\star(\tilde{\boldsymbol{\xi}}, \boldsymbol{\theta}), \tilde{\boldsymbol{\xi}}, \boldsymbol{\theta})\Big| \ \leq \ 3L_{\mathcal{R}} \|\psi^\star(\tilde{\boldsymbol{\xi}}, \boldsymbol{\theta}) - \psi^\star(\tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{\theta}})\|_2^2 \ \leq \ K_\psi \|\tilde{\boldsymbol{\theta}} - \boldsymbol{\theta}\|_2^2. \tag{3.187}$$

This completes the proof of the lemma. □

Proof of Lemma 22. Proof of the probability bound in (3.70). We first look at the expectation bound in (3.70). For any $T \geq 2$, we let π be the SRI policy and $\{\boldsymbol{p}_s : s = 1, \ldots, T\}$ be the price process induced by the sellers under policy π. By the design of the SRI policy (see Algorithm 2), for any $t \in \{1, \ldots, \lceil \sqrt{T \log(T)} \rceil\}$ and $i \in \{1, \ldots, n\}$, the exploration profile $\boldsymbol{\mathcal{W}}_t^0$ induces a price profile $\boldsymbol{p}_t$ that satisfies $p_{it} = l + (u - l)X_{it}$ where $X_{it} \sim \textit{Bernoulli}(\frac{1}{2})$ are i.i.d across i, t. It is worth noting that the induced price process under the exploration reward contract follows the same distribution as that in the RI policy in the first $\lceil \sqrt{T \log(T)} \rceil$ periods. Thus, we can directly replicate the same arguments as in Proposition 22 to establish that there exists $K_2 > 0$ such that for any $T \geq 2$, $t \in \{1, \ldots, T\}$ and $\tau(t) = \min\{t, \sqrt{T \log(T)}\}$,

$$\mathbb{E}_{\boldsymbol{\theta}}^\pi \Big\{ \|\tilde{\boldsymbol{\theta}}_t - \boldsymbol{\theta}\|_2^2 \Big\} \ \leq \ K_2 \frac{\log(t)}{\tau(t)}. \tag{3.188}$$

.

Proof of Claim (i). For any $(\tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{\theta}}) \in \Xi \times \Theta$, we let $\Delta(\tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{\theta}}) = \mathcal{R}^\star(\tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{\theta}}) - \mathcal{R}(\bar{\boldsymbol{p}}^{\tilde{\boldsymbol{\theta}}}, \tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{\theta}})$. Our goal is to establish that $\Delta(\tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{\theta}})$ is continuous in $(\tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{\theta}}) \in \Xi \times \Theta$. Given the expression of $\mathcal{R}^\star(\tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{\theta}})$ in (3.16), since $Q_i(\tilde{\boldsymbol{p}}, \tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{\theta}})$ is continuous in the compact set $[l, u]^n \times \Xi \times \Theta$, we can leverage the maximum theorem to deduce that $\mathcal{R}^\star(\tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{\theta}})$ is continuous in $\tilde{\boldsymbol{\theta}} \in \Theta$. Next, we establish that $\mathcal{R}(\bar{\boldsymbol{p}}^{\tilde{\boldsymbol{\theta}}}, \tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{\theta}})$ is also continuous in $\tilde{\boldsymbol{\theta}} \in \Theta$. Fix any $\tilde{\boldsymbol{\theta}} \in \Theta$, it is sufficient to show that for any $\tilde{\boldsymbol{\theta}}_k \to \tilde{\boldsymbol{\theta}}$, we have $\bar{\boldsymbol{p}}^{\tilde{\boldsymbol{\theta}}_k} \to \bar{\boldsymbol{p}}^{\tilde{\boldsymbol{\theta}}}$. Suppose towards a contradiction that this is not the case. We can find subsequence $\{n_k : k = 1, 2, \dots\}$ such that $\bar{\boldsymbol{p}}^{\tilde{\boldsymbol{\theta}}_{n_k}} \to \bar{\boldsymbol{p}}$ where $\bar{\boldsymbol{p}} \neq \bar{\boldsymbol{p}}^{\tilde{\boldsymbol{\theta}}}$. From the expression in (3.5), we obtain that $R_{s_i}(\tilde{\boldsymbol{p}}, \tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{\theta}}) = \gamma \tilde{p}_i Q_i(\tilde{\boldsymbol{p}}, \tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{\theta}})$ is differentiable in $(\tilde{\boldsymbol{p}}, \tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{\theta}}) \in [l, u]^n \times \Xi \times \Theta$. We further let $\bar{\mathcal{R}}_{s_i}(\tilde{\boldsymbol{p}}, \tilde{\boldsymbol{\theta}}) = \mathbb{E}_{\boldsymbol{\xi},\boldsymbol{\varepsilon}}\{R_{s_i}(\tilde{\boldsymbol{p}}, \tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{\theta}})\}$ in which we recall that $\mathbb{E}_{\boldsymbol{\xi},\boldsymbol{\varepsilon}}\{\cdot\}$ is the expectation taken over the i.i.d feature vector $\tilde{\boldsymbol{\xi}}$. It readily follows that $\bar{\mathcal{R}}_{s_i}(\tilde{\boldsymbol{p}}, \tilde{\boldsymbol{\theta}})$ is differentiable in $(\tilde{\boldsymbol{p}}, \tilde{\boldsymbol{\theta}}) \in [l, u]^n \times \Theta$. We next consider the value function $W_i(\tilde{\boldsymbol{p}}_{-i}, \tilde{\boldsymbol{\theta}}) = \max_{p_i \in [l,u]} \bar{\mathcal{R}}_{s_i}(p_i, \tilde{\boldsymbol{p}}_{-i}, \tilde{\boldsymbol{\theta}})$ for any $(\tilde{\boldsymbol{p}}_{-i}, \tilde{\boldsymbol{\theta}}) \in [l, u]^{n-1} \times \Xi$ and $i \in \{1, \dots, n\}$. By the maximum theorem, $W_i(\tilde{\boldsymbol{p}}_{-i}, \tilde{\boldsymbol{\theta}})$ is continuous in $(\tilde{\boldsymbol{p}}_{-i}, \tilde{\boldsymbol{\theta}}) \in [l, u]^{n-1} \times \Theta$. Given that $\bar{\boldsymbol{p}}^{\tilde{\boldsymbol{\theta}}_{n_k}} \to \bar{\boldsymbol{p}}$, we obtain that $W_i(\bar{\boldsymbol{p}}_{-i}^{\tilde{\boldsymbol{\theta}}_{n_k}}, \tilde{\boldsymbol{\theta}}_{n_k}) \to W_i(\bar{\boldsymbol{p}}_{-i}, \boldsymbol{\theta})$ as $n_k \to \infty$ for any $i \in \{1, \dots, n\}$. Note that by definition, we have $W_i(\bar{\boldsymbol{p}}_{-i}^{\tilde{\boldsymbol{\theta}}_{n_k}}, \tilde{\boldsymbol{\theta}}_{n_k}) = \max_{p_i \in [l,u]} \bar{\mathcal{R}}_{s_i}(p_i, \bar{\boldsymbol{p}}_{-i}^{\tilde{\boldsymbol{\theta}}_{n_k}}, \tilde{\boldsymbol{\theta}}_{n_k}) = \bar{\mathcal{R}}_{s_i}(\bar{p}_i^{\tilde{\boldsymbol{\theta}}_{n_k}}, \bar{\boldsymbol{p}}_{-i}^{\tilde{\boldsymbol{\theta}}_{n_k}}, \tilde{\boldsymbol{\theta}}_{n_k})$, which by the continuity property, further implies that $W_i(\bar{\boldsymbol{p}}_{-i}, \tilde{\boldsymbol{\theta}}) = \max_{p_i \in [l,u]} \bar{\mathcal{R}}_{s_i}(p_i, \bar{\boldsymbol{p}}_{-i}, \tilde{\boldsymbol{\theta}}) = \bar{\mathcal{R}}_{s_i}(\bar{p}_i, \bar{\boldsymbol{p}}_{-i}, \tilde{\boldsymbol{\theta}})$ for any $i \in \{1, \dots, n\}$. Thus, $\bar{\boldsymbol{p}}$ is also a DN Nash equilibrium. This is a contradiction to the observation that $\bar{\boldsymbol{p}}^{\tilde{\boldsymbol{\theta}}}$ is the unique DN Nash equilibrium given $\tilde{\boldsymbol{\theta}}$. Thus, we conclude that $\bar{\boldsymbol{p}}^{\tilde{\boldsymbol{\theta}}_k} \to \bar{\boldsymbol{p}}^{\tilde{\boldsymbol{\theta}}}$ as $\tilde{\boldsymbol{\theta}}_k \to \tilde{\boldsymbol{\theta}}$ and moreover, $\mathcal{R}(\bar{\boldsymbol{p}}^{\tilde{\boldsymbol{\theta}}}, \tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{\theta}})$ is continuous in $\tilde{\boldsymbol{\theta}} \in \Theta$. Summarizing the observations above, we establish that $\Delta(\tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{\theta}})$ is continuous in $\tilde{\boldsymbol{\theta}} \in \Theta$.

To proceed, given that the demand parameter vector satisfies $\boldsymbol{\theta} \in \mathrm{int}(\tilde{\Theta})$, for any $\delta > 0$ that is small enough, we consider $B_\delta = \{\tilde{\boldsymbol{\theta}} : \|\tilde{\boldsymbol{\theta}} - \boldsymbol{\theta}\|_2^2 \le \delta\}$ where $B_\delta \subset \mathrm{int}(\tilde{\Theta})$, and B_δ^c be the complement of set B_δ. There exists $\varepsilon > 0$ such that $\mathbb{E}_{\boldsymbol{\xi},\boldsymbol{\varepsilon}}\{\Delta(\boldsymbol{\xi}, \boldsymbol{\theta})\} + \varepsilon = 0$. Since $\Delta(\tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{\theta}})$ is continuous in $\tilde{\boldsymbol{\theta}} \in \Theta$, we can find $\epsilon_0 \in (0, \frac{\varepsilon}{3})$ and $\delta_0 > 0$ such that for any $\tilde{\boldsymbol{\theta}} \in B_{\delta_0}$, we have $\left|\mathbb{E}_{\boldsymbol{\xi},\boldsymbol{\varepsilon}}\{\Delta(\boldsymbol{\xi}, \tilde{\boldsymbol{\theta}})\} + \epsilon\right| < \epsilon_0$. This allows us to establish that, for any $\tilde{\boldsymbol{\theta}} \in B_{\delta_0}$, we have $\mathbb{E}_{\boldsymbol{\xi},\boldsymbol{\varepsilon}}\{\Delta(\boldsymbol{\xi}, \tilde{\boldsymbol{\theta}})\} \le \varepsilon_0 - \varepsilon \le -\frac{2}{3}\varepsilon \le -\varepsilon_0$, which further suggests that

$$\left\{\tilde{\boldsymbol{\theta}} : \mathbb{E}_{\boldsymbol{\xi},\boldsymbol{\varepsilon}}\{\Delta(\boldsymbol{\xi}, \tilde{\boldsymbol{\theta}})\} \ge -\epsilon_0\right\} \subset \left\{\tilde{\boldsymbol{\theta}} : \tilde{\boldsymbol{\theta}} \in B_{\delta_0}^c\right\} = \left\{\tilde{\boldsymbol{\theta}} : \|\tilde{\boldsymbol{\theta}} - \boldsymbol{\theta}\|_2^2 \ge \delta_0\right\}. \tag{3.189}$$

Next, for simplicity of notation, we define the probability measure $d\mu_t(\tilde{\boldsymbol{\theta}}) = \mathbb{P}_{\boldsymbol{\theta}}^{\pi}(\tilde{\boldsymbol{\theta}}_t = \tilde{\boldsymbol{\theta}})$. Recall that $\{\Delta(\boldsymbol{\xi}_s, \tilde{\boldsymbol{\theta}}) : s = 1, \dots, t\}$ are i.i.d and that $\Delta(\tilde{\boldsymbol{\xi}}, \boldsymbol{\theta})$ is uniformly bounded by $2\bar{\mathcal{R}}$

for any $(\tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{\theta}}) \in \Xi \times \Theta$.

Under the SRI policy π, we develop an upper bound on $\mathbb{P}^{\pi}_{\boldsymbol{\theta}}\Big\{\|\tilde{\boldsymbol{\theta}}_{\lceil\sqrt{T\log(T)}\rceil+1} - \boldsymbol{\theta}\|_2^2 \geq \delta_0\Big\}$. For any simplicity of notation, we let $\tau_0 = \lceil\sqrt{T\log(T)}\rceil$. For any $T \geq 2$, given the maximum likelihood estimator $\tilde{\boldsymbol{\theta}}_{\tau_0+1}$, the unconstrained estimator $\tilde{\boldsymbol{\theta}}^u_{\tau_0+1}$ and event $A = \{\|\nabla\ell_{\tau_0}(\boldsymbol{\theta})\|_2 \leq \kappa_1\tau_0\}$ for some positive constant $\kappa_1 > \frac{1}{2}\kappa_\theta\bar{\lambda}_I$ where κ_θ is the dimension of the demand parameter vector $\boldsymbol{\theta}$ and $\bar{\lambda}_I$ is the positive constant in Assumption 13, we can establish that

$$\begin{aligned}
\mathbb{P}^{\pi}_{\boldsymbol{\theta}}\left\{\|\tilde{\boldsymbol{\theta}}_{\tau_0+1} - \boldsymbol{\theta}\|_2^2 > \delta_0\right\} &\overset{(a)}{\leq} \mathbb{P}^{\pi}_{\boldsymbol{\theta}}\Big\{\lambda_{min}(\mathcal{I}_{\tau_0}(\boldsymbol{\theta})) \leq \lambda_0\tau_0\Big\} \\
&\quad + \mathbb{P}^{\pi}_{\boldsymbol{\theta}}\left\{\|\tilde{\boldsymbol{\theta}}^u_{\tau_0+1} - \boldsymbol{\theta}\|_2^2 > \delta_0, \lambda_{min}(\mathcal{I}_{\tau_0}(\boldsymbol{\theta})) \geq \lambda_0\tau_0, A\right\} \\
&\quad + \mathbb{P}^{\pi}_{\boldsymbol{\theta}}\left\{\|\tilde{\boldsymbol{\theta}}^u_{\tau_0+1} - \boldsymbol{\theta}\|_2^2 > \delta_0, \lambda_{min}(\mathcal{I}_{\tau_0}(\boldsymbol{\theta})) \geq \lambda_0\tau_0, A^c\right\} \\
&\overset{(b)}{\leq} \frac{k_0}{\tau_0} \\
&\quad + K'_1(\tau_0)^{\kappa_\theta}\delta_0^{\frac{\kappa_\theta}{2}} \exp\left\{-K'_2\frac{\delta_0}{\max\{1,\sqrt{\delta_0}\}}\tau_0\right\} \\
&\quad + K'_3 \exp\Bigg\{-K'_4\tau_0\Bigg\} \\
&\overset{(c)}{\leq} \frac{M_2}{\tau_0}, \qquad (3.190)
\end{aligned}$$

where in step (a), we leverage the same arguments as in (3.63) of Proposition 22. Similarly, in step (b), we can again replicate the same proof arguments in (3.64) (by Assumption 13), in (3.65) (by Lemma 19), and in (3.66) (by Lemma 20) of Proposition 22 to establish the inequality in the right-hand-side expression. Step (c) follows from the observation that the highest order in the previous step is $\mathcal{O}(1/tau_0)$ where $\tau_0 = \lceil\sqrt{T\log(T)}\rceil$. Thus, we can pick a positive constant M_2 large enough such that the inequality in this step holds.

To proceed, given that $\tau_0 = \lceil\sqrt{T\log(T)}\rceil$, we complete the proof of this claim by showing that

$$\begin{aligned}
\mathbb{P}^{\pi}_{\boldsymbol{\theta}}\Big\{\tau=\tau_0+1\Big\} &\overset{(d)}{=} \int_{\tilde{\boldsymbol{\theta}}\in\Theta} \mathbb{P}^{\pi}_{\boldsymbol{\theta}}\left\{\frac{1}{\tau_0}\sum_{s=1}^{\tau_0}\Delta(\boldsymbol{\xi}_s,\tilde{\boldsymbol{\theta}})\geq 0\middle|\tilde{\boldsymbol{\theta}}_{\tau_0+1}=\tilde{\boldsymbol{\theta}}\right\}d\mu_{\tau_0+1}(\tilde{\boldsymbol{\theta}})\\
&\overset{(e)}{=} \int_{\tilde{\boldsymbol{\theta}}\in\Theta} \mathbb{P}^{\pi}_{\boldsymbol{\theta}}\Bigg\{\Big|\frac{1}{\tau_0}\sum_{s=1}^{\tau_0}\Delta(\boldsymbol{\xi}_s,\tilde{\boldsymbol{\theta}})-\mathbb{E}_{\boldsymbol{\xi},\boldsymbol{\varepsilon}}\{\Delta(\boldsymbol{\xi},\tilde{\boldsymbol{\theta}})\}\Big|\geq\epsilon_0,\\
&\qquad\qquad \frac{1}{\tau_0}\sum_{s=1}^{\tau_0}\Delta(\boldsymbol{\xi}_s,\tilde{\boldsymbol{\theta}})\geq 0\Big|\tilde{\boldsymbol{\theta}}_{\tau_0+1}=\tilde{\boldsymbol{\theta}}\Bigg\}d\mu_{\tau_0+1}(\tilde{\boldsymbol{\theta}})\\
&\quad + \int_{\tilde{\boldsymbol{\theta}}\in\Theta} \mathbb{P}^{\pi}_{\boldsymbol{\theta}}\Bigg\{\Big|\frac{1}{\tau_0}\sum_{s=1}^{\tau_0}\Delta(\boldsymbol{\xi}_s,\tilde{\boldsymbol{\theta}})-\mathbb{E}_{\boldsymbol{\xi},\boldsymbol{\varepsilon}}\{\Delta(\boldsymbol{\xi},\tilde{\boldsymbol{\theta}})\}\Big|\leq\epsilon_0,\\
&\qquad\qquad \frac{1}{\tau_0}\sum_{s=1}^{\tau_0}\Delta(\boldsymbol{\xi}_s,\tilde{\boldsymbol{\theta}})\geq 0\Big|\tilde{\boldsymbol{\theta}}_{\tau_0+1}=\tilde{\boldsymbol{\theta}}\Bigg\}d\mu_{\tau_0+1}(\tilde{\boldsymbol{\theta}})\\
&\overset{(f)}{\leq} \int_{\tilde{\boldsymbol{\theta}}\in\Theta} \mathbb{P}^{\pi}_{\boldsymbol{\theta}}\left\{\Big|\frac{1}{\tau_0}\sum_{s=1}^{\tau_0}\Delta(\boldsymbol{\xi}_s,\tilde{\boldsymbol{\theta}})-\mathbb{E}_{\boldsymbol{\xi},\boldsymbol{\varepsilon}}\{\Delta(\boldsymbol{\xi},\tilde{\boldsymbol{\theta}})\}\Big|\geq\epsilon_0\middle|\tilde{\boldsymbol{\theta}}_{\tau_0+1}=\tilde{\boldsymbol{\theta}}\right\}d\mu_{\tau_0+1}(\tilde{\boldsymbol{\theta}})\\
&\qquad\qquad + \int_{\tilde{\boldsymbol{\theta}}\in\Theta} \mathbb{P}^{\pi}_{\boldsymbol{\theta}}\left\{\mathbb{E}_{\boldsymbol{\xi},\boldsymbol{\varepsilon}}\{\Delta(\boldsymbol{\xi},\tilde{\boldsymbol{\theta}})\}\geq-\epsilon_0\middle|\tilde{\boldsymbol{\theta}}_{\tau_0+1}=\tilde{\boldsymbol{\theta}}\right\}d\mu_{\tau_0+1}(\tilde{\boldsymbol{\theta}})\\
&\overset{(g)}{\leq} \int_{\tilde{\boldsymbol{\theta}}\in\Theta}\frac{4\bar{\mathcal{R}}^2}{\tau_0\epsilon_0^2}d\mu_{\tau_0+1}(\tilde{\boldsymbol{\theta}}) + \int_{\tilde{\boldsymbol{\theta}}\in\Theta}\mathbb{P}^{\pi}_{\boldsymbol{\theta}}\left\{\|\tilde{\boldsymbol{\theta}}-\boldsymbol{\theta}\|_2^2\geq\delta_0\middle|\tilde{\boldsymbol{\theta}}_{\tau_0+1}=\tilde{\boldsymbol{\theta}}\right\}d\mu_{\tau_0+1}(\tilde{\boldsymbol{\theta}})\\
&\overset{(h)}{\leq} \frac{4\bar{\mathcal{R}}^2}{\tau_0\epsilon_0^2}+\mathbb{P}^{\pi}_{\boldsymbol{\theta}}\Big\{\|\tilde{\boldsymbol{\theta}}_{\tau_0+1}-\boldsymbol{\theta}\|_2^2\geq\delta_0\Big\}\\
&\overset{(i)}{\leq} \frac{4\bar{\mathcal{R}}^2}{\tau_0\epsilon_0^2}+\frac{M_2}{\tau_0}\\
&\overset{(j)}{\leq} \frac{\bar{K}_0}{\tau_0}, \qquad (3.191)
\end{aligned}$$

where step (d) follows directly from the stopping rule on τ under the SRI policy in period τ_0+1 (see Algorithm 2). In step (e), we decompose the event on the left hand side of the step into two on the right hand side. Step (f) follows from the following two observations: (1) event $\Big\{\ \Big|\frac{1}{\tau_0}\sum_{s=1}^{\tau_0}\Delta(\boldsymbol{\xi}_s,\tilde{\boldsymbol{\theta}})-\mathbb{E}_{\boldsymbol{\xi},\boldsymbol{\varepsilon}}\{\Delta(\boldsymbol{\xi},\tilde{\boldsymbol{\theta}})\}\Big|\geq\epsilon_0,\ \frac{1}{\tau_0}\sum_{s=1}^{\tau_0}\Delta(\boldsymbol{\xi}_s,\tilde{\boldsymbol{\theta}})\geq 0\Big\}$ implies event $\Big\{\Big|\frac{1}{\tau_0}\sum_{s=1}^{\tau_0}\Delta(\boldsymbol{\xi}_s,\tilde{\boldsymbol{\theta}})-\mathbb{E}_{\boldsymbol{\xi},\boldsymbol{\varepsilon}}\{\Delta(\boldsymbol{\xi},\tilde{\boldsymbol{\theta}})\}\Big|\geq\epsilon_0\Big\}$, and (2) event $\Big\{\Big|\frac{1}{\tau_0}\sum_{s=1}^{\tau_0}\Delta(\boldsymbol{\xi}_s,\tilde{\boldsymbol{\theta}})-\mathbb{E}_{\boldsymbol{\xi},\boldsymbol{\varepsilon}}\{\Delta(\boldsymbol{\xi},\tilde{\boldsymbol{\theta}})\}\Big|\leq\epsilon_0,\ \frac{1}{\tau_0}\sum_{s=1}^{\tau_0}\Delta(\boldsymbol{\xi}_s,\tilde{\boldsymbol{\theta}})\geq 0\Big\}$ implies event $\Big\{\mathbb{E}_{\boldsymbol{\xi},\boldsymbol{\varepsilon}}\{\Delta(\boldsymbol{\xi},\tilde{\boldsymbol{\theta}})\}\geq-\epsilon_0\Big\}$. In step (g), the first term follows directly from applying Chebyshev's inequality on the random variable $\frac{1}{\tau_0}\sum_{s=1}^{\tau_0}\Delta(\boldsymbol{\xi}_s,\tilde{\boldsymbol{\theta}})$. In the second term, we leverage the observation in (3.189). Step (h) follows from direct integration. In step (i), we leverage the inequality in (3.190) to establish the right-hand-side expression. In step (j), we pick a positive constant $\bar{K}_0 > \frac{4\bar{\mathcal{R}}^2}{\epsilon_0^2} + M_2$ such that the inequality in this step follows.

With $\tau_0 = \lceil \sqrt{T \log(T)} \rceil$ and (3.191), this completes the proof of Claim (i).

To prove Claim (ii), we can replicate the same arguments as in Claim (i). For any $\delta > 0$, we consider $B_\delta = \{\tilde{\boldsymbol{\theta}} : \|\tilde{\boldsymbol{\theta}} - \boldsymbol{\theta}\|_2^2 \leq \delta\}$ where $B_\delta \subset \mathrm{int}(\tilde{\Theta}^c)$, and let B_δ^c be the complement of set B_δ. Given that the demand parameter vector satisfies $\boldsymbol{\theta} \in \mathrm{int}(\tilde{\Theta}^c)$, there exists $\varepsilon > 0$ such that $\mathbb{E}_{\boldsymbol{\xi},\boldsymbol{\varepsilon}}\{\Delta(\boldsymbol{\xi}, \boldsymbol{\theta})\} - \varepsilon = 0$. Since $\Delta(\tilde{\boldsymbol{\xi}}, \tilde{\boldsymbol{\theta}})$ is continuous in $\tilde{\boldsymbol{\theta}} \in \Theta$, we can find $\epsilon_0 \in (0, \frac{\varepsilon}{3})$ and $\delta_0 > 0$ such that for any $\tilde{\boldsymbol{\theta}} \in B_{\delta_0}$, we have $\left|\mathbb{E}_{\boldsymbol{\xi},\boldsymbol{\varepsilon}}\{\Delta(\boldsymbol{\xi}, \tilde{\boldsymbol{\theta}})\} - \epsilon\right| < \epsilon_0$. This allows us to establish that, for any $\tilde{\boldsymbol{\theta}} \in B_{\delta_0}$, we have $\mathbb{E}_{\boldsymbol{\xi},\boldsymbol{\varepsilon}}\{\Delta(\boldsymbol{\xi}, \tilde{\boldsymbol{\theta}})\} \geq -\varepsilon_0 + \varepsilon \geq \frac{2}{3}\varepsilon \geq \varepsilon_0$, which further suggests that

$$\left\{\tilde{\boldsymbol{\theta}} : \mathbb{E}_{\boldsymbol{\xi},\boldsymbol{\varepsilon}}\{\Delta(\boldsymbol{\xi}, \tilde{\boldsymbol{\theta}})\} \leq \epsilon_0\right\} \subset \left\{\tilde{\boldsymbol{\theta}} : \tilde{\boldsymbol{\theta}} \in B_{\delta_0}^c\right\} = \left\{\tilde{\boldsymbol{\theta}} : \|\tilde{\boldsymbol{\theta}} - \boldsymbol{\theta}\|_2^2 \geq \delta_0\right\}. \tag{3.192}$$

We again let $\tau_0 = \lceil \sqrt{T \log(T)} \rceil$ for simplicity of notation. By using the same probability upper bound on $\mathbb{P}_{\boldsymbol{\theta}}^{\pi}\{\|\tilde{\boldsymbol{\theta}}_{\tau_0+1} - \boldsymbol{\theta}\|_2^2 > \delta_0\}$ as in (3.190), we repeat the same arguments as in

(3.191) to establish that there exists $\bar{K}_1 > 0$ such that for any $T \geq 2$,

$$\begin{aligned}
\mathbb{P}^{\pi}_{\boldsymbol{\theta}}\Big\{\tau = \infty\Big\} &\overset{(k)}{=} \int_{\tilde{\boldsymbol{\theta}} \in \Theta} \mathbb{P}^{\pi}_{\boldsymbol{\theta}} \left\{ \frac{1}{\tau_0} \sum_{s=1}^{\tau_0} \Delta(\boldsymbol{\xi}_s, \tilde{\boldsymbol{\theta}}) \leq 0 \middle| \tilde{\boldsymbol{\theta}}_{\tau_0+1} = \tilde{\boldsymbol{\theta}} \right\} d\mu_{\tau_0+1}(\tilde{\boldsymbol{\theta}}) \\
&\overset{(l)}{=} \int_{\tilde{\boldsymbol{\theta}} \in \Theta} \mathbb{P}^{\pi}_{\boldsymbol{\theta}} \Bigg\{ \Big| \frac{1}{\tau_0} \sum_{s=1}^{\tau_0} \Delta(\boldsymbol{\xi}_s, \tilde{\boldsymbol{\theta}}) - \mathbb{E}_{\boldsymbol{\xi},\boldsymbol{\varepsilon}}\{\Delta(\boldsymbol{\xi}, \tilde{\boldsymbol{\theta}})\} \Big| \geq \epsilon_0, \\
&\qquad \frac{1}{\tau_0} \sum_{s=1}^{\tau_0} \Delta(\boldsymbol{\xi}_s, \tilde{\boldsymbol{\theta}}) \leq 0 \Bigg| \tilde{\boldsymbol{\theta}}_{\tau_0+1} = \tilde{\boldsymbol{\theta}} \Bigg\} d\mu_{\tau_0+1}(\tilde{\boldsymbol{\theta}}) \\
&\quad + \int_{\tilde{\boldsymbol{\theta}} \in \Theta} \mathbb{P}^{\pi}_{\boldsymbol{\theta}} \Bigg\{ \Big| \frac{1}{\tau_0} \sum_{s=1}^{\tau_0} \Delta(\boldsymbol{\xi}_s, \tilde{\boldsymbol{\theta}}) - \mathbb{E}_{\boldsymbol{\xi},\boldsymbol{\varepsilon}}\{\Delta(\boldsymbol{\xi}, \tilde{\boldsymbol{\theta}})\} \Big| \leq \epsilon_0, \\
&\qquad \frac{1}{\tau_0} \sum_{s=1}^{\tau_0} \Delta(\boldsymbol{\xi}_s, \tilde{\boldsymbol{\theta}}) \leq 0 \Bigg| \tilde{\boldsymbol{\theta}}_{\tau_0+1} = \tilde{\boldsymbol{\theta}} \Bigg\} d\mu_{\tau_0+1}(\tilde{\boldsymbol{\theta}}) \\
&\overset{(m)}{\leq} \int_{\tilde{\boldsymbol{\theta}} \in \Theta} \mathbb{P}^{\pi}_{\boldsymbol{\theta}} \left\{ \Big| \frac{1}{\tau_0} \sum_{s=1}^{\tau_0} \Delta(\boldsymbol{\xi}_s, \tilde{\boldsymbol{\theta}}) - \mathbb{E}_{\boldsymbol{\xi},\boldsymbol{\varepsilon}}\{\Delta(\boldsymbol{\xi}, \tilde{\boldsymbol{\theta}})\} \Big| \geq \epsilon_0 \middle| \tilde{\boldsymbol{\theta}}_{\tau_0+1} = \tilde{\boldsymbol{\theta}} \right\} d\mu_{\tau_0+1}(\tilde{\boldsymbol{\theta}}) \\
&\qquad + \int_{\tilde{\boldsymbol{\theta}} \in \Theta} \mathbb{P}^{\pi}_{\boldsymbol{\theta}} \left\{ \mathbb{E}_{\boldsymbol{\xi},\boldsymbol{\varepsilon}}\{\Delta(\boldsymbol{\xi}, \tilde{\boldsymbol{\theta}})\} \leq \epsilon_0 \middle| \tilde{\boldsymbol{\theta}}_{\tau_0+1} = \tilde{\boldsymbol{\theta}} \right\} d\mu_{\tau_0+1}(\tilde{\boldsymbol{\theta}}) \\
&\overset{(n)}{\leq} \int_{\tilde{\boldsymbol{\theta}} \in \Theta} \frac{4\bar{\mathcal{R}}^2}{\tau_0 \epsilon_0^2} d\mu_{\tau_0+1}(\tilde{\boldsymbol{\theta}}) + \int_{\tilde{\boldsymbol{\theta}} \in \Theta} \mathbb{P}^{\pi}_{\boldsymbol{\theta}} \left\{ \|\tilde{\boldsymbol{\theta}} - \boldsymbol{\theta}\|_2^2 \geq \delta_0 \middle| \tilde{\boldsymbol{\theta}}_{\tau_0+1} = \tilde{\boldsymbol{\theta}} \right\} d\mu_{\tau_0+1}(\tilde{\boldsymbol{\theta}}) \\
&\overset{(o)}{\leq} \frac{4\bar{\mathcal{R}}^2}{\tau_0 \epsilon_0^2} + \mathbb{P}^{\pi}_{\boldsymbol{\theta}}\Big\{ \|\tilde{\boldsymbol{\theta}}_{\tau_0+1} - \boldsymbol{\theta}\|_2^2 \geq \delta_0 \Big\} \\
&\overset{(p)}{\leq} \frac{\bar{K}_1}{\tau_0},
\end{aligned} \tag{3.193}$$

where step (k) - (l) follow from the same arguments as in step (d) - (e) in (3.191). Similar to step (f) of (3.191), step (m) follows from the following two observations: (1) event $\Big\{ \Big|\frac{1}{\tau_0} \sum_{s=1}^{\tau_0} \Delta(\boldsymbol{\xi}_s, \tilde{\boldsymbol{\theta}}) - \mathbb{E}_{\boldsymbol{\xi},\boldsymbol{\varepsilon}}\{\Delta(\boldsymbol{\xi}, \tilde{\boldsymbol{\theta}})\}\Big| \geq \epsilon_0,\ \frac{1}{\tau_0} \sum_{s=1}^{\tau_0} \Delta(\boldsymbol{\xi}_s, \tilde{\boldsymbol{\theta}}) < 0 \Big\}$ implies event $\Big\{\Big|\frac{1}{\tau_0} \sum_{s=1}^{\tau_0} \Delta(\boldsymbol{\xi}_s, \tilde{\boldsymbol{\theta}}) - \mathbb{E}_{\boldsymbol{\xi},\boldsymbol{\varepsilon}}\{\Delta(\boldsymbol{\xi}, \tilde{\boldsymbol{\theta}})\}\Big| \geq \epsilon_0\Big\}$, and (2) event $\Big\{\Big|\frac{1}{\tau_0} \sum_{s=1}^{\tau_0} \Delta(\boldsymbol{\xi}_s, \tilde{\boldsymbol{\theta}}) - \mathbb{E}_{\boldsymbol{\xi},\boldsymbol{\varepsilon}}\{\Delta(\boldsymbol{\xi}, \tilde{\boldsymbol{\theta}})\}\Big| \leq \epsilon_0,\ \frac{1}{\tau_0} \sum_{s=1}^{\tau_0} \Delta(\boldsymbol{\xi}_s, \tilde{\boldsymbol{\theta}}) < 0\Big\}$ implies event $\Big\{\mathbb{E}_{\boldsymbol{\xi},\boldsymbol{\varepsilon}}\{\Delta(\boldsymbol{\xi}, \tilde{\boldsymbol{\theta}})\} \leq \epsilon_0\Big\}$. In step (n), the first term follows directly from applying Chebyshev's inequality on the random variable $\frac{1}{\tau_0} \sum_{s=1}^{\tau_0} \Delta(\boldsymbol{\xi}_s, \tilde{\boldsymbol{\theta}})$. The second term follows directly from the observation in (3.192). Step (n) follows from direct integration. In step (o), we leverage the inequality in (3.190) to establish the right-hand-side expression. In step (p), we pick a positive constant $\bar{K}_1 > \frac{4\bar{\mathcal{R}}^2}{\epsilon_0^2} + M_2$ such that the inequality in this step follows.

With $\tau_0 = \lceil \sqrt{T \log(T)} \rceil$, this completes the proof of Claim (ii). $\square$

Proof of Lemma 23. To prove the claim, we deduce that for all $p, v \in \mathbb{R}$ and $i \in \{1, \ldots, n\}$,

$$\begin{aligned} F_i(p,v) &= h_i(p) + h_i^\star(v) - vp \\ &\overset{(a)}{\leq} h_i(p) + h_i^\star(v') - vp + (v-v')[h_i^\star]'(v') + \frac{1}{2l_h}(v-v')^2 \\ &\overset{(b)}{=} h_i(p) + h_i^\star(v') - v'p - (v-v')p + (v-v')[h_i^\star]'(v') + \frac{1}{2l_h}(v-v')^2 \\ &\overset{(c)}{=} F_i(p,v') + (v-v')([h_i^\star]'(v') - p) + \frac{1}{2l_h}(v-v')^2, \end{aligned} \tag{3.194}$$

where step (a) follows from inequality $h_i^\star(v) \leq h_i^\star(v') + (v-v')[h_i^\star]'(v') + \frac{1}{2l_h}(v-v')^2$ induced from $\frac{1}{l_h}$-strongly smoothness of the Fenchel conjugate $h_i^\star(v)$ for all $v \in \mathbb{R}$. In step (b), the equality follows readily from subtracting and adding the same term $v'p$ to both sides of the expressions. In step (c), we apply the definition $F_i(p,v') = h_i(p) + h_i^\star(v') - v'p$ in (3.125) to simplify the expression. $\square$

Proof of Lemma 24. Proof of Claim (i). By definition of $F_i(p,v)$ in (3.125), it is sufficient to show $h_i^\star(h_i'(p_{i(t+1)})) - \bar{p}_i^{\boldsymbol{\theta}} h_i'(p_{i(t+1)}) \leq h_i^\star(\tilde{y}_{i(t+1)}) - \bar{p}_i^{\boldsymbol{\theta}} \tilde{y}_{i(t+1)}$. Note that we have

$$\begin{aligned} h_i^\star(\tilde{y}_{i(t+1)}) - h_i^\star(h_i'(p_{i(t+1)})) &\overset{(a)}{\geq} [h_i^\star]'(h_i'(p_{i(t+1)}))(\tilde{y}_{i(t+1)} - h_i'(p_{i(t+1)})) \\ &\overset{(b)}{=} p_{i(t+1)}(\tilde{y}_{i(t+1)} - h_i'(p_{i(t+1)})), \end{aligned} \tag{3.195}$$

where (a) follows from the convexity of the Fenchel conjugate $h_i^\star(v)$ for all $v \in \mathbb{R}$. Step (b) follows readily from the fact that $[h_i^\star]'(h_i'(p_{i(t+1)})) = p_{i(t+1)}$.

For any $i \in \{1, \ldots, n\}$, there are three possibilities for $(y_{i(t+1)}, h_i'(p_{i(t+1)}))$ given the location of $p_{i(t+1)}$ in the interval $[l, u]$: (1) if $p_{i(t+1)} \in (l, u)$, then we have $y_{i(t+1)} = h_i'(p_{i(t+1)})$ by the first order condition at the interior point in (3.124); (2) if $p_{i(t+1)} = u$, then $y_{i(t+1)} \geq$

$h_i'(p_{i(t+1)})$; (3) if $p_{i(t+1)} = l$, then $y_{t+1} \leq h_i'(p_{i(t+1)})$. Given that $\bar{\boldsymbol{p}}^{\boldsymbol{\theta}}$ is in the interior of $[l,u]^n$ (by Assumption 11), in all three scenarios, we have

$$p_{i(t+1)}(\tilde{y}_{i(t+1)} - h_i'(p_{i(t+1)})) \geq \bar{p}_i^{\boldsymbol{\theta}}(\tilde{y}_{i(t+1)} - h_i'(p_{i(t+1)})). \tag{3.196}$$

Summarizing (3.195) and (3.196), we have $h_i^\star(\tilde{y}_{i(t+1)}) - h_i^\star(h_i'(p_{i(t+1)})) \geq \bar{p}_i^{\boldsymbol{\theta}}(\tilde{y}_{i(t+1)} - h_i'(p_{i(t+1)}))$, which from the definition of $F_i(p,v)$ in (3.125), immediately implies that

$$F_i(\bar{p}_i^{\boldsymbol{\theta}}, h_i'(p_{i(t+1)})) \ \leq \ F_i(\bar{p}_i^{\boldsymbol{\theta}}, \tilde{y}_{i(t+1)}). \tag{3.197}$$

This completes the proof of Claim (i).

<u>Proof of Claim (ii).</u> Given the platform's DN policy π, the expectation operator $\mathbb{E}_{\boldsymbol{\xi},\boldsymbol{\varepsilon}}\{\cdot\}$ can be replaced with $\mathbb{E}^\pi_{\boldsymbol{\theta}}\{\cdot\}$. To prove this claim, we first establish the following inequality

$$\begin{aligned}
\mathbb{E}^\pi_{\boldsymbol{\theta}}\left\{F_i(\bar{p}_i^{\boldsymbol{\theta}}, h_i'(p_{i(t+1)}))\middle|\mathcal{F}^0_{t-1}\right\} &\overset{(c)}{\leq} \mathbb{E}^\pi_{\boldsymbol{\theta}}\left\{F_i(\bar{p}_i^{\boldsymbol{\theta}}, \tilde{y}_{i(t+1)})\middle|\mathcal{F}^0_{t-1}\right\} \\
&\overset{(d)}{\leq} \mathbb{E}^\pi_{\boldsymbol{\theta}}\left\{F_i(\bar{p}_i^{\boldsymbol{\theta}}, h_i'(p_{it}))|\mathcal{F}^0_{t-1}\right\} \\
&\quad + \mathbb{E}^\pi_{\boldsymbol{\theta}}\left\{\left(\tilde{y}_{i(t+1)} - h_i'(p_{it})\right)(p_{it} - \bar{p}_i^{\boldsymbol{\theta}})\middle|\mathcal{F}^0_{t-1}\right\} \\
&\quad + \mathbb{E}^\pi_{\boldsymbol{\theta}}\left\{\frac{1}{2l_h}(\tilde{y}_{i(t+1)} - h_i'(p_{it})^2\middle|\mathcal{F}^0_{t}\right\}, \\
&\overset{(e)}{=} F_i(\bar{p}_i^{\boldsymbol{\theta}}, h_i'(p_{it})) + \eta_t\bar{g}_{it}(p_{it} - \bar{p}_i^{\boldsymbol{\theta}}) + \frac{1}{2l_h}\mathbb{E}^\pi_{\boldsymbol{\theta}}\left\{\eta_t^2\tilde{g}_{it}^2\middle|\mathcal{F}^0_{t-1}\right\},
\end{aligned} \tag{3.198}$$

where step (c) follows directly from the inequality in (3.197). In step (d), we implement inequality (3.194) (from Lemma 23). In step (e), conditional on $\mathcal{F}^0_{t-1}$, $\boldsymbol{p}_t$ is known. By implementing the equation that $\tilde{y}_{i(t+1)} = h_i'(p_{it}) + \eta_t\tilde{g}_{it}$ (from (3.124)) and the fact that $\bar{g}_{it} = \mathbb{E}^\pi_{\boldsymbol{\theta}}\{\tilde{g}_{it}|\mathcal{F}^0_{t-1}\}$, we obtain the right-hand-side expression in step (e).

This completes the proof of Claim (ii). □

Proof of Lemma 25. Let π be the platform's DN policy. For simplicity of notation, we define $\mathbb{P}^{\pi,0}_{\boldsymbol{\theta}}\{\cdot\} = \mathbb{P}^{\pi}_{\boldsymbol{\theta}}\{\cdot|\boldsymbol{Q}(\cdot,\boldsymbol{\theta}) = E_{\boldsymbol{\xi},\boldsymbol{\varepsilon}}[\boldsymbol{Q}(\cdot,\tilde{\boldsymbol{\xi}},\boldsymbol{\theta}^{(0)})]\}$. We complete the proof in three steps.

<u>Step 1: Derive a multi-period belief updating inequality.</u> Let $k \in \{1,\dots,K\}$ and $t \in \{1,2,\dots\}$. Thus, under $\boldsymbol{Q}^{(0)}(\boldsymbol{p},\boldsymbol{\theta}^{(0)}) = \mathbb{E}_{\boldsymbol{\xi},\boldsymbol{\varepsilon}}\{\boldsymbol{Q}(\boldsymbol{p},\tilde{\boldsymbol{\xi}},\boldsymbol{\theta})\}$, we have for all $s \in \{1,2,\dots\}$, and

$$\begin{aligned}
b^{(k)}_{i(t+1)} &= \frac{b^{(k)}_{i1}\prod_{s=1}^{t}\mathcal{L}^{(k)}_{is}(D_{is},Q^{(k)}_i(\boldsymbol{p}_s,\boldsymbol{\theta}^{(k)}))|\mathcal{F}^0_{s-1})}{\sum_{k'=0}^{K} b^{(k')}_{i1}\prod_{s=1}^{t}\mathcal{L}^{(k')}_{is}(D_{is},Q^{(k')}_i(\boldsymbol{p}_s,\boldsymbol{\theta}^{(k')}))|\mathcal{F}^0_{s-1})}\\
&= \frac{1}{1+\sum_{k'\neq k}\frac{b^{(k')}_{i1}}{b^{(k)}_{i1}}\exp\Big(\sum_{s=1}^{t}\log\frac{\mathcal{L}^{(k')}_{is}(D_{is},Q^{(k')}_i(\boldsymbol{p}_s,\boldsymbol{\theta}^{(k')})|\mathcal{F}^0_{s-1})}{\mathcal{L}^{(k)}_{is}(D_{is},Q^{(k)}_i(\boldsymbol{p}_s,\boldsymbol{\theta}^{(k)})|\mathcal{F}^0_{s-1})}\Big)}\\
&\leq \frac{1}{1+\frac{b^{(0)}_{i1}}{b^{(k)}_{i1}}\exp\Big(\sum_{s=1}^{t}\log\frac{\mathcal{L}^{(0)}_{is}(D_{is},Q^{(0)}_i(\boldsymbol{p}_s,\boldsymbol{\theta}^{(0)})|\mathcal{F}^0_{s-1})}{\mathcal{L}^{(k)}_{is}(D_{is},Q^{(k)}_i(\boldsymbol{p}_s,\boldsymbol{\theta}^{(k)})|\mathcal{F}^0_{s-1})}\Big)}.
\end{aligned} \tag{3.199}$$

<u>Step 2: Construct a martingale.</u> For $s = 1,2,\dots$, let $X_s = \mathbb{E}^{\pi}_{\boldsymbol{\theta}}\Big[\log\frac{\mathcal{L}^{(0)}_{is}(D_{is},Q^{(k)}_i(\boldsymbol{p}_s,\boldsymbol{\theta}^{(0)})|\mathcal{F}^0_{s-1})}{\mathcal{L}^{(k)}_{is}(D_{is},Q^{(k)}_i(\boldsymbol{p}_s,\boldsymbol{\theta}^{(k)})|\mathcal{F}^0_{s-1})}\Big]$ and $Z_s = \log\frac{\mathcal{L}^{(0)}_{is}(D_{is},Q^{(k)}_i(\boldsymbol{p}_s,\boldsymbol{\theta}^{(0)})|\mathcal{F}^0_{s-1})}{\mathcal{L}^{(k)}_{is}(D_{is},Q^{(k)}_i(\boldsymbol{p}_s,\boldsymbol{\theta}^{0})|\mathcal{F}^0_{s-1})} - X_s$. Since the demand realization D_{is} is uniformly bounded, we let $\underline{d},\overline{d}$ be constants such that $D_{is} \in [\underline{d},\overline{d}]$ for all $i \in \{1,\dots,n\}$ and $s \in \{1,2\dots,\}$. We define an identity function $I : [\underline{d},\overline{d}] \to [\underline{d},\overline{d}]$ satisfying $I(x) = x$ for all $x \in [\underline{d},\overline{d}]$, and let $u = \max\{|\underline{d}|,|\overline{d}|\}$. Set $\gamma = \frac{\delta_d^2}{2u^2}$, where δ_d is as in the statement of the lemma. Note that, for any pair of probability density functions $\mu,\nu : [\underline{d},\overline{d}] \to \mathbb{R}^+$ satisfying $\mu(x),\nu(x) \in [\underline{f},\overline{f}]$ for $x \in [\underline{d},\overline{d}]$ and $|\int_{\underline{d}}^{\overline{d}} x\mu(x)dx - \int_{\underline{d}}^{\overline{d}} x\nu(x)dx| \geq \delta_d$, we have the following:

$$\begin{aligned}
\int_{\underline{d}}^{\overline{d}}\log\frac{\mu(x)}{\nu(x)}\mu(x)dx &\overset{\text{(a)}}{\geq} \frac{1}{2}\|\mu-\nu\|_1^2\\
&\overset{\text{(b)}}{\geq} \frac{\big\|I(\mu-\nu)\big\|_1^2}{2\|I\|_\infty^2}\\
&\overset{\text{(c)}}{=} \frac{1}{2u^2}\bigg|\int_{\underline{d}}^{\overline{d}}\Big|x\Big(\mu(x)-\nu(x)\Big)\Big|dx\bigg|^2\\
&\overset{\text{(d)}}{\geq} \frac{1}{2u^2}\bigg|\int_{\underline{d}}^{\overline{d}}x\mu(x)dx - \int_{\underline{d}}^{\overline{d}}x\nu(x)dx\bigg|^2 \geq \frac{\delta_d^2}{2u^2},
\end{aligned} \tag{3.200}$$

where: $\|\mu-\nu\|_1 = \int_{\underline{d}}^{\overline{d}} |\mu(x)-\nu(x)|dx$, $\|I(\mu-\nu)\|_1 = \int_{\underline{d}}^{\overline{d}} \Big|I(x)\Big(\mu(x)-\nu(x)\Big)\Big|dx$, $\|I\|_\infty = \sup_{x\in[\underline{d},\overline{d}]} |I(x)|$, (a) follows by Pinsker's inequality, (b) follows by Hölder's inequality, (c) follows because $\|I\|_\infty = u$, and (d) follows by Minkowski's inequality. Since $\mathcal{L}_{is}^{(0)}(\cdot, Q_i^{(0)}(\boldsymbol{p}_s, \boldsymbol{\theta}^{(0)})|\mathcal{F}_{s-1}^0)$ and $\mathcal{L}_{is}^{(k)}(\cdot, Q_i^{(k)}(\boldsymbol{p}_s, \boldsymbol{\theta}^{(k)})|\mathcal{F}_{s-1}^0)$ are two probability density functions satisfying the above properties of $\mu(\cdot)$ and $\nu(\cdot)$, we deduce that (3.200) holds for $\mu(\cdot) = \mathcal{L}_{is}^{(0)}(\cdot, Q_i^{(0)}(\boldsymbol{p}_s, \boldsymbol{\theta}^{(0)})|\mathcal{F}_{s-1}^0)$ and $\nu(\cdot) = \mathcal{L}_{is}^{(k)}(\cdot, Q_i^{(k)}(\boldsymbol{p}_s, \boldsymbol{\theta}^{(k)})|\mathcal{F}_{s-1}^0)$. That is, $X_s \geq \gamma = \frac{\delta_d^2}{2u^2}$ for all s. Moreover, Z_s satisfies the following properties for all s: (i) $|Z_s| \leq \bar{z}$, where $\bar{z} = 2\log\frac{\overline{f}}{\underline{f}} > 0$, and (ii) $\mathbb{E}_{\boldsymbol{\theta}}^{\pi}\Big\{Z_s|\mathcal{F}_{s-1}^0\Big\} = \mathbb{E}_{\boldsymbol{\theta}}^{\pi}\Big[\log\frac{\mathcal{L}_{is}^{(0)}(D_{is}, Q_i^{(0)}(\boldsymbol{p}_s, \boldsymbol{\theta}^{(0)}|\mathcal{F}_{s-1}^0)}{\mathcal{L}_{is}^{(k)}(D_{is}, Q_i^{(k)}(\boldsymbol{p}_s, \boldsymbol{\theta}^{(k)})|\mathcal{F}_{s-1})} - X_s\Big|\mathcal{F}_{s-1}\Big]$ $= \mathbb{E}_{\boldsymbol{\theta}}^{\pi}\Big[\log\frac{\mathcal{L}_{is}^{(0)}(D_{is}, Q_i^{(0)}(\boldsymbol{p}_s, \boldsymbol{\theta}^{(0)}|\mathcal{F}_{s-1}^0)}{\mathcal{L}_{is}^{(k)}(D_{is}, Q_i^{(k)}(\boldsymbol{p}_s, \boldsymbol{\theta}^{(k)}|\mathcal{F}_{s-1}^0)}\Big|\mathcal{F}_{s-1}\Big] - \mathbb{E}_{\boldsymbol{\theta}}^{\pi}\Big[\log\frac{\mathcal{L}_{is}^{(0)}(D_{is}, Q_i^{(0)}(\boldsymbol{p}_s, \boldsymbol{\theta}^{(0)}|\mathcal{F}_{s-1}^0)}{\mathcal{L}_{is}^{(k)}(D_{is}, Q_i^{(k)}(\boldsymbol{p}_s, \boldsymbol{\theta}^{(k)}|\mathcal{F}_{s-1}^0)}\Big|\mathcal{F}_{s-1}^0\Big] = 0$. As a result, $\{Z_s\}$ is a bounded martingale difference sequence adapted to $\mathcal{F}_s$.

Step 3: Derive an upper bound on the convergence rate. Let $\pi \in \Pi$ be such that, under π, we have $|Q_i^{(0)}(\boldsymbol{p}_s, \boldsymbol{\theta}^{(0)}) - Q_i^{(k)}(\boldsymbol{p}_s, \boldsymbol{\theta}^{(k)})| \geq \delta_d > 0$ for all $s \in \{1, 2, \ldots, t\}$. Let $A_t = \sum_{s=1}^t X_s$ and $W_t = \sum_{s=1}^t Z_s$. By (3.199), we have

$$\mathbb{E}_{\boldsymbol{\theta}}^{\pi}\{b_{i(t+1)}^{(k)}\} \leq \mathbb{E}_{\boldsymbol{\theta}}^{\pi}\left\{\frac{1}{1+\frac{b_{i1}^{(0)}}{b_{i1}^{(k)}}\exp(A_t+W_t)}\right\}.$$

Letting $B_t = \{W_t \leq -\frac{1}{2}A_t\}$, we deduce from the preceding inequality that

$$\begin{aligned}
\mathbb{E}_{\boldsymbol{\theta}}^{\pi}\{b_{i(t+1)}^{(k)}\} &\leq \mathbb{E}_{\boldsymbol{\theta}}^{\pi}\left\{\frac{1}{1+\frac{b_{i1}^{(0)}}{b_{i1}^{(k)}}\exp(A_t+W_t)}\, I\{B_t\}\right\} + \mathbb{E}_{\boldsymbol{\theta}}^{\pi}\left\{\frac{1}{1+\frac{b_{i1}^{(0)}}{b_{i1}^{(k)}}\exp(A_t+W_t)}\, I\{B_t^c\}\right\} \\
&\overset{\text{(e)}}{\leq} \mathbb{P}_{\boldsymbol{\theta}}^{\pi}\{B_t\} + \mathbb{E}_{\boldsymbol{\theta}}^{\pi}\left\{\frac{1}{1+\frac{b_{i1}^{(0)}}{b_{i1}^{(k)}}\exp(A_t+W_t)}\, I\{B_t^c\}\right\} \\
&\overset{\text{(f)}}{\leq} \mathbb{P}_{\boldsymbol{\theta}}^{\pi}\{B_t\} + \mathbb{E}_{\boldsymbol{\theta}}^{\pi}\left\{\frac{1}{1+\frac{b_{i1}^{(0)}}{b_{i1}^{(k)}}\exp(\frac{1}{2}A_t)}\, I\{B_t^c\}\right\}, \qquad (3.201)
\end{aligned}$$

where: $I\{\cdot\}$ is the indicator function (i.e., given condition A, $I\{A\} = 1$ if A holds, and 0 otherwise); (e) follows because $\frac{b_{i1}^{(0)}}{b_{i1}^{(k)}}\exp(A_t + W_t) \geq 0$, and $\mathbb{E}_{\boldsymbol{\theta}}^{\pi}\{I\{B_t\}\} = \mathbb{P}_{\boldsymbol{\theta}}^{\pi}\{B_t\}$; and (f) follows because $A_t + W_t \geq \frac{1}{2}A_t$ on B_t^c. Under π, $A_t \geq \gamma t$. Therefore, (3.201) implies that

$$\mathbb{E}_{\boldsymbol{\theta}}^{\pi}\{b_{i(t+1)}^{(k)}\} \leq \mathbb{P}_{\boldsymbol{\theta}}^{\pi}\{B_t\} + \mathbb{E}_{\boldsymbol{\theta}}^{\pi}\left\{\frac{1}{1 + \frac{b_{i1}^{(0)}}{b_{i1}^{(k)}}\exp(\frac{1}{2}\gamma t)} I\{B_t^c\}\right\} \overset{\text{(g)}}{\leq} \mathbb{P}_{\boldsymbol{\theta}}^{\pi}\{B_t\} + \frac{b_{i1}^{(k)}}{b_{i1}^{(0)}}\exp(-\frac{1}{2}\gamma t), \tag{3.202}$$

where (g) follows because $I\{B_t^c\} \leq 1$. Note that

$$\begin{aligned} \mathbb{P}_{\boldsymbol{\theta}}^{\pi}\{B_t\} = \mathbb{P}_{\boldsymbol{\theta}}^{\pi}\{W_t \leq -\frac{1}{2}A_t\} &\overset{\text{(h)}}{\leq} \mathbb{P}_{\boldsymbol{\theta}}^{\pi}\{W_t \leq -\frac{1}{2}\gamma t\} \\ &\overset{\text{(i)}}{\leq} \mathbb{P}_{\boldsymbol{\theta}}^{\pi}\{|W_t| \geq \frac{1}{2}\gamma t\} \\ &\overset{\text{(j)}}{\leq} 2\exp(-\frac{\gamma^2 t}{2\bar{z}}), \end{aligned} \tag{3.203}$$

where: (h) follows because $A_t \geq \gamma t$, (i) follows because $\{W_t \geq -\frac{1}{2}\gamma t\} \subset \{|W_t| \geq \frac{1}{2}\gamma t\}$, (j) follows by the Azuma-Hoeffding inequality and the fact that $|W_s - W_{s-1}| = |Z_s| \leq \bar{z}$ for all s. Combining (3.202) and (3.203), we deduce that

$$\mathbb{E}_{\boldsymbol{\theta}}^{\pi}\{b_{i(t+1)}^{(k)}\} \leq 2\exp(-\frac{\gamma^2 t}{2\bar{z}}) + \frac{b_{i1}^{(k)}}{b_{i1}^{(0)}}\exp(-\frac{1}{2}\gamma t) \leq \left(2 + \frac{b_{i1}^{(0)}}{b_{i1}^{(k)}}\right)\exp(-\eta t).$$

As a result, $\mathbb{E}_{\boldsymbol{\theta}}^{\pi}\{1 - b_{t+1}^{i}\} = \sum_{j\neq i}\mathbb{E}_{\boldsymbol{\theta}}^{\pi}\{b_{i(t+1)}^{(k)}\} \leq \zeta e^{-\eta t}$, where $\zeta = \max_i\left\{\sum_{j\neq i}\left(2 + \frac{b_{i1}^{(0)}}{b_{i1}^{(k)}}\right)\right\}$, and $\eta = \frac{1}{2}\gamma\min\{1, \frac{\gamma}{\bar{z}}\}$. $\square$

REFERENCES

[1] Alexander Schrijver. *Combinatorial optimization: polyhedra and efficiency*, volume 24. Springer Science & Business Media, 2003.

[2] Jean-Charles Rochet and Jean Tirole. Platform competition in two-sided markets. *Journal of European Economic Association*, 1(4):990–1029, 2003.

[3] Jean-Charles Rochet and Jean Tirole. Two-sided markets: A progress report. *The RAND Journal of Economics*, 37(3):645–667, 2006.

[4] Mark Armstrong. Competition in two-sided markets. *The RAND Journal of Economics*, 37(3):668–691, 2006.

[5] E Glen Weyl. A price theory of multi-sided platforms. *American Economic Review*, 100(4):1642–1672, 2010.

[6] Lloyd S Shapley and Martin Shubik. The assignment game I: The core. *International Journal of Game Theory*, 1(1):111–130, 1971.

[7] Rachel E Kranton and Deborah F Minehart. A theory of buyer-seller networks. *The American Economic Review*, 91(3):485–508, 2001.

[8] Sham M Kakade, Michael Kearns, and Luis E Ortiz. Graphical economics. *International Conference on Computational Learning Theory*, pages 17–32, 2004.

[9] Moshe Babaioff, Noam Nisan, and Elan Pavlov. Mechanisms for a spatially distributed market. *Games and Economic Behavior*, 66:660–684, 2009.

[10] Larry Blume, David Easley, Jon Kleinberg, and Eva Tardos. Trading networks with price-setting agents. *Games and Economic Behavior*, 67(1):36–50, 2009.

[11] Dilip Abreu and Mihai Manea. Bargaining and efficiency in networks. *Journal of Economic Theory*, 147:43–70, 2012.

[12] Daniele Condorelli, Andrea Galeotti, and Ludovic Renou. Bilateral trading in networks. *Review of Economic Studies*, 84(1):82–105, 2017.

[13] Mihai Manea. Intermediation and resale in networks. *Journal of Political Economy*, 126(3):To appear, 2018.

[14] Kostas Bimpikis, Shayan Ehsani, and Rahmi Ilkilic. Cournot competition in networked mar-

kets. *Working paper*, page, 2015.

[15] Weixuan Lin, John Pang, Eilyan Bitar, and Adam Wierman. Networked cournot competition in platform markets access control and efficiency loss. In *2017 IEEE 56th Annual Conference on Decision and Control (CDC)*. IEEE, 2017.

[16] Desmond W H Cai, Subhonmesh Bose, and Adam Wierman. On the role of a market maker in networked cournot competition. *Working paper*, page, 2017.

[17] John Pang, Weixuan Lin, Hu Fu, Jack Kleeman, Eilyan Bitar, and Adam Wierman. Transparency and control in platforms for networked markets. *Working paper*, page, 2019.

[18] Michael Gofman. A network-based analysis of over-the-counter markets. *Working paper*, page, 2014.

[19] Matthew Elliot. Inefficiencies in networked markets. *American Economic Journal: Microeconomics*, 7(4):43–82, 2015.

[20] Michael Gofman. Efficiency and stability of a financial architecture with too-interconnected-to-fail institutions. *Journal of Financial Economics*, 124, 2017.

[21] Sanjeev Goyal. Networks and markets. In *Advances in Economics and Econometrics: Volume 1: Eleventh World Congress*, volume 58, page 215. Cambridge University Press, 2017.

[22] Siddhartha Banerjee, Sreenivas Gollapudi, Kostas Kollias, and Kamesh Munagala. Segmenting two-sided markets. *Proceedings of the 26th International Conference on World Wide Web*, pages 63–72, 2017.

[23] Gad Allon, Achal Bassamboo, and Eren B Çil. Large-scale service marketplaces: The role of the moderating firm. *Management Science*, 58(10):1854–1872, 2012.

[24] Yash Kanoria and Daniela Saban. Facilitating the search for partners on matching platforms: Restricting agents' actions. *Working Paper*, page, 2017.

[25] Nick Arnosti, Ramesh Johari, and Yash Kanoria. Managing congestion in decentralized matching markets. In *EC '14 Proceedings of the fifteenth ACM conference on Economics and computation*, pages 451 – 451. ACM, 2014.

[26] Saif Benjaafar, Guangwen Kong, Xiang Li, and Costas Courcoubetis. Peer-to-peer product sharing: Implications for ownership, usage and social welfare in the sharing economy. *Management Science*, page Articles in Advance, 2018.

[27] Ming Hu and Yun Zhou. Price, wage and fixed commission in on-demand matching. *Working paper*, page, 2019.

[28] Siddhartha Banerjee, Ramesh Johari, and Carlos Riquelme. Pricing in ride-sharing platforms: A queueing-theoretic approach. In *Proceedings of the Sixteenth ACM Conference on Economics and Computation*, pages 639–639. ACM, 2015.

[29] Gerard P Cachon, Kaitlin M Daniels, and Ruben Lobel. The role of surge pricing on a service platform with self-scheduling capacity. *Manufacturing Service Operations Management*, 19(3), 2017.

[30] Itai Gurvich, Martin Lariviere, and Antonio Moreno. Operations in the on-demand economy: Staffing services with self-scheduling capacity. *Working paper*, page, 2016.

[31] Philipp Afeche, Liu Zhe, and Costis Maglaras. The impact of platform control capabilities on the performance of rideshare networks. *Working paper*, page, 2018.

[32] Terry Taylor. On-demand service platforms. *Manufacturing Service Operations Management*, 20(4), 2018.

[33] Christopher S Tang, Jiaru Bai, Kut C So, Xiqun Michael Chen, and Hai Wang. Coordinating supply and demand on an on-demand platform: Price, wage, and payout ratio. *Working paper*, page, 2017.

[34] Erhun Ozkan and Amy Ward. Dynamic matching for real-time ridesharing. *Working paper*, page, 2019.

[35] Kostas Bimpikis, Ozan Candogan, and Daniela Saban. Spatial pricing in ride-sharing networks. *Operations Research*, page To appear, 2018.

[36] Zhixuan Fang, Longbo Huang, and Adam Wierman. Prices and subsidies in the sharing economy. *Proceedings of the 26th international conference on World Wide Web*, pages 53–62, 2017.

[37] Maxime Cohen and Renyu Zhang. Coopetition and profit sharing for ride-sharing platforms. *Working paper*, page, 2017.

[38] Francisco Castro, Omar Besbes, and Ilan Lobel. Surge pricing and its spatial supply response. *Working paper*, page, 2018.

[39] Ming Hu and Yun Zhou. Dynamic type matching. *Working paper*, page, 2016.

[40] John Birge, Ozan Candogan, Hongfan Chen, and Daniela Saban. E-companion for "Optimal Commissions and Subscriptions in Networked Markets". Technical report, 2018.

[41] Nimrod Megiddo. Optimal flows in networks with multiple sources and sinks. *Mathematical Programming*, 7:97–107, 1974.

[42] Satoru Fujishige. Lexicographically optimal base of a polymatroid with respect to a weight vector. *Mathematics of Operations Research*, 5(2):186–196, 1980.

[43] Alexander Shapiro. Directional differentiability of the optimal value function in convex semi-infinite programming. 70:149–157, 1995.

[44] Airbnb. Overview of the Airbnb community in Chicago, 2017.

[45] Airbnb. What is the Airbnb service fee?, 2017.

[46] InsideAirbnb. About inside Airbnb, 2017.

[47] Tim Roughgarden and Éva Tardos. How bad is selfish routing? *Journal of the ACM (JACM)*, 49(2):236–259, 2002.

[48] Jiawei Zhang. Joint replenishment game and maximizing an H-Schur concave function over a polymatroid. *Working paper*, page, 2008.

[49] Claude Berge. *Topological Spaces: Including a Treatment of Multi-valued Functions, Vector Spaces, and Convexity*. 1963.

[50] R. Tyrrell Rockafellar. *Convex Analysis*. 1970.

[51] Stephen Boyd and Lieven Vandenberghe. *Convex Optimization*. 2004.

[52] M Marn, E Roegner, and C Zawada. Pricing new products. *McKinsey Quarterly*, 3:40–49, 2003.

[53] IBM. IBM markdown optimization. `https://www.ibm.com/downloads/cas/J4XO2BVL`, 2019.

[54] Oracle. Oracle retail markdown optimization. `http://www.oracle.com/us/products/applications/062075.pdf`, 2019.

[55] SAS. SAS markdown optimization. `https://www.sas.com/content/dam/SAS/en_us/doc/productbrief/sas-markdown-optimization-105878.pdf`, 2019.

[56] Victor Araman and Rene Caldentey. Dynamic pricing for nonperishable products with demand learning. *Operations Research*, 57(5):1169–1188, 2009.

[57] Vivek F. Farias and Benjamin van Roy. Dynamic pricing with a prior on market response. *Operations Research*, 58(1):16–29, 2010.

[58] J. Michael Harrison, Bora N. Keskin, and Assaf Zeevi. Bayesian dynamic pricing policies: Learning and earning under a binary prior distribution. *Management Science*, 58(3):570–586, 2012.

[59] Arnold den Boer and Bert Zwart. Simultaneously learning and optimizing using controlled variance pricing. *Management Science*, 60(3):770–783, 2014.

[60] David Besanko and Wayne L. Winston. Optimal price skimming by a monopolist facing rational consumers. *Management Science*, 36(5):555–567, 1990.

[61] R.H. Coase. Durability and monopoly. *Journal of Law and Economics*, 15(1):143–149, 1972.

[62] Nancy Stokey. Intertemporal price discrimination. *Quarterly Journal of Economics*, 93(3):355–371, 1979.

[63] Nancy Stokey. Rational expectations and durable goods pricing. *Bell Journal of Economics*, 12(1):112–128, 1981.

[64] Jeremy Bulow. Durable-goods monopolists. *Journal of Political Economy*, 90(2):314–332, 1982.

[65] Edward P Lazear. Retail pricing and clearance sales. *American Economic Review*, 76(1):14–32, 1986.

[66] Harikesh Nair. Intertemporal price discrimination with forward-looking consumers: Application to the us market for console video-games. *Quantitative Marketing and Economics*, 5(3):239–292, 2007.

[67] Xuanming Su. Intertemporal pricing with strategic customer behavior. *Management Science*, 53(5):726–741, 2007.

[68] Yossi Aviv and Aviv Pazgal. Optimal pricing of seasonal products in the presence of forward-looking consumers. *Manufacturing Service Operations Management*, 10(3):339–359, 2008.

[69] Wedad Elmaghraby, Atlan Gulcu, and Pinar Keskinocak. Designing optimal preannounced markdowns in the presence of rational customers with multiunit demands. *Manufacturing and Service Operations Management*, 10(1):126–148, 2008.

[70] R. Yin, Y. Aviv, A. Pazgal, and C. Tang. Optimal markdown pricing: Implications of inven-

tory display formats in the presence of strategic customers. *Management Science*, 55(8):1391–1408, 2009.

[71] Adam J. Mersereau and Dan Zhang. Markdown pricing with unknown fraction of strategic customers. *Manufacturing and Service Operations Management*, 14(3):355–370, 2012.

[72] Goker Aydin and Serhan Ziya. Personalized dynamic pricing of limited inventories. *Operations Research*, 57(6):1523–1531, 2009.

[73] Gerard Cachon and Robert Swinney. Purchasing, pricing, and quick response in the presence of strategic consumers. *Management Science*, 55(3):497–511, 2009.

[74] Robert Swinney. Selling to strategic consumers when product value is uncertain: The value of matching supply and demand. *Management Science*, 57(10):1737–1751, 2011.

[75] Omar Besbes and Ilan Lobel. Intertemporal price discrimination: Structure and computation of optimal policies. *Management Science*, 61(1):92–110, 2015.

[76] Yuri Levin, Jeff McGill, and Mikhail Nediak. Dynamic pricing in the presence of strategic consumers and oligopolistic competition. *Management Science*, 55(1):32–46, 2009.

[77] Ali K. Parlakturk. The value of product variety when selling to strategic consumers. *Manufacturing and Service Operations Management*, 14(3):371–385, 2012.

[78] N. Surasvadi, C. Tang, and G. Vulcano. Using contingent markdown with reservation to profit from strategic consumer behavior. *Production of Operations Management*, 26(12):2226–2246, 2017.

[79] K. Moon, K. Bimpikis, and H. Mendelson. Randomized markdowns and online monitoring. *Management Science*, 2017.

[80] Yossi Aviv and Amit Pazgal. A partially observed markov decision process for dynamic pricing. *Management Science*, 51(9):1400–1416, 2005.

[81] T. Levina, Y. Levin, J. McGill, and M. Nediak. Dynamic pricing with online learning and strategic consumers: An application of the aggregating algorithm. *Operations Research*, 57(2):327–341, 2009.

[82] J. Broder and P. Rusmevichientong. Dynamic pricing under a general parametric choice model. *Operations Research*, 60(4):965–980, 2012.

[83] N. B. Keskin and A. Zeevi. Dynamic pricing with an unknown demand model: Asymptotically

optimal semi-myopic policies. *Operations Research*, 62(5):1142–1167, 2014.

[84] H.Dharma Kwon, Steven A. Lippman, and Tangm Christopher S. Optimal markdown pricing strategy with demand learning. *Probability in the Engineering and Informational Sciences*, 26(1):77–104, 2012.

[85] Wang Chi Cheung, David Simchi-Levi, and He Wang. Dynamic pricing and demand learning with limited price experimentation. *Operations Research*, 65(6):1722–1731, 2017.

[86] Tamer Boyaci and Ozalp Ozer. Information acquisition for capacity planning via pricing and advance selling: When to stop and act? *Operations Research*, 58(5):1328–1349, 2010.

[87] Bora N. Keskin and Birge Birge. Dynamic selling mechanisms for product differentiation and learning. 2019. Forthcoming in *Operations Research*.

[88] T. Erdem, S. Imai, and M. Keane. Brand and quantity choice dynamics under price uncertainty. *Quantitative Marketing and Economics*, 1:5–64, 2003.

[89] I. Hendel and A. Nevo. Measuring the implications of sales and consumer inventory behavior. *Econometrica*, 74(6):1637–1673, 2006.

[90] J. Li, N. Granados, and S. Netessine. Are consumers strategic? structural estimation from the air-travel industry. *Management Science*, 60(9):2114–2137, 2014.

[91] Thomas Cover. Universal portfolios. *Mathematical Finance*, 1(1):1–29, 1991.

[92] John F. Muth. Rational expectations and the theory of price movements. *Econometrica*, 29(3):315–335, 1961.

[93] Roy Radner. Rational expectations equilibrium: Generic existence and the information revealed by prices. *Econometrica*, 47(3):655–678, 1979.

[94] N Bora Keskin and Assaf Zeevi. On incomplete learning and certainty-equivalence control. *Operations Research*, 66(4):1136–1167, 2018.

[95] A. Tsybakov. *Introduction to Nonparametric Estimation*. Springer, New York, 2009.

[96] Olivier Morand, Kevin Reffett, and Suchismita Tarafdar. A nonsmooth approach to envelope theorems. *Journal of Mathematical Economics*, 61:157–165, 2015.

[97] Airbnb. How you make money on airbnb. `https://www.airbnb.com/d/financials`, 2019.

[98] Upwork. Upwork pricing. `https://www.upwork.com/i/pricing/`, 2019.

[99] Amazon. Create a pricing rule. `https://sellercentral.amazon.com/gp/help/external/`

`help.html?itemID=201995750&ref=efph_201995750_cont_home`, 2019.

[100] Airbnb. How does the airbnb plus incentive program work. `https://www.airbnb.com/help/article/2365/how-does-the-airbnb-plus-incentive-program-work`, 2019.

[101] Arnoud V den Boer and Bert Zwart. Simultaneously learning and optimizing using controlled variance pricing. *Management Science*, 60(3):770–783, 2013.

[102] Sentao Miao, Xi Chen, Xiuli Chao, Jiaxi Liu, and Yidong Zhang. Context-based dynamic pricing with online clustering. *Working paper*, page, 2019.

[103] Gah-Yi Ban and N. Bora Keskin. Personalized dynamic pricing with machine learning: High dimensional features and heterogeneous elasticity. *Working Paper*, 2019.

[104] Siddhartha Banerjee, Ramesh Johari, and Carlos Riquelme. Pricing in ride-sharing platforms: A queueing-theoretic approach. In *Proceedings of the Sixteenth ACM Conference on Economics and Computation*, pages 639–639. ACM, 2015.

[105] Jiaru Bai, Kut C So, Christopher S Tang, Xiqun Chen, and Hai Wang. Coordinating supply and demand on on-demand service platform with impatient customers. *Manufacturing and Service Operations Management*, 21(3):556 – 570, 2018.

[106] Terry Taylor. On-demand service platforms. *Manufacturing Service Operations Management*, 20(4), 2018.

[107] Kostas Bimpikis, Ozan Candogan, and Daniela Saban. Spatial pricing in ride-sharing networks. *Operations Research*, 67(3):744–769, 2019.

[108] Gad Allon, Achal Bassamboo, and Eren B Çil. Large-scale service marketplaces: The role of the moderating firm. *Management Science*, 58(10):1854–1872, 2012.

[109] Nick Arnosti, Ramesh Johari, and Yash Kanoria. Managing congestion in decentralized matching markets. In *EC '14 Proceedings of the fifteenth ACM conference on Economics and computation*, pages 451 – 451. ACM, 2014.

[110] Erhun Ozkan and Amy Ward. Dynamic matching for real-time ridesharing. *Working paper*, page, 2019.

[111] Yash Kanoria and Daniela Saban. Facilitating the search for partners on matching platforms: Restricting agents' actions. *Working Paper*, page, 2019.

[112] Philipp Afeche, Liu Zhe, and Costis Maglaras. The impact of platform control capabilities

on the performance of rideshare networks. *Working paper*, page, 2018.

[113] John Birge, Ozan Candogan, Hongfan Chen, and Daniela Saban. Optimal commissions and subscriptions in networked markets. *Manufacturing and Service Operations Management*, page To appear, 2019.

[114] Luis Rayo and Ilya Segal. Optimal information disclosure. *Journal of Political Economy*, 118(5):949–987, 2010.

[115] Andrei Hagiu and Bruno Jullien. Why do intermediaries divert search. *RAND Journal of Economies*, 42(2):337–362, 2011.

[116] Dina Mayzlin, Yaniv Dover, and Judith Chavalier. Promotional reviews: An empirical investigation of online review manipulation. *American Economic Review*, 104(8):2421Ű2455, 2014.

[117] Matthieu Bouvard and Raphael Levy. Two-sided reputation in certification markets. *Management Science*, 64(10), 2018.

[118] Yishay Mansour, Aleksandrs Slivkins, and Vasilis Syrgkanis. Bayesian incentive-compatible bandit exploration. *Proceedings of the fifteenth ACM conference on Economics and computation*, pages 565–582, 2015.

[119] Yishay Mansour, Aleksandrs Slivkins, Vasilis Syrgkanis, and Zhiwei Steven Wu. Bayesian exploration: Incentivizing exploration in bayesian games. *Proceedings of the 2016 ACM Conference on Economics and Computation*, pages 661–661, 2016.

[120] Yiangos Papanastasiou, Kostas Bimpikis, and Nicos Savva. Crowdsourcing exploration. *Management Science*, 64(4):1727–1746, 2017.

[121] Yeon-Koo Che and Johannes Horner. Recommender systems as mechanisms for social learning. *The Quarterly Journal of Economics*, 133(2), 2018.

[122] Peter Frazier, David Kempe, Jon Kleinberg, and Robert Kleinberg. Incentivizing exploration. *Proceedings of the fifteenth ACM conference on Economics and computation*, pages 5–22, 2014.

[123] Bangrui Chen, Peter I Frazier, and David Kempe. Incentivizing exploration by heterogeneous users. *Proceedings of Machine Learning Research*, 75:1–21, 2018.

[124] Michael Bowling. Convergence and no-regret in multiagent learning. *Proceedings of the 17th International Conference on Neural Information Processing Systems*, pages 209–216, 2004.

[125] Sherief Abdallah and Victor Lesser. A multiagent reinforcement learning algorithm with non-linear dynamics. *Journal of Artificial Intelligence Research*, 33:521 – 549, 2008.

[126] Panayotis Mertikopoulos and William Sandholm. Learning in games via reinforcement and regularization. *Mathematics of Operations Research*, 41(4):1297 – 1324, 2016.

[127] Hoi-To Wai, Zhuoran Yang, Zhaoran Wang, and Mengyi Hong. Multi-agent reinforcement learning via double averaging primal-dual optimization. *32nd Conference on Neural Information Processing Systems*, pages 9672–9683, 2018.

[128] Mario Bravo, David Leslie, and Panayotis Mertikopoulos. Bandit learning in concave n-person games. *NIPS*, page To appear, 2018.

[129] Panayotis Mertikopoulos and Zhengyuan Zhou. Learning in games with continuous action sets and unknown payoff functions. *Mathematical Programming*, 120:221–259, 2018.

[130] Zhengyuan Zhou, Panayotis Mertikopoulos, Glynn Bambos, Nicholas, Peter, and Claire Tomlin. Multi-agent online learning with imperfect information. Working paper, 2019.

[131] Caroline Claus and Craig Boutilier. The dynamics of reinforcement learning in cooperative multiagent systems. *Proceedings of the fifteenth national/tenth conference on Artificial intelligence/Innovative applications of artificial intelligence*, pages 746–752, 1998.

[132] Xiaofeng Wang and Sandholm Tuomas. Reinforcement learning to play an optimal nash equilibrium in team markov games. *Proceedings of the 17th International Conference on Neural Information Processing Systems*, pages 1603–1610, 2002.

[133] Junling Hu and Michael P. Wellman. Nash q-learning for general-sum stochastic games. *Journal of Machine Learning Research*, 4:1039 – 1069, 2003.

[134] Amy Greenwald and Keith Hall. Correlated-q learning. *Proceedings of the Twentieth International Conference on Machine Learning*, pages 242–249, 2003.

[135] Ryszard Kowalczyk and Eduardo Rodrigues Gomes. Dynamic analysis of multiagent q-learning with ϵ-greedy exploration. *Proceedings of the 26th Annual International Conference on Machine Learning*, pages 369 – 376, 2009.

[136] Sherief Abdallah and Michael Kaisers. Addressing environment non-stationarity by repeating q-learning updates. *Journal of Machine Learning Research*, 17:1–31, 2016.

[137] Drew Fudenberg and David Kreps. Learning mixed equilibria. *Games nd Economic Behavior*,

5:320–367, 1993.

[138] Ido Erev and Alvin E Roth. Predicting how people play games: Reinforcement learning in experimental games with unique, mixed strategy equilibria. *American Economic Association*, 88(4):848–881, 1998.

[139] Colin Camerer and Tech-Hua Ho. Experienced-weighted attraction learning in normal form games. *Econometrica*, 67(4):827 – 874, 1999.

[140] Timothy C Salmon. An evaluation of econometric models of adaptive learning. *Econometrica*, 69(6):1597 – 1628, 2001.

[141] Ed Hopkins. Two competing models of how people learn in games. *Econometrica*, 70(6):2141 – 2166, 2002.

[142] David Kreps and Robert Wilson. Sequential equilibria. *Econometrica*, 50(4):863 – 894, 1982.

[143] Drew Fudenberg and Jean Tirole. Perfect bayesian equilibrium and sequential equilibrium. *Journal of Economic Theory*, 53:236 – 260, 1991.

[144] Pierpaolo Battigalli. Strategic independence and perfect bayesian equilibria. *Journal of Economic Theory*, 70:201–234, 1996.

[145] Joel Watson. Alternating-offer bargaining with two-sided incomplete information. *Review of Economic Studies*, 65(4):573–594, 1998.

[146] J B Rosen. Existence and uniqueness of equilibrium points for concave n-person games. *Econometrica*, 33(3):520–534, 1965.

[147] Thodoris Lykouris, Vasilis Syrgkanis, and Eva Tardos. Learning and efficiency in games with dynamic population. *Working Paper*, 2016.

[148] Ilan Kremer, Yishay Mansour, and Motty Perry. Implementing the Şwisdom of the crowdŤ. *Journal of Political Economy*, 122(5):988–1012, 2014.

[149] Yuri Nesterov. Primal-dual subgradient methods for convex problems. *Mathematical Programming*, 120:221–259, 2009.

[150] Josef Haobauer and William H Sandholm. On the global convergence of stochastic fictitious play. *Econometrica*, 70(6):2265 – 2294, 2002.

[151] Steven Perkins and David S Leslie. Stochastic fictitious play with continuous action sets. *Journal of Economic Theory*, 152:179–213, 2014.

[152] Ratul Lahkar and Frank Riedel. The logit dynamic for games with continuous strategy sets. *Games and Economic Behavior*, 91:268–282, 2015.

[153] Steven Perkins, Panayotis Mertikopoulos, and David S Leslie. Mixed-strategy learning with continuous action sets. *IEEE Transactions on Automatic Control*, 62:379 – 384, 2017.

[154] Dov Monderer and Lloyd S Shapley. Potential games. *Games nd Economic Behavior*, 14(44):124–143, 1996.

[155] Paul Milgrom and Ilya Segal. Envelope theorem for arbitrary choice set. *Econometrica*, 70(2):583–601, 2002.

[156] Alexander Shapiro. Sensitivity analysis of nonlinear programs and differentiability properties of metric projections. *Siam Journal of Control and Optimization*, 26(3):628 – 645, 1988.

[157] Dov Monderer and Lloyd S Shapley. Fictitious play property for games with identical interests. *Journal of Economic Theory*, 68:258–265, 1996.