Supplementary Information for

A nonlinear, geometric Hall effect without magnetic field

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Supplementary text
Figs. S1 to S3

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Supporting Information Text

Narrow-wire limit

Equation 3 in the main text is the full expression for the transverse potential ΔV_{geom} for a curved wire of arbitrary width $w = r_{\text{out}} - r_{\text{in}}$. Equation 5 shows how this expression simplifies in the limit of a narrow wire, when $w \ll r_{\text{in}}$. We consider the ratio of the full expression for the transverse potential, ΔV_{geom}, to this narrow-wire approximation, as a function of $x \equiv r_{\text{out}}/r_{\text{in}}$.

Dividing Eq. 3 by Eq. 5, we obtain:

$$\frac{\Delta V_{\text{geom}}(x)}{\Delta V_{\text{geom}}(x=1)} = \frac{(x-1)\left(1-\frac{1}{x^2}\right)}{2\,(\ln x)^2}. \quad [1]$$

As shown in Fig. S1, this approximation becomes less good with increasing x. For the geometry of the circuits that we use in our experiments, $x \approx 10$ and $\frac{\Delta V_{\text{geom}}(x)}{\Delta V_{\text{geom}}(x=1)} \approx 0.84$. We use the exact expression for ΔV_{geom} from Eq. 3 throughout our discussion in the main text.

Diffusive transport analysis

In the geometric Hall effect the charge carriers in the wire may be experiencing diffusive transport due to frequent collisions that randomize their momenta. Thus the trajectory of any single carrier will not be in a perfectly circular orbit, but instead will resemble a random walk with steps of length equal to the electronic mean free path. One might ask whether the calculation leading to Eq. 3 is relevant to such a trajectory.

To answer this question, we consider a small region of the wire, with dimension not much larger than the mean free path. There may be carriers in this region whose instantaneous velocities point in different directions due to the diffusive scattering. However, by definition the *average* velocity of the carriers in this region must equal the drift velocity. If the wire's material properties are uniform, then the drift velocity must follow the wire. In the curved wire geometry, symmetry requires that the drift velocity is purely azimuthal. This means that the average momentum of the carriers in a small region of the wire is also directed azimuthally. If we consider another location farther along the wire, we find that the drift velocity's direction has changed in order to follow the wire curvature. This change in the average momentum of the carriers along the wire must be due to a force acting on them. In the absence of a magnetic field, the only force is the one due to a radial component of the electric field.

To see the origin of this radial electric field component in the diffusive transport regime, we consider two different starting points from the Drude model: (1) the average displacements of individual charge carriers, and (2) the electronic equation of motion.

Formulation in terms of average displacements of charge carriers. We consider the motion of individual charge carriers in our curved wire geometry using a Drude model. We assume that the charge carriers experience collisions with a typical frequency $1/\tau$, where τ is the mean free time between collisions. Each collision randomizes the carrier's momentum, and the carrier then accelerates due to the electric field,

$$\mathbf{a} = \frac{q}{m}\mathbf{E}, \quad [2]$$

until it experiences another collision. Shortly after a collision at time t_0, the velocity of the carrier is

$$\mathbf{v}\left(t_0 + t\right) = \mathbf{v}_0 + \frac{q}{m}\mathbf{E}t, \quad [3]$$

for $t < \tau$ and where $\mathbf{v}_0 = \mathbf{v}\left(t_0\right)$. The displacement of the charge carrier between collisions is

$$\Delta\mathbf{x} = \int_0^\tau \mathbf{v}\left(t_0 + t\right) dt \quad [4]$$

$$= \int_0^\tau \left(\mathbf{v}_0 + \frac{q\mathbf{E}t}{m}\right) dt \quad [5]$$

$$= \mathbf{v}_0\tau + \frac{q\mathbf{E}\tau^2}{2m}. \quad [6]$$

Due to randomization of the carriers' momentum after each collision, $\langle\mathbf{v}_0\rangle = 0$. If we average over all the charge carriers in a small portion of the wire, we find the average displacement of the carriers between collisions,

$$\langle\Delta\mathbf{x}\rangle = \frac{q\mathbf{E}\tau^2}{2m}. \quad [7]$$

Because the current follows the circular path of the wire, the average displacement between collisions must change direction as the carriers drift along the wire. In particular, the endpoints of the average displacement vectors must lie along circular paths that are concentric with the curved wire, as illustrated in Fig. S2a. The electric field $\mathbf{E}$ that determines the direction of both the acceleration and the average displacement in Eq. 7 must therefore have a radial component.

We express the local electric field in terms of an azimuthal component E_θ and a radial component E_r. The drift velocity follows the wire and is thus purely azimuthal, $\mathbf{v}_\mathrm{D} = v_\mathrm{D}\hat{\theta}$. During time τ, the azimuthal component of the electric field accelerates the average carrier through a displacement $\mathbf{v}_\mathrm{D}\tau \approx qE_\theta\tau^2/2m$ while the radial component accelerates the average carrier towards the center of the circle through a displacement $qE_r\tau^2/2m$, illustrated in Fig. S2b. The tip and tail of the resultant vector lie along a circular path and thus the distance of each from the center of the circle must be r. We can apply the Pythagorean theorem to this vector sum to obtain the equation

$$\left(r - \frac{qE_r\tau^2}{2m}\right)^2 + \left(\frac{qE_\theta\tau^2}{2m}\right)^2 = r^2. \qquad [8]$$

Neglecting terms of order τ^4 yields

$$E_r = \frac{qE_\theta^2\tau^2}{4mr}. \qquad [9]$$

Assuming $v_\mathrm{D}\tau \approx \frac{qE_\theta\tau^2}{2m} \ll r$, we simplify to find

$$qE_r = \frac{mv_\mathrm{D}^2}{r}. \qquad [10]$$

This shows that the radial component of the electric field may be computed in terms of the drift velocity of the charge carriers.

Formulation in terms of electronic equation of motion. An alternative way to see the role of the radial electric field component in terms of the Drude model is to consider the electronic equation of motion. In the absence of a magnetic field, the Drude model gives

$$\frac{d}{dt}\langle \mathbf{p}(t)\rangle = q\mathbf{E} - \frac{\langle \mathbf{p}(t)\rangle}{\tau}, \qquad [11]$$

where $\langle \mathbf{p}(t)\rangle$ is the average momentum of the charge carriers in a small region of the wire. In steady state, the time derivative of $\langle \mathbf{p}(t)\rangle$ is nonzero because the current is not flowing along a straight path. The time derivative of $\langle \mathbf{p}(t)\rangle$ is the net force on the carriers necessary to keep them moving along a circular path,

$$\frac{d}{dt}\langle \mathbf{p}(t)\rangle = \frac{m\langle v\rangle^2}{r}(-\hat{r}), \qquad [12]$$

where $\hat{r}$ is a unit vector pointing outward from the center of the circle. Substitution of this expression into Eq. 11 yields

$$\frac{m\langle v\rangle^2}{r}(-\hat{r}) = q\mathbf{E} - \frac{\langle \mathbf{p}(t)\rangle}{\tau}. \qquad [13]$$

Using the relationships $\langle \mathbf{p}\rangle = m\langle \mathbf{v}\rangle$ and $\mathbf{j} = j\hat{\theta} = nq\langle \mathbf{v}\rangle$, leads to the result

$$\mathbf{E} = \left(\frac{mj}{nq^2\tau}\right)\hat{\theta} + \left(\frac{mj^2}{n^2q^3r}\right)(-\hat{r}). \qquad [14]$$

The first expression in parentheses on the right side of this equation is the usual longitudinal component of the wire's internal electric field, $E_\theta = j/\sigma$, where $\sigma = nq^2\tau/m$ is the conductivity. The second term is the geometric Hall effect term for E_r, equivalent to the expression we derive in Eq. 2 of the main text.

Joule heating oscillations in longitudinal potential

By using a lock-in amplifier to measure the potential at 2ω, we filter out Ohmic potential drops along the wire as a possible source of error because they occur at 1ω. However, one might ask whether Joule heating could lead to oscillations in the longitudinal potential that could appear at 2ω. The resistance R of a wire will fluctuate with temperature T relative to its steady-state resistance R_0 and temperature T_0 as

$$R(T) \approx R_0\left[1 + \alpha_1(T - T_0) + \alpha_2(T - T_0)^2 + \ldots\right], \qquad [15]$$

where α_1 is the material's temperature coefficient of resistance at the ambient temperature, T_0. The temperature fluctuates in time t with power dissipation P due to Joule heating,

$$T(t) \approx T_0 + \beta P(t), \qquad [16]$$

for some coefficient β. The power dissipation, in turn, is related to the resistance by

$$P(t) = i^2 R(T). \qquad [17]$$

Substitution of these relationships into Ohm's Law, $\mathcal{E} = iR$, leads to higher-order terms in the longitudinal potential drop:

$$\mathcal{E} \approx iR_0 + \alpha_1\beta i^3 R_0^2 + \ldots \quad [18]$$

With ac current at frequency ω, the i^3 term in Eq. 18 affects only odd harmonics, so there is no interference with a measurement at 2ω. This is also true for all the higher-order (nonlinear) terms in the expansion in Eq. 15.

Seebeck effect

Joule heating can raise the temperature of one side of the wire more than the other if, for example, the current density, $j(r)$, depends on radius or is otherwise not homogeneous. This can induce a Seebeck voltage *across* the wire where the graphene makes contact with a conducting lead of a different material. The temperature at any point in the graphene wire will oscillate due to Joule heating:

$$\begin{aligned} T(\mathbf{r},t) &\approx T_0(\mathbf{r}) + \alpha(\mathbf{r}) j^2(\mathbf{r},t) \\ &\approx T_0(\mathbf{r}) + \alpha(\mathbf{r}) j_0^2(\mathbf{r}) \left[\frac{1-\cos(2\omega t)}{2}\right], \end{aligned} \quad [19]$$

where the function $\alpha(\mathbf{r})$ depends on the frequency as well as the material's specific heat capacity and resistivity, which could vary spatially. The dependence on j^2 means that the local temperature oscillates at 2ω in our experiments. The temperature *difference* ΔT between two points in the wire, $\mathbf{r}_1$ and $\mathbf{r}_2$, will oscillate at 2ω with an amplitude that depends on the differences between the current-density amplitudes $j_0(\mathbf{r})$ and the local values of $\alpha(\mathbf{r})$ at those two points in the wire:

$$\Delta T(\mathbf{r}_1,\mathbf{r}_2,t) \approx T_0(\mathbf{r}_1) - T_0(\mathbf{r}_2) + \left[\alpha(\mathbf{r}_1) j_0^2(\mathbf{r}_1) - \alpha(\mathbf{r}_2) j_0^2(\mathbf{r}_2)\right] \left[\frac{1-\cos(2\omega t)}{2}\right]. \quad [20]$$

This will create a transverse signal at 2ω. For this reason we keep our metal measurement leads a distance of at least one wire width, w, away from the conducting path of the wire, so that $j(\mathbf{r}_1) \approx j(\mathbf{r}_2) \approx 0$. The rest of the measurement lead is made of the same graphene as the wire itself. By moving the point of connection away from where there is any temperature variation, we can minimize the influence of this Seebeck effect. We have varied the length L of the graphene leads between $1w$ and $5w$ and have not found any systematic variation of the 2ω signal with their length.

Hall voltage due to a current-induced magnetic field

The most important source of an extraneous signal at 2ω is the Hall effect due to the magnetic field generated by the current itself. An oscillating electric current at ω through the circuit contributes to a time-varying magnetic field $\mathbf{B}(\mathbf{r}) \propto i$. Eq. 1 (main text) shows that this produces a Hall potential that is quadratic in the current and which therefore appears at 2ω. We note that the magnetic field created by such a current is proportional to the *current*, while ΔV_{geom} is dependent only on the *current density* (or i/t) as shown in Eq. 3 (main text). Thus, we can minimize this extraneous source of a 2ω signal by reducing the thickness, t, of the conducting wire; by making the wire thin, we reduce the current without changing the current density. By using atomically thin, monolayer graphene, we have minimized the thickness, t.

We can express the magnetic field as

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0 i C_{\text{self}}(\mathbf{r})}{r}. \quad [21]$$

The dimensionless function $C_{\text{self}}(\mathbf{r})$ depends on how the entire circuit is arranged. We include the r in the denominator of Eq. 21 because the denominator must have units of length and is thus set by a length scale in the system.

Estimation of effect size for graphene circuits. The Hall potential between our measurement leads is determined by the component of the magnetic field perpendicular to the plane of the circuit pattern. The largest contribution to this component of the magnetic field comes from the wires that are actually in that plane — the wires on the substrate itself. Therefore, we choose $r \equiv w = (r_{\text{out}} - r_{\text{in}})$ for the denominator, since w is the length scale that characterizes our graphene wire within the plane, and w is also the approximate distance that the curved wire in our design is offset from the straight portions of the circuit pattern.

We next need to make an assumption about the value of the prefactor $C_{\text{self}}(\mathbf{r})$ at the locations in our circuit where we measure transverse potentials. We note that in the case of a point that is a distance w away from an infinite wire carrying a current i, Ampère's Law tells us that $C_{\text{self}} = 1/(2\pi) \approx 0.16$. In order to ensure that we are not underestimating the magnitude of $\mathbf{B}$, we set $C_{\text{self}} = 1$. We can then substitute for $\mathbf{B}$ to estimate the self-induced Hall potential:

$$\Delta V_{\text{Hall}} = C_{\text{self}} \frac{\mu_0 i^2}{ntq(r_{\text{out}} - r_{\text{in}})} \leq \frac{\mu_0 i^2}{n_{\text{2D}} q(r_{\text{out}} - r_{\text{in}})}, \quad [22]$$

where t in this equation represents the thickness of the wire out of plane. The two-dimensional carrier density $n_{\text{2D}} = nt$ is given by

$$n_{\text{2D}} = \frac{\varepsilon_0 \varepsilon V_{\text{bg}}}{dq}, \quad [23]$$

as described by Novoselov *et al* [17]. Here ε_0 and ε are the permittivities of free space and SiO_2, V_{bg} is the back-gate voltage relative to the Dirac point, and d is the thickness of the SiO_2 layer.

For any given value of the current, we find that the theoretical value of ΔV_{geom} (assuming uniform material properties in the graphene) is more than an order of magnitude larger than ΔV_{Hall}, as shown in Fig. S3. For this calculation we assume the graphene is sufficiently doped that the charge neutrality point is 100 V away in back-gate voltage.

Alternative options. Minimizing the wire thickness t is not the only means of minimizing the current-induced Hall effect relative to ΔV_{geom}. One option is to use a superconductor, which will expel any magnetic fields as long as they do not exceed the critical field of the superconductor. This effectively places an upper bound on the current or current density that one may use for the measurement in the superconductor, and one must ensure that the noise floor of the lock-in amplifier does not exceed the expected size of the signal ΔV_{geom} when using that amount of current.

Another option is to intentionally design the circuit's geometry to minimize the function $C_{\mathrm{self}}(\mathbf{r})$ at the location of the curved wire, as a way to minimize ΔV_{Hall} in Eq. 22. One way to accomplish this would be a two-layer circuit design. In the first layer, a thin, flat wire follows a path through a curved section and a straight section, similar to the graphene circuit design that we have used. At one end of the pattern, however, the wire could instead connect to a second conducting layer, directly on top of the first, with a thin insulating spacer, such that the wire then traces out an identical path but such that the current will travel through the second layer in the opposite direction. The contribution of both layers to the out-of-plane magnetic field would be minimized, for the same reason that the magnetic field is minimized outside of a coaxial cable or a pair of twisted cables that carries current in both directions.

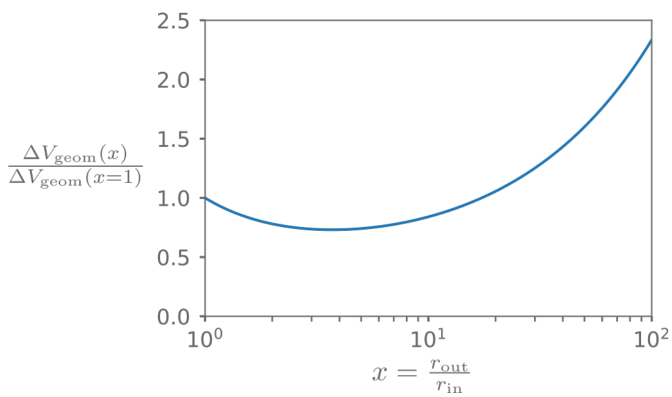

Fig. S1. The full expression for the transverse potential across a curved wire, ΔV_{geom}, deviates from the narrow-wire approximation, $\Delta V_{\mathrm{geom}}(x = 1)$, as x increases.

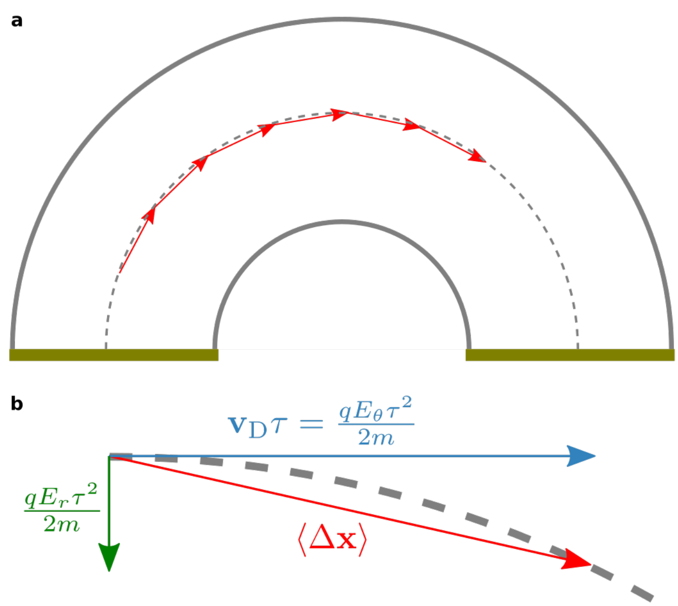

Fig. S2. (**a**) Average displacement vectors of charge carriers (red, not to scale) must follow circular paths in order for the current to follow the path of the wire. The fact that their endpoints lie along circles means that they are not tangent to the circular path itself, and thus the acceleration of the carriers after each collision must not be strictly azimuthal. (**b**) A single average displacement step (red) is the result of acceleration due to orthogonal components of the electric field: an azimuthal field component (blue, in the direction of the drift velocity) and a radial field component (green). The tip and tail of the resultant average displacement vector lie along a circular path.

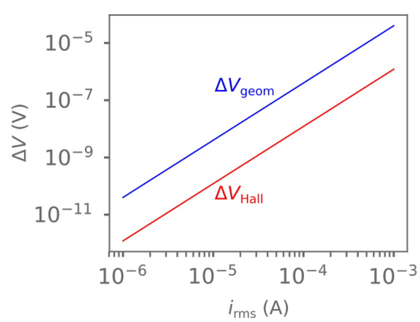

Fig. S3. Predicted values of transverse potentials due to centripetal acceleration (blue) and due to the Hall effect from the magnetic field generated by the current itself (red).