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THE TOPOLOGY OF SURFACE BUNDLES: COHOMOLOGY AND ENUMERATIONS OF FIBERINGS

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For my family

"Maybe you should thank your wife or something." - *Margaret Nichols*

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ABSTRACT

This thesis undertakes a study of surface bundles, especially 4-manifolds equipped with one or more surface bundle structures. A central theme is the interplay between the number of surface bundle structures on a manifold, the properties of the associated monodromy representations, and the algebro-topological invariants of the manifold. In Chapter 2, we show that any non-trivial surface bundle with monodromy in the Johnson kernel has a unique fibering. In Chapter 3, we provide the first examples of 4-manifolds admitting 3 or more surface bundle structures. In Chapter 4, we study how the cohomology algebra of a surface bundle can be computed from the monodromy representation, and relate this problem to the cohomology of the mapping class group and the Torelli subgroup.

CHAPTER 1
INTRODUCTION

A *surface bundle* $p : E \to B$ is a fiber bundle with fiber F a (finite-type, oriented) surface. A standing assumption of this thesis is that the genus of F is at least 2. The theory of surface bundles is the "simplest non-linear bundle theory"[1], and many basic aspects of the topology and classification of surface bundles are still poorly understood. This thesis offers several results along these lines. The first two chapters address issues related to the existence and (non)uniqueness of surface bundle structures on 4-manifolds. These problems turn out to be intimately related to the study of the classical topological invariants of surface bundles (the ring structure of the cohomology algebra, in particular), and in the third chapter, we determine this ring structure in general, and find a relationship between this and the cohomology of the classifying space of surface bundles.

The simplest examples of surface bundles occur in the setting of 3-manifold theory, where fibered 3-manifolds form a fantastically rich class of objects. The theory of the Thurston norm gives a detailed picture of the set of possible ways that a compact, oriented 3-manifold M can fiber as a surface bundle.[2] If $b_1(M) > 1$, then M admits infinitely many such fibrations $\Sigma_g \to M \to S^1$; finitely many for each $g \geq 2$. The main goal of this thesis is to take up a similar sort of inquiry for 4-manifolds $\Sigma_g \to E \to \Sigma_h$ fibering as a surface bundle over a surface of genus $g \geq 2$.

When $h = 1$ (i.e. the base surface is a torus), a similar story as in the 3-manifold setting unfolds; if M^3 is a 3-manifold admitting infinitely many fiberings $p : M \to S^1$, then $p \times \mathrm{id} : M^3 \times S^1 \to S^1 \times S^1$ admits infinitely many fiberings as well. However, in stark contrast to the 3-manifold setting, F.E.A. Johnson showed that every surface bundle over a surface $\Sigma_g \to E^4 \to \Sigma_h$ with $g, h \geq 2$ has at most finitely many fiberings (see [18], [14], [29] or Proposition 3.2.1 of Chapter 3 for various accounts). It is possible to deduce from Johnson's work that there is a universal upper bound on the number of fiberings that any surface bundle over a surface E^4 can have, as a function of the Euler characteristic $\chi(E)$. Specifically, Proposition 3.2.1 of Chapter 3 shows that if E^4 satisfies $\chi(E) = 4d$, then E has at most $\sigma_0(d)(d+1)^{2d+6}$ fiberings as a surface bundle over a surface, where $\sigma_0(d)$ denotes the number of positive divisors of d.

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1. One would of course argue that the theory of S^1-bundles is simpler, but in light of the homotopy equivalence of classifying spaces $\mathrm{BHomeo}^+(S^1) \simeq \mathrm{BSO}(2)$, many aspects of the theory of S^1 bundles are essentially linear.

2. While the theory of the Thurston norm gives the most complete picture of the ways in which a 3-manifold fibers over S^1, earlier examples of 3-manifolds with multiple surface bundle structures were found by J. Tollefson [37] and D. Neumann [28].

The simplest example of a surface bundle over a surface with multiple fiberings[3] is that of a product $\Sigma_g \times \Sigma_h$, which has the two projections onto the factors Σ_g and Σ_h. Prior to the results of this thesis, there was essentially one general method for constructing nontrivial examples of surface bundles over surfaces with multiple fiberings, and they all yielded bundles with only two known fiberings (although it is in theory possible that these examples could admit three or more, cf Chapter 3 Question 3.2.4). Such examples were first constructed by Atiyah and Kodaira (see [1], [21], as well as the account in [27]), and proceeded by taking a fiberwise branched covering of particular "diagonally embedded" submanifolds of products of surfaces.

It is worth remarking that if one is willing to relax the requirement that both the base and fiber surface have negative Euler characteristic, then it is possible to construct examples of 4-manifolds E admitting infinitely many fibrations over the torus T^2. If M^3 is a 3-manifold admitting infinitely many fibrations over S^1, then $E = M^3 \times S^1$ has the required properties. However, Johnson's result indicates that when $g, h \geq 2$, the situation is necessarily much more rigid and correspondingly richer. The mechanism by which $E = M^3 \times S^1$ admits infinitely many fiberings is completely understood via the theory of the Thurston norm. In contrast, in the case $g, h \geq 2$, entirely new phenomena will necessarily occur.

Let $p : E \to B$ be a surface bundle with fiber $F \cong \Sigma_g$ a closed surface of genus g (here and throughout, if left unspecified, B is assumed to be paracompact and Hausdorff). The fundamental invariant associated to p is the *monodromy representation*. This is a homomorphism

$$\rho : \pi_1(B) \to \mathrm{Mod}(\Sigma_g),$$

where $\mathrm{Mod}(\Sigma_g) := \pi_0(\mathrm{Diff}(\Sigma_g))$ is the *mapping class group*[4]. The monodromy representation records how the topology of a fiber is altered after being transported around a given loop in B. Let $\mathrm{Mod}(\Sigma_{g,*}) := \pi_0(\mathrm{Diff}(\Sigma_g, *))$ denote the group of isotopy classes of diffeomorphisms relative to a fixed point $* \in \Sigma_g$. When $p : E \to B$ is equipped with a section, ρ lifts to a homomorphism $\rho : \pi_1 B \to \mathrm{Mod}(\Sigma_{g,*})$.

There is a classifying space $\mathrm{BDiff}(\Sigma_g)$ (resp. $\mathrm{BDiff}(\Sigma_g, *)$) for surface bundles (resp. for surface bundles equipped with a section). A fundamental theorem of Earle-Eells [7], in com-

3. The most straightforward notion of "distinction" for fiberings is that of fiberwise diffeomorphism. In this thesis, we will also have occasion to consider a strictly stronger notion known as "π_1-fiberwise diffeomorphism". See Section 2.2.1 for the precise definition of π_1-fiberwise diffeomorphism, and see Proposition 3.3.1.2, as well as Remark 3.3.1.4, for a discussion of why we adopt this convention.

4. This group will also at times be written Mod_g.

bination with some basic algebraic topology, implies that there are homotopy equivalences

$$\mathrm{BDiff}(\Sigma_g) \simeq K(\mathrm{Mod}_g, 1)$$
$$\mathrm{BDiff}(\Sigma_g, *) \simeq K(\mathrm{Mod}_{g,*}, 1).$$

This implies that, given a group extension

$$1 \to \pi_1(\Sigma_g) \to \Pi_* \to \Pi \to 1, \tag{1.1}$$

there is an associated Σ_g-bundle $\pi : K(\Pi_*, 1) \to K(\Pi, 1)$ for which the monodromy representation $\rho : \Pi \to \mathrm{Mod}(\Sigma_g)$ coincides with the map $\Pi \to \mathrm{Out}(\pi_1(\Sigma_g)) \cong \mathrm{Mod}(\Sigma_g)$ attached to the group extension (1.1). The extension (1.1) splits if and only if ρ lifts to $\rho : \Pi \to \mathrm{Aut}(\pi_1\Sigma_g) \cong \mathrm{Mod}(\Sigma_{g,*})$. Because of this equivalence, we will be somewhat lax in passing between the setting of surface bundles and the setting of group extensions with surface group kernel.

In light of the homotopy equivalences above, one can interpret elements of $H^*(\mathrm{Mod}_g; M)$ (for an arbitrary $\mathbb{Q}\,\mathrm{Mod}_g$-module M) as "M-valued characteristic classes of Σ_g-bundles".

The monodromy is in fact a *complete* invariant of surface bundles, as a consequence of the well-known correspondence (see, e.g. [8, Section 5.6])

$$\left\{ \begin{array}{c} \text{Bundle-isomorphism classes of} \\ \text{oriented } \Sigma_g\text{-bundles over } B \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{Conjugacy classes of representations} \\ \pi_1(B) \to \mathrm{Mod}(\Sigma_g) \end{array} \right\}.$$

This raises the question of translating between topological and geometric properties of surface bundles on the one hand, and on the other, algebraic or geometric properties of the monodromy representation, a problem which goes by the name of the "monodromy-topology dictionary". Certain entries in this dictionary are well-established, for instance Thurston's landmark result that a fibered 3-manifold $\Sigma_g \to M_\phi \to S^1$ admits a complete hyperbolic metric if and only if the monodromy is a so-called "pseudo-Anosov" element of $\mathrm{Mod}(\Sigma_g)$.

One of the central themes of this thesis is the interplay between the topology of surface bundles and the algebraic structure of the mapping class group, especially with regards to the so-called *Johnson filtration*. The Johnson filtration is a decreasing filtration

$$\mathcal{I}_g(1) \supset \mathcal{I}_g(2) \supset \mathcal{I}_g(3) \supset \dots$$

on the mapping class group $\mathrm{Mod}(\Sigma_g) = \mathcal{I}_g(1)$. The terms $\mathcal{I}_g(2)$ and $\mathcal{I}_g(3)$ will be of particular interest. The term $\mathcal{I}_g(2)$ is known as the *Torelli group*, and is defined to be the kernel of the symplectic representation $\Psi : \mathrm{Mod}(\Sigma_g) \to \mathrm{Sp}_{2g}(\mathbb{Z})$. Typically the Torelli group will be written $\mathcal{I}_g$. The term $\mathcal{I}_g(3)$ is known as the *Johnson kernel*, and is defined as the group generated by Dehn twists T_γ with γ a *separating* curve.[5] The Johnson kernel will typically be referred to as $\mathcal{K}_g$. There are versions of the Johnson filtration for the pointed mapping class group $\mathrm{Mod}(\Sigma_{g,*})$ as well as the mapping class group of a surface with boundary $\mathrm{Mod}(\Sigma_g^1)$, and these groups will be notated with a decoration $*$ or ^{1} as necessary.

In the remainder of this introduction we will outline the body of the thesis. Chapter 2 studies surface bundles over surfaces with monodromy contained in the Johnson kernel $\mathcal{K}_g$. The main result of the chapter gives an entry in the monodromy-topology dictionary, relating the Johnson kernel monodromy assumption to the uniqueness of the surface bundle structure.

Theorem A. *Let $\pi : E \to B$ be a surface bundle over a surface with monodromy in the Johnson kernel $\mathcal{K}_g$. If E admits two distinct fiberings then E is diffeomorphic to $B \times B'$, the product of the base spaces. In other words, any nontrivial surface bundle over a surface with monodromy in $\mathcal{K}_g$ admits a unique fibering.*

In Chapter 3, we will give a method for constructing surface bundles over surfaces with multiple fiberings, including the first examples of bundles admitting an arbitrarily large number of fiberings. The results of Chapter 2 are complimentary in that our concern here is in addressing the question of when surface bundles over surfaces admit *unique* fiberings. The surface bundles over surfaces of Chapter 3 can be constructed so as to have monodromy contained in $\mathcal{I}_g$. It follows that the hypothesis in Theorem A that the monodromy be contained in $\mathcal{K}_g$ is effectively *sharp* with respect to the Johnson filtration.

Theorem A is proved by first relating the monodromy representation of a surface bundle over a surface $E^4 \to B^2$ to the cohomology ring $H^*(E)$. This analysis will show that the integral cohomology of a surface bundle over a surface with monodromy in $\mathcal{K}_g$ is as simple as possible. It is then shown that in these circumstances, obstructions to possessing alternative fiberings can be extracted from $H^*(E)$.

In a similar spirit we also have the following general criterion which we believe to be of independent interest, for a surface bundle over a surface to possess a unique fibering. It can be viewed as the 4-manifold analogue of a well-known fact about fibered 3-manifolds (see Remark 2.2.6).

Theorem 2.2.5. *Let $p : E \to B$ be a surface bundle over a surface B of genus $g \geq 2$ with monodromy representation $\rho : \pi_1 B \to \mathrm{Mod}(\Sigma_g)$. Suppose that the space of invariant*

5. As discussed further in section 2.4.1, there is an alternative characterization of $\mathcal{K}_g$ as the kernel of the Johnson homomorphism (to be defined in section 2.4.1). We will pass between these two perspectives as the situation dictates.

cohomology $(H^1(F, \mathbb{Q}))^\rho$ *(equivalently, the coinvariant homology of the fiber* $(H_1(F, \mathbb{Q}))_\rho$*) vanishes. Then* E *admits a unique fibering.*

The chapter is organized as follows. In Section 2.1, we give various characterizations of the notion of equivalence under consideration. In Section 2.2, we prove Theorem 2.2.5. Sections 2.3 - 2.6 are devoted to the proof of Theorem A. Section 2.3 is devoted to a lemma in differential topology that features in later stages of the proof of Theorem A. The technical heart of the chapter is Section 2.4. In it, we first give an overview of the classical description of the Johnson homomorphism τ in terms of the intersection theory of surfaces in 3-manifolds that fiber over S^1. Using this description of τ we then carry out a construction of 3-manifolds embedded in surface bundles over surfaces that realizes the relationship between the Johnson homomorphism and the intersection product in the homology of the surface bundle. We give a complete description of the product structure in (co)homology for a surface bundle over a surface with monodromy in $\mathcal{I}_g$. These methods of Section 2.4 extend to an arbitrary surface bundle over a surface, but we do not state them in this level of generality since we have no need for them here.

Section 2.5 is devoted to some technical results concerning multisections of surface bundles, and their connection to splittings on rational cohomology. These results are used in the course of proving Theorem A.

In Section 2.6 we turn finally to the proof of Theorem A. The result follows from an analysis of the intersection product structure in $H_*(E)$ for a surface bundle over a surface $\Sigma_g \to E \to \Sigma_h$ with monodromy in $\mathcal{K}_g$. The results of Section 2.4 are applied to show that when the monodromy of $\Sigma_g \to E \to \Sigma_h$ is contained in $\mathcal{K}_g$, then E, which necessarily has $H^*E \approx H^*\Sigma_g \otimes H^*\Sigma_h$ as an additive group, in fact has $H^*E \approx H^*\Sigma_g \otimes H^*\Sigma_h$ (with $\mathbb{Z}$ coefficients) as a graded ring. This condition is then exploited to prove Theorem A.

In Chapter 3, we give the first examples of surface bundles over surfaces that admit 3 or more fiberings; in fact, we give examples admitting arbitrarily many fiberings. The construction is a variant of the Atiyah-Kodaira method. Both constructions start with a trivial bundle equipped with a "diagonal" section. Where Atiyah-Kodaira use this section to construct a fiberwise branched cover, we will use the section to perform a "fiberwise connect sum". The notable features of the construction are summarized below.

Theorem B (Existence of multiple fiberings)**.**

1. *For each* $n \ge 3$ *and each* $g_1 \ge 2$ *there exists a 4-manifold* E*, integers* $g_2, \ldots, g_n$ *(which can be chosen so that* $g_1, \ldots g_n$ *are pairwise distinct), and maps* $p_i : E \to \Sigma_{g_i} (i = 1, \ldots, n)$ *realizing* E *as the total space of a surface bundle over a surface in at least* n *ways, distinct up to* π_1*-fiberwise diffeomorphism. If* $g_i \neq g_j$*, the fibers of* p_i *and* p_j *have*

distinct genera; consequently p_i and p_j are inequivalent up to fiberwise diffeomorphism whenever $g_i \neq g_j$.

2. *There exist constructions as in (1) for which at least one of the monodromy representations $\rho_i : \pi_1 \Sigma_{g_i} \to \mathrm{Mod}_{h_i}$ has image contained in the Torelli group $\mathcal{I}_{h_i} \le \mathrm{Mod}_{h_i}$.*
3. *There exists a sequence of surface bundles over surfaces E_n for which $\chi(E_n) = 24n - 8$ and such that E_n admits 2^n fiberings as a surface bundle over a surface, distinct up to π_1-fiberwise diffeomorphism.*

The bound of Proposition 3.2.1 makes it sensible to define the following function:

$$N(d) := \max \left\{ n \;\middle|\; \begin{array}{c} \text{there exists } E^4, \chi(E) \le 4d, \, E \text{ admits } n \text{ surface bundle structures} \\ \text{distinct up to } \pi_1\text{-fiberwise diffeomorphism.} \end{array} \right\}$$

Phrased in these terms, (3) of Theorem B, in combination with the upper bound of Proposition 3.2.1 implies that

$$2^{(d+2)/6} \le N(d) \le \sigma_0(d)(d+1)^{2d+6},$$

where $\sigma_0(d)$ denotes the number of positive divisors of d. This should be compared to the previous lower bound $N(d) \ge 2$.

The theme of Chapter 4 is the central role that the structure of the cup product in surface bundles plays in the understanding of the cohomology of the mapping class group and its subgroups. We use this perspective to gain a new understanding of the relationships between several well-known cohomology classes, and we also use these ideas to study the topology of surface bundles.

For $i \ge 1$, there is a class $e_i \in H^{2i}(\mathrm{Mod}_g)$ known as the i^{th} *Mumford-Morita-Miller class* (hereafter abbreviated to MMM class). See Definition 4.2.1. The Madsen-Weiss theorem [22] asserts that the so-called "stable" rational cohomology of Mod_g is generated by the MMM classes, and apart from a few sporadic low-genus examples, the algebra generated by the classes e_i are the only known elements of $H^*(\mathrm{Mod}_g)$. In [19], N. Kawazumi introduced a generalization of the MMM classes, defining classes $m_{ij} \in H^{2i+j-2}(\mathrm{Mod}_{g,*}; H_1^{\otimes j})$, specializing to $m_{i,0} = e_{i-1}$. Again, see Definition 4.2.1.

The content of Theorem C below is that the cup product form on the total space E gives a characteristic class for surface bundles. Theorem C also gives an "intrinsic meaning" to the twisted MMM class $m_{0,k}$ in much the same way that the first MMM class $e_1 \in H^2(\mathrm{Mod}_g)$ has an interpretation as the signature of the total space of a surface bundle over a surface (see [27, Proposition 4.11]).

Theorem C (Cup product as characteristic class)**.** *For all $k \geq 2$ and $g \geq 2$, the twisted MMM class $m_{0,k} \in H^{k-2}(\mathrm{Mod}_{g,*}; \wedge^k H_1)$ computes the cup product in surface bundles in the following sense:*

Suppose B is a paracompact Hausdorff space and $f : B \to K(\mathrm{Mod}_{g,*}, 1)$ is a map classifying a surface-bundle-with-section $\pi : E \to B$. Then for all $i \geq 0$ there is a splitting of vector spaces

$$H^i(E) \cong H^{i-2}(B) \oplus H^{i-1}(B; H_1) \oplus H^i(B). \tag{1.2}$$

Let $\varepsilon : H^{i-1}(B; H_1) \to H^i(E)$ denote the inclusion associated to this splitting. For $1 \leq i \leq k$ and any $d_1, \ldots, d_k$, let $x_i \in H^{d_i-1}(B; H_1)$; for convenience set $D := \sum d_i$. Then there are the following expressions for the components of the product $\varepsilon(x_1)\ldots\varepsilon(x_k) \in H^D(E)$ in the splitting (1.2):

$$H^{D-2}(B)\text{- component:} \qquad (-1)^{\gamma} m_{0,k} \lrcorner (x_1, \ldots, x_k) \tag{1.3}$$

$$H^{D-1}(B; H_1)\text{- component:} \qquad (-1)^{\gamma+1} \varepsilon(m_{0,k+1} \lrcorner (x_1, \ldots, x_k)) \tag{1.4}$$

$$H^{D}(B)\text{- component:} \qquad 0 \tag{1.5}$$

(see Definition 4.2.4 for the meaning of $m_{0,j} \lrcorner (x_1, \ldots, x_k)$, and Equation (4.8) for the definition of γ).

The line of thought culminating in Theorem C begins with D. Sullivan [35], who showed that every element of $\wedge^3 V$ (for V an arbitrary finitely generated torsion-free $\mathbb{Z}$-module) arises as the cup product form $\wedge^3 H^1(M; \mathbb{Z}) \to \mathbb{Z}$ for some 3-manifold M. Johnson [15] incorporated some of these ideas in his far-reaching theory of the Johnson homomorphism $\tau : H_1(\mathcal{I}_{g,*}) \to \wedge^3 H_1$, one definition of which is by means of the cup product form in a 3-manifold fibering as a surface bundle over S^1.

In one direction, the Johnson homomorphism was generalized by S. Morita [25], who constructed an extension of τ by means of a class $\tilde{k} \in H^1(\mathrm{Mod}_{g,*}; \wedge^3 H_1)$ restricting to τ on $\mathcal{I}_{g,*}$. In [26], he showed that all of the MMM classes e_i can be expressed in terms of $\tilde{k}$.

Another generalization of the Johnson homomorphism was given by Johnson himself [16], who gave a definition of "higher Johnson invariants" $\tau_k : H_k(\mathcal{I}_{g,*}) \to \wedge^{k+2} H_1$ (see Definition 4.4.1), but his definition was formulated as a generalization of a different method of constructing the Johnson homomorphism. Theorem D below can be viewed as a synthesis of Morita's and Johnson's perspectives, in that it shows that the twisted MMM classes $m_{0,k}$ restrict on $\mathcal{I}_{g,*}$ to (a multiple of) τ_{k-2}.

Theorem D (Extending the higher Johnson invariants)**.** *There is an equality for all $g \geq 2$ and $k \geq 2$*

$$m_{0,k} = (-1)^k k!\ \tau_{k-2}$$

as elements of $H^{k-2}(\mathcal{I}_{g,*};\wedge^k H_1) \cong \mathrm{Hom}(H_{k-2}(\mathcal{I}_{g,*};\mathbb{Q}),\wedge^k H_1)$.

The cases $k = 2, 3$ were established by Kawazumi and Morita in [20]. In [6], T. Church and B. Farb developed a method for studying the map τ_k. A central component of their computation is the principle that, when viewed as a homomorphism $H_k(\mathcal{I}_{g,*}) \to \wedge^{k+2} H_1$, the Johnson invariant τ_k is a map of representations of $\mathrm{Sp}(2g,\mathbb{Q})$. Johnson showed in [15] that $\tau = \tau_1$ is a rational isomorphism and in [16, Question C], asked if the same was true for all τ_k. In [10], R. Hain showed that τ_2 was not injective. Church and Farb later used their methods to show that τ_k is not injective for any $2 \le k < g$. This leaves the question of surjectivity of τ_k as an unresolved aspect of the theory of the cohomology of $\mathcal{I}_{g,*}$. Church and Farb showed that $\tau_2 : H_2(\mathcal{I}_{g,*}) \to \wedge^4 H_1$ is a surjection, but did not address higher k, or the behavior of τ_2 on $\mathcal{I}_g^1$.

In the following theorem, we show that the question of surjectivity of τ_k (when pulled back to $\mathcal{I}_g^1$) is intimately related to another well-known open question about the homology of the Torelli group. It is well-known (see, e.g. the introduction to [26]) that the MMM classes e_{2i+1} of *odd* index vanish when restricted to $\mathcal{I}_g$. However, the behavior of the *even*-index classes e_{2i} on $\mathcal{I}_g$ is completely unknown.

Theorem E (Higher Johnson invariants detect MMM classes)**.** *For all i, the restriction of e_i to $H^{2i}(\mathcal{I}_g^1;\mathbb{Q})$ is nonzero if and only if the* $\mathrm{Sp}(2g,\mathbb{Q})$*-representation* $\mathrm{Im}(\tau_{2i} : H_{2i}(\mathcal{I}_g^1) \to \wedge^{2i+2} H_1)$ *contains a copy of the trivial representation $V(\lambda_0)$.*

The primary case of interest is of course i even, but as a corollary of Theorem E and the vanishing of e_{2i-1} on $\mathcal{I}_g^1$, it follows that for all $i \ge 1$, the map $\tau_{4i-2} : H_{4i-2}(\mathcal{I}_g^1) \to \wedge^{4i} H_1$ fails to contain a copy of $V(\lambda_0)$, even though $\wedge^{4i} H_1$ always does. This gives a partial resolution of Johnson's question.

Theorem F (Non-surjectivity of τ_{4k-2})**.** *For all $k \ge 1$, the map*

$$\tau_{4k-2} : H_{4k-2}(\mathcal{I}_g^1) \to \wedge^{4k} H_1$$

is not surjective.

As an application of Theorems C and D, we obtain some results concerning the topology of surface bundles. If $\pi : E \to B$ is a Σ_g-bundle with monodromy contained in $\mathcal{I}_g$, it is well-known that $H_*(E) \cong H_*(B) \otimes H_*(\Sigma_g)$, an isomorphism of graded vector spaces (see Section **??** for the relevant terminology). Briefly put, surface bundles with Torelli monodromy are "homology products". In general, the additive isomorphism $H^*(E) \cong H^*(B) \otimes H^*(\Sigma_g)$ is very far from being an isomorphism of rings, as the rich theory of the Johnson homomorphism attests to. For individual elements $\phi \in \mathrm{Mod}_g$, it is well-understood when a mapping torus

$\pi : M^3_\phi \to S^1$ satisfies a multiplicative isomorphism $H^*(M_\phi) \cong H^*(S^1) \otimes H^*(\Sigma_g)$: such an isomorphism holds if and only if $\phi \in \mathcal{K}_g$, the so-called *Johnson kernel* (see the beginning of Section 4.6 and in particular (4.11)). However, if $\pi : E \to B$ is a Σ_g-bundle over a higher-dimensional B with monodromy contained in $\mathcal{K}_g$, it is not *a priori* clear whether a multiplicative isomorphism $H^*(E) \cong H^*(B) \otimes H^*(\Sigma_g)$ must hold. We show that this is the case.

Theorem G (Künneth formula). *Let $\pi : E \to B$ be a Σ_g-bundle over a paracompact Hausdorff space B with monodromy $\rho : \pi_1 B \to \mathcal{K}_{g,*}$ contained in the Johnson kernel. Then there is an isomorphism of rings*

$$H^*(E) \cong H^*(B) \otimes H^*(\Sigma_g).$$

The case $B = S^1$ is essentially a *definition* of $\mathcal{K}_{g,*}$. The case $B = \Sigma_h$ a surface was dealt with by explicit construction in Chapter 2.

A final corollary of this theorem is the vanishing of *all* higher Johnson invariants on $\mathcal{K}_{g,*}$. As remarked above, the vanishing of $\tau = \tau_1$ on $\mathcal{K}_{g,*}$ is a definition, but it is not *a priori* clear that this implies the vanishing of higher invariants. Nonetheless, the results of the chapter combine to show that this is the case.

Theorem H (Vanishing of τ_k on $\mathcal{K}_{g,*}$). *For each $k \geq 1$, the restriction of $\tau_k \in H^k(\mathcal{I}_{g,*}, \wedge^{k+2} H_1)$ to $\mathcal{K}_{g,*}$ is zero.*

The methods of the chapter are primarily homological and make heavy use of the theory of the Gysin homomorphism. As the central objects of study are the twisted MMM classes m_{ij} introduced by Kawazumi in [19], we will frequently make reference to their theory, especially some later developments by Kawazumi-Morita in [20].

In Section 4.1, we review some preliminary material, including the relationship between surface bundles and the mapping class group, some constructions from multilinear algebra and symplectic representation theory, and a primer on the Gysin homomorphism. Section 4.2 is a primer on Kawazumi and Morita's work on the twisted MMM classes. The latter four sections are devoted to the proofs of theorems C, D, E, G respectively.

CHAPTER 2
THE JOHNSON KERNEL AND SURFACE BUNDLES OVER SURFACES WITH UNIQUE FIBERINGS

2.1 Equivalence

If E is a smooth n-manifold and $p_i : E \to B_i,\ i = 1, \ldots, k$ are projection maps for various fiber bundle structures on E, we can consider the product of all the projection maps:

$$p_1 \times \cdots \times p_k : E \to B_1 \times \cdots \times B_k.$$

In particular, if E^4 is the total space of a surface bundle over a surface with two fiberings, the *bi-projection* $p_1 \times p_2 : E \to B_2 \times B_2$ is defined. As remarked in the introduction, ultimately we are concerned with fiberwise-diffeomorphism classes of surface bundles. However, it is convenient to consider a more restrictive notion of equivalence which will turn out to have the advantage of being describable purely on the level of the fundamental group.

We say that two fiberings $p_1 : E \to B_1$, $p_2 : E \to B_2$ are π_1*-fiberwise diffeomorphic* if (1) - they are fiberwise diffeomorphic, i.e. there exists a commutative diagram

$$\begin{array}{ccc} E & \xrightarrow{\ \phi\ } & E \\ {\scriptstyle p_1}\big\downarrow & & \big\downarrow{\scriptstyle p_2} \\ B_1 & \xrightarrow[\ \alpha\ ]{} & B_2 \end{array}$$

with ϕ, α diffeomorphisms, and (2) - $\phi_*(\pi_1 F_1) = \pi_1 F_1$ (here, as always, F_i denotes a fiber of p_i). Certainly if p_1, p_2 are π_1-fiberwise diffeomorphic bundle structures, then they are fiberwise-diffeomorphic in the usual sense. We are interested in this notion because we want to always regard the trivial bundle $\Sigma_g \times \Sigma_h$ as having two distinct fiberings. In the setting of fiberwise-diffeomorphism, the projections onto either factor of $\Sigma_g \times \Sigma_g$ yield equivalent fiberings via the factor-swapping map $\phi(x, y) = (y, x)$, which covers the identity on Σ_g, but $\phi_*(\pi_1(\Sigma_g \times \{p\})) \neq \pi_1(\Sigma_g \times \{p\})$. The following proposition asserts that π_1-fiberwise diffeomorphism classes are in correspondence with the fiber subgroups $\pi_1 F \lhd \pi_1 E$. Recall that this is the setting in which F.E.A. Johnson proved his finiteness result (see [18]).

Proposition 2.1.1. *Suppose E is the total space of a surface bundle over a surface in two ways: $p_1 : E \to B_1$ and $p_2 : E \to B_2$. Let F_1, F_2 denote fibers of p_1, p_2 respectively. Then the following are equivalent:*

1. *The fiberings p_1, p_2 are π_1-fiberwise diffeomorphic.*

2. The fiber subgroups $\pi_1 F_1, \pi_1 F_2 \leq \pi_1 E$ are equal.

If $\deg(p_1 \times p_2) \neq 0$ *then the bundle structures* p_1 *and* p_2 *are distinct.*

Proof. First suppose that p_1 and p_2 are equivalent. Appealing to the long exact sequence in homotopy, we see that

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \pi_1 F_1 & \longrightarrow & \pi_1 E & \longrightarrow & \pi_1 B_1 & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \phi_*} & & \downarrow{\scriptstyle \phi_*} & & \downarrow{\scriptstyle \alpha_*} & & \\ 1 & \longrightarrow & \pi_1 F_2 & \longrightarrow & \pi_1 E & \longrightarrow & \pi_1 B_2 & \longrightarrow & 1. \end{array}$$

By assumption $\phi_*(\pi_1 F_1) = \pi_1 F_1$, so that (3.1.2.1) $\implies$ (3.1.2.2) as claimed.

Conversely, suppose that $\pi_1 F_1 = \pi_1 F_2$. Therefore the bundle structures p_1 and p_2 give rise to the same splitting

$$1 \to \pi_1 F \to \pi_1 E \to \pi_1 B \to 1$$

on fundamental group. The monodromy for each bundle can be obtained from this sequence via the map $\pi_1 B \to \mathrm{Out}(\pi_1 F) \approx \mathrm{Mod}(\Sigma_g)$. This shows that the monodromies for the two bundle structures are conjugate, and so via the correspondence (1), there is a bundle-isomorphism $\phi : E \to E$ covering the identity on B. To see that $\phi_*(\pi_1 F_1) = \pi_1 F_1$, consider the induced map on the long exact sequence in homotopy coming from ϕ:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \pi_1 F_1 & \longrightarrow & \pi_1 E & \longrightarrow & \pi_1 B & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \phi_*} & & \downarrow{\scriptstyle \phi_*} & & \| & & \\ 1 & \longrightarrow & \pi_1 F_2 & \longrightarrow & \pi_1 E & \longrightarrow & \pi_1 B & \longrightarrow & 1. \end{array}$$

This shows $\phi_*(\pi_1 F_1) = \pi_1 F_2$, and $\pi_1 F_1 = \pi_1 F_2$ by assumption.

Having established the equivalence of (3.1.2.1) and (3.1.2.2), it remains to show that if $\deg(p_1 \times p_2) \neq 0$, then p_1 and p_2 are distinct. We establish the contrapositive. Suppose that $\pi_1 F_1 = \pi_1 F_2$. For $i = 1, 2$, we view $\pi_1 B_i$ as the quotient $\pi_1 B_i \approx \pi_1 E / \pi_1 F_i$. If $p_1 \times p_2$ is the bi-projection, then in this notation,

$$(p_1 \times p_2)_* : \pi_1 E \to \pi_1 B_1 \times \pi_1 B_2$$

is given by

$$(p_1 \times p_2)_*(x) = (x\, \pi_1 F_1, x\, \pi_1 F_2) = ([x], [x]),$$

where $[x] = x \pmod{\pi_1 F_1} = x \pmod{\pi_1 F_2}$. As $\pi_1 F_1 = \pi_1 F_2$, the quotients $\pi_1 B_1$ and $\pi_1 B_2$

are isomorphic, and as they are $K(G,1)$'s, there is a homotopy equivalence

$$f : B_1 \to B_2.$$

Let g be the map

$$g = (f \times \mathrm{id}) \circ (p_1 \times p_2) : E \to B_2 \times B_2.$$

By the above,

$$\mathrm{Im}(g) = \Delta = \{(x, x) \mid x \in B_2\}.$$

Being non-surjective, g has degree 0. As $p_1 \times p_2$ is the composition of g with a homotopy equivalence, we conclude that also $\deg(p_1 \times p_2) = 0$. □

In general the condition $\deg(p_1 \times p_2) = 0$ on a bi-projection does not imply that the associated fiberings are equivalent. However, in the setting of the Johnson kernel, this is indeed the case.

Proposition 2.1.2. *Suppose E is the total space of a surface bundle over a surface in two ways: $p_1 : E \to B_1$ and $p_2 : E \to B_2$. Let F_1, F_2 denote fibers of p_1, p_2 respectively. Suppose that $\rho_1 : \pi_1 B_1 \to \mathrm{Mod}(F_1)$ is contained in the Johnson kernel $\mathcal{K}_g$. Then the following are equivalent:*

1. *The fiberings p_1, p_2 are not π_1-fiberwise diffeomorphic.*
2. *The fiber subgroups $\pi_1 F_1, \pi_1 F_2 \leq \pi_1 E$ are distinct.*
3. $\deg(p_1 \times p_2) \neq 0$.
4. *E is diffeomorphic to $B_1 \times B_2$.*

The additional assertions in Proposition 2.1.2 will be proved in the course of establishing Theorem A (see Remark 2.6.6).

2.2 Surface bundles over surfaces with unique fiberings

In this section, we prove Theorem 2.2.5. The additive structure of H^*E is central to everything that follows in this chapter, and so we begin with a review of the relevant results. The following theorem was formulated and proved by Morita in [23] for the case of field coefficients of characteristic not dividing $\chi(F)$; subsequently this was improved to integral coefficients in the cohomological setting by Cavicchioli, Hegenbarth and Repovš in [5].

Proposition 2.2.1 (Morita, Cavicchioli - Hegenbarth - Repovš). *Let F be a closed surface of genus $g \geq 2$. The Serre spectral sequence (with twisted coefficients) of any surface bundle $F \to E \to B$ collapses at the E_2 page. Consequently, there are noncanonical isomorphisms for all k*

$$H_k(E, \mathbb{Q}) = H_k(B, \mathbb{Q}) \oplus H_{k-1}(B, H_1(F, \mathbb{Q})) \oplus H_{k-2}(B, \mathbb{Q})$$
$$H^k(E, \mathbb{Z}) = H^k(B, \mathbb{Z}) \oplus H^{k-1}(B, H^1(F, \mathbb{Z})) \oplus H^{k-2}(B, \mathbb{Z})$$

The $H_{k-2}B$ summand of H_kE is canonical, and is realized by the Gysin map $p^!$ which associates to a homology class $x \in B$ the induced sub-bundle E_x sitting over x. Similarly, the H^kB summand is canonical via the pullback map $p^* : H^kB \to H^kE$.

If $F \to E \to B$ has monodromy in $\mathcal{I}_g$, then the coefficient system is untwisted and $H^*(E, \mathbb{Z}) \approx H^*(B, \mathbb{Z}) \otimes H^*(F, \mathbb{Z})$ additively. In particular, $H^*(E, \mathbb{Z})$ is torsion free, and so by the universal coefficients theorem, there is also an isomorphism $H_*(E, \mathbb{Z}) \approx H_*(B, \mathbb{Z}) \otimes H_*(F, \mathbb{Z})$.

Because the surface bundles we will be considering in this chapter have monodromy lying in $\mathcal{I}_g$, we will subsequently take all coefficients to be $\mathbb{Z}$ without further mention. A remark which is obvious from Proposition 2.2.1 is that if $*$ generates $H_0(B)$, then $p^!(*)$ is a primitive class; we will use this fact later on. Here and throughout, we will use the notation

$$[F] = p^!(*) \in H_2(E)$$

to denote the (pushforward of the) fundamental class of the fiber.

The following result is a well-known application of the theory of the Gysin homomorphism, and we state it without proof.

Proposition 2.2.2. *Let $p : E \to B$ be a surface bundle with fiber F. If $\chi(F) \neq 0$, then there are injections*

$$p^* : H^*(B, \mathbb{Q}) \to H^*(E, \mathbb{Q})$$
$$p^! : H_k(B, \mathbb{Q}) \to H_{k+2}(E, \mathbb{Q}).$$

In the case where $H_*(E, \mathbb{Z})$ is torsion-free, the same statements hold with $\mathbb{Z}$ coefficients. In particular, this is true whenever E has monodromy lying in $\mathcal{I}_g$, since in this case $H^*(E, \mathbb{Z})$ is isomorphic to $H^*(F, \mathbb{Z}) \otimes H^*(B, \mathbb{Z})$ as an abelian group (see Proposition 2.2.1).

For surface bundles over surfaces with multiple fiberings, there is an extension of the previous result.

Lemma 2.2.3. *Let E be a 4-manifold with two distinct surface bundle structures $p_1 : E \to B_1$ and $p_2 : E \to B_2$. Then the intersection*

$$p_1^*(H^1(B_1,\mathbb{Q})) \cap p_2^*(H^1(B_2,\mathbb{Q})) = \{0\},$$

and so by Proposition 4.1.1, there is a canonical injection

$$p_1^* \times p_2^* : H^1(B_1,\mathbb{Q})) \oplus H^1(B_2,\mathbb{Q}) \hookrightarrow H^1(E,\mathbb{Q}).$$

Proof. By the universal coefficients theorem, for any space X there is an identification

$$H^1(X,\mathbb{Q}) \approx \mathrm{Hom}(\pi_1 X,\mathbb{Q}).$$

Under this identification, a character $\alpha \in \mathrm{Hom}(\pi_1 B_i,\mathbb{Q})$ is pulled back to $p_i^*(\alpha) \in \mathrm{Hom}(\pi_1 E,\mathbb{Q})$ by precomposition with $(p_i)_*$. In particular, $p_i^*(\alpha)$ vanishes on $\pi_1 F_i = \ker(p_i)_*$. Therefore, any character $\alpha \in p_1^*(H^1(B_1,\mathbb{Q})) \cap p_2^*(H^1(B_2,\mathbb{Q}))$ must vanish on the subgroup generated by $(\pi_1 F_1)(\pi_1 F_2)$.

By Lemma 2.2.4 below, $(\pi_1 F_1)(\pi_1 F_2)$ has finite index in $\pi_1 E$. For any group Γ, any character $\alpha : \Gamma \to \mathbb{Q}$ vanishing on a finite-index subgroup must vanish identically, proving the claim. □

Lemma 2.2.4. *Let E be a surface bundle over a surface with two distinct fiberings $p_i : E \to B_i$; let the fibers be denoted F_1 and F_2, respectively. Then $(\pi_1 F_1)(\pi_1 F_2)$ has finite index in $\pi_1 E$.*

Proof. Consider the cross-projection $\pi_1 F_1 \to \pi_1 B_2$. Let the image of $\pi_1 F_1$ in $\pi_1 B_2$ be denoted Γ. This is a finitely-generated normal subgroup of $\pi_1 B_2$. For any surface group of genus $g \geq 2$, any nontrivial finitely-generated normal subgroup has finite index (see Property ($\mathcal{D}$6) in [18]). If Γ is the trivial group, then $\pi_1 F_1 \leq \pi_1 F_2$, necessarily again of finite index. In this case, the image of $\pi_1 F_2$ in $\pi_1 B_1$ is therefore finite, but $\pi_1 B_1$ is torsion-free. We conclude that $\Gamma \leq \pi_1 B_2$ has finite index. The kernel of the map $\pi_1 E \to (\pi_1 B_2/\Gamma)$ is exactly $(\pi_1 F_1)(\pi_1 F_2)$. □

Recall that if $\rho : G \to \mathrm{GL}(V)$ is a representation, then the *invariant* space V^ρ is defined via

$$V^\rho = \{v \in V : \rho(g)(v) = v \text{ for all } g \in G\}.$$

The space of *co-invariants* V_ρ of the representation is defined as

$$V_\rho = V/W, \qquad \text{where} \qquad W = \{v - \rho(g)(v) | v \in V, g \in G\}.$$

Theorem 2.2.5. *Let $p : E \to B$ be a surface bundle over a surface B of genus $g \geq 2$ with monodromy representation $\rho : \pi_1 B \to \mathrm{Mod}(\Sigma_g)$. Suppose that the space of invariant cohomology $(H^1(F, \mathbb{Q}))^\rho$ (equivalently, the coinvariant homology of the fiber $(H_1(F, \mathbb{Q}))_\rho$) vanishes. Then E admits a unique fibering.*

Proof. For any surface bundle $p : E \to B$ with monodromy ρ and any choice of coefficients, there is a (noncanonical) splitting

$$H^1(E) = p^*(H^1(B)) \oplus (H^1(F))^\rho.$$

(see Proposition 2.2.1). If $(H^1(F, \mathbb{Q}))^\rho = 0$, then this reduces to

$$H^1(E, \mathbb{Q}) = p^* H^1(B, \mathbb{Q}).$$

If $p_2 : E \to B_2$ is a second, distinct fibering, the above shows that $p_2^*(H^1(B_2, \mathbb{Q})) \leq p^* H^1(B, \mathbb{Q})$. However, this contradicts Lemma 2.2.3. ☐

Remark 2.2.6. Recall that a surface bundle over S^1, viewed as the mapping torus M of some diffeomorphism ϕ of a surface F, admits a unique fibering if and only if $b_1(M) = 1$. This is the case exactly when $(H_1(F, \mathbb{Q}))_\phi = 0$, so Theorem 2.2.5 is the counterpart to this fact in dimension 4. Moreover, a random element $\phi \in \mathrm{Mod}(\Sigma_g)$ satisfies $(H_1(F, \mathbb{Q}))_\phi = 0$ (see [30]). It easily follows that a generic monodromy representation will also have $(H_1(F, \mathbb{Q}))_\rho = 0$: "most" surface bundles over surfaces have a single fibering. The proof of Theorem 2.2.5 is special to the case of surface bundles over surfaces and it is not clear if Theorem 2.2.5 is true in greater generality.

2.3 Bi-projections

In this section we state and prove the key lemma from differential topology needed for the proof of Theorem A.

Proposition 2.3.1. *Let E be a 4-manifold with surface bundle structures $p_1 : E \to B_1$ and $p_2 : E \to B_2$. Let F_1, F_2 denote fibers of p_1, p_2 lying over a regular value of $p_1 \times p_2$. If the mod-2 degree $\deg_2(p_1 \times p_2 : E \to B_1 \times B_2) \neq 0$, then the following four quantities are equal as elements of $\mathbb{Z}/2\mathbb{Z}$:*

1. $\deg_2(p_1 \times p_2 : E \to B_1 \times B_2)$
2. $\deg_2(p_1|_{F_2} : F_2 \to B_1)$
3. $\deg_2(p_2|_{F_1} : F_1 \to B_2)$

4. $I_E(F_1, F_2)$ *(the mod-2 algebraic intersection number)*

Proof. As p_1 and p_2 are projection maps for fiber bundle structures on E, they are everywhere regular, and $\ker(dp_1)_x$ is identified with the tangent space to the fiber of p_1 through x. Let $z = (b_1, b_2) \in B_1 \times B_2$ be a regular value for $p_1 \times p_2$. It follows from the assumption that $\deg_2(p_1 \times p_2 : E \to B_1 \times B_2) \neq 0$ that $d(p_1 \times p_2)_x$ is an isomorphism for all $x \in (p_1 \times p_2)^{-1}(z)$ (and that this preimage is non-empty). The kernel of $d(p_1 \times p_2)_x$ is just the intersection of the kernels of $d(p_1)_x$ and $d(p_2)_x$. It follows that for all $x \in (p_1 \times p_2)^{-1}(z)$,

$$T_x E \approx T_x F_1 \oplus T_x F_2. \tag{2.1}$$

Note that this shows that the fibers F_1, F_2 over b_1, b_2 respectively are transverse.

Recall that if $f : X^n \to Y^n$ is a smooth map of oriented closed n-manifolds, then

$$\deg_2(f) = \# f^{-1}(y),$$

where y is any regular value of f. If Y, Z are smoothly embedded and transversely intersecting oriented submanifolds of the oriented manifold X such that $\dim(X) = \dim(Y) + \dim(Z)$, then the mod-2 algebraic intersection number of Y and Z is computed as

$$I_X(Y, Z) = \#(Y \cap Z).$$

It follows from the definitions that

$$(p_1 \times p_2)^{-1}(b_1, b_2) = p_1|_{F_2}^{-1}(b_1) = p_2|_{F_1}^{-1}(b_2) = F_1 \cap F_2.$$

Therefore each of the sums computing (2.3.1.1) – (2.3.1.4) take place over the same set of points, and the result follows. □

2.4 Cup products and the Johnson homomorphism

The goal of this section is to give a construction of embedded submanifolds in a surface bundle over a surface E that will be explicit enough to compute the intersection form on homology, or dually the cup product structure in cohomology. One of the original definitions of the Johnson homomorphism was via the cup product structure in surface bundles over S^1. In this section we turn this perspective on its head and explain how the Johnson homomorphism computes the cup product structure in a surface bundle over a surface (in fact, these methods extend to surface bundles over arbitrary manifolds). The submanifolds we construct will be codimension-1 (i.e. 3-manifolds), and built so that their intersection theory is explicitly connected to the Johnson homomorphism.

To this end, in Section 2.4.1 we give a discussion of the definition of the Johnson homomorphism in the setting of the cup product in surface bundles over S^1. The centerpiece of this is the construction of geometric representatives for classes in H^1, via embedded surfaces which we call "tube-and-cap surfaces". Then in Section 2.4.2, we return to the original problem of constructing representatives for classes in H^1 of a surface bundle over a surface as embedded 3-manifolds. The construction is carried out so that the intersection of particular pairs of these 3-manifolds is a tube-and-cap surface, thereby realizing the link between cup products in surface bundles over surfaces and the Johnson homomorphism.

2.4.1 From the intersection form to the Johnson homomorphism, and back again

In this subsection we will begin to dive into the theory of the Torelli group in earnest, so we begin with a brief review of the relevant defintions. The *Torelli group* $\mathcal{I}_g$ is the kernel of the symplectic representation $\Psi : \mathrm{Mod}(\Sigma_g) \to \mathrm{Sp}_{2g}(\mathbb{Z})$. The *Johnson kernel* $\mathcal{K}_g$ is the subgroup of $\mathcal{I}_g$ generated by all Dehn twists T_γ about *separating* curves γ. It is a deep theorem of D. Johnson that $\mathcal{K}_g$ can alternately be characterized as the kernel of the Johnson homomorphism τ to be defined below.

Let $\phi \in \mathcal{I}_g$ be a Torelli mapping class, and build the mapping torus $M_\phi = \Sigma_g \times I / \{(x,1) \sim (\phi(x), 0)\}$. As $\phi \in \mathcal{I}_g$ for any curve $\gamma \subset \Sigma_g$, the homology class $[\gamma] - \phi_*[\gamma]$ is zero. Thus there exists a map of a surface $i : S \to \Sigma_g$ which cobounds $\gamma \cup \phi(\gamma)$. Indeed, there exists an *embedded* surface $S \le \Sigma_g \times I$ whose boundary is given by

$$\partial S = \gamma \times \{1\} \cup \phi(\gamma) \times \{0\}.$$

To see this, recall that since S^1 is a $K(\mathbb{Z}, 1)$, there is a correspondence

$$H^1(\Sigma_g, \mathbb{Z}) \approx [\Sigma_g, S^1].$$

Via Poincaré duality,

$$H^1(\Sigma_g, \mathbb{Z}) \approx H_1(\Sigma_g, \mathbb{Z}).$$

The induced correspondence

$$H_1(\Sigma_g, \mathbb{Z}) \approx [\Sigma_g, S^1]$$

is realized by taking the preimage of a regular value, which will be an embedded submanifold. Under this correspondence, homotopic maps $f, g : \Sigma_g \to S^1$ yield homologous submanifolds,

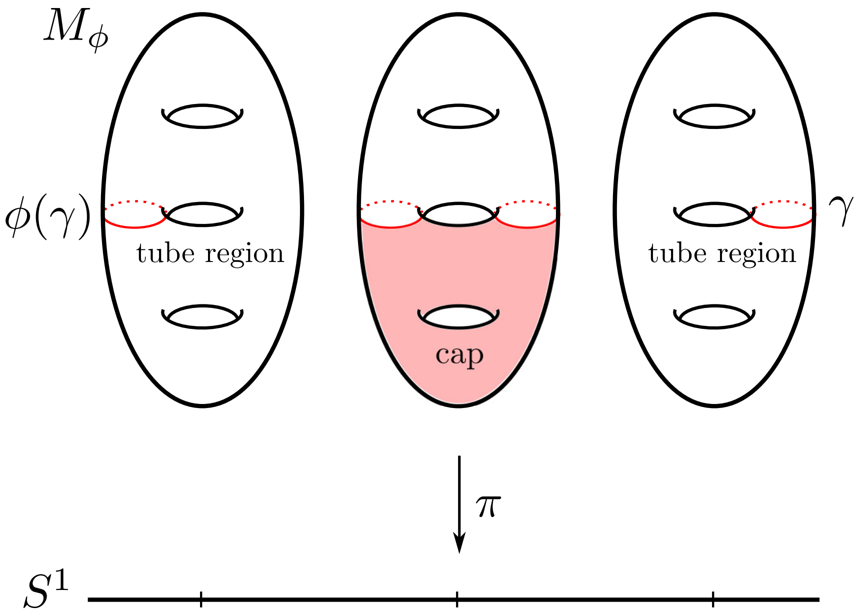

Figure 2.1: A tube surface

and conversely. Therefore, the maps $f, g : \Sigma_g \to S^1$ which determine $\gamma, \phi(\gamma)$ are homotopic. This gives the desired map $F : \Sigma_g \times I \to S^1$ such that the preimage of a regular value is an embedded surface S cobounding γ and $\phi(\gamma)$.

In fact, the choice of S is not unique. Let $i' : S' \to M_\phi$ be any map of a *closed* surface to M_ϕ. Then the chain $S + S'$ satisfies $\partial(S + S') = \partial S = \gamma - \phi(\gamma)$. Nonetheless, given any S satisfying $\partial(S) = \gamma - \phi(\gamma)$, we can form a closed submanifold of M_ϕ in the following way. We begin with a tube, diffeomorphic to $S^1 \times I$, embedded into M_ϕ as $\phi(\gamma) \times [0, 1/3] \cup \gamma \times [2/3, 1]$. We may then glue in S to $\Sigma_g \times [1/3, 2/3]$. The result is a smoothly-embedded oriented submanifold $\Sigma_\gamma \subset M_\phi$, which will descend to a homology class Σ_z (here $z = [\gamma]$). See Figure 2.1.

For convenience, we introduce the following terminology for these surfaces, which we will refer to as *tube surfaces*. The *tube* of a tube surface is the cylinder $S^1 \times I = \phi(\gamma) \times [0, 1/3] \cup \gamma \times [2/3, 1]$, and the *cap* is the subsurface S.

We assign an orientation to Σ_γ as follows. The tangent space to a point x contained in the tube has a direct sum decomposition via

$$T_x\Sigma_\gamma = V \oplus T_x\gamma, \tag{2.2}$$

where V is any preimage of $T_{\pi(x)}S^1$ and $T_x\gamma$ is interpreted as the tangent space to the copy of γ sitting in the fiber containing x. Both of the summands in (2.2) have orientations induced from those on S^1 and γ respectively, and this endows $T_x\Sigma$ with an orientation. This can then be extended over the cap surface in a coherent way, since S was chosen to be a boundary for $[\gamma] - [\phi(\gamma)]$ with $\mathbb{Z}$ coefficients.

Recall however that the choice of S was not unique. Any closed surface mapping into Σ_g is homologous to some multiple of the fundamental class, and so the above procedure really defines a homomorphism $H_1(\Sigma_g) \to H_2(M_\phi)/[F]$, where $[F]$ is the fundamental class of the fiber. If the bundle has a section $\sigma : S^1 \to M_\phi$, then we can choose S so that $\operatorname{Im}\sigma$ and Σ_z have zero algebraic intersection, which gives a canonical lift $H_1(\Sigma_g) \to H_2(M_\phi)$. In the absence of such auxiliary data, we instead just choose an *arbitrary* lift, and we will account for the consequences later.

Having chosen an embedding $i : H_1(\Sigma_g) \hookrightarrow H_2(M_\phi)$ such that $z \mapsto \Sigma_z$, there is an associated direct sum decomposition of $H_2(M_\phi)$, namely

$$H_2(M_\phi) = \langle [F] \rangle \oplus \operatorname{Im} i.$$

Relative to such an embedding, we form the map $\tau(\phi) \in \operatorname{Hom}(\wedge^3 H_1(\Sigma_g), \mathbb{Z})$ by

$$\tau(\phi)(x \wedge y \wedge z) = \Sigma_x \cdot \Sigma_y \cdot \Sigma_z,$$

the term on the right being interpreted as the triple algebraic intersection of the given homology classes. Suppose a section exists, and that the Σ_x have been constructed accordingly. In this case, D. Johnson showed that the map

$$\begin{aligned} \tau : \mathcal{I}_{g,*} &\to \operatorname{Hom}(\wedge^3 H_1(\Sigma_g), \mathbb{Z}) \\ \phi &\mapsto \tau(\phi) \end{aligned}$$

is a surjective homomorphism. See [8, Chapter 6] for a summary of the Johnson homomorphism, including two alternative definitions. The (pointed) *Johnson kernel* $\mathcal{K}_{g,*}$ is defined analogously to the case of closed surfaces, as the subgroup of $\operatorname{Mod}(\Sigma_{g,*})$ generated by Dehn twists about separating simple closed curves. As in the closed case, D. Johnson established that $\mathcal{K}_{g,*}$ coincides with the kernel of τ. In our context this exactly means that all triple intersections between the various Σ_x vanish.

Having fixed a family of Σ_x, it is then easy to compute the entire intersection form on $\wedge^3 H_2(M_\phi)$. Certainly $[F]^2 = 0$. It is also fairly easy to see that

$$[F] \cdot \Sigma_x \cdot \Sigma_y = i(x, y),$$

where $i(x, y)$ denotes the algebraic intersection pairing in $H_1(\Sigma_g)$. Indeed, by picking the choice of fiber to intersect Σ_x on the tube, it is clear that the result is simply the curve x, so that $[F] \cdot \Sigma_x \cdot \Sigma_y$ computes the intersection of x, y on F, at least up to a sign that may be introduced by the (non)compatibilities of the various orientation conventions in play. A quick check reveals this sign to be positive.

We will now be able to account for the ambiguity introduced by our choice of embedding $i : H_1(\Sigma_g) \hookrightarrow H_2(M_\phi)$, which will in turn lead to the definition of the Johnson homomorphism on the closed Torelli group $\mathcal{I}_g$. Suppose that $\Sigma'_w = \Sigma_w + k_w[F]$ is some other set of choices which is *coherent* in the sense that $\Sigma'_w + \Sigma'_z = \Sigma'_{w+z}$ (i.e. $x \mapsto k_x \in H^1(\Sigma_g)$). By linearity,

$$\begin{aligned}\Sigma'_x \cdot \Sigma'_y \cdot \Sigma'_z &= \Sigma_x \cdot \Sigma_y \cdot \Sigma_z + k_x i(y,z) + k_y i(z,x) + k_z i(x,y) \\ &= \tau(\phi)(x \wedge y \wedge z) + k_x i(y,z) + k_y i(z,x) + k_z i(x,y) \\ &= \tau(\phi)(x \wedge y \wedge z) + C^*(k);\end{aligned}$$

here $C : \wedge^3 H_1(\Sigma_g) \to H_1(\Sigma_g)$ is the contraction with the symplectic form $i(\cdot,\cdot)$, and $k \in \mathrm{Hom}(H_1(\Sigma_g), \mathbb{Z})$ is the form such that $k(w) = k_w$. The upshot of this calculation is that $\tau(\phi)$ is well-defined as an element of $\mathrm{Hom}(\wedge^3 H_1(\Sigma_g), \mathbb{Z}) / \operatorname{Im} C^*$, which can be identified with the more familiar space $\wedge^3 H/H$ (here we adopt the usual convention that $H = H_1(\Sigma_g)$). The *Johnson homomorphism* on the closed Torelli group is then defined via

$$\begin{aligned}\tau : \mathcal{I}_g &\to \mathrm{Hom}(\wedge^3 H_1(\Sigma_g), \mathbb{Z}) / \operatorname{Im} C^* \approx \wedge^3 H/H \\ \phi &\mapsto \tau(\phi).\end{aligned}$$

As mentioned above, work of D. Johnson shows that the kernel of τ coincides with the previously-defined subgroup

$$\mathcal{K}_g = \langle T_\gamma \mid \gamma \text{ separating scc} \rangle.$$

Remark 2.4.1. The construction given above with the tube-cap surfaces is a concrete realization of the isomorphism $H_1(\Sigma_g) \approx H_2(M_\phi)/[F]$ coming from the Serre spectral sequence for $p : M_\phi \to S^1$. In fact, this same construction will work for an arbitrary $\phi \in \mathrm{Mod}(\Sigma_g)$, yielding an isomorphism $(H_1(\Sigma_g))^\phi \approx H_2(M_\phi)/[F]$, but we do not pursue this here.

The above discussion shows how to construct the Johnson homomorphism in terms of the intersection form on M_ϕ. Conversely, we will show next how to reconstruct the intersection form on M_ϕ from the data of the Johnson homomorphism $\tau(\phi) \in \wedge^3 H/H \approx \mathrm{Hom}(\wedge^3 H\Sigma_g, \mathbb{Z}) / \operatorname{Im} C^*$. Begin by selecting an *arbitrary* lift $\tilde{\tau}(\phi)$ of $\tau(\phi)$ (of course, the presence of a section gives a canonical such choice). Next, construct a coherent family of homology classes Σ'_x by making choices arbitrarily. Define $\tau'(\phi) \in \mathrm{Hom}(\wedge^3 H, \mathbb{Z})$ by

$$\tau'(\phi)(x \wedge y \wedge z) = \Sigma'_x \cdot \Sigma'_y \cdot \Sigma'_z.$$

There is no reason to suspect that $\tau'(\phi) = \tilde{\tau}(\phi)$. However, as we saw above, we do know that $\tau'(\phi) - \tilde{\tau}(\phi) \in \mathrm{Im}\, C^*$, and so there is some functional $\alpha \in H^1(\Sigma_g)$ such that $\tau'(\phi) - \tilde{\tau}(\phi) = C^*(\alpha)$. This functional α will allow us to choose the correct set of Σ_x so that the triple intersections are computed by our choice of $\tilde{\tau}(\phi)$.

Lemma 2.4.2. *We assume the notation of the above setting. By taking*

$$\Sigma_x = \Sigma'_x - \alpha(x)[F],$$

there is an equality for all x, y, z:

$$\Sigma_x \cdot \Sigma_y \cdot \Sigma_z = \tilde{\tau}(\phi)(x \wedge y \wedge z).$$

Proof. Compute:

$$\begin{aligned}\Sigma_x \cdot \Sigma_y \cdot \Sigma_z &= \Sigma'_x \cdot \Sigma'_y \cdot \Sigma'_z - \alpha(x)i(y,z) - \alpha(y)i(z,x) - \alpha(z)i(x,y) \\ &= \tau'(\phi)(x \wedge y \wedge z) - C^*(\alpha)(x \wedge y \wedge z) \\ &= \tilde{\tau}(\phi).\end{aligned}$$

□

2.4.2 Intersections in surface bundles over surfaces, and beyond

The methods of the previous subsection can be adapted to give a description of certain cup products in $H^1(E)$, where $p : E^{n+2} \to B^n$ has monodromy lying in $\mathcal{I}_g$. The idea will be to define an embedding, as before,

$$i : H_1(\Sigma_g) \hookrightarrow H_{n+1}(E),$$

by constructing submanifolds M_γ for curves $\gamma \subset \Sigma_g$ by means of a higher-dimensional "tubing construction". Then the triple intersections of collections of M_x will be partially computable via the Johnson homomorphism in a certain sense to be described below. In this subsection we will first briefly sketch the properties we require of the submanifolds M_γ, then we will give the construction. Then in Section 2.4.3, we will determine much of the intersection pairing in $H_*(E, \mathbb{Z})$.

Our construction will provide, for each simple closed curve $\gamma \subset F$, a submanifold M_γ, such that if $[\gamma] = [\gamma']$, then also $[M_\gamma] = [M_{\gamma'}]$. If $[\gamma] = x$, we write M_x in place of $[M_\gamma]$. Let $p : E \to B$ be a surface bundle with monodromy in $\mathcal{I}_g$, and let $\rho : \pi_1 B \to \mathcal{I}_g$ be the monodromy. By post-composing with $\tau : \mathcal{I}_g \to \wedge^3 H/H$, we obtain a map from $\pi_1 B$ to an abelian group, and so $\tau \circ \rho$ factors through $H_1(B)$. By an abuse of notation we will write $\tau(b)$ for $b \in H_1(B)$.

This map computes (most of) the intersection form in $H_*(E)$. Recall the notation from Proposition 2.2.1: given a curve $\alpha \subset B$, there is an induced bundle E_α over α, which determines a homology class E_a. A given M_γ can be intersected with E_α to yield a surface $\Sigma_{\alpha,\gamma}$ inside E_α. Our construction will be set up so that

$$M_x \cdot M_y \cdot M_z \cdot E_b = \tau(b)(x \wedge y \wedge z),$$

possibly up to a sign. This is the sense in which $M_x \cdot M_y \cdot M_z$ is partially computable. As a remark, the intersections $M_x \cdot M_y \cdot M_z \cdot X$ for arbitrary $X \in H_3 E$ will all involve intersections with further M_w, and are describable (at least in the case of bundles with section) in terms of the higher Johnson invariants

$$\tau : H_i(\mathcal{I}_{g,*}) \to \wedge^{i+2} H,$$

but we will not pursue this point of view further in this chapter. In Chapter 4 we will return to this theme.

The construction. As usual, let $\pi : E \to B$ be a surface bundle over a surface with monodromy $\rho : \pi_1 B \to \mathcal{I}_g$. We turn now to the question of constructing suitable homology classes $M_x \in H_3(E)$, for $x \in H_1(\Sigma_g)$. The construction will be a higher-dimensional analogue of the construction of tube-and-cap surfaces given in the previous subsection. The reader may find it helpful to consult Figure 2.2 as they read this subsection.

When the base space B has dimension 2, a new layer of complexity is introduced by the potential absence of sections $\sigma : B \to E$, and this will require some additional preparatory work in order to construct geometric representatives for homology classes. Our construction method proceeds by exploiting the fact that it is always possible to find sections defined on $B' := \overline{B \setminus D^2}$. We define $E' := \pi^{-1}(B')$, and refer to a section $\sigma : B' \to E'$ as a *partial section* of the bundle E. We say that two sections σ_0, σ_1 of a fiber bundle are *homotopic through sections* if there exists a homotopy σ_t between σ_0 and σ_1 such that σ_t is a section for each fixed t.

Lemma 2.4.3. *Let $\pi : E \to \Sigma_h$ be a surface bundle over a surface with monodromy $\rho : \pi_1 \Sigma_h \to \mathrm{Mod}(\Sigma_g)$. Let $E' = \pi^{-1}(\overline{\Sigma_h \setminus D^2})$, and note that π restricts to give E' the* *structure of a Σ_g-bundle over $\overline{\Sigma_h \setminus D^2}$. Then there is a one-one correspondence between the set of classes of partial sections $\sigma : \overline{\Sigma_h \setminus D^2} \to E'$ up to homotopy through sections, and homomorphisms $\tilde{\rho} : F_{2h} \to \mathrm{Mod}(\Sigma_{g,*})$ making the diagram below commute:*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & K & \longrightarrow & F_{2h} & \longrightarrow & \pi_1\Sigma_h & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle\tilde{\rho}} & & \downarrow{\scriptstyle\tilde{\rho}} & & \downarrow{\scriptstyle\rho} & & \\ 1 & \longrightarrow & \pi_1\Sigma_g & \longrightarrow & \mathrm{Mod}(\Sigma_{g,*}) & \longrightarrow & \mathrm{Mod}(\Sigma_g) & \longrightarrow & 1. \end{array}$$

Proof. This follows immediately from the well-known fact that there is a homotopy equivalence

$$K(\mathrm{Mod}(\Sigma_{g,*}), 1) \simeq B(\mathrm{Diff}(\Sigma_g, *)),$$

the latter space being the classifying space of Σ_g-bundles with section. □

The kernel $K \lhd F_{2h}$ is normally generated by a single element ω, represented geometrically by the boundary of $\overline{\Sigma_h \setminus D^2}$. The element $\tilde{\rho}(\omega) \in \pi_1\Sigma_g$ associated to a section σ will be denoted by ω_σ. It is called the *index curve.* The following lemma is immediate from the definitions.

Lemma 2.4.4. *Assume the notation of Lemma 2.4.3. Let σ be a partial section of E, and let $\omega_\sigma \in \pi_1\Sigma_g$ be the corresponding index curve. Then there exists a local trivialization of E*

$$t : \pi^{-1}(D^2) \to D^2 \times \Sigma_g$$

relative to which $\sigma(\partial D^2)$ is in the free homotopy class of ω_σ.

The next lemma will be used in the course of the construction in Proposition 2.4.6.

Lemma 2.4.5. *Let $S \subset \Sigma_g \times S^1$ be an embedded closed oriented subsurface. Suppose $\gamma : S^1 \to \Sigma_g \times S^1$ is a section of the projection $\Sigma_g \times S^1 \to S^1$, and that $p_*[\gamma] = 0 \in H_1(\Sigma_g, \mathbb{Z})$ (where $p : \Sigma_g \times S^1 \to \Sigma_g$ is the obvious projection). Let $i : \Sigma_g \times S^1 \to \Sigma_g \times D^2$ be the natural inclusion. If the algebraic intersection number $[\gamma] \cdot [S] = 0$ (computed in $\Sigma_g \times S^1$), then there exists an oriented, properly-embedded 3-manifold $M \subset \Sigma_g \times D^2$, such that $\partial M = S$.*

Proof. The first step is to establish that $i_*[S] = 0$ in $H_2(\Sigma_g \times D^2)$. The Künneth formula establishes natural splittings

$$\begin{aligned} H_1(\Sigma_g \times S^1) &\approx H_1(\Sigma_g) \oplus H_1(S^1) \\ H_2(\Sigma_g \times S^1) &\approx H_2(\Sigma_g) \oplus \Big(H_1(\Sigma_g) \otimes H_1(S^1)\Big). \end{aligned}$$

In these coordinates, the map $i_* : H_2(\Sigma_g \times S^1) \to H_2(\Sigma_g \times D^2) \approx H^2(\Sigma_g)$ is given simply by projection onto the $H_2(\Sigma_g)$ factor. The assumptions on γ imply that $[\gamma]$ generates $H_1(S^1) \leq H_1(\Sigma_g \times S^1)$. Under the intersection pairing, $H_1(S^1)$ is orthogonal to $H_1(\Sigma_g) \otimes H_1(S^1)$. From the assumption $[\gamma] \cdot [S] = 0$, it then follows easily that $i_*[S] = 0$. Consequently, there exists a 3-chain C_p in $\Sigma_g \times D^2$ with $\partial C_p = S$.

It remains to explain why C_p can be replaced with a smooth, oriented, properly-embedded 3-manifold. This will follow from general results on representing (relative) codimension-one homology classes by smooth submanifolds (with boundary). The argument proceeds along very similar lines to the construction of embedded cap surfaces in fibered 3-manifolds described above. For an oriented manifold X with boundary, Lefschetz duality gives an isomorphism

$$H_{n-1}(X, \partial X, \mathbb{Z}) \approx H^1(X, \mathbb{Z}) \approx [X, S^1]$$

In our setting, the surface $S \subset \Sigma_g \times S^1$ is represented by a map

$$f : \Sigma_g \times S^1 \to S^1,$$

such that $S = f^{-1}(*)$ for some regular value $* \in S^1$. Similarly, the (relative) homology class of C_p in $H_3(\Sigma_g \times D^2, \Sigma_g \times S^1, \mathbb{Z})$ corresponds to a map

$$F : \Sigma_g \times D^2 \to S^1.$$

Moreover, as $\partial C_p = S$, they represent the same homology class in $H_2(\Sigma_g \times S^1, \mathbb{Z})$. This means that the maps f and $F \mid_{\Sigma_g \times S^1}$ are homotopic. We can therefore concatenate this homotopy with F, to obtain a map

$$\tilde{F} : \Sigma_g \times D^2 \to S^1.$$

On the boundary, $\tilde{F} = f$, and is therefore transverse to $* \subset S^1$. In order to replace C_p by a smooth submanifold such that $\partial C_p = C$, we must therefore perturb $\tilde{F}$ away from a neighborhood of $\partial(\Sigma_g \times D^2)$ and make the result everywhere transverse to $* \subset S^1$. The Extension Theorem (see [9, p. 72]) asserts that we can do precisely this. □

The theory of index curves established above will allow us to construct embedded representatives of homology classes in surface bundles over surfaces, when suitable conditions on the monodromy are satisfied.

Proposition 2.4.6. *Let $\pi : E \to B$ be a surface bundle over a surface with monodromy $\rho : \pi_1 B \to \mathcal{I}_g$ contained in the Torelli group. Suppose there is a partial section $\sigma : B' \to E'$ for which the associated index curve ω_σ lies in the commutator subgroup $[\pi_1\Sigma_g, \pi_1\Sigma_g]$. Then* *there is an embedding*

$$\iota : H_1(F, \mathbb{Z}) \to H_3(E, \mathbb{Z})$$

constructed so that if $c \in H_1(F, \mathbb{Z})$ is a primitive class, then $\iota(c)$ can be represented by some embedded, oriented, piecewise-smooth 3-submanifold M_c of E.

Proof. Let $c \in H_1(F, \mathbb{Z})$ be given. By assumption, c is primitive, so that there exists a simple closed curve $\gamma \subset \Sigma_g$ with $[\gamma] = c$. We will use this to construct a 3-manifold M_γ.

Consider a cell decomposition

$$B = B^0 \subset B^1 \subset B^2$$

of B, where B^0 consists of the single point p, there are $2g$ one-cells $\{a_1, b_1, \ldots, a_h, b_h\}$, and a single two-cell D. For each one-cell e, there is an associated element of the monodromy, $\rho(e)$, such that the effect of transporting a curve γ across e (from the negative to the positive side, relative to orientations of B and e) sends the isotopy class of γ to $\rho(e)\gamma$. For a one-cell e, let $N(e) \approx e \times I$ be a (closed) regular neighborhood in B. We also let $N(p)$ be a small closed neighborhood of p. If necessary, shrink the $N(e)$ so that

$$N := N(a_1) \cup \cdots \cup N(b_h) \setminus N(p)$$

is a union of $2h$ disjoint rectangles.

Let $\gamma \subset F$ be a simple closed curve on a fiber F over a point in

$$D' := \overline{D \setminus (N(p) \cup N(a_1) \cup \cdots \cup N(b_h))}.$$

By construction, D' is nothing more than a closed disk (in the upper-left portion of Figure 2.2, D' is the closure of the complement of the shaded regions). The submanifold M_γ will be constructed in three stages, denoted M_γ^i for $i = 1, 2, 3$: first over D', then over N, and finally over $N(p)$. Choose a trivialization $\pi^{-1}(D') \approx D' \times F$, and define $M_\gamma^1 = \gamma \times D'$ relative to this trivialization. Then $\partial(M_\gamma^1) \subset \pi^{-1}(\partial D')$. We specify an orientation on M_γ^1 as follows: a point $x \in M_\gamma^1$ has a decomposition of the tangent space via

$$T_x M_\gamma^1 \approx T_{\pi(x)} B \oplus T_x \gamma. \tag{2.3}$$

Both of these two summands carry pre-existing orientations, and M_γ^1 is then oriented by specifying the above isomorphism to be orientation-preserving. By analogy with the construction of tube surfaces, we refer to M_γ^1 as the *tube region* of M_γ.

Next we construct M_γ^2. Let e be a one-cell, and consider the intersection $M_\gamma^1 \cap \pi^{-1}(N(e) \cap N)$. The base space $N(e) \cap N$ is just a rectangle, and so the bundle $\pi^{-1}(N(e) \cap N)$ is trivializable. We can therefore find a diffeomorphism

$$\psi : \pi^{-1}(N(e) \cap N) \approx I \times I \times \Sigma_g$$

under which $M^1_\gamma \cap \pi^{-1}(N(e) \cap N)$ is identified with

$$(I \times \{0\} \times \gamma) \cup \left(I \times \{1\} \times \gamma'\right),$$

where γ' is some curve in the isotopy class of $\rho(e)(\gamma)$. As we saw in the previous subsection, for each e there exists a family of properly-embedded surfaces S_e in $I \times \Sigma_g$ such that $\partial S_e = \{0\} \times \gamma \cup \{1\} \times \gamma'$.

Our choice of S_e will be dictated by the section σ. Applying ψ, the image of σ in $\{t\} \times I \times \Sigma_g$ is a properly-embedded arc α_σ. This determines a preferred homology class in $H_2(I \times \Sigma_g, \partial(I \times \Sigma_g), \mathbb{Z})$ among the set of possible S_e, by the relation $[\alpha_\sigma] \cdot [S_e] = 0$.

Let S_e be any properly-embedded subsurface of $I \times \Sigma_g$ satisfying the conditions $\partial S_e = \{0\} \times \gamma \cup \{1\} \times \gamma'$ and $[\alpha_\sigma] \cdot [S_e] = 0$. We can then fill in $\pi^{-1}(N(e) \cap N)$ with $I \times S_e$ for each e, creating M^2_γ. As in the case of a tube surface, the orientation for M^1_γ can be extended over each of these pieces coherently. We refer to $M^2_\gamma \setminus M^1_\gamma$ as the *cap region* of M_γ.

It therefore remains to construct $M^3_\gamma = M_\gamma$. By construction, $\partial M^2_\gamma \subset \pi^{-1}(\partial N(p))$. We would like to be able to fill this boundary in by inserting a "plug" contained in $\pi^{-1}(N(p))$. *A priori*, there is a homological obstruction to this: if $[\partial M^2_\gamma] \neq 0$ in $H_2(\pi^{-1}(N(p)))$ then this problem is not solvable even on the chain level.

However, the assumption that the index curve $\omega_\sigma \in [\pi_1\Sigma_g, \pi_1\Sigma_g]$ will imply that this obstruction vanishes. Let $t : \pi^{-1}(N(p)) \to D^2 \times \Sigma_g$ be the trivialization of Lemma 2.4.4, and define $\gamma = t(\sigma(\partial(N(p))))$. Set $S = t(\partial(M^2_\gamma))$. By Lemma 2.4.4, $[\gamma] = 0 \in H_1(\pi^{-1}(N(p))) \approx H_1(\Sigma_g)$. We wish to show that $[\gamma] \cdot [S] = 0$. By construction, $\partial(M^2_\gamma)$ consists of $4g$ subsurfaces, corresponding to the $2g$ surfaces $S_{a_1}, \ldots, S_{b_g}$, each appearing twice (one for each component of $N(e) \cap N(p)$). Similarly, γ is comprised of $4g$ segments, again indexed by the components of $N(e) \cap N(p)$. On each one of these components, the relevant S_e was selected to have zero algebraic intersection with the relevant portion of γ, and so the same holds true globally: $[\gamma] \cdot [S] = 0$.

Applying Lemma 2.4.5, we obtain a 3-manifold $M_p \subset N(p) \times \Sigma_g$, such that $\partial M_p = t(\partial(M^2_\gamma))$. Extending the orientation of M^2_γ over M_p, the result is an oriented, piecewise-smooth submanifold $M_\gamma \subset E$. □

While in general, not every surface bundle over a surface satisfies the hypotheses of Proposition 2.4.6 (specifically, the requirement that there exist a partial section with $[\omega_\sigma] =$

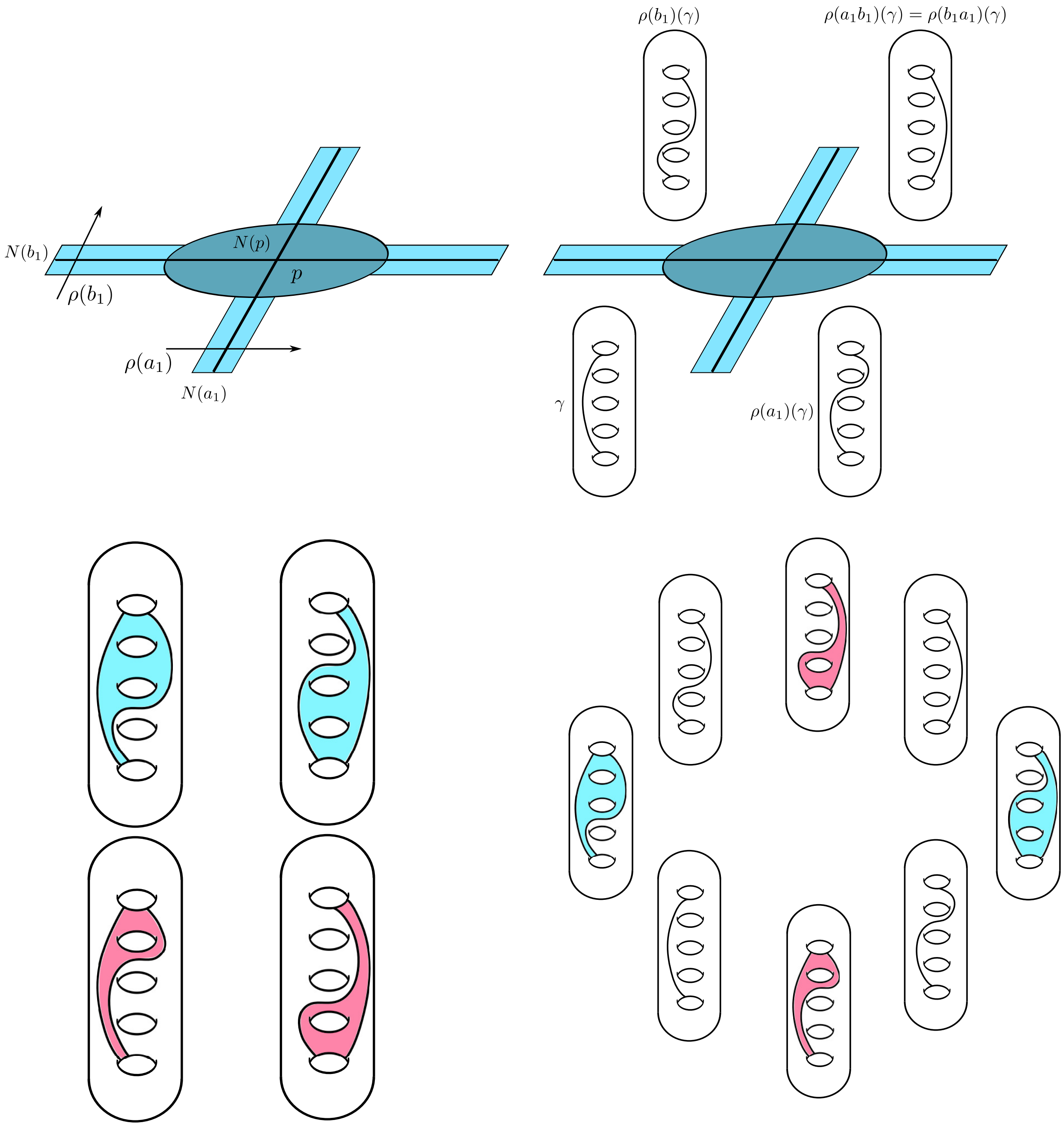

Figure 2.2: Upper left: The neighborhoods $N(e)$ and $N(p)$. Upper right: M^1_γ intersected with four different fibers. Lower left: Cap surfaces, lying over different portions of N. Lower right: A depiction of $M^2_\gamma \cap \pi^{-1}(\partial N)$.

$0 \in H_1(\Sigma_g, \mathbb{Z})$), it turns out that this is always the case for surface bundles over surfaces with monodromy in $\mathcal{K}_g$.

Lemma 2.4.7. *Let $\rho : \pi_1\Sigma_h \to \mathcal{K}_g$ be given. Then for any lift $\tilde{\rho} : F_{2h} \to \mathcal{K}_{g,*}$ of ρ, the index curve satisfies $\omega_\sigma \in [\pi_1\Sigma_g, \pi_1\Sigma_g]$.*

Proof. When restricted to $\mathcal{K}_g$, the Birman exact sequence takes the form

$$1 \to [\pi_1\Sigma_g, \pi_1\Sigma_g] \to \mathcal{K}_{g,*} \to \mathcal{K}_g \to 1.$$

The result follows. □

An essential feature of the above construction is the relationship between an M_γ and a sub-bundle E_α lying over a curve $\alpha \subset B$. Suppose α is chosen so that relative to the cell decomposition of B used in constructing M_γ, α is transverse to all the one-cells e, and does not pass through $N(p)$. Then a little visual imagination reveals that the intersection of M_γ and E_α is given by a tube surface for γ sitting inside E_α. We call the resulting surface $\Sigma_{\alpha,\gamma}$, and then $[\Sigma_{\alpha,\gamma}]$ is denoted by $\Sigma_{a,x}$, where $[\alpha] = a$ and $[\gamma] = x$.

We define a *family* of M_x to be a set of M_x for each $x \in H_1(F)$ such that for all $c \in \mathbb{Z}$ and $x, y \in H_1(F)$,

$$M_{cx+y} = cM_x + M_y.$$

Different choices of M_x lead to different spaces of $\Sigma_{b,x}$, but conversely, a choice of a family of M_x leads to a corresponding distinguished summand of $H_2(E)$.

2.4.3 Determination of the intersection form

From this point onwards, we assume without further comment that our surface bundle over a surface $\pi : E \to B$ satisfy the hypotheses of Proposition 2.4.6 (as a special case, these results apply to all surface bundles over surfaces with monodromy in $\mathcal{K}_g$, by Lemma 2.4.7). The purpose of this subsection is to give a description of the cup product structure on $H^*(E, \mathbb{Z})$; equivalently, we will describe the intersection form. By Poincaré duality, it suffices to determine, for each X, the set of pairings $X \cdot Y$.

Proposition 2.4.8. *Let i_B and i_F denote the algebraic intersection pairing on the homology of the base and on the fiber, respectively.*

1. *There exists a unique class $C \in H_2(E)$ such that $C \cdot \Sigma_{b,z} = 0$ for all $b \in H_1(B)$, $z \in H_1(\Sigma_g)$, and $C \cdot [F] = 1$. The intersection pairing $H_2(E) \otimes H_2(E) \to \mathbb{Z}$ is given*

as follows, where e = C^2 by definition.

	C	$[F]$	$\Sigma_{a,z}$
C	e	1	0
$[F]$	1	0	0
$\Sigma_{b,w}$	0	0	$-i_B(a,b)i_F(z,w)$

In the case where the monodromy is contained in the Johnson kernel, we have e = 0.

2. *For any family of M_x, we have*

$$E_a \cdot E_b = i_B(a,b)[F]$$
$$M_x \cdot E_b = \Sigma_{b,x}$$
$$M_z \cdot M_w \cdot [F] = i_F(z,w).$$

3. *Let $\sigma : B' \to E'$ be a partial section for which $[\omega_\sigma] = 0 \in H_1(F)$. Associated to such a section is a lift of $\tau : H_1(B) \to \wedge^3 H/H$ to $\tilde{\tau} : H_1(B) \to \wedge^3 H$. The choice of σ gives rise to a splitting*

$$H_3(E) = \pi^!(H_1(B)) \oplus H_1(M) = \{E_b, b \in H_1(B)\} \oplus \{M_z, z \in H_1(F)\}$$

relative to which

$$M_x \cdot M_y \cdot M_z \cdot E_b = M_x \cdot M_y \cdot \Sigma_{b,z} = \tilde{\tau}(b)(x \wedge y \wedge z).$$

In the case where the monodromy is contained in the Johnson kernel, we can take the canonical lift $\tilde{\tau} \equiv 0$, and for this family of M_x we have

$$C \cdot M_x = 0$$
$$C^2 = 0$$

for all $x \in H_1(\Sigma_g)$.

Remark. The intersection pairing $H_{n-k}E \otimes H_k E \to \mathbb{Z}$ identifies $H_{n-k}E$ with $\mathrm{Hom}(H_k E, \mathbb{Z})$ and hence with $H^k E$ by the universal coefficients theorem, since the homology of a surface bundle over a surface with monodromy in $\mathcal{I}_g$ is torsion-free (see Proposition 2.2.1). Therefore, Proposition 2.4.8 can also be viewed as a description of the cup product in $H^*(E)$.

Proof. Before beginning with the proof of the statements, a comment on orientations is in order. Recall that if X, Y are embedded surfaces intersecting transversely, then $X \cap Y$ is oriented via the convention that

$$N(X) \oplus N(Y) \oplus T(X \cap Y)$$

should be positively oriented, where, for $W = X$ or $W = Y$, $N(W)$ is oriented by the convention that $N(W) \oplus T(W)$ be positively oriented with respect to the orientation fixed on W. Note that relative to this convention, if X is of odd codimension, then $X \cdot X = 0$; we will often employ this fact without comment in the sequel.

Recall that the submanifolds $\Sigma_x \subset M_\phi$ and $M_z \subset E$ have been oriented using a "base-first" convention; see (2.2) and (2.3). We orient E by selecting orientations for B and F. It is a somewhat tedious process to go through and verify the signs on all of the intersections being asserted in this theorem, and we omit the full verification of these results. At the same time, the reader who is interested in verifying the calculations should have no trouble doing so by carefully tracking the orientation conventions we have laid out.

It will turn out to be most natural to construct C after verifying the other statements not involving C. We begin with computing $\Sigma_{a,z} \cdot \Sigma_{b,w}$. These are represented by surfaces contained in some E_α, E_β respectively, where they are tube surfaces constructed from curves γ, δ. We can arrange it so that α, β intersect transversely, and such that over these points, the surfaces intersect in their tube regions. Following the orientation conventions as above, one verifies that the local intersection at such a point (p, q), written $I_{(p,q)}$ is equal to $-I_p I_q$, where I_p denotes the local intersection of α, β relative to the orientation on B, and I_q is the local intersection of γ, δ relative to the orientation on F. Summing over all local intersections gives the result in the lower-right hand corner of the table in Proposition 2.4.8.1.

The relation $[F] \cdot \Sigma_{a,z} = 0$ is easy to verify, by taking $[F]$ to be represented by a fiber not contained in the E_α containing $\Sigma_{a,z}$. This same idea also shows $[F]^2 = 0$, by picking representative fibers over distinct points.

Let us turn now to Proposition 2.4.8.2. If E_α, E_β intersect transversely at a point, then $E_\alpha \cap E_\beta = F$, the fiber over the point of intersection; a check of the orientation conventions shows that the orientation on F given by the intersection convention agrees with the predetermined orientation, so that

$$E_a \cdot E_b = i_B(a, b)[F]$$

as asserted.

The manifolds M_γ were constructed so as to intersect each E_b in a tube surface, and so the relation

$$M_z \cdot E_b = \Sigma_{b,z}$$

can be taken as a definition of the orientation on $\Sigma_{b,z}$. We choose this over the alternative because it can be verified that under this convention, the orientation on $\Sigma_{b,z}$ agrees with the "base first" convention discussed above.

Now let M_x, M_y be given, and consider $M_x \cdot M_y \cdot [F]$. By perturbing the one-skeleton of B, it can be arranged so that the plugs for M_x and M_y are disjoint and so that the cap regions intersect transversely, and so that the representative fiber intersects M_x, M_y in their tube regions. The local picture therefore becomes the intersection of x and y on F. A check of the orientation convention then shows

$$M_x \cdot M_y \cdot [F] = i_F(x, y).$$

Turning to Proposition 2.4.8.3, consider now a four-fold intersection

$$M_x \cdot M_y \cdot M_z \cdot [E_\beta].$$

We will assume without further comment that the intersection of representative submanifolds has been made suitably transverse by choosing one-skeleta wisely. The M_w were constructed so that the problem of computing $M_x \cdot M_y \cdot M_z \cdot [E_\beta]$ is exactly the same as the problem of computing the corresponding $\Sigma_x \cdot \Sigma_y \cdot \Sigma_z$ inside the 3-manifold E_β, up to a sign which records whether the orientation on $M_x \cdot [E_\beta]$ agrees with the orientation on the corresponding $\Sigma_x \subset E_\beta$; the convention $M_x \cdot E_b = \Sigma_{x,b}$ makes this sign positive. Lemma 2.4.2 shows that within E_b, there exist choices of homology classes Σ_x such that

$$\Sigma_x \cdot \Sigma_y \cdot \Sigma_z = \tilde{\tau}(b)(x \wedge y \wedge z).$$

Recall from Lemma 2.4.2 that the Σ_x's are obtained by starting with an arbitrary family Σ_x', and adding appropriate multiples of $[F]$. By the preceding, if $a \in B$ satisfies $i_B(a, b) = 1$, then

$$(M_z + E_a) \cdot E_b = M_z \cdot E_b + [F].$$

This shows that by adding appropriate multiples of E_a to M_z (as specified by the formulas in Lemma 2.4.2), for a given b, the formula

$$M_x \cdot M_y \cdot M_z \cdot [E_\beta] = \tilde{\tau}(b)(x \wedge y \wedge z) \tag{2.4}$$

can be made to hold. By choosing a symplectic basis for $H_1(B)$, this can be made to hold for all $b \in H_1(B)$ simultaneously.

It therefore remains to construct the class C. If $x, y \in H_1(\Sigma_g)$ satisfy $i_F(x,y) = 1$, then $[F] \cdot M_x \cdot M_y = 1$. Similarly, if α, β are loops in B intersecting transversely exactly once, and M_x, M_y are as above, then

$$\Sigma_{\alpha,x} \cdot \Sigma_{\beta,y} = \Sigma_{\alpha,x} \cdot M_x \cdot E_\beta = \pm 1. \tag{2.5}$$

As the space spanned by $[F]$ and the $\Sigma_{b,x}$ classes has codimension one in $H_2(E)$, (2.4) and (2.5) together show that the space of classes in $H_2(E)$ pairing trivially with the space of M_x has dimension at most one. We claim that

$$C = M_{x_1} \cdot M_{y_1} + \sum_{(b,z) \in \mathcal{B} \times \mathcal{F}} \tilde{\tau}(b)(x_1 \wedge y_1 \wedge z) \Sigma_{\hat{b}\hat{z}}$$

has all the required properties; here $\mathcal{B}, \mathcal{F}$ are symplectic bases for $H_1(B), H_1(F)$, respectively, the map $x \mapsto \hat{x}$ satisfies $i(x, \hat{x}) = 1$, $x_1 \in \mathcal{B}$, and $\hat{x}_1 = y_1$. Recall that C is asserted to have the following properties: $C \cdot [F] = 1$ and $C \cdot \Sigma_{b,z} = 0$ for all $b \in H_1(B), z \in H_1(\Sigma_g)$. Additionally, when the monodromy of E is contained in the Johnson kernel, we require $C^2 = 0$ and $C \cdot M_x = 0$ for M_x in the family associated to the lift of τ to the zero homomorphism. The proof is a direct calculation. For $C \cdot [F]$, one has by Proposition 2.4.8.1 and then Proposition 2.4.8.2

$$\begin{aligned} C \cdot [F] &= \left(M_{x_1} \cdot M_{y_1} + \sum_{(b,z) \in \mathcal{B} \times \mathcal{F}} \tilde{\tau}(b)(x_1 \wedge y_1 z) \Sigma_{\hat{b}\hat{z}} \right) \cdot [F] \\ &= M_{x_1} \cdot M_{y_1} \cdot [F] \\ &= 1. \end{aligned}$$

Computation of $C \cdot \Sigma_{b,z}$ proceeds by Proposition 2.4.8.3 and Proposition 2.4.8.1 respectively.

$$\begin{aligned} C \cdot \Sigma_{b,z} &= M_{x_1} \cdot M_{y_1} \cdot \Sigma_{b,z} + \tilde{\tau}(b)(x_1 \wedge y_1 \wedge z)(\Sigma_{\hat{b}\hat{z}}) \cdot \Sigma_{b,z} \\ &= \tilde{\tau}(b)(x_1 \wedge y_1 \wedge z) - \tilde{\tau}(b)(x_1 \wedge y_1 \wedge z) \\ &= 0. \end{aligned}$$

When the monodromy of E is contained in $\mathcal{K}_g$, the above formula for C simplifies to $C = M_{x_1} \cdot M_{y_1}$, from which it is apparent that $C^2 = 0$. To see that $C \cdot M_x = 0$ for all x, we will apply Poincaré duality to see that it suffices to show that

$$C \cdot M_x \cdot Y = 0$$

for all classes $Y \in H_3E$. Since $M_x \cdot E_b = \Sigma_{bx}$ and we have shown $C \cdot \Sigma_{bx} = 0$, it remains only to consider $C \cdot M_z \cdot M_w$. Expanding $M_z \cdot M_w$ in the additive basis for $H_2(E)$,

$$M_z \cdot M_w = \alpha[F] + \beta C + \sum_{(b,z)\in\mathcal{B}\times\mathcal{F}} \gamma_{b,z}\Sigma_{\hat{b},\hat{z}}.$$

As the monodromy of E is contained in $\mathcal{K}_g$, we have $M_z \cdot M_w \cdot \Sigma_{b,x} = 0$; applying this in coordinates for some $(b,x) \in \mathcal{B} \times \mathcal{F}$ gives, by applying the prior formulas,

$$\begin{aligned} 0 &= \left(\alpha[F] + \beta C + \sum_{(b,z)\in\mathcal{B}\times\mathcal{F}} \gamma_{b,z}\Sigma_{\hat{b},\hat{z}}\right) \cdot \Sigma_{b,x} \\ &= -\gamma_{b,x}, \end{aligned}$$

so that all $\gamma_{b,z} = 0$. Consequently, $M_z \cdot M_w = \alpha[F] + \beta C$. Recalling that $[F]^2 = C^2 = 0$ and that $(M_z \cdot M_w)^2 = 0$, this implies $\alpha\beta = 0$.

Also,

$$i_F(z,w) = M_z \cdot M_w \cdot [F] = \beta.$$

Therefore, we conclude that in the case $i_F(z,w) \neq 0$,

$$M_z \cdot M_w = i_F(z,w)C.$$

As $C^2 = 0$ this shows the result in this case. Now suppose that $i_F(z,w) = 0$. Then we can find z' such that $M_z \cdot M_{z'} = cC$ by above, with $c \neq 0$, and then

$$0 = M_z \cdot M_w \cdot M_z \cdot M_{z'} = cM_z \cdot M_w \cdot C.$$

This shows that $M_z \cdot M_w \cdot C = 0$ for all z, w, finishing the proof of Proposition 2.4.8. □

2.5 Multisections and splittings on rational cohomology

Let $p : E \to B$ be a surface bundle over an arbitrary base space B equipped with a section $\sigma : B \to E$. Then there is an associated splitting of $H^1(E,\mathbb{Z})$ as a direct sum, via

$$H^1(E,\mathbb{Z}) = \operatorname{Im} p^* \oplus \ker \sigma^*. \tag{2.6}$$

The condition that $p : E \to B$ admit a section is restrictive. However, recent work of Hamenstädt shows that all surface bundles over surfaces with zero signature admit *multisections* (see Theorem 2.5.2). In this section, we develop some necessary machinery showing how a multisection of a surface bundle gives rise to a splitting of $H^1(E, \mathbb{Q})$, similarly to (2.6). The results of this section will be required in the proof of Theorem A.

Remark 2.5.1. Theorem 2.5.2 is the only result in this section that requires the base space B to be a surface of genus $g \geq 2$. Lemma 2.5.3 and Proposition 2.5.4 are valid for any base space B.

Let $\mathrm{Conf}_n(E)$ denote the configuration space of n unordered distinct points in E, and let $\mathrm{PConf}_n(E)$ denote the space of n ordered distinct points in E. The symmetric group on n letters S_n acts freely on $\mathrm{PConf}_n(E)$ by permuting the order of the points, and $\mathrm{PConf}_n(E)/S_n = \mathrm{Conf}_n(E)$.

By a *multisection* of $p : E \to B$, we mean a map

$$\sigma : B \to \mathrm{Conf}_n(E)$$

for some $n \geq 1$, such that the composition

$$B \to \mathrm{Conf}_n(E) \to B^n/S_n$$

is given by $x \mapsto [x, \ldots, x]$. In other words, a multisection selects n distinct unordered points in each fiber. A *pure multisection* is a map

$$\sigma : B \to \mathrm{PConf}_n(E)$$

such that the composition

$$B \to \mathrm{PConf}_n(E) \to B^n$$

is given by $x \mapsto (x, \ldots, x)$. Our interest in multisections is due to the following result of Hamenstädt (see [11], [12]):

Theorem 2.5.2. (Hamenstädt) *Let $p : E \to B$ be a surface bundle over a surface such that the signature of E is zero (e.g. a bundle with at least one fibering with monodromy lying in $\mathcal{I}_g$). Then $p : E \to B$ has a multisection σ of cardinality $2g - 2$.*

We will use this result to obtain a splitting on $H^*(E, \mathbb{Q})$. As (2.6) indicates, this is straightforward when the multisection is pure; the work will be to obtain the required maps for general multisections. First note that by taking a finite cover $\tilde{B} \to B$, we can pull the bundle back to $\tilde{p} : \tilde{E} \to \tilde{B}$, such that the multisection pulls back to a pure multisection:

$$\psi : \tilde{B} \to \mathrm{PConf}_n(\tilde{E}).$$

Moreover, we can assume that the covering $\tilde{B} \to B$ is normal, with deck group Γ. By pulling back the Γ action on $\tilde{B}$, we see that Γ also acts on $\tilde{E}$, by sending the fiber over b to the fiber over $\gamma(b)$. Then the multisection ψ is in fact Γ-equivariant. This suggests the following lemma.

Lemma 2.5.3. *Let $\tilde{\sigma} : \tilde{B} \to \tilde{E}$ be a Γ-equivariant section. Then there is an induced map on Γ-invariant cohomology:*

$$\tilde{\sigma}^* : H^*(\tilde{E}, \mathbb{Q})^\Gamma \to H^*(\tilde{B}, \mathbb{Q})^\Gamma.$$

As a result, the transfer map

$$\tau^* : H^*(\tilde{B}, \mathbb{Q}) \to H^*(B, \mathbb{Q})$$

is injective when restricted to $\tilde{\sigma}^*(H^*(\tilde{E}, \mathbb{Q})^\Gamma)$.

Proof. If $f : X \to Y$ is any Γ-equivariant map of topological spaces, then $f^* : H^*(Y) \to H^*(X)$ will be equivariant, and so will restrict to a map on the Γ-invariant subspaces. Transfer (see [13]) gives an identification $H^*(\tilde{B}, \mathbb{Q})^\Gamma \approx H^*(B, \mathbb{Q})$, and the remaining statement follows. □

We now come to the main result of the section. This asserts that when $p : E \to B$ is a surface bundle with a multisection $\sigma : B \to \mathrm{Conf}_n(E)$, there exists a map $\hat{\sigma}^* : H^*(B, \mathbb{Q}) \to H^*(E, \mathbb{Q})$ with many of the same properties as (the pullback of) an actual section map.

Proposition 2.5.4. *Suppose $\sigma : B \to \mathrm{Conf}_n(E)$ is a multisection. Then there exist maps*

$$\hat{\sigma}^* : H^*(E, \mathbb{Q}) \to H^*(B, \mathbb{Q})$$
$$\hat{\sigma}_* : H_*(B, \mathbb{Q}) \to H_*(E, \mathbb{Q})$$

with the following properties:

1. *The composition*
$$\hat{\sigma}^* \circ p^* : H^*(B) \to H^*(B) = \mathrm{id}$$
and similarly
$$p_* \circ \hat{\sigma}_* : H_*(B) \to H_*(B) = \mathrm{id}\,.$$
2. *The maps $\hat{\sigma}^*$ and $\hat{\sigma}_*$ are adjoint under the evaluation pairing. That is, for all $\alpha \in H^*(E), x \in H_*(B)$,*
$$\langle \alpha, \hat{\sigma}_* x \rangle = \langle \hat{\sigma}^* \alpha, x \rangle.$$
3. *If $\alpha \in \ker \hat{\sigma}^*$, then for any $\beta \in H^*(E, \mathbb{Q})$ and any $x \in H_*(B, \mathbb{Q})$,*
$$\langle \alpha \smile \beta, \hat{\sigma}_*(x) \rangle = 0.$$

Consequently, $\hat{\sigma}^*$ *induces a splitting*

$$H^1(E, \mathbb{Q}) = \operatorname{Im} p^* \oplus \ker \hat{\sigma}^*. \tag{2.7}$$

Proof. Begin by assuming that the multisection is pure. For $i = 1, \ldots, n$ let $p_i : \mathrm{PConf}_n(E) \to E$ be the projection onto the i^{th} coordinate. We define

$$\hat{\sigma}^*(\alpha) = \frac{1}{n}\sum_{i=1}^{n} \sigma^*(p_i^*(\alpha))$$
$$\hat{\sigma}_*(x) = \frac{1}{n}\sum_{i=1}^{n} (p_i)_*(\sigma_*(x)).$$

Then properties (2.5.4.1) - (2.5.4.3) follow by direct verification.

In the general case, let $c : \tilde{B} \to B$ be a normal covering such that σ pulls back to a pure multisection ψ. We will use $\bar{c}$ to denote the covering $\tilde{E} \to E$. Let $\tau^* : H^*(\tilde{B}, \mathbb{Q}) \to H^*(B, \mathbb{Q})$ be the transfer map, normalized so that $c^* \circ \tau^* = \mathrm{id}$. Then define $\hat{\sigma}^* : H^*(E, \mathbb{Q}) \to H^*(B, \mathbb{Q})$ by

$$\hat{\sigma}^* = \tau^* \circ \hat{\psi}^* \circ \bar{c}^*.$$

Similarly, define $\hat{\sigma}_* : H_*(B, \mathbb{Q}) \to H_*(E, \mathbb{Q})$ by

$$\hat{\sigma}_* = \bar{c}_* \circ \hat{\psi}_* \circ \tau_*.$$

For what follows, it will be useful to refer to the following diagram.

$$\begin{array}{ccc} H^*(\tilde{E}) & \underset{\bar{c}^*}{\overset{\tau^*}{\rightleftarrows}} & H^*(E) \\ \tilde{p}^* \uparrow \; \downarrow \hat{\psi}^* & & p^* \uparrow \; \downarrow \hat{\sigma} \\ H^*(\tilde{B}) & \underset{c^*}{\overset{\tau^*}{\rightleftarrows}} & H^*(B) \end{array}$$

By definition,

$$\hat{\sigma}^* \circ p^* = \tau^* \circ \hat{\psi}^* \circ \bar{c}^* \circ p^*.$$

By commutativity, $\bar{c}^* \circ p^* = \tilde{p}^* \circ c^*$. Then

$$\begin{aligned} \tau^* \circ \hat{\psi}^* \circ \bar{c}^* \circ p^* &= \tau^* \circ \hat{\psi}^* \circ \tilde{p}^* \circ c^* \\ &= \tau^* \circ c^* \\ &= \mathrm{id}\,. \end{aligned}$$

Here, we have used the property $\hat{\psi}^* \circ \tilde{p}^* = \mathrm{id}$ for the pure multisection ψ, as well as our normalization convention $\tau^* \circ c^* = \mathrm{id}$ for the transfer map. A similar calculation proves the corresponding result for $\hat{\psi}_*$, and (2.5.4.1) follows.

Statement 2.5.4.2 follows from the observation that the cohomology and homology transfer maps are adjoint under the evaluation pairing. That is, if $\tilde{X} \to X$ is a normal covering space with deck group Γ, then for $x \in H_*(X)$ and $\alpha \in H^*(\tilde{X})$,

$$\langle \alpha, \tau_*(x) \rangle = \langle \tau^*(\alpha), x \rangle.$$

As $\hat{\psi}^*$ and $\bar{c}^*$ certainly also enjoy this adjointness property, so does $\hat{\sigma}^*$, and (2.5.4.2) follows.

To establish (2.5.4.3), suppose $\alpha \in \ker \hat{\sigma}^*$, and take $\beta \in H^*(E, \mathbb{Q}), x \in H_*(B, \mathbb{Q})$. As the transfer map is not a ring homomorphism, (2.5.4.3) does not follow immediately from (2.5.4.2). However, we see that

$$\begin{aligned} \langle \alpha \smile \beta, \hat{\sigma}_*(x) \rangle &= \langle \hat{\sigma}^*(\alpha \smile \beta), x \rangle \\ &= \langle \tau^*((\hat{\psi}^* \circ \bar{c}^*)(\alpha) \smile (\hat{\psi}^* \circ \bar{c}^*)(\beta)), x \rangle. \end{aligned}$$

It therefore suffices to show that $\hat{\psi}^* \circ \bar{c}^*(\alpha) = 0$. This follows by Lemma 2.5.3. Indeed, $\bar{c}^*(\alpha) \in H^*(\tilde{E}, \mathbb{Q})^\Gamma$, and $\hat{\psi}^*$, being a sum of Γ-equivariant maps, is itself Γ-equivariant, and so $\hat{\psi}^* \circ \bar{c}^*$ takes image in $H^*(\tilde{B}, \mathbb{Q})^\Gamma$. On the one hand, we have

$$0 = \hat{\sigma}^* \alpha = \tau^* \circ \hat{\psi}^* \circ \bar{c}^*(\alpha)$$

by assumption. By Lemma 2.5.3, τ^* is injective on the image of $\hat{\psi}^* \circ \bar{c}^*$, so that $\hat{\psi}^* \circ \bar{c}^*(\alpha) = 0$ as desired. □

2.6 Unique fibering in the Johnson kernel

This section is devoted to the proof of Theorem A. The outline is as follows. Let $p_1 : E \to B_1$ be a surface bundle with monodromy in the Torelli group $\mathcal{I}_g$, and suppose there is a second distinct fibering $p_2 : E \to B_2$ with fiber F_2. The proof proceeds by analyzing $[F_2]$ in the coordinates on $H_*(E)$ coming from the Torelli fibering p_1. On the one hand, the intersection form in these coordinates is completely understood by virtue of Proposition 2.4.8. On the other, $[F_2]$ is realizable as an intersection of classes induced from $H_1(B_2)$. Under the assumption that the monodromy of p_1 is contained in $\mathcal{K}_g$ and not merely $\mathcal{I}_g$, it will follow that there is a unique possibility for $[F_2]$. The final step will be to extract the condition that the genera of F_2 and B_1 must be equal from the cohomology ring $H^*(E)$ and to argue that this enforces the triviality of either bundle structure.

The fundamental class of a second fiber. In this subsection we will compute $[F_2]$ in the coordinates on H_2 coming from the fibering p_1. The results are formulated under the more general assumption that the monodromy of p_1 lie in $\mathcal{I}_g$ rather than $\mathcal{K}_g$, because we feel that the arguments are clearer in this larger context. The main objective is Lemma 2.6.3.

Suppose that $p_1 : E \to B_1$ is a bundle with monodromy lying in $\mathcal{I}_g$. Suppose there is a partial section $\sigma : B' \to E'$ such that $[\omega_\sigma] = 0 \in H_1(F)$, giving rise to a lift $\tilde{\tau}$ of the Johnson homomorphism to $\wedge^3 H$; then by Proposition 2.4.8.3, there is a natural splitting

$$H_3(E) \approx p_1^! H_1(B_1) \oplus H_1(F_1)$$

We use this direct sum decomposition to define the projections

$$P : H_3(E) \to p_1^! H_1(B_1) \qquad \text{and} \qquad Q : H_3(E) \to H_1(F),$$

and we consider the restrictions of P and Q to $p_2^!(H_1(B_2))$ for a second fibering $p_2 : E \to B_2$. Where convenient, we will also define P and Q on $H_1(B_2)$ directly, by precomposing with the injection $p^!$.

Lemma 2.6.1. *For any second fibering $p_2 : E \to B_2$, the restriction of Q to $H_1(B_2)$ is a symplectic mapping, with respect to $d\, i_{F_1}$ on $H_1(F_1)$ and i_{B_2} on $H_1(B_2)$, where $d = [F_1] \cdot [F_2]$ is the algebraic intersection number of the two fibers.*

Proof. There exist classes $x, y \in H_1(B_2)$ such that $x \cdot y = 1 \in H_0(B_2)$, so that $[F_2] = p_2^! x \cdot p_2^! y$, and there are expressions

$$p_2^! x = Px + Qx, \qquad p_2^! y = Py + Qy.$$

Consequently,

$$[F_2] = Px \cdot Py + Px \cdot Qy - Py \cdot Qx + Qx \cdot Qy.$$

By Proposition 2.4.8, $[F_1] \cdot Pz = 0$ for all $z \in H_1(B_2)$, so that

$$d = [F_1] \cdot [F_2] = [F_1] \cdot Qx \cdot Qy,$$

with the first equality holding by assumption. The condition $[F_2] = p_2^! x \cdot p_2^! y$ is equivalent to $i_{B_2}(x, y) = 1$. By Proposition 2.4.8,

$$d = [F_1] \cdot Qx \cdot Qy = i_{F_1}(Qx, Qy),$$

proving the claim. □

As in the above proof, let $x, y \in H_1(B_2)$ satisfy $x \cdot y = 1$. By Poincaré duality, in order to determine $[F_2]$ it suffices to determine the collection of cup products $[F_2] \cdot Z$ for $Z \in H_2(E)$. Relative to the splitting of $H_2(E)$ coming from p_1 (where the monodromy lies in $\mathcal{I}_g$), in particular we must determine $[F_2] \cdot \Sigma_{b,z}$, where $b \in H_1(B_1)$ and $z \in H_1(F_1)$.

Lemma 2.6.2. *Take $x, y \in H_1(B_2)$ satisfying $x \cdot y = 1$. For $b \in H_1(B_1)$ and $z \in H_1(F_1)$, let $\Sigma_{b,z}$ be the associated element of $H_2(E)$. Then*

$$[F_2] \cdot \Sigma_{b,z} = i_{B_1}(Px, b) i_{F_1}(Qy, z) - i_{B_1}(Py, b) i_{F_1}(Qx, z) + \tau(b)(Qx \wedge Qy \wedge z). \tag{2.8}$$

In particular, if $z \in \langle Qx, Qy \rangle^{\perp}$, then (2.8) simplifies to

$$[F_2] \cdot \Sigma_{b,z} = \tau(b)(Qx \wedge Qy \wedge z). \tag{2.9}$$

In fact, for all $z \in H_1(F_1)$, there exists a pair $x_z, y_z \in H_1(B_2)$ such that $z \in \langle Qx_z, Qy_z \rangle^{\perp}$ holds, so that for all b, z, (2.9) is satisfied for this choice of x_z, y_z.

Proof. The formulas in (2.8) and (2.9) follow directly from the description of the intersection form given in Proposition 2.4.8. The existence of a suitable x, y for a given z is nothing but a matter of symplectic linear algebra. Since we will use some features of the construction later on, we give a detailed explanation. Lemma 2.6.1 shows that $W = \operatorname{Im} Q$ is a symplectic subspace of $H_1(F_1)$, and so we can take a symplectic complement $W^{\perp}$. Any z can therefore be written as $w + w'$ with $w \in W$ and $w' \in W^{\perp}$. If $w = 0$ there is nothing to show. Otherwise, extend w to a symplectic basis for W so that $w = x_1$. As B_2 has genus ≥ 2, this basis includes x_2, y_2, and as $W = \operatorname{Im} Q$, we can select x_z, y_z in $H_1(B_2)$ with $Qx_z = x_2$ and $Qy_z = y_2$. □

We conclude this subsection by amalgamating the work we have done in the previous two propositions in order to give a description of $[F_2]$.

Lemma 2.6.3. *Let $p_2 : E \to B_2$ be a second fibering. The choice of partial section $\sigma : B' \to E'$ furnishes $H_2(E)$ with the following splitting*

$$H_2(E) = \langle [F_1] \rangle \oplus (H_1(B_1) \otimes H_1(F_1)) \oplus H_2(B_1),$$

with $H_1(B_1) \otimes H_1(F_1)$ spanned by the set of $\Sigma_{b,z}$ where b, z range in symplectic bases $\mathcal{B}, \mathcal{F}$ for $H_1(B_1), H_1(F_1)$ respectively, and $H_2(B_1)$ spanned by C as in Proposition 2.4.8. Relative to this splitting of $H_2(E)$ there is the following expression for $[F_2]$:

$$[F_2] = (\delta - 2de)[F_1] + dC + \sum_{b \in \mathcal{B}, z \in \mathcal{F}} \tilde{\tau}(b)(Qx_z \wedge Qy_z \wedge z) \Sigma_{\hat{b}\hat{z}}. \tag{2.10}$$

Here, $\delta = i_{B_1}(Px, Py) + Qx \cdot Qy \cdot C$ *for any choice of* $x, y \in H_1(B_2)$ *satisfying* $x \cdot y = 1$, $e = C^2$, *and* $d = [F_1] \cdot [F_2]$ *(the algebraic intersection of the two fibers). Also* $\hat{x}$ *denotes the symplectic dual of* x *relative to the chosen symplectic basis.*

Proof. Suppose V is a free $\mathbb{Z}$-module equipped with a nondegenerate symmetric bilinear pairing $\langle \cdot, \cdot \rangle$. Suppose moreover that there exists a generating set $\mathcal{A} = \{a_1, \ldots, a_k, b_1, \ldots, b_k\}$ with the property that $\langle a_i, a_j \rangle = \langle b_i, b_j \rangle = 0$ for all i, j, $\langle a_i, b_j \rangle = 0$ for $i \neq j$, and $\langle a_i, b_i \rangle = 1$. Then any element $x \in V$ is expressible in the form

$$x = \sum_{i=1}^{k} \langle x, a_i \rangle b_i + \sum_{i=1}^{k} \langle x, b_i \rangle a_i. \tag{2.11}$$

We will apply this to $V = H_2(E)$ with the intersection pairing; in order to do this we must find a suitable generating set $\mathcal{A}$. Via Proposition 2.4.8, the space $H_1(B_1) \otimes H_1(F_1)$ is orthogonal under $\cdot$ to $H_2(B_2)$ and to $H_2(F_1)$, and moreover, the collection of $\Sigma_{b,z}$ for $(b, z) \in \mathcal{B} \times \mathcal{F}$ is such a generating set on this subspace. We also have $[F_1] \cdot C = 1$, as well as $([F_1])^2 = 0$ and $C^2 = e$. Therefore, we can take

$$\mathcal{A} = \{[F_1], C - e[F_1]\} \cup \{\Sigma_{b,z} \mid (b, z) \in \mathcal{B} \times \mathcal{F}\}.$$

The only intersection that remains to be computed is $[F_2] \cdot C$. As $Px \cdot Py = i_{B_1}(Px, Py)[F_1]$, a direct computation gives

$$\begin{aligned} [F_2] \cdot C &= (Px \cdot Py + Px \cdot Qy - Py \cdot Qx + Qx \cdot Qy) \cdot C \\ &= Px \cdot Py \cdot C + Qx \cdot Qy \cdot C \\ &= i_{B_1}(Px, Py) + Qx \cdot Qy \cdot C = \delta. \end{aligned}$$

By assumption, $[F_1] \cdot [F_2] = d$, and Formula (2.9) computes $[F_2] \cdot \Sigma_{b,z}$. Therefore we may insert these computations into Formula (2.11) to obtain (2.10). □

Rigidity in the Johnson kernel. We now assume, as is required for Theorem A, that the monodromy of p_1 is contained in $\mathcal{K}_g$. As noted in the previous section, the closed Johnson kernel $\mathcal{K}_g$ coincides with the kernel of $\tau : \mathcal{I}_g \to \wedge^3 H/H$; similarly the pointed Johnson kernel $\mathcal{K}_{g,*}$ is the kernel of $\tau : \mathcal{I}_{g,*} \to \wedge^3 H$. We also noted above that if $\tau \circ \rho : H_1(B) \to \wedge^3 H/H$ is identically zero then there is a *canonical* lift $\tilde{\tau} : H_1(B) \to \wedge^3 H$, namely zero. This furnishes the (co)homology of E with a canonical splitting in which all cup products in (2.9) vanish.

In order to prove the main result of this section, we will compute $[F_2]$ and see that it is "as simple as possible" in the coordinates coming from p_1, the fibering with monodromy in $\mathcal{K}_g$. This will be accomplished via Lemma 2.6.3. Per our choice of lift $\tilde{\tau}$, the terms expressed via the Johnson homomorphism all vanish, so that

$$[F_2] = a[F_1] + dC,$$

for some $a \in \mathbb{Z}$. The coefficient a is determined by $[F_2] \cdot C$, or equivalently $\delta = i_{B_2}(Px, Py)$ (by Proposition 2.4.8.3, the term $Qx \cdot Qy \cdot C = 0$). This can be determined from Lemma 2.6.2.

Lemma 2.6.4. *Let E be a 4-manifold with two fiberings as a surface bundle over a surface: $p_1 : E \to B_1$ and $p_2 : E \to B_2$. Define the projection $P : H_1(B_2) \to H_1(B_1)$. Suppose the monodromy for the bundle structure associated to p_1 lies in $\mathcal{K}_g$. Then $P \equiv 0$, and consequently $\delta = 0$.*

Proof. Returning to (2.8), in the Johnson kernel setting, both $[F_2] \cdot \Sigma_{b,z}$ and $\tilde{\tau}(b)(Qx \wedge Qy \wedge z)$ are zero for all x, y, z. Taking z to be any element satisfying $i_{F_1}(Qy, z) \neq 0$ and $i_{F_1}(Qx, z) = 0$, (2.8) simplifies to $i_{B_1}(Px, b) = 0$. Since this is true for all b, we conclude that $Px = 0$, and since any $x \in H_1(B_2)$ has a suitable y so that (2.8) holds, we conclude that $P \equiv 0$ and $\delta = 0$ as claimed. □

With this in hand, we can apply Lemma 2.6.3 (recalling from Proposition 2.4.8.3 that $e = 0$) to see that $[F_2]$ is as simple as possible:

$$[F_2] = dC. \tag{2.12}$$

As was noted following the statement of Proposition 2.2.1, $[F_2]$ must be a primitive class, and so $d = \pm 1$. We record this fact for later reference:

Lemma 2.6.5. *Let $p_1 : E \to B_1$ be a surface bundle over a surface with monodromy in $\mathcal{K}_g$. Suppose there is a second fibering $p_2 : E \to B_2$. Then*

$$\deg(p_1 \times p_2) = \pm 1.$$

Proposition 2.3.1 asserts the equality of $\deg_2(p_1 \times p_2)$ *with* $\deg_2(p_2|_{F_1} : F_1 \to B_2)$ *and with* $\deg_2(p_1|_{F_2} : F_2 \to B_1)$. *Consequently*

$$\deg_2(p_2|_{F_1} : F_1 \to B_2) = \deg_2(p_1|_{F_2} : F_2 \to B_1) = 1 \pmod 2.$$

Remark 2.6.6. Observe that Lemma 2.6.5 supplies a proof of the missing assertion (2.1.2.1) $\Longrightarrow$ (2.1.2.3) in Proposition 2.1.2, namely that if E is a surface bundle over a surface with monodromy in the Johnson kernel, then any second fibering necessarily yields a bi-projection with nonzero degree. Of course, the assertion that any of the conditions (2.1.2.1), (2.1.2.2), (2.1.2.3), are equivalent to the bundle E being a product is the content of Theorem A.

Cohomology - splittings coming from multisections. In order to complete the proof of Theorem A, we will combine the work we have done above with an analysis of what the (co)homology of E looks like with respect to the coordinates coming from the second fibering (where the monodromy need not be contained in $\mathcal{I}_g$). The most convenient setting for this portion of the argument is in the *cohomology* ring, so we pause briefly to establish some preliminaries.

Most of what we have established vis a vis the intersection pairing on $H_*(E)$ is directly portable to the setting of the cup product in cohomology. In particular, the maps

$$p_i^* : H^*(B_i) \to H^*(E)$$

for $i = 1, 2$ are injections. We let $\eta_i \in H^2(B_i)$ be an integral generator compatible with the chosen orientations; it is easy to see that $p_i^*(\eta_i)$ is Poincaré dual to $[F_i]$. Relative to a choice of splitting

$$H^1(E) = p_1^* H^1(B_1) \oplus H^1(F_1),$$

there are the projection maps $P : H^1(B_2) \to H^1(B_1)$ and $Q : H^1(B_2) \to H^1(F_1)$, and Lemma 2.6.4 carries over to show that $P \equiv 0$. We can also transport our analysis of the intersection form on $H_*(E)$. In the cohomological setting, we have proved:

Proposition 2.6.7. *Let $F_1 \to E \to B_1$ be a surface bundle over a surface with monodromy in the Johnson kernel $\mathcal{K}_g$. Then E is an integral cohomology $B_1 \times F_1$, i.e. there exists a canonical isomorphism*

$$H^*(E) \approx H^*(B_1) \otimes H^*(F_1)$$

as graded rings.

We now continue with the proof of Theorem A.

Lemma 2.6.8. *Suppose that the genus of B_2 is strictly smaller than that of F_1. Then there exist classes $x, y \in H^1(E)$ annihilating $p_2^* H^1 B_2$ (that is, $x \smile p_2^* z = y \smile p_2^* z = 0$ for all $z \in H^1(B)$), such that $x \smile y = \Phi_1$, where $\Phi_1 \in H^2(F_1)$ is a generator.*

Proof. The cohomological formulation of Lemma 2.6.4 shows that

$$p_2^* H^1(B_2) \leq H^1(F_1).$$

By (the cohomological reformulation of) Lemma 2.6.1, $p_2^* H^1(B_2)$ is in fact a *symplectic* subspace of $H^1(F)$, and so there exists a symplectic complement. We can then take the desired x, y to be suitable elements of this complement. □

To finish the proof of Theorem A, we will examine where x, y must live, relative to coordinates on $H^*(E)$ coming from the fibering p_2. At this point, the results of Section 2.5 come into play. In particular, (2.7) endows $H^1(E, \mathbb{Q})$ with a splitting via

$$H^1(E, \mathbb{Q}) = \operatorname{Im} p^* \oplus \ker \hat{\sigma}^*.$$

For the remainder of the proof, we will assume that all of our cohomology groups have rational coefficients.

Lemma 2.6.9. *Let $p : E \to B$ be any surface bundle over a surface with multisection σ. Suppose that there exists $x \in H^1(E)$ annihilating $p^* H^1(B)$. Then $x \in \ker \hat{\sigma}^*$.*

Proof. Write

$$x = v + p^* b,$$

with $v \in \ker \hat{\sigma}^*$ and $b \in H^1(B)$. If $b \neq 0$, then there exists $c \in H^1(B)$ with $b \smile c \neq 0$. On the one hand, $x \smile p^* c = 0$ by assumption. On the other, letting $[B] \in H_2(B)$ denote the fundamental class, we have by Proposition 2.5.4

$$\begin{aligned}
\langle x \smile p^* c, \hat{\sigma}_*[B] \rangle &= \langle (v + p^* b) \smile p^* c, \hat{\sigma}_*[B] \rangle \\
&= \langle v \smile p^* c, \hat{\sigma}_*[B] \rangle + \langle p^*(b \smile c), \hat{\sigma}_*[B] \rangle \\
&= 0 + \langle \hat{\sigma}^* p^*(b \smile c), [B] \rangle \\
&= \langle b \smile c, [B] \rangle \neq 0,
\end{aligned}$$

since $v \in \ker \hat{\sigma}^*$. In this case we have reached a contradiction, and so $b = 0$ as desired. □

Lemma 2.6.10. *Let $F_1 \to E \to B_1$ be a surface bundle over a surface with monodromy in $\mathcal{K}_g$, and suppose there is a second fibering $p_2 : E \to B_2$. Let g denote the genus of F_1, and h denote the genus of B_2. Then $g = h$.*

Proof. We have already established (Lemma 2.6.5) that

$$\deg_2(p_2|_{F_1}) = 1 \pmod 2.$$

As p_2 has nonzero degree, we conclude immediately that $g \geq h$. Suppose $g > h$. Then there exist classes $x, y \in H^1(E)$ as in the statement of Lemma 2.6.8. We will make use of the existence of a multisection σ of $p_2 : E \to B_2$, so that by Lemma 2.6.9, we must have $x, y \in \ker \hat{\sigma}^*$. So by Proposition 2.5.4,

$$\langle x \smile y, \hat{\sigma}_*[B_2] \rangle = 0.$$

In the notation of Proposition 2.6.7, both $p_2^*H^1(B_2)$ and the classes x, y are contained in $H^1(F_1)$, and as the image of

$$\smile : \wedge^2 H^1(F_1) \to H^2(F_1)$$

is one-dimensional (since F_1 is a surface), we conclude that $x \smile y = p_2^*(\eta_2)$, where $\eta_2 \in H_2(B_2)$ is a generator. So then

$$\langle x \smile y, \hat{\sigma}_*[B_2]\rangle = \langle p_2^*(\eta_2), \hat{\sigma}^*[B_2]\rangle = \langle \eta_2, [B_2]\rangle = 1.$$

This is a contradiction; necessarily $g = h$. □

This shows that $p_2|_{F_1}$ is a map of nonzero degree between surfaces of the same genus, and as is well-known, therefore

$$(p_2)_* : \pi_1 F_1 \to \pi_1 B_2$$

must be an isomorphism.

Finishing Theorem A. At this point, we turn to an analysis of the fundamental group. Via the long exact sequence in homotopy for a fibration, there is an exact sequence

$$1 \to \pi_1 F_i \to \pi_1 E \to \pi_1 B_i \to 1,$$

for $i = 1, 2$. Consequently, the kernel of

$$(p_1 \times p_2)_* : \pi_1 E \to \pi_1 B_1 \times \pi_1 B_2$$

is given by $\pi_1 F_1 \cap \pi_1 F_2$. On the other hand, this is also the kernel of the cross-projection

$$\pi_1 F_1 \to \pi_1 B_2$$

which was just shown to be an isomorphism. We conclude that $(p_1 \times p_2)_*$ is an isomorphism.

The monodromy of the bundle E can be read off from the fundamental group, as the map $\pi_1 B_1 \to \mathrm{Out}(\pi_1 F_1) \approx \mathrm{Mod}(\Sigma_g)$ (the latter isomorphism coming from the theorem of Dehn, Nielsen, and Baer). Since $\pi_1 E$ is a product, this map is trivial. The correspondence (1) then shows that E, being a surface bundle with trivial monodromy, is diffeomorphic to $B_1 \times B_2$. This completes the proof of Theorem A. □

CHAPTER 3
SURFACE BUNDLES OVER SURFACES WITH ARBITRARILY MANY FIBERINGS

3.1 The examples

The basic construction. To illustrate our general method we start by describing a construction of a surface bundle over a surface E admitting four fiberings $p_1, p_2, p_3, p_4 : E \to \Sigma_g$. The monodromy of this bundle was first considered by Korkmaz[1], as an example of an embedding of a surface group inside the Torelli group. Related constructions were also used by Baykur and Margalit to construct Lefschetz fibrations that are not fiber-sums of holomorphic ones in [3]. For what follows it will be necessary to give a direct topological construction of the total space.

The method of construction is to perform a "section sum" of two surface bundles over surfaces (see [2] for a discussion of the section sum operation, including an equivalent description on the level of the monodromy representation). Let $\Sigma_{g_1} \to M_1 \to \Sigma_h$ and $\Sigma_{g_2} \to M_2 \to \Sigma_h$ be two surface bundles over a base space Σ_h, and for $i = 1, 2$ let $\sigma_i : \Sigma_h \to M_i$ be sections of M_1, M_2. If the Euler numbers of σ_1, σ_2 are equal up to sign, then it is possible to perform a fiberwise connect-sum of M_1, M_2 along tubular neighborhoods of $\mathrm{Im}(\sigma_i)$ (possibly after reversing orientation), giving rise to a surface bundle $\Sigma_{g_1+g_2} \to M \to \Sigma_h$. In what follows, we will give a more detailed description of this construction and explain how it can be used to produce surface bundles over surfaces with many fiberings.

Remark 3.1.1. We have chosen to present an example here where all of the fiberings have the same genus. In fact, the four fiberings presented here are equivalent up to fiberwise diffeomorphism, but *not* up to π_1-fiberwise diffeomorphism. We stress here that this is *not* an essential feature of the general method of construction described in the chapter, but merely the simplest example which requires the least amount of cumbersome notation. See Remark 3.1.4 for more on why π_1-fiberwise diffeomorphism is an important notion of equivalence for our purposes, and see Theorem 3.1.12 for the most general method of construction, which can produce 4-manifolds that fiber as surface bundles in arbitrarily many ways with surfaces of distinct genera. It is worth noting that if E^4 fibers as a Σ_g-bundle and a Σ_h-bundle, for $g \neq h$, then clearly these two fiberings are distinct, up to bundle isomorphism, fiberwise diffeomorphism, or π_1-fiberwise diffeomorphism, since the fibers are not even homeomorphic!

For $g \geq 2$, consider the product bundle $E_1 = \Sigma_g \times \Sigma_g$ with projection maps $p_V, p_H : E_1 \to \Sigma_g$ onto the first (resp. second) factor. Let N be an open tubular neighborhood of

1. Unpublished; communicated to the author by D. Margalit.

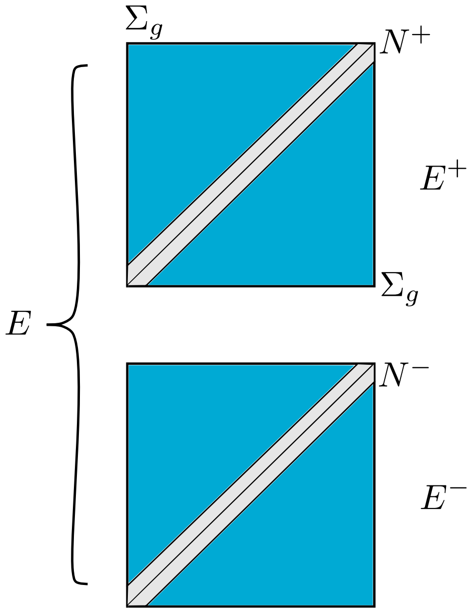

Figure 3.1: A cartoon rendering of E, depicted as shaded. The boundaries are identified.

the diagonal Δ. The manifold E is then constructed as the double

$$E = (E_1 \setminus N) \cup_{\partial \overline{N}} (E_1 \setminus N),$$

where the boundary components ∂N are identified via the identity map. We let E^+, E^- denote the "upper" and "lower" copies of $E_1 \setminus N$ contained in E. See Figure 3.1.

E is equipped with four fiberings $p_1, p_2, p_3, p_4 : E \to \Sigma_g$. These correspond to the four combinations of horizontal and vertical fiberings on E^+ and E^-. For each p_i, we will exhibit collar neighborhoods of $\partial E^{\pm}$ relative to which the given p_i will be smooth.

To describe these collar neighborhoods, we endow Σ_g with the structure of a Riemann surface. Via uniformization, this gives rise to a Riemannian metric, inducing a path metric d on Σ_g. Relative to d, there is a neighborhood N of the diagonal Δ, for suitably small ε, given via

$$N = \{(z, w) \in \Sigma_g \times \Sigma_g \mid d(z, w) < \varepsilon\}.$$

The boundary $\partial \overline{N}$ is parameterized via the Riemannian exponential map $\exp_z$ at each $z \in \Sigma_g$ (for convenience we locally parameterize the circle of radius ε about $0 \in T_z\Sigma_g$ using the complex exponential):

$$\begin{aligned}\partial\overline{N} &= \{(z,w) \mid |z-w| = \varepsilon\}\\ &= \{(z, \exp_z(\varepsilon e^{i\theta})) \mid \theta \in [0, 2\pi)\}\\ &= \{(\exp_z(-\varepsilon e^{i\theta}), z) \mid \theta \in [0, 2\pi)\}.\end{aligned}$$

p_1 is defined using the vertical projection p_V on each component. A suitable collar neighborhood (on either component) is given locally (for $t \in [1, 2)$) by

$$\theta_V(z, \varepsilon, t) = (z, \exp_z(t\varepsilon e^{i\theta})).$$

Similarly p_2 is defined using the horizontal projection p_H on each component. A suitable collar neighborhood of either boundary component is now given locally (again for $t \in [1, 2)$) by

$$\theta_H(z, \varepsilon, t) = (\exp_z(-t\varepsilon e^{i\theta}), z).$$

The remaining projections p_3, p_4 are defined using p_V on one component and p_H on the other. To realize these as smooth maps it will be necessary to modify the choice of boundary identification made in the construction of E. Consider the isotopy $h_t : \partial\overline{N} \times [0, 1] \to \partial\overline{N}$ given locally by

$$h_t(z, \exp_z(\varepsilon e^{i\theta})) = (\exp_z(-t\varepsilon e^{i\theta}), \exp_z((1-t)\varepsilon e^{i\theta}))).$$

More intrinsically, h_t acts by rigidly translating the pair (z, w) a distance $t\varepsilon$ along the geodesic ray from w to z; from this point of view it is clear that h_t is a diffeomorphism, and so h is indeed an isotopy.

As $h_0 = \text{id}$, there is a diffeomorphism

$$f : E \to (E_1 \setminus N) \cup_{h_1} (E_1 \setminus N).$$

p_3 is defined on $(E_1 \setminus N) \cup_{h_1} (E_1 \setminus N)$ using p_V on the first component and p_H on the second. Note that

$$p_V(z, \exp_z(\varepsilon e^{i\theta})) = (p_H \circ h_1)(z, \exp_z(\varepsilon e^{i\theta})) = z,$$

so p_3 is well-defined. Moreover, p_3 is smooth relative to the collar neighborhoods θ_V on the first component and θ_H on the second.

Completely analogously, p_4 is defined on $(E_1 \setminus N) \cup_{h_1} (E_1 \setminus N)$ using p_H on the first component and p_V on the second. See Figures 3.2 and 3.3 for some depictions of the fibering

p_4.

It is clear that each p_i is a proper surjective submersion; consequently by Ehresmann's theorem each p_i realizes E as the total space of a fiber bundle. In each case the base space is Σ_g, while the fiber is $\Sigma_g \# \Sigma_g \cong \Sigma_{2g}$.

We next recall the notion of π_1-*fiberwise diffeomorphism* from [32]. We say that two fiberings $p_1 : E \to B_1$, $p_2 : E \to B_2$ of a surface bundle are π_1-*fiberwise diffeomorphic* if

1. The bundles $p_1 : E \to B_1$ and $p_2 : E \to B_2$ are fiberwise diffeomorphic. That is, there exists a commutative diagram
$$\begin{array}{ccc} E & \xrightarrow{\phi} & E \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p_2} \\ B_1 & \xrightarrow[\alpha]{} & B_2 \end{array}$$
with ϕ, α diffeomorphisms.
2. The induced map ϕ_* preserves $\pi_1 F_1$, i.e. $\phi_*(\pi_1 F_1) = \pi_1 F_1$ (here, as always, F_i denotes a fiber of p_i).

In Chapter 2, we gave the following criterion for two bundle structures to be distinct up to π_1-fiberwise diffeomorphisms (Proposition 2.2.1):

Proposition 3.1.2. *Suppose E is the total space of a surface bundle over a surface in two ways: $p_1 : E \to B_1$ and $p_2 : E \to B_2$. Let F_1, F_2 denote fibers of p_1, p_2 respectively. Then the following are equivalent:*

1. *The fiberings p_1, p_2 are π_1-fiberwise diffeomorphic.*

2. *The fiber subgroups $\pi_1 F_1, \pi_1 F_2 \leq \pi_1 E$ are equal.*

If $\deg(p_1 \times p_2) \neq 0$ *then the bundle structures p_1 and p_2 are distinct.*

With this characterization in mind, we will establish the following theorem.

Theorem 3.1.3. *The fiberings $p_i : E \to \Sigma_g$ for $i = 1, 2, 3, 4$ constructed above are pairwise distinct up to π_1-fiberwise diffeomorphisms.*

Proof. To show that the projections p_i as defined are pairwise distinct, we will appeal to condition (2) of Proposition 3.1.2. For each i, the long exact sequence in homotopy of a fibration reduces to a short exact sequence

$$1 \longrightarrow \pi_1 F_i \longrightarrow \pi_1 E \xrightarrow{p_{i,*}} \pi_1 \Sigma_g \longrightarrow 1.$$

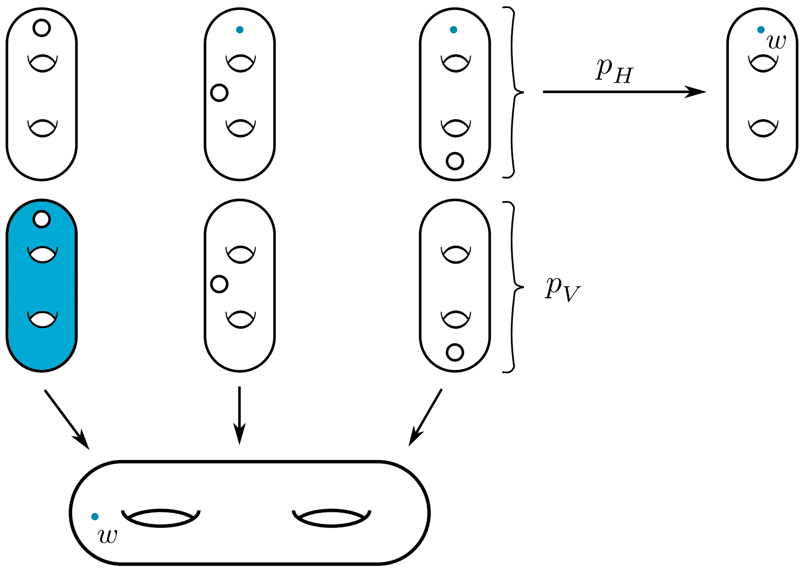

Figure 3.2: The fibering $p_4 : E \to \Sigma_g$. The fiber over $w \in \Sigma_g$ is shaded. On the upper portion of the bundle it intersects each of the p_V-fibers in a single point.

To show that $\pi_1 F_i$ and $\pi_1 F_j$ are distinct for distinct i, j, it therefore suffices to produce an element $x \in \pi_1 F_i$ such that $p_{j,*}(x) \neq 1$ in $\pi_1 \Sigma_g$. Let i and j be distinct. Without loss of generality, suppose that p_i is defined via p_V on E^+, while p_j is defined on E^+ via p_H. Let F_i and F_j denote generic fibers of p_i, p_j respectively. Both of $F_i \cap E^+$ and $F_j \cap E^+$ are homeomorphic to Σ_g^1, the surface of genus g and one boundary component.

Let $\gamma \subset \Sigma_g^1$ be a non-peripheral loop representing a nontrivial element of $\pi_1 \Sigma_g^1$, and identify γ with a loop in F_i. Then $[\gamma] \in \pi_1 F_i$ by construction (and is nontrivial), while $p_j(\gamma) = p_H(\gamma) = \gamma$. Here γ is viewed as a loop in Σ_g under the natural inclusion of Σ_g^1. As γ was chosen to be non-peripheral and essential in Σ_g^1, it remains homotopically nontrivial in Σ_g. It follows that $\pi_1 F_i$ and $\pi_1 F_j$ are distinct for all distinct $i, j \in \{1, 2, 3, 4\}$. Per Proposition 3.1.2, p_i and p_j are not π_1-fiberwise diffeomorphic as claimed. $\square$

Remark 3.1.4. As remarked above, the four fiberings constructed above are in fact fiberwise diffeomorphic, by applying factor-swapping involutions $(x, y) \to (y, x)$ on one or more of the components $E^{\pm}$. This same phenomenon appears for trivial bundles $\Sigma_g \times \Sigma_h$. When $g \neq h$ the projections onto the first and second factors clearly yield inequivalent bundles, as the fibers are not even the same manifold. On the other hand, when $g = h$, the factor-swapping involution yields a bundle isomorphism between the horizontal and vertical projections of

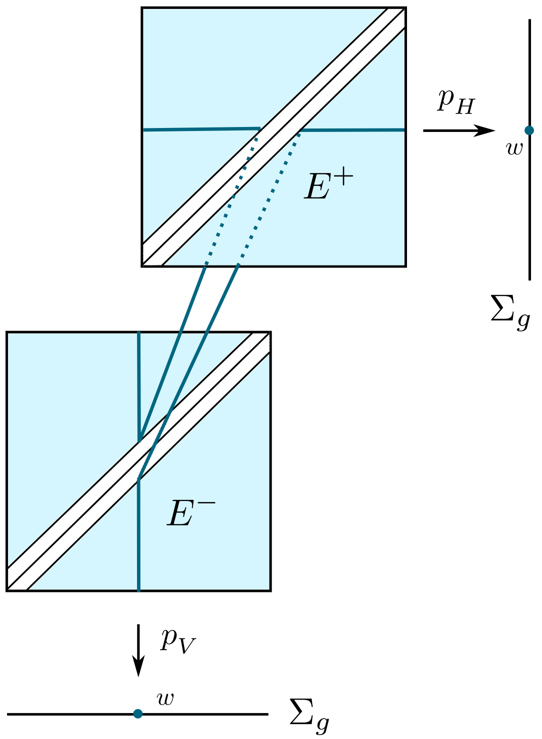

Figure 3.3: A second cartoon sketch of the fibering p_4.

$\Sigma_g \times \Sigma_g$. However, in both of these examples the fiberings are not π_1-fiberwise diffeomorphic. Moreover, Proposition 3.1.2 shows that π_1-fiberwise diffeomorphism is equivalent to the natural notion of equivalence on the group-theoretic level. For this reason, we believe that π_1-fiberwise diffeomorphism is an important notion of equivalence for surface bundles over surfaces. By using the techniques of Theorem 3.1.12, one can construct surface bundles over surfaces with arbitrarily many fiberings for which the fibers all have distinct genera, and therefore certainly give examples of bundles where the fiberings are not fiberwise diffeomorphic.

Remark 3.1.5. Via the Seifert-van Kampen theorem, it is possible to compute

$$\pi_1 E \cong \Gamma *_{\pi_1 UT\Sigma_g} \Gamma, \tag{3.1}$$

where $\Gamma = \pi_1(\Sigma_g \times \Sigma_g \setminus N)$ and $UT\Sigma_g$ denotes the unit tangent bundle. Let

$$\varpi_1 : \Sigma_g \times \Sigma_g \setminus N \to \Sigma_g$$

denote the vertical projection, and define ϖ_2 similarly as the horizontal projection. Relative to the isomorphism of (3.1), the induced maps of the four fiberings $(p_i)_* : \pi_1 E \to \pi_1 \Sigma_g$ correspond to the four amalgamations $(\varpi_i)_* * (\varpi_j)_* : \Gamma *_{\pi_1 UT\Sigma_g} \Gamma \to \pi_1 \Sigma_g$.

As remarked above, the bundle $p_1 : E \to \Sigma_g$ was originally considered by Korkmaz (see Footnote 1 of [3]), who constructed its monodromy representation as an example of an embedding $\rho : \pi_1\Sigma_g \to \mathcal{I}_{2g}$. We now give a description of this embedding. Let Mod_g^1 denote the mapping class group of a surface with one boundary component (where as usual the isotopies are required to fix the boundary component pointwise). We will denote this boundary curve by η. Consider the embedding

$$f : \pi_1(UT(\Sigma_g)) \to \mathrm{Mod}_g^1 \times \mathrm{Mod}_g^1$$
$$\alpha \mapsto (\mathrm{Push}(\alpha), F^{-1} \circ \mathrm{Push}(\alpha) \circ F),$$

where $F : \Sigma_g^1 \to \Sigma_g^1$ is any orientation-reversing diffeomorphism. Compose this with the map

$$h : \mathrm{Mod}_g^1 \times \mathrm{Mod}_g^1 \to \mathrm{Mod}_{2g}$$

obtained by juxtaposing the mapping classes (x, y) on the two halves of Σ_{2g}. Let $\gamma \in \pi_1(UT(\Sigma_g))$ denote the loop around the circle fiber in $UT\Sigma_g$ in the positive direction as specified by the orientation on Σ_g. The map $\mathrm{Push}(\gamma) \in \mathrm{Mod}(\Sigma_g^1)$ corresponds to a positive twist about η. We claim that $h(f(\gamma)) = \mathrm{id}$. Indeed, the notion of "positive" twist is relative to a choice of orientation, and after the boundary components of the two copies of Σ_g^1 have been identified, the two twists correspond to a positive and negative twist about η, and so the result is isotopic to the identity.

The element $\gamma \in \pi_1(UT(\Sigma_g))$ generates a normal subgroup, and the quotient $\pi_1(UT(\Sigma_g))/\langle\gamma\rangle \approx \pi_1\Sigma_g$. Therefore, we arrive at an embedding $\rho : \pi_1\Sigma_g \to \mathrm{Mod}_{2g}$ as follows.

$$\begin{array}{ccccc} \pi_1(UT(\Sigma_g)) & \xrightarrow{f} & \mathrm{Mod}_g^1 \times \mathrm{Mod}_g^1 & \xrightarrow{h} & \mathrm{Mod}_{2g} \\ \downarrow & & & \overset{\rho}{\nearrow} & \\ \pi_1\Sigma_g & & & & \end{array}$$

Lemma 3.1.6. *The image of ρ is contained in the Torelli group $\mathcal{I}_{2g}$.*

Proof. Let $\{\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g\}$ be a collection of simple closed curves for which the homology classes $\{[\alpha_1], \ldots, [\beta_g]\}$ comprise a generating set for $H_1(\Sigma_g^1)$. Let $F : \Sigma_g^1 \to \Sigma_g^1$ be the orientation-reversing map in the definition of f. We can then view Σ_{2g} as $\Sigma_g^1 \cup_{\partial\Sigma_g^1} F(\Sigma_g^1)$. Define

$$\mathcal{B} = \{\alpha_1, \ldots, \beta_g, F(\alpha_1), \ldots, F(\beta_g)\}.$$

It follows that the homology classes $\{[\alpha_1], \ldots, [\beta_g], [F(\alpha_1)], \ldots, [F(\beta_g)]\}$ comprise a generating set for $H_1(\Sigma_{2g})$. To determine whether a mapping class $\phi \in \mathrm{Mod}(\Sigma_{2g})$ is contained in $\mathcal{I}_{2g}$, it suffices to show that the homology class of each $\alpha_i, \beta_i, F(\alpha_i), F(\beta_i)$ is preserved by ϕ. Up to isotopy, η is preserved by the action of $\pi_1\Sigma_g$ via ρ, so it suffices to consider how $\pi_1\Sigma_g$ acts on both copies of Σ_g^1. If $x \in \pi_1\Sigma_g$ is given, then on Σ_g^1, the effect of $\rho(x)$ is to push the boundary component around a loop in Σ_g in the homotopy class of x. As is well-known (see, for example, [8], section 6.5.2), the curves η and $\rho(x)(\eta)$ are homologous, for any choice of $x \in \pi_1\Sigma_g$ and η a simple closed curve on Σ_g^1. In particular,

$$[\rho(x)(\alpha_1)] = [\alpha_1], \ldots, [\rho(x)(\beta_g)] = [\beta_g],$$

where these homologies hold in Σ_g^1 and so necessarily also in Σ_{2g}. The element $x \in \pi_1\Sigma_g$ acts on the other half of Σ_{2g} via conjugation by F, and so similarly the curves $F(\alpha_1), \ldots, F(\beta_g)$ are preserved on the level of homology. As we have shown that each homology class of a generating set for $H_1(\Sigma_{2g})$ is preserved under $\mathrm{Im}(\rho)$, it follows that $\mathrm{Im}(\rho) \leq \mathcal{I}_{2g}$ as claimed. □

Theorem 3.1.7. *The monodromy of any of the surface bundle structures $p_i : E \to \Sigma_g$ ($i = 1,2,3,4$) is the map $\rho : \pi_1\Sigma_g \to \mathcal{I}_{2g}$ described above.*

Proof. We begin by considering p_1. Let $x \in \pi_1\Sigma_g$ be given. The image of the monodromy representation $\mu(x) \in \mathrm{Mod}_{2g}$ is computed by selecting some immersed representative γ for x, considering the pullback of the bundle $E \to \Sigma_g$ along the immersion map $S^1 \to \Sigma_g$ specified by γ, and determining the monodromy of this fibered 3-manifold.

The bundle $p_1 : E \to \Sigma_g$ is constructed so that the fiber over $w \in \Sigma_g$ consists of two disjoint copies of Σ_g connect-summed along disks centered at w. This means that as one traverses a loop $\gamma \subset \Sigma_g$, the effect of the monodromy is to drag the cylinder connecting the two halves along the loops in either half corresponding to γ. As a mapping class, this is exactly the map $\rho(x)$ described above.

Now let $\pi_1 E = \Gamma *_{\pi_1 UT\Sigma_g} \Gamma$ as in Remark 3.1.5. There is an involution $\iota : \Gamma \to \Gamma$ induced from the factor-swapping map on $\Sigma_g \times \Sigma_g \setminus \nu(\Delta)$. Let ϖ_1, ϖ_2 denote the vertical (resp. horizontal) projection $\Sigma_g \times \Sigma_g \setminus (\nu(\Delta)) \to \Sigma_g$. Then $(\varpi_i)_* \circ \iota = (\varpi_{i+1})_*$ for $i = 1, 2$ interpreted mod 2. As ι preserves $\pi_1 UT\Sigma_g$, it can be extended to an automorphism of either factor of $\pi_1 E = \Gamma *_{\pi_1 UT\Sigma_g} \Gamma$. In other words, the four surface-by-surface group extension structures on $\pi_1 E$ are in the same orbit of the action of $\mathrm{Aut}(\pi_1 E)$. Consequently, the monodromy representations $r : \pi_1\Sigma_g \to \mathrm{Out}(\pi_1\Sigma_{2g})$ are the same. As r is identified with the topological monodromy representation $\rho : \pi_1\Sigma_g \to \mathrm{Mod}_{2g}$ under the Dehn-Nielsen-Baer isomorphism $\mathrm{Mod}_{2g} \approx \mathrm{Out}^+(\pi_1\Sigma_{2g})$, this shows that any of the four monodromy representations are equal. □

We summarize the results of the basic construction in the following theorem.

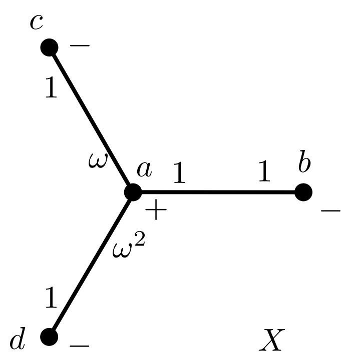

Figure 3.4: An example of a graph X equipped with a labeling of the half-edges by elements of $G = \mathbb{Z}/3 \approx \{1, \omega, \omega^2\}$ the group of third roots of unity.

Theorem 3.1.8. *For any $g \geq 2$, there exists a 4-manifold E which admits four fiberings $p_i : E \to \Sigma_g, i = 1, 2, 3, 4$ as a Σ_{2g}-bundle over Σ_g that are pairwise distinct up to π_1-fiberwise diffeomorphism. For each i, the monodromy $\rho_i : \pi_1\Sigma_g \to \mathrm{Mod}_{2g}$ of $p_i : E \to \Sigma_g$ is contained in the Torelli group $\mathcal{I}_{2g}$.*

Surface bundles over surfaces with n distinct fiberings. We next extend the construction given in the previous subsection to yield examples of surface bundles over surfaces with n distinct (up to π_1-fiberwise diffeomorphism) fiberings for arbitrary n. Let X be a connected bipartite graph with vertex set $V(X)$ and edge set $E(X)$ of cardinalities C, D respectively. As X is bipartite, it admits a coloring $c : V(X) \to \{+, -\}$ in such a way that if v is colored with $\pm$, then all the vertices w adjacent to v are colored $\mp$. Consequently we define $\delta^{\pm} : E(X) \to V(X)$ be the map which sends e to the vertex $v \in e$ colored $\pm$.

Let G be a finite group with $|G| = n$, where n is an integer such that every $v \in V(X)$ has valence at most n. Assign labelings $g^{\pm} : E(X) \to G$ to the half-edges of X, subject to the restriction that $g^{\pm}$ is an injection when restricted to

$$\{e \in E(X) \mid \delta^{\pm}(e) = v\}$$

for any $v \in V(X)$. In other words, the set of half-edges adjacent to any vertex must have distinct labelings. See Figure 3.4.

Let Σ be a surface admitting a free action of G, such as the one depicted in Figure 3.5. For each $v \in V(X)$, consider the 4-manifold $E_1^v = \Sigma \times \Sigma$, oriented so that the orientations on E_1^v and E_1^w disagree whenever $c(v) \neq c(w)$. Each E_1^v admits two projections $p^{v,1}, p^{v,2} : E_1^v \to \Sigma_g$ onto the first (resp. second) factor.

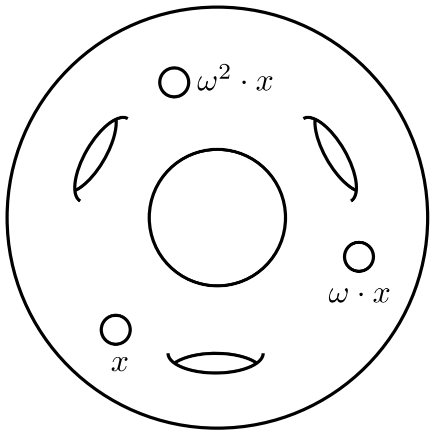

Figure 3.5: A surface Σ admitting a free action of $G = \{1, \omega, \omega^2\}$. With respect to the labeling in Figure 3.4, the fiber of E_2^a over $x \in \Sigma$ has neighborhoods of $x, \omega \cdot x$, and $\omega^2 \cdot x$ removed.

For $x \in G$, let

$$\Delta^x = \{(w, x \cdot w) \mid w \in \Sigma\} \subset \Sigma \times \Sigma$$

be the graph of $x : \Sigma \to \Sigma$. By abuse of notation we can view Δ^x as embedded in any of the E_1^v. Let Δ be the disconnected surface embedded in $E_1 = \bigcup_{v \in V(X)} E_1^v$ for which

$$\Delta \cap E_1^v = \bigcup_{v \in e} \Delta^{g^{c(v)}(e)}.$$

Let N denote the ε-neighborhood of Δ. There is a decomposition

$$N = \bigcup_{e \in E(X)} N^e$$

and a further decomposition

$$N^e = N^{e,+} \cup N^{e,-} \qquad \text{with} \qquad N^{e,\pm} \subset E_1^{\delta^{\pm}(e)}.$$

Each $N^{e,\pm}$ is the ε-neighborhood of a single component of Δ.

Define

$$E_2 = E_1 \setminus \operatorname{int}(N)$$

and, for $v \in V(X)$,

$$E_2^v = E_2 \cap E_1^v.$$

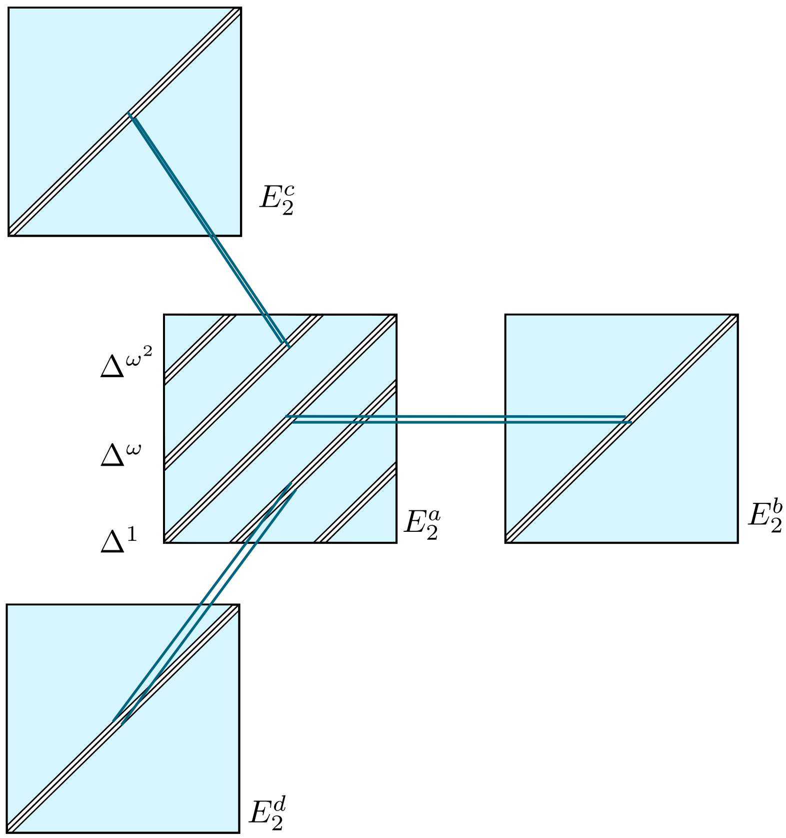

Figure 3.6: A schematic rendering of the 4-manifold E_X associated to the graph X of Figure 3.4 and the surface Σ of Figure 3.5. The lines connecting the components indicate how the various $\tilde{N}^e$ are attached.

The orientation convention ensures that for each $e \in E$, the Euler numbers of the disk bundles $N^{e,\pm}$ are given by $\pm\chi(\Sigma)$. Their boundaries can therefore be identified via an orientation-reversing diffeomorphism. As in the previous construction, it will be convenient to specify the gluing maps only up to isotopy, and as before we will take the isotopy class of the identity.

With these conventions in place, we define the (connected oriented) 4-manifold

$$E_X = \bigcup_{v \in V(X)} E_2^v$$

glued together as prescribed by the labeled graph X with all identifications of boundary components in the isotopy class of the identity. Figure 3.5 depicts a portion of the fiber of E_X for the graph X of Figure 3.4. Figure 3.6 depicts the total space of E_X. The portion of the fiber shown in Figure 3.5 is the portion contained in the central component of Figure 3.6.

Theorem 3.1.9. *Let X be a connected finite bipartite graph, possibly with multiple edges, with vertex set $V(X)$ and edge set $E(X)$ of cardinalities C, D respectively. Then,*

1. *There is a surface Σ_X such that the manifold E_X constructed above admits 2^C fiberings $p^f : E \to \Sigma_X$ as a surface bundle over a surface, indexed by the set of maps $f : V(X) \to \{1, 2\}$. The fiberings are pairwise-inequivalent up to π_1-fiberwise diffeomorphism.*
2. *When X is a tree we can take $\Sigma_X = \Sigma$. Otherwise, Σ_X admits a finite-sheeted regular covering by Σ.*
3. *The total space E_X has the structure of a graph of groups modeled on X where the vertex groups are free-by-surface group extensions Γ and the edge groups are given by $\pi_1 UT\Sigma$ (with notation as in Remark 3.1.5).*

Proof. The construction of p^f is now complicated by the fact that one must take care to make the restrictions of p^f to adjacent components E_2^v match up along the identification sites. The construction is simplest in the case where X is a tree. Choose some root vertex v. Given $f : V(X) \to \{1, 2\}$, define p^f on the root component E_2^v via $p^{v,f(v)}$. Next consider an adjacent vertex w. Then there is a unique element $\tau \in G$ for which $p^{v,f(v)}$ and $\tau \circ p^{w,f(w)}$ agree on the intersection $E_2^v \cap E_2^w$, and we define p^f on E_2^w as this $\tau \circ p^{2,f(w)}$. In this way, we can continue through the tree X defining p^f one component at a time.

For a general bipartite graph X, it is possible that the τ as above need not be unique. To rectify this, we define a subgroup $G' \leq G$ as follows. Choose a root vertex v as above. The procedure above assigns an element $\tau_\gamma \in G$ to each simplicial path γ from v to some vertex w. Define $G' \leq G$ as the group generated by elements $\tau_\gamma \tau_{\gamma'}^{-1}$ as γ, γ' range over all simplicial paths in X ending at a common vertex. Define $\Sigma_X = \Sigma/G'$, and denote the projection $\Sigma \to \Sigma_X$ by q. Then, defining p^f on each component E_2^v as $q \circ p^{v,f(v)}$ yields a well-defined map.

To realize p^f as a smooth map, it is necessary to specify gluing maps identifying the various components of E_2, as well as appropriate collar neighborhoods. We proceed exactly as in Theorem 3.1.8. For each $x \in G$, there is an identification of (neighborhoods of) Δ^x with Δ^1 via the action of the diffeomorphism $\mathrm{id} \times x^{-1}$ of $\Sigma \times \Sigma$. Relative to these identifications, we will speak of identifying $\partial(N^{e,+})$ and $\partial(N^{e,-})$ via id or by h_1 as in Theorem 3.1.8. Likewise, we will speak of the collar neighborhoods θ_1 and θ_2 of $\partial(N^{e,\pm})$ (referred to as θ_V and θ_H respectively in Theorem 3.1.8).

The identifications are indexed via $E(X)$. As in Theorem 3.1.8, identify $\partial(N^{e,+})$ and $\partial(N^{e,-})$ via id if $f(\delta^+(e)) = f(\delta^-(e))$ and via h_1 otherwise. Then a collar neighborhood of $\partial(N^{e,\pm})$ for which p^f is smooth is given by $\theta_{f(\delta^\pm(e))}$.

The argument that each of the fiberings are distinct up to π_1-fiberwise diffeomorphism proceeds along the same lines as in Theorem 3.1.3. If $f_1, f_2 : V(X) \to \{1, 2\}$ are distinct, then there exists at least one v for which $f_1(v) \neq f_2(v)$. Arguing as in Theorem 3.1.3, one produces an essential loop $\gamma \subset E_2^v$ contained in the fiber of f_1 that projects onto an essential loop under f_2.

By definition, a graph of groups on a graph X is constructed by connecting Eilenberg-Mac Lane spaces $K(\Gamma_v, 1)$ indexed by the vertices, along mapping cylinders induced from homomorphisms $\phi_e : \Gamma_e \to \Gamma_v$. In our setting, for each $v \in V(X)$, the space E_2^v is a $K(\pi_1 E_2^v, 1)$ space, since it is the total space of a fibration $\Sigma' \to E_2^v \to \Sigma$, where Σ' is obtained from Σ by removing n open disks, one for each edge incident to v. As the base and the fiber of this fibration are both aspherical, it follows from the homotopy long exact sequence that E_2^v is aspherical as well. The edge spaces are given by $\partial(N^{e,\pm})$, each of which is diffeomorphic to the aspherical space $UT\Sigma$. It follows that E_X is indeed a graph of groups. □

Remark 3.1.10. In contrast with the construction in Theorem 3.1.8, the monodromy representations associated to an arbitrary E_X need not be contained in the Torelli group. For example, let X be a graph with two vertices and two edges connecting them. We can take Σ to be a surface of genus 3. Then it is easy to find elements of the monodromy that do not preserve the homology of the fiber. See Figure 3.7.

It can also be seen from this point of view that the images of the monodromy representations will be contained in the *Lagrangian mapping class group* $\mathcal{L}_g$, defined as follows. The algebraic intersection pairing endows $H_1(\Sigma_g, \mathbb{Z})$ with a symplectic structure, and there is a decomposition

$$H_1(\Sigma_g, \mathbb{Z}) = L_x \oplus L_y$$

as a direct sum, with the property that the algebraic intersection pairing restricts trivially to L_x and to L_y. Then

$$\mathcal{L}_g := \{f \in \mathrm{Mod}_g \mid f(L_x) = L_x\}.$$

Suppose $\tilde{\Sigma}$ has been constructed from a finite graph X as in Theorem 3.1.9. Let $\rho : \pi_1\Sigma \to \mathrm{Mod}(\tilde{\Sigma})$ be the associated monodromy. There is a Lagrangian subspace of $H_1(\tilde{\Sigma})$ of the form

$$L = \bigoplus_{v \in V(X)} L_v \oplus \mathcal{C},$$

where L_v is a Lagrangian subspace of the fiber of E_2^v, and $\mathcal{C} \leq H_1(\tilde{\Sigma})$ is the (possibly empty) subspace generated by the homology classes of the former boundary components in $\tilde{\Sigma}$. By

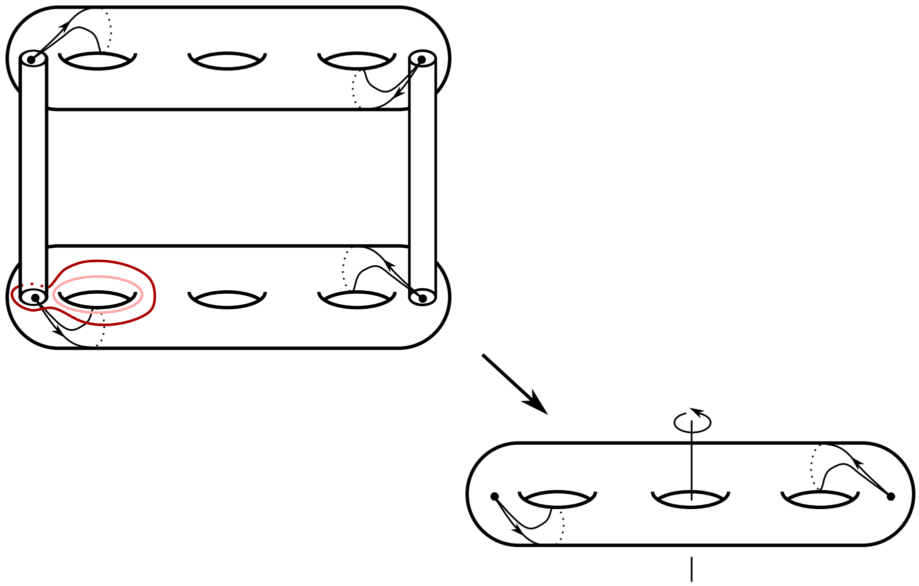

Figure 3.7: The lighter curve is taken to the darker one under the monodromy action associated to the loop on the base surface. The dark and the light curves are not homologous. The identifications of the boundary components have been indicated by cylinders.

construction, for all $x \in L_v$ and all $g \in \pi_1(\Sigma)$, the equation

$$\rho(g)(x) = x + c$$

holds in $H_1(\tilde{\Sigma})$, for some appropriate $c \in \mathcal{C}$. As $\mathcal{C}$ is fixed elementwise by the action of ρ, it follows that L is indeed a ρ-invariant Lagrangian subspace.

In [31], Sakasai showed that the first MMM class $e_1 \in H^2(\mathrm{Mod}_g, \mathbb{Z})$ vanishes when restricted to $\mathcal{L}_g$. It follows that the surface bundles over surfaces constructed in this section all have signature zero. More generally, suppose $\Sigma_g \to E \to \Sigma_h$ is a surface bundle over a surface with monodromy representation $\rho : \pi_1 \Sigma_h \to \Gamma$, where $\Gamma \leq \mathrm{Mod}_g$ is a subgroup. We can view the bundle $E \to \Sigma_h$ as giving rise to a homology class $[E] \in H_2(\Gamma, \mathbb{Z})$, e.g. by taking the pushforward $\rho_*([\Sigma_h])$ of the fundamental class.

Question 3.1.11. *Do the examples of surface bundles over surfaces given in Theorem 3.1.9 determine nonzero classes in $\mathcal{L}_g$? For a fixed g, what is the dimension of the space spanned in $H_2(\mathcal{L}_g, \mathbb{Q})$ by the examples in Theorem 3.1.9 with fiber genus g?*

Further constructions. It is possible to extend the constructions in Theorem 3.1.8 and Theorem 3.1.9 to obtain examples where the base and fibers of distinct fiberings do not all have the same genus. The author is grateful to D. Margalit for suggesting the basic idea underlying the constructions in this subsection.

Theorem 3.1.12. *Let Σ be a surface admitting a free action by a finite group G of order n, let X be a connected bipartite graph of maximal valence n, and let $f^v : \tilde{\Sigma} \to \Sigma^v$ for $v \in V(X)$ be covering maps, not necessarily distinct. Then there exists a 4-manifold E_X admitting $|V(X)| + 1$ fiberings $p^0, p^v(v \in V(X))$, with $p^0 : E_X \to \Sigma$ and $p^v : E_X \to \Sigma^v$ all projection maps for surface bundle structures on E, distinct up to π_1-fiberwise diffeomorphism. If the surfaces Σ^v and Σ^w have distinct genera, the fiberings p^v, p^w are distinct up to fiberwise diffeomorphism.*

Proof. Let Σ^0 be a closed surface of genus g that admits coverings $f^1 : \Sigma^0 \to \Sigma^1$ and $f^2 : \Sigma^0 \to \Sigma^2$ of degree d_1, d_2 respectively. For $i = 1, 2$, consider the graphs $\Gamma_i \subset \Sigma^0 \times \Sigma^i$ of the coverings f^i. Thicken these to tubular neighborhoods N^i. Each ∂N^i is an S^1-bundle over Σ^0 with Euler number $\chi(\Sigma^0)$. By reversing the orientation on one of the components, it is therefore possible to fiberwise connect-sum $\Sigma^0 \times \Sigma^1$ and $\Sigma^0 \times \Sigma^2$ along N^1 and N^2 to make the 4-manifold E.

Let $p_V : E_2 \to \Sigma^0$ and $p_H^i : E_2^i \to \Sigma^i$ be the vertical and horizontal projections. These can be combined in various ways to define three distinct fiberings on E. The first fibering $p_0 : E \to \Sigma^0$ is given by the projection onto the first factor on both coordinates of E_2, so that the fiber is $\Sigma^1 \# \Sigma^2$. The second fibering $p_1 : E \to \Sigma^1$ is given by p_H^1 on E_2^1, and by $f^1 \circ p_V$ on E_2^2. Let F_1 denote the fiber of p_1 over $w \in \Sigma^1$. Then (relative to an appropriate metric d and a suitable $\varepsilon > 0$)

$$F_1 \cap E_2^1 = \{(y, w) \in \Sigma^0 \times \Sigma^1 \mid d(f^1(y), w) \geq \varepsilon\}$$

is a copy of Σ^0 with d_1 disks removed (recall that d_i is the degree of the covering $f^i : \Sigma^0 \to \Sigma^i$). In turn,

$$F_1 \cap E_2^2 = \{(v, y) \in \Sigma^0 \times \Sigma^2 \mid f^1(v) = w, d(f^2(v), y) \geq \varepsilon\}$$

consists of d_1 copies of Σ^2, each with one boundary component. In total then,

$$F_1 = \Sigma^0 \# \left(\Sigma^2\right)^{\# d_1}.$$

When $d_1 > 1$, the monodromy of p_1 is not contained in the Torelli group $\mathcal{I}_g$. Let γ be a loop on Σ^1 which lifts to an arc $\tilde{\gamma} \subset \Sigma^0$ with endpoints v_1, v_2. Then the component of $F_1 \cap E_2^2$ lying over $v_1 \in \Sigma^0$ is sent to the component lying over v_2. If x is a loop in the first component representing some nontrivial homology class in F_1, then $\rho(\gamma)(x)$ is a distinct homology class in F_1, and so the monodromy of p_1 has a nontrivial action on $H_1(\Sigma_g, \mathbb{Z})$.

The construction of $p_2 : E \to \Sigma^2$ is completely analogous. The fibering p_2 is given by $f^2 \circ p_V$ on E_2^1 and by p_H^2 on E_2^2. The fiber is of the form

$$F_2 = \Sigma^0 \# \left(\Sigma^1\right)^{\# d_2}.$$

As in the previous constructions it is necessary to specify the precise identification maps as well as collar neighborhoods. The internal details proceed along similar lines as before, except that the boundary identifications require some further comment. Realize ∂N^i as a subset of $\Sigma^0 \times \Sigma^i$. Then ∂N^i is the total space of two different fiber bundle structures inherited respectively from p_V and p_H^i. The identification maps for the various p_i will be constructed so as to preserve fibers of these various fiberings.

For p_0, identify ∂N^1 and ∂N^2 in a fiber-preserving way with respect to p_V on both ∂N_1 and ∂N_2. For p_1, identify ∂N^1 and ∂N^2 so that p_H^1-fibers on ∂N_1 correspond to p_V-fibers ∂N_2. More precisely, given $z \in \Sigma^1$, the p_H^1-fiber of z consists of d_1 disjoint circles projecting down to circles in Σ^0 centered at the points of $(f^1)^{-1}(z)$. For every $x \in \Sigma^0$, the identification of ∂N_1 and ∂N_2 identifies $p_V^{-1}(x)$ with the component of $(p_H^1)^{-1}(f^1(x))$ centered over x. The identification of $\partial N^1, \partial N^2$ appropriate for p_2 is constructed analogously, matching p_V-fibers of ∂N_1 with p_H^2-fibers of ∂N_2.

The straight-line isotopy h_t constructed in the course of Theorem 3.1.8 was purely local in its definition. The same formulas as before show that the three gluing maps constructed in the above paragraph are mutually isotopic, and the construction proceeds as before.

It is also possible to generalize the construction of Theorem 3.1.9, so that the surfaces used in the construction of E_X are all covered by Σ. For $v \in V(X)$, let $f^v : \Sigma \to \Sigma_v$ be a covering. Suppose that each Σ_v admits a free action of a group G_v, such that $|G_v|$ is at least the valence of v. We may then repeat the construction of Theorem 3.1.9, taking $E_1^v = \Sigma \times \Sigma_v$. Since G_v acts freely, for $g, h \in G_v$, the graphs of $g \circ f^v$ and $h \circ f^v$ are disjoint as submanifolds of E_1^v. We may then remove neighborhoods of these graphs to produce E_2^v and connect the boundaries as in Theorem 3.1.9. The resulting E_X has at least $|V(X)| + 1$ fiberings $p_0, p_v (v \in V(X))$. The first fibering p_0 is defined on each E_2^v via p_V, and the result is a fiber bundle $p_0 : E_X \to \Sigma$. For $v \in V(X)$, define p_v on the components E_2^v via

$$p_v|_{E_2^w} \begin{cases} p_H^v & w = v \\ f^v \circ p_V & w \neq v. \end{cases}$$

The result is a fibering $p_v : E_X \to \Sigma^v$. □

Example 3.1.13. Let Σ be a surface admitting a free action of $\mathbb{Z}/2^n$ for some n. For

$0 \leq k \leq n$ define $\Sigma^k = \Sigma/(\mathbb{Z}/2^k)$. Let $f^k : \Sigma \to \Sigma^k$ be the associated covering. Each Σ^k admits an action of $\mathbb{Z}/2^{n-k}$, so that for $k \leq n-1$, each Σ^k admits a free involution τ^k. Let X be the "line graph" with vertex set $V(X) = \{0, 1, \ldots, n\}$, such that $\{i, j\} \in E(X)$ whenever $|i - j| = 1$.

In this setting, the construction of Theorem 3.1.12 produces a 4-manifold E^4 which fibers as a surface bundle over Σ^k for each $0 \leq k \leq n$. In more detail, define $E_1^k = \Sigma \times \Sigma^k$. For $0 \leq k \leq n-1$, the graphs of f^k and $\tau^k \circ f^k$ are disjoint, and we attach E_1^k to E_1^{k+1} by joining the graph of $\tau^k \circ f^k \subset E_1^k$ to the graph of $f^{k+1} \subset E_1^{k+1}$. Although E_1^n does not necessarily admit a free involution, the vertex $n \in X$ has valence 1, and E_1^{n-1} can still be joined to E_1^n using the rule described above, resulting in a 4-manifold E_X.

For $0 \leq k \leq n$, there are fiberings $p_k : E_X \to \Sigma^k$ defined on components $E_2^j \subset E_X$ via

$$p_k|_{E_2^j} = \begin{cases} p_H^k & j = k \\ f^k \circ p_V & j \neq k \end{cases}$$

Together, these realize E_X as the total space of a surface bundle over Σ^k for each $0 \leq k \leq n$.

3.2 Further questions

In this final section we collect together some questions about surface bundles over surfaces with multiple fiberings. Our first line of inquiry concerns the number of possible fiberings that surface bundles over a surface with given Euler characteristic can admit.

Proposition 3.2.1. *Let E^4 be a 4-manifold with $\chi(E) = 4d$. Then E admits at most*[2]

$$F(d) = \sigma_0(d)(d+1)^{2d+6}$$

fiberings as a surface bundle over a surface which are distinct up to π_1-fiberwise diffeomorphism, where $\sigma_0(d)$ denotes the number of divisors of d.

Proof. To obtain the explicit bound given above, we will first reproduce F.E.A. Johnson's original argument, incorporating some improvements suggested by J. Hillman. Let $p : E \to \Sigma_h$ be the projection for a Σ_g-bundle structure on E. There is an associated short exact sequence of fundamental groups

$$1 \to K \to \pi_1 E \to \pi_1 \Sigma_h \to 1, \tag{3.2}$$

2. In fact, an additional argument, such as the one given in section 5.2 of [14], can be used to obtain the slightly better bound $\sigma_0(d)d^{2d+6}$. The bound given here is good enough for our purposes.

with $K \approx \pi_1\Sigma_g$ the fundamental group of the fiber.

We will first show that if $g < h$, then p determines the unique Σ_g-bundle structure on E, up to π_1-fiberwise diffeomorphism. Equivalently (by Proposition 3.1.2), it suffices to show that (3.2) is the unique splitting of $\pi_1 E$ as an extension of $\pi_1\Sigma_h$ by $\pi_1\Sigma_g$.

Suppose $p' : E \to \Sigma_h$ is a second fibering, giving rise to a short exact sequence

$$1 \to K' \to \pi_1 E \to \Sigma_h \to 1.$$

Consider the projection $p_*|_{K'}$. Suppose first that $p_*(K') = \{1\}$, or equivalently $K' \leq \ker p_* = K$. As K and K' are both isomorphic to $\pi_1\Sigma_g$, in this case $K = K'$.

Suppose next that $\mathrm{Im}(p_*|_{K'})$ is nontrivial. In this case, the image $p_*(K')$ is a nontrivial finitely generated normal subgroup of the surface group $\pi_1\Sigma_h$. It is a general fact that if $N \lhd \pi_1\Sigma_h$ is any nontrivial finitely-generated normal subgroup, then N has finite index in $\pi_1\Sigma_h$ (cf Theorem 3.1 of [29]). No finite-index subgroup of $\pi_1\Sigma_h$ is generated by strictly fewer than $2h$ generators. On the other hand, K' is generated by $2g$ generators by assumption. This is a contradiction, and it follows that $\mathrm{Im}(p_*|_{K'}) = \{1\}$. By the argument of the previous paragraph, this shows that necessarily $K = K'$, and so $p : E \to \Sigma_h$ is the unique Σ_g-bundle structure on E as claimed.

Returning to the general setting, suppose $p : E \to \Sigma_h$ is a Σ_g-bundle over Σ_h. As before, let $K \approx \pi_1\Sigma_g$ denote the fundamental group of the fiber. The Euler characteristic is multiplicative for fiber bundles:

$$\chi(E) = \chi(\Sigma_g)\chi(\Sigma_h) = 4(g-1)(h-1).$$

Let $d = (g-1)(h-1)$, so that $\chi(E) = 4d$. Any $d+1$-sheeted cover of Σ_h has genus $(h-1)d + h = (h-1)^2(g-1) + h$, and this quantity is strictly larger than g. Let $\tilde{\Sigma} \to \Sigma_h$ be such a cover, and let $\tilde{p} : \tilde{E} \to \tilde{\Sigma}$ denote the pullback of p along this cover. Then $\tilde{p}$ has the property that the genus of the fiber is strictly smaller than the genus of the base. By the above argument, K is the unique normal subgroup of $\pi_1\tilde{E}$ isomorphic to $\pi_1\Sigma_g$ with surface group quotient.

Let $\tilde{\alpha} : \pi_1 E \to \mathbb{Z}/(d+1)\mathbb{Z}$ be an epimorphism. If $\tilde{\alpha}(K) = 0$, then $\tilde{\alpha}$ is induced from a map $\alpha : \pi_1\Sigma_h \to \mathbb{Z}/(d+1)\mathbb{Z}$. Let $\tilde{\Sigma}$ denote the cover of Σ_h associated to α. Carrying out the construction of the previous paragraph, it follows that to each such $\tilde{\alpha}$ there is at most one Σ_g-bundle structure on E. As $\chi(\Sigma_g)$ must divide $\chi(E)$, it follows that E can be the total space of a Σ_g-bundle for only finitely many g. As $\mathrm{Hom}(\pi_1 E, \mathbb{Z}/(d+1)\mathbb{Z})$ is finite, this completes the portion of the argument due to F.E.A. Johnson.

Our own extremely modest contribution to Proposition 3.2.1 is to determine an explicit upper bound on the maximal cardinality of $\mathrm{Hom}(\pi_1 E, \mathbb{Z}/(d+1)\mathbb{Z})$ over all possible surface bundles E of a fixed Euler characteristic $4d$. It follows from (3.2) that a surface bundle $\Sigma_g \to E \to \Sigma_h$ admits a generating set for $\pi_1 E$ of size $2g + 2h$. As g, h range over all possible pairs such that $(g-1)(h-1) = d$, the largest value of $2g + 2h$ is obtained for $g = d+1, h = 2$. This shows that any surface bundle over a surface E with $\chi(E) = 4d$ has a generating set with at most $2d + 6$ generators. It follows that

$$|\mathrm{Hom}(\pi_1 E, \mathbb{Z}/(d+1)\mathbb{Z})| \le (d+1)^{2d+6}.$$

As noted above, for each $\alpha \in \mathrm{Hom}(\pi_1 E, \mathbb{Z}/(d+1)\mathbb{Z})$, the corresponding cover $\tilde{E}$ has at most one Σ_g-bundle structure for each $g \ge 2$ such that $g - 1$ divides d. The bound in the statement of the Proposition follows. □

We defined the function $N(d)$ in the Introduction,

$$N(d) := \max \left\{ n \;\middle|\; \begin{array}{c} \text{there exists } E^4, \chi(E) \le 4d,\ E \text{ admits } n \text{ surface bundle structures} \\ \text{distinct up to } \pi_1\text{-fiberwise diffeomorphism.} \end{array} \right\}$$

Proposition 3.2.1 shows that $N(d) \le \sigma_0(d)(d+1)^{2d+6}$. Prior to the results of this chapter, the best known lower bound on $N(d)$ was $N(d) \ge 2$. Drastic improvements can be made by making use of the construction of Theorem 3.1.9. Let Σ be a surface of genus 3 admitting a free involution τ, and let X be the "line graph" with vertex set $V(X) = \{1, 2, \ldots, n\}$, such that $\{i, j\} \in E(X)$ whenever $|i - j| = 1$. According to Theorem 3.1.9, the corresponding E_X has 2^n fiberings. For each fibering, the base has genus 3 and the fiber has genus $3n$; consequently $\chi(E_X) = 4 \cdot 2 \cdot (3n - 1)$. This shows that

$$N(6n - 2) \ge 2^n.$$

Combining this with Johnson's upper bound, we obtain

$$2^{(d+2)/6} \le N(d) \le \sigma_0(d)(d+1)^{2d+6}.$$

Problem 3.2.2. *Study the function $N(d)$. Sharpen the known upper bounds on N, and construct new examples of surface bundles over surfaces to improve the lower bounds.*

One feature of the constructions given here is that they all take place within the smooth category, and cannot be given complex or algebraic structures. Indeed, all of the monodromy representations of the constructions of Section 3.1 globally fix the isotopy class of a curve contained in the fiber (one of the former boundary components). H. Shiga has shown ([34]) that if E is a 4-manifold with a complex structure, B a Riemann surface, and $p : E \to B$ a holomorphic map realizing E as the total space of a holomorphic family of Riemann surfaces, then the monodromy cannot globally fix the isotopy class of any curve. On the other hand, it has been shown independently by J. Hillman, M. Kapovich, and D. Kotschick (cf. Theorem 13.7 of [14]) that if E and B are as above and $p : E \to B$ is a *smooth* fibration of E over B, then there exists a holomorphic map $p' : E \to B$ that realizes E as the total space of a holomorphic family of Riemann surfaces. Combining these results with the known reducibility of the monodromies of the examples in this chapter, one sees that our examples cannot be given complex structures. On the other hand, the examples of Atiyah and Kodaira that admit two fiberings take place in the algebraic category, prompting the following.

Question 3.2.3. *Let E^4 be a complex surface that is the total space of a surface bundle over a surface $p : E \to X$. Can such an E admit three or more such fiberings? More generally, can a* 4*-manifold with nonzero signature admit three or more structures as a surface bundle over a surface?*

This question is closely related to a point raised briefly in the introduction, and we remark that it is possible that the list of known fiberings of a given 4-manifold need not be exhaustive. There can be "hidden" fiberings that are not immediately apparent.

Question 3.2.4. *Are the two known fiberings of surface bundles over surfaces of the Atiyah-Kodaira type the only surface bundle structures on these manifolds? Do the manifolds constructed in Section 3.1 admit more fiberings than described in this chapter? Is there some finite-sheeted cover of an Atiyah-Kodaira manifold that admits three or more fiberings?*

CHAPTER 4
CUP PRODUCTS IN SURFACE BUNDLES

4.1 Preliminaries

4.1.1 Symplectic multilinear algebra

In this subsection, we lay out some basic facts concerning multilinear algebra over the $\mathbb{Q}$-vector space $H_1(\Sigma_g;\mathbb{Q})$, as well as the representation theory of the symplectic group.

We recall the definitions $H_1 := H_1(\Sigma_g;\mathbb{Q})$ and $H^1 := H^1(\Sigma_g;\mathbb{Q})$. The intersection pairing furnishes a nondegenerate alternating $\mathrm{Sp}(2g,\mathbb{Q})$-invariant form $\mu : H_1^{\otimes 2} \to \mathbb{Q}$. This form extends to a nondegenerate pairing $C_k : (H_1^{\otimes k})^{\otimes 2} \to \mathbb{Q}$ given by

$$(a_1 \otimes \cdots \otimes a_k) \otimes (b_1 \otimes \cdots \otimes b_k) \mapsto \prod_{i=1}^{k} \mu(a_i \otimes b_i). \tag{4.1}$$

For $u, v \in H_1^{\otimes k}$, the pairing satisfies $C_k(u \otimes v) = (-1)^k C_k(v \otimes u)$.

By convention, given a vector space V, the k^{th} exterior power $\wedge^k V$ will always be defined as a *quotient* of $V^{\otimes k}$ by imposing the skew-symmetry relations. Define the projection $q : V^{\otimes k} \to \wedge^k V$. There is a lift $L : \wedge^k V \to V^{\otimes k}$ given by

$$L(a_1 \wedge \cdots \wedge a_k) = \sum_{\tau \in S_k} (-1)^\tau a_{\tau(1)} \otimes \cdots \otimes a_{\tau(k)} \tag{4.2}$$

(to lighten the notational load, we will omit reference to k, which should be clear from context). By construction, $q \circ L = k!\,\mathrm{id}$.

There is a natural pairing $C_k' : (\wedge^k H_1)^{\otimes 2} \to \mathbb{Q}$ given by

$$(a_1 \wedge \cdots \wedge a_k) \otimes (b_1 \wedge \cdots \wedge b_k) \mapsto \det(\mu(a_i \otimes b_j)). \tag{4.3}$$

The pairings C_k and C_k' are related via

$$\begin{aligned} C_k'(q(a_1 \otimes \cdots \otimes a_k) \otimes q(b_1 \otimes \cdots \otimes b_k)) &= C_k(L(a_1 \wedge \cdots \wedge a_k) \otimes (b_1 \otimes \cdots \otimes b_k)) \\ &= C_k((a_1 \otimes \cdots \otimes a_k) \otimes L(b_1 \wedge \cdots \wedge b_k)) \\ &= \frac{1}{k!} C_k(L(a_1 \wedge \cdots \wedge a_k) \otimes L(b_1 \wedge \cdots \wedge b_k)). \end{aligned}$$

The map $C_k' : \wedge^{2k} H_1 \to \mathbb{Q}$ is $\mathrm{Sp}(2g,\mathbb{Q})$-equivariant (with respect to the trivial action on $\mathbb{Q}$), and it is a standard fact from representation theory that the invariant space $(\wedge^{2k} H_1)^{\mathrm{Sp}(2g,\mathbb{Q})} \cong \mathbb{Q}$, so that up to scalars, C_k' is the *only* such map.

4.1.2 The Gysin homomorphism

In this subsection, we collect some basic information on the Gysin homomorphism in the setting of local systems. Let X be a topological space. A *local system* on X is a $\mathbb{Z}[\pi_1 X]$-module M. If M and N are local systems on X, there is an evaluation pairing

$$H^p(X;M) \otimes H_p(X;N) \to M \otimes_{\mathbb{Z}[\pi_1 X]} N$$

written $\alpha \otimes x \mapsto \langle \alpha, x \rangle$; see [4, Section V.3]. While the Gysin homomorphism in the local coefficient setting appears to not be treated in the literature, the construction is the same as in the constant coefficient case; see for instance [27, Section 4.2.3]. The following proposition collects the properties of the Gysin homomorphism that will be used throughout the rest of the chapter.

Proposition 4.1.1 (Gysin basics)**.** *Suppose that $\pi : E \to B$ is a fibration with F^n a closed oriented n-manifold; let $\iota : F \to E$ denote the inclusion of a fiber. Let M be a local system on B, determining by pullback a local system (also denoted M) on E, which restricts to a constant system of coefficients on F.*

1. *There are homomorphisms*

   $$\pi_! : H^*(E;M) \to H^{*-n}(B;M)$$

   and

   $$\pi^! : H_*(B;M) \to H_{*+n}(E;M),$$

   called Gysin homomorphisms*. For $u \in H^n(E;M)$, the Gysin homomorphism coincides with the edge map $\iota^* : H^n(E;M) \to H^0(B;H^n(F;M))$. If π is an* oriented *F-bundle in the sense that $H^0(B;H^n(F;M)) \cong H^n(F;M) \cong M$, then there is an expression*

   $$\pi_!(u) = \langle \iota^*(u), [F] \rangle,$$

   where $[F] \in H_n(F)$ denotes the fundamental class.

2. *If N is another local system on B and $f : M \to N$ is a map of local systems, then f_* and $\pi_!$ commute.*

3. *Let $u \in H^i(E;M)$ and $v \in H^j(B;N)$ be given. Then there is an equality of elements in $H^{i+j-n}(B;M \otimes N)$*

   $$\pi_!(u\pi^*(v)) = \pi_!(u)v.$$

4. If $u \in H^i(E;M)$ and $x \in H_i(B;N)$ are given, there is an adjunction formula

$$\langle \pi_!(u), x\rangle = \langle u, \pi^!(x)\rangle$$

of elements in $M \otimes_{\mathbb{Z}[\pi_1 B]} N$.

4.2 Twisted MMM classes

In this section, we review the theory of twisted MMM classes, drawing on the work of Kawazumi and Morita in [20]. As above, let Mod_g denote the mapping class group of a closed surface, and let $\mathrm{Mod}_{g,*}$ denote the mapping class group of a closed surface with a marked point. There is the projection $\pi : \mathrm{Mod}_{g,*} \to \mathrm{Mod}_g$ giving rise to the Birman exact sequence

$$1 \longrightarrow \pi_1(\Sigma_g) \xrightarrow{\iota} \mathrm{Mod}_{g,*} \xrightarrow{\pi} \mathrm{Mod}_g \longrightarrow 1.$$

Form the fiber product $\overline{\mathrm{Mod}}_{g,*}$ via the diagram

$$\begin{array}{ccc} \overline{\mathrm{Mod}}_{g,*} & \xrightarrow{\bar{\pi}} & \mathrm{Mod}_{g,*} \\ {\scriptstyle \bar{\pi}}\downarrow\uparrow{\scriptstyle \sigma} & & \downarrow \\ \mathrm{Mod}_{g,*} & \longrightarrow & \mathrm{Mod}_g \end{array}$$

The section $\sigma : \mathrm{Mod}_{g,*} \to \overline{\mathrm{Mod}}_{g,*}$ is given by $\sigma(\phi) = (\phi, \phi)$. There is an isomorphism

$$\overline{\mathrm{Mod}}_{g,*} \cong \pi_1(\Sigma_g) \rtimes \mathrm{Mod}_{g,*}$$

via

$$(\phi, \psi) \mapsto (\psi\phi^{-1}, \phi).$$

Under this isomorphism, σ is given by $\sigma(\phi) = (1, \phi)$. This semi-direct product decomposition gives rise to a cocycle $k_0 \in Z^1(\overline{\mathrm{Mod}}_{g,*}, H_1)$ via

$$k_0((x, \phi)) = [x].$$

By an abuse of notation we will also use k_0 to denote the associated element of $H^1(\overline{\mathrm{Mod}}_{g,*}; H_1)$. By construction, $\iota^* k_0 = \mathrm{id} \in H^1(\pi_1\Sigma_g; H_1)$, and it is also clear that $\sigma^*(k_0) = 0$.

Let $e \in H^2(\mathrm{Mod}_{g,*})$ denote the Euler class of the vertical tangent bundle. For convenience, let $\bar{e} \in H^2(\overline{\mathrm{Mod}}_{g,*})$ denote $\bar{\pi}^*(e)$. The twisted MMM classes defined below were introduced by Kawazumi in [19].

Definition 4.2.1 (Twisted MMM classes)**.** Let $i, j \geq 0$. The *twisted MMM class* $m_{ij} \in H^{2i+j-2}(\mathrm{Mod}_{g,*}; H_1^{\otimes k})$ is defined as

$$m_{ij} = \pi_!(\bar{e}^i k_0^j).$$

For $j = 0$, this definition specializes to $m_{i,0} = \pi_!(\bar{e}^i) = e_{i-1}$, the $(i-1)^{st}$ (classical) MMM class.

Remark 4.2.2. Via the graded-commutativity of the cup product, the class

$$k_0^j \in H^j(\overline{\mathrm{Mod}}_{g,*}; H_1^{\otimes j})$$

in fact is valued in the subspace $L(\wedge^j H_1)$, and the same is therefore true of m_{ij}. In accordance with our convention that $\wedge^j H_1$ is a quotient of $H_1^{\otimes j}$, we will avoid writing $m_{ij} \in H^{2i+j-2}(\mathrm{Mod}_{g,*}; \wedge^j H_1)$.

The formulas at the heart of the present chapter are best expressed using a sort of "interior product". It will be convenient to first introduce the following piece of notation.

Definition 4.2.3. Let $i \geq 2j$ be given. Let $T_{i,j} \in \mathrm{End}\, H_1^{\otimes i}$ be the automorphism induced by permuting the factors via the permutation $f_{ij} \in S_i$ given by

$$f_{ij}(k) = \begin{cases} 2k-1 & k \leq j \\ 2(k-j) & j+1 \leq k \leq 2j \\ k & k > 2j \end{cases}$$

The effect of f_{ij} is to "interlace" the first $2j$ factors, making the k^{th} factor adjacent to the $(k+j)^{th}$ factor. f_{ij} factors as a composition of $\binom{j-1}{2}$ transpositions of adjacent factors. When $i = 2j$, the notation will be abbreviated to $T_j := T_{2j,j}$.

Definition 4.2.4. Let $\alpha \in H^m(\mathrm{Mod}_{g,*}; H_1^{\otimes n})$ and $x_i \in H^{d_i}(\mathrm{Mod}_{g,*}; H_1)$ be given for $1 \leq i \leq k \leq n$. Define the class

$$\alpha \lrcorner (x_1, \ldots, x_k) \in H^{m+\sum d_i}(\mathrm{Mod}_{g,*}; H_1^{\otimes n-k})$$

by the formula

$$\alpha \lrcorner (x_1, \ldots, x_k) = ((\mu^{\otimes k} \otimes \mathrm{id}^{\otimes n-k}) \circ T_{n+k,k})_*(x_1 \ldots x_k\ \alpha).$$

This formula can be equivalently expressed using C_k:

$$\alpha\lrcorner(x_1, \ldots, x_k) = (C_k \otimes \mathrm{id}^{\otimes n-k})_*(x_1 \ldots x_k\ \alpha).$$

Let $f : \Pi \to \mathrm{Mod}_g$ be a homomorphism from a group Π to the mapping class group. The fiber product $\Pi_* = \Pi \times_{\mathrm{Mod}_g} \mathrm{Mod}_{g,*}$ admits an extension of groups

$$1 \longrightarrow \pi_1(\Sigma_g) \xrightarrow{\iota} \Pi_* \xrightarrow{\pi} \Pi \longrightarrow 1. \tag{4.4}$$

The following proposition gives a canonical splitting on $H^*(\Pi_*)$. It appears as [20, Proposition 5.2].

Proposition 4.2.5 (Kawazumi-Morita)**.** *Suppose that there exists a cohomology class $\theta \in H^2(\Pi_*)$ such that*

$$\pi_!(\theta) = \langle \iota^*\theta, [\Sigma_g] \rangle = 1 \in H^0(\Pi).$$

Let

$$\theta' = \theta - \pi^*\pi_!(\theta^2)$$

which also satisfies $\pi_!(\theta') = 1$. The following statements hold:

1. *For any $\mathbb{Q}\Pi$-module M, the Lyndon-Hochschild-Serre spectral sequence of the extension (4.4) collapses at the E_2-term, and the cohomology group $H^*(\Pi_*; M)$ naturally decomposes as*

$$H^*(\Pi_*; M) \cong H^{*-2}(\Pi; M) \oplus H^{*-1}(\Pi; H_1 \otimes M) \oplus H^*(\Pi; M).$$

2. *There exists a unique element $\chi \in H^1(\Pi_*; H_1)$ satisfying*

$$\iota^*\chi = \mathrm{id} \in H^1(\pi_1(\Sigma_g); H_1), \quad \textit{and} \ \ \pi_!(\theta\chi) = \pi_!(\theta'\chi) = 0.$$

3. *The homomorphism $\varepsilon : H^{*-1}(\Pi; H_1 \otimes M) \to H^*(\Pi_*; M)$ given by*

$$\varepsilon(v) = (\mu \otimes \mathrm{id}_M)_*(\pi^* v\ \chi) \qquad (v \in H^{*-1}(\Pi; H_1 \otimes M)) \tag{4.5}$$

   is a left inverse of the edge homomorphism $\pi_\sharp : \ker \pi_! \to E_\infty^{*-1,1} = H^{*-1}(\Pi; H_1 \otimes M)$.

4. *Explicitly, for any $u \in H^*(\Pi_*; M)$:*

$$u = \theta' \pi^* \pi_!(u) - \mu_*(\pi^*\pi_!(u\chi)\ \chi) + \pi^*\pi_!(\theta u). \tag{4.6}$$

Remark 4.2.6. The primary case of interest will be the "universal" one, taking $\Pi = \mathrm{Mod}_{g,*}$ and $\Pi_* = \overline{\mathrm{Mod}}_{g,*}$. In [24], Morita constructs a class $\nu \in H^2(\overline{\mathrm{Mod}}_{g,*})$ satisfying the properties of θ listed in Proposition 4.2.5. Letting χ_ν denote the element χ associated to ν given by (2) of Proposition 4.2.5, Kawazumi-Morita show in [20] that $\chi_\nu = k_0$.

As was established by Kawazumi-Morita, the class $\nu \in H^2(\overline{\mathrm{Mod}}_{g,*})$ satisfies certain additional useful formulae; in essence, it behaves like a "Thom class" for surface bundles with section. These results are taken from [20, Theorem 5.1].

Theorem 4.2.7 (Kawazumi-Morita)**.** *There is a class $\nu \in H^2(\overline{\mathrm{Mod}}_{g,*})$ satisfying the following properties.*

1. $\pi_! \nu = 1$.

2. *For any $u \in H^*(\overline{\mathrm{Mod}}_{g,*}; M)$, there is an equality*

$$\nu u = \nu \pi^* \sigma^* u.$$

   Consequently,

$$\pi_!(\nu u) = \sigma^* u. \tag{4.7}$$

3. $\pi_!(\nu^2) = \sigma^* \nu = e$.

The following lemma gives a useful alternative characterization of $\mathrm{Im}\,\varepsilon$.

Lemma 4.2.8. *For all $p \geq 1$, there is an equality*

$$\mathrm{Im}\,\varepsilon = \ker \pi_! \cap \ker \sigma^*$$

of subspaces of $H^p(\overline{\mathrm{Mod}}_{g,*})$.

Proof. The containment $\mathrm{Im}\,\varepsilon \subset \ker \pi_!$ follows from the calculation

$$\begin{aligned} \pi_!(\mu_*(\pi^* u\ k_0)) &= \mu_*(\pi_!(\pi^* u\ k_0)) \\ &= \mu_*(u \pi_!(k_0)) \\ &= 0, \end{aligned}$$

with the equality $\pi_!(k_0) = 0$ holding for degree reasons.

To establish the containment $\mathrm{Im}\,\varepsilon \subset \ker \sigma^*$, recall the formula (4.7). Applied to $u =$

$\mu_*(k_0\ \pi^* u) \in \operatorname{Im}\varepsilon$, the formula gives

$$\begin{aligned}\sigma^* v &= \pi_!(\nu\ \mu_*(\pi^* u\ k_0)) \\ &= \pi_!(\mu_*(\nu\ \pi^* u\ k_0)) \\ &= \mu_*(u\ \pi_!(\nu k_0)) \\ &= 0,\end{aligned}$$

with the equality $\mu_*(u\ \pi_!(\nu k_0)) = 0$ coming from Proposition 4.2.5.2.

The reverse containment is a consequence of the explicit form of the splitting on $H^p(\overline{\mathrm{Mod}}_{g,*})$ given by Proposition 4.2.5.4. If $u \in \ker \pi_! \cap \ker \sigma^*$, then the first and third components in this splitting vanish (recalling that $\pi_!(\nu u) = \sigma^* u$), and so $u \in \operatorname{Im}\varepsilon$ as desired. □

4.3 Proof of Theorem C

The first part of Theorem C asserts the existence of a splitting on $H^*(E)$. This is precisely the content of Proposition 4.2.5.1. It remains to establish the formulas for the components given in (1.3, 1.4, 1.5).

Per Proposition 4.2.5.4, the $H^{D-2}(B)$-component of $\varepsilon(x_1)\dots\varepsilon(x_k)$ is given by $\pi_!(\varepsilon(x_1)\dots\varepsilon(x_k))$. Consider the element

$$\pi^*(x_1\dots x_k)k_0^k \in H^D(E; H_1^{\otimes 2k}).$$

Recall the interlacing operator T_k of Definition 4.2.3. As an automorphism of $H_1^{\otimes 2k}$, it is the composition of $\binom{k-1}{2}$ transpositions of adjacent factors. Via the graded-commutativity of the cup product,

$$T_{k,*}(\pi^*(x_1\dots x_k)k_0^k) = (-1)^\gamma(\pi^* x_1\ k_0)\dots(\pi^* x_k\ k_0),$$

where

$$\gamma = \sum_{i=1}^{k-1}(k-i)(d_i - 1). \tag{4.8}$$

From the definition of ε given in Proposition 4.2.5.3,

$$\varepsilon(x_i) = \mu_*(\pi^* x_i\ k_0).$$

It follows that

$$(\mu^{\otimes k} \circ T_k)_*(\pi^*(x_1\dots x_k)k_0^k) = (-1)^\gamma \varepsilon(x_1)\dots\varepsilon(x_k).$$

Via the commutativity of $(\mu^{\otimes k} \circ T_k)_*$ with $\pi_!$ (Proposition 4.1.1.2),

$$\begin{aligned}\pi_!(\varepsilon(x_1)\ldots\varepsilon(x_k)) &= (-1)^\gamma(\mu^{\otimes k} \circ T_k)_*(\pi_!(\pi^*(x_1 \ldots x_k)k_0^k)) \\ &= (-1)^\gamma(\mu^{\otimes k} \circ T_k)_*(x_1 \ldots x_k \; m_{0,k}) \\ &= (-1)^\gamma m_{0,k} \lrcorner (x_1, \ldots, x_k)\end{aligned}$$

with the penultimate equality holding as a consequence of the property (4.1.1.3) of the Gysin homomorphism and the definition of $m_{0,k}$. This establishes (1.3).

Per Proposition 4.2.5.4, the $H^{D-1}(B; H_1)$-component of $\varepsilon(x_1)\ldots\varepsilon(x_k)$ is given by

$$-\mu_*(\pi^*\pi_!(\varepsilon(x_1)\ldots\varepsilon(x_k)k_0)k_0) = -\varepsilon(\pi_!(\varepsilon(x_1)\ldots\varepsilon(x_k)k_0))$$

Arguing as in the previous paragraph,

$$\pi_!(\varepsilon(x_1)\ldots\varepsilon(x_k)k_0) = (-1)^\gamma m_{0,k+1} \lrcorner (x_1, \ldots, x_k).$$

(1.4) follows.

It remains to show that the $H^D(B)$-component of $\varepsilon(x_1)\ldots\varepsilon(x_k)$ is 0. From Proposition 4.2.5.4, this amounts to showing that

$$\pi_!(\nu\varepsilon(x_1)\ldots\varepsilon(x_k)) = 0.$$

From (4.7) and Lemma 4.2.8,

$$\pi_!(\nu\varepsilon(x_1)\ldots\varepsilon(x_k)) = \sigma^*(\varepsilon(x_1)\ldots\varepsilon(x_k)) = 0.$$

This establishes (1.5). □

4.4 The restriction of $m_{0,k}$ to $\mathcal{I}_{g,*}$

We begin this section with a review of the construction of the higher Johnson invariants. Let B be a paracompact Hausdorff space equipped with a distinguished class $[B] \in H_k(B)$. As the notation suggests, a primary case of interest will be when B is a closed oriented k-manifold. Let $f : B \to K(\mathcal{I}_{g,*}, 1)$ be a map classifying a surface bundle $\pi : E \to B$. Then $f_*([B])$ determines an element of $H_k(K(\mathcal{I}_{g,*}, 1))$. The space $K(\mathcal{I}_{g,*}, 1)$ is the base space for a "universal surface bundle with Torelli monodromy"; i.e. there is a space denoted $K(\overline{\mathcal{I}}_{g,*}, 1)$ and a map $\pi : K(\overline{\mathcal{I}}_{g,*}, 1) \to K(\mathcal{I}_{g,*}, 1)$ giving $K(\overline{\mathcal{I}}_{g,*}, 1)$ the structure of a Σ_g-bundle over

$K(\mathcal{I}_{g,*},1)$. The total space E therefore determines a $k+2$-cycle

$$[E] = \pi^! f_*[B] \in H_{k+2}(\overline{\mathcal{I}}_{g,*}).$$

By hypothesis, the monodromy representation $\rho : \pi_1(B) \to \mathcal{I}_{g,*}$ is valued in $\mathcal{I}_{g,*}$, so that $H^0(B; H_1(\Sigma_g, \mathbb{Z})) \cong H_1(\Sigma_g; \mathbb{Z})$, and there is a section $\sigma : B \to E$. Let $\mathrm{Jac}(E) \to B$ be the T^{2g}-bundle obtained by replacing each fiber $\pi^{-1}(b)$ of $E \to B$ with its Jacobian $\mathrm{Jac}(\pi^{-1}(b)) = H_1(\Sigma_g; \mathbb{R})/H_1(\Sigma_g; \mathbb{Z})$. The section σ endows each fiber $\pi^{-1}(b)$ with a base-point $\sigma(b)$; consequently there is a fiberwise embedding

$$J : E \to \mathrm{Jac}(E).$$

It follows from the equality

$$H^0(B; H_1(\Sigma_g; \mathbb{Z})) \cong H_1(\Sigma_g; \mathbb{Z})$$

that $\mathrm{Jac}(E) \cong B \times T^{2g}$ is a trivial bundle, so that there is a projection map $p : \mathrm{Jac}(E) \to T^{2g}$.

Definition 4.4.1 (Higher Johnson invariants)**.** With notation as above, the k^{th} *higher Johnson invariant* $\tau_k(B) \in \wedge^{k+2} H_1$ is the element

$$p_* J_*[E] \in H_{k+2}(T^{2g}) \cong \wedge^{k+2} H_1.$$

It is clear from the constructions that if B, B' are homologous k-cycles in $K(\mathcal{I}_{g,*},1)$, then $\tau_k(B) = \tau_k(B')$ and that τ_k is additive. Consequently, τ_k descends to a homomorphism

$$\tau_k : H_k(\mathcal{I}_{g,*}) \to \wedge^{k+2} H_1;$$

in view of the Universal Coefficient Theorem, this is equivalent to the description

$$\tau_k \in H^k(\mathcal{I}_{g,*}; \wedge^{k+2} H_1).$$

Proof of Theorem D. The proof will proceed in two steps. The first step is to understand the relationship between τ_{k-2} and the structure of the cup product form $\wedge^k H^1(E) \to H^k(E) \to \mathbb{Q}$ (this last map is obtained by the pairing $\alpha \mapsto \langle \alpha, [E] \rangle$). Once this is established, the second step is to compare this to the relationship between $m_{0,k}$ and the cup product form established by Theorem C.

Step 1: The higher Johnson invariants record the cup product form.

Proposition 4.4.2. *Let $f : B \to K(\mathcal{I}_{g,*},1)$ determine a $k-2$-cycle $[B]$ in $K(\mathcal{I}_{g,*},1)$ and let $[E]$ be the associated k-cycle in $K(\overline{\mathcal{I}}_{g,*},1)$. Let $\varepsilon : H^{*-1}(B; H_1) \to H^*(E)$ be the map* *defined in Proposition 4.2.5.3, and let $a_1, \ldots, a_k \in H_1$ be given. Then*

$$\langle \varepsilon(a_1)\ldots\varepsilon(a_k), [E]\rangle = (-1)^k C_k'((a_1 \wedge \cdots \wedge a_k) \otimes \tau_{k-2}[B]).$$

Proof. The symplectic pairing $\mu : H_1^{\otimes 2} \to \mathbb{Q}$ induces an isomorphism $\cdot^\vee : H_1 \to H^1$ given by $w^\vee(u) = \mu(u \otimes w)$. By pullback, any $w \in H_1 \cong H_1(T^{2g})$ determines the class $J^*p^*w^\vee \in H^1(E)$.

We claim that there is an equality for any $w \in H_1$,

$$\varepsilon(w) = J^*p^*w^\vee.$$

The first step is to show that $\mathrm{Im}(J^*p^*) \subseteq \mathrm{Im}\,\varepsilon$. This will follow from Lemma 4.2.8. For degree reasons, $\pi_!(J^*p^*w^\vee) = 0$. It remains to show that $\sigma^*(J^*p^*w^\vee) = 0$. By construction, $p \circ J \circ \sigma : B \to T^{2g}$ is the constant map sending B to $0 \in T^{2g}$; the result follows.

Given $w \in H_1$, we have shown that there is some $v \in H_1 = H^0(B; H_1)$ such that $J^*p^*w^\vee = \varepsilon(v)$. It remains to show that $v = w$. Let $\iota : \Sigma_g \to E$ be the inclusion of a fiber. The composition $p \circ J \circ \iota : \Sigma_g \to T^{2g}$ coincides with the Jacobian mapping. Consequently, $\iota^*(J^*p^*w^\vee) = w^\vee$.

On the other hand,

$$\iota^*(\varepsilon(v)) = \iota^*(\mu_*(\pi^*v\ k_0)) = \mu_*(\iota^*(\pi^*v\ k_0)).$$

Let $u \in H_1$ be arbitrary. Then

$$\langle \mu_*(\iota^*(\pi^*v\ k_0)), u\rangle = \mu(\langle \iota^*(\pi^*v\ k_0), u\rangle).$$

As $\iota^*k_0 = \mathrm{id}$, the above formula simplifies to

$$\mu(\langle \iota^*(\pi^*v\ k_0), u\rangle) = -\mu(v \otimes u) = v^\vee(u).$$

Consequently, $w^\vee = v^\vee$, from which the equality $w = v$ follows.

From the above, there is an expression

$$\begin{aligned}\langle \varepsilon(a_1)\ldots\varepsilon(a_k), [E]\rangle &= \langle J^*p^*(a_1^\vee \ldots a_k^\vee), [E]\rangle \\ &= \langle a_1^\vee \ldots a_k^\vee, p_*J_*[E]\rangle \\ &= \langle a_1^\vee \ldots a_k^\vee, \tau_{k-2}[B]\rangle.\end{aligned}$$

Under the isomorphisms $H_k(T^{2g}) \cong \wedge^k H_1$ and $H^k(T^{2g}) \cong \wedge^k H^1$, the evaluation pairing

$H^k(T^{2g}) \otimes H_k(T^{2g}) \to \mathbb{Q}$ is mapped to the pairing

$$(\alpha_1 \wedge \cdots \wedge \alpha_k) \otimes (a_1 \wedge \cdots \wedge a_k) \mapsto \det(\alpha_i(a_j)). \tag{4.9}$$

Under the embedding

$$\wedge^k(\cdot^\vee)^{-1} \otimes \mathrm{id} : \wedge^k H^1 \otimes \wedge^k H_1 \to (\wedge^k H_1)^{\otimes 2},$$

the pairing (4.9) corresponds to $(-1)^k C'_k$. Consequently,

$$\langle a_1^\vee \dots a_k^\vee, \tau_{k-2}[B]\rangle = (-1)^k C'_k((a_1 \wedge \cdots \wedge a_k) \otimes \tau_{k-2}[B])$$

as was to be shown. □

Step 2: Comparison with $m_{0,k}$**.** Suppose that B determines a $(k-2)$-cycle in $K(\mathcal{I}_{g,*}, 1)$. We must show that

$$q(\langle m_{0,k}, [B]\rangle) = (-1)^k k!\ \tau_{k-2}[B],$$

where, as in Section 4.1.1, the map $q : H_1^{\otimes k} \to \wedge^k H_1$ is the projection. As the pairing $C'_k : (\wedge^k H_1)^{\otimes 2} \to \mathbb{Q}$ of (4.3) is nondegenerate, it suffices to show the equality of the forms:

$$a_1 \wedge \cdots \wedge a_k \mapsto (-1)^k C'_k((a_1 \wedge \cdots \wedge a_k) \otimes \tau_{k-2}[B])$$

and

$$a_1 \wedge \cdots \wedge a_k \mapsto (\frac{1}{k!} C'_k((a_1 \wedge \cdots \wedge a_k) \otimes q(\langle m_{0,k}, [B]\rangle))).$$

Proposition 4.4.2 asserts that for $a_1, \dots, a_k \in H_1$, there is an equality

$$\langle \varepsilon(a_1) \dots \varepsilon(a_k), [E]\rangle = (-1)^k C'_k((a_1 \wedge \cdots \wedge a_k) \otimes \tau_{k-2}[B])$$

Proposition 4.1.1.4 implies:

$$\begin{aligned}\langle \varepsilon(a_1) \dots \varepsilon(a_k), [E]\rangle &= \langle \varepsilon(a_1) \dots \varepsilon(a_k), \pi^![B]\rangle \\ &= \langle \pi_!(\varepsilon(a_1) \dots \varepsilon(a_k)), [B]\rangle.\end{aligned}$$

Theorem C implies:

$$\begin{aligned}\langle \pi_!(\varepsilon(a_1)\dots\varepsilon(a_k)),[B]\rangle &= \langle m_{0,k}\lrcorner(a_1,\dots,a_k),[B]\rangle \\ &= \langle C_{k,*}(a_1\dots a_k\ m_{0,k}),[B]\rangle \\ &= C_k(\langle a_1\dots a_k\ m_{0,k},[B]\rangle) \\ &= C_k((a_1\otimes\cdots\otimes a_k)\otimes\langle m_{0,k},[B]\rangle) \\ &= C_k((a_1\otimes\cdots\otimes a_k)\otimes\langle m_{0,k},[B]\rangle)\end{aligned}$$

(here $\gamma = 0$ as each $d_i = 1$). As $m_{0,k} \in H^{k-2}(\mathrm{Mod}_{g,*}; L(\wedge^k H_1))$, there is an expression of the form

$$\langle m_{0,k},[B]\rangle = L(\zeta)$$

for some $\zeta \in \wedge^k H_1$. It follows that $q(\langle m_{0,k},[B]\rangle) = k!\zeta$. The results of Section 4.1.1 imply:

$$C_k((a_1\otimes\cdots\otimes a_k)\otimes\langle m_{0,k},[B]\rangle) = \frac{1}{k!}C'_k((a_1\wedge\cdots\wedge a_k)\otimes q(\langle m_{0,k},[B]\rangle)).$$

The result follows. □

4.5 Relation to MMM classes: Theorem E

This section is devoted to the proof of Theorem E. This will be divided into two steps. The first step is to establish a contraction formula for $\mu_{0,2n}$. The second step will be to relate this to the representation theory of $\mathrm{Sp}(2g,\mathbb{Q})$.

Step 1: Contraction formula. The first step is to calculate $\mu_*^{\otimes n}(m_{0,2n}) \in H^{2n-2}(\mathrm{Mod}_{g,*})$. We claim that the following formula holds:

$$\mu_*^{\otimes n}(m_{0,2n}) = (-1)^{n-1}2^n e^{n-1} + (-1)^n\sum_{i=1}^{n}\binom{n}{i}e^{n-i}e_{i-1}. \tag{4.10}$$

By convention, $e_0 = 2-2g \in H^0(\mathrm{Mod}_{g,*})$.

According to [24, Theorem 1.3], there is an expression for $\mu_*(k_0^2) \in H^2(\overline{\mathrm{Mod}}_{g,*})$ of the form

$$\mu_*(k_0^2) = 2\nu - e - \bar{e}$$

(this also appears as [20, Theorem 6.1] with the notation matching that of this chapter). Therefore,

$$\mu_*^{\otimes n}(k_0^{2n}) = (2\nu - e - \bar{e})^n.$$

It follows from Proposition 4.1.1.2 that

$$\mu_*^{\otimes n}(m_{0,2n}) = \pi_!((2\nu - e - \bar{e})^n).$$

Recall that $e \in H^2(\overline{\mathrm{Mod}}_{g,*})$ is defined as $\pi^*(e)$ for $e \in H^2(\mathrm{Mod}_{g,*})$, and $\bar{e}$ is defined as $\bar{\pi}^*(e), e \in H^2(\mathrm{Mod}_{g,*})$. Equation (4.7) of Theorem 4.2.7 asserts that $\pi_!(\nu x) = \sigma^*(x)$. The composition $\bar{\pi} \circ \sigma = \mathrm{id}$, and so $\sigma^*(e) = \sigma^*(\bar{e}) = e$. Theorem 4.2.7.3 implies that $\sigma^*(\nu) = e$.

Expand $(2\nu - e - \bar{e})^n$ as

$$(2\nu - e - \bar{e})^n = 2\nu(2\nu - e - \bar{e})^{n-1} - (e + \bar{e})(2\nu - e - \bar{e})^{n-1}.$$

For $n \geq 2$, the above discussion shows that $\pi_!(2\nu(2\nu - e - \bar{e})^{n-1}) = 2\sigma^*(2\nu - e - \bar{e})^n = 0$. It follows that

$$\pi_!((2\nu - e - \bar{e})^n) = -\pi_!((e + \bar{e})(2\nu - e - \bar{e})^{n-1}),$$

and that in general, for $j \leq n - 2$,

$$\pi_!((e + \bar{e})^j(2\nu - e - \bar{e})^{n-j}) = -\pi_!((e + \bar{e})^{j+1}(2\nu - e - \bar{e})^{n-j-1}).$$

Applying this formula repeatedly,

$$\begin{aligned}
\pi_!((2\nu - e - \bar{e})^n) &= (-1)^{n-1}\pi_!((e + \bar{e})^{n-1}(2\nu - e - \bar{e})) \\
&= (-1)^{n-1}\pi_!(2\nu(e + \bar{e})^{n-1}) + (-1)^n\pi_!((e + \bar{e})^n) \\
&= (-1)^{n-1}2^n e^{n-1} + (-1)^n \sum_{i=1}^{n} \binom{n}{i} e^{n-i} e_{i-1}.
\end{aligned}$$

In the last equality, we have applied Proposition 4.1.1.3, recalling that e is the pullback $\pi^*(e), e \in H^2(\mathrm{Mod}_{g,*})$.

Step 2: Contractions in symplectic representation theory. As the restriction of e to $H^2(\mathcal{I}_g^1)$ is zero, Step 1 implies that the pullback of e_i to $H^{2i}(\mathcal{I}_g^1)$ is zero if and only if $\mu_*^{\otimes i+1}(m_{2i+2})$ vanishes in $H^{2i}(\mathcal{I}_g^1)$. Theorem D implies that this is in turn equivalent to the vanishing of $\mu_*^{\otimes i+1}(\tau_{2i})$.

In the notation of Section 4.1.1, there is a decomposition

$$\wedge^{2i+2}H_1 = V(\lambda_{2i+2}) \oplus V(\lambda_{2i}) \oplus \cdots \oplus V(\lambda_0).$$

Treating $\wedge^{2i+2}H_1$ as a subspace of $(H_1^{\otimes 2})^{\otimes i+1}$, the contraction $\mu^{\otimes i+1}$ is a map of $\mathrm{Sp}(2g, \mathbb{Q})$-representations projecting onto $V(\lambda_0) \cong \mathbb{Q}$. Viewed as an element of $\mathrm{Hom}(H_{2i}(\mathcal{I}_g^1), \mathbb{Q})$, the class $\mu_*^{\otimes i+1}(\tau_{2i})$ is therefore nonzero if and only if

$$V(\lambda_0) \subseteq \mathrm{Im}(\tau_{2i}).$$

This completes the proof of Theorem E. □

4.6 Applications to surface bundles

In this last section, we turn from a study of global cohomology classes on Mod_g and $\mathcal{I}_g$ in favor of a study of $H^*(E)$ for $\pi : E \to B$ a particular Σ_g-bundle over a paracompact Hausdorff space B. The particular bundles under consideration will have an additional constraint on their monodromy representations, namely that $\rho : \pi_1 B \to \mathcal{K}_{g,*}$ is valued in the *Johnson kernel* $\mathcal{K}_{g,*} = \ker(\tau : \mathcal{I}_{g,*} \to \wedge^3 H_1)$. It is a deep fact due to Johnson [17] that

$$\mathcal{K}_{g,*} = \langle T_\gamma \mid \gamma \text{ separating} \rangle, \tag{4.11}$$

i.e. that the Johnson kernel is the group generated by all Dehn twists about *separating* simple closed curves. There is an analogous definition of $\mathcal{K}_g \leq \mathrm{Mod}_g$ and a statement analogous to (4.11); see [17, Section 7].

Proof of Theorem G: The method will be to exploit Theorem C. We will show that under the splitting of graded vector spaces

$$H^*(E) \cong H^*(B) \otimes H^*(\Sigma_g),$$

the multiplication on $H^*(E)$ induced by the cup product agrees with the ring structure on $H^*(B) \otimes H^*(\Sigma_g)$ induced by the cup products on B and Σ_g. This will be accomplished by a separate verification on the six different pairs of subspaces $(H^*(B) \otimes H^i(\Sigma_g)) \otimes (H^*(B) \otimes H^j(\Sigma_g))$ of $H^*(E)^{\otimes 2}$ for $0 \leq i \leq j \leq 2$.

For the readers convenience we list below the inclusions $F : H^m(B) \otimes H^i(\Sigma_g) \to H^{m+i}(E)$ of Theorem C that will yield the ring isomorphism. We have identified $H^1(\Sigma_g) \cong H_1(\Sigma_g)$ by means of μ. A generator of $H^2(\Sigma_g)$ will be denoted ω.

$$\begin{aligned} F(u \otimes 1) &= \pi^* u & &(H^m(B) \otimes H^0(\Sigma_g) \to H^m(E)) \\ F(u \otimes x) &= \mu_*(\pi^*(u \otimes x) k_0) & &(H^m(B) \otimes H^1(\Sigma_g) \to H^{m+1}(E)) \\ F(u \otimes \omega) &= \pi^* u\ \nu' & &(H^m(B) \otimes H^2(\Sigma_g) \to H^{m+2}(E)) \end{aligned}$$

The table below records the multiplicative structure on $H^*(B) \otimes H^*(\Sigma_g)$ induced by the cup products on B and Σ_g. Under the identification $H^1(\Sigma_g) \cong H_1(\Sigma_g)$, the cup product is given by $xy = \mu(x,y)\omega$.

	$v \otimes 1$	$v \otimes y$	$v \otimes \omega$
$u \otimes 1$	$uv \otimes 1$	$uv \otimes y$	$uv \otimes \omega$
$u \otimes x$		$(-1)^{\lvert v \rvert}\mu(x,y)uv \otimes \omega$	0
$u \otimes \omega$			0

Passing the entries in this table through F yields a table of values for $F(ab)$ (for $a, b \in H^*(B) \otimes H^*(\Sigma_g)$):

	$v \otimes 1$	$v \otimes y$	$v \otimes \omega$
$u \otimes 1$	$\pi^*(uv)$	$\mu_*(\pi^*(uv \otimes y)k_0)$	$\pi^*(uv)\ \nu'$
$u \otimes x$		$(-1)^{\lvert v \rvert}\mu(x,y)\pi^*(uv)\ \nu'$	0
$u \otimes \omega$			0

Showing that F is a ring isomorphism reduces to showing that this table matches the table of values for $F(a)F(b)$, given below.

	$\pi^* v$	$\mu_*(\pi^*(v \otimes y)k_0)$	$\pi^* v\ \nu'$
$\pi^* u$	$\pi^*(uv)$	$\pi^* u\ \mu_*(\pi^*(v \otimes y)k_0)$	$\pi^*(uv)\ \nu'$
$\mu_*(\pi^*(u \otimes x)k_0)$		$\mu_*(\pi^*(u \otimes x)k_0)\ \mu_*(\pi^*(v \otimes y)k_0)$	$\mu_*(\pi^*(u \otimes x)k_0)\ \pi^* v\ \nu'$
$\pi^* u\ \nu'$			$\pi^*(uv)(\nu')^2$

The first pair of entries to reconcile is $\mu_*(\pi^*(uv \otimes y)k_0)$ and $\pi^* u\ \mu_*(\pi^*(v \otimes y)k_0)$. This is essentially immediate. We must next show the equality

$$(-1)^{\lvert v \rvert}\mu(x,y)\pi^*(uv)\ \nu' = \mu_*(\pi^*(u \otimes x)k_0)\ \mu_*(\pi^*(v \otimes y)k_0).$$

Calculating,

$$\begin{aligned} \mu_*(\pi^*(u \otimes x)k_0)\ \mu_*(\pi^*(v \otimes y)k_0) &= (-1)^{\lvert v \rvert} C_{2,*}(\pi^*(u \otimes x)\pi^*(v \otimes y)k_0^2) \\ &= (-1)^{\lvert v \rvert}\pi^*(uv) C_{2,*}(\pi^*(x \otimes y)\ k_0^2). \end{aligned}$$

Here, $(x \otimes y)$ is to be interpreted as an element of $H^0(B; H_1^{\otimes 2})$. Clearly the equality will be established if the statement

$$C_{2,*}(\pi^*(x \otimes y)\ k_0^2) = \mu(x,y)\nu'$$

is shown to hold. To do this, the components of $C_{2,*}(\pi^*(x\otimes y)\ k_0^2)$ will be computed for the splitting on $H^*(E)$ given by F. To compute $\pi_!(C_{2,*}(\pi^*(x\otimes y)\ k_0^2))$, observe that

$$\begin{aligned}\pi_!(C_{2,*}(\pi^*(x\otimes y)\ k_0^2)) &= C_{2,*}((x\otimes y)\pi_!(k_0^2))\\ &= C_{2,*}((x\otimes y)\iota^*(k_0^2))\\ &= C_{2,*}((x\otimes y)\,\mathrm{id}^2).\end{aligned}$$

The last equality holds in light of the fact that $\iota^* k_0 = \mathrm{id} \in H^1(\Sigma_g; H_1)$. From here, an examination of the definition of $C_{2,*}$ shows that $C_{2,*}((x\otimes y)\,\mathrm{id}^2) = \mu(x,y)$.

The next step is to compute the $H^*(B;H_1)$-component of $C_{2,*}(\pi^*(x\otimes y)\ k_0^2)$; the goal is to show this is zero. This is computed as follows:

$$\mu_*(\pi^*\pi_!(C_{2,*}(\pi^*(x\otimes y)\ k_0^3))k_0) = \mu_*(\pi^*(m_{0,3}\lrcorner(x,y))k_0).$$

Theorem D asserts that $m_{0,3} = -6\tau_1$. Therefore $m_{0,3} = 0$ when restricted to $\mathcal{K}_{g,*}$, showing that the $H^*(B;H_1)$-component of $C_{2,*}(\pi^*(x\otimes y)\ k_0^2)$ is zero as desired.

The final step is to show that

$$\pi_!(\nu C_{2,*}(\pi^*(x\otimes y)\ k_0^2)) = 0,$$

or equivalently that $\sigma^*(C_{2,*}(\pi^*(x\otimes y)\ k_0^2)) = 0$. This latter expression is divisible by $\sigma^*(k_0) = 0$, and the result follows.

To complete the proof of Theorem G, it remains to show the vanishing of $\mu_*(\pi^*(u\otimes x)k_0)\ \pi^* v\ \nu'$ and of $\pi^*(uv)(\nu')^2$. To show the former, it suffices to show that $k_0\nu' = 0$ when restricted to $\mathcal{K}_{g,*}$. This will be shown by computing the components of $k_0\nu'$ in the splitting given by F. $\pi_!(k_0\nu') = 0$ is seen to hold immediately by properties of ν' and k_0. It must next be shown that

$$\mu_*(\pi^*\pi_!(\nu' k_0^2)k_0) = 0. \tag{4.12}$$

Recall that

$$\nu' = \nu - \pi^*\pi_!(\nu^2) = \nu - \pi^*\sigma^*(\nu) = \nu - e.$$

According to [26, Theorem 5.1], the Euler class $e \in H^2(\mathrm{Mod}_{g,*})$ is in the image of the pullback ρ_1^*, where ρ_1 is the map

$$\rho_1 : \mathrm{Mod}_{g,*} \to \frac{1}{2}\wedge^3 H_1 \rtimes \mathrm{Sp}(2g,\mathbb{Z})$$

given by $\rho_1(\phi) = (\tilde{k}(\phi), \Psi(\phi))$. Restricted to $\mathcal{I}_{g,*}$, the map ρ_1 simplifies to the Johnson homomorphism τ_1, and so ρ_1^* has zero image when pulled back to $H^2(\mathcal{K}_{g,*})$. It follows that $e = 0$, and so, when restricted to $\mathcal{K}_{g,*}$, there is an equality $\nu' = \nu$. Therefore, the term $\pi_!(\nu' k_0^2)$ in (4.12) simplifies to $\pi_!(\nu k_0^2) = \sigma^*(k_0^2) = 0$. Likewise,

$$\pi_!(\nu\nu' k_0) = \pi_!(\nu^2 k_0) = \sigma^*(\nu k_0) = 0,$$

and the final component of $\nu' k_0$ is seen to vanish.

It remains only to show $\pi^*(uv)(\nu')^2 = 0$, which is obviously implied by showing $(\nu')^2 = 0$. As was remarked in the previous step, $\nu' = \nu$ on $\mathcal{K}_{g,*}$. As before, we will show $\nu^2 = 0$ by computing the components of ν^2. The first of these is divisible by the factor

$$\pi_!(\nu^2) = \sigma^*(\nu) = e = 0,$$

while the third is

$$\pi_!(\nu^3) = \sigma^*(\nu^2) = e^2 = 0.$$

The remaining step is to show

$$\mu_*(\pi^*\pi_!(\nu^2 k_0)k_0) = 0.$$

This follows from the vanishing $\pi_!(\nu^2 k_0) = 0$ established above. □

Finally, Theorem H follows as a corollary.

Proof of Theorem H: Let $f : B \to K(\mathcal{K}_{g,*}, 1)$ determine a Σ_g-bundle $\pi : E \to B$ with monodromy contained in $\mathcal{K}_{g,*}$; let B be equipped with the distinguished homology class $[B] \in H_k(B)$. Proposition 4.4.2 asserts that for any $a_1, \dots, a_{k+2} \in H_1$, there is an equality

$$\langle \varepsilon(a_1) \dots \varepsilon(a_{k+2}), [E] \rangle = (-1)^k C_k'((a_1 \wedge \dots \wedge a_{k+2}) \otimes \tau_k[B]).$$

As C_k' is nondenegerate, it suffices to show that $\langle \varepsilon(a_1) \dots \varepsilon(a_{k+2}), [E] \rangle = 0$ for all $k+2$-tuples $a_1, \dots, a_{k+2} \in H_1$. From Theorem G, there is an expression

$$\varepsilon(a_1)\varepsilon(a_2) = \mu(a_1, a_2)\nu.$$

Theorem G also asserts that $\nu \ \varepsilon(a_3) = 0$, so that the triple product $\varepsilon(a_1)\varepsilon(a_2)\varepsilon(a_3) = 0$. The result follows. □

REFERENCES

[1] M. F. Atiyah. The signature of fibre-bundles. In *Global Analysis (Papers in Honor of K. Kodaira)*, pages 73–84. Univ. Tokyo Press, Tokyo, 1969.

[2] R.I. Baykur and D. Margalit. Indecomposable surface bundles over surfaces. *J. Topol. Anal.*, 5(2):161–181, 2013.

[3] R.I. Baykur and D. Margalit. Lefschetz fibrations and Torelli groups. *Geom. Dedicata*, 177:275–291, 2015.

[4] Kenneth S. Brown. *Cohomology of groups*, volume 87 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1994. Corrected reprint of the 1982 original.

[5] A. Cavicchioli, F. Hegenbarth, and D. Repovš. On four-manifolds fibering over surfaces. *Tsukuba J. Math.*, 22(2):333–342, 1998.

[6] T. Church and B. Farb. Parameterized Abel-Jacobi maps and abelian cycles in the Torelli group. *J. Topol.*, 5(1):15–38, 2012.

[7] C. J. Earle and J. Eells. A fibre bundle description of Teichmüller theory. *J. Differential Geometry*, 3:19–43, 1969.

[8] B. Farb and D. Margalit. *A primer on mapping class groups*, volume 49 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 2012.

[9] V. Guillemin and A. Pollack. *Differential topology*. AMS Chelsea Publishing, Providence, RI, 2010. Reprint of the 1974 original.

[10] R. Hain. Infinitesimal presentations of the Torelli groups. *J. Amer. Math. Soc.*, 10(3):597–651, 1997.

[11] U. Hamenstädt. On surface subgroups of mapping class groups. MSRI workshop video, https://www.msri.org/workshops/723/schedules/16639, March 2013.

[12] U. Hamenstädt. Personal communication, March 2015.

[13] A. Hatcher. *Algebraic topology*. Cambridge University Press, Cambridge, 2002.

[14] J. A. Hillman. *Four-manifolds, geometries and knots*, volume 5 of *Geometry & Topology Monographs*. Geometry & Topology Publications, Coventry, 2002.

[15] D. Johnson. An abelian quotient of the mapping class group $\mathcal{I}_g$. *Math. Ann.*, 249(3):225–242, 1980.

[16] D. Johnson. A survey of the Torelli group. In *Low-dimensional topology (San Francisco, Calif., 1981)*, volume 20 of *Contemp. Math.*, pages 165–179. Amer. Math. Soc., Providence, RI, 1983.

[17] D. Johnson. The structure of the Torelli group. II. A characterization of the group generated by twists on bounding curves. *Topology*, 24(2):113–126, 1985.

[18] F. E. A. Johnson. A rigidity theorem for group extensions. *Arch. Math. (Basel)*, 73(2):81–89, 1999.

[19] N. Kawazumi. A generalization of the Morita-Mumford classes to extended mapping class groups for surfaces. *Invent. Math.*, 131(1):137–149, 1998.

[20] N. Kawazumi and S. Morita. The primary approximation to the cohomology of the moduli space of curves and cocycles for the Mumford-Morita-Miller classes. Preprint; http://kyokan.ms.u-tokyo.ac.jp/users/preprint/pdf/2001-13.pdf.

[21] K. Kodaira. A certain type of irregular algebraic surfaces. *J. Analyse Math.*, 19:207–215, 1967.

[22] I. Madsen and M. Weiss. The stable moduli space of Riemann surfaces: Mumford's conjecture. *Ann. of Math. (2)*, 165(3):843–941, 2007.

[23] S. Morita. Characteristic classes of surface bundles. *Invent. Math.*, 90(3):551–577, 1987.

[24] S. Morita. Families of Jacobian manifolds and characteristic classes of surface bundles. I. *Ann. Inst. Fourier (Grenoble)*, 39(3):777–810, 1989.

[25] S. Morita. The extension of Johnson's homomorphism from the Torelli group to the mapping class group. *Invent. Math.*, 111(1):197–224, 1993.

[26] S. Morita. A linear representation of the mapping class group of orientable surfaces and characteristic classes of surface bundles. In *Topology and Teichmüller spaces (Katinkulta, 1995)*, pages 159–186. World Sci. Publ., River Edge, NJ, 1996.

[27] S. Morita. *Geometry of characteristic classes*, volume 199 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, RI, 2001. Translated from the 1999 Japanese original, Iwanami Series in Modern Mathematics.

[28] D. Neumann. 3-manifolds fibering over S^1. *Proc. Amer. Math. Soc.*, 58:353–356, 1976.

[29] I. Rivin. Rigidity of fibering. Preprint, http://arxiv.org/pdf/1106.4595v2.pdf, 2011.

[30] I. Rivin. Statistics of 3-manifolds fibering over the circle. Preprint, http://arxiv.org/pdf/1401.5736v2.pdf, 2014.

[31] T. Sakasai. Lagrangian mapping class groups from a group homological point of view. *Algebr. Geom. Topol.*, 12(1):267–291, 2012.

[32] N. Salter. Cup products, the Johnson homomorphism, and surface bundles over surfaces with multiple fiberings. *Algebr. Geom. Topol.*, 15:3613 – 3652, 2015.

[33] N. Salter. Surface bundles over surfaces with arbitrarily many fiberings. *Geom. Topol.*, 19(5):2901–2923, 2015.

[34] H. Shiga. On monodromies of holomorphic families of Riemann surfaces and modular transformations. *Math. Proc. Cambridge Philos. Soc.*, 122(3):541–549, 1997.

[35] D. Sullivan. On the intersection ring of compact three manifolds. *Topology*, 14(3):275–277, 1975.

[36] W. Thurston. A norm for the homology of 3-manifolds. *Mem. Amer. Math. Soc.*, 59(339):i–vi and 99–130, 1986.

[37] J. Tollefson. 3-manifolds fibering over S^1 with nonunique connected fiber. *Proc. Amer. Math. Soc.*, 21:79–80, 1969.