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LOCAL WELL-POSEDNESS OF THE MODIFIED KADOMTSEV-PETVIASHVILI I EQUATIONS

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To my caring parents who supported me endlessly at every moment of this journey. To everyone who showed me love, friendship and respect.

Mathematics, rightly viewed, possesses not only truth, but supreme beauty - a beauty cold and austere, like that of a sculpture, without appeal to any part of our weaker nature, without the gorgeous trappings of painting or music, yet sublimely pure, and capable of a stern perfection such as only the greatest art can show.

Bertrand Russell

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ABSTRACT

We define the anisotropic Sobolev spaces as $H^{s_1,s_2}(M \times N) = \{g \in L^2(M \times N) : \|g\|_{H^{s_1,s_2}} = \|\widehat{g}(\xi,\eta)[(1+\xi^2)^{\frac{s_1}{2}} + (1+\eta^2)^{\frac{s_2}{2}}]\|_{L^2(M^* \times N^*)} < \infty\}$, where M or N can be either the real line $\mathbb{R}$ or the torus $\mathbb{T}$. We prove local well-posedness of modified KP-I equations in the KP hierarchy, namely for $\partial_t u + (-1)^{\frac{l+1}{2}} \partial_x^l u - \partial_x^{-1} \partial_y^2 u + u^2 \partial_x u = 0$ in the anisotropic Sobolev space $H^{s,0}(\mathbb{R} \times \mathbb{R})$ if $l = 3$ and $s > 2$, in $H^{s,s}(\mathbb{R} \times \mathbb{T})$ if $l = 3$ and $s > 2$, in $H^{s,s}(\mathbb{T} \times \mathbb{T})$ if $l = 3$ and $s > \frac{19}{8}$, and in $H^{s,s}(\mathbb{R} \times \mathbb{T})$ if $l = 5$ and $s > \frac{5}{2}$. All four results require the initial data to be small.

CHAPTER 1

INTRODUCTION

1.1 Background

My main area of research is in Dispersive Partial Differential Equations. These differential equations come from models of physical phenomena. Famous equations in this area include the wave equation, Schrödinger equation, Korteweg-de Vries equation, Kadomtsev-Petviashvili equation, Boussinesq equation, Zakharov-Kuznetsov equation, Klein-Gordon equation, Benjamin-Ono equation, Intermediate Long Wave equation.

An evolution partial differential equation is dispersive if, when no boundary conditions are imposed, its wave solutions spread out in space as they evolve in time. Plane waves with large wave number travel faster than those with a smaller one. This is the reason why there is "spreading". In mathematical terms, this phenomenon is called broadening of the wave packet. My research is around the class of equations called Kadomtsev-Petviashvili equations.

The classical KP equations

$$\partial_x(\partial_t u + \partial_x^3 u + u\partial_x u) \pm \partial_y^2 u = 0 \tag{1.1}$$

were introduced by Kadomtsev and Petviashvili [25] to study the transverse stability of the solitary wave solution of the Korteweg-de Vries (KdV) equation, which reads in the context of water waves

$$\partial_t u + \partial_x u + u\partial_x u + \left(\frac{1}{3} - T\right)\partial_x^3 u = 0. \tag{1.2}$$

Here $T \geq 0$ is the Bond number, which measures surface tension effects in the context of surface hydrodynamical waves. If $T = 0$, this corresponds to the absence of surface tension. The KdV equation (1.2) was derived by Boussinesq [7] and Korteweg and de Vries

[31] and the KdV approximation presented there describes the evolution of one-dimensional surface waves which means that these waves do not depend on the transverse direction. Thus, a natural question comes up: how about the more realistic case of waves that depend on the transverse variable, but with a weak dependence? This corresponds exactly to the Kadomtsev-Petviashvili (KP) regime. The analysis of Kadomtsev and Petviashvili [25] (see also [1]) consists of looking for a weakly transverse perturbation of the one-dimensional transport equation

$$\partial_t u + \partial_x u = 0.$$

This perturbation is obtained by a Taylor expansion of dispersion relation $\omega(k_1, k_2) = \sqrt{k_1^2 + k_2^2}$ of the two-dimensional linear wave equation, assuming that $|k_1| \ll 1$ and $\frac{|k_1|}{|k_2|} \ll 1$. Namely, one writes $\omega(k_1, k_2) \sim \pm k_1 \left(1 + \frac{1}{2}\frac{k_2^2}{k_1^2}\right)$. This perturbation amounts to adding a nonlocal term, leading to

$$\partial_t u + \partial_x u + \partial_x^{-1}\partial_y^2 u = 0.$$

Here the operator ∂_x^{-1} is defined via Fourier transform, $\widehat{\partial_x^{-1} f}(\xi, \eta) = \frac{1}{i\xi}\widehat{f}(\xi, \eta)$.

The same formal procedure is applied to the KdV equation (1.2) in [25], assuming that the transverse dispersive effects are of the same order as the x-dispersive and nonlinear terms, yielding the KP equation in the form

$$\partial_t u + \partial_x u + u\partial_x u + \left(\frac{1}{3} - T\right)\partial_x^3 u + \frac{1}{2}\partial_x^{-1}\partial_y^2 u = 0. \tag{1.3}$$

By change of frame and scaling, (1.3) reduces to (1.1) with the + sign (KP-II) when $T < \frac{1}{3}$ and the − sign (KP-I) when $T > \frac{1}{3}$.

Note, however, that $T > \frac{1}{3}$ corresponds to a layer of fluid of depth smaller than 0.46 cm, and in this situation viscous effects due to the boundary layer at the bottom cannot be ignored. On the other hand, Druyma [11] found a Lax pair to the KP-I/II equations, proving the "integrability" of the KP equations. In [41] and [47], it is included a description of the inverse scattering aspects of the KP equations.

1.2 Third-order KP-I equation

In the past 30 years, the KP-I and KP-II equations have been well studied. The first local well-posedness result of the IVP (1.1) was obtained by Ukai [53] for initial data $\phi \in H^s(\mathbb{R}^2)$, for $s \geq 3$. The KP-II equation is well understood in regard with well-posedness, due to the groundbreaking work of Bourgain [6], proving local and global well-posedness in $L^2(\mathbb{R}^2)$ (and in $L^2(\mathbb{T}^2)$) by introducing the $X^{s,b} = \{f \in \mathcal{S}'(\mathbb{R}^2) : \int_{\mathbb{R}^3} \langle \tau - \xi^3 + \frac{\eta^2}{\xi} \rangle^{2b} \langle \xi \rangle^{2s} |\widehat{f}(\xi, \eta, \tau)|^2 d\xi d\eta d\tau < \infty\}$ spaces and devising an iterative Picard scheme (here, $\langle \cdot \rangle = \sqrt{1 + |\cdot|^2}$). This result was later improved by Takaoka and Tzvetkov [49] and Isaza and Mejía [23] proving local well-posedness in the anisotropic Sobolev space $H^{s_1,s_2}(\mathbb{R}^2)$, with $s_1 > -\frac{1}{3}, s_2 \geq 0$, building up on previous results as in [48], [52], [51]. Hadac [17] and Hadac, Herr and Koch [18],[19] further improved for negative order Sobolev spaces, by proving local well-posedness in $H^{s_1,s_2}(\mathbb{R}^2)$, with $s_1 \geq -\frac{1}{2}, s_2 \geq 0$ and global well-posedness for the homogeneous anisotropic Sobolev space $\dot{H}^{-\frac{1}{2},0}(\mathbb{R}^2)$ for small data. Isaza and Mejía [24] showed global well-posedness in $H^{s_1,0}(\mathbb{R}^2)$ for $s_1 > -\frac{1}{14}$.

For the KP-I equation, the problem is more delicate. This stems from the fact that we cannot apply the Picard iterative methods since the flow map fails to be real-analytic (more precisely C^2) at the origin in these spaces, as shown by Molinet, Saut and Tzvetkov [38]. The KP-I equation can be written in the Lax pair form [47] and thus it shares many features with the integrable PDE. There is in fact an infinite sequence of formal conservation laws associated to the KP-I equation. However, as noticed in [37] and [39], it is hard to find a suitable framework of distributions on $\mathbb{R}^2$ where these conservation laws make sense. The same authors in [38] and [37] use the first three conservation laws, namely the momentum L^2-conservation law $M(\phi) = \int_{\mathbb{R}^2} |\phi|^2$, the energy $E(\phi) = \frac{1}{2} \int_{\mathbb{R}^2} (\partial_x \phi)^2 + \frac{1}{2} \int_{\mathbb{R}^2} (\partial_x^{-1} \partial_y \phi)^2 -$

$\frac{1}{6}\int_{\mathbb{R}^2}|\phi|^3$ and

$$\begin{aligned} F(\phi) =& \frac{3}{2}\int_{\mathbb{R}^2}(\partial_x\phi)^2 + 5\int_{\mathbb{R}^2}(\partial_y\phi)^2 + \frac{5}{6}\int_{\mathbb{R}^2}(\partial_x^{-2}\partial_y^2\phi)^2 \\ & - \frac{5}{6}\int_{\mathbb{R}^2}\phi^2\partial_x^{-2}\partial_y^2\phi - \frac{5}{6}\int_{\mathbb{R}^2}\phi(\partial_x^{-1}\partial_y\phi)^2 + \frac{5}{4}\int_{\mathbb{R}^2}\phi^2\partial_x^2\phi + \frac{5}{24}\int_{\mathbb{R}^2}\phi^4 \end{aligned} \tag{1.4}$$

to show global well-posedness in the space $Z = \{\phi \in L^2(\mathbb{R}^2) : \|\phi\|_Z < \infty\}$ where

$$\|\phi\|_Z = \|\phi\|_{L^2_{xy}} + \|\partial_x^3\phi\|_{L^2_{xy}} + \|\partial_y\phi\|_{L^2_{xy}} + \|\partial_x\partial_y\phi\|_{L^2_{xy}} + \|\partial_x^{-1}\partial_y\phi\|_{L^2_{xy}} + \|\partial_x^{-2}\partial_y^2\phi\|_{L^2_{xy}}.$$

Their proof uses an useful anisotropic inequality, first appeared in [50] but inspired from [3], [4], that for $2 \leq p \leq 6$, there exists a C such that for every $H^s_{-1}(\mathbb{R}^2) = \{\phi \in L^2 : \|(1+|\xi|^{-1})\langle\xi\rangle^s\langle\eta\rangle^s\widehat{\phi}(\xi,\eta)\|_{L^2_{\xi,\eta}} < \infty\}$,

$$\|u\|_{L^p_{xy}} \leq \|u\|_{L^2_{xy}}^{\frac{6-p}{2p}} \|\partial_x u\|_{L^2_{xy}}^{\frac{p-2}{p}} \|\partial_x^{-1}\partial_y u\|_{L^2_{xy}}^{\frac{p-2}{2p}}. \tag{1.5}$$

Later, Kenig [28] improved the result by showing global well-posedness in the "second energy space" $Z^2_{(3)} = \{\phi \in L^2(\mathbb{R}^2) : \|\phi\|_{L^2_{xy}} + \|\partial_x^2\phi\|_{L^2_{xy}} + \|\partial_x^{-1}\partial_y\phi\|_{L^2_{xy}} + \|\partial_x^{-2}\partial_y^2\phi\|_{L^2_{xy}} < \infty\}$. The best result was achieved by Ionescu, Kenig and Tataru [21], proving global well-posedness in the first energy space $Z^1_{(3)} = \{\phi \in L^2(\mathbb{R}^2) : \|\phi\|_{L^2_{xy}} + \|\partial_x\phi\|_{L^2_{xy}} + \|\partial_x^{-1}\partial_y\phi\|_{L^2_{xy}} < \infty\}$. In the latter paper, the authors introduced the short time Fourier restriction norm method, by exploiting the symmetries of the KP-I equation. Using the same methods, Guo, Peng and Wang [16] showed local well-posedness in $H^{1,0}(\mathbb{R}^2)$. Recently, Linares, Pilod and Saut [35] showed a local well-posedness result for the fractional KP-I equation (fKP-I):

$$\partial_t u - D_x^\alpha\partial_x u + u\partial_x u - \partial_x^{-1}\partial_y^2 u = 0 \text{ for } 0 < \alpha \leq 2. \tag{1.6}$$

The result states that equation (1.6) is locally well-posed in $X^s(\mathbb{R}^2)$, for $s > 2 - \frac{\alpha}{4}$, where

$X^s(\mathbb{R}^2) = \{f \in L^2(\mathbb{R}^2) : \|J_x^s f\|_{L^2_{xy}} + \|\partial_x^{-1}\partial_y f\|_{L^2_{xy}} < \infty\}$. The authors also proved that for $0 < \alpha \leq 2$, equation (1.6) is quasi-linear, while for $\alpha = 4$ is semilinear.

Part of my research was dedicated to the study of the Cauchy problem for the third-order modified Kadomtsev-Petviashvili I equation (mKP-I)

$$\begin{cases} \partial_t u + \partial_x^3 u - \partial_x^{-1}\partial_y^2 u + u^2 \partial_x u = 0, \\ u(0,x,y) = u_0(x,y) \end{cases} \tag{1.7}$$

in the anisotropic Sobolev space $H^{s_1,s_2}(\mathbb{R} \times \mathbb{R})$. The third order mKP-I equation (1.7) is the modified version of the third order KP-I equation (1.1). The modified KP equations appear in [13] which describe the evolution of sound waves in antiferromagnetics. There are several known results about mKP-I equation and its relative, the mKP-II equation, where the sign of the KP term $\partial_x^{-1}\partial_y^2$ in (1.7) is $+$. For the third order modified KP-I and KP-II equation, Saut [44] showed that the generalized KP-I/KP-II equation ($p \geq 1$)

$$\begin{cases} \partial_t u + \partial_x^3 u + \varepsilon \partial_x^{-1}\partial_y^2 u + u^p \partial_x u = 0, \\ u(0,x,y) = u_0(x,y) \end{cases} \tag{1.8}$$

($\varepsilon = \pm 1$) is locally well-posed in $C([-T,T]; H^s(\mathbb{R}^2)) \cap C^1([-T,T]; H^{s-3}(\mathbb{R}^2))$ for $s \geq 3$ with the momentum $M(u)(t) = \int_{\mathbb{R}^2} u^2(t)dxdy$ and energy

$$E(u)(t) = \int_{\mathbb{R}^2} \frac{(\partial_x u)^2}{2} - \varepsilon \frac{(\partial_x^{-1}\partial_y u)^2}{2} - \frac{u^{p+2}}{(p+1)(p+2)} dxdy$$

being conserved quantities. There is no known equivalent third conservation law (1.4) for this case. Several blow-up result were found as well for the mKP-I equations. In [44], if $p \geq 4$, the corresponding solution u in (1.8) blows up in finite time, i.e. there exists $\infty > T > 0$ such that $\lim_{t \to T^-} \|\partial_y u(\cdot, y)\|_{L^2} = +\infty$. Liu [36] improved the blow-up result for $\frac{4}{3} \leq p < 4$, also by showing that $\lim_{t\to T^-}\|\partial_y u(\cdot,y)\|_{L^2}=+\infty$. Both proofs are based on some virial-type identities. The threshold $\frac{4}{3}$ comes from the observation that in order to bound the energy norm of a solution u by the conserved momentum and conserved Hamiltonian, we can use the anisotropic inequality (1.5) only if $p<\frac{4}{3}$.

In the realm of the modified version of the equations, we mention several results and developments: Kenig and Martel [29] showed global well-posedness for the modified KP-II equation in the energy space $Z=\{u\in L^2 | \|u\|_{H^1}+\|\partial_x^{-1}\partial_y u\|_{L^2}<\infty\}$.Moreover, Kenig and Ziesler [30] proved local well-posedness of the modified KP-I equation in $Y_s=\{u_0\in \mathscr{S}'(\mathbb{R}\times\mathbb{R}) : \|u_0\|_{L^2}+\|(1+D_x)^s u_0\|_{L^2}+\|(D_yD_x^{-1})u_0\|_{L^2}<\infty\}$ for $s>3/2$, with Grünrock [14] sharpening their result. For our case of the third order modified KP-I, I prove the following theorem

Theorem 1.2.1. *[8] Assume $\phi\in H^{s,0}(\mathbb{R}\times\mathbb{R})$ with $s>2$. Then the initial value problem*

$$\begin{cases}\partial_t u+\partial_x^3 u-\partial_x^{-1}\partial_y^2 u+u^2\partial_x u=0,\\ u(0,x,y)=\phi(x,y)\end{cases}$$

admits a unique solution in $C([-T,T]:H^{s,0}(\mathbb{R}\times\mathbb{R}))$ with $T=T(\|\phi\|_{H^{s,0}})$ with $u,\partial_x u\in L_T^2L_{xy}^\infty$ if $\|\phi\|_{H^{s,0}}$ is sufficiently small. Moreover, the mapping $\phi\to u$ is continuous from $H^{s,0}(\mathbb{R}\times\mathbb{R})$ to $C([-T,T];H^{s,0}(\mathbb{R}\times\mathbb{R}))$.

My proof follows the ideas of Kenig [28], by trying to obtain an a priori bound on the quantity $\|u\|_{L_T^2L_{xy}^\infty}+\|\partial_x u\|_{L_T^2L_{xy}^\infty}$. The modification of the L_T^1 that appears in the latter paper is that this norm is not sufficient to obtain an estimate on the nonlinear cubic term. We are using the dispersive estimates that appear in [40], which require only smoothness in x. The smallness assumption for the initial data comes from the fact that the scale invariance for the mKP-I equation does not behave well for the $H^{s,0}$ norm. Using the a priori estimate, we use a local well-posedness result by Iorío and Nunes [22] for the gKP-I equation in $\mathbb{R}^2$. As an extra tool, we apply the Kato-Ponce commutator estimates [27] and the fractional Leibniz rule that appear in [28]. For the continuity of the flow map we use a standard Bona-Smith argument [5].

The KP-I can be considered in different settings, as for example if the y-variable is located in $\mathbb{T}$. For the case of the KP-I equation on $\mathbb{R}\times\mathbb{T}$, Ionescu and Kenig [20] showed global well-posedness in the second energy space, i.e. $Z^2_{(3)} = \{\phi \in L^2 : \|(1+\xi^2+\frac{n^2}{\xi^2})\widehat{\phi}(\xi,n)\|_{L^2_{\xi,n}} < \infty\}$ and Robert [42] proved global well-posedness in the first energy space $Z^1_{(3)} = \{\phi \in L^2 : \|(1+\xi+\frac{n}{\xi})\widehat{\phi}(\xi,n)\|_{L^2_{\xi,n}} < \infty\}$. The latter uses the method of short time Fourier restriction norm from [21]. For this case, I proved the following result:

Theorem 1.2.2. *[9] Assume $\phi \in H^{s,s}(\mathbb{R}\times\mathbb{T})$ with $s > 2$. Then the initial value problem*

$$\begin{cases} \partial_t u + \partial_x^3 u - \partial_x^{-1}\partial_y^2 u + u^2\partial_x u = 0, \\ u(0,x,y) = \phi(x,y) \end{cases} \tag{1.9}$$

admits a unique solution in $C([-T,T] : H^{s,s}(\mathbb{R}\times\mathbb{T}))$ with $T = T(\|\phi\|_{H^{s,s}})$ with $u, \partial_x u, \partial_y u \in L^2_T L^\infty_{xy}$ if $\|\phi\|_{H^{s,s}}$ is sufficiently small. Moreover, the mapping $\phi \to u$ is continuous from $H^{s,s}(\mathbb{R}\times\mathbb{T})$ to $C([-T,T]; H^{s,s}(\mathbb{R}\times\mathbb{T}))$.

In the case $\mathbb{T}\times\mathbb{T}$, Ionescu and Kenig showed in [20] global well-posedness in the second energy space $Z^2_{(3)}$ as defined above. Zhang showed in [55] that the periodic KP-I equation is locally well-posed in a Besov type space, namely in $B^1_{2,1}(\mathbb{T}^2) = \{\phi : \mathbb{T}^2 \to \mathbb{R} : \widehat{\phi}(0,n) = 0,\ \forall n \in \mathbb{Z}\backslash\{0\}, \|\phi\|_{B^1_{2,1}} = \sum_{k=0}^{\infty} 2^k \|1_{[2^{k-1},2^{k+1}]}(m)\widehat{\phi}(m,n)\left(1+\frac{|n|}{|m|(1+|m|)}\right)\|_{l^2_{m,n}} < \infty\}$. For this case, I proved the following result:

Theorem 1.2.3. *[9] Assume $\phi \in H^{s,s}(\mathbb{T}\times\mathbb{T})$ with $s > \frac{19}{8}$. Then the initial value problem*

$$\begin{cases} \partial_t u + \partial_x^3 u - \partial_x^{-1}\partial_y^2 u + u^2\partial_x u = 0, \\ u(0,x,y) = \phi(x,y) \end{cases} \tag{1.10}$$

admits a unique solution in $C([-T,T]:H^{s,s}(\mathbb{T}\times\mathbb{T}))$ *with* $T=T(\|\phi\|_{H^{s,s}})$ *with* $u,\partial_x u,\partial_y u\in L^2_T L^\infty_{xy}$ *if* $\|\phi\|_{H^{s,s}}$ *is sufficiently small. Moreover, the mapping* $\phi\to u$ *is continuous from* $H^{s,s}(\mathbb{T}\times\mathbb{T})$ *to* $C([-T,T];H^{s,s}(\mathbb{T}\times\mathbb{T}))$.

The proofs of the latter local-posedness results, in the partially periodic and periodic setting, follow the ideas of [20]. In these cases, we are looking to find an a priori estimate for $\|u\|_{L^2_T L^\infty_{xy}}+\|\partial_x u\|_{L^2_T L^\infty_{xy}}+\|\partial_y u\|_{L^2_T L^\infty_{xy}}$. Again, we are constrained to consider the L^2_T norm instead of the L^1_T as we deal with a cubic nonlinearity. The fact that we need to add the term $\|\partial_y u\|_{L^2_T L^\infty_{xy}}$ is more subtle. The dispersive estimates on $\mathbb{R}\times\mathbb{T}$ and $\mathbb{T}\times\mathbb{T}$ require more smoothness in the y-variable. The mKP-I with nonlinearity $u^p\partial_x u$, $p=2$ is a limiting case for these dispersive equations, therefore in the linear estimate that we need more smoothness in y for the cubic nonlinearity than for the quadratic nonlinearity. For the continuity of the flow map we use a Bona-Smith argument [5] adapted to the periodic setting of the variable y.

1.3 Fifth-order KP-I equation

The fifth order KP-I equation

$$\partial_t u-\partial_x^5 u-\partial_x^{-1}\partial_y^2 u+u\partial_x u=0 \tag{1.11}$$

appears when modeling certain long dispersive waves with weak transverse effects, as we see in [2], [25], [26]. By the work of Saut and Tzvetkov in [45], we know that the fifth order KP-II initial value problem is globally wellposed in L^2 on both $\mathbb{R}\times\mathbb{R}$ and $\mathbb{T}\times\mathbb{T}$. In [32], local wellposedness in $H^{s,0}(\mathbb{R}\times\mathbb{T})$ for $s>-\frac{3}{4}$ and global wellposedness in L^2. The fifth order KP-I initial value problem is known to be globally well-posed in the energy spaces $Z^1_{(5)}=\{\phi\in L^2:\|(1+\xi^2+\frac{\eta}{\xi})\widehat{\phi}(\xi,\eta)\|_{L^2_{\xi\eta}}<\infty\}$ on both $\mathbb{R}\times\mathbb{R}$ and $\mathbb{T}\times\mathbb{R}$ from the work of Saut and Tzetkov in [45] and [46], using Picard iterative methods (see also [10]). Using the Fourier restriction norm method and sufficiently exploiting the geometric structure of the resonant set of (1.11) to deal with the high-high frequency interaction, Li and Xiao established in [33] the global well-posedness in $L^2(\mathbb{R}^2)$. Guo et al. [15] established the local well-posedness of the Cauchy problem in $H^{s,0}(\mathbb{R}\times\mathbb{R})$ for $s \geq -\frac{3}{4}$, Yan et al [54] showed global well-posedness in $H^{s,0}(\mathbb{R}\times\mathbb{R})$ for $s > -\frac{6}{23}$ and finally Li et al. [34] proved global well-posedness in $H^{s,0}(\mathbb{R}\times\mathbb{R})$ for $s > -\frac{4}{7}$ and local well-posedness for $s > -\frac{9}{8}$. We conclude with the result from [20] which proves global well-posedness on $\mathbb{R}\times\mathbb{T}$ and from [43] which proves global well-posedness on $\mathbb{T}\times\mathbb{T}$, both results in $Z^1_{(5)}(\mathbb{R}\times\mathbb{T})$, resepctively $Z^1_{(5)}(\mathbb{T}\times\mathbb{T})$, the natural energy spaces in these cases. Esfahani [12] showed that the generalized fifth-order KP-I equation

$$\begin{cases} \partial_t u - \partial_x^5 u - \partial_x^{-1}\partial_y^2 u + u^p \partial_x u = 0, \\ u(0,x,y) = u_0(x,y) \end{cases} \tag{1.12}$$

is locally well-posed in $C([-T,T];H^s(\mathbb{R}^2)) \cap C^1([-T,T];H^{s-5}(\mathbb{R}))$ for $s \geq 5$. In the same paper, if $p \geq 4$, the corresponding solution u in (1.12) blows up in finite time, i.e. there exists $\infty > T > 0$ such that $\lim_{t\to T^-}\|\partial_y u(\cdot,y)\|_{L^2} = +\infty$. For our case of the fifth order partially periodic modified KP-I we prove the following theorem

Theorem 1.3.1. *[9] Assume $\phi \in H^{s,s}(\mathbb{R}\times\mathbb{T})$ with $s > \frac{5}{2}$. Then the initial value problem*

$$\begin{cases} \partial_t u - \partial_x^5 u - \partial_x^{-1}\partial_y^2 u + u^2 \partial_x u = 0, \\ u(0,x,y) = \phi(x,y) \end{cases}$$

admits a unique solution in $C([-T,T]:H^{s,s}(\mathbb{R}\times\mathbb{T}))$ with $T = T(\|\phi\|_{H^{s,s}})$ with $u, \partial_x u, \partial_y u \in L^2_T L^\infty_{xy}$ if $\|\phi\|_{H^{s,s}}$ is sufficiently small. Moreover, the mapping $\phi \to u$ is continuous from $H^{s,s}(\mathbb{R}\times\mathbb{T})$ to $C([-T,T];H^{s,s}(\mathbb{R}\times\mathbb{T}))$.

The proof is in the same spirit as the one for the third order mKP-I equation on $\mathbb{R}\times\mathbb{T}$. Since $p = 2$ is the limit case for the dispersive estimates for the fifth-order dispersion operator

$W_{(5)}(t)$ defined by the Fourier multiplier $(\xi, n) \to e^{i(\xi^5 + \frac{n^2}{\xi})}$, we require more smoothness in the linear estimate in y for the cubic nonlinearity.

The rest of the thesis is organized as follows: in Chapter 2 we consider the well-posedness of the modified KP-I equation in $\mathbb{R} \times \mathbb{R}$, namely proving 1.2.1. The latter chapters include the proofs of the periodic and partially periodic cases of the modified KP-I equations, more precisely in Chapter 3 we present the proof of 1.2.2 and 1.2.3 and in Chapter 4 we present the proof of 1.3.1.

CHAPTER 2

LOCAL WELL-POSEDNESS FOR THE THIRD ORDER MODIFIED KADOMTSEV-PETVIASHVILI I EQUATION ON $\mathbb{R} \times \mathbb{R}$

2.1 Notation

We start by defining, for $g \in L^2(\mathbb{R} \times \mathbb{R})$, $\widehat{g}(\xi, \eta)$ denote its Fourier transform in both x and y. We consider the equation $\partial_t u + \partial_x^3 u - \partial_x^{-1}\partial_y^2 u + u^2 \partial_x u = 0$. We define the Sobolev spaces which we will consider from now on: for $s_1, s_2 \geq 0$

$$H^{s_1,s_2}(\mathbb{R}\times\mathbb{R}) = \{g \in L^2(\mathbb{R}\times\mathbb{R}) : \|g\|_{H^{s_1,s_2}} = \|\widehat{g}(\xi, n)[(1+\xi^2)^{\frac{s_1}{2}} + (1+\eta^2)^{\frac{s_2}{2}}]\|_{L^2(\mathbb{R}\times\mathbb{R})} < \infty\}$$

and for $s \geq 0$

$$H^s(\mathbb{R} \times \mathbb{R}) = \{g \in L^2(\mathbb{R} \times \mathbb{R}) : \|g\|_{H^s} = \|\widehat{g}(\xi, \eta)[(1 + \xi^2 + \eta^2)^{\frac{s}{2}}]\|_{L^2(\mathbb{R}\times\mathbb{R})} < \infty\}$$

and so

$$H^\infty(\mathbb{R} \times \mathbb{R}) = \cap_{k=0}^{\infty} H^k(\mathbb{R} \times \mathbb{R}).$$

For $s \in \mathbb{R}$ we define the operators J_x^s, J_y^s by

$$\widehat{J_x^s g}(\xi, \eta) = (1 + \xi^2)^{\frac{s}{2}} \widehat{g}(\xi, \eta);$$

$$\widehat{J_y^s g}(\xi, \eta) = (1 + \eta^2)^{\frac{s}{2}} \widehat{g}(\xi, \eta)$$

on $\mathcal{S}'(\mathbb{R} \times \mathbb{R})$. For any set A let $\mathbf{1}_A$ denote its the characteristic function. Given a Banach space X, a measurable function $u : \mathbb{R} \to X$, and an exponent $p \in [1, \infty]$, we define

$$\|u\|_{L^p X} = \Big[\int_{\mathbb{R}} (\|u(t)\|_X^p) dt \Big]^{\frac{1}{p}} \text{ if } p \in [1, \infty) \text{ and}$$
$$\|u\|_{L^\infty X} = \operatorname{esssup}_{t \in \mathbb{R}} \|u(t)\|_X$$

Also, if $I \subseteq \mathbb{R}$ is a measurable set, and $u : I \to X$ is a measurable function, we define

$$\|u\|_{L^p_I X} = \|\mathbf{1}_I(t) u\|_{L^p X}.$$

For $T \geq 0$, we define $\|u\|_{L^p_T X} = \|u\|_{L^p_{[-T,T]} X}$

We also introduce the Kato-Ponce commutator estimates (as in Lemma XI from [27] and Appendix 9.A from [20]):

Lemma 1. *Let $m \geq 0$ and $f, g \in H^m(\mathbb{R})$. If $s \geq 1$ then*

$$\|J^s_{\mathbb{R}}(fg) - f J^s_{\mathbb{R}} g\|_{L^2} \leq C_s [\|J^s_{\mathbb{R}} f\|_{L^2} \|g\|_{L^\infty} + (\|f\|_{L^\infty} + \|\partial f\|_{L^\infty}) \|J^{s-1}_{\mathbb{R}} g\|_{L^2}]$$

and if $s \in (0, 1)$ then

$$\|J^s_{\mathbb{R}}(fg) - f J^s_{\mathbb{R}} g\|_{L^2} \leq C_s \|J^s_{\mathbb{R}} f\|_{L^2} \|g\|_{L^\infty}.$$

We have the following corollary which we will use later.

Corollary. *If $s \geq 1$, then we have $\|J^s_{\mathbb{R}}(u^3)\| \lesssim \|u\|^2_{L^\infty} \|J^s_{\mathbb{R}} u\|_{L^2} + \|u\|_{L^\infty} \|\partial_x u\|_{L^\infty} \|J^{s-1}_{\mathbb{R}} u\|_{L^2}$.*

2.2 Dispersive Estimates

We use the dispersive estimates that appear in Molinet, Saut and Tzvetkov [40].

Let $U(t) = \exp(-t(-\partial_x^3 - \partial_x^{-1} \partial_y^2))$ be the unitary group in $H^s(\mathbb{R}^2)$ defining the free KP-I equation. D_x has Fourier symbol $|\xi|$.

Theorem 2.2.1. *Let $T > 0$. Then for every $0 \leq \varepsilon \leq \frac{1}{2}$, we have the estimates*

$$\|D_x^{\frac{-\varepsilon\delta(r)}{2}} U(t)\varphi\|_{L_T^q L_{xy}^r} \lesssim \|\varphi\|_{L_{xy}^2}$$

and

$$\left\| \int_0^t D_x^{\frac{-\varepsilon\delta(r)}{2}} U(t-t')F(t')dt' \right\|_{L_T^q L_{xy}^r} \lesssim \|F\|_{L_T^1 L_{xy}^2}$$

provided that $r \in [2, +\infty]$ and $0 \leq \frac{2}{q} = (1 - \frac{\varepsilon}{3})\delta(r) < 1$ with $\delta(r) := 1 - \frac{2}{r}$.

2.3 Linear Estimates

Theorem 2.3.1. *Let $v \in C([0, T] : H_{-1}^{\infty}(\mathbb{R}^2))$ be a solution of*

$$\partial_t v + \partial_x^3 v - \partial_x^{-1} \partial_y^2 v = \partial_x F.$$

Then, for every $\varepsilon > 0$, there exists C_ε such that

$$\|v\|_{L_T^2 L_{xy}^\infty} \lesssim C_\varepsilon (1+T) \|J_x^{1+\varepsilon} v\|_{L_T^\infty L_{xy}^2} + C_\varepsilon \|J_x^{1+\varepsilon} F\|_{L_T^1 L_{xy}^2}$$

.

Proof. We consider a Littlewood-Paley decomposition in the x-variable

$$v = \sum_{\lambda-\text{dyadic}} v_\lambda \text{ where } v_\lambda := \Delta_\lambda v \text{ are defined as}$$

$$\widehat{\Delta_\lambda v}(t, \xi, \eta) = \begin{cases} \varphi(\frac{\xi}{\lambda}) \cdot \hat{v}(t, \xi, \eta), & \lambda = 2^k, k \geq 1 \\ \chi(\xi) \cdot \hat{v}(t, \xi, \eta), & \lambda = 1. \end{cases}$$

where $\chi, \varphi \in C_0^\infty(\mathbb{R}^n)$ are non-negative functions such that $\chi(\xi) + \sum_{\lambda>1} \varphi(\frac{\xi}{\lambda}) = 1$ and

$$\varphi(\xi) = \begin{cases} 0 & \text{if } |\xi| \leq \frac{5}{8} \text{ or } |\xi| \geq 2 \\ 1 & \text{if } 1 \leq |\xi| \leq \frac{5}{4} \end{cases}$$

For $\lambda \geq 2$ fixed, we write a natural splitting $[0,T] = \cup_j I_j$, where $I_j = [a_j, b_j]$ have disjoint interiors and $|I_j| \lesssim \frac{1}{\lambda}$. We can suppose that the number of intervals is bounded by $C(1+T)\lambda$. Therefore, by applying Hölder inequality in time,

$$\|v_\lambda\|_{L_T^2 L_{xy}^\infty} \lesssim \sum_j \|v_\lambda\|_{L_{I_j}^2 L_{xy}^\infty} \lesssim \lambda^{-\frac{\varepsilon}{6}} \sum_j \|v_\lambda\|_{L_{I_j}^{q_\varepsilon} L_{xy}^\infty} \tag{2.1}$$

where $\frac{1}{q_\varepsilon} = \frac{1}{2} - \frac{\varepsilon}{6}$.

Next, we apply the Duhamel formula in each I_j to obtain,

$$v_\lambda(t) = D_x^{-\frac{\varepsilon}{2}} U(t-a_j) D_x^{\frac{\varepsilon}{2}} v_\lambda(a_j) - \int_{a_j}^t D_x^{-\varepsilon} U(t-t')[\Delta_\lambda D_x^\varepsilon \partial_x F](t')dt'$$

where we used that D_x and U commute.

By the Dispersive Estimates with $r = \infty$ and $q = \frac{1}{2} - \frac{\varepsilon}{6}$, we have

$$\begin{aligned} \|v_\lambda(t)\|_{L_{I_j}^{q_\varepsilon} L_{xy}^\infty} &\lesssim \|D_x^{-\frac{\varepsilon}{2}} U(t-a_j) D_x^{\frac{\varepsilon}{2}} v_\lambda(a_j)\|_{L_{I_j}^{q_\varepsilon} L_{xy}^\infty} + \|\int_{a_j}^t D_x^\varepsilon U(t-t')[\Delta_\lambda D_x^\varepsilon \partial_x F](t')dt'\|_{L_{I_j}^{q_\varepsilon} L_{xy}^\infty} \\ &\lesssim \|D_x^{\frac{\varepsilon}{2}} v_\lambda(a_j)\|_{L_{xy}^2} + \|\Delta_\lambda D_x^{1+\varepsilon} F\|_{L_{I_j}^1 L_{xy}^2}. \end{aligned}$$

Therefore, by equation (2.1), $\|v_\lambda\|_{L_{I_j}^2 L_{xy}^\infty} \lesssim \lambda^{\frac{\varepsilon}{3}} \|v_\lambda(a_j)\|_{L_{xy}^2} + \lambda^{1+\frac{5\varepsilon}{6}} \|\Delta_\lambda F\|_{L_{I_j}^1 L_{xy}^2}$ and summing over j,

$$
\begin{aligned}
\|v_\lambda\|_{L^2_T L^\infty_{xy}} &\lesssim \lambda^{\frac{\varepsilon}{3}} \sum_j \|v_\lambda(a_j)\|_{L^2_{xy}} + \sum_j \lambda^{1+\frac{5\varepsilon}{6}} \|\Delta_\lambda F\|_{L^1_{I_j} L^2_{xy}} \\
&\lesssim \lambda^{1+\frac{\varepsilon}{3}} (1+T) \|v_\lambda\|_{L^\infty_T L^2_{xy}} + \lambda^{1+\frac{5\varepsilon}{6}} \|\Delta_\lambda F\|_{L^1_T L^2_{xy}}
\end{aligned}
$$

Moreover, again, by Duhamel formula and the Dispersive Estimates,

$$
\|\Delta_1 v\|_{L^2_T L^\infty_{xy}} \lesssim (1+T)(\|\Delta_1 v(0)\|_{L^2_{xy}} + \|\Delta_1 F\|_{L^1_T L^2_{xy}})
$$

Hence, for any $0 < \alpha$, we get

$$
\begin{aligned}
\|v\|_{L^2_T L^\infty_{xy}} &\lesssim \sum_\lambda \|v_\lambda\|_{L^2_T L^\infty_{xy}} \lesssim \|\Delta_1 v\|_{L^2_T L^\infty_{xy}} + \sum_{\lambda \geq 2} \lambda^{-\alpha} \cdot \lambda^\alpha \|v_\lambda\|_{L^2_T L^\infty_{xy}} \\
&\lesssim \|\Delta_1 v\|_{L^2_T L^\infty_{xy}} + \sup_{\lambda \geq 2} \lambda^\alpha \|v_\lambda\|_{L^2_T L^\infty_{xy}} (\sum_{\lambda \geq 2} \lambda^{-\alpha}) \\
&\lesssim (1+T) \|J_x^{\alpha+1+\frac{\varepsilon}{3}} v\|_{L^\infty_T L^2_{xy}} + \|J_x^{\alpha+1+\frac{5\varepsilon}{6}} F\|_{L^1_T L^2_{xy}}
\end{aligned}
$$

where in the last inequality we used Bernstein's inequality. □

2.4 A Priori Estimates

Suppose u solves the equation

$$
\begin{cases}
\partial_t u + \partial_x^3 u - \partial_x^{-1} \partial_y^2 + u^2 \partial_x u = 0 \\
u(0) = \phi \in \mathrm{H}^{2+\varepsilon,0}(\mathbb{R}^2)
\end{cases}
$$

We are going to bound $f_u(T) = \|u\|_{L^2_T L^\infty_{xy}} + \|\partial_x u\|_{L^2_T L^\infty_{xy}}$.

Proposition 1. *Suppose u satisfies the IVP with initial data ϕ and let $s > 2$.*

Then $u, \partial_x u \in L^2([-T,T]; L^\infty(\mathbb{R}\times\mathbb{R}))$. *Moreover,*

$$f_u(T) = \|u\|_{L^2_T L^\infty_{xy}} + \|\partial_x u\|_{L^2_T L^\infty_{xy}} \leq C_T$$

for a suitable small T*, if* $\|\phi\|_{H^{s,0}}$ *is small enough.*

Proof. By the linear estimate 2.3.1, we have, by taking now $F = u^3$

$$\|u\|_{L^2_T L^\infty_{xy}} \lesssim \|J_x^{1+\varepsilon} u\|_{L^\infty_T L^2_{xy}} + \|J_x^{1+\varepsilon}(u^3)\|_{L^1_T L^2_{xy}}$$

and

$$\|\partial_x u\|_{L^2_T L^\infty_{xy}} \lesssim \|J_x^{2+\varepsilon} u\|_{L^\infty_T L^2_{xy}} + \|J_x^{2+\varepsilon}(u^3)\|_{L^1_T L^2_{xy}}.$$

By the corollary of the Kato-Ponce commutator estimates

$$\|J_x^{1+\varepsilon}(u^3)\|_{L^2_{xy}} \lesssim \|J_x^{1+\varepsilon} u\|_{L^2_{xy}} \|u\|^2_{L^\infty_{xy}} + \|J_x^{1+\varepsilon} u\|_{L^2_{xy}} \|u\|_{L^\infty_{xy}} \|\partial_x u\|_{L^\infty_{xy}} \lesssim \beta_u(t) \|J_x^{1+\varepsilon} u\|_{L^2_{xy}}$$

and

$$\|J_x^{2+\varepsilon}(u^3)\|_{L^2_{xy}} \lesssim \|J_x^{2+\varepsilon} u\|_{L^2_{xy}} \|u\|^2_{L^\infty_{xy}} + \|J_x^{2+\varepsilon} u\|_{L^2_{xy}} \|u\|_{L^\infty_{xy}} \|\partial_x u\|_{L^\infty_{xy}} \lesssim \beta_u(t) \|J_x^{2+\varepsilon} u\|_{L^2_{xy}}$$

therefore

$$\|u\|_{L^2_T L^\infty_{xy}} + \|\partial_x u\|_{L^2_T L^\infty_{xy}} \leq \|J_x^{2+\varepsilon} u\|_{L^\infty_T L^2_{xy}} (1 + f_u(T)^2).$$

Then we have

$$f_u(T) \leq \|J_x^{2+\varepsilon} u\|_{L^\infty_T L^2_{xy}} \cdot (1 + f_u(T)^2). \tag{2.2}$$

First, if we apply to the equation (1.7) the operator J_x^p and then we multiply by $J_x^p u$, we get

$$\begin{aligned}
\frac{d}{dt}\|J_x^p u\|_{L^2_{xy}}^2 &= \int J_x^p u J_x^p(u^2\partial_x u) \\
&= \int J_x^p u[J_x^p(u^2\partial_x u) - u^2 J_x^p \partial_x u] + \int u^2 J_x^p u J_x^p \partial_x u \\
&\lesssim \|J_x^p u\|_{L^2_{xy}}^2 (\|u\|_{L^\infty_{xy}}^2 + \|u\|_{L^\infty_{xy}} \|\partial_x u\|_{L^\infty_{xy}}) \lesssim \|J_x^p u\|_{L^2_{xy}}^2 \beta_u(t)
\end{aligned}$$

therefore, by Grönwall's inequality, we get that

$$\|J_x^p u\|_{L^\infty_T L^2_{xy}} \lesssim \|J_x^p \phi\|_{L^2_{xy}} \exp(f_u(T)^2). \tag{2.3}$$

Hence, by 2.2 and 2.3, we get

$$f_u(T) \lesssim \|J_x^{2+\varepsilon}\phi\|_{L^2_{xy}}^2 \cdot \exp(f_u(T)^2)(1 + f_u(T)^2).$$

If we choose ϕ such that $\|J_x^{2+\varepsilon}\phi\|_{L^2_{xy}}$ is small, then by a continuity argument, we get that $f_u(T) \leq C(T, \|J_x^{2+\varepsilon}\phi\|_{L^2_{xy}})$. Hence $\|u\|_{L^2_T L^\infty_{xy}} + \|\partial_x u\|_{L^2_T L^\infty_{xy}} \leq C(T, \|J_x^{2+\varepsilon}\phi\|_{L^2_{xy}})$. □

2.5 Local Well-posedness

We start by stating a well-known local-wellposedness result from Iorio and Nunes (see [22], Section 4):

Lemma 2. *Assume $\phi \in H^\infty$. Then there is $T = T(\|\phi\|_{H^3}) > 0$ and a solution $u \in C([-T,T] : H^\infty)$ of the initial value problem*

$$\begin{cases} \partial_t u + \partial_x^3 u - \partial_x^{-1}\partial_y^2 u + u^2 \partial_x u = 0, \\ u(0,x,y) = u_0(x,y). \end{cases}$$

We proceed to prove the local well-posedness result. In this section, the proof will include just the existence and uniqueness of the solution.

Theorem 2.5.1. *The IVP* $\partial_t u+\partial_x^3 u-\partial_x^{-1}\partial_y^2 u+u^2\partial_x u=0$ *is locally well-posed in* $H^{s,0}(\mathbb{R}^2)$, $s>2$. *More precisely, given* $u_0\in H^{s,0}(\mathbb{R}^2)$, $s>2$, *there exists* $T=T(\|u_0\|_{H^{s,0}})$ *and a unique solution* u *to the IVP such that* $u\in u\in C([0,T];H^{s,0}(\mathbb{R}^2))$, *with* $u,\partial_x u\in L_T^2L_{xy}^\infty$. *Moreover, the mapping* $u_0\mapsto u\in C([0,T];H^{s,0}(\mathbb{R}^2))$ *is continuous.*

Proof. Let $u_0\in H^{s,0}(\mathbb{R}^2)$ and find $u_{0,\varepsilon}\in H^{s,0}(\mathbb{R}^2)\cap H_{-1}^\infty(\mathbb{R}^2)$ such that $\|u_0-u_{0,\varepsilon}\|_{H^{s,0}}\to 0$ and $\|u_{0,\varepsilon}\|_{H^{s,0}}\leq 2\|u_0\|_{H^{s,0}}$. We know by the Iorio-Nunes local well-posedness result that $u_{0,\varepsilon}$ gives a unique solution u_ε. By applying the a priori bound, we get

$$\|\partial_x u_\varepsilon\|_{L_T^2L_{xy}^\infty}+\|u_\varepsilon\|_{L_T^2L_{xy}^\infty}\leq C_T.$$

By the previous result, $\sup_{0<t<T}\|u_\varepsilon\|_{H^{s,0}}\leq C_T$.

$$\partial_t(u_\varepsilon-u_{\varepsilon'})+\partial_x^3(u_\varepsilon-u_{\varepsilon'})-\partial_x^{-1}\partial_y^2(u_\varepsilon-u_{\varepsilon'})+\partial_x\left(\frac{u_\varepsilon^3}{3}-\frac{u_{\varepsilon'}^3}{3}\right)=0$$

with $\varepsilon,\varepsilon'\to 0$. Henceforth,

$$\begin{aligned}\partial_t\|u_\varepsilon-u_{\varepsilon'}\|_{L^2}^2&=\int(u_\varepsilon-u_{\varepsilon'})\partial_x\left(\frac{u_\varepsilon^3}{3}-\frac{u_{\varepsilon'}^3}{3}\right)\\&=\int\partial_x(u_\varepsilon-u_{\varepsilon'})\cdot(u_\varepsilon-u_{\varepsilon'})\frac{u_\varepsilon^2+u_\varepsilon u_{\varepsilon'}+u_{\varepsilon'}^2}{3}=\\&=\int(u_\varepsilon-u_{\varepsilon'})^2\partial_x\left[\frac{u_\varepsilon^2+u_\varepsilon u_{\varepsilon'}+u_{\varepsilon'}^2}{3}\right]\\&\leq\|u_\varepsilon-u_{\varepsilon'}\|_{L^2}^2(\|u_\varepsilon\|_{L_{xy}^\infty}^2+\|\partial_x u_\varepsilon\|_{L_{xy}^\infty}^2+\|u_{\varepsilon'}\|_{L_{xy}^\infty}^2+\|\partial_x u_{\varepsilon'}\|_{L_{xy}^\infty}^2)\\&\leq(\beta_{u_\varepsilon}(t)+\beta_{u_{\varepsilon'}}(t))\|u_\varepsilon-u_{\varepsilon'}\|_{L^2}^2.\end{aligned}$$

By Grönwall's inequality and the a priori estimate we get $\|u_\varepsilon-u_{\varepsilon'}\|_{L^2}^2\lesssim_T\|u_{0,\varepsilon}-u_{0,\varepsilon'}\|_{L^2}^2$, implying that $\sup_{0<t<T}\|u_\varepsilon-u_{\varepsilon'}\|_{L^2}\to 0$, hence we can find $u\in C([0,T],H^{s',0}(\mathbb{R}^2))\cap$

$L^\infty([0,T]:H^{s,0}(\mathbb{R}^2))$. The fact that u is a solution of the IVP is clear now. Uniqueness also comes from the previous Grönwall's inequality. $\square$

2.6 Continuity with respect to time

We are completing the proof by showing continuity with respect to time.

Let u be a fixed solution of $\partial_t u+\partial_x^3 u-\partial_x^{-1}\partial_y^2 u+u^2\partial_x u=0$ with initial data in $H^{s,0}(\mathbb{R}^2)$, with $s>2$. Define φ^ε is defined by its Fourier transform namely,

$$\widehat{\varphi^\varepsilon}(\xi,\eta):=\begin{cases}1 \text{ if } \varepsilon<\xi<\frac{1}{\varepsilon} \text{ and if } \varepsilon<\eta<\frac{1}{\varepsilon},\\ 0 \text{ otherwise}\end{cases} \tag{2.4}$$

Then u_ε is a solution to the IVP with $\phi_\varepsilon=\varphi^\varepsilon*\phi$ as initial data. We will show that $\{u_\varepsilon\}$ is in fact a Cauchy sequence in $C([0,T]:H^{s,0}(\mathbb{R}^2))$. By straightforward calculations in the Fourier space, one can show that for $\phi\in H^{s,0}(\mathbb{R}^2)$, $0<\varepsilon<1$ and $r\geq 0$, we have

$$\|\varphi^\varepsilon*\phi\|_{H^{s+r,0}(\mathbb{R}^2)}\lesssim \varepsilon^{-r}\|\phi\|_{H^{s,0}(\mathbb{R}^2)}$$

and

$$\|\varphi^\varepsilon*\phi-\phi\|_{H^{s-r,0}(\mathbb{R}^2)}\lesssim o(\varepsilon^r)\|\phi\|_{H^{s,0}(\mathbb{R}^2)}$$

as $\varepsilon\to 0$.

From the definition, we have that, if $p\geq s$, then $\|J_x^p\phi_\varepsilon\|_{L^2_{xy}}\lesssim C(T,\|\phi\|_{H^{s,0}})\varepsilon^{s-p}$. Since $\phi_\varepsilon\in H^\infty$, by local well-posedness result of Iorio and Nunes, they give rise to unique solutions u_ε in H^∞. The above estimates together with (2.3), if $p\geq s$, we also have

$$\|J_x^p u_k\|_{L^\infty_T L^2_{xy}}\leq C(T,\|\phi\|_{H^{s,0}})\varepsilon^{s-p}. \tag{2.5}$$

Denote $\omega=u_{\varepsilon_1}-u_{\varepsilon_2}$ with $0<\varepsilon_1<\varepsilon_2$. Now choose $0\leq q\leq s$. By definition, there exists a function $h_\phi^{(3)}(\varepsilon)\to 0$ as $\varepsilon\to 0$ such that $\|\phi-\phi_\varepsilon\|_{L^2_{xy}}\lesssim \varepsilon^s h_\phi^{(3)}(\varepsilon)$. From the interpolation inequality,

$$\|J_x^q\omega\|_{L^\infty_T L^2_{xy}}\leq \|J_x^s\omega\|^{\frac{q}{s}}_{L^\infty_T L^2_{xy}}\|\omega\|^{1-\frac{q}{s}}_{L^\infty_T L^2_{xy}}\lesssim \|\omega\|^{1-\frac{q}{s}}_{L^\infty_T L^2_{xy}}$$

it yields

$$\|J_x^q\omega\|_{L^\infty_T L^2_{xy}}\lesssim \varepsilon^{s-q}h_\phi^{(3)}(\varepsilon)^{1-\frac{q}{s}}. \tag{2.6}$$

Lemma 3. *We have the following estimates:*

$$\begin{aligned}\|J_x^s\omega\|_{L^\infty_T L^2_{xy}} \lesssim exp(\frac{1}{2}f_\omega(T)^2+\frac{1}{2}f_{u_{\varepsilon_1}}(T)^2)\Big[&\|J_x^s\omega(0)\|_{L^2_{xy}}\\ &+\|J_x^s\omega\|_{L^\infty_T L^2_{xy}}\|J_x^{s+1}u_{\varepsilon_1}\|_{L^\infty_T L^2_{xy}}(\|\omega\|^2_{L^2_T L^\infty_{xy}}+\|\omega\|_{L^2_T L^\infty_{xy}}\|u_{\varepsilon_1}\|_{L^2_T L^\infty_{xy}})\\ &+\|J_x^s\omega\|_{L^\infty_T L^2_{xy}}\|J_x^s u_{\varepsilon_1}\|_{L^\infty_T L^2_{xy}}(\|\omega\|_{L^2_T L^\infty_{xy}}+\|\partial_x\omega\|_{L^2_T L^\infty_{xy}})\|u_{\varepsilon_1}\|_{L^2_T L^\infty_{xy}}\\ &+\|J_x^s\omega\|_{L^\infty_T L^2_{xy}}\|J_x^s u_{\varepsilon_1}\|_{L^\infty_T L^2_{xy}}\|\omega\|_{L^2_T L^\infty_{xy}}(\|\partial_x u_{\varepsilon_1}\|_{L^2_T L^\infty_{xy}}+\|\partial_x\omega\|_{L^2_T L^\infty_{xy}})\\ &+\|J_x^s\omega\|_{L^\infty_T L^2_{xy}}\|J_x^s u_{\varepsilon_1}\|_{L^\infty_T L^2_{xy}}\|\omega\|^2_{L^2_T L^\infty_{xy}}\Big].\end{aligned}$$

Proof.

$$\partial_t\omega+\partial_x^3\omega-\partial_x^{-1}\partial_y^2\omega+\omega^2\partial_x\omega+3u_{\varepsilon_1}^2\partial_x\omega+3u_{\varepsilon_1}\omega\partial_x u_{\varepsilon_1}-3u_{\varepsilon_1}\omega\partial_x\omega-3\omega^2\partial_x u_{\varepsilon_1}=0. \tag{2.7}$$

We apply J_x^s to (2.7) and then we multiply by $J_x^s\omega$, in order to get

$$\begin{aligned}\frac{d}{dt}\|J_x^s\omega\|^2_{L^2}=&\int J_x^s(\omega^2\partial_x\omega)J_x^s\omega+3\int J_x^s(u_{\varepsilon_1}^2\partial_x\omega)J_x^s\omega+3\int J_x^s(u_{\varepsilon_1}\omega\partial_x u_{\varepsilon_1})J_x^s\omega\\ &-3\int J_x^s(u_{\varepsilon_1}\omega\partial_x\omega)J_x^s\omega-3\int J_x^s(\omega^2\partial_x u_{\varepsilon_1})J_x^s\omega\end{aligned}$$

and we will analyze each term in the sum.

We have $(I)=\int J_x^s(\omega^2\partial_x\omega)J_x^s\omega$, $(II)=\int J_x^s(u_{\varepsilon_1}^2\partial_x\omega)J_x^s\omega$, $(III)=\int J_x^s(u_{\varepsilon_1}\omega\partial_x u_{\varepsilon_1})J_x^s\omega$, $(IV)=\int J_x^s(u_{\varepsilon_1}\omega\partial_x\omega)J_x^s\omega$ and $(V)=\int J_x^s(\omega^2\partial_x u_{\varepsilon_1})J_x^s\omega$.

For $(I)=\int J_x^s(\omega^2\partial_x\omega)J_x^s\omega=\int[J_x^s(\omega^2\partial_x\omega)-\omega^2J_x^s\partial_x\omega]J_x^s\omega+\int\omega^2J_x^s\partial_x\omega J_x^s\omega$, and we will denote $(I)_1 = \int [J_x^s(\omega^2 \partial_x \omega) - \omega^2 J_x^s \partial_x \omega] J_x^s \omega$ and $(I)_2 = \int \omega^2 J_x^s \partial_x \omega J_x^s \omega$. For the first one, we have by the Kato-Ponce commutator estimate

$$
\begin{aligned}
(I)_1 &\le \|J_x^s \omega\|_{L^2_{xy}} \|J_x^s(\omega^2 \partial_x \omega) - \omega^2 J_x^s \partial_x \omega\|_{L^2_{xy}} \\
&\le \|J_x^s \omega\|_{L^2_{xy}} \|\omega\|_{L^\infty_{xy}} \Big[\|\partial_x \omega\|_{L^\infty_{xy}} \cdot \|J_x^s \omega\|_{L^2_{xy}} + (\|\omega\|_{L^\infty_{xy}} + \|\partial_x \omega\|_{L^\infty_{xy}}) \|J_x^{s-1} \partial_x \omega\|_{L^2_{xy}} \Big] \\
&\le \|J_x^s \omega\|^2_{L^2_{xy}} \cdot \beta_\omega(t)
\end{aligned}
$$

and

$$
(I)_2 \le \|J_x^s \omega\|^2_{L^2_{xy}} \|\omega\|_{L^\infty_{xy}} \|\partial_x \omega\|_{L^\infty_{xy}} \le \|J_x^s \omega\|^2_{L^2_{xy}} \beta_\omega(t)
$$

so $(I) \lesssim \|J_x^s \omega\|^2_{L^2_{xy}} \beta_\omega(t)$.

Now, $(II) = \int J_x^s(u_{\varepsilon_1}^2 \partial_x \omega) J_x^s \omega = \int [J_x^s(u_{\varepsilon_1}^2 \partial_x \omega) - u_{\varepsilon_1}^2 J_x^s \partial_x \omega] J_x^s \omega + \int u_{\varepsilon_1}^2 J_x^s \partial_x \omega J_x^s \omega$ and we denote $(II)_1 = \int [J_x^s(u_{\varepsilon_1}^2 \partial_x \omega) - u_{\varepsilon_1}^2 J_x^s \partial_x \omega] J_x^s \omega$ and $(II)_2 = \int u_{\varepsilon_1}^2 J_x^s \partial_x \omega J_x^s \omega$. For the first term we have by the Kato-Ponce commutator estimate

$$
\begin{aligned}
(II)_1 &\lesssim \|J_x^s \omega\|_{L^2_{xy}} \cdot \|J_x^s(u_{\varepsilon_1}^2 \partial_x \omega) - u_{\varepsilon_1}^2 J_x^s \partial_x \omega\|_{L^2_{xy}} \\
&\lesssim \|J_x^s \omega\|_{L^2_{xy}} \|u_{\varepsilon_1}\|_{L^\infty_{xy}} (\|u_{\varepsilon_1}\|_{L^\infty_{xy}} + \|\partial_x u_{\varepsilon_1}\|_{L^\infty_{xy}}) \|J_x^{s-1} \partial_x \omega\|_{L^2_{xy}} \\
&+ \|J_x^s \omega\|_{L^2_{xy}} \|u_{\varepsilon_1}\|_{L^\infty_{xy}} \|\partial_x \omega\|_{L^\infty_{xy}} \|J_x^s u_{\varepsilon_1}\|_{L^2_{xy}} \\
&\lesssim \|J_x^s \omega\|^2_{L^2_{xy}} \beta_{u_{\varepsilon_1}}(t) + \|J_x^s \omega\|_{L^2_{xy}} \|J_x^s u_{\varepsilon_1}\|_{L^2_{xy}} \|\partial_x \omega\|_{L^\infty_{xy}} \|u_{\varepsilon_1}\|_{L^\infty_{xy}}
\end{aligned}
$$

Also, we have

$$
(II)_2 \lesssim \|J_x^s \omega\|^2_{L^2_{xy}} \|u_{\varepsilon_1}\|_{L^\infty} \|\partial_x u_{\varepsilon_1}\|_{L^\infty} \lesssim \|J_x^s \omega\|^2_{L^2_{xy}} \beta_{u_{\varepsilon_1}}(t)
$$

Therefore,

$$
(II) \lesssim \|J_x^s \omega\|^2_{L^2_{xy}} \beta_{u_{\varepsilon_1}}(t) + \|J_x^s \omega\|_{L^2_{xy}} \|J_x^s u_{\varepsilon_1}\|_{L^2_{xy}} \|\partial_x \omega\|_{L^\infty_{xy}} \|u_{\varepsilon_1}\|_{L^\infty_{xy}}.
$$

Let

$$
\begin{aligned}
(III) &= \int J_x^s(u_{\varepsilon_1}\omega\partial_x u_{\varepsilon_1})J_x^s\omega \\
&= \int [J_x^s(u_{\varepsilon_1}\omega\partial_x u_{\varepsilon_1}) - u_{\varepsilon_1}\omega J_x^s\partial_x u_{\varepsilon_1}]J_x^s\omega + \int u_{\varepsilon_1}\omega J_x^s\partial_x u_{\varepsilon_1} J_x^s\omega
\end{aligned}
$$

and denote $(III)_1 = \int[J_x^s(u_{\varepsilon_1}\omega\partial_x u_{\varepsilon_1}) - u_{\varepsilon_1}\omega J_x^s\partial_x u_{\varepsilon_1}]J_x^s\omega$ and

$$
(III)_2 = \int u_{\varepsilon_1}\omega J_x^s\partial_x u_{\varepsilon_1} J_x^s\omega.
$$

We have by the Kato-Ponce commutator estimate

$$
\begin{aligned}
(III)_1 &\lesssim \|J_x^s\omega\|_{L^2_{xy}}\|J_x^s(u_{\varepsilon_1}\omega\partial_x u_{\varepsilon_1}) - u_{\varepsilon_1}\omega J_x^s\partial_x u_{\varepsilon_1}\|_{L^2_{xy}} \\
&\lesssim \|J_x^s\omega\|_{L^2_{xy}}\Big[\|J_x^s u_{\varepsilon_1}\|_{L^2_{xy}}\|\omega\|_{L^\infty_{xy}}\|\partial_x u_{\varepsilon_1}\|_{L^\infty_{xy}} \\
&+ \|J_x^{s-1}\omega\|_{L^2_{xy}}(\|u_{\varepsilon_1}\|_{L^\infty_{xy}}\|\partial_x u_{\varepsilon_1}\|_{L^\infty_{xy}}+\|\partial_x u_{\varepsilon_1}\|^2_{L^\infty_{xy}}) \\
&+ \|J_x^{s-1}\partial_x u_{\varepsilon_1}\|_{L^2_{xy}}(\|\omega\|_{L^\infty_{xy}}\|\partial_x u_{\varepsilon_1}\|_{L^\infty_{xy}}+\|\omega\|_{L^\infty_{xy}}\|u_{\varepsilon_1}\|_{L^\infty_{xy}}+\|\partial_x\omega\|_{L^\infty_{xy}}\|u_{\varepsilon_1}\|_{L^\infty_{xy}})\Big] \\
&\lesssim \|J_x^s\omega\|^2_{L^2_{xy}}\beta_{u_{\varepsilon_1}}(t) + \|J_x^s\omega\|_{L^2_{xy}}\|J_x^s u_{\varepsilon_1}\|_{L^2_{xy}}\|\omega\|_{L^\infty_{xy}}\|\partial_x u_{\varepsilon_1}\|_{L^\infty_{xy}} \\
&+ \|J_x^s\omega\|_{L^2_{xy}}\|J_x^s u_{\varepsilon_1}\|_{L^2_{xy}}\|u_{\varepsilon_1}\|_{L^\infty_{xy}}(\|\omega\|_{L^\infty_{xy}}+\|\partial_x\omega\|_{L^\infty_{xy}})
\end{aligned}
$$

Also, $(III)_2 \lesssim \|J_x^s\omega\|_{L^2_{xy}}\|J_x^{s+1}u_{\varepsilon_1}\|_{L^2_{xy}}\|\omega\|_{L^\infty_{xy}}\|u_{\varepsilon_1}\|_{L^\infty_{xy}}$ and so therefore

$$
\begin{aligned}
(III) &\lesssim \|J_x^s\omega\|^2_{L^2_{xy}}\beta_{u_{\varepsilon_1}}(t) + \|J_x^s\omega\|_{L^2_{xy}}\|J_x^s u_{\varepsilon_1}\|_{L^2_{xy}}\|\omega\|_{L^\infty_{xy}}\|\partial_x u_{\varepsilon_1}\|_{L^\infty_{xy}} \\
&+ \|J_x^s\omega\|_{L^2_{xy}}\|J_x^s u_{\varepsilon_1}\|_{L^2_{xy}}\|\omega\|_{L^\infty_{xy}}\|u_{\varepsilon_1}\|_{L^\infty_{xy}}+\|J_x^s\omega\|_{L^2_{xy}}\|J_x^s u_{\varepsilon_1}\|_{L^2_{xy}}\|\partial_x\omega\|_{L^\infty_{xy}}\|u_{\varepsilon_1}\|_{L^\infty_{xy}} \\
&+ \|J_x^s\omega\|_{L^2_{xy}}\|J_x^{s+1}u_{\varepsilon_1}\|_{L^2_{xy}}\|\omega\|_{L^\infty_{xy}}\|u_{\varepsilon_1}\|_{L^\infty_{xy}}.
\end{aligned}
$$

Again,

$$
(IV) = \int J_x^s(u_{\varepsilon_1}\omega\partial_x\omega)J_x^s\omega = \int[J_x^s(u_{\varepsilon_1}\omega\partial_x\omega) - u_{\varepsilon_1}\omega J_x^s\partial_x\omega]J_x^s\omega + \int u_{\varepsilon_1}\omega J_x^s\partial_x\omega J_x^s\omega
$$

and we denote $(IV)_1 = \int [J_x^s(u_{\varepsilon_1}\omega\partial_x\omega) - u_{\varepsilon_1}\omega J_x^s\partial_x\omega]J_x^s\omega$ and

$$(IV)_2 = \int u_{\varepsilon_1}\omega J_x^s\partial_x\omega J_x^s\omega.$$

We have by the Kato-Ponce commutator estimate

$$\begin{aligned}
(IV)_1 &\lesssim \|J_x^s\omega\|_{L^2_{xy}}\|J_x^s(u_{\varepsilon_1}\omega\partial_x\omega) - u_{\varepsilon_1}\omega J_x^s\partial_x\omega\|_{L^2_{xy}} \\
&\lesssim \|J_x^s\omega\|_{L^2_{xy}}\|\partial_x\omega\|_{L^\infty_{xy}}\Big[\|J_x^s\omega\|_{L^2_{xy}}(\|u_{\varepsilon_1}\|_{L^\infty_{xy}}+\|\partial_x u_{\varepsilon_1}\|_{L^\infty_{xy}}) + \|J_x^s u_{\varepsilon_1}\|_{L^2_{xy}}\|\omega\|_{L^\infty_{xy}}\Big] \\
&+ \|J_x^s\omega\|_{L^2_{xy}}\Big[\|u_{\varepsilon_1}\|_{L^\infty_{xy}}(\|\omega\|_{L^\infty_{xy}}+\|\partial_x\omega\|_{L^\infty_{xy}}) + \|\partial_x u_{\varepsilon_1}\|_{L^\infty_{xy}}\|\omega\|_{L^\infty_{xy}})\|J_x^{s-1}\partial_x\omega\|_{L^2_{xy}} \\
&\lesssim \|J_x^s\omega\|^2_{L^2_{xy}}(\|u_{\varepsilon_1}\|_{L^\infty_{xy}}+\|\partial_x u_{\varepsilon_1}\|_{L^\infty_{xy}})(\|\partial_x\omega\|_{L^\infty_{xy}}+\|\omega\|_{L^\infty_{xy}}) \\
&+ \|J_x^s\omega\|_{L^2_{xy}}\|J_x^s u_{\varepsilon_1}\|_{L^2_{xy}}\|\omega\|_{L^\infty_{xy}}\|\partial_x\omega\|_{L^\infty_{xy}}.
\end{aligned}$$

Also, $(IV)_2 \lesssim \|J_x^s\omega\|^2_{L^2_{xy}}(\|\partial_x u_{\varepsilon_1}\|_{L^\infty_{xy}}\|\omega\|_{L^\infty_{xy}}+\|\partial_x\omega\|_{L^\infty_{xy}}\|u_{\varepsilon_1}\|_{L^\infty_{xy}})$ and so therefore

$$\begin{aligned}
(IV) &\lesssim \|J_x^s\omega\|^2_{L^2_{xy}}(\|u_{\varepsilon_1}\|_{L^\infty_{xy}}+\|\partial_x u_{\varepsilon_1}\|_{L^\infty_{xy}})(\|\partial_x\omega\|_{L^\infty_{xy}}+\|\omega\|_{L^\infty_{xy}}) \\
&+ \|J_x^s\omega\|_{L^2_{xy}}\|J_x^s u_{\varepsilon_1}\|_{L^2_{xy}}\|\omega\|_{L^\infty_{xy}}\|\partial_x\omega\|_{L^\infty_{xy}}.
\end{aligned}$$

Again, $(V) = \int J_x^s(\omega^2\partial_x u_{\varepsilon_1})J_x^s\omega = \int [J_x^s(\omega^2\partial_x u_{\varepsilon_1}) - \omega^2 J_x^s\partial_x u_{\varepsilon_1}]J_x^s\omega + \int \omega^2 J_x^s\partial_x u_{\varepsilon_1}J_x^s\omega$ and we denote $(V)_1 = \int [J_x^s(\omega^2\partial_x u_{\varepsilon_1}) - \omega^2 J_x^s\partial_x u_{\varepsilon_1}]J_x^s\omega$ and $(V)_2 = \int \omega^2 J_x^s\partial_x u_{\varepsilon_1}J_x^s\omega$. We have by the Kato-Ponce commutator estimate

$$\begin{aligned}
(V)_1 &\lesssim \|J_x^s\omega\|_{L^2_{xy}}\|J_x^s(\omega^2\partial_x u_{\varepsilon_1}) - \omega^2 J_x^s\partial_x u_{\varepsilon_1}\|_{L^2_{xy}} \\
&\lesssim \|J_x^s\omega\|_{L^2_{xy}}\|\omega\|_{L^\infty_{xy}}\Big[\|\partial_x u_{\varepsilon_1}\|_{L^\infty_{xy}}\|J_x^s\omega\|_{L^2_{xy}}+(\|\omega\|_{L^\infty_{xy}}+\|\partial_x\omega\|_{L^\infty_{xy}})\|J_x^{s-1}\partial_x u_{\varepsilon_1}\|_{L^2_{xy}}\Big] \\
&\lesssim \|J_x^s\omega\|^2_{L^2_{xy}}\|\omega\|_{L^\infty_{xy}}\|\partial_x u_{\varepsilon_1}\|_{L^\infty_{xy}}+\|J_x^s\omega\|_{L^2_{xy}}\|J_x^s u_{\varepsilon_1}\|_{L^2_{xy}}\|\omega\|_{L^\infty_{xy}}(\|\omega\|_{L^\infty_{xy}}+\|\partial_x\omega\|_{L^\infty_{xy}}).
\end{aligned}$$

Also, $(V)_2 \lesssim \|J_x^s\omega\|_{L^2_{xy}}\|J_x^{s+1}u_{\varepsilon 1}\|_{L^2_{xy}}\|\omega\|^2_{L^\infty_{xy}}$ and so therefore

$$
\begin{aligned}
(V) &\lesssim \|J_x^s\omega\|^2_{L^2_{xy}}(\beta_{u_{\varepsilon_1}}(t)+\beta_\omega(t)) \\
&\quad + \|J_x^s\omega\|_{L^2_{xy}}\Big(\|J_x^s u_{\varepsilon 1}\|_{L^2_{xy}}\|\omega\|_{L^\infty_{xy}}(\|\omega\|_{L^\infty_{xy}}+\|\partial_x\omega\|_{L^\infty_{xy}})+\|J_x^{s+1}u_{\varepsilon 1}\|_{L^2_{xy}}\|\omega\|^2_{L^\infty_{xy}}\Big).
\end{aligned}
$$

Now, putting together all the terms we get that

$$
\begin{aligned}
\frac{d}{dt}\|J_x^s\omega\|^2_{L^2_{xy}} \lesssim{}& (\|J_x^s\omega\|^2_{L^2_{xy}})(\beta_\omega(t)+\beta_{u_{\varepsilon_1}}(t)) \\
&+\|J_x^s\omega\|_{L^2_{xy}}\|J_x^{s+1}u_{\varepsilon 1}\|_{L^2_{xy}}(\|\omega\|^2_{L^\infty_{xy}}+\|\omega\|_{L^\infty_{xy}}\|u_{\varepsilon 1}\|_{L^\infty_{xy}}) \\
&+\|J_x^s\omega\|_{L^2_{xy}}\|J_x^s u_{\varepsilon 1}\|_{L^2_{xy}}(\|\omega\|_{L^\infty_{xy}}\|u_{\varepsilon 1}\|_{L^\infty_{xy}}+\|\partial_x\omega\|_{L^\infty_{xy}}\|u_{\varepsilon 1}\|_{L^\infty_{xy}}) \\
&+\|J_x^s\omega\|_{L^2_{xy}}\|J_x^s u_{\varepsilon 1}\|_{L^2_{xy}}(\|\omega\|_{L^\infty_{xy}}\|\partial_x u_{\varepsilon 1}\|_{L^\infty_{xy}}+\|\omega\|_{L^\infty_{xy}}\|\partial_x\omega\|_{L^\infty_{xy}}+\|\omega\|^2_{L^\infty_{xy}})
\end{aligned}
\tag{2.8}
$$

We are using the following variant of Grönwall's inequality:

Lemma 4. *If $\alpha(t), \beta(t)$ are two non-negative functions, and $\frac{d}{dt}u(t) \leq u(t)\beta(t)+\alpha(t)$ for all $t\in[0,T]$ then*

$$u(t) \leq e^{\int_0^t \beta(s)ds}\Big(u(0)+\int_0^t \alpha(s)ds\Big).$$

By putting $u(t) = \|J_x^s\omega\|_{L^2_{xy}}$, $\beta(t) = \beta_\omega(t)+\beta_{u_{\varepsilon_1}}(t) \geq 0$ and

$$
\begin{aligned}
\alpha(t) ={}& \|J_x^{s+1}u_{\varepsilon 1}\|_{L^2_{xy}}(\|\omega\|^2_{L^\infty_{xy}}+\|\omega\|_{L^\infty_{xy}}\|u_{\varepsilon 1}\|_{L^\infty_{xy}}) \\
&+\|J_x^s u_{\varepsilon 1}\|_{L^2_{xy}}(\|\omega\|_{L^\infty_{xy}}\|u_{\varepsilon 1}\|_{L^\infty_{xy}}+\|\partial_x\omega\|_{L^\infty_{xy}}\|u_{\varepsilon 1}\|_{L^\infty_{xy}}) \\
&+\|J_x^s u_{\varepsilon 1}\|_{L^2_{xy}}(\|\omega\|_{L^\infty_{xy}}\|\partial_x u_{\varepsilon 1}\|_{L^\infty_{xy}}+\|\omega\|_{L^\infty_{xy}}\|\partial_x\omega\|_{L^\infty_{xy}}+\|\omega\|^2_{L^\infty_{xy}}) \geq 0
\end{aligned}
$$

by applying the lemma to (2.8) together with Cauchy-Schwarz we get

$$
\begin{aligned}
\|J_x^s\omega\|_{L_T^\infty L_{xy}^2} \lesssim \exp(\frac{1}{2}f_\omega(T)^2 + \frac{1}{2}f_{u_{\varepsilon_1}}(T)^2)\Big[&\|J_x^s\omega(0)\|_{L_{xy}^2} \\
&+ \|J_x^s\omega\|_{L_T^\infty L_{xy}^2}\|J_x^{s+1}u_{\varepsilon_1}\|_{L_T^\infty L_{xy}^2}(\|\omega\|^2_{L_T^2 L_{xy}^\infty} + \|\omega\|_{L_T^2 L_{xy}^\infty}\|u_{\varepsilon_1}\|_{L_T^2 L_{xy}^\infty}) \\
&+ \|J_x^s\omega\|_{L_T^\infty L_{xy}^2}\|J_x^s u_{\varepsilon_1}\|_{L_T^\infty L_{xy}^2}(\|\omega\|_{L_T^2 L_{xy}^\infty} + \|\partial_x\omega\|_{L_T^2 L_{xy}^\infty})\|u_{\varepsilon_1}\|_{L_T^2 L_{xy}^\infty} \\
&+ \|J_x^s\omega\|_{L_T^\infty L_{xy}^2}\|J_x^s u_{\varepsilon_1}\|_{L_T^\infty L_{xy}^2}\|\omega\|_{L_T^2 L_{xy}^\infty}(\|\partial_x u_{\varepsilon_1}\|_{L_T^2 L_{xy}^\infty} + \|\partial_x\omega\|_{L_T^2 L_{xy}^\infty}) \\
&+ \|J_x^s\omega\|_{L_T^\infty L_{xy}^2}\|J_x^s u_{\varepsilon_1}\|_{L_T^\infty L_{xy}^2}\|\omega\|^2_{L_T^2 L_{xy}^\infty}\Big].
\end{aligned}
$$

□

Lemma 5. *For $p \leq s$, we have the following estimates:*

$$
\begin{aligned}
\|J_x^p[\omega(u_{\varepsilon_1}^2 + u_{\varepsilon_1}u_{\varepsilon_2} + u_{\varepsilon_2}^2)]\|_{L_T^1 L_{xy}^2} \lesssim\ &\|J_x^p\omega\|_{L_T^\infty L_{xy}^2}(\|u_{\varepsilon_1}\|_{L_T^2 L_{xy}^\infty} + \|u_{\varepsilon_2}\|_{L_T^2 L_{xy}^\infty}) \\
&+ \|\omega\|_{L_T^2 L_{xy}^\infty}\|\phi\|_{H^{s,s}}(\|u_{\varepsilon_1}\|_{L_T^2 L_{xy}^\infty} + \|u_{\varepsilon_2}\|_{L_T^2 L_{xy}^\infty}).
\end{aligned}
$$

Proof. By using 7 part (c), we get that

$$
\begin{aligned}
\|J_x^p[\omega(u_{\varepsilon_1}^2 + u_{\varepsilon_1}u_{\varepsilon_2} + u_{\varepsilon_2}^2)]\|_{L_{xy}^2} \lesssim\ &\|J_x^p\omega\|_{L_T^\infty L_{xy}^2}\|u_{\varepsilon_1}^2 + u_{\varepsilon_1}u_{\varepsilon_2} + u_{\varepsilon_2}^2\|_{L_{xy}^\infty} \\
&+ \|\omega\|_{L_{xy}^\infty}\|J_x^p(u_{\varepsilon_1}^2 + u_{\varepsilon_1}u_{\varepsilon_2} + u_{\varepsilon_2}^2)\|_{L_{xy}^2}.
\end{aligned} \tag{2.9}
$$

Observe that $\|u_{\varepsilon_1}^2 + u_{\varepsilon_1}u_{\varepsilon_2} + u_{\varepsilon_2}^2\|_{L_{xy}^\infty} \lesssim \|u_{\varepsilon_1}\|^2_{L_{xy}^\infty} + \|u_{\varepsilon_2}\|^2_{L_{xy}^\infty}$. Also, by 7 part (c) again, we have

$$
\|J_x^p(u_{\varepsilon_1}^2 + u_{\varepsilon_1}u_{\varepsilon_2} + u_{\varepsilon_2}^2)\|_{L_{xy}^2} \lesssim (\|J_x^p u_{\varepsilon_1}\|_{L_{xy}^2} + \|J_x^p u_{\varepsilon_2}\|_{L_{xy}^2})(\|u_{\varepsilon_1}\|_{L^\infty} + \|u_{\varepsilon_2}\|_{L_{xy}^\infty}).
$$

By 2.3, we get $\|J_x^p u_{\varepsilon_1}\|_{L_{xy}^2} + \|J_x^p u_{\varepsilon_2}\|_{L_{xy}^2} \lesssim \|J_x^p\phi_k\|_{L_{xy}^2} + \|J_x^p\phi_{k'}\|_{L_{xy}^2} \lesssim \|\phi\|_{H^{s,0}}$. Combining all the above observations together with 2.9, we get

$$\|J_x^p[\omega(u_{\varepsilon_1}^2+u_{\varepsilon_1}u_{\varepsilon_2}+u_{\varepsilon_2}^2)]\|_{L^2_{xy}} \lesssim \|J_x^p\omega\|_{L^2_{xy}}(\|u_{\varepsilon_1}\|_{L^\infty_{xy}}+\|u_{\varepsilon_2}\|_{L^\infty_{xy}})$$
$$+\|\omega\|_{L^2_TL^\infty_{xy}}\|\phi\|_{H^{s,0}}(\|u_{\varepsilon_1}\|_{L^\infty_{xy}}+\|u_{\varepsilon_2}\|_{L^\infty_{xy}}).$$

Integrating both sides from 0 to T and applying Cauchy-Schwarz inequality, we obtain the conclusion of the lemma. □

Lemma 6. *Suppose u_{ε_1} satisfies the IVP (1.9) with initial data ϕ_{ε_1}. We have $\|\omega\|_{L^2_TL^\infty_{xy}}\lesssim \varepsilon_1^{1+}$, $\|\partial_x\omega\|_{L^2_TL^\infty_{xy}}\lesssim \varepsilon_1^{0+}$ and $\|\partial_y\omega\|_{L^2_TL^\infty_{xy}}\lesssim \varepsilon_1^{0+}$ as $\varepsilon_1\to 0$. In particular, $f_\omega(T)\lesssim \varepsilon_1^{0+}$ as $\varepsilon_1\to 0$.*

Proof. Take $\delta<\frac{s-2}{2}$. By the linear estimate in Proposition 2 applied to 2.7,

$$\|\omega\|_{L^2_TL^\infty_{xy}}\lesssim C_\delta(1+T)\|J_x^{1+\delta}\omega\|_{L^\infty_TL^2_{xy}}+\|J_x^{1+\delta}[\omega(u_{\varepsilon_1}^2+u_{\varepsilon_1}u_{\varepsilon_2}+u_{\varepsilon_2}^2)]\|_{L^1_TL^2_{xy}}.$$

From 2.9 we have $\|J_x^{1+\delta}\omega\|_{L^\infty_TL^2_{xy}}\lesssim \varepsilon_1^{s-1-\delta}h_\phi^{(3)}(\varepsilon_1)^{1-\frac{1+\delta}{s}}$. By Lemma 5 we get that

$$\|J_x^{1+\delta}[\omega(u_{\varepsilon_1}^2+u_{\varepsilon_1}u_{\varepsilon_2}+u_{\varepsilon_2}^2)]\|_{L^1_TL^2_{xy}} \lesssim \|J_x^{1+\delta}\omega\|_{L^\infty_TL^2_{xy}}(\|u_{\varepsilon_1}\|_{L^2_TL^\infty_{xy}}+\|u_{\varepsilon_2}\|_{L^2_TL^\infty_{xy}})$$
$$+\|\omega\|_{L^2_TL^\infty_{xy}}\|\phi\|_{H^{s,0}}(\|u_{\varepsilon_1}\|_{L^2_TL^\infty_{xy}}+\|u_{\varepsilon_2}\|_{L^2_TL^\infty_{xy}}).$$

By 2.9 we have $\|J_x^{1+\delta}\omega\|_{L^\infty_TL^2_{xy}}\lesssim \varepsilon_1^{s-1-\delta}h_\phi^{(3)}(\varepsilon_1)^{1-\frac{1+\delta}{s}}$. By combining the previous observations, we obtain

$$\|\omega\|_{L^2_TL^\infty_{xy}}\lesssim \varepsilon_1^{s-1-\delta}h_\phi^{(3)}(\varepsilon_1)^{1-\frac{1+\delta}{s}}$$
$$+\|\omega\|_{L^2_TL^\infty_{xy}}\|\phi\|_{H^{s,0}}(\|u_{\varepsilon_1}\|_{L^2_TL^\infty_{xy}}+\|u_{\varepsilon_2}\|_{L^2_TL^\infty_{xy}})$$

Since we consider that $\|\phi\|_{H^{s,0}}$ is small, such that $\|\phi\|_{H^{s,0}}(\|u_{\varepsilon_1}\|_{L^2_TL^\infty_{xy}}+\|u_{\varepsilon_2}\|_{L^2_TL^\infty_{xy}})\leq\frac{1}{2}$, we get that

$$\|\omega\|_{L^2_T L^\infty_{xy}} \lesssim \varepsilon_1^{s-1-\delta} h^{(3)}_\phi(\varepsilon_1)^{1-\frac{1+\delta}{s}} \to 0$$

as $\varepsilon_1 \to 0$ since $1+\delta < s$.

The linear estimate 2.3.1 applied to $\partial_x \omega$ results in

$$\|\partial_x \omega\|_{L^2_T L^\infty_{xy}} \lesssim \|J^{2+\delta}_x \omega\|_{L^\infty_T L^2_{xy}} + \|J^{2+\delta}_x [\omega(u^2_{\varepsilon_1} + u_{\varepsilon_1} u_{\varepsilon_2} + u^2_{\varepsilon_2})]\|_{L^1_T L^2_{xy}}.$$

and by the same reasoning as above

$$\begin{aligned}\|\partial_x \omega\|_{L^2_T L^\infty_{xy}} &\lesssim \varepsilon_1^{s-2-\delta} h^{(3)}_\phi(\varepsilon_1)^{1-\frac{2+\delta}{s}} \\ &\quad + \|\omega\|_{L^2_T L^\infty_{xy}} \|\phi\|_{H^{s,0}} (\|u_{\varepsilon_1}\|_{L^2_T L^\infty_{xy}} + \|u_{\varepsilon_2}\|_{L^2_T L^\infty_{xy}})\end{aligned}$$

which, combined with the above fact that $\|\omega\|_{L^2_T L^\infty_{xy}} \lesssim \varepsilon_1^{s-1-\delta} h^{(3)}_\phi(\varepsilon_1)^{1-\frac{1+\delta}{s}}$, for ε_1 small enough, it gives us $\|\partial_x \omega\|_{L^2_T L^\infty_{xy}} \lesssim \varepsilon_1^{s-2-\delta} \to 0$ as $\varepsilon_1 \to 0$ since $2+\delta < s$. $\square$

Corollary. *We have $\|\omega\|_{L^\infty_T H^{s,0}} \to 0$ as $\varepsilon_1 \to 0$, where $s > 2$ for the initial value problem (1.7).*

Proof. From (2.5) and Lemmas 6 we get $\|J^{s+1}_x u_k\|_{L^\infty_T L^2_{xy}} \|\omega\|_{L^2_T L^\infty_{xy}} \lesssim \varepsilon_1^{-1} \cdot \varepsilon_1^{1+} = \varepsilon_1^{0+}$ and $\varepsilon_1^{0+} \to 0$ as $\varepsilon_1 \to 0$. From Lemma 6 used in Lemma 3 we obtain

$$\|J^s_x \omega\|_{L^\infty_T L^2_{xy}} \lesssim \exp(\frac{1}{2} f_{u_k}(T)^2 + \frac{1}{2} f_\omega(T)^2)(\|J^s_x \omega(0)\|_{L^\infty_T L^2_{xy}} + C\varepsilon_1^{0+}) \to 0$$

as $\varepsilon_1 \to 0$, where we used that $\|J^s_x \omega(0)\|_{L^\infty_T L^2_{xy}} \to 0$ as $\varepsilon_1 \to 0$ and the boundedness of $f_{u_k}(T)$ and $f_\omega(T)$ by 1.

Therefore, as $\|J^s_x \omega\|_{L^\infty_T L^2_{xy}} \to 0$ as $\varepsilon_1 \to 0$, it means that $u \in C([0,T] : H^{s,0}(\mathbb{R}^2))$.

Hence u_ε is Cauchy in $H^{s,0}(\mathbb{R}^2)$, meaning that $u \in C([0,T]; H^{s,0}(\mathbb{R}^2))$. $\square$

2.7 Continuity of the flow map

We assume that $T \in [0,\infty)$ and $\phi^l \to \phi$ in $H^{s,0}(\mathbb{R}\times\mathbb{T})$ as $l\to\infty$. We are going to prove that $u^l \to u$ in $C([-T,T]:H^{s,0}(\mathbb{R}\times\mathbb{T}))$ as $l\to\infty$, where u^l and u are solutions of the the initial value problem $\partial_t u + \partial_x^3 u - \partial_x^{-1}\partial_y^2 u + u^2\partial_x u = 0$ corresponding to initial data ϕ^l and ϕ, for $s>2$.

For $\varepsilon>0$, let as before, ϕ_ε^l and $u_\varepsilon^l \in C([-T,T]:H^\infty)$ the corresponding solutions. Denote by $\omega_\varepsilon = u_\varepsilon - u$. By the same estimates from Lemma 3 and Lemma 6 applied to ω_ε we get

$$\|u_\varepsilon - u\|_{H^{s,0}} = \lesssim \exp(\frac{1}{2}f_{\omega_\varepsilon}(T)^2 + \frac{1}{2}f_{u_\varepsilon}(T)^2)(\|\phi_\varepsilon - \phi\|_{H^{s,0}} + C(T,\|\phi_\varepsilon\|_{H^{s,0}},\|\phi\|_{H^{s,0}})\varepsilon^{0+}).$$

By the same reasoning, we have that

$$\|u_\varepsilon^l - u^l\|_{H^{s,0}} \lesssim \exp(\frac{1}{2}f_{\omega_\varepsilon^l}(T)^2 + \frac{1}{2}f_{u_\varepsilon^l}(T)^2)(\|\phi_\varepsilon^l - \phi^l\|_{H^{s,0}} + C(T,\|\phi_\varepsilon^l\|_{H^{s,0}},\|\phi^l\|_{H^{s,0}})\varepsilon^{0+}).$$

Now, denote $\omega_\varepsilon^l = u_\varepsilon^l - u_\varepsilon$. By the same estimates from Lemma 3 and Lemma 6 applied to ω_ε^l

$$\|u_\varepsilon^l - u_\varepsilon\|_{H^{s,0}} \lesssim \exp(\frac{1}{2}f_{\omega_\varepsilon^l}(T)^2 + \frac{1}{2}f_{u_\varepsilon^l}(T)^2)(\|\phi_\varepsilon^l - \phi_\varepsilon\|_{H^{s,0}} + C(T,\|\phi_\varepsilon^l\|_{H^{s,0}},\|\phi_\varepsilon\|_{H^{s,0}})\varepsilon^{0+}).$$

By the boundedness of $f_{u_\varepsilon}(T)$, $f_{u_\varepsilon^l}(T)$, $f_{\omega_\varepsilon}(T)$ and $f_{\omega_\varepsilon^l}(T)$ by 1 and the triangle inequality, we get

$$\begin{aligned}
\|u^l - u\|_{H^{s,0}} &\le \|u_\varepsilon - u\|_{H^{s,0}} + \|u_\varepsilon^l - u_\varepsilon\|_{H^{s,0}} + \|u_\varepsilon^l - u^l\|_{H^{s,0}} \\
&\lesssim \|\phi_\varepsilon - \phi\|_{H^{s,0}} + \|\phi_\varepsilon^l - \phi_\varepsilon\|_{H^{s,0}} + \|\phi_\varepsilon^l - \phi^l\|_{H^{s,0}} \\
&+ C(T,\|\phi\|_{H^{s,0}},\|\phi_\varepsilon\|_{H^{s,0}},\|\phi^l\|_{H^{s,0}},\|\phi_\varepsilon^l\|_{H^{s,0}})\varepsilon^{0+}
\end{aligned}$$

which, by letting $\varepsilon \to 0$, we get $\|u^l - u\|_{H^{s,0}} \lesssim \|\phi^l - \phi\|_{H^{s,0}}$ and proves the continuity of the flow map.

CHAPTER 3

LOCAL WELL-POSEDNESS FOR THE PERIODIC AND PARTIALLY PERIODIC THIRD ORDER MODIFIED KP-I EQUATIONS

3.1 Notation and Preliminaries

We start by defining, for $g \in L^2(\mathbb{R} \times \mathbb{T})$, $\widehat{g}(\xi, n)$ denote its Fourier transform in both x and y. We define the Sobolev spaces which we will consider from now on: for $s_1, s_2 \geq 0$

$$H^{s_1,s_2}(\mathbb{R}\times\mathbb{T}) = \{g \in L^2(\mathbb{R}\times\mathbb{T}) : \|g\|_{H^{s_1,s_2}} = \|\widehat{g}(\xi,n)[(1+\xi^2)^{\frac{s_1}{2}} + (1+n^2)^{\frac{s_2}{2}}]\|_{L^2(\mathbb{R}\times\mathbb{Z})} < \infty\}$$

and for $s \geq 0$

$$H^s(\mathbb{R} \times \mathbb{T}) = \{g \in L^2(\mathbb{R} \times \mathbb{T}) : \|g\|_{H^s} = \|\widehat{g}(\xi, n)[(1 + \xi^2 + n^2)^{\frac{s}{2}}]\|_{L^2(\mathbb{R}\times\mathbb{Z})} < \infty\}$$

and so

$$H^\infty(\mathbb{R} \times \mathbb{T}) = \cap_{k=0}^{\infty} H^k(\mathbb{R} \times \mathbb{T}).$$

For $s \in \mathbb{R}$ we define the operators J_x^s, J_y^s by

$$\widehat{J_x^s g}(\xi, n) = (1 + \xi^2)^{\frac{s}{2}} \widehat{g}(\xi, n);$$

$$\widehat{J_y^s g}(\xi, n) = (1 + n^2)^{\frac{s}{2}} \widehat{g}(\xi, n)$$

on $\mathcal{S}'(\mathbb{R} \times \mathbb{T})$.

For $g \in L^2(\mathbb{T} \times \mathbb{T})$, $\widehat{g}(m, n)$ denote its Fourier transform in both x and y. In this case, we define similarly the Sobolev spaces which we will consider from now on: for $s_1, s_2 \geq 0$

$$H^{s_1,s_2}(\mathbb{T}\times\mathbb{T}) = \{g \in L^2(\mathbb{T}\times\mathbb{T}) : \|g\|_{H^{s_1,s_2}} = \|\widehat{g}(m,n)[(1+m^2)^{\frac{s_1}{2}} + (1+n^2)^{\frac{s_2}{2}}]\|_{L^2(\mathbb{Z}\times\mathbb{Z})} < \infty\}$$

and for $s \geq 0$

$$H^s(\mathbb{T}\times\mathbb{T}) = \{g \in L^2(\mathbb{T}\times\mathbb{T}) : \|g\|_{H^s} = \|\widehat{g}(m,n)[(1+m^2+n^2)^{\frac{s}{2}}]\|_{L^2(\mathbb{Z}\times\mathbb{Z})} < \infty\}$$

and so

$$H^\infty(\mathbb{T}\times\mathbb{T}) = \cap_{k=0}^{\infty} H^k(\mathbb{T}\times\mathbb{T}).$$

By slight abuse of notation, for $s \in \mathbb{R}$ we define the operators J_x^s, J_y^s by

$$\widehat{J_x^s g}(m,n) = (1+m^2)^{\frac{s}{2}}\widehat{g}(m,n);$$

$$\widehat{J_y^s g}(m,n) = (1+n^2)^{\frac{s}{2}}\widehat{g}(m,n)$$

on $\mathcal{S}'(\mathbb{T}\times\mathbb{T})$.

For any set A let $\mathbf{1}_A$ denote its the characteristic function. Given a Banach space X, a measurable function $u : \mathbb{R} \to X$, and an exponent $p \in [1,\infty]$, we define

$$\|u\|_{L^pX} = \Big[\int_{\mathbb{R}} (\|u(t)\|_X^p) dt\Big]^{\frac{1}{p}} \text{ if } p \in [1,\infty) \text{ and}$$
$$\|u\|_{L^\infty X} = \operatorname{esssup}_{t\in\mathbb{R}} \|u(t)\|_X$$

Also, if $I \subseteq \mathbb{R}$ is a measurable set, and $u : I \to X$ is a measurable function, we define

$$\|u\|_{L_I^pX} = \|\mathbf{1}_I(t)u\|_{L^pX}.$$

For $T \geq 0$, we define $\|u\|_{L_T^pX} = \|u\|_{L^p_{[-T,T]}X}$

We also introduce the Kato-Ponce commutator estimates (as in Lemma XI from [27] and Appendix 9.A from [20]):

Lemma 7. *(a) Let $m \geq 0$ and $f, g \in H^m(\mathbb{R})$. If $s \geq 1$ then*

$$\|J_{\mathbb{R}}^s(fg) - fJ_{\mathbb{R}}^s g\|_{L^2} \leq C_s[\|J_{\mathbb{R}}^s f\|_{L^2}\|g\|_{L^\infty} + (\|f\|_{L^\infty} + \|\partial f\|_{L^\infty})\|J_{\mathbb{R}}^{s-1} g\|_{L^2}].$$

and if $s \in (0,1)$ then

$$\|J_{\mathbb{R}}^s(fg) - fJ_{\mathbb{R}}^s g\|_{L^2} \leq C_s\|J_{\mathbb{R}}^s f\|_{L^2}\|g\|_{L^\infty}.$$

(b) Let $m > 0$ and $f, g \in H^m(\mathbb{T})$. If $s \geq 1$, then

$$\|J_{\mathbb{T}}^s(fg) - fJ_{\mathbb{T}}^s g\|_{L^2} \leq C_s[\|J_{\mathbb{T}}^s f\|_{L^2}\|g\|_{L^\infty} + (\|f\|_{L^\infty} + \|\partial f\|_{L^\infty})\|J_{\mathbb{T}}^{s-1} g\|_{L^2}].$$

and if $s \in (0,1)$ then

$$\|J_{\mathbb{T}}^s(fg) - fJ_{\mathbb{T}}^s g\|_{L^2} \leq C_s\|J_{\mathbb{T}}^s f\|_{L^2}\|g\|_{L^\infty}.$$

(c) Let $m \geq 0$ and $f, g \in H^m(M)$, where M is either $\mathbb{R}$ or $\mathbb{R}$. If $s > 0$ then

$$\|J_M^s(fg)\|_{L^2_{xy}} \leq C_s\|J_M^s f\|_{L^2}\|g\|_{L^\infty} + \|J_M^s g\|_{L^2}\|f\|_{L^\infty}.$$

We have the following corollary which we will use later.

Corollary. *Let M be either $\mathbb{R}$ or $\mathbb{T}$.*

(a) If $s \geq 1$, then we have $\|J_M^s(u^3)\| \lesssim \|u\|_{L^\infty}^2\|J_M^s u\|_{L^2} + \|u\|_{L^\infty}\|\partial u\|_{L^\infty}\|J_M^{s-1} u\|_{L^2}$.

(b) If $s \in (0,1)$, then we have $\|J_M^s(u^3)\| \lesssim \|u\|_{L^\infty}^2\|J_M^s u\|_{L^2}$.

3.2 Dispersive Estimates

For integers $k = 0, 1, \ldots$ we define the operators $Q_x^k, Q_y^k, \widetilde{Q}_x^k, \widetilde{Q}_y^k$ on $H^\infty(\mathbb{R} \times \mathbb{T})$ by

$$\widehat{Q_x^k g}(\xi, n) = \mathbf{1}_{[2^{k-1}, 2^k)}(|\xi|) \text{ if } k \geq 1$$

with

$$\widehat{Q_x^0 g}(\xi, n) = \mathbf{1}_{[0,1)}(|\xi|)$$

and

$$\widehat{Q_y^k g}(\xi, n) = \mathbf{1}_{[2^{k-1}, 2^k)}(|n|) \text{ if } k \geq 1$$

with

$$\widehat{Q_y^0 g}(\xi, n) = \mathbf{1}_{[0,1)}(|n|).$$

Also, $\widetilde{Q}_x^k = \sum_{k'=0}^k Q_x^{k'}, \widetilde{Q}_y^k = \sum_{k'=0}^k Q_y^{k'}, k \geq 1$.

By slight abuse of notation, we define the operators $Q_x^k, Q_y^k, \widetilde{Q}_x^k, \widetilde{Q}_y^k$ on $H^\infty(\mathbb{T} \times \mathbb{T})$ by

$$\widehat{Q_x^k g}(m, n) = \mathbf{1}_{[2^{k-1}, 2^k)}(|m|) \text{ if } k \geq 1$$

with

$$\widehat{Q_x^0 g}(m, n) = \mathbf{1}_{[0,1)}(|m|)$$

and

$$\widehat{Q_y^k g}(m, n) = \mathbf{1}_{[2^{k-1}, 2^k)}(|n|) \text{ if } k \geq 1$$

with

$$\widehat{Q_y^0 g}(m, n) = \mathbf{1}_{[2^{k-1}, 2^k)}(|n|).$$

Also, $\widetilde{Q}_x^k = \sum_{k'=0}^k Q_x^{k'}, \widetilde{Q}_y^k = \sum_{k'=0}^k Q_y^{k'}, k \geq 1$.

We are stating the dispersion estimates for the partially periodic and fully periodic cases that appear in Kenig and Ionescu [20].

Theorem 3.2.1. *For $t \in \mathbb{R}$ let $W_{(3)}(t)$ denote the operator on $H^\infty(\mathbb{R} \times \mathbb{T})$ defined by the Fourier multiplier $(\xi, n) \mapsto e^{i(\xi^3 + \frac{n^2}{\xi})t}$. Assume $\phi \in H^\infty(\mathbb{R} \times \mathbb{T})$. Then for any $\epsilon > 0$, we have*

$$\|W_{(3)}(t)\widetilde{Q}_y^{2j} Q_x^j \phi\|_{L^2_{2^{-j}} L^\infty_{xy}} \leq C_\epsilon 2^{\epsilon j} \|\tilde{Q}_y^{2j} Q_x^j \phi\|_{L^2_{xy}} \tag{3.1}$$

and

$$\|W_{(3)}(t) Q_y^{2j+k} Q_x^j \phi\|_{L^2_{2^{-j-k}} L^\infty_{xy}} \leq C_\epsilon 2^{\epsilon j} \|Q_y^{2j+k} Q_x^j \phi\|_{L^2_{xy}} \tag{3.2}$$

for any integers $j \geq 0$ and $k \geq 1$.

Theorem 3.2.2. *For $t \in \mathbb{R}$ let $\widetilde{W}_{(3)}(t)$ denote the operator on $H^\infty(\mathbb{T} \times \mathbb{T})$ defined by the Fourier multiplier $(m, n) \mapsto e^{i(m^3 + \frac{n^2}{m})t}$.Assume $\phi \in H^\infty(\mathbb{T} \times \mathbb{T})$. Then for any $\epsilon > 0$, we have*

$$\|\widetilde{W}_{(3)}(t)\widetilde{Q}_y^{2j} Q_x^j \phi\|_{L^2_{2^{-j}} L^\infty_{xy}} \leq C_\epsilon 2^{(\frac{3}{8} + \frac{\epsilon}{2})j} \|\tilde{Q}_y^{2j} Q_x^j \phi\|_{L^2_{xy}} \tag{3.3}$$

and

$$\|\widetilde{W}_{(3)}(t) Q_y^{2j+k} Q_x^j \phi\|_{L^2_{2^{-j-k}} L^\infty_{xy}} \leq C_\epsilon 2^{(\frac{3}{8} + \frac{\epsilon}{2})j} \|Q_y^{2j+k} Q_x^j \phi\|_{L^2_{xy}} \tag{3.4}$$

for any integers $j \geq 0$ and $k \geq 1$.

3.3 Linear Estimate

We continue by adapting the argument in [20] to get the linear estimates.

Proposition 2. *Assume $N \geq 4$, $u \in C^1([-T, T] : H^{-N-1}(\mathbb{R} \times \mathbb{T}))$, $f \in C([-T, T] : H^{-N}(\mathbb{R} \times \mathbb{T}))$ with $T \in [0, \frac{1}{2}]$ and*

$$[\partial_t + \partial_x^3 - \partial_x^{-1} \partial_y^2] u = \partial_x f \ \text{ on } \mathbb{R} \times \mathbb{T} \times [-T, T].$$

Then for any $\epsilon > 0$, we have

$$\|u\|_{L^2_T L^\infty_{xy}} \leq C_\epsilon [\|J_x^{1+\epsilon} u\|_{L^\infty_T L^2_{xy}} + \|J_x^{-1} J_y^{1+\epsilon} u\|_{L^\infty_T L^2_{xy}} + \|J_x^{1+\epsilon} J_y^\epsilon f\|_{L^1_T L^2_{xy}}].$$

Proof. Without loss of generality, we may assume that $u \in C^1([-T,T] : H^\infty(\mathbb{R}\times\mathbb{T}))$ and $f \in C([-T,T] : H^\infty(\mathbb{R}\times\mathbb{T}))$. It suffices to prove that if, for $\epsilon > 0$,

$$\|\widetilde{Q}_y^{2j} Q_x^j u\|_{L^2_T L^\infty_{xy}} \leq C_\epsilon 2^{-\frac{\epsilon j}{2}} \left[\|J_x^{1+\epsilon} u\|_{L^\infty_T L^2_{xy}} + \|J_x^{1+\epsilon} f\|_{L^1_T L^2_{xy}}\right] \tag{3.5}$$

and

$$\|Q_y^{2j+k} Q_x^j u\|_{L^2_T L^\infty_{xy}} \leq C_\epsilon 2^{-\frac{\epsilon(j+k)}{2}} \left[\|J_x^{-1} J_y^{1+\epsilon} u\|_{L^\infty_T L^2_{xy}} + \|J_x^1 J_y^\epsilon f\|_{L^1_T L^2_{xy}}\right] \tag{3.6}$$

for any integers $j \leq 0$ and $k \leq 1$. For (3.5), we partition the interval $[-T,T]$ into 2^j equal subintervals of length $2T2^{-j}$, denoted by $[a_{j,l}, a_{j,l+1}), l = 1,\ldots,2^j$. By Duhamel's formula, for $t \in [a_{j,l}, a_{j,l+1}]$,

$$u(t) = W_{(3)}(t - a_{j,l})[u(a_{j,l})] + \int_{a_{j,l}}^t W_{(3)}(t-s)[\partial_x f(s)]ds.$$

It follows from the dispersive estimate (3.1) that

$$\begin{aligned}
&\|\mathbf{1}_{[a_{j,l},a_{j,l+1})}(t)\widetilde{Q}_y^{2j} Q_x^j u\|_{L^2_T L^\infty_{xy}} \\
&\qquad \leq C_\epsilon \|\mathbf{1}_{[a_{j,l},a_{j,l+1})}(t) W_{(3)}(t-a_{j,l})\widetilde{Q}_y^{2j} Q_x^j u(a_{j,l})\|_{L^2_T L^\infty_{xy}} \\
&\qquad + C_\epsilon \|\mathbf{1}_{[a_{j,l},a_{j,l+1})}(t) \int_{a_{j,l}}^t W_{(3)}(s)\widetilde{Q}_y^{2j} Q_x^j \partial_x f(s) ds\|_{L^2_T L^\infty_{xy}} \\
&\qquad \lesssim C_\epsilon 2^{\frac{\epsilon j}{2}} \|\widetilde{Q}_y^{2j} Q_x^j u(a_{j,l})\|_{L^2_{xy}} \\
&\qquad + C_\epsilon 2^{\frac{\epsilon j}{2}} 2^j \|\mathbf{1}_{[a_{j,l},a_{j,l+1})}(t)\widetilde{Q}_y^{2j} Q_x^j f\|_{L^1_T L^2_{xy}}.
\end{aligned} \tag{3.7}$$

For the first term of the right-hand side of (3.7), we have

$$
\begin{aligned}
&\sum_{l=1}^{2^j} 2^{\frac{\epsilon j}{2}} \|\mathbf{1}_{[a_{j,l},a_{j,l+1})}(t)\widetilde{Q}_y^{2j} Q_x^j u(a_{j,l})\|_{L^2_{xy}} \\
&\qquad \lesssim \sum_{l=1}^{2^j} 2^{\frac{\epsilon j}{2}} 2^{-(1+\epsilon)j} \|\mathbf{1}_{[a_{j,l},a_{j,l+1})}(t)\widetilde{Q}_y^{2j} Q_x^j J_x^{1+\epsilon} u(a_{j,l})\|_{L^2_{xy}} \\
&\qquad \lesssim 2^j 2^{\frac{\epsilon j}{2}} 2^{-(1+\epsilon)j} \|\widetilde{Q}_y^{2j} Q_x^j J_x^{1+\epsilon} u\|_{L^\infty_T L^2_{xy}} \\
&\qquad \lesssim 2^{-\frac{\epsilon j}{2}} \|J_x^{1+\epsilon} u\|_{L^\infty_T L^2_{xy}}.
\end{aligned} \tag{3.8}
$$

For the second term of the right-hand side of (3.7) we have

$$
\begin{aligned}
&\sum_{l=1}^{2^j} 2^{\frac{\epsilon j}{2}} \|\mathbf{1}_{[a_{j,l},a_{j,l+1})}(t)\widetilde{Q}_y^{2j} Q_x^j f\|_{L^1_T L^2_{xy}} \\
&\qquad \lesssim \sum_{l=1}^{2^j} 2^{\frac{\epsilon j}{2}} 2^j 2^{-(1+\epsilon)j} \|\mathbf{1}_{[a_{j,l},a_{j,l+1})}(t)\widetilde{Q}_y^{2j} Q_x^j J_x^{1+\epsilon} f\|_{L^1_T L^2_{xy}} \\
&\qquad \lesssim 2^{-\frac{\epsilon j}{2}} \sum_{l=1}^{2^j} \|\mathbf{1}_{[a_{j,l},a_{j,l+1})}(t)\widetilde{Q}_y^{2j} Q_x^j J_x^{1+\epsilon} f\|_{L^1_T L^2_{xy}} \\
&\qquad \lesssim 2^{-\frac{\epsilon j}{2}} \|\widetilde{Q}_y^{2j} Q_x^j J_x^{1+\epsilon} f\|_{L^1_T L^2_{xy}} \\
&\qquad \lesssim 2^{-\frac{\epsilon j}{2}} \|J_x^{1+\epsilon} f\|_{L^1_T L^2_{xy}}.
\end{aligned} \tag{3.9}
$$

Therefore, (3.8) and (3.9) give (3.5).

For (3.6), we partition the interval $[-T,T]$ into 2^{j+k} equal subintervals of length $2T2^{-j-k}$, denoted by $[b_{j,l}, b_{j,l+1}), l = 1, \ldots, 2^{j+k}$. By Duhamel's formula, for $t \in [b_{j,l}, b_{j,l+1}]$,

$$
u(t) = W_{(3)}(t - b_{j,l})[u(b_{j,l})] + \int_{b_{j,l}}^t W_{(3)}(t-s)[\partial_x f(s)]ds.
$$

It follows from the dispersive estimate (3.2) that

$$\begin{aligned}
&\|\mathbf{1}_{[b_{j,l},b_{j,l+1})}(t)Q_y^{2j+k}Q_x^j u\|_{L_T^2L_{xy}^\infty} \\
&\qquad \le C_\epsilon\|\mathbf{1}_{[b_{j,l},b_{j,l+1})}(t)W_{(3)}(t-b_{j,l})Q_y^{2j+k}Q_x^j u(b_{j,l})\|_{L_T^2L_{xy}^\infty} \\
&\qquad + \|\mathbf{1}_{[b_{j,l},b_{j,l+1})}(t)\int_{b_{j,l}}^t W_{(3)}(s)Q_y^{2j+k}Q_x^j\partial_x f(s)ds\|_{L_T^2L_{xy}^\infty} \\
&\qquad \lesssim C_\epsilon 2^{\frac{\epsilon j}{2}}\|\mathbf{1}_{[b_{j,l},b_{j,l+1})}(t)Q_y^{2j+k}Q_x^j u(b_{j,l})\|_{L_{xy}^2} \\
&\qquad + C_\epsilon 2^{\frac{\epsilon j}{2}}2^j\|\mathbf{1}_{[b_{j,l},b_{j,l+1})}(t)Q_y^{2j+k}Q_x^j f\|_{L_T^1L_{xy}^2}.
\end{aligned} \tag{3.10}$$

For the first term of the right-hand side of (3.10), we have

$$\begin{aligned}
&\sum_{l=1}^{2^{j+k}} 2^{\frac{\epsilon j}{2}}\|\mathbf{1}_{[b_{j,l},b_{j,l+1})}(t)Q_y^{2j+k}Q_x^j u(b_{j,l})\|_{L_{xy}^2} \\
&\quad \lesssim \sum_{l=1}^{2^{j+k}} 2^{\frac{\epsilon j}{2}}2^j2^{-(1+\frac{\epsilon}{2})(2j+k)}\|\mathbf{1}_{[b_{j,l},b_{j,l+1})}(t)Q_y^{2j+k}Q_x^jJ_x^{-1}J_y^{1+\frac{\epsilon}{2}}u(b_{j,l})\|_{L_{xy}^2} \\
&\quad \lesssim 2^{j+k}2^{\frac{\epsilon j}{2}}2^j2^{-(1+\frac{\epsilon}{2})(2j+k)}\|Q_y^{2j+k}Q_x^jJ_x^{-1}J_y^{1+\frac{\epsilon}{2}}u\|_{L_T^\infty L_{xy}^2} \\
&\quad \lesssim 2^{-\frac{\epsilon(j+k)}{2}}\|J_x^{-1}J_y^{1+\frac{\epsilon}{2}}u\|_{L_T^\infty L_{xy}^2}.
\end{aligned} \tag{3.11}$$

For the second term of the right-hand side of (3.10) we have

$$\begin{aligned}
&\sum_{l=1}^{2^{j+k}} 2^{\frac{\epsilon j}{2}}2^j\|\mathbf{1}_{[b_{j,l},b_{j,l+1})}(t)Q_y^{2j+k}Q_x^j f\|_{L_T^1L_{xy}^2} \\
&\qquad \lesssim \sum_{l=1}^{2^{j+k}} 2^{\frac{\epsilon j}{2}}2^j2^{-j}2^{-\frac{\epsilon}{2}(2j+k)}\|\mathbf{1}_{[b_{j,l},b_{j,l+1})}(t)Q_y^{2j+k}Q_x^jJ_x^1J_y^{\frac{\epsilon}{2}}f\|_{L_T^1L_{xy}^2} \\
&\qquad \lesssim 2^{-\frac{\epsilon(j+k)}{2}}\sum_{l=1}^{2^{j+k}}\|\mathbf{1}_{[b_{j,l},b_{j,l+1})}(t)Q_y^{2j+k}Q_x^jJ_x^1J_y^{\frac{\epsilon}{2}}f\|_{L_T^1L_{xy}^2} \\
&\qquad \lesssim 2^{-\frac{\epsilon(j+k)}{2}}\|Q_y^{2j+k}Q_x^jJ_x^1J_y^{\frac{\epsilon}{2}}f\|_{L_T^1L_{xy}^2} \\
&\qquad \lesssim 2^{-\frac{\epsilon(j+k)}{2}}\|J_x^1J_y^{\frac{\epsilon}{2}}f\|_{L_T^1L_{xy}^2}.
\end{aligned} \tag{3.12}$$

Therefore, (3.11) and (3.12) give (3.6). $\square$

Proposition 3. *Assume* $N \geq 4$, $u \in C^1([-T,T] : H^{-N-1}(\mathbb{T}\times\mathbb{T}))$, $f \in C([-T,T] : H^{-N}(\mathbb{T}\times\mathbb{T}))$ *with* $T \in [0, \frac{1}{2}]$ *and*

$$[\partial_t + \partial_x^3 - \partial_x^{-1}\partial_y^2]u = \partial_x f \text{ on } \mathbb{T}\times\mathbb{T}\times[-T,T].$$

Then for any $\epsilon > 0$, *we have*

$$\|u\|_{L_T^2 L_{xy}^\infty} \leq C_\epsilon[\|J_x^{\frac{11}{8}+\epsilon}u\|_{L_T^\infty L_{xy}^2} + \|J_x^{-\frac{5}{8}}J_y^{1+\epsilon}u\|_{L_T^\infty L_{xy}^2} + \|J_x^{\frac{11}{8}+\epsilon}J_y^\epsilon f\|_{L_T^1 L_{xy}^2}].$$

Proof. Without loss of generality, we may assume that $u \in C^1([-T,T] : H^\infty(\mathbb{T}\times\mathbb{T}))$ and $f \in C([-T,T] : H^\infty(\mathbb{T}\times\mathbb{T}))$. It suffices to prove that if, for $\epsilon > 0$,

$$\|\widetilde{Q}_y^{2j}Q_x^j u\|_{L_T^2 L_{xy}^\infty} \leq C_\epsilon 2^{-\frac{\epsilon j}{2}}\left[\|J_x^{\frac{11}{8}+\epsilon}u\|_{L_T^\infty L_{xy}^2} + \|J_x^{\frac{11}{8}+\epsilon}f\|_{L_T^1 L_{xy}^2}\right] \tag{3.13}$$

and

$$\|Q_y^{2j+k}Q_x^j u\|_{L_T^2 L_{xy}^\infty} \leq C_\epsilon 2^{-\frac{\epsilon(j+k)}{2}}\left[\|J_x^{-\frac{5}{8}}J_y^{1+\epsilon}u\|_{L_T^\infty L_{xy}^2} + \|J_x^{\frac{11}{8}}J_y^\epsilon f\|_{L_T^1 L_{xy}^2}\right] \tag{3.14}$$

for any integers $j \leq 0$ and $k \leq 1$. For (3.13), we partition the interval $[-T,T]$ into 2^j equal subintervals of length $2T2^{-j}$, denoted by $[a_{j,l}, a_{j,l+1}), l = 1, \ldots, 2^j$. By Duhamel's formula, for $t \in [a_{j,l}, a_{j,l+1}]$,

$$u(t) = \widetilde{W}_{(3)}(t - a_{j,l})[u(a_{j,l})] + \int_{a_{j,l}}^t \widetilde{W}_{(3)}(t-s)[\partial_x f(s)]ds.$$

It follows from the dispersive estimate (3.3) that

$$\begin{aligned}
&\|\mathbf{1}_{[a_{j,l},a_{j,l+1})}(t)\widetilde{Q}_y^{2j}Q_x^j u\|_{L_T^2L_{xy}^\infty}\\
&\qquad\le C_\epsilon\|\mathbf{1}_{[a_{j,l},a_{j,l+1})}(t)\widetilde{W}_{(3)}(t-a_{j,l})\widetilde{Q}_y^{2j}Q_x^j u(a_{j,l})\|_{L_T^2L_{xy}^\infty}\\
&\qquad+C_\epsilon\|\mathbf{1}_{[a_{j,l},a_{j,l+1})}(t)\int_{a_{j,l}}^t \widetilde{W}_{(3)}(s)\widetilde{Q}_y^{2j}Q_x^j\partial_x f(s)ds\|_{L_T^2L_{xy}^\infty}\\
&\qquad\lesssim C_\epsilon 2^{(\frac{3}{8}+\frac{\epsilon}{2})j}\|\widetilde{Q}_y^{2j}Q_x^j u(a_{j,l})\|_{L_{xy}^2}\\
&\qquad+C_\epsilon 2^{(\frac{3}{8}+\frac{\epsilon}{2})j}2^j\|\mathbf{1}_{[a_{j,l},a_{j,l+1})}(t)\widetilde{Q}_y^{2j}Q_x^j f\|_{L_T^1L_{xy}^2}.
\end{aligned}\tag{3.15}$$

For the first term of the right-hand side of (3.15), we have

$$\begin{aligned}
&\sum_{l=1}^{2^j}2^{(\frac{3}{8}+\frac{\epsilon}{2})j}\|\mathbf{1}_{[a_{j,l},a_{j,l+1})}(t)\widetilde{Q}_y^{2j}Q_x^j u(a_{j,l})\|_{L_{xy}^2}\\
&\qquad\lesssim\sum_{l=1}^{2^j}2^{(\frac{3}{8}+\frac{\epsilon}{2})j}2^{-(\frac{11}{8}+\epsilon)j}\|\mathbf{1}_{[a_{j,l},a_{j,l+1})}(t)\widetilde{Q}_y^{2j}Q_x^jJ_x^{\frac{11}{8}+\epsilon}u(a_{j,l})\|_{L_{xy}^2}\\
&\qquad\lesssim 2^j2^{(\frac{3}{8}+\frac{\epsilon}{2})j}2^{-(\frac{11}{8}+\epsilon)j}\|\widetilde{Q}_y^{2j}Q_x^jJ_x^{\frac{11}{8}+\epsilon}u\|_{L_T^\infty L_{xy}^2}\\
&\qquad\lesssim 2^{-\frac{\epsilon j}{2}}\|J_x^{\frac{11}{8}+\epsilon}u\|_{L_T^\infty L_{xy}^2}.
\end{aligned}\tag{3.16}$$

For the second term of the right-hand side of (3.15) we have

$$\begin{aligned}
&\sum_{l=1}^{2^j}2^{(\frac{3}{8}+\frac{\epsilon}{2})j}\|\mathbf{1}_{[a_{j,l},a_{j,l+1})}(t)\widetilde{Q}_y^{2j}Q_x^j f\|_{L_T^1L_{xy}^2}\\
&\qquad\lesssim\sum_{l=1}^{2^j}2^{(\frac{3}{8}+\frac{\epsilon}{2})j}2^j2^{-(\frac{11}{8}+\epsilon)j}\|\mathbf{1}_{[a_{j,l},a_{j,l+1})}(t)\widetilde{Q}_y^{2j}Q_x^jJ_x^{\frac{11}{8}+\epsilon}f\|_{L_T^1L_{xy}^2}\\
&\qquad\lesssim 2^{-\frac{\epsilon j}{2}}\sum_{l=1}^{2^j}\|\mathbf{1}_{[a_{j,l},a_{j,l+1})}(t)\widetilde{Q}_y^{2j}Q_x^jJ_x^{\frac{11}{8}+\epsilon}f\|_{L_T^1L_{xy}^2}\\
&\qquad\lesssim 2^{-\frac{\epsilon j}{2}}\|\widetilde{Q}_y^{2j}Q_x^jJ_x^{\frac{11}{8}+\epsilon}f\|_{L_T^1L_{xy}^2}\\
&\qquad\lesssim 2^{-\frac{\epsilon j}{2}}\|J_x^{\frac{11}{8}+\epsilon}f\|_{L_T^1L_{xy}^2}.
\end{aligned}\tag{3.17}$$

Therefore, (3.16) and (3.17) give (3.13).

For (3.14), we partition the interval $[-T,T]$ into 2^{j+k} equal subintervals of length $2T2^{-j-k}$, denoted by $[b_{j,l}, b_{j,l+1}), l = 1, \dots, 2^{j+k}$. By Duhamel's formula, for $t \in [b_{j,l}, b_{j,l+1}]$,

$$u(t) = \widetilde{W}_{(3)}(t-b_{j,l})[u(b_{j,l})] + \int_{b_{j,l}}^{t} \widetilde{W}_{(3)}(t-s)[\partial_x f(s)]ds.$$

It follows from the dispersive estimate (3.4) that

$$\begin{aligned}
&\|\mathbf{1}_{[b_{j,l},b_{j,l+1})}(t)Q_y^{2j+k}Q_x^j u\|_{L_T^2 L_{xy}^\infty} \\
&\qquad \le C_\epsilon \|\mathbf{1}_{[b_{j,l},b_{j,l+1})}(t)\widetilde{W}_{(3)}(t-b_{j,l})Q_y^{2j+k}Q_x^j u(b_{j,l})\|_{L_T^2 L_{xy}^\infty} \\
&\qquad + \|\mathbf{1}_{[b_{j,l},b_{j,l+1})}(t)\int_{b_{j,l}}^{t} \widetilde{W}_{(3)}(s)Q_y^{2j+k}Q_x^j \partial_x f(s)ds\|_{L_T^2 L_{xy}^\infty} \\
&\qquad \lesssim C_\epsilon 2^{(\frac{3}{8}+\frac{\epsilon}{2})j}\|\mathbf{1}_{[b_{j,l},b_{j,l+1})}(t)Q_y^{2j+k}Q_x^j u(b_{j,l})\|_{L_{xy}^2} \\
&\qquad + C_\epsilon 2^{(\frac{3}{8}+\frac{\epsilon}{2})j}2^j\|\mathbf{1}_{[b_{j,l},b_{j,l+1})}(t)Q_y^{2j+k}Q_x^j f\|_{L_T^1 L_{xy}^2}.
\end{aligned} \tag{3.18}$$

For the first term of the right-hand side of (3.18), we have

$$\begin{aligned}
&\sum_{l=1}^{2^{j+k}} 2^{(\frac{3}{8}+\frac{\epsilon}{2})j}\|\mathbf{1}_{[b_{j,l},b_{j,l+1})}(t)Q_y^{2j+k}Q_x^j u(b_{j,l})\|_{L_{xy}^2} \\
&\quad \lesssim \sum_{l=1}^{2^{j+k}} 2^{(\frac{3}{8}+\frac{\epsilon}{2})j}2^{\frac{5}{8}j}2^{-(1+\frac{\epsilon}{2})(2j+k)}\|\mathbf{1}_{[b_{j,l},b_{j,l+1})}(t)Q_y^{2j+k}Q_x^j J_x^{-\frac{5}{8}}J_y^{1+\frac{\epsilon}{2}}u(b_{j,l})\|_{L_{xy}^2} \\
&\quad \lesssim 2^{j+k}2^{(\frac{3}{8}+\frac{\epsilon}{2})j}2^{\frac{5}{8}j}2^{-(1+\frac{\epsilon}{2})(2j+k)}\|Q_y^{2j+k}Q_x^j J_x^{-\frac{5}{8}}J_y^{1+\frac{\epsilon}{2}}u\|_{L_T^\infty L_{xy}^2} \\
&\quad \lesssim 2^{-\frac{\epsilon(j+k)}{2}}\|J_x^{-\frac{5}{8}}J_y^{1+\frac{\epsilon}{2}}u\|_{L_T^\infty L_{xy}^2}.
\end{aligned} \tag{3.19}$$

For the second term of the right-hand side of (3.18) we have

$$
\begin{aligned}
&\sum_{l=1}^{2^{j+k}} 2^{(\frac{3}{8}+\frac{\epsilon}{2})j}2^j\|\mathbf{1}_{[b_{j,l},b_{j,l+1})}(t)Q_y^{2j+k}Q_x^j f\|_{L_T^1L_{xy}^2}\\
&\qquad\lesssim \sum_{l=1}^{2^{j+k}} 2^{(\frac{3}{8}+\frac{\epsilon}{2})j}2^j2^{-\frac{11}{8}j}2^{-\frac{\epsilon}{2}(2j+k)}\|\mathbf{1}_{[b_{j,l},b_{j,l+1})}(t)Q_y^{2j+k}Q_x^jJ_x^{\frac{11}{8}}J_y^{\frac{\epsilon}{2}}f\|_{L_T^1L_{xy}^2}\\
&\qquad\lesssim 2^{-\frac{\epsilon(j+k)}{2}}\sum_{l=1}^{2^{j+k}}\|\mathbf{1}_{[b_{j,l},b_{j,l+1})}(t)Q_y^{2j+k}Q_x^jJ_x^{\frac{11}{8}}J_y^{\frac{\epsilon}{2}}f\|_{L_T^1L_{xy}^2}\\
&\qquad\lesssim 2^{-\frac{\epsilon(j+k)}{2}}\|Q_y^{2j+k}Q_x^jJ_x^{\frac{11}{8}}J_y^{\frac{\epsilon}{2}}f\|_{L_T^1L_{xy}^2}\\
&\qquad\lesssim 2^{-\frac{\epsilon(j+k)}{2}}\|J_x^{\frac{11}{8}}J_y^{\frac{\epsilon}{2}}f\|_{L_T^1L_{xy}^2}.
\end{aligned}
\tag{3.20}
$$

Therefore, (3.19) and (3.20) give (3.14). □

3.4 A Priori Estimates

We are going to bound $f_u(T)=\|u\|_{L_T^2L_{xy}^\infty}+\|\partial_xu\|_{L_T^2L_{xy}^\infty}+\|\partial_yu\|_{L_T^2L_{xy}^\infty}$.

Lemma 8. *Suppose $u\in C([-T,T]:H^{s,s}(M\times\mathbb{T}))$ satisfies the initial value problems (1.9) or (1.10), with initial data $\phi\in H^{s,s}(M\times\mathbb{T})$ (here, M is either $\mathbb{R}$ or $\mathbb{T}$). Then we have*

$$\|u\|_{L_T^\infty H^{s,s}}\lesssim\|\phi\|_{H^{s,s}}\,exp(3f_u(T)).$$

Proof. First,

$$\beta_f(t)=\|f(t)\|_{L_{xy}^\infty}^2+\|\partial_xf(t)\|_{L_{xy}^\infty}^2+\|\partial_yf(t)\|_{L_{xy}^\infty}^2$$

for a function f. If we apply to any of (1.9), (1.10), the operator J_x^s and then we multiply by $J_x^s u$, we get by integration by parts

$$\frac{d}{dt}\|J_x^s u\|_{L^2_{xy}}^2 = \int J_x^s u J_x^s(u^2\partial_x u) = \int J_x^s u[J_x^s(u^2\partial_x u) - u^2 J_x^s \partial_x u] + \int u^2 J_x^s u J_x^s \partial_x u$$
$$\lesssim \|J_x^s u\|_{L^2_{xy}}^2(\|u\|_{L^\infty_{xy}}^2 + \|u\|_{L^\infty_{xy}}\|\partial_x u\|_{L^\infty_{xy}}) \lesssim \|J_x^s u\|_{L^2_{xy}}^2 \beta_u(t)$$

therefore, by Grönwall's inequality, we get that

$$\|J_x^s u\|_{L^\infty_T L^2_{xy}} \lesssim \|J_x^s \phi\|_{L^2_{xy}} \exp(f_u(T)^2). \tag{3.21}$$

Again, if we apply to any of (1.9), (1.10), the operator J_y^s and then we multiply by $J_y^s u$, we obtain integrating by parts,

$$\frac{d}{dt}\|J_y^s u\|_{L^2_{xy}}^2 = \int J_y^s u J_y^s(u^2\partial_x u) = \int J_y^s u[J_y^s(u^2\partial_x u) - u^2 J_y^s \partial_x u] + \int u^2 J_y^s u J_y^s \partial_x u$$

and we denote $(I) = \int J_y^s u[J_y^s(u^2\partial_x u) - u^2 J_y^s \partial_x u]$ and $(II) = \int u^2 J_y^s u J_y^s \partial_x u$. For the first term, by the Kato-Ponce commutator estimates we have

$$\begin{aligned}
(I) &\lesssim \|J_y^s u\|_{L^2_{xy}} \|J_y^s(u^2\partial_x u) - u^2 J_y^s \partial_x u\|_{L^2_{xy}} \\
&\lesssim \|J_y^s u\|_{L^2_{xy}}[\|\partial_x u\|_{L^2_{xy}}\|J_y^s u\|_{L^2_{xy}}\|u\|_{L^\infty_{xy}} + (\|u\|_{L^\infty_{xy}}^2 + \|u\|_{L^\infty_{xy}}\|\partial_y u\|_{L^\infty_{xy}})\|J_y^{s-1}\partial_x u\|_{L^2_{xy}}] \\
&\lesssim \|J_y^s u\|_{L^2_{xy}}^2(\|\partial_x u\|_{L^\infty_{xy}}\|u\|_{L^\infty_{xy}} + \|u\|_{L^\infty_{xy}}^2 + \|u\|_{L^\infty_{xy}}\|\partial_y u\|_{L^\infty_{xy}}) \\
&+ \|J_y^s u\|_{L^2_{xy}}\|J_x^s u\|_{L^2_{xy}}(\|u\|_{L^\infty_{xy}}^2 + \|u\|_{L^\infty_{xy}}\|\partial_y u\|_{L^\infty_{xy}}) \\
&\lesssim \|J_y^s\|_{L^2_{xy}}^2 \beta_u(t) + \|J_y^s u\|_{L^2_{xy}}\|J_x^s u\|_{L^2_{xy}}\beta_u(t) \\
&\lesssim \|J_y^s u\|_{L^2_{xy}}^2 \beta_u(t) + \|J_y^s u\|_{L^2_{xy}}\|\phi\|_{H^{s,s}} \exp(f_u(T)^2)\beta_u(t)
\end{aligned}$$

By integration by parts, we get that

$$(II) \lesssim \|J_y^s u\|_{L^2_{xy}}^2 \beta_u(t).$$

Therefore, we get

$$\frac{d}{dt}\|J_y^s u\|_{L^2_{xy}}^2 \lesssim \|J_y^s u\|^2 \beta_u(t) + \|J_y^s u\|_{L^2_{xy}} \|\phi\|_{H^{s,s}} \exp(f_u(T)^2)\beta_u(t)$$

so

$$\frac{d}{dt}\|J_y^s u\|_{L^2_{xy}} \lesssim (\|J_y^s u\|_{L^2_{xy}} + \|\phi\|_{H^{s,s}} \exp(f_u(T)^2)\beta_u(t)$$

hence, by Grönwall's inequality, we get

$$\begin{aligned} \|J_y^s u\|_{L^2_{xy}} &\lesssim (\|J_y^s \phi\|_{L^2_{xy}} + \|\phi\|_{H^{s,s}} \exp(f_u(T)^2)\exp(f_u(T)^2) \\ &\lesssim \|\phi\|_{H^{s,s}}(1+\exp(f_u(T)^2))\exp(f_u(T)^2) \end{aligned} \tag{3.22}$$

which yields that $\|u\|_{L^\infty_T H^{s,s}} \lesssim \|\phi\|_{H^{s,s}_{xy}} \exp(2f_u(T)^2)$. □

Proposition 4. *Let $s > 2$ and $u_0 \in H^{s,s}(\mathbb{R}\times\mathbb{T})$. Suppose $u \in C([-T,T] : H^{s,s}(\mathbb{R}\times\mathbb{T}))$ satisfies the IVP (1.9). Then $u, \partial_x u, \partial_y u \in L^2([-T,T]; L^\infty(\mathbb{R}\times\mathbb{T}))$. Moreover,*

$$f_u(T) = \|u\|_{L^2_T L^\infty_{xy}} + \|\partial_x u\|_{L^2_T L^\infty_{xy}} + \|\partial_y u\|_{L^2_T L^\infty_{xy}} \le C_T$$

for a suitable small T, if $\|u_0\|_{H^{s,s}}$ is small enough.

Proof. From now on $s > 2+2\delta$. By the linear estimate in Proposition 2, we have

$$\|u\|_{L^2_T L^\infty_{xy}} \lesssim \|J_x^{1+\delta} u\|_{L^\infty_T L^2_{xy}} + \|J_x^{-1} J_y^{1+\delta} u\|_{L^\infty_T L^2_{xy}} + \|J_x^{1+\delta} J_y^{\delta}(u^3)\|_{L^1_T L^2_{xy}}$$

and

$$\|\partial_x u\|_{L^2_T L^\infty_{xy}} \lesssim \|J_x^{2+\delta} u\|_{L^\infty_T L^2_{xy}} + \|J_y^{1+\delta} u\|_{L^\infty_T L^2_{xy}} + \|J_x^{2+\delta} J_y^{\delta}(u^3)\|_{L^1_T L^2_{xy}}.$$

We observe that

$$\|J_x^{2+\delta} u\|_{L^\infty_T L^2_{xy}} \le \|J_x^s u\|_{L^\infty_T L^2_{xy}} \le \|u\|_{L^\infty_T H^{s,s}_{xy}},$$

$$\|J_y^{1+\delta}u\|_{L_T^\infty L_{xy}^2}\lesssim \|J_y^s u\|_{L_T^\infty L_{xy}^2}\lesssim \|u\|_{L_T^\infty H_{xy}^{s,s}},$$

and

$$\begin{aligned}\|J_x^{2+\delta}J_y^\delta(u^3)\|_{L_T^\infty L_{xy}^2} &\lesssim \|J_x^{2+2\delta}(u^3)\|_{L_T^\infty L_{xy}^2}+\|J_y^{2+2\delta}(u^3)\|_{L_T^\infty L_{xy}^2}\\ &\lesssim \|J_x^s(u^3)\|_{L_T^\infty L_{xy}^2}+\|J_y^s(u^3)\|_{L_T^\infty L_{xy}^2}\end{aligned}$$

so by the corollary of the Kato-Ponce commutator estimates

$$\|J_x^s(u^3)\|_{L_{xy}^2}\lesssim \|J_x^s u\|_{L_{xy}^2}\|u\|_{L_{xy}^\infty}^2+\|J_x^s u\|_{L_{xy}^2}\|u\|_{L_{xy}^\infty}\|\partial_x u\|_{L_{xy}^\infty}\lesssim \beta_u(t)\|J_x^s u\|_{L_{xy}^2}$$

and

$$\|J_y^s(u^3)\|_{L_{xy}^2}\lesssim \|J_y^s u\|_{L_{xy}^2}\|u\|_{L_{xy}^\infty}^2+\|J_y^s u\|_{L_{xy}^2}\|u\|_{L_{xy}^\infty}\|\partial_y u\|_{L_{xy}^\infty}\lesssim \beta_u(t)\|J_y^s u\|_{L_{xy}^2}$$

so therefore

$$\|J_x^{2+\delta}J_y^\delta(u^3)\|_{L_T^1 L_{xy}^2}\lesssim f_u(T)^2\|u\|_{L_T^\infty H_{xy}^{s,s}}.$$

Also,

$$\|\partial_y u\|_{L_T^2 L_{xy}^\infty}\lesssim \|J_x^{1+\delta}\partial_y u\|_{L_T^\infty L_{xy}^2}+\|J_x^{-1}J_y^{2+\delta}u\|_{L_T^\infty L_{xy}^2}+\|J_x^{1+\delta}J_y^\delta\partial_y(u^3)\|_{L_T^1 L_{xy}^2}.$$

By the arithmetic-geometric inequality, we also have that

$$\begin{aligned}\|J_x^{1+\delta}\partial_y u\|_{L_T^\infty L_{xy}^2} &\lesssim \|J_x^s u\|_{L_T^\infty L_{xy}^2}+\|J_y^{\overline{\frac{s}{s-1-\delta}}}u\|_{L_T^\infty L_{xy}^2}\lesssim \|J_x^s u\|_{L_T^\infty L_{xy}^2}+\|J_y^s u\|_{L_T^\infty L_{xy}^2}\\ &\lesssim \|u\|_{L_T^\infty H_{xy}^{s,s}}\end{aligned}$$

the second inequality being true as $s-1-\delta>1$. We also have

$$\|J_x^{-1}J_y^{2+\delta}u\|_{L_T^\infty L_{xy}^2}\lesssim \|J_y^s u\|_{L_T^\infty L_{xy}^2}\lesssim \|u\|_{L_T^\infty H_{xy}^{s,s}}.$$

Since

$$\|J_x^{1+\delta}J_y^{\delta}\partial_y(u^3)\|_{L^2_{xy}} \lesssim \|J_x^{1+\delta}J_y^{1+\delta}(u^3)\|_{L^2_{xy}} \lesssim \|J_x^{2+2\delta}(u^3)\|_{L^\infty_T L^2_{xy}} + \|J_y^{2+2\delta}(u^3)\|_{L^\infty_T L^2_{xy}}$$

the corollary of the Kato-Ponce commutator estimates gives

$$\|J_x^{1+\delta}J_y^{\delta}\partial_y(u^3)\|_{L^2_{xy}} \lesssim \beta_u(t)(\|J_x^s u\|_{L^2_{xy}} + \|J_x^s u\|_{L^2_{xy}}).$$

So we get from the previous estimates $\|J_x^{1+\delta}\partial_y(u^3)\|_{L^1_T L^2_{xy}} \lesssim f_u(T)^2\|u\|_{L^\infty_T H^{s,s}_{xy}}$.

Hence,

$$\begin{aligned} f_u(T) &= \|u\|_{L^2_T L^\infty_{xy}} + \|\partial_x u\|_{L^2_T L^\infty_{xy}} + \|\partial_y u\|_{L^2_T L^\infty_{xy}} \\ &\lesssim \|u\|_{L^\infty_T H^{s,s}_{xy}}(1 + f_u(T)^2). \end{aligned}$$

Together by the previous lemma,

$$f_u(T) \lesssim \|\phi\|_{H^{s,s}}(1 + f_u(T)^2)\exp(2f_u(T)^2)$$

and therefore, if $\|\phi\|_{H^{s,s}_{xy}}$ is small enough, by a continuity argument, we get $f_u(T) \lesssim C$ for T sufficiently small. $\square$

Proposition 5. *Let $s > \frac{19}{8}$ and $u_0 \in H^{s,s}(\mathbb{T}\times\mathbb{T})$. Suppose $u \in C([-T,T] : H^{s,s}(\mathbb{T}\times\mathbb{T}))$ satisfies the IVP (1.10). Then $u, \partial_x u, \partial_y u \in L^2([-T,T]; L^\infty(\mathbb{T}\times\mathbb{T}))$. Moreover,*

$$f_u(T) = \|u\|_{L^2_T L^\infty_{xy}} + \|\partial_x u\|_{L^2_T L^\infty_{xy}} + \|\partial_y u\|_{L^2_T L^\infty_{xy}} \leq C_T$$

for a suitable small T, if $\|u_0\|_{H^{s,s}}$ is small enough.

Proof. From now on $s > \frac{19}{8} + 2\delta$.

By the linear estimate in Proposition 3, we have

$$\|u\|_{L^2_T L^\infty_{xy}} \lesssim \|J_x^{\frac{11}{8}+\delta} u\|_{L^\infty_T L^2_{xy}} + \|J_x^{-\frac{5}{8}} J_y^{1+\delta} u\|_{L^\infty_T L^2_{xy}} + \|J_x^{\frac{11}{8}+\delta} J_y^{\delta}(u^3)\|_{L^1_T L^2_{xy}}$$

and

$$\|\partial_x u\|_{L^2_T L^\infty_{xy}} \lesssim \|J_x^{\frac{19}{8}+\delta} u\|_{L^\infty_T L^2_{xy}} + \|J_x^{\frac{3}{8}} J_y^{1+\delta} u\|_{L^\infty_T L^2_{xy}} + \|J_x^{\frac{19}{8}+\delta} J_y^{\delta}(u^3)\|_{L^1_T L^2_{xy}}.$$

By the estimates,

$$\|J_x^{\frac{19}{8}+\delta} u\|_{L^\infty_T L^2_{xy}} \le \|J_x^s u\|_{L^\infty_T L^2_{xy}} \le \|u\|_{L^\infty_T H^{s,s}_{xy}},$$

$$\|J_x^{\frac{3}{8}} J_y^{1+\delta} u\|_{L^\infty_T L^2_{xy}} \lesssim \|J_x^{\frac{11}{8}+\delta} u\|_{L^\infty_T L^2_{xy}} + \|J_y^{\frac{11}{8}+\delta} u\|_{L^\infty_T L^2_{xy}} \lesssim \|u\|_{L^\infty_T H^{s,s}_{xy}},$$

and

$$\begin{aligned}\|J_x^{\frac{19}{8}+\delta} J_y^{\delta}(u^3)\|_{L^\infty_T L^2_{xy}} &\lesssim \|J_x^{\frac{19}{8}+2\delta}(u^3)\|_{L^\infty_T L^2_{xy}} + \|J_y^{\frac{19}{8}+2\delta}(u^3)\|_{L^\infty_T L^2_{xy}} \\ &\lesssim \|J_x^s(u^3)\|_{L^\infty_T L^2_{xy}} + \|J_y^s(u^3)\|_{L^\infty_T L^2_{xy}}\end{aligned}$$

so by the corollary of the Kato-Ponce commutator estimates

$$\|J_x^{\frac{19}{8}+2\delta}(u^3)\|_{L^2_{xy}} \lesssim \|J_x^s u\|_{L^2_{xy}} \|u\|^2_{L^\infty_{xy}} + \|J_x^s u\|_{L^2_{xy}} \|u\|_{L^\infty_{xy}} \|\partial_x u\|_{L^\infty_{xy}} \lesssim \beta_u(t) \|J_x^s u\|_{L^2_{xy}}$$

and

$$\|J_x^{\frac{19}{8}+2\delta}(u^3)\|_{L^2_{xy}} \lesssim \|J_y^s u\|_{L^2_{xy}} \|u\|^2_{L^\infty_{xy}} + \|J_y^s u\|_{L^2_{xy}} \|u\|_{L^\infty_{xy}} \|\partial_y u\|_{L^\infty_{xy}} \lesssim \beta_u(t) \|J_y^s u\|_{L^2_{xy}}$$

so therefore

$$\|J_x^{\frac{19}{8}+\delta} J_y^{\delta}(u^3)\|_{L^1_T L^2_{xy}} \lesssim f_u(T)^2 \|u\|_{L^\infty_T H^{s,s}_{xy}}.$$

Lastly,

$$\|\partial_y u\|_{L^2_T L^\infty_{xy}} \lesssim \|J_x^{\frac{11}{8}+\delta}\partial_y u\|_{L^\infty_T L^2_{xy}} + \|J_x^{-\frac{5}{8}} J_y^{2+\delta} u\|_{L^\infty_T L^2_{xy}} + \|J_x^{\frac{11}{8}+\delta} J_y^{\delta}\partial_y(u^3)\|_{L^1_T L^2_{xy}}.$$

By the arithmetic-geometric inequality, we also have that

$$\begin{aligned}\|J_x^{\frac{11}{8}+\delta}\partial_y u\|_{L^\infty_T L^2_{xy}} &\lesssim \|J_x^s u\|_{L^\infty_T L^2_{xy}} + \|J_y^{\overline{s-\frac{11}{8}-\delta}^{\,s}} u\|_{L^\infty_T L^2_{xy}} \\ &\lesssim \|J_x^s u\|_{L^\infty_T L^2_{xy}} + \|J_y^s u\|_{L^\infty_T L^2_{xy}} \lesssim \|u\|_{L^\infty_T H^{s,s}_{xy}}\end{aligned}$$

the second inequality being true as $s - \frac{11}{8} - \delta > 1$. We also have

$$\|J_x^{-\frac{5}{8}} J_y^{2+\delta} u\|_{L^\infty_T L^2_{xy}} \lesssim \|J_y^s u\|_{L^\infty_T L^2_{xy}} \lesssim \|u\|_{L^\infty_T H^{s,s}_{xy}}.$$

Since

$$\|J_x^{\frac{11}{8}+\delta} J_y^{\delta}\partial_y(u^3)\|_{L^2_{xy}} \lesssim \|J_x^{\frac{11}{8}+\delta} J_y^{1+\delta}(u^3)\|_{L^2_{xy}} \lesssim \|J_x^{\frac{19}{8}+2\delta}(u^3)\|_{L^\infty_T L^2_{xy}} + \|J_y^{\frac{19}{8}+2\delta}(u^3)\|_{L^\infty_T L^2_{xy}}$$

the corollary of the Kato-Ponce commutator estimates gives

$$\|J_x^{\frac{11}{8}+\delta} J_y^{\delta}\partial_y(u^3)\|_{L^2_{xy}} \lesssim \beta_u(t)(\|J_x^s u\|_{L^2_{xy}} + \|J_x^s u\|_{L^2_{xy}}).$$

So we get from the previous estimates $\|J_x^{\frac{11}{8}+\delta}\partial_y(u^3)\|_{L^1_T L^2_{xy}} \lesssim f_u(T)^2 \|u\|_{L^\infty_T H^{s,s}_{xy}}$.

Hence,

$$\begin{aligned} f_u(T) &= \|u\|_{L^2_T L^\infty_{xy}} + \|\partial_x u\|_{L^2_T L^\infty_{xy}} + \|\partial_y u\|_{L^2_T L^\infty_{xy}} \\ &\lesssim \|u\|_{L^\infty_T H^{s,s}_{xy}}(1 + f_u(T)^2).\end{aligned}$$

Together with the previous lemma

$$f_u(T) \lesssim \|\phi\|_{H^{s,s}}(1 + f_u(T)^2)\exp(2f_u(T)^2)$$

and therefore, if $\|\phi\|_{H^{s,s}_{xy}}$ is small enough, by a continuity argument, we get $f_u(T) \lesssim C$ for T sufficiently small. □

3.5 Existence and Uniqueness

We start by stating a well-known local-wellposedness result from Iorio and Nunes (see [22], Section 4):

Lemma 9. *Assume $\phi \in H^\infty(M \times \mathbb{T})$, where M is either $\mathbb{R}$ or $\mathbb{T}$. Then there is $T = T(\|\phi\|_{H^3}) > 0$ and a solution $u \in C([-T, T] : H^\infty(\mathbb{M} \times \mathbb{T}))$ of the initial value problem*

$$\begin{cases} \partial_t u + \partial_x^3 u - \partial_x^{-1}\partial_y^2 u + u^2\partial_x u = 0, \\ u(0, x, y) = \phi(x, y). \end{cases}$$

The proof for $\mathbb{R} \times \mathbb{T}$ and $\mathbb{T} \times \mathbb{T}$ is the same as the proof in [22] for $\mathbb{R} \times \mathbb{R}$.

We proceed to prove the local well-posedness result.

Theorem 3.5.1. *The initial value problem (1.9) is locally well-posed in $H^{s,s}(\mathbb{R} \times \mathbb{T}), s > 2$. More precisely, given $u_0 \in H^{s,s}(\mathbb{R} \times \mathbb{T}), s > 2$, there exists $T = T(\|u_0\|_{H^{s,s}})$ and a unique solution u to the IVP such that $u \in C([0, T] : H^{s,s}(\mathbb{R} \times \mathbb{T}))$, $u, \partial_x u, \partial_y u \in L^2_T L^\infty_{xy}$. Moreover, the mapping $u_0 \to u$ in $C([0, T] : H^{s,s}(\mathbb{R} \times \mathbb{T}))$ is continuous.*

Theorem 3.5.2. *The initial value problem (1.10) is locally well-posedness in $H^{s,s}(\mathbb{T}\times\mathbb{T}), s > \frac{19}{8}$. More precisely, given $u_0 \in H^{s,s}(\mathbb{T} \times \mathbb{T}), s > \frac{19}{8}$, there exists $T = T(\|u_0\|_{H^{s,s}})$ and a unique solution u to the IVP such that $u \in C([0, T] : H^{s,s}(\mathbb{T} \times \mathbb{T}))$, $u, \partial_x u, \partial_y u \in L^2_T L^\infty_{xy}$. Moreover, the mapping $u_0 \to u$ in $C([0, T] : H^{s,s}(\mathbb{T} \times \mathbb{T}))$ is continuous.*

We present the proof for existence and uniqueness in the case of the third order mKP-I on $\mathbb{R} \times \mathbb{T}$, since the other case is similar.

Proof. Let $u_0 \in H^{s,s}(\mathbb{R} \times \mathbb{T})$ and fixed $u_{0,\epsilon} \in H^{s,s}(\mathbb{R} \times \mathbb{T}) \cap H^{\infty}_{-1}(\mathbb{R} \times \mathbb{T})$ such that $\|u_0 - u_{0,\epsilon}\|_{H^{s,s}} \to 0$ and $\|u_{0,\epsilon}\|_{H^{s,s}} \leq 2\|u_0\|_{H^{s,s}}$.

We know by the Iorio-Nunes local well-posedness result that $u_{0,\epsilon}$ gives a unique solution u_ϵ. We have by the a priori bound that $\|u_\epsilon\|_{L^2_T L^\infty_{xy}} + \|\partial_x u_\epsilon\|_{L^2_T L^\infty_{xy}} + \|\partial_y u_\epsilon\|_{L^2_T L^\infty_{xy}} \leq C_T$ and by the previous result, $\sup_{0<t<T}\|u_\epsilon\|_{H^{s,s}} \leq C_T$.

Henceforth,

$$
\begin{aligned}
\partial_t \|u_\varepsilon - u_{\varepsilon'}\|^2_{L^2} &= \int (u_\varepsilon - u_{\varepsilon'})\partial_x\left(\frac{u_\varepsilon^3}{3} - \frac{u_{\varepsilon'}^3}{3}\right) \\
&= \int \partial_x(u_\varepsilon - u_{\varepsilon'}) \cdot (u_\varepsilon - u_{\varepsilon'}) \frac{u_\varepsilon^2 + u_\varepsilon u_{\varepsilon'} + u_{\varepsilon'}^2}{3} = \\
&= \int (u_\varepsilon - u_{\varepsilon'})^2 \partial_x\left[\frac{u_\varepsilon^2 + u_\varepsilon u_{\varepsilon'} + u_{\varepsilon'}^2}{3}\right] \\
&\leq \|u_\varepsilon - u_{\varepsilon'}\|^2_{L^2}(\|u_\varepsilon\|^2_{L^\infty_{xy}} + \|\partial_x u_\varepsilon\|^2_{L^\infty_{xy}} + \|u_{\varepsilon'}\|^2_{L^\infty_{xy}} + \|\partial_x u_{\varepsilon'}\|^2_{L^\infty_{xy}}) \\
&\leq (\beta_{u_\varepsilon}(t) + \beta_{u_{\varepsilon'}}(t))\|u_\varepsilon - u_{\varepsilon'}\|^2_{L^2}.
\end{aligned}
$$

and by Grönwall's inequality and the a priori estimate

$$
\|u_\epsilon - u_{\epsilon'}\|^2_{L^\infty_T L^2_{xy}} \lesssim_T \|u_{0,\epsilon} - u_{0,\epsilon'}\|^2_{L^2_{xy}},
$$

hence $\sup_{0<t<T}\|u_\epsilon - u_{\epsilon'}\|_{L^2_{xy}} \to 0$, hence we can find $u \in C([0,T] : H^{s',s'}(\mathbb{R}\times\mathbb{T})) \cap L^\infty([0,T] : H^{s,s}(\mathbb{R} \times \mathbb{T}))$ with $s' < s$. The fact that u is a solution of the IVP is clear now. Uniqueness also comes from the previous Grönwall's inequality. $\square$

3.6 Continuity with respect to time

We proceed by a standard Bona-Smith argument ([5]).

Definition 3.6.1. *For $\phi \in H^{s,s}(\mathbb{R}\times\mathbb{T})$ with $s > 2$, let $\phi_k = P^k_{(3)}\phi$ where $\widehat{P^k_{(3)}g}(\xi,n) = \widehat{g}(\xi,n)\cdot 1_{[0,k]}(|\xi|)\cdot 1_{[0,k]}(|n|)$. Let*

$$h^{(3)}_\phi(k) = \Big[\sum_{n\in\mathbb{Z}}\int_{|\xi|+|n|\geq k}|\widehat{\phi}(\xi,n)|^2[(1+\xi^2)^s+(1+n^2)^s]d\xi\Big]^{\frac{1}{2}}.$$

Clearly, $h^{(3)}_\phi$ is nondecreasing in k and $\lim_{k\to\infty} h^{(3)}_\phi(k) = 0$. By Plancherel,

$$\begin{aligned}\|\phi-\phi_k\|_{L^2_{xy}} = \|\widehat{\phi}-\widehat{\phi}_k\|_{L^2_{xy}} &= \Big[\sum_{|n|\geq k}\int_{|\xi|\geq k}|\widehat{\phi}(\xi,n)|^2 d\xi\Big]^{\frac{1}{2}}\\ &\leq \Big[\sum_{n\in\mathbb{Z}}\int_{|\xi|+|n|\geq k}|\widehat{\phi}(\xi,n)|\frac{(1+\xi^2)^s+(1+n^2)^s}{k^{2s}}\Big]^{\frac{1}{2}} \lesssim k^{-s}h^{(3)}_\phi(k)\end{aligned}$$

Definition 3.6.2. *For $\phi \in H^{s,s}(\mathbb{T}\times\mathbb{T})$ with $s > \frac{19}{8}$, let $\phi_k = \widetilde{P}^k_{(3)}\phi$ where $\widehat{\widetilde{P}^k_{(3)}g}(m,n) = \widehat{g}(m,n)\cdot 1_{[0,k]}(|m|)\cdot 1_{[0,k]}(|n|)$. Let $\widetilde{h}^{(3)}_\phi(k) = \Big[\sum_{m\in\mathbb{Z}}\sum_{\substack{n\in\mathbb{Z}\\ |m|+|n|\geq k}}|\widehat{\phi}(m,n)|^2[(1+m^2)^s+(1+n^2)^s]\Big]^{\frac{1}{2}}$.*

Clearly, $\widetilde{h}^{(3)}_\phi$ is nondecreasing in k and $\lim_{k\to\infty}\widetilde{h}^{(3)}_\phi(k) = 0$. By Plancherel,

$$\begin{aligned}\|\phi-\phi_k\|_{L^2_{xy}} = \|\widehat{\phi}-\widehat{\phi}_k\|_{L^2_{xy}} &= \Big[\sum_{|n|\geq k}\sum_{|m|\geq k}|\widehat{\phi}(m,n)|^2\Big]^{\frac{1}{2}}\\ &\leq \Big[\sum_{m,n\in\mathbb{Z},}\sum_{|m|+|n|\geq k}|\widehat{\phi}(m,n)|\frac{(1+m^2)^s+(1+n^2)^s}{k^{2s}}\Big]^{\frac{1}{2}}\\ &\lesssim k^{-s}\widetilde{h}^{(3)}_\phi(k)\end{aligned}$$

In all the cases, from their respective definitions, we have that, if $p \geq s$, then

$$\|J^p_x\phi_k\|_{L^2_{xy}} \lesssim C(T,\|\phi\|_{H^{s,s}})k^{p-s} \text{ and } \|J^p_y\phi_k\|_{L^2_{xy}} \lesssim C(T,\|\phi\|_{H^{s,s}})k^{p-s}.$$

Since $\phi_k \in H^\infty$, by local well-posedness result of Iorio and Nunes, they give rise to unique solutions u_k in H^∞. The above estimates together with (3.21) and (3.22), if $p \geq s$, we also have

$$\|J_x^p u_k\|_{L_T^\infty L_{xy}^2} \leq C(T, \|\phi\|_{H^{s,s}}) k^{p-s} \tag{3.23}$$

and

$$\|J_y^p u_k\|_{L_T^\infty L_{xy}^2} \leq C(T, \|\phi\|_{H^{s,s}}) k^{p-s}. \tag{3.24}$$

Denote $\omega = u_k - u_{k'}$ with $k < k'$. Now choose $0 \leq q \leq s$. By using that $\|\phi - \phi_k\|_{L_{xy}^2} \lesssim k^{-s} h_\phi^{(3)}(k)$ for the $\mathbb{R} \times \mathbb{T}$ case and $\|\phi - \phi_k\|_{L_{xy}^2} \lesssim k^{-s} \widetilde{h}_\phi^{(3)}(k)$ for the $\mathbb{T} \times \mathbb{T}$ case, together with the interpolation inequality,

$$\|J_x^q \omega\|_{L_T^\infty L_{xy}^2} \leq \|J_x^s \omega\|_{L_T^\infty L_{xy}^2}^{\frac{q}{s}} \|\omega\|_{L_T^\infty L_{xy}^2}^{1-\frac{q}{s}} \lesssim \|\omega\|_{L_T^\infty L_{xy}^2}^{1-\frac{q}{s}}$$

it yields

$$\|J_x^q \omega\|_{L_T^\infty L_{xy}^2} \lesssim k^{q-s} h_\phi^{(3)}(k)^{1-\frac{q}{s}} \tag{3.25}$$

respectively,

$$\|J_x^q \omega\|_{L_T^\infty L_{xy}^2} \lesssim k^{q-s} \widetilde{h}_\phi^{(3)}(k)^{1-\frac{q}{s}}. \tag{3.26}$$

Similarly, we get results for J_y, more precisely,

$$\|J_y^q \omega\|_{L_T^\infty L_{xy}^2} \lesssim k^{q-s} h_\phi^{(3)}(k)^{1-\frac{q}{s}} \tag{3.27}$$

respectively,

$$\|J_y^q \omega\|_{L_T^\infty L_{xy}^2} \lesssim k^{q-s} \widetilde{h}_\phi^{(3)}(k)^{1-\frac{q}{s}}. \tag{3.28}$$

Lemma 10. *We have the following estimates:*

a)

$$\begin{aligned}\|J_x^s\omega\|_{L_T^\infty L_{xy}^2} \lesssim exp(\frac{1}{2}f_\omega(T)^2+\frac{1}{2}f_{u_k}(T)^2)\Big[&\|J_x^s\omega(0)\|_{L_{xy}^2}\\
&+\|J_x^s\omega\|_{L_T^\infty L_{xy}^2}\|J_x^{s+1}u_k\|_{L_T^\infty L_{xy}^2}(\|\omega\|^2_{L_T^2L_{xy}^\infty}+\|\omega\|_{L_T^2L_{xy}^\infty}\|u_k\|_{L_T^2L_{xy}^\infty})\\
&+\|J_x^s\omega\|_{L_T^\infty L_{xy}^2}\|J_x^s u_k\|_{L_T^\infty L_{xy}^2}(\|\omega\|_{L_T^2L_{xy}^\infty}+\|\partial_x\omega\|_{L_T^2L_{xy}^\infty})\|u_k\|_{L_T^2L_{xy}^\infty}\\
&+\|J_x^s\omega\|_{L_T^\infty L_{xy}^2}\|J_x^s u_k\|_{L_T^\infty L_{xy}^2}\|\omega\|_{L_T^2L_{xy}^\infty}(\|\partial_x u_k\|_{L_T^2L_{xy}^\infty}+\|\partial_x\omega\|_{L_T^2L_{xy}^\infty})\\
&+\|J_x^s\omega\|_{L_T^\infty L_{xy}^2}\|J_x^s u_k\|_{L_T^\infty L_{xy}^2}\|\omega\|^2_{L_T^2L_{xy}^\infty}\Big].\end{aligned}$$

b)

$$\begin{aligned}\|J_y^s\omega\|_{L_T^\infty L_{xy}^2} \lesssim exp(\frac{1}{2}f_\omega(T)^2+\frac{1}{2}f_{u_k}(T)^2)\Big[&\|J_y^s\omega(0)\|_{L_{xy}^2}\\
&+\|J_y^s\omega\|_{L_T^\infty L_{xy}^2}\|J_y^s u_k\|_{L_T^\infty L_{xy}^2}\|\omega\|_{L_T^2L_{xy}^\infty}(\|\partial_x u_k\|_{L_T^2L_{xy}^\infty}+\|\partial_y u_k\|_{L_T^2L_{xy}^\infty})\\
&+\|J_y^s\omega\|_{L_T^\infty L_{xy}^2}\|J_y^s u_k\|_{L_T^\infty L_{xy}^2}\Big(\|u_k\|_{L_T^2L_{xy}^\infty}\|\partial_y\omega\|_{L_T^2L_{xy}^\infty}+f_\omega(T)^2\Big)\\
&+\|J_y^s\omega\|_{L_T^\infty L_{xy}^2}\|J_x^s u_k\|_{L_T^\infty L_{xy}^2}\|\omega\|_{L_T^2L_{xy}^\infty}(\|u_k\|_{L_T^2L_{xy}^\infty}+\|\partial_y u_k\|_{L_T^2L_{xy}^\infty})\\
&+\|J_y^s\omega\|_{L_T^\infty L_{xy}^2}\|J_x^s u_k\|_{L_T^\infty L_{xy}^2}\Big(\|u_k\|_{L_T^2L_{xy}^\infty}\|\partial_y\omega\|_{L_T^2L_{xy}^\infty}+f_\omega(T)^2\Big)\\
&+\|J_y^s\omega\|_{L_T^\infty L_{xy}^2}\|J_x^s\omega\|_{L_T^\infty L_{xy}^2}\|\omega\|_{L_T^2L_{xy}^\infty}(\|u_k\|_{L_T^2L_{xy}^\infty}+\|\partial_y u_k\|_{L_T^2L_{xy}^\infty})\\
&+\|J_y^s\omega\|_{L_T^\infty L_{xy}^2}\|J_x^s\omega\|_{L_T^\infty L_{xy}^2}\Big(\|u_k\|_{L_T^2L_{xy}^\infty}\|\partial_y\omega\|_{L_T^2L_{xy}^\infty}+f_\omega(T)^2\Big)\\
&+\|J_y^s\omega\|_{L_T^\infty L_{xy}^2}\|J_x^s\omega\|_{L_T^\infty L_{xy}^2}\|u_k\|_{L_T^2L_{xy}^\infty}\|\partial_y u_k\|_{L_T^2L_{xy}^\infty}\\
&+\|J_y^{s+1}u_k\|_{L_T^\infty L_{xy}^2}(\|\omega\|^2_{L_T^\infty L_{xy}^2}+\|\omega\|_{L_T^\infty L_{xy}^2}\|u_k\|_{L_T^\infty L_{xy}^2})\Big].\end{aligned}$$

Proof.

$$\partial_t\omega+\partial_x^3\omega-\partial_x^{-1}\partial_y^2\omega+\omega^2\partial_x\omega+3u_k^2\partial_x\omega+3u_k\omega\partial_x u_k-3u_k\omega\partial_x\omega-3\omega^2\partial_x u_k=0. \quad (3.29)$$

a) We apply J_x^s to (3.29) and then we multiply by $J_x^s\omega$, in order to get

$$\begin{aligned}\frac{d}{dt}\|J_x^s\omega\|_{L^2}^2 =& \int J_x^s(\omega^2\partial_x\omega)J_x^s\omega + 3\int J_x^s(u_k^2\partial_x\omega)J_x^s\omega + 3\int J_x^s(u_k\omega\partial_x u_k)J_x^s\omega \\ &- 3\int J_x^s(u_k\omega\partial_x\omega)J_x^s\omega - 3\int J_x^s(\omega^2\partial_x u_k)J_x^s\omega\end{aligned}$$

and we will analyze each term in the sum.

We have $(I) = \int J_x^s(\omega^2\partial_x\omega)J_x^s\omega$, $(II) = \int J_x^s(u_k^2\partial_x\omega)J_x^s\omega$, $(III) = \int J_x^s(u_k\omega\partial_x u_k)J_x^s\omega$, $(IV) = \int J_x^s(u_k\omega\partial_x\omega)J_x^s\omega$ and $(V) = \int J_x^s(\omega^2\partial_x u_k)J_x^s\omega$.

For $(I) = \int J_x^s(\omega^2\partial_x\omega)J_x^s\omega = \int[J_x^s(\omega^2\partial_x\omega) - \omega^2 J_x^s\partial_x\omega]J_x^s\omega + \int \omega^2 J_x^s\partial_x\omega J_x^s\omega$, and we will denote $(I)_1 = \int[J_x^s(\omega^2\partial_x\omega) - \omega^2 J_x^s\partial_x\omega]J_x^s\omega$ and $(I)_2 = \int \omega^2 J_x^s\partial_x\omega J_x^s\omega$. For the first one, we have by the Kato-Ponce commutator estimate

$$\begin{aligned}(I)_1 &\le \|J_x^s\omega\|_{L^2_{xy}}\|J_x^s(\omega^2\partial_x\omega) - \omega^2 J_x^s\partial_x\omega\|_{L^2_{xy}} \\ &\le \|J_x^s\omega\|_{L^2_{xy}}\|\omega\|_{L^\infty_{xy}}\Big[\|\partial_x\omega\|_{L^\infty_{xy}}\cdot\|J_x^s\omega\|_{L^2_{xy}} + (\|\omega\|_{L^\infty_{xy}} + \|\partial_x\omega\|_{L^\infty_{xy}})\|J_x^{s-1}\partial_x\omega\|_{L^2_{xy}}\Big] \\ &\le \|J_x^s\omega\|_{L^2_{xy}}^2\cdot\beta_\omega(t)\end{aligned}$$

and

$$(I)_2 \le \|J_x^s\omega\|_{L^2_{xy}}^2\|\omega\|_{L^\infty_{xy}}\|\partial_x\omega\|_{L^\infty_{xy}} \le \|J_x^s\omega\|_{L^2_{xy}}^2\beta_\omega(t)$$

so $(I) \lesssim \|J_x^s\omega\|_{L^2_{xy}}^2\beta_\omega(t)$.

Now, $(II) = \int J_x^s(u_k^2\partial_x\omega)J_x^s\omega = \int[J_x^s(u_k^2\partial_x\omega) - u_k^2 J_x^s\partial_x\omega]J_x^s\omega + \int u_k^2 J_x^s\partial_x\omega J_x^s\omega$ and we denote $(II)_1 = \int[J_x^s(u_k^2\partial_x\omega) - u_k^2 J_x^s\partial_x\omega]J_x^s\omega$ and $(II)_2 = \int u_k^2 J_x^s\partial_x\omega J_x^s\omega$. For the first term we have by the Kato-Ponce commutator estimate

$$\begin{aligned}(II)_1 &\lesssim \|J_x^s\omega\|_{L^2_{xy}}\cdot\|J_x^s(u_k^2\partial_x\omega) - u_k^2 J_x^s\partial_x\omega\|_{L^2_{xy}} \\ &\lesssim \|J_x^s\omega\|_{L^2_{xy}}\|u_k\|_{L^\infty_{xy}}\Big[\|\partial_x\omega\|_{L^\infty_{xy}}\|J_x^s u_k\|_{L^2_{xy}} + (\|u_k\|_{L^\infty_{xy}} + \|\partial_x u_k\|_{L^\infty_{xy}})\|J_x^{s-1}\partial_x\omega\|_{L^2_{xy}}\Big] \\ &\lesssim \|J_x^s\omega\|_{L^2_{xy}}^2\beta_{u_k}(t) + \|J_x^s\omega\|_{L^2_{xy}}\|J_x^s u_k\|_{L^2_{xy}}\|\partial_x\omega\|_{L^\infty_{xy}}\|u_k\|_{L^\infty_{xy}}\end{aligned}$$

Also, we have

$$(II)_2 \lesssim \|J_x^s\omega\|_{L^2_{xy}}^2 \|u_k\|_{L^\infty}\|\partial_x u_k\|_{L^\infty} \lesssim \|J_x^s\omega\|_{L^2_{xy}}^2 \beta_{u_k}(t)$$

Therefore,

$$(II) \lesssim \|J_x^s\omega\|_{L^2_{xy}}^2 \beta_{u_k}(t) + \|J_x^s\omega\|_{L^2_{xy}}\|J_x^s u_k\|_{L^2_{xy}}\|\partial_x\omega\|_{L^\infty_{xy}}\|u_k\|_{L^\infty_{xy}}.$$

Let $(III) = \int J_x^s(u_k\omega\partial_x u_k)J_x^s\omega = \int [J_x^s(u_k\omega\partial_x u_k) - u_k\omega J_x^s\partial_x u_k]J_x^s\omega + \int u_k\omega J_x^s\partial_x u_k J_x^s\omega$ and denote $(III)_1 = \int [J_x^s(u_k\omega\partial_x u_k) - u_k\omega J_x^s\partial_x u_k]J_x^s\omega$ and $(III)_2 = \int u_k\omega J_x^s\partial_x u_k J_x^s\omega$. We have by the Kato-Ponce commutator estimate

$$\begin{aligned}
(III)_1 &\lesssim \|J_x^s\omega\|_{L^2_{xy}}\|J_x^s(u_k\omega\partial_x u_k) - u_k\omega J_x^s\partial_x u_k\|_{L^2_{xy}} \\
&\lesssim \|J_x^s\omega\|_{L^2_{xy}}\Big[\|J_x^s u_k\|_{L^2_{xy}}\|\omega\|_{L^\infty_{xy}}\|\partial_x u_k\|_{L^\infty_{xy}} \\
&+ \|J_x^{s-1}\omega\|_{L^2_{xy}}(\|u_k\|_{L^\infty_{xy}}\|\partial_x u_k\|_{L^\infty_{xy}} + \|\partial_x u_k\|_{L^\infty_{xy}}^2) \\
&+ \|J_x^{s-1}\partial_x u_k\|_{L^2_{xy}}(\|\omega\|_{L^\infty_{xy}}\|\partial_x u_k\|_{L^\infty_{xy}} + \|\omega\|_{L^\infty_{xy}}\|u_k\|_{L^\infty_{xy}} + \|\partial_x\omega\|_{L^\infty_{xy}}\|u_k\|_{L^\infty_{xy}})\Big] \\
&\lesssim \|J_x^s\omega\|_{L^2_{xy}}^2\beta_{u_k}(t) + \|J_x^s\omega\|_{L^2_{xy}}\|J_x^s u_k\|_{L^2_{xy}}\|\omega\|_{L^\infty_{xy}}\|\partial_x u_k\|_{L^\infty_{xy}} \\
&+ \|J_x^s\omega\|_{L^2_{xy}}\|J_x^s u_k\|_{L^2_{xy}}\|\omega\|_{L^\infty_{xy}}\|u_k\|_{L^\infty_{xy}} + \|J_x^s\omega\|_{L^2_{xy}}\|J_x^s u_k\|_{L^2_{xy}}\|\partial_x\omega\|_{L^\infty_{xy}}\|u_k\|_{L^\infty_{xy}}
\end{aligned}$$

Also, $(III)_2 \lesssim \|J_x^s\omega\|_{L^2_{xy}}\|J_x^{s+1}u_k\|_{L^2_{xy}}\|\omega\|_{L^\infty_{xy}}\|u_k\|_{L^\infty_{xy}}$ and so therefore

$$\begin{aligned}
(III) &\lesssim \|J_x^s\omega\|_{L^2_{xy}}^2\beta_{u_k}(t) + \|J_x^s\omega\|_{L^2_{xy}}\|J_x^s u_k\|_{L^2_{xy}}\|\omega\|_{L^\infty_{xy}}\|\partial_x u_k\|_{L^\infty_{xy}} \\
&+ \|J_x^s\omega\|_{L^2_{xy}}\|J_x^s u_k\|_{L^2_{xy}}\|\omega\|_{L^\infty_{xy}}\|u_k\|_{L^\infty_{xy}} + \|J_x^s\omega\|_{L^2_{xy}}\|J_x^s u_k\|_{L^2_{xy}}\|\partial_x\omega\|_{L^\infty_{xy}}\|u_k\|_{L^\infty_{xy}} \\
&+ \|J_x^s\omega\|_{L^2_{xy}}\|J_x^{s+1}u_k\|_{L^2_{xy}}\|\omega\|_{L^\infty_{xy}}\|u_k\|_{L^\infty_{xy}}.
\end{aligned}$$

Again, $(IV) = \int J_x^s(u_k\omega\partial_x\omega)J_x^s\omega = \int [J_x^s(u_k\omega\partial_x\omega) - u_k\omega J_x^s\partial_x\omega]J_x^s\omega + \int u_k\omega J_x^s\partial_x\omega J_x^s\omega$

and we denote $(IV)_1 = \int [J_x^s(u_k\omega\partial_x\omega) - u_k\omega J_x^s\partial_x\omega]J_x^s\omega$ and $(IV)_2 = \int u_k\omega J_x^s\partial_x\omega J_x^s\omega$. We have by the Kato-Ponce commutator estimate

$$\begin{aligned}
(IV)_1 &\lesssim \|J_x^s\omega\|_{L^2_{xy}}\|J_x^s(u_k\omega\partial_x\omega) - u_k\omega J_x^s\partial_x\omega\|_{L^2_{xy}} \\
&\lesssim \|J_x^s\omega\|_{L^2_{xy}}\|\partial_x\omega\|_{L^\infty_{xy}}\Big[\|J_x^s\omega\|_{L^2_{xy}}(\|u_k\|_{L^\infty_{xy}}+\|\partial_x u_k\|_{L^\infty_{xy}}) + \|J_x^s u_k\|_{L^2_{xy}}\|\omega\|_{L^\infty_{xy}}\Big] \\
&+ \|J_x^s\omega\|_{L^2_{xy}}\Big[\|u_k\|_{L^\infty_{xy}}(\|\omega\|_{L^\infty_{xy}}+\|\partial_x\omega\|_{L^\infty_{xy}}) + \|\partial_x u_k\|_{L^\infty_{xy}}\|\omega\|_{L^\infty_{xy}})\|J_x^{s-1}\partial_x\omega\|_{L^2_{xy}} \\
&\lesssim \|J_x^s\omega\|^2_{L^2_{xy}}(\|u_k\|_{L^\infty_{xy}}+\|\partial_x u_k\|_{L^\infty_{xy}})(\|\partial_x\omega\|_{L^\infty_{xy}}+\|\omega\|_{L^\infty_{xy}}) \\
&+ \|J_x^s\omega\|_{L^2_{xy}}\|J_x^s u_k\|_{L^2_{xy}}\|\omega\|_{L^\infty_{xy}}\|\partial_x\omega\|_{L^\infty_{xy}}.
\end{aligned}$$

Also, $(IV)_2 \lesssim \|J_x^s\omega\|^2_{L^2_{xy}}(\|\partial_x u_k\|_{L^\infty_{xy}}\|\omega\|_{L^\infty_{xy}}+\|\partial_x\omega\|_{L^\infty_{xy}}\|u_k\|_{L^\infty_{xy}})$ and so therefore

$$\begin{aligned}
(IV) &\lesssim \|J_x^s\omega\|^2_{L^2_{xy}}(\|u_k\|_{L^\infty_{xy}}+\|\partial_x u_k\|_{L^\infty_{xy}})(\|\partial_x\omega\|_{L^\infty_{xy}}+\|\omega\|_{L^\infty_{xy}}) \\
&+ \|J_x^s\omega\|_{L^2_{xy}}\|J_x^s u_k\|_{L^2_{xy}}\|\omega\|_{L^\infty_{xy}}\|\partial_x\omega\|_{L^\infty_{xy}}.
\end{aligned}$$

Again, $(V) = \int J_x^s(\omega^2\partial_x u_k)J_x^s\omega = \int [J_x^s(\omega^2\partial_x u_k) - \omega^2 J_x^s\partial_x u_k]J_x^s\omega + \int \omega^2 J_x^s\partial_x u_k J_x^s\omega$ and we denote $(V)_1 = \int [J_x^s(\omega^2\partial_x u_k) - \omega^2 J_x^s\partial_x u_k]J_x^s\omega$ and $(V)_2 = \int \omega^2 J_x^s\partial_x u_k J_x^s\omega$. We have by the Kato-Ponce commutator estimate

$$\begin{aligned}
(V)_1 &\lesssim \|J_x^s\omega\|_{L^2_{xy}}\|J_x^s(\omega^2\partial_x u_k) - \omega^2 J_x^s\partial_x u_k\|_{L^2_{xy}} \\
&\lesssim \|J_x^s\omega\|_{L^2_{xy}}\|\omega\|_{L^\infty_{xy}}\Big[\|\partial_x u_k\|_{L^\infty_{xy}}\|J_x^s\omega\|_{L^2_{xy}}+(\|\omega\|_{L^\infty_{xy}}+\|\partial_x\omega\|_{L^\infty_{xy}})\|J_x^{s-1}\partial_x u_k\|_{L^2_{xy}}\Big] \\
&\lesssim \|J_x^s\omega\|^2_{L^2_{xy}}\|\omega\|_{L^\infty_{xy}}\|\partial_x u_k\|_{L^\infty_{xy}}+\|J_x^s\omega\|_{L^2_{xy}}\|J_x^s u_k\|_{L^2_{xy}}\|\omega\|_{L^\infty_{xy}}(\|\omega\|_{L^\infty_{xy}}+\|\partial_x\omega\|_{L^\infty_{xy}}).
\end{aligned}$$

Also, $(V)_2 \lesssim \|J_x^s\omega\|_{L^2_{xy}}\|J_x^{s+1}u_k\|_{L^2_{xy}}\|\omega\|^2_{L^\infty_{xy}}$ and so therefore

$$\begin{aligned}
(V) &\lesssim \|J_x^s\omega\|^2_{L^2_{xy}}(\beta_{u_k}(t) + \beta_\omega(t)) \\
&+ \|J_x^s\omega\|_{L^2_{xy}}\Big(\|J_x^s u_k\|_{L^2_{xy}}\|\omega\|_{L^\infty_{xy}}(\|\omega\|_{L^\infty_{xy}}+\|\partial_x\omega\|_{L^\infty_{xy}}) + \|J_x^{s+1}u_k\|_{L^2_{xy}}\|\omega\|^2_{L^\infty_{xy}}\Big).
\end{aligned}$$

Now, putting together all the terms we get that

$$\begin{aligned}
\frac{d}{dt}\|J_x^s\omega\|_{L^2_{xy}}^2 \lesssim (\|J_x^s\omega\|_{L^2_{xy}}^2)&(\beta_\omega(t)+\beta_{u_k}(t)) \\
&+\|J_x^s\omega\|_{L^2_{xy}}\|J_x^{s+1}u_k\|_{L^2_{xy}}(\|\omega\|_{L^\infty_{xy}}^2+\|\omega\|_{L^\infty_{xy}}\|u_k\|_{L^\infty_{xy}}) \\
&+\|J_x^s\omega\|_{L^2_{xy}}\|J_x^s u_k\|_{L^2_{xy}}(\|\omega\|_{L^\infty_{xy}}\|u_k\|_{L^\infty_{xy}}+\|\partial_x\omega\|_{L^\infty_{xy}}\|u_k\|_{L^\infty_{xy}}) \\
&+\|J_x^s\omega\|_{L^2_{xy}}\|J_x^s u_k\|_{L^2_{xy}}(\|\omega\|_{L^\infty_{xy}}\|\partial_x u_k\|_{L^\infty_{xy}}+\|\omega\|_{L^\infty_{xy}}\|\partial_x\omega\|_{L^\infty_{xy}}+\|\omega\|_{L^\infty_{xy}}^2)
\end{aligned} \tag{3.30}$$

We are using the following variant of Grönwall's inequality:

Lemma 11. *If $\alpha(t), \beta(t)$ are two non-negative functions, and $\frac{d}{dt}u(t) \leq u(t)\beta(t) + \alpha(t)$ for all $t \in [0,T]$ then*

$$u(t) \leq e^{\int_0^t \beta(s)ds}\Big(u(0) + \int_0^t \alpha(s)ds\Big).$$

By putting $u(t) = \|J_x^s\omega\|_{L^2_{xy}}$, $\beta(t) = \beta_\omega(t) + \beta_{u_k}(t) \geq 0$ and

$$\begin{aligned}
\alpha(t) = \|J_x^{s+1}u_k\|_{L^2_{xy}}&(\|\omega\|_{L^\infty_{xy}}^2+\|\omega\|_{L^\infty_{xy}}\|u_k\|_{L^\infty_{xy}}) \\
&+\|J_x^s u_k\|_{L^2_{xy}}(\|\omega\|_{L^\infty_{xy}}\|u_k\|_{L^\infty_{xy}}+\|\partial_x\omega\|_{L^\infty_{xy}}\|u_k\|_{L^\infty_{xy}}) \\
&+\|J_x^s u_k\|_{L^2_{xy}}(\|\omega\|_{L^\infty_{xy}}\|\partial_x u_k\|_{L^\infty_{xy}}+\|\omega\|_{L^\infty_{xy}}\|\partial_x\omega\|_{L^\infty_{xy}}+\|\omega\|_{L^\infty_{xy}}^2) \geq 0
\end{aligned}$$

by applying the lemma to (3.30) together with Cauchy-Schwarz we get

$$\begin{aligned}\|J_x^s\omega\|_{L_T^\infty L^2_{xy}} \lesssim \exp(\frac{1}{2}f_\omega(T)^2+\frac{1}{2}f_{u_k}(T)^2)\Big[&\|J_x^s\omega(0)\|_{L^2_{xy}}\\
&+\|J_x^s\omega\|_{L_T^\infty L^2_{xy}}\|J_x^{s+1}u_k\|_{L_T^\infty L^2_{xy}}(\|\omega\|^2_{L_T^2L^\infty_{xy}}+\|\omega\|_{L_T^2L^\infty_{xy}}\|u_k\|_{L_T^2L^\infty_{xy}})\\
&+\|J_x^s\omega\|_{L_T^\infty L^2_{xy}}\|J_x^s u_k\|_{L_T^\infty L^2_{xy}}(\|\omega\|_{L_T^2L^\infty_{xy}}+\|\partial_x\omega\|_{L_T^2L^\infty_{xy}})\|u_k\|_{L_T^2L^\infty_{xy}}\\
&+\|J_x^s\omega\|_{L_T^\infty L^2_{xy}}\|J_x^s u_k\|_{L_T^\infty L^2_{xy}}\|\omega\|_{L_T^2L^\infty_{xy}}(\|\partial_x u_k\|_{L_T^2L^\infty_{xy}}+\|\partial_x\omega\|_{L_T^2L^\infty_{xy}})\\
&+\|J_x^s\omega\|_{L_T^\infty L^2_{xy}}\|J_x^s u_k\|_{L_T^\infty L^2_{xy}}\|\omega\|^2_{L_T^2L^\infty_{xy}}\Big].\end{aligned}$$

b) We apply J_y^s to (3.29) and then we multiply by $J_y^s\omega$, in order to get

$$\begin{aligned}\frac{d}{dt}\|J_y^s\omega\|^2_{L^2}=&\int J_y^s(\omega^2\partial_x\omega)J_y^s\omega+3\int J_y^s(u_k^2\partial_x\omega)J_y^s\omega+3\int J_y^s(u_k\omega\partial_x u_k)J_y^s\omega\\
&-3\int J_y^s(u_k\omega\partial_x\omega)J_y^s\omega-3\int J_y^s(\omega^2\partial_x u_k)J_y^s\omega\end{aligned}$$

and we will analyze each term in the sum.

We have $(I)=\int J_y^s(\omega^2\partial_x\omega)J_y^s\omega$, $(II)=\int J_y^s(u_k^2\partial_x\omega)J_y^s\omega$, $(III)=\int J_y^s(u_k\omega\partial_x u_k)J_y^s\omega$, $(IV)=\int J_y^s(u_k\omega\partial_x\omega)J_y^s\omega$ and $(V)=\int J_y^s(\omega^2\partial_x u_k)J_y^s\omega$.

For $(I)=\int J_y^s(\omega^2\partial_x\omega)J_y^s\omega=\int[J_y^s(\omega^2\partial_x\omega)-\omega^2J_y^s\partial_x\omega]J_y^s\omega+\int\omega^2J_y^s\partial_x\omega J_y^s\omega$, and we will denote $(I)_1=\int[J_y^s(\omega^2\partial_x\omega)-\omega^2J_y^s\partial_x\omega]J_y^s\omega$ and $(I)_2=\int\omega^2J_y^s\partial_x\omega J_y^s\omega$. For the first one, we have by the Kato-Ponce commutator estimate

$$\begin{aligned}(I)_1&\leq\|J_y^s\omega\|_{L^2_{xy}}\|J_y^s(\omega^2\partial_x\omega)-\omega^2J_y^s\partial_x\omega\|_{L^2_{xy}}\\
&\leq\|J_y^s\omega\|_{L^2_{xy}}\|\omega\|_{L^\infty_{xy}}\Big[\|\partial_x\omega\|_{L^\infty_{xy}}\cdot\|J_y^s\omega\|_{L^2_{xy}}+(\|\omega\|_{L^\infty_{xy}}+\|\partial_y\omega\|_{L^\infty_{xy}})\|J_y^{s-1}\partial_x\omega\|_{L^2_{xy}}\Big]\\
&\leq\|J_y^s\omega\|_{L^2_{xy}}\Big[\|\partial_x\omega\|_{L^\infty_{xy}}\cdot\|J_y^s\omega\|_{L^2_{xy}}\|\omega\|_{L^\infty_{xy}}\\
&+(\|\omega\|^2_{L^\infty_{xy}}+\|\omega\|_{L^\infty}\|\partial_y\omega\|_{L^\infty_{xy}})(\|J_y^s\omega\|_{L^2_{xy}}+\|J_x^s\omega\|_{L^2_{xy}})\Big]\\
&\leq(\|J_y^s\omega\|^2_{L^2_{xy}}+\|J_y^s\omega\|_{L^2_{xy}}\|J_x^s\omega\|_{L^2_{xy}})\cdot\beta_\omega(t)\end{aligned}$$

and

$$(I)_2 \le \|J_y^s\omega\|_{L^2_{xy}}^2 \|\omega\|_{L^\infty_{xy}} \|\partial_x\omega\|_{L^\infty_{xy}} \le \|J_y^s\omega\|_{L^2_{xy}}^2 \beta_\omega(t)$$

so $(I) \lesssim (\|J_y^s\omega\|_{L^2_{xy}}^2 + \|J_y^s\omega\|_{L^2_{xy}} \|J_x^s\omega\|_{L^2_{xy}})\beta_\omega(t)$.

Now, $(II) = \int J_y^s(u_k^2\partial_x\omega)J_y^s\omega = \int [J_y^s(u_k^2\partial_x\omega) - u_k^2 J_y^s\partial_x\omega]J_y^s\omega + \int u_k^2 J_y^s\partial_x\omega J_y^s\omega$ and we denote $(II)_1 = \int [J_y^s(u_k^2\partial_x\omega) - u_k^2 J_y^s\partial_x\omega]J_y^s\omega$ and $(II)_2 = \int u_k^2 J_y^s\partial_x\omega J_y^s\omega$. For the first term we have by the Kato-Ponce commutator estimate

$$\begin{aligned}
(II)_1 &\lesssim \|J_y^s\omega\|_{L^2_{xy}} \cdot \|J_y^s(u_k^2\partial_x\omega) - u_k^2 J_y^s\partial_x\omega\|_{L^2_{xy}} \\
&\lesssim \|J_y^s\omega\|_{L^2_{xy}} \|u_k\|_{L^\infty_{xy}} \Big[\|\partial_x\omega\|_{L^\infty_{xy}} \|J_y^s u_k\|_{L^2_{xy}} + (\|u_k\|_{L^\infty_{xy}} + \|\partial_y u_k\|_{L^\infty}) \|J_x^{s-1}\partial_x\omega\|_{L^2_{xy}}\Big] \\
&\lesssim \|J_y^s\omega\|_{L^2_{xy}}^2 \beta_{u_k}(t) + \|J_y^s\omega\|_{L^2_{xy}} \|J_y^s u_k\|_{L^2_{xy}} \|\partial_y\omega\|_{L^\infty_{xy}} \|u_k\|_{L^\infty_{xy}} \\
&+ \|J_x^s\omega\|_{L^2_{xy}} \|J_y^s\omega\|_{L^2_{xy}} \|\partial_y u_k\|_{L^\infty_{xy}} \|u_k\|_{L^\infty_{xy}}
\end{aligned}$$

where here we used that $\|J_x^{s-1}\partial_x\omega\|_{L^2_{xy}} \le |J_y^s\omega\|_{L^2_{xy}} + |J_x^s\omega\|_{L^2_{xy}}$. Also, we have

$$(II)_2 \lesssim \|J_y^s\omega\|_{L^2_{xy}}^2 \|u_k\|_{L^\infty} \|\partial_x u_k\|_{L^\infty} \lesssim \|J_y^s\omega\|_{L^2_{xy}}^2 \beta_{u_k}(t)$$

Therefore,

$$\begin{aligned}
(II) &\lesssim \|J_y^s\omega\|_{L^2_{xy}}^2 \beta_{u_k}(t) + \|J_y^s\omega\|_{L^2_{xy}} \|J_y^s u_k\|_{L^2_{xy}} \|\partial_y\omega\|_{L^\infty_{xy}} \|u_k\|_{L^\infty_{xy}} \\
&+ \|J_x^s\omega\|_{L^2_{xy}} \|J_y^s\omega\|_{L^2_{xy}} \|\partial_y u_k\|_{L^\infty_{xy}} \|u_k\|_{L^\infty_{xy}}
\end{aligned}$$

Let $(III) = \int J_y^s(u_k\omega\partial_x u_k)J_y^s\omega = \int [J_y^s(u_k\omega\partial_x u_k) - u_k\omega J_y^s\partial_x u_k]J_y^s\omega + \int u_k\omega J_y^s\partial_x u_k J_y^s\omega$ and denote $(III)_1 = \int [J_y^s(u_k\omega\partial_x u_k) - u_k\omega J_y^s\partial_x u_k]J_y^s\omega$ and $(III)_2 = \int u_k\omega J_y^s\partial_x u_k J_y^s\omega$.

We have by the Kato-Ponce commutator estimate

$$
\begin{aligned}
(III)_1 &\lesssim \|J_y^s\omega\|_{L^2_{xy}}\|J_y^s(u_k\omega\partial_x u_k) - u_k\omega J_y^s\partial_x u_k\|_{L^2_{xy}} \\
&\lesssim \|J_y^s\omega\|_{L^2_{xy}}\Big[\|J_y^s u_k\|_{L^2_{xy}}\|\omega\|_{L^\infty_{xy}}\|\partial_x u_k\|_{L^\infty_{xy}} \\
&+ \|J_y^{s-1}\omega\|_{L^2_{xy}}(\|\partial_y u_k\|_{L^\infty_{xy}}\|\partial_x u_k\|_{L^\infty_{xy}}+\|u_k\|_{L^\infty_{xy}}\|\partial_x u_k\|_{L^\infty_{xy}}) \\
&+ \|J_y^{s-1}\partial_x u_k\|_{L^2_{xy}}(\|\omega\|_{L^\infty_{xy}}\|\partial_y u_k\|_{L^\infty_{xy}}+\|u_k\|_{L^\infty_{xy}}\|\partial_y\omega\|_{L^\infty_{xy}}) \\
&+ \|J_y^s\omega\|_{L^2_{xy}}\|u_k\|_{L^\infty_{xy}}\|\partial_x u_k\|_{L^\infty_{xy}}\Big] \\
&\lesssim \|J_y^s\omega\|^2_{L^2_{xy}}\beta_{u_k}(t) + \|J_y^s\omega\|_{L^2_{xy}}\|J_x^s u_k\|_{L^2_{xy}}(\|\omega\|_{L^\infty_{xy}}\|\partial_y u_k\|_{L^\infty_{xy}}+\|u_k\|_{L^\infty_{xy}}\|\partial_y\omega\|_{L^\infty_{xy}}) \\
&+ \|J_y^s\omega\|_{L^2_{xy}}\|J_y^s u_k\|_{L^2_{xy}}\Big[\|\omega\|_{L^\infty_{xy}}(\|\partial_x u_k\|_{L^\infty_{xy}}+\|\|\partial_y u_k\|_{L^\infty_{xy}}) + \|u_k\|_{L^\infty_{xy}}\|\partial_y\omega\|_{L^\infty_{xy}}\Big].
\end{aligned}
$$

Also,

$$
(III)_2 \lesssim \|J_y^s\omega\|_{L^2_{xy}}\|J_y^{s+1}u_k\|_{L^2_{xy}}\|\omega\|_{L^\infty_{xy}}\|u_k\|_{L^\infty_{xy}}+\|J_y^s\omega\|_{L^2_{xy}}\|J_x^s u_k\|_{L^2_{xy}}\|\omega\|_{L^\infty_{xy}}\|u_k\|_{L^\infty_{xy}}
$$

and so therefore

$$
\begin{aligned}
(III) &\lesssim \|J_y^s\omega\|^2_{L^2_{xy}}\beta_{u_k}(t) + \|J_y^s\omega\|_{L^2_{xy}}\|J_y^{s+1}u_k\|_{L^2_{xy}}\|\omega\|_{L^\infty_{xy}}\|u_k\|_{L^\infty_{xy}} \\
&+ \|J_y^s\omega\|_{L^2_{xy}}\|J_y^s u_k\|_{L^2_{xy}}(\|\omega\|_{L^\infty_{xy}}\|\partial_x u_k\|_{L^\infty_{xy}}+\|\omega\|_{L^\infty_{xy}}\|\partial_y u_k\|_{L^\infty_{xy}}+\|u_k\|_{L^\infty_{xy}}\|\partial_y\omega\|_{L^\infty_{xy}}) \\
&+ \|J_y^s\omega\|_{L^2_{xy}}\|J_x^s u_k\|_{L^2_{xy}}(\|\omega\|_{L^\infty_{xy}}\|\partial_y u_k\|_{L^\infty_{xy}}+\|u_k\|_{L^\infty_{xy}}\|\partial_y\omega\|_{L^\infty_{xy}}+\|\omega\|_{L^\infty_{xy}}\|u_k\|_{L^\infty_{xy}}).
\end{aligned}
$$

Again, $(IV) = \int J_y^s(u_k\omega\partial_x\omega)J_y^s\omega = \int[J_y^s(u_k\omega\partial_x\omega) - u_k\omega J_y^s\partial_x\omega]J_y^s\omega + \int u_k\omega J_y^s\partial_x\omega J_y^s\omega$ and we denote $(IV)_1 = \int[J_y^s(u_k\omega\partial_x\omega) - u_k\omega J_y^s\partial_x\omega]J_y^s\omega$ and $(IV)_2 = \int u_k\omega J_y^s\partial_x\omega J_y^s\omega$.

We have by the Kato-Ponce commutator estimate

$$
\begin{aligned}
(IV)_1 &\lesssim \|J_y^s\omega\|_{L^2_{xy}}\|J_y^s(u_k\omega\partial_x\omega) - u_k\omega J_y^s\partial_x\omega\|_{L^2_{xy}} \\
&\lesssim \|J_y^s\omega\|_{L^2_{xy}}\Big[\|\partial_x\omega\|_{L^\infty_{xy}}\|J_y^{s-1}\omega\|_{L^2_{xy}}(\|u_k\|_{L^\infty_{xy}}+\|\partial_y u_k\|_{L^\infty_{xy}}) \\
&+ (\|u_k\|_{L^\infty_{xy}}\|\omega\|_{L^\infty_{xy}}+\|u_k\|_{L^\infty_{xy}}\|\partial_y\omega\|_{L^\infty_{xy}}+\|\partial_y u_k\|_{L^\infty_{xy}}\|\omega\|_{L^\infty_{xy}})\|J_y^{s-1}\partial_x\omega\|_{L^2_{xy}} \\
&+ \|\partial_x\omega\|_{L^\infty_{xy}}\|J_y^s u_k\|_{L^2_{xy}}\|\omega\|_{L^\infty_{xy}}\Big] \\
&\lesssim \|J_y^s\omega\|^2_{L^2_{xy}}(\beta_{u_k}(t)+\beta_\omega(t)) + \|J_y^s\omega\|_{L^2_{xy}}\|J_y^s u_k\|_{L^2_{xy}}\|\omega\|_{L^\infty_{xy}}\|\partial_x\omega\|_{L^\infty_{xy}} \\
&+ \|J_y^s\omega\|_{L^2_{xy}}\|J_x^s\omega\|_{L^2_{xy}}(\|u_k\|_{L^\infty_{xy}}\|\omega\|_{L^\infty_{xy}}+\|u_k\|_{L^\infty_{xy}}\|\partial_y\omega\|_{L^\infty_{xy}}+\|\partial_y u_k\|_{L^\infty_{xy}}\|\omega\|_{L^\infty_{xy}}).
\end{aligned}
$$

Also, $(IV)_2 \lesssim \|J_y^s\omega\|^2_{L^2_{xy}}(\|\partial_x u_k\|_{L^\infty_{xy}}\|\omega\|_{L^\infty_{xy}}+\|\partial_x\omega\|_{L^\infty_{xy}}\|u_k\|_{L^\infty_{xy}}) \lesssim \|J_y^s\omega\|^2_{L^2_{xy}}(\beta_{u_k}(t) + \beta_\omega(t))$ and so therefore

$$
\begin{aligned}
(IV) &\lesssim \|J_y^s\omega\|^2_{L^2_{xy}}(\beta_{u_k}(t)+\beta_\omega(t)) + \|J_y^s\omega\|_{L^2_{xy}}\|J_y^s u_k\|_{L^2_{xy}}\|\omega\|_{L^\infty_{xy}}\|\partial_x\omega\|_{L^\infty_{xy}} \\
&+ \|J_y^s\omega\|_{L^2_{xy}}\|J_x^s\omega\|_{L^2_{xy}}(\|u_k\|_{L^\infty_{xy}}\|\omega\|_{L^\infty_{xy}}+\|u_k\|_{L^\infty_{xy}}\|\partial_y\omega\|_{L^\infty_{xy}}+\|\partial_y u_k\|_{L^\infty_{xy}}\|\omega\|_{L^\infty_{xy}}).
\end{aligned}
$$

Again, $(V) = \int J_y^s(\omega^2\partial_x u_k)J_y^s\omega = \int[J_y^s(\omega^2\partial_x u_k) - \omega^2 J_y^s\partial_x u_k]J_y^s\omega + \int \omega^2 J_y^s\partial_x u_k J_y^s\omega$ and we denote $(V)_1 = \int[J_y^s(\omega^2\partial_x u_k) - \omega^2 J_y^s\partial_x u_k]J_y^s\omega$ and $(V)_2 = \int \omega^2 J_y^s\partial_x u_k J_y^s\omega$. We have by the Kato-Ponce commutator estimate

$$
\begin{aligned}
(V)_1 &\lesssim \|J_y^s\omega\|_{L^2_{xy}}\|J_y^s(\omega^2\partial_x u_k) - \omega^2 J_y^s\partial_x u_k\|_{L^2_{xy}} \\
&\lesssim \|J_y^s\omega\|_{L^2_{xy}}\|\omega\|_{L^\infty_{xy}}\Big[\|\partial_x u_k\|_{L^\infty_{xy}}\|J_y^s\omega\|_{L^2_{xy}}+(\|\omega\|_{L^\infty_{xy}}+\|\partial_y\omega\|_{L^\infty_{xy}})\|J_y^{s-1}\partial_x u_k\|_{L^2_{xy}}\Big] \\
&\lesssim \|J_y^s\omega\|^2_{L^2_{xy}}(\beta_{u_k}(t)+\beta_\omega(t)) + \|J_y^s\omega\|_{L^2_{xy}}(\|J_y^s u_k\|_{L^2_{xy}}+\|J_x^s u_k\|_{L^2_{xy}})\beta_\omega(t).
\end{aligned}
$$

Also,

$$
(V)_2 \lesssim \|J_y^s\omega\|_{L^2_{xy}}\|J_y^{s+1}u_k\|_{L^2_{xy}}\|\omega\|^2_{L^\infty_{xy}}+\|J_y^s\omega\|_{L^2_{xy}}\|J_x^s u_k\|_{L^2_{xy}}\|\omega\|^2_{L^\infty_{xy}}
$$

and so therefore

$$
\begin{aligned}
(V) \lesssim \|J_y^s\omega\|_{L^2_{xy}}^2(\beta_{u_k}(t)+\beta_\omega(t)) + \|J_y^s\omega\|_{L^2_{xy}}\|J_y^s u_k\|_{L^2_{xy}}\beta_\omega(t) \\
+ \|J_y^s\omega\|_{L^2_{xy}}\|J_x^s u_k\|_{L^2_{xy}}\beta_\omega(t) + \|J_y^s\omega\|_{L^2_{xy}}\|J_x^{s+1} u_k\|_{L^2_{xy}}\|\omega\|_{L^\infty_{xy}}^2.
\end{aligned}
$$

We make the following notation:

$$
a(\omega,u_k) = \|\omega\|_{L^\infty_{xy}}(\|\partial_x u_k\|_{L^\infty_{xy}}+\|\partial_y u_k\|_{L^\infty_{xy}}) + \|u_k\|_{L^\infty_{xy}}\|\partial_y\omega\|_{L^\infty_{xy}}+\beta_\omega(t),
$$

$$
b(\omega,u_k) = \|\omega\|_{L^\infty_{xy}}(\|\partial_y u_k\|_{L^\infty_{xy}}+\|u_k\|_{L^\infty_{xy}}) + \|u_k\|_{L^\infty_{xy}}\|\partial_y\omega\|_{L^\infty_{xy}}+\beta_\omega(t),
$$

$$
c(\omega,u_k) = \|\omega\|_{L^\infty_{xy}}(\|\partial_y u_k\|_{L^\infty_{xy}}+\|u_k\|_{L^\infty_{xy}})+\|u_k\|_{L^\infty_{xy}}\|\partial_y\omega\|_{L^\infty_{xy}}+\|u_k\|_{L^\infty_{xy}}\|\partial_y u_k\|_{L^\infty_{xy}}+\beta_\omega(t).
$$

Now, putting together all the terms we get that

$$
\begin{aligned}
\frac{d}{dt}\|J_y^s\omega\|_{L^2_{xy}}^2 \lesssim\ & \|J_y^s\omega\|_{L^2_{xy}}^2(\beta_\omega(t)+\beta_{u_k}(t)) \\
& + \|J_y^s\omega\|_{L^2_{xy}}\|J_y^s u_k\|_{L^2_{xy}}a(\omega,u_k) \\
& + \|J_y^s\omega\|_{L^2_{xy}}\|J_x^s u_k\|_{L^2_{xy}}b(\omega,u_k) \\
& + \|J_y^s\omega\|_{L^2_{xy}}\|J_x^s \omega\|_{L^2_{xy}}c(\omega,u_k) \\
& + \|J_y^s\omega\|_{L^2_{xy}}\|J_x^{s+1} u_k\|_{L^2_{xy}}(\|\omega\|_{L^\infty_{xy}}^2+\|\omega\|_{L^\infty_{xy}}\|u_k\|_{L^\infty_{xy}}).
\end{aligned} \tag{3.31}
$$

Using the variant of Grönwall's inequality from part a) and applying it to (3.31) with $u(t) = \|J_y^s\omega\|_{L^2_{xy}}$, $\beta(t) = \beta_\omega(t)+\beta_{u_k}(t) \geq 0$ and

$$
\begin{aligned}
\alpha(t) =\ & \|J_y^s\omega\|_{L^2_{xy}}\|J_y^s u_k\|_{L^2_{xy}}a(\omega,u_k) \\
& + \|J_y^s\omega\|_{L^2_{xy}}\|J_x^s u_k\|_{L^2_{xy}}b(\omega,u_k) \\
& + \|J_y^s\omega\|_{L^2_{xy}}\|J_x^s \omega\|_{L^2_{xy}}c(\omega,u_k) \\
& + \|J_y^s\omega\|_{L^2_{xy}}\|J_x^{s+1} u_k\|_{L^2_{xy}}(\|\omega\|_{L^\infty_{xy}}^2+\|\omega\|_{L^\infty_{xy}}\|u_k\|_{L^\infty_{xy}}).
\end{aligned}
$$

we obtain

$$\begin{aligned}
\|J_y^s\omega\|_{L_T^\infty L_{xy}^2} \lesssim \exp(\frac{1}{2}f_\omega(T)^2 + \frac{1}{2}f_{u_k}(T)^2)\Big[&\|J_y^s\omega(0)\|_{L_{xy}^2} \\
&+ \|J_y^s\omega\|_{L_T^\infty L_{xy}^2}\|J_y^s u_k\|_{L_{xy}^2}\|\omega\|_{L_T^2 L_{xy}^\infty}(\|\partial_x u_k\|_{L_T^2 L_{xy}^\infty}+\|\partial_y u_k\|_{L_T^2 L_{xy}^\infty}) \\
&+ \|J_y^s\omega\|_{L_T^\infty L_{xy}^2}\|J_y^s u_k\|_{L_{xy}^2}\Big(\|u_k\|_{L_T^2 L_{xy}^\infty}\|\partial_y\omega\|_{L_T^2 L_{xy}^\infty}+f_\omega(T)^2\Big) \\
&+ \|J_y^s\omega\|_{L_T^\infty L_{xy}^2}\|J_x^s u_k\|_{L_{xy}^2}\|\omega\|_{L_T^2 L_{xy}^\infty}(\|u_k\|_{L_T^2 L_{xy}^\infty}+\|\partial_y u_k\|_{L_T^2 L_{xy}^\infty}) \\
&+ \|J_y^s\omega\|_{L_T^\infty L_{xy}^2}\|J_x^s u_k\|_{L_{xy}^2}\Big(\|u_k\|_{L_T^2 L_{xy}^\infty}\|\partial_y\omega\|_{L_T^2 L_{xy}^\infty}+f_\omega(T)^2\Big) \\
&+ \|J_y^s\omega\|_{L_T^\infty L_{xy}^2}\|J_x^s \omega\|_{L_{xy}^2}\|\omega\|_{L_T^2 L_{xy}^\infty}(\|u_k\|_{L_T^2 L_{xy}^\infty}+\|\partial_y u_k\|_{L_T^2 L_{xy}^\infty}) \\
&+ \|J_y^s\omega\|_{L_T^\infty L_{xy}^2}\|J_x^s \omega\|_{L_{xy}^2}\Big(\|u_k\|_{L_T^2 L_{xy}^\infty}\|\partial_y\omega\|_{L_T^2 L_{xy}^\infty}+f_\omega(T)^2\Big) \\
&+ \|J_y^s\omega\|_{L_T^\infty L_{xy}^2}\|J_x^s \omega\|_{L_{xy}^2}\|u_k\|_{L_T^2 L_{xy}^\infty}\|\partial_y u_k\|_{L_T^2 L_{xy}^\infty} \\
&+ \|J_y^s\omega\|_{L_T^\infty L_{xy}^2}\|J_y^{s+1} u_k\|_{L_T^\infty L_{xy}^2}(\|\omega\|_{L_T^\infty L_{xy}^2}^2+\|\omega\|_{L_T^\infty L_{xy}^2}\|u_k\|_{L_T^\infty L_{xy}^2})\Big].
\end{aligned}$$

□

Lemma 12. *For $p \le s$, we have the following estimates:*

(a)

$$\begin{aligned}
\|J_x^p[\omega(u_k^2 + u_k u_{k'} + u_{k'}^2)]\|_{L_T^1 L_{xy}^2} \lesssim\ &\|J_x^p\omega\|_{L_T^\infty L_{xy}^2}(\|u_k\|_{L_T^2 L_{xy}^\infty}+\|u_{k'}\|_{L_T^2 L_{xy}^\infty}) \\
&+ \|\omega\|_{L_T^2 L_{xy}^\infty}\|\phi\|_{H^{s,s}}(\|u_k\|_{L_T^2 L_{xy}^\infty}+\|u_{k'}\|_{L_T^2 L_{xy}^\infty}).
\end{aligned}$$

(b)

$$\begin{aligned}
\|J_y^p[\omega(u_k^2 + u_k u_{k'} + u_{k'}^2)]\|_{L_T^1 L_{xy}^2} \lesssim\ &\|J_y^p\omega\|_{L_T^\infty L_{xy}^2}(\|u_k\|_{L_T^2 L_{xy}^\infty}+\|u_{k'}\|_{L_T^2 L_{xy}^\infty}) \\
&+ \|\omega\|_{L_T^2 L_{xy}^\infty}\|\phi\|_{H^{s,s}}(\|u_k\|_{L_T^2 L_{xy}^\infty}+\|u_{k'}\|_{L_T^2 L_{xy}^\infty}).
\end{aligned}$$

Proof. By using 7 part (c), we get that

$$\|J_x^p[\omega(u_k^2 + u_k u_{k'} + u_{k'}^2)]\|_{L^2_{xy}} \lesssim \|J_x^p \omega\|_{L_T^\infty L^2_{xy}} (\|u_k^2 + u_k u_{k'} + u_{k'}^2\|_{L^\infty_{xy}} \\ \|\omega\|_{L^\infty_{xy}} \|J_x^p(u_k^2 + u_k u_{k'} + u_{k'}^2)\|_{L^2_{xy}}. \tag{3.32}$$

Observe that $\|u_k^2 + u_k u_{k'} + u_{k'}^2\|_{L^\infty_{xy}} \lesssim \|u_k\|^2_{L^\infty_{xy}} + \|u_{k'}\|^2_{L^\infty_{xy}}$. Also, by 7 part (c) again, we have

$$\|J_x^p(u_k^2 + u_k u_{k'} + u_{k'}^2)\|_{L^2_{xy}} \lesssim (\|J_x^p u_k\|_{L^2_{xy}} + \|J_x^p u_{k'}\|_{L^2_{xy}})(\|u_k\|_{L^\infty} + \|u_{k'}^2\|_{L^\infty_{xy}}).$$

By 3.21, we get $\|J_x^p u_k\|_{L^2_{xy}} + \|J_x^p u_{k'}\|_{L^2_{xy}} \lesssim \|J_x^p \phi_k\|_{L^2_{xy}} + \|J_x^p \phi_{k'}\|_{L^2_{xy}} \lesssim \|\phi\|_{H^{s,s}}$. Combining all the above observation together with 3.32, we get

$$\|J_x^p[\omega(u_k^2 + u_k u_{k'} + u_{k'}^2)]\|_{L^2_{xy}} \lesssim \|J_x^p \omega\|_{L^2_{xy}} (\|u_k\|_{L^\infty_{xy}} + \|u_{k'}\|_{L^\infty_{xy}}) \\ + \|\omega\|_{L^2_T L^\infty_{xy}} \|\phi\|_{H^{s,s}} (\|u_k\|_{L^\infty_{xy}} + \|u_{k'}\|_{L^\infty_{xy}}).$$

Integrating both sides from 0 to T and applying Cauchy-Schwarz inequality, we obtain the conclusion of the lemma for J_x. The proof for J_y goes the same way. □

Lemma 13. *Suppose u_k satisfies the IVP (1.9) with initial data $\phi_k = P^k_{(3)}\phi$. We have $\|\omega\|_{L^2_T L^\infty_{xy}} \lesssim k^{(-1)-}$, $\|\partial_x \omega\|_{L^2_T L^\infty_{xy}} \lesssim k^{0-}$ and $\|\partial_y \omega\|_{L^2_T L^\infty_{xy}} \lesssim k^{0-}$ as $k \to \infty$. In particular, $f_\omega(T) \lesssim k^{0-}$ as $k \to \infty$.*

Proof. Take $\delta < \frac{s-2}{2}$. By the linear estimate in Proposition 2 applied to 3.29,

$$\|\omega\|_{L^2_T L^\infty_{xy}} \lesssim \|J_x^{1+\delta}\omega\|_{L^\infty_T L^2_{xy}} + \|J_x^{-1} J_y^{1+\delta}\omega\|_{L^\infty_T L^2_{xy}} + \|J_x^{1+\delta} J_y^{\delta}[\omega(u_k^2 + u_k u_{k'} + u_{k'}^2)]\|_{L^1_T L^2_{xy}}.$$

From 3.25 and 3.27 we have $\|J_x^{1+\delta}\omega\|_{L^\infty_T L^2_{xy}} \lesssim k^{1+\delta-s} h^{(3)}_\phi(k)^{1-\frac{1+\delta}{s}}$, together with

$$\|J_x^{-1} J_y^{1+\delta}\omega\|_{L^\infty_T L^2_{xy}} \lesssim \|J_y^{1+\delta}\omega\|_{L^\infty_T L^2_{xy}} \lesssim k^{1+\delta-s} h^{(3)}_\phi(k)^{1-\frac{1+\delta}{s}}.$$

For the last term, we observe

$$\|J_x^{1+\delta}J_y^{\delta}[\omega(u_k^2+u_ku_{k'}+u_{k'}^2)]\|_{L_T^1L_{xy}^2}\lesssim \|J_x^{1+2\delta}[\omega(u_k^2+u_ku_{k'}+u_{k'}^2)]\|_{L_T^1L_{xy}^2} + \|J_y^{1+2\delta}[\omega(u_k^2+u_ku_{k'}+u_{k'}^2)]\|_{L_T^1L_{xy}^2}$$

By Lemma 12 we get that

$$\|J_x^{1+2\delta}[\omega(u_k^2+u_ku_{k'}+u_{k'}^2)]\|_{L_T^1L_{xy}^2}\lesssim \|J_x^{1+2\delta}\omega\|_{L_T^\infty L_{xy}^2}(\|u_k\|_{L_T^2L_{xy}^\infty}+\|u_{k'}\|_{L_T^2L_{xy}^\infty}) + \|\omega\|_{L_T^2L_{xy}^\infty}\|\phi\|_{H^{s,s}}(\|u_k\|_{L_T^2L_{xy}^\infty}+\|u_{k'}\|_{L_T^2L_{xy}^\infty})$$

and

$$\|J_y^{1+2\delta}[\omega(u_k^2+u_ku_{k'}+u_{k'}^2)]\|_{L_T^1L_{xy}^2}\lesssim \|J_y^{1+2\delta}\omega\|_{L_T^\infty L_{xy}^2}(\|u_k\|_{L_T^2L_{xy}^\infty}+\|u_{k'}\|_{L_T^2L_{xy}^\infty}) + \|\omega\|_{L_T^2L_{xy}^\infty}\|\phi\|_{H^{s,s}}(\|u_k\|_{L_T^2L_{xy}^\infty}+\|u_{k'}\|_{L_T^2L_{xy}^\infty}).$$

By 3.25 and 3.27 we have $\|J_x^{1+2\delta}\omega\|_{L_T^\infty L_{xy}^2}\lesssim k^{1+2\delta-s}h_\phi^{(3)}(k)^{1-\frac{1+2\delta}{s}}$ and $\|J_y^{1+2\delta}\omega\|_{L_T^\infty L_{xy}^2}\lesssim k^{1+2\delta-s}h_\phi^{(3)}(k)^{1-\frac{1+2\delta}{s}}$. By combining the previous observations, we obtain

$$\|\omega\|_{L_T^2L_{xy}^\infty}\lesssim k^{1+2\delta-s}h_\phi^{(3)}(k)^{1-\frac{1+2\delta}{s}}\max(1,h_\phi^{(3)}(k)^{\frac{\delta}{s}}) + \|\omega\|_{L_T^2L_{xy}^\infty}\|\phi\|_{H^{s,s}}(\|u_k\|_{L_T^2L_{xy}^\infty}+\|u_{k'}\|_{L_T^2L_{xy}^\infty})$$

Since we consider that $\|\phi\|_{H^{s,s}}$ is small enough, such that $\|\phi\|_{H^{s,s}}(\|u_k\|_{L_T^2L_{xy}^\infty}+\|u_{k'}\|_{L_T^2L_{xy}^\infty})\leq \frac{1}{2}$, we get that

$$\|\omega\|_{L_T^2L_{xy}^\infty}\lesssim k^{1+2\delta-s}h_\phi^{(3)}(k)^{1-\frac{1+2\delta}{s}}\max(1,h_\phi^{(3)}(k)^{\frac{\delta}{s}})\to 0$$

as $k\to\infty$ since $1+2\delta<s$.

The linear estimate 2 applied to $\partial_x\omega$ results in

$$\|\partial_x\omega\|_{L^2_TL^\infty_{xy}} \lesssim \|J_x^{2+\delta}\omega\|_{L^\infty_TL^2_{xy}} + \|J_y^{1+\delta}\omega\|_{L^\infty_TL^2_{xy}} + \|J_x^{2+\delta}J_y^{\delta}[\omega(u_k^2+u_ku_{k'}+u_{k'}^2)]\|_{L^1_TL^2_{xy}}.$$

and by the same reasoning as above

$$\begin{aligned}\|\partial_x\omega\|_{L^2_TL^\infty_{xy}} &\lesssim k^{2+2\delta-s}h_\phi^{(3)}(k)^{1-\frac{2+2\delta}{s}}\max(1,h_\phi^{(3)}(k)^{\frac{\delta}{s}})\\ &\quad + \|\omega\|_{L^2_TL^\infty_{xy}}\|\phi\|_{H^{s,s}}(\|u_k\|_{L^2_TL^\infty_{xy}}+\|u_{k'}\|_{L^2_TL^\infty_{xy}})\end{aligned}$$

which, combined with the above fact that $\|\omega\|_{L^2_TL^\infty_{xy}} \lesssim k^{1+2\delta-s}h_\phi^{(3)}(k)^{1-\frac{1+2\delta}{s}}\max(1,h_\phi^{(3)}(k)^{\frac{\delta}{s}})$, for k large enough, it gives us $\|\partial_x\omega\|_{L^2_TL^\infty_{xy}} \lesssim k^{2+2\delta-s} \to 0$ as $k\to\infty$ since $2+2\delta<s$.

Lastly, the linear estimate 2 applied to $\partial_y\omega$ results in

$$\|\partial_y\omega\|_{L^2_TL^\infty_{xy}} \lesssim \|J_x^{1+\delta}J_y^{1}\omega\|_{L^\infty_TL^2_{xy}} + \|J_y^{2+\delta}\omega\|_{L^\infty_TL^2_{xy}} + \|J_x^{1+\delta}J_y^{1+\delta}[\omega(u_k^2+u_ku_{k'}+u_{k'}^2)]\|_{L^1_TL^2_{xy}}.$$

and by the same reasoning as above

$$\begin{aligned}\|\partial_y\omega\|_{L^2_TL^\infty_{xy}} &\lesssim k^{2+2\delta-s}h_\phi^{(3)}(k)^{1-\frac{2+2\delta}{s}}\max(1,h_\phi^{(3)}(k)^{\frac{\delta}{s}})\\ &\quad + \|\omega\|_{L^2_TL^\infty_{xy}}\|\phi\|_{H^{s,s}}(\|u_k\|_{L^2_TL^\infty_{xy}}+\|u_{k'}\|_{L^2_TL^\infty_{xy}})\end{aligned}$$

which, combined with the above fact that $\|\omega\|_{L^2_TL^\infty_{xy}} \lesssim k^{1+2\delta-s}h_\phi^{(3)}(k)^{1-\frac{1+2\delta}{s}}\max(1,h_\phi^{(3)}(k)^{\frac{\delta}{s}})$, for k large enough, it gives us $\|\partial_x\omega\|_{L^2_TL^\infty_{xy}} \lesssim k^{2+2\delta-s} \to 0$ as $k\to\infty$ since $2+2\delta<s$. $\square$

Lemma 14. *Suppose u_k satisfies the IVP (1.10) with initial data $\phi_k = \widetilde{P}^k_{(3)}\phi$. We have $\|\omega\|_{L^2_TL^\infty_{xy}} \lesssim k^{(-1)-}$, $\|\partial_x\omega\|_{L^2_TL^\infty_{xy}} \lesssim k^{0-}$ and $\|\partial_y\omega\|_{L^2_TL^\infty_{xy}} \lesssim k^{0-}$ as $k\to\infty$. In particular, $f_\omega(T) \lesssim k^{0-}$ as $k\to\infty$.*

Proof. Take $\delta < \frac{s-\frac{19}{8}}{2}$. By the linear estimate in Proposition 3 applied to 3.29,

$$\|u\|_{L^2_T L^\infty_{xy}} \lesssim [\|J_x^{\frac{11}{8}+\delta} u\|_{L^\infty_T L^2_{xy}} + \|J_x^{-\frac{5}{8}} J_y^{1+\delta} u\|_{L^\infty_T L^2_{xy}} + \|J_x^{\frac{11}{8}+\delta} J_y^{\delta} f\|_{L^1_T L^2_{xy}}].$$

From 3.26 and 3.28 we have $\|J_x^{\frac{11}{8}+\delta}\omega\|_{L^\infty_T L^2_{xy}} \lesssim k^{\frac{11}{8}+\delta-s}\widetilde{h}^{(3)}_{\phi}(k)^{1-\frac{\frac{11}{8}+\delta}{s}}$, together with

$$\|J_x^{-\frac{5}{8}} J_y^{1+\delta}\omega\|_{L^\infty_T L^2_{xy}} \lesssim \|J_y^{1+\delta}\omega\|_{L^\infty_T L^2_{xy}} \lesssim k^{1+\delta-s}\widetilde{h}^{(3)}_{\phi}(k)^{1-\frac{1+\delta}{s}}.$$

For the last term, we observe

$$\begin{aligned}\|J_x^{\frac{11}{8}+\delta} J_y^{\delta}[\omega(u_k^2 + u_k u_{k'} + u_{k'}^2)]\|_{L^1_T L^2_{xy}} \lesssim& \|J_x^{\frac{11}{8}+2\delta}[\omega(u_k^2 + u_k u_{k'} + u_{k'}^2)]\|_{L^1_T L^2_{xy}} \\ &+ \|J_y^{\frac{11}{8}+2\delta}[\omega(u_k^2 + u_k u_{k'} + u_{k'}^2)]\|_{L^1_T L^2_{xy}}\end{aligned}$$

By Lemma 12 we get that

$$\begin{aligned}\|J_x^{\frac{11}{8}+2\delta}[\omega(u_k^2 + u_k u_{k'} + u_{k'}^2)]\|_{L^1_T L^2_{xy}} \lesssim& \|J_x^{\frac{11}{8}+2\delta}\omega\|_{L^\infty_T L^2_{xy}}(\|u_k\|_{L^2_T L^\infty_{xy}} + \|u_{k'}\|_{L^2_T L^\infty_{xy}}) \\ &+ \|\omega\|_{L^2_T L^\infty_{xy}}\|\phi\|_{H^{s,s}}(\|u_k\|_{L^2_T L^\infty_{xy}} + \|u_{k'}\|_{L^2_T L^\infty_{xy}})\end{aligned}$$

and

$$\begin{aligned}\|J_y^{\frac{11}{8}+2\delta}[\omega(u_k^2 + u_k u_{k'} + u_{k'}^2)]\|_{L^1_T L^2_{xy}} \lesssim& \|J_y^{\frac{11}{8}+2\delta}\omega\|_{L^\infty_T L^2_{xy}}(\|u_k\|_{L^2_T L^\infty_{xy}} + \|u_{k'}\|_{L^2_T L^\infty_{xy}}) \\ &+ \|\omega\|_{L^2_T L^\infty_{xy}}\|\phi\|_{H^{s,s}}(\|u_k\|_{L^2_T L^\infty_{xy}} + \|u_{k'}\|_{L^2_T L^\infty_{xy}}).\end{aligned}$$

By 3.26 and 3.28 we have $\|J_x^{\frac{11}{8}+2\delta}\omega\|_{L^\infty_T L^2_{xy}} \lesssim k^{\frac{11}{8}+2\delta-s}\widetilde{h}^{(3)}_{\phi}(k)^{1-\frac{\frac{11}{8}+2\delta}{s}}$ and $\|J_y^{\frac{11}{8}+\delta}\omega\|_{L^\infty_T L^2_{xy}} \lesssim$

$k^{\frac{11}{8}+2\delta-s}\widetilde{h}_{\phi}^{(3)}(k)^{1-\frac{\frac{11}{8}+2\delta}{s}}$. By combining the previous observations, we obtain

$$\begin{aligned}\|\omega\|_{L_T^2L_{xy}^\infty} &\lesssim k^{\frac{11}{8}+2\delta-s}\widetilde{h}_{\phi}^{(3)}(k)^{1-\frac{\frac{11}{8}+2\delta}{s}}\max(1,\widetilde{h}_{\phi}^{(3)}(k)^{\frac{\delta}{s}}) \\ &\quad + \|\omega\|_{L_T^2L_{xy}^\infty}\|\phi\|_{H^{s,s}}(\|u_k\|_{L_T^2L_{xy}^\infty}+\|u_{k'}\|_{L_T^2L_{xy}^\infty})\end{aligned}$$

Since we consider that $\|\phi\|_{H^{s,s}}$ is small enough, such that $\|\phi\|_{H^{s,s}}(\|u_k\|_{L_T^2L_{xy}^\infty}+\|u_{k'}\|_{L_T^2L_{xy}^\infty}) \leq \frac{1}{2}$, we get that

$$\|\omega\|_{L_T^2L_{xy}^\infty}\lesssim k^{\frac{11}{8}+2\delta-s}\widetilde{h}_{\phi}^{(3)}(k)^{1-\frac{\frac{11}{8}+2\delta}{s}}\max(1,\widetilde{h}_{\phi}^{(3)}(k)^{\frac{\delta}{s}}) \to 0$$

as $k\to\infty$ since $\frac{11}{8}+2\delta<s$.

The linear estimate 3 applied to $\partial_x\omega$ results in

$$\|\partial_x\omega\|_{L_T^2L_{xy}^\infty}\lesssim \|J_x^{\frac{19}{8}+\delta}\omega\|_{L_T^\infty L_{xy}^2}+\|J_x^{\frac{3}{8}}J_y^{1+\delta}\omega\|_{L_T^\infty L_{xy}^2}+\|J_x^{\frac{19}{8}+\delta}J_y^{\delta}[\omega(u_k^2+u_ku_{k'}+u_{k'}^2)]\|_{L_T^1L_{xy}^2}.$$

and by the same reasoning as above

$$\begin{aligned}\|\partial_x\omega\|_{L_T^2L_{xy}^\infty} &\lesssim k^{\frac{19}{8}+2\delta-s}\widetilde{h}_{\phi}^{(3)}(k)^{1-\frac{\frac{19}{8}+2\delta}{s}}\max(1,\widetilde{h}_{\phi}^{(3)}(k)^{\frac{\delta}{s}}) \\ &\quad + \|\omega\|_{L_T^2L_{xy}^\infty}\|\phi\|_{H^{s,s}}(\|u_k\|_{L_T^2L_{xy}^\infty}+\|u_{k'}\|_{L_T^2L_{xy}^\infty})\end{aligned}$$

which, combined with the above fact that

$$\|\omega\|_{L_T^2L_{xy}^\infty}\lesssim k^{\frac{11}{8}+2\delta-s}\widetilde{h}_{\phi}^{(3)}(k)^{1-\frac{\frac{11}{8}+2\delta}{s}}\max(1,\widetilde{h}_{\phi}^{(3)}(k)^{\frac{\delta}{s}}),$$

for k large enough, it gives us $\|\partial_x\omega\|_{L_T^2L_{xy}^\infty}\lesssim k^{\frac{19}{8}+2\delta-s}\to 0$ as $k\to\infty$ since $\frac{19}{8}+2\delta<s$.

Lastly, the linear estimate 3 applied to $\partial_y\omega$ results in

$$\|\partial_y\omega\|_{L_T^2L_{xy}^\infty}\lesssim \|J_x^{\frac{11}{8}+\delta}J_y^1\omega\|_{L_T^\infty L_{xy}^2}+\|J_y^{2+\delta}\omega\|_{L_T^\infty L_{xy}^2}+\|J_x^{\frac{11}{8}+\delta}J_y^{1+\delta}[\omega(u_k^2+u_ku_{k'}+u_{k'}^2)]\|_{L_T^1L_{xy}^2}.$$

and by the same reasoning as above

$$\|\partial_y \omega\|_{L^2_T L^\infty_{xy}} \lesssim k^{\frac{19}{8}+2\delta-s} \widetilde{h}^{(3)}_\phi(k)^{1-\frac{\frac{19}{8}+2\delta}{s}} \max(1, \widetilde{h}^{(3)}_\phi(k)^{\frac{\delta}{s}})$$
$$+ \|\omega\|_{L^2_T L^\infty_{xy}} \|\phi\|_{H^{s,s}} (\|u_k\|_{L^2_T L^\infty_{xy}} + \|u_{k'}\|_{L^2_T L^\infty_{xy}})$$

which, combined with the above fact that

$$\|\omega\|_{L^2_T L^\infty_{xy}} \lesssim k^{\frac{19}{8}+2\delta-s} h^{(3)}_\phi(k)^{1-\frac{\frac{19}{8}+2\delta}{s}} \max(1, h^{(3)}_\phi(k)^{\frac{\delta}{s}}),$$

for k large enough, it gives us $\|\partial_y \omega\|_{L^2_T L^\infty_{xy}} \lesssim k^{\frac{19}{8}+2\delta-s} \to 0$ as $k \to \infty$ since $\frac{19}{8} + 2\delta < s$.

$\square$

Corollary. *We have $\|\omega\|_{H^{s,s}} \to 0$ as $k \to \infty$, where $s > 2$ for the initial value problem (1.9) and $s > \frac{19}{8}$ for the initial value problem (1.10).*

Proof. From (3.23) and Lemmas 13 and 14 we get $\|J^{s+1}_x u_k\|_{L^\infty_T L^2_{xy}} \|\omega\|_{L^2_T L^\infty_{xy}} \lesssim k^1 \cdot k^{(-1)-} = k^{0-}$ and $k^{0-} \to 0$ as $k \to \infty$. From the Lemmas 13, 14 used in Lemma 10 we obtain

$$\|J^s_x \omega\|_{L^\infty_T L^2_{xy}} \lesssim \exp(\frac{1}{2} f_{u_k}(T)^2 + \frac{1}{2} f_\omega(T)^2)(\|J^s_x \omega(0)\|_{L^\infty_T L^2_{xy}} + Ck^{0-}) \to 0$$

as $k \to \infty$, where we used that $\|J^s_x \omega(0)\|_{L^\infty_T L^2_{xy}} \to 0$ as $k \to \infty$ and the boundedness of $f_{u_k}(T)$ and $f_\omega(T)$ by 4 and 5.

From (3.24) and Lemmas 13 and 14 we get $\|J^{s+1}_y u_k\|_{L^\infty_T L^2_{xy}} \|\omega\|_{L^2_T L^\infty_{xy}} \lesssim k^1 \cdot k^{(-1)-} = k^{0-}$ and $k^{0-} \to 0$ as $k \to \infty$. From the Lemmas 13, 14 used in Lemma 10 together with the fact we just proved, $\|J^s_x \omega\|_{L^\infty_T L^2_{xy}} \to 0$, we obtain

$$\|J^s_y \omega\|_{L^\infty_T L^2_{xy}} \lesssim \exp(\frac{1}{2} f_{u_k}(T)^2 + \frac{1}{2} f_\omega(T)^2)(\|J^s_y \omega(0)\|_{L^\infty_T L^2_{xy}} + Ck^{0-}) \to 0$$

as $k \to \infty$ and the boundedness of $f_{u_k}(T)$ and $f_\omega(T)$ by 4 and 5.

Therefore, as $\|J_x^s\omega\|_{L_T^\infty L_{xy}^2}+\|J_y^s\omega\|_{L_T^\infty L_{xy}^2}\to 0$ as $k\to\infty$, it means that $u\in C([0,T]:H^{s,s})$. $\square$

3.7 Continuity of the flow map

We assume that $T\in[0,\infty)$ and $\phi^l\to\phi$ in $H^{s,s}(M\times\mathbb{T})$ as $l\to\infty$. We are going to prove that $u^l\to u$ in $C([-T,T]:H^{s,s}(M\times\mathbb{T}))$ as $l\to\infty$, where u^l and u are solutions of the the initial value problem $\partial_t u+\partial_x^3 u-\partial_x^{-1}\partial_y^2 u+u^2\partial_x u=0$ corresponding to initial data ϕ^l and ϕ, for $M=\mathbb{R}$ and $s>2$ and for $M=\mathbb{T}$ and $s>\frac{19}{8}$.

For $k\geq 1$, let as before, $\phi_k^l=P^k\phi^l$ and $u_k^l\in C([-T,T]:H^\infty)$ the corresponding solutions. Denote by $\omega_k=u_k-u$. By the same estimates from Lemma 10, Lemma 13, Lemma 14 applied to ω_k we get

$$\|u_k-u\|_{H^{s,s}}\lesssim \exp(\frac{1}{2}f_{\omega_k}(T)^2+\frac{1}{2}f_{u_k}(T)^2)(\|\phi_k-\phi\|_{H^{s,s}}+C(T,\|\phi_k\|_{H^{s,s}},\|\phi\|_{H^{s,s}})k^{0-}).$$

By the same reasoning, we have that

$$\|u_k^l-u^l\|_{H^{s,s}}\lesssim \exp(\frac{1}{2}f_{\omega_k^l}(T)^2+\frac{1}{2}f_{u_k^l}(T)^2)(\|\phi_k^l-\phi^l\|_{H^{s,s}}+C(T,\|\phi_k^l\|_{H^{s,s}},\|\phi^l\|_{H^{s,s}})k^{0-}).$$

Now, denote $\omega_k^l=u_k^l-u_k$. By the same estimates from Lemma 10, Lemma 13, Lemma 14 applied to ω_k^l

$$\|u_k^l-u_k\|_{H^{s,s}}\lesssim \exp(\frac{1}{2}f_{\omega_k^l}(T)^2+\frac{1}{2}f_{u_k^l}(T)^2)(\|\phi_k^l-\phi_k\|_{H^{s,s}}+C(T,\|\phi_k^l\|_{H^{s,s}},\|\phi_k\|_{H^{s,s}})k^{0-}).$$

By the boundedness of $f_{u_k}(T)$, $f_{u_k^l}(T)$, $f_{\omega_k}(T)$ and $f_{\omega_k^l}(T)$ by 4 and by 5 and the triangle inequality, we get

$$\begin{aligned}
\|u^l - u\|_{H^{s,s}} &\le \|u_k - u\|_{H^{s,s}} + \|u_k^l - u_k\|_{H^{s,s}} + \|u_k^l - u^l\|_{H^{s,s}} \\
&\lesssim \|\phi_k - \phi\|_{H^{s,s}} + \|\phi_k^l - \phi_k\|_{H^{s,s}} + \|\phi_k^l - \phi^l\|_{H^{s,s}} \\
&+ C(T, \|\phi\|_{H^{s,s}}, \|\phi_k\|_{H^{s,s}}, \|\phi^l\|_{H^{s,s}}, \|\phi_k^l\|_{H^{s,s}}) k^{0-}
\end{aligned}$$

which, by letting $k \to \infty$, we get $\|u^l - u\|_{H^{s,s}} \lesssim \|\phi^l - \phi\|_{H^{s,s}}$ and proves the continuity of the flow map.

CHAPTER 4

LOCAL WELL-POSEDNESS FOR THE PARTIALLY PERIODIC FIFTH ORDER MODIFIED KADOMTSEV-PETVIASHVILI I EQUATION

4.1 Dispersive Estimates

Throughout this chapter, we will use the same notations from 3. For $t \in \mathbb{R}$ let $W_{(5)}(t)$ denote the operator on $H^\infty(\mathbb{R} \times \mathbb{T})$ defined by the Fourier multiplier $(\xi, n) \mapsto e^{i(\xi^5 + \frac{n^2}{\xi})t}$. For integers $k = 0, 1, \ldots$ we define the operators $Q_x^k, Q_y^k, \widetilde{Q}_x^k, \widetilde{Q}_y^k$ on $H^\infty(\mathbb{R} \times \mathbb{T})$ by

$$\widehat{Q_x^k g}(\xi, n) = \mathbf{1}_{[2^{k-1}, 2^k)}(|\xi|) \text{ if } k \geq 1$$

with

$$\widehat{Q_x^0 g}(\xi, n) = \mathbf{1}_{[0,1)}(|\xi|)$$

and

$$\widehat{Q_y^k g}(\xi, n) = \mathbf{1}_{[2^{k-1}, 2^k)}(|n|) \text{ if } k \geq 1$$

with

$$\widehat{Q_y^0 g}(\xi, n) = \mathbf{1}_{[2^{k-1}, 2^k)}(|n|).$$

Also, $\widetilde{Q}_x^k = \sum_{k'=0}^k Q_x^{k'}, \widetilde{Q}_y^k = \sum_{k'=0}^k Q_y^{k'}, k = 1, \ldots$

We are stating the dispersion estimates for the partially periodic case that appear in Kenig and Ionescu [20].

Theorem 4.1.1. *Assume $\phi \in H^\infty(\mathbb{R} \times \mathbb{T})$. Then, for any $\epsilon > 0$,*

$$\|W_{(5)}(t)\tilde{Q}_y^{3j} Q_x^j \phi\|_{L^2_{2^{-j}} L^\infty_{xy}} \leq C_\epsilon 2^{(-\frac{1}{2}+\epsilon)j} \|\tilde{Q}_y^{2j} Q_x^j \phi\|_{L^2_{xy}} \tag{4.1}$$

and

$$\|W_{(5)}(t)Q_y^{3j+k}Q_x^j\phi\|_{L^2_{2^{-2j-k}}L^\infty_{xy}} \leq C_\epsilon 2^{(-\frac{1}{2}+\epsilon)j}\|Q_y^{3j+k}Q_x^j\phi\|_{L^2_{xy}} \tag{4.2}$$

for any integers $j \geq 0$ and $k \geq 1$.

Remark. *The same dispersive estimates are true for the operator $\widetilde{W}_{(5)}(t)$ defined by the Fourier multiplier $(\xi, n) \mapsto e^{i(\xi^5+\alpha\xi^3+\frac{n^2}{\xi})t}$.*

4.2 Linear Estimate

Proposition 6. *Assume $N \geq 4$, $u \in C^1([-T,T] : H^{-N-1}(\mathbb{R}\times\mathbb{T}))$, $f \in C([-T,T] : H^{-N}(\mathbb{R}\times\mathbb{T}))$, $T \in [0, \frac{1}{2}]$ and $[\partial_t - \partial_x^5 - \partial_x^{-1}\partial_y^2]u = \partial_x f$ on $\mathbb{R}\times\mathbb{T}\times[-T,T]$. Then, for any $1 \leq p \leq 2$, we have*

$$\|u\|_{L^p_T L^\infty_{xy}} \lesssim C_p T^{\frac{2-p}{2p}}[\|J_x^{\frac{5p-4}{2p}+\epsilon}u\|_{L^\infty_T L^2_{xy}} + \|J_x^{-\frac{2p-1}{p}+\epsilon}J_y^{\frac{3p-2}{2p}+\epsilon}u\|_{L^\infty_T L^2_{xy}} + \|J_x^{p'}J_y^{\frac{p-2}{2p}+\epsilon}u\|_{L^1_T L^2_{xy}}]$$

where $p' = max(\frac{3p-4}{2p}+\epsilon, \frac{1}{p}-\epsilon)$.

Proof. Without loss of generality, we may assume that $u \in C([-T,T] : Y^\infty(\mathbb{R}\times\mathbb{T})) \cap C^1([-T,T] : H^\infty(\mathbb{R}\times\mathbb{T}))$ and $f \in C([-T,T] : H^\infty(\mathbb{R}\times\mathbb{T}))$. It suffices to prove that if, for $\epsilon > 0$,

$$\|\widetilde{Q}_y^{3j}Q_x^j u\|_{L^p_T L^\infty_{xy}} \leq C_\epsilon 2^{-\frac{\epsilon j}{2}}T^{\frac{2-p}{2p}}[\|J_x^{\frac{5p-4}{2p}+\epsilon}u\|_{L^\infty_T L^2_{xy}} + \|J_x^{\frac{3p-4}{2p}}f\|_{L^1_T L^2_{xy}}] \tag{4.3}$$

and

$$\|Q_y^{3j+k}Q_x^j u\|_{L^p_T L^\infty_{xy}} \leq C_\epsilon 2^{-\frac{\epsilon(j+2k)}{2}}T^{\frac{2-p}{2p}}[\|J_x^{-\frac{2p-1}{2p}+\epsilon}J_y^{\frac{3p-2}{2p}+\epsilon}u\|_{L^\infty_T L^2_{xy}} + \|J_x^{\frac{3p-4}{2p}}f\|_{L^1_T L^2_{xy}} \tag{4.4}$$

for any integers $j \leq 0$ and $k \leq 1$. For 4.3, we partition the interval $[-T,T]$ into 2^{2j} equal subintervals of length $2T2^{-2j}$, denoted by $[a_{j,l}, a_{j,l+1}), l = 1, \ldots, 2^{2j}$. The term in the left-hand side of 4.3, using Hölder's inequality, is dominated by

$$
\begin{aligned}
&\sum_{l=1}^{2^{2j}} \|\mathbf{1}_{[a_{j,l},a_{j,l+1})}(t)\widetilde{Q}_y^{3j} Q_x^j u\|_{L_T^p L_{xy}^\infty} \\
&\qquad \leq C\|\mathbf{1}_{[a_{j,l},a_{j,l+1})}(t)\|_{L_T^{\frac{2-p}{2p}} L_{xy}^\infty} \|\sum_{l=1}^{2^{2j}} \|\mathbf{1}_{[a_{j,l},a_{j,l+1})}(t)\widetilde{Q}_y^{3j} Q_x^j u\|_{L_T^2 L_{xy}^\infty} \qquad (4.5)\\
&\qquad \leq C 2^{-\frac{2-p}{2p}2j} T^{\frac{2-p}{2p}} \sum_{l=1}^{2^{2j}} \|\mathbf{1}_{[a_{j,l},a_{j,l+1})}(t)\widetilde{Q}_y^{3j} Q_x^j u\|_{L_T^2 L_{xy}^\infty}.
\end{aligned}
$$

By Duhamel's formula, for $t \in [a_{j,l}, a_{j,l+1}]$,

$$
u(t) = W_{(5)}(t - a_{j,l})[u(a_{j,l})] + \int_{a_{j,l}}^t W_{(5)}(t-s)[\partial_x f(s)]ds.
$$

It follows from the dispersive estimate 4.1 that

$$
\begin{aligned}
&\|\mathbf{1}_{[a_{j,l},a_{j,l+1})}(t)\widetilde{Q}_y^{3j} Q_x^j u\|_{L_T^2 L_{xy}^\infty} \\
&\qquad \leq C_\epsilon \|\mathbf{1}_{[a_{j,l},a_{j,l+1})}(t) W_{(5)}(t-a_{j,l})\widetilde{Q}_y^{3j} Q_x^j u(a_{j,l})\|_{L_T^2 L_{xy}^\infty} \\
&\qquad + \|\mathbf{1}_{[a_{j,l},a_{j,l+1})}(t)\int_{a_{j,l}}^t W_{(5)}(s)\widetilde{Q}_y^{3j} Q_x^j \partial_x f(s) ds\|_{L_T^2 L_{xy}^\infty} \qquad (4.6)\\
&\qquad \lesssim C_\epsilon 2^{(-\frac{1}{2}+\frac{\epsilon}{2})j} \|\widetilde{Q}_y^{3j} Q_x^j u(a_{j,l})\|_{L_{xy}^2} \\
&\qquad + C_\epsilon 2^{(-\frac{1}{2}+\frac{\epsilon}{2})j} 2^j \|\mathbf{1}_{[a_{j,l},a_{j,l+1})}(t)\widetilde{Q}_y^{3j} Q_x^j f\|_{L_T^1 L_{xy}^2}.
\end{aligned}
$$

For the first term of the right-hand side of 4.6, we have

$$\begin{aligned}
&\sum_{l=1}^{2^{2j}} 2^{-\frac{2-p}{2p}2j} 2^{(-\frac{1}{2}+\frac{\epsilon}{2})j} \|\mathbf{1}_{[a_{j,l},a_{j,l+1})}(t)\widetilde{Q}_y^{3j} Q_x^j u(a_{j,l})\|_{L^2_{xy}} \\
&\qquad\qquad \lesssim \sum_{l=1}^{2^{2j}} 2^{-\frac{2-p}{2p}2j} 2^{(-\frac{1}{2}+\frac{\epsilon}{2})j} 2^{-(\frac{5p-4}{2p}+\epsilon)j} \|\mathbf{1}_{[a_{j,l},a_{j,l+1})}(t)\widetilde{Q}_y^{3j} Q_x^j J_x^{\frac{5p-4}{2p}} u(a_{j,l})\|_{L^2_{xy}} \\
&\qquad\qquad \lesssim 2^{2j} 2^{-\frac{2-p}{2p}2j} 2^{(-\frac{1}{2}+\frac{\epsilon}{2})j} 2^{-(\frac{5p-4}{2p}+\epsilon)j} \|\widetilde{Q}_y^{3j} Q_x^j J_x^{\frac{5p-4}{2p}} u\|_{L^\infty_T L^2_{xy}} \\
&\qquad\qquad \lesssim 2^{-\frac{\epsilon j}{2}} \|J_x^{\frac{5p-4}{2p}} u\|_{L^\infty_T L^2_{xy}}.
\end{aligned} \tag{4.7}$$

For the second term of the right-hand side of 4.6 we have

$$\begin{aligned}
&\sum_{l=1}^{2^{2j}} 2^{-\frac{2-p}{2p}2j} 2^{(-\frac{1}{2}+\frac{\epsilon}{2})j} \|\mathbf{1}_{[a_{j,l},a_{j,l+1})}(t)\widetilde{Q}_y^{3j} Q_x^j f\|_{L^1_T L^2_{xy}} \\
&\qquad\qquad \lesssim \sum_{l=1}^{2^{2j}} 2^{-\frac{2-p}{2p}2j} 2^{(-\frac{1}{2}+\frac{\epsilon}{2})j} 2^{2j} 2^{-(\frac{3p-4}{2p}+\epsilon)j} \|\mathbf{1}_{[a_{j,l},a_{j,l+1})}(t)\widetilde{Q}_y^{3j} Q_x^j J_x^{\frac{3p-4}{2p}+\epsilon} f\|_{L^1_T L^2_{xy}} \\
&\qquad\qquad \lesssim 2^{-\frac{\epsilon j}{2}} \sum_{l=1}^{2^{2j}} \|\mathbf{1}_{[a_{j,l},a_{j,l+1})}(t)\widetilde{Q}_y^{3j} Q_x^j J_x^{\frac{3p-4}{2p}+\epsilon} f\|_{L^1_T L^2_{xy}} \\
&\qquad\qquad \lesssim 2^{-\frac{\epsilon j}{2}} \|\widetilde{Q}_y^{3j} Q_x^j J_x^{\frac{3p-4}{2p}+\epsilon} f\|_{L^1_T L^2_{xy}} \\
&\qquad\qquad \lesssim 2^{-\frac{\epsilon j}{2}} \|J_x^{\frac{3p-4}{2p}+\epsilon} f\|_{L^1_T L^2_{xy}}.
\end{aligned} \tag{4.8}$$

Therefore, 4.7 and 4.8 give 4.3.

For 4.4, we partition the interval $[-T,T]$ into 2^{2j+k} equal subintervals of length $2T2^{-2j-k}$, denoted by $[b_{j,l}, b_{j,l+1}), l = 1, \ldots, 2^{2j+k}$. The term in the left-hand side of 4.4, using Hölder's inequality, is dominated by

$$\sum_{l=1}^{2^{2j+k}} \|\mathbf{1}_{[b_{j,l},b_{j,l+1})}(t)Q_y^{3j+k}Q_x^j u\|_{L_T^p L_{xy}^\infty}$$
$$\leq C\|\mathbf{1}_{[b_{j,l},b_{j,l+1})}(t)\|_{L_T^{\frac{2-p}{2p}} L_{xy}^\infty} \|\sum_{l=1}^{2^{2j+k}} \|\mathbf{1}_{[b_{j,l},b_{j,l+1})}(t)Q_y^{3j+k}Q_x^j u\|_{L_T^2 L_{xy}^\infty}$$
$$\leq C2^{-\frac{2-p}{2p}(2j+k)}T^{\frac{2-p}{2p}} \sum_{l=1}^{2^{2j+k}} \|\mathbf{1}_{[a_{j,l},a_{j,l+1})}(t)Q_y^{3j+k}Q_x^j u\|_{L_T^2 L_{xy}^\infty}.$$

By Duhamel's formula, for $t \in [b_{j,l}, b_{j,l+1}]$,

$$u(t) = W_{(5)}(t - b_{j,l})[u(b_{j,l})] + \int_{b_{j,l}}^t W_{(5)}(t-s)[\partial_x f(s)]ds.$$

It follows from the dispersive estimate 4.2 that

$$\begin{aligned} &\|\mathbf{1}_{[b_{j,l},b_{j,l+1})}(t)Q_y^{3j+k}Q_x^j u\|_{L_T^2 L_{xy}^\infty} \\ &\quad \leq C_\epsilon \|\mathbf{1}_{[b_{j,l},b_{j,l+1})}(t)W_{(5)}(t-b_{j,l})Q_y^{3j+k}Q_x^j u(b_{j,l})\|_{L_T^2 L_{xy}^\infty} \\ &\quad + \|\mathbf{1}_{[b_{j,l},b_{j,l+1})}(t)\int_{b_{j,l}}^t W_{(5)}(s)Q_y^{3j+k}Q_x^j \partial_x f(s)ds\|_{L_T^2 L_{xy}^\infty} \\ &\quad \lesssim C_\epsilon 2^{(-\frac{1}{2}+\frac{\epsilon}{2})j}\|Q_y^{3j+k}Q_x^j u(b_{j,l})\|_{L_{xy}^2} \\ &\quad + C_\epsilon 2^{(-\frac{1}{2}+\frac{\epsilon}{2})j}2^j\|\mathbf{1}_{[b_{j,l},b_{j,l+1})}(t)Q_y^{3j+k}Q_x^j f\|_{L_T^1 L_{xy}^2}. \end{aligned} \tag{4.9}$$

Denote $C^{\epsilon}_{j,k} = 2^{-\frac{2-p}{2p}(2j+k)}2^{(-\frac{1}{2}+\frac{\epsilon}{2})j}$. For the first term of the right-hand side of 4.9, we have

$$\begin{aligned}
&\sum_{l=1}^{2^{2j+k}} C^{\epsilon}_{j,k}\|\mathbf{1}_{[b_{j,l},b_{j,l+1})}(t)Q_y^{3j+k}Q_x^j u(b_{j,l})\|_{L^2_{xy}} \\
&\lesssim \sum_{l=1}^{2^{2j+k}} C^{\epsilon}_{j,k}2^{(\frac{2p-1}{p}+\epsilon)j}2^{-(\frac{3p-2}{2p}+\epsilon)(3j+k)}\|\mathbf{1}_{[b_{j,l},b_{j,l+1})}(t)Q_y^{3j+k}Q_x^j J_x^{-\frac{2p-1}{p}-\epsilon}J_y^{\frac{3p-2}{2p}+\epsilon}u(b_{j,l})\|_{L^2_{xy}} \\
&\lesssim 2^{2j+k}C^{\epsilon}_{j,k}2^{(\frac{2p-1}{p}+\epsilon)j}2^{-(\frac{3p-2}{2p}+\epsilon)(3j+k)}\|Q_y^{3j+k}Q_x^j J_x^{-\frac{2p-1}{p}-\epsilon}J_y^{\frac{3p-2}{2p}+\epsilon}u\|_{L^\infty_T L^2_{xy}} \\
&\lesssim 2^{-\frac{\epsilon(3j+2k)}{2}}\|J_x^{-\frac{2p-1}{p}-\epsilon}J_y^{\frac{3p-2}{2p}+\epsilon}u\|_{L^\infty_T L^2_{xy}}.
\end{aligned} \tag{4.10}$$

For the second term of the right-hand side of 4.9 we have

$$\begin{aligned}
&\sum_{l=1}^{2^{2j+k}} 2^{-\frac{2-p}{2p}(2j+k)}2^{(-\frac{1}{2}+\frac{\epsilon}{2})j}2^j\|\mathbf{1}_{[b_{j,l},b_{j,l+1})}(t)Q_y^{3j+k}Q_x^j f\|_{L^1_T L^2_{xy}} \\
&\lesssim \sum_{l=1}^{2^{2j+k}} C^{\epsilon}_{j,k}2^j 2^{-(\frac{1}{p}-\epsilon)j}2^{(\frac{2-p}{2p}-\epsilon)(3j+k)}\|\mathbf{1}_{[b_{j,l},b_{j,l+1})}(t)Q_y^{3j+k}Q_x^j J_x^{\frac{1}{p}+\epsilon}J_y^{-\frac{2-p}{2p}+\epsilon}f\|_{L^1_T L^2_{xy}} \\
&\lesssim 2^{-\frac{\epsilon(3j+2k)}{2}}\sum_{l=1}^{2^{2j+k}}\|\mathbf{1}_{[b_{j,l},b_{j,l+1})}(t)Q_y^{3j+k}Q_x^j J_x^{\frac{1}{p}-\epsilon}J_y^{-\frac{2-p}{2p}+\epsilon}f\|_{L^1_T L^2_{xy}} \\
&\lesssim 2^{-\frac{\epsilon(3j+2k)}{2}}\|Q_y^{3j+k}Q_x^j J_x^{\frac{1}{p}-\epsilon}J_y^{-\frac{2-p}{2p}+\epsilon}f\|_{L^1_T L^2_{xy}} \\
&\lesssim 2^{-\frac{\epsilon(3j+2k)}{2}}\|J_x^{\frac{1}{p}-\epsilon}J_y^{-\frac{2-p}{2p}+\epsilon}f\|_{L^1_T L^2_{xy}}.
\end{aligned} \tag{4.11}$$

Therefore, 4.10 and 4.11 give 4.4.

□

For our purpose, we will use the linear estimates

$$\|u\|_{L^2_T L^\infty_{xy}} \lesssim \|J_x^{\frac{3}{2}+\delta}u\|_{L^\infty_T L^2_{xy}} + \|J_x^{-\frac{3}{2}+\delta}J_y^{1+\delta}u\|_{L^\infty_T L^2_{xy}} + \|J_x^{\frac{1}{2}+\delta}J_y^{\delta}f\|_{L^1_T L^2_{xy}}.$$

4.3 A Priori Estimate

We are going to bound $f_u(T) = \|u\|_{L_T^2 L_{xy}^\infty} + \|\partial_x u\|_{L_T^2 L_{xy}^\infty} + \|\partial_y u\|_{L_T^2 L_{xy}^\infty}$.

Proposition 7. *Suppose u satisfies the IVP (1.11) with initial data u_0 and let $s > \frac{5}{2}$. Then $u, \partial_x u, \partial_y u \in L^2([-T,T]; L^\infty(\mathbb{R} \times \mathbb{T}))$. Moreover,*

$$f_u(T) = \|u\|_{L_T^2 L_{xy}^\infty} + \|\partial_x u\|_{L_T^2 L_{xy}^\infty} + \|\partial_y u\|_{L_T^2 L_{xy}^\infty} \leq C_T$$

for a suitable small enough T, if $\|u_0\|_{H^{s,s}}$ is small enough.

Proof. First, $\beta_f(t) = \|f(t)\|_{L_{xy}^\infty}^2 + \|\partial_x f(t)\|_{L_{xy}^\infty}^2 + \|\partial_y f(t)\|_{L_{xy}^\infty}^2$ for a function f and from now on we consider $s > \frac{5}{2}$.

First, apply the operator J_x^s to (1.11) and then multiply it by $J_x^s u$. We have that, by the Kato-Ponce commutator estimates and integration by parts,

$$\begin{aligned}\frac{d}{dt}\|J_x^s u\|_{L_{xy}^2}^2 &= \int (J_x^s u J_x^s(u^2 \partial_x u) = \int J_x^s u[J_x^s(u^2\partial_x u) - u^2 J_x^s \partial_x u] + \int u^2 J_x^s u J_x^s \partial_x u \\ &\lesssim \|J_x^s u\|_{L_{xy}^2}^2 (\|u\|_{L_{xy}^\infty} + \|u\|_{L_{xy}^\infty} \|\partial_x u\|_{L_{xy}^\infty}) \lesssim \|J_x^s u\|_{L_{xy}^2}^2 \beta_u(t)\end{aligned}$$

therefore, by Grönwall's inequality, we get that

$$\|J_x^s u\|_{L_T^\infty L_{xy}^2} \lesssim \|J_x^s \phi\|_{L_{xy}^2} \exp(f_u(T)^2). \tag{4.12}$$

Similarly, apply the operator J_y^s to (1.11) and then multiply it by $J_y^s u$. We get that, by the Kato-Ponce commutator estimates and integration by parts,

$$\begin{aligned}\frac{d}{dt}\|J_y^s u\|_{L^2_{xy}}^2 &= \int J_y^s u J_y^s(u^2\partial_x u) = \int J_y^s u[J_y^s(u^2\partial_x u) - u^2 J_x^s \partial_x u)] + \int u^2 J_y^s u J_y^s \partial_x u \\ &\lesssim \|J_y^s u\|_{L^2_{xy}}\Big[\|\partial_x u\|_{L^\infty_{xy}}\|J_y^s(u^2)\|_{L^2_{xy}} + (\|u\|_{L^\infty_{xy}}^2 + \|u\|_{L^\infty_{xy}}\|\partial_y u\|_{L^\infty_{xy}})\|J_y^s u\|_{L^2_{xy}}\Big] \\ &+ \|u\|_{L^\infty_{xy}}\|\partial_x u\|_{L^\infty_{xy}}\|J_y^s u\|_{L^2_{xy}}^2 \\ &\lesssim \|J_y^s u\|_{L^2_{xy}}^2(\|u\|_{L^\infty_{xy}}\|\partial_x u\|_{L^\infty_{xy}} + \|u\|_{L^\infty_{xy}}^2 + \|u\|_{L^\infty_{xy}}\|\partial_y u\|_{L^\infty_{xy}} + \|\partial_x u\|_{L^\infty_{xy}}\|\partial_y u\|_{L^\infty_{xy}}) \\ &\lesssim \|J_y^s u\|_{L^2_{xy}}^2 \beta_u(t)\end{aligned}$$

therefore, by Grönwall's inequality, we get that

$$\|J_y^s u\|_{L^\infty_T L^2_{xy}} \lesssim \|J_y^s \phi\|_{L^2_{xy}} \exp(f_u(T)^2). \tag{4.13}$$

This means that $\|u\|_{L^\infty_T H^{s,s}_{xy}} \lesssim \|\phi\|_{H^{s,s}_{xy}} \exp(f_u(T)^2)$.

Now, we bound $f_u(T)$. We take the first term, $\|u\|_{L^2_T L^\infty_{xy}}$. We take $0 < \delta < s - \frac{5}{2}$. By the linear estimate, we have

$$\|u\|_{L^2_T L^\infty_{xy}} \lesssim \|J_x^{\frac{3}{2}+\delta} u\|_{L^\infty_T L^2_{xy}} + \|J_x^{-\frac{3}{2}+\delta} J_y^{1+\delta} u\|_{L^\infty_T L^2_{xy}} + \|J_x^{\frac{1}{2}+\delta} J_y^{\delta}(u^3)\|_{L^1_T L^2_{xy}}$$

with $\|J_x^{\frac{3}{2}+\delta} u\|_{L^\infty_T L^2_{xy}} \lesssim \|u\|_{L^\infty_T H^{s,s}_{xy}}$ and $\|J_x^{-\frac{3}{2}+\delta} J_y^{1+\delta}\|_{L^\infty_T L^2_{xy}} \lesssim \|u\|_{L^\infty_T H^{s,s}_{xy}}$. For the third term of the linear estimate we have $\|J_x^{\frac{1}{2}+\delta} J_y^{\delta}(u^3)\|_{L^2_{xy}} \leq \|J_x^{\frac{1}{2}+2\delta}(u^3)\|_{L^2_{xy}} + \|J_y^{\frac{1}{2}+2\delta}(u^3)\|_{L^2_{xy}}$ so by the Kato-Ponce commutator estimates,

$$\|J_x^{\frac{1}{2}+2\delta}(u^3)\|_{L^2_{xy}} \lesssim \|J_x^{\frac{1}{2}+2\delta} u\|_{L^2_{xy}} \|u\|_{L^\infty_{xy}}^2 \text{ and } \|J_y^{\frac{1}{2}+2\delta}(u^3)\|_{L^2_{xy}} \lesssim \|J_y^{\frac{1}{2}+\delta} u\|_{L^2_{xy}} \|u\|_{L^\infty_{xy}}^2$$

and so

$$\|J_x^{\frac{1}{2}+\delta}(u^3)\|_{L^1_T L^2_{xy}} \lesssim \|J_x^{\frac{1}{2}+\delta} u\|_{L^\infty_T L^2_{xy}} \|u\|_{L^2_T L^\infty_{xy}}^2 \lesssim f_u(T)^2 \|u\|_{L^\infty_T H^{s,s}_{xy}}$$

and finally we get

$$\|u\|_{L^2_T L^\infty_{xy}} \lesssim (1 + f_u(T)^2)\|u\|_{L^\infty_T H^{s,s}_{xy}} \lesssim \|\phi\|_{H^{s,s}_{xy}} \exp(f_u(T)^2)(1 + f_u(T)^2).$$

Now, we look at the second term of $f_u(T)$. By the linear estimate applied to $\partial_x u$, we have

$$\|\partial_x u\|_{L^2_T L^\infty_{xy}} \lesssim \|J_x^{\frac{5}{2}+\delta} u\|_{L^\infty_T L^2_{xy}} + \|J_x^{-\frac{1}{2}+\delta} J_y^{1+\delta} u\|_{L^\infty_T L^2_{xy}} + \|J_x^{\frac{3}{2}+\delta} J_y^{\delta}(u^3)\|_{L^1_T L^2_{xy}}$$

with $\|J_x^{\frac{5}{2}+\delta} u\|_{L^\infty_T L^2_{xy}} \lesssim \|u\|_{L^\infty_T H^{s,s}_{xy}}$ and $\|J_x^{-\frac{1}{2}+\delta} J_y^{1+\delta}\|_{L^\infty_T L^2_{xy}} \lesssim \|u\|_{L^\infty_T H^{s,s}_{xy}}$. For the third term of the linear estimate we have, $\|J_x^{\frac{3}{2}+\delta} J_y^{\delta}(u^3)\|_{L^2_{xy}} \leq \|J_x^{\frac{3}{2}+2\delta}(u^3)\|_{L^2_{xy}} + \|J_y^{\frac{3}{2}+2\delta}(u^3)\|_{L^2_{xy}}$ by the Kato-Ponce commutator estimates,

$$\|J_x^{\frac{3}{2}+\delta}(u^3)\|_{L^2_{xy}} \lesssim \|J_x^{\frac{3}{2}+\delta} u\|_{L^2_{xy}} \|u\|^2_{L^\infty_{xy}} \text{ and } \|J_y^{\frac{3}{2}+\delta}(u^3)\|_{L^2_{xy}} \lesssim \|J_y^{\frac{3}{2}+\delta} u\|_{L^2_{xy}} \|u\|^2_{L^\infty_{xy}}$$

and so

$$\|J_x^{\frac{3}{2}+\delta} J_y^{\delta}(u^3)\|_{L^1_T L^2_{xy}} \lesssim f_u(T)^2 \|u\|_{L^\infty_T H^{s,s}_{xy}}.$$

Finally we get

$$\|\partial_x u\|_{L^2_T L^\infty_{xy}} \lesssim (1 + f_u(T)^2)\|u\|_{L^\infty_T H^{s,s}_{xy}} \lesssim \|\phi\|_{H^{s,s}_{xy}} \exp(f_u(T)^2)(1 + f_u(T)^2).$$

Now, for the final term of $f_u(T)$, by the linear estimate applied to $\partial_y u$, we have

$$\|\partial_y u\|_{L^2_T L^\infty_{xy}} \lesssim \|J_x^{\frac{1}{2}+\delta} J_y^1 u\|_{L^\infty_T L^2_{xy}} + \|J_x^{-\frac{3}{2}+\delta} J_y^{2+\delta} u\|_{L^\infty_T L^2_{xy}} + \|J_x^{\frac{1}{2}+\delta} \partial_y(u^3)\|_{L^1_T L^2_{xy}}$$

with

$$\|J_x^{\frac{1}{2}+\delta} J_y^1 u\|_{L^\infty_T L^2_{xy}} \lesssim \|J_x^{\frac{5}{2}+\delta} u\|_{L^\infty_T L^2_{xy}} + \|J_y^{\frac{5}{4}+\frac{\delta}{2}} u\|_{L^\infty_T L^2_{xy}} \lesssim \|u\|_{L^\infty_T H^{s,s}_{xy}}$$

and $\|J_x^{-\frac{3}{2}+\delta}J_y^{2+\delta}u\|_{L^\infty_T L^2_{xy}} \lesssim \|u\|_{L^\infty_T H^{s,s}_{xy}}$. For the third term in the linear estimate, we have

$$\|J_x^{\frac{1}{2}+\delta}J_y^{\delta}\partial_y(u^3)\|_{L^2_{xy}} \leq \|J_x^{\frac{1}{2}+\delta}J_y^{1+\delta}(u^3)\|_{L^2_{xy}} \leq \|J_x^{\frac{3}{2}+2\delta}(u^3)\|_{L^2_{xy}} + \|J_y^{\frac{3}{2}+2\delta}(u^3)\|_{L^2_{xy}}$$

so by the same reasoning, after applying the Kato-Ponce commutator estimates we get

$$\|J_x^{1+\delta}J_y^{\delta}(u^2\partial_y u)\|_{L^\infty_T L^2_{xy}} \lesssim \|u\|_{L^\infty_T H^{s,s}_{xy}} f_u(T)^2.$$

Finally, we get

$$\|\partial_y u\|_{L^2_T L^\infty_{xy}} \lesssim (1+f_u(T)^2)\|u\|_{L^\infty_T H^{s,s}_{xy}} \lesssim \|\phi\|_{H^{s,s}_{xy}} \exp(f_u(T)^2)(1+f_u(T)^2).$$

All in all, we have that $f_u(T) \lesssim \|\phi\|_{H^{s,s}_{xy}} \exp(f_u(T)^2)(1+f_u(T)^2)$, and if $\|\phi\|_{H^{s,s}_{xy}}$ is small, then by a continuity argument we get that $f_u(T) \leq C$ if T is sufficiently small. $\square$

4.4 Local Well-Posedness

We start by stating a well-known local-wellposedness result from Iorio and Nunes (see [22], Section 4):

Lemma 15. *Assume $\phi \in H^\infty$. Then there is $T = T(\|\phi\|_{H^3}) > 0$ and a solution $u \in C([-T,T] : H^\infty)$ of the initial value problem*

$$\begin{cases} \partial_t u - \partial_x^5 u - \partial_x^{-1}\partial_y^2 u + u^2\partial_x u = 0, \\ u(0,x,y) = u_0(x,y). \end{cases}$$

The proof for $\mathbb{R}\times\mathbb{T}$ is the same as the proof in [22] for $\mathbb{R}\times\mathbb{R}$.

We proceed to prove the local well-posedness result. In this section, the proof will include just the existence and uniqueness of the solution.

Theorem 4.4.1. *The IVP $\partial_t u - \partial_x^5 u - \partial_x^{-1}\partial_y^2 u + u^2\partial_x u = 0$ is locally well-posed in $H^{s,s}(\mathbb{R}\times\mathbb{T}), s > \frac{5}{2}$. More precisely, given $u_0 \in H^{s,s}(\mathbb{R}\times\mathbb{T}), s > \frac{5}{2}$, there exists $T = T(\|u_0\|_{H^{s,s}})$ and a unique solution u to the IVP such that $u \in C([0,T] : H^{s,s}(\mathbb{R}\times\mathbb{T}))$, $u, \partial_x u, \partial_y u \in L^2_T L^\infty_{xy}$. Moreover, the mapping $u_0 \to\in C([0,T] : H^{s,s}(\mathbb{R}\times\mathbb{T}))$ is continuous.*

Proof. Let $u_0 \in H^{s,s}(\mathbb{R}\times\mathbb{T})$ and fixed $u_{0,\epsilon} \in H^{s,s}(\mathbb{R}\times\mathbb{T}) \cap H^\infty_{-1}(\mathbb{R}\times\mathbb{T})$ such that $\|u_0 - u_{0,\epsilon}\|_{H^{s,s}}\to 0$ and $\|u_{0,\epsilon}\|_{H^{s,s}}\leq 2\|u_0\|_{H^{s,s}}$.

We know by the Iorio-Nunes local well-posedness result that $u_{0,\epsilon}$ gives a unique solution u_ϵ. We have by the a priori bound that $\|u_\epsilon\|_{L^2_T L^\infty_{xy}}+\|\partial_x u_\epsilon\|_{L^2_T L^\infty_{xy}}+\|\partial_y u_\epsilon\|_{L^2_T L^\infty_{xy}}\leq C_T$ and by the previous result, $\sup_{0<t<T}\|u_\epsilon\|_{H^{s,s}}\leq C_T$. Henceforth,

$$
\begin{aligned}
\partial_t \|u_\varepsilon - u_{\varepsilon'}\|^2_{L^2} &= \int (u_\varepsilon - u_{\varepsilon'})\partial_x(\frac{u_\varepsilon^3}{3} - \frac{u_{\varepsilon'}^3}{3}) \\
&= \int \partial_x(u_\varepsilon - u_{\varepsilon'})\cdot(u_\varepsilon - u_{\varepsilon'})\frac{u_\varepsilon^2 + u_\varepsilon u_{\varepsilon'} + u_{\varepsilon'}^2}{3} = \\
&= \int (u_\varepsilon - u_{\varepsilon'})^2\partial_x[\frac{u_\varepsilon^2 + u_\varepsilon u_{\varepsilon'} + u_{\varepsilon'}^2}{3}] \\
&\leq \|u_\varepsilon - u_{\varepsilon'}\|^2_{L^2}(\|u_\varepsilon\|^2_{L^\infty_{xy}}+\|\partial_x u_\varepsilon\|^2_{L^\infty_{xy}}+\|u_{\varepsilon'}\|^2_{L^\infty_{xy}}+\|\partial_x u_{\varepsilon'}\|^2_{L^\infty_{xy}}) \\
&\leq (\beta_{u_\varepsilon}(t) + \beta_{u_{\varepsilon'}}(t))\|u_\varepsilon - u_{\varepsilon'}\|^2_{L^2}.
\end{aligned}
$$

and by Grönwall's inequality and the a priori estimate

$$
\|u_\epsilon - u_{\epsilon'}\|^2_{L^\infty_T L^2_{xy}} \lesssim_T \|u_{0,\epsilon} - u_{0,\epsilon'}\|^2_{L^2_{xy}},
$$

hence $\sup_{0<t<T}\|u_\epsilon - u_{\epsilon'}\|_{L^2_{xy}} \to 0$, hence we can find $u \in C([0,T] : H^{s',s'}(\mathbb{R}\times\mathbb{T})) \cap L^\infty([0,T] : H^{s,s}(\mathbb{R}\times\mathbb{T}))$ with $s' < s$. The fact that u is a solution of the IVP is clear now. Uniqueness also comes from the previous Grönwall's inequality. $\square$

4.5 Continuity with respect to time

We proceed by a standard Bona-Smith argument ([5]).

Let $h_\phi^{(5)}(k) = [\sum_{n\in\mathbb{Z}} \int_{|\xi|+|n|\geq k} |\hat{\phi}(\xi,n)|^2[(1+\xi^2)^s + (1+n^2)^s] d\xi]^{\frac{1}{2}}$.

For $\phi \in H^{s,s}$, let $\phi_k = P_{(5)}^k \phi$ where $\widehat{P_{(5)}^k g}(\xi,n) = \widehat{g}(\xi,n) \cdot 1_{[0,k]}(|\xi|) \cdot 1_{[0,k]}(|n|)$. Clearly, $h_\phi^{(5)}$ is nondecreasing in k and $\lim_{k\to\infty} h_k^{(5)}(\phi) = 0$. By Plancherel,

$$\begin{aligned}\|\phi - \phi_k\|_{L^2_{xy}} = \|\widehat{\phi} - \widehat{\phi}_k\|_{L^2_{xy}} &= \Big[\sum_{|n|\geq k} \int_{|\xi|\geq k} |\hat{\phi}(\xi,n)|^2 d\xi\Big]^{\frac{1}{2}} \\ &\leq \Big[\sum_{n\in\mathbb{Z}} \int_{|\xi|+|n|\geq k} |\hat{\phi}(\xi,n)| \frac{(1+\xi^2)^s + (1+n^2)^s}{k^{2s}}\Big]^{\frac{1}{2}} \lesssim k^{-s} h_\phi^{(5)}(k)\end{aligned}$$

and also,

$$\|J_x^p \phi_k\|_{L^2_{xy}} \lesssim C(T, \|\phi\|_{H^{s,s}}) k^{p-s} \text{ and } \|J_y^p \phi_k\|_{L^2_{xy}} \lesssim C(T, \|\phi\|_{H^{s,s}}) k^{p-s}.$$

Denote $\omega = u_k - u_{k'}$ with $k < k'$.

In all the cases, from their respective definitions, we have that, if $p \geq s$, then

$$\|J_x^p \phi_k\|_{L^2_{xy}} \lesssim C(T, \|\phi\|_{H^{s,s}}) k^{p-s} \text{ and } \|J_y^p \phi_k\|_{L^2_{xy}} \lesssim C(T, \|\phi\|_{H^{s,s}}) k^{p-s}.$$

Since $\phi_k \in H^\infty$, by local well-posedness result of Iorio and Nunes, they give rise to unique solutions u_k in H^∞. The above estimates together with (4.12) and (4.13), if $p \geq s$, we also have

$$\|J_x^p u_k\|_{L_T^\infty L^2_{xy}} \leq C(T, \|\phi\|_{H^{s,s}}) k^{p-s} \tag{4.14}$$

and

$$\|J_y^p u_k\|_{L_T^\infty L^2_{xy}} \leq C(T, \|\phi\|_{H^{s,s}}) k^{p-s}. \tag{4.15}$$

Denote $\omega = u_k - u_{k'}$ with $k < k'$. Now choose $0 \leq q \leq s$. By using that $\|\phi - \phi_k\|_{L^2_{xy}} \lesssim$

$k^{-s}h_{\phi}^{(5)}(k)$ for the $\mathbb{R}\times\mathbb{T}$ case, together with the interpolation inequality,

$$\|J_x^q\omega\|_{L_T^\infty L_{xy}^2}\leq \|J_x^s\omega\|_{L_T^\infty L_{xy}^2}^{\frac{q}{s}}\|\omega\|_{L_T^\infty L_{xy}^2}^{1-\frac{q}{s}}\lesssim \|\omega\|_{L_T^\infty L_{xy}^2}^{1-\frac{q}{s}}$$

it yields

$$\|J_x^q\omega\|_{L_T^\infty L_{xy}^2}\lesssim k^{q-s}h_{\phi}^{(5)}(k)^{1-\frac{q}{s}}. \tag{4.16}$$

Similarly, we get results for J_y, more precisely,

$$\|J_y^q\omega\|_{L_T^\infty L_{xy}^2}\lesssim k^{q-s}h_{\phi}^{(5)}(k)^{1-\frac{q}{s}}. \tag{4.17}$$

Lemma 16. *We have the following estimates:*

a)

$$\begin{aligned}\|J_x^s\omega\|_{L_T^\infty L_{xy}^2} &\lesssim exp(\frac{1}{2}f_\omega(T)^2+\frac{1}{2}f_{u_k}(T)^2)\Big[\|J_x^s\omega(0)\|_{L_{xy}^2}\\
&+\|J_x^s\omega\|_{L_T^\infty L_{xy}^2}\|J_x^{s+1}u_k\|_{L_T^\infty L_{xy}^2}(\|\omega\|_{L_T^2L_{xy}^\infty}^2+\|\omega\|_{L_T^2L_{xy}^\infty}\|u_k\|_{L_T^2L_{xy}^\infty})\\
&+\|J_x^s\omega\|_{L_T^\infty L_{xy}^2}\|J_x^su_k\|_{L_T^\infty L_{xy}^2}(\|\omega\|_{L_T^2L_{xy}^\infty}+\|\partial_x\omega\|_{L_T^2L_{xy}^\infty})\|u_k\|_{L_T^2L_{xy}^\infty}\\
&+\|J_x^s\omega\|_{L_T^\infty L_{xy}^2}\|J_x^su_k\|_{L_T^\infty L_{xy}^2}\|\omega\|_{L_T^2L_{xy}^\infty}(\|\partial_xu_k\|_{L_T^2L_{xy}^\infty}+\|\partial_x\omega\|_{L_T^2L_{xy}^\infty})\\
&+\|J_x^s\omega\|_{L_T^\infty L_{xy}^2}\|J_x^su_k\|_{L_T^\infty L_{xy}^2}\|\omega\|_{L_T^2L_{xy}^\infty}^2\Big].\end{aligned}$$

b)

$$\|J_y^s\omega\|_{L_T^\infty L_{xy}^2} \lesssim exp(\frac{1}{2}f_\omega(T)^2+\frac{1}{2}f_{u_k}(T)^2)\Big[\|J_y^s\omega(0)\|_{L_{xy}^2}$$
$$+\|J_y^s\omega\|_{L_T^\infty L_{xy}^2}\|J_y^s u_k\|_{L_{xy}^2}\|\omega\|_{L_T^2L_{xy}^\infty}(\|\partial_x u_k\|_{L_T^2L_{xy}^\infty}+\|\partial_y u_k\|_{L_T^2L_{xy}^\infty})$$
$$+\|J_y^s\omega\|_{L_T^\infty L_{xy}^2}\|J_y^s u_k\|_{L_{xy}^2}\Big(\|u_k\|_{L_T^2L_{xy}^\infty}\|\partial_y\omega\|_{L_T^2L_{xy}^\infty}+f_\omega(T)^2\Big)$$
$$+\|J_y^s\omega\|_{L_T^\infty L_{xy}^2}\|J_x^s u_k\|_{L_{xy}^2}\|\omega\|_{L_T^2L_{xy}^\infty}(\|u_k\|_{L_T^2L_{xy}^\infty}+\|\partial_y u_k\|_{L_T^2L_{xy}^\infty})$$
$$+\|J_y^s\omega\|_{L_T^\infty L_{xy}^2}\|J_x^s u_k\|_{L_{xy}^2}\Big(\|u_k\|_{L_T^2L_{xy}^\infty}\|\partial_y\omega\|_{L_T^2L_{xy}^\infty}+f_\omega(T)^2\Big)$$
$$+\|J_y^s\omega\|_{L_T^\infty L_{xy}^2}\|J_x^s \omega\|_{L_{xy}^2}\|\omega\|_{L_T^2L_{xy}^\infty}(\|u_k\|_{L_T^2L_{xy}^\infty}+\|\partial_y u_k\|_{L_T^2L_{xy}^\infty})$$
$$+\|J_y^s\omega\|_{L_T^\infty L_{xy}^2}\|J_x^s \omega\|_{L_{xy}^2}\Big(\|u_k\|_{L_T^2L_{xy}^\infty}\|\partial_y\omega\|_{L_T^2L_{xy}^\infty}+f_\omega(T)^2\Big)$$
$$+\|J_y^s\omega\|_{L_T^\infty L_{xy}^2}\|J_x^s \omega\|_{L_{xy}^2}\|u_k\|_{L_T^2L_{xy}^\infty}\|\partial_y u_k\|_{L_T^2L_{xy}^\infty}$$
$$+\|J_y^{s+1}u_k\|_{L_T^\infty L_{xy}^2}(\|\omega\|^2_{L_T^\infty L_{xy}^2}+\|\omega\|_{L_T^\infty L_{xy}^2}\|u_k\|_{L_T^\infty L_{xy}^2})\Big].$$

Proof.

$$\partial_t\omega-\partial_x^5\omega-\partial_x^{-1}\partial_y^2\omega+\omega^2\partial_x\omega+3u_k^2\partial_x\omega+3u_k\omega\partial_x u_k-3u_k\omega\partial_x\omega-3\omega^2\partial_x u_k=0. \quad (4.18)$$

a) We apply J_x^s to (4.18) and then we multiply by $J_x^s\omega$, in order to get

$$\frac{d}{dt}\|J_x^s\omega\|^2_{L^2}=\int J_x^s(\omega^2\partial_x\omega)J_x^s\omega+3\int J_x^s(u_k^2\partial_x\omega)J_x^s\omega+3\int J_x^s(u_k\omega\partial_x u_k)J_x^s\omega$$
$$-3\int J_x^s(u_k\omega\partial_x\omega)J_x^s\omega-3\int J_x^s(\omega^2\partial_x u_k)J_x^s\omega$$

and we will analyze each term in the sum.

We have $(I)=\int J_x^s(\omega^2\partial_x\omega)J_x^s\omega$, $(II)=\int J_x^s(u_k^2\partial_x\omega)J_x^s\omega$, $(III)=\int J_x^s(u_k\omega\partial_x u_k)J_x^s\omega$, $(IV)=\int J_x^s(u_k\omega\partial_x\omega)J_x^s\omega$ and $(V)=\int J_x^s(\omega^2\partial_x u_k)J_x^s\omega$.

For $(I)=\int J_x^s(\omega^2\partial_x\omega)J_x^s\omega=\int[J_x^s(\omega^2\partial_x\omega)-\omega^2J_x^s\partial_x\omega]J_x^s\omega+\int\omega^2J_x^s\partial_x\omega J_x^s\omega$, and we will denote $(I)_1=\int[J_x^s(\omega^2\partial_x\omega)-\omega^2J_x^s\partial_x\omega]J_x^s\omega$ and $(I)_2=\int\omega^2J_x^s\partial_x\omega J_x^s\omega$. For the first one, we have by the Kato-Ponce commutator estimate

$$
\begin{aligned}
(I)_1 &\le \|J_x^s\omega\|_{L^2_{xy}}\|J_x^s(\omega^2\partial_x\omega)-\omega^2 J_x^s\partial_x\omega\|_{L^2_{xy}}\\
&\le \|J_x^s\omega\|_{L^2_{xy}}\|\omega\|_{L^\infty_{xy}}\Big[\|\partial_x\omega\|_{L^\infty_{xy}}\cdot\|J_x^s\omega\|_{L^2_{xy}}+(\|\omega\|_{L^\infty_{xy}}+\|\partial_x\omega\|_{L^\infty_{xy}})\|J_x^{s-1}\partial_x\omega\|_{L^2_{xy}}\Big]\\
&\le \|J_x^s\omega\|^2_{L^2_{xy}}\cdot\beta_\omega(t)
\end{aligned}
$$

and

$$
(I)_2 \le \|J_x^s\omega\|^2_{L^2_{xy}}\|\omega\|_{L^\infty_{xy}}\|\partial_x\omega\|_{L^\infty_{xy}}\le \|J_x^s\omega\|^2_{L^2_{xy}}\beta_\omega(t)
$$

so $(I)\lesssim \|J_x^s\omega\|^2_{L^2_{xy}}\beta_\omega(t)$.

Now, $(II)=\int J_x^s(u_k^2\partial_x\omega)J_x^s\omega=\int[J_x^s(u_k^2\partial_x\omega)-u_k^2J_x^s\partial_x\omega]J_x^s\omega+\int u_k^2J_x^s\partial_x\omega J_x^s\omega$ and we denote $(II)_1=\int[J_x^s(u_k^2\partial_x\omega)-u_k^2J_x^s\partial_x\omega]J_x^s\omega$ and $(II)_2=\int u_k^2J_x^s\partial_x\omega J_x^s\omega$. For the first term we have by the Kato-Ponce commutator estimate

$$
\begin{aligned}
(II)_1 &\lesssim \|J_x^s\omega\|_{L^2_{xy}}\cdot\|J_x^s(u_k^2\partial_x\omega)-u_k^2J_x^s\partial_x\omega\|_{L^2_{xy}}\\
&\lesssim \|J_x^s\omega\|_{L^2_{xy}}\|u_k\|_{L^\infty_{xy}}\Big[\|\partial_x\omega\|_{L^\infty_{xy}}\|J_x^su_k\|_{L^2_{xy}}+(\|u_k\|_{L^\infty_{xy}}+\|\partial_xu_k\|_{L^\infty_{xy}})\|J_x^{s-1}\partial_x\omega\|_{L^2_{xy}}\Big]\\
&\lesssim \|J_x^s\omega\|^2_{L^2_{xy}}\beta_{u_k}(t)+\|J_x^s\omega\|_{L^2_{xy}}\|J_x^su_k\|_{L^2_{xy}}\|\partial_x\omega\|_{L^\infty_{xy}}\|u_k\|_{L^\infty_{xy}}
\end{aligned}
$$

Also, we have

$$
(II)_2\lesssim \|J_x^s\omega\|^2_{L^2_{xy}}\|u_k\|_{L^\infty}\|\partial_xu_k\|_{L^\infty}\lesssim \|J_x^s\omega\|^2_{L^2_{xy}}\beta_{u_k}(t)
$$

Therefore,

$$
(II)\lesssim \|J_x^s\omega\|^2_{L^2_{xy}}\beta_{u_k}(t)+\|J_x^s\omega\|_{L^2_{xy}}\|J_x^su_k\|_{L^2_{xy}}\|\partial_x\omega\|_{L^\infty_{xy}}\|u_k\|_{L^\infty_{xy}}.
$$

Let $(III)=\int J_x^s(u_k\omega\partial_xu_k)J_x^s\omega=\int[J_x^s(u_k\omega\partial_xu_k)-u_k\omega J_x^s\partial_xu_k]J_x^s\omega+\int u_k\omega J_x^s\partial_xu_kJ_x^s\omega$ and denote $(III)_1=\int[J_x^s(u_k\omega\partial_xu_k)-u_k\omega J_x^s\partial_xu_k]J_x^s\omega$ and $(III)_2=\int u_k\omega J_x^s\partial_xu_kJ_x^s\omega$.

We have by the Kato-Ponce commutator estimate

$$
\begin{aligned}
(III)_1 &\lesssim \|J_x^s\omega\|_{L^2_{xy}}\|J_x^s(u_k\omega\partial_x u_k) - u_k\omega J_x^s\partial_x u_k\|_{L^2_{xy}} \\
&\lesssim \|J_x^s\omega\|_{L^2_{xy}}\Big[\|J_x^s u_k\|_{L^2_{xy}}\|\omega\|_{L^\infty_{xy}}\|\partial_x u_k\|_{L^\infty_{xy}} \\
&+ \|J_x^{s-1}\omega\|_{L^2_{xy}}(\|u_k\|_{L^\infty_{xy}}\|\partial_x u_k\|_{L^\infty_{xy}}+\|\partial_x u_k\|^2_{L^\infty_{xy}}) \\
&+ \|J_x^{s-1}\partial_x u_k\|_{L^2_{xy}}(\|\omega\|_{L^\infty_{xy}}\|\partial_x u_k\|_{L^\infty_{xy}}+\|\omega\|_{L^\infty_{xy}}\|u_k\|_{L^\infty_{xy}}+\|\partial_x\omega\|_{L^\infty_{xy}}\|u_k\|_{L^\infty_{xy}})\Big] \\
&\lesssim \|J_x^s\omega\|^2_{L^2_{xy}}\beta_{u_k}(t) + \|J_x^s\omega\|_{L^2_{xy}}\|J_x^s u_k\|_{L^2_{xy}}\|\omega\|_{L^\infty_{xy}}\|\partial_x u_k\|_{L^\infty_{xy}} \\
&+ \|J_x^s\omega\|_{L^2_{xy}}\|J_x^s u_k\|_{L^2_{xy}}\|\omega\|_{L^\infty_{xy}}\|u_k\|_{L^\infty_{xy}}+\|J_x^s\omega\|_{L^2_{xy}}\|J_x^s u_k\|_{L^2_{xy}}\|\partial_x\omega\|_{L^\infty_{xy}}\|u_k\|_{L^\infty_{xy}}
\end{aligned}
$$

Also, $(III)_2 \lesssim \|J_x^s\omega\|_{L^2_{xy}}\|J_x^{s+1}u_k\|_{L^2_{xy}}\|\omega\|_{L^\infty_{xy}}\|u_k\|_{L^\infty_{xy}}$ and so therefore

$$
\begin{aligned}
(III) &\lesssim \|J_x^s\omega\|^2_{L^2_{xy}}\beta_{u_k}(t) + \|J_x^s\omega\|_{L^2_{xy}}\|J_x^s u_k\|_{L^2_{xy}}\|\omega\|_{L^\infty_{xy}}\|\partial_x u_k\|_{L^\infty_{xy}} \\
&+ \|J_x^s\omega\|_{L^2_{xy}}\|J_x^s u_k\|_{L^2_{xy}}\|\omega\|_{L^\infty_{xy}}\|u_k\|_{L^\infty_{xy}}+\|J_x^s\omega\|_{L^2_{xy}}\|J_x^s u_k\|_{L^2_{xy}}\|\partial_x\omega\|_{L^\infty_{xy}}\|u_k\|_{L^\infty_{xy}} \\
&+ \|J_x^s\omega\|_{L^2_{xy}}\|J_x^{s+1}u_k\|_{L^2_{xy}}\|\omega\|_{L^\infty_{xy}}\|u_k\|_{L^\infty_{xy}}.
\end{aligned}
$$

Again, $(IV) = \int J_x^s(u_k\omega\partial_x\omega)J_x^s\omega = \int[J_x^s(u_k\omega\partial_x\omega)-u_k\omega J_x^s\partial_x\omega]J_x^s\omega+\int u_k\omega J_x^s\partial_x\omega J_x^s\omega$ and we denote $(IV)_1 = \int[J_x^s(u_k\omega\partial_x\omega)-u_k\omega J_x^s\partial_x\omega]J_x^s\omega$ and $(IV)_2 = \int u_k\omega J_x^s\partial_x\omega J_x^s\omega$. We have by the Kato-Ponce commutator estimate

$$
\begin{aligned}
(IV)_1 &\lesssim \|J_x^s\omega\|_{L^2_{xy}}\|J_x^s(u_k\omega\partial_x\omega) - u_k\omega J_x^s\partial_x\omega\|_{L^2_{xy}} \\
&\lesssim \|J_x^s\omega\|_{L^2_{xy}}\|\partial_x\omega\|_{L^\infty_{xy}}\Big[\|J_x^s\omega\|_{L^2_{xy}}(\|u_k\|_{L^\infty_{xy}}+\|\partial_x u_k\|_{L^\infty_{xy}}) + \|J_x^s u_k\|_{L^2_{xy}}\|\omega\|_{L^\infty_{xy}}\Big] \\
&+ \|J_x^s\omega\|_{L^2_{xy}}\Big[\|u_k\|_{L^\infty_{xy}}(\|\omega\|_{L^\infty_{xy}}+\|\partial_x\omega\|_{L^\infty_{xy}}) + \|\partial_x u_k\|_{L^\infty_{xy}}\|\omega\|_{L^\infty_{xy}})\|J_x^{s-1}\partial_x\omega\|_{L^2_{xy}} \\
&\lesssim \|J_x^s\omega\|^2_{L^2_{xy}}(\|u_k\|_{L^\infty_{xy}}+\|\partial_x u_k\|_{L^\infty_{xy}})(\|\partial_x\omega\|_{L^\infty_{xy}}+\|\omega\|_{L^\infty_{xy}}) \\
&+ \|J_x^s\omega\|_{L^2_{xy}}\|J_x^s u_k\|_{L^2_{xy}}\|\omega\|_{L^\infty_{xy}}\|\partial_x\omega\|_{L^\infty_{xy}}.
\end{aligned}
$$

Also, $(IV)_2 \lesssim \|J_x^s\omega\|_{L^2_{xy}}^2(\|\partial_x u_k\|_{L^\infty_{xy}}\|\omega\|_{L^\infty_{xy}}+\|\partial_x\omega\|_{L^\infty_{xy}}\|u_k\|_{L^\infty_{xy}})$ and so therefore

$$
\begin{aligned}
(IV) &\lesssim \|J_x^s\omega\|_{L^2_{xy}}^2(\|u_k\|_{L^\infty_{xy}}+\|\partial_x u_k\|_{L^\infty_{xy}})(\|\partial_x\omega\|_{L^\infty_{xy}}+\|\omega\|_{L^\infty_{xy}}) \\
&\quad + \|J_x^s\omega\|_{L^2_{xy}}\|J_x^s u_k\|_{L^2_{xy}}\|\omega\|_{L^\infty_{xy}}\|\partial_x\omega\|_{L^\infty_{xy}}.
\end{aligned}
$$

Again, $(V) = \int J_x^s(\omega^2\partial_x u_k)J_x^s\omega = \int[J_x^s(\omega^2\partial_x u_k) - \omega^2 J_x^s\partial_x u_k]J_x^s\omega + \int \omega^2 J_x^s\partial_x u_k J_x^s\omega$ and we denote $(V)_1 = \int[J_x^s(\omega^2\partial_x u_k) - \omega^2 J_x^s\partial_x u_k]J_x^s\omega$ and $(V)_2 = \int \omega^2 J_x^s\partial_x u_k J_x^s\omega$. We have by the Kato-Ponce commutator estimate

$$
\begin{aligned}
(V)_1 &\lesssim \|J_x^s\omega\|_{L^2_{xy}}\|J_x^s(\omega^2\partial_x u_k) - \omega^2 J_x^s\partial_x u_k\|_{L^2_{xy}} \\
&\lesssim \|J_x^s\omega\|_{L^2_{xy}}\|\omega\|_{L^\infty_{xy}}\Big[\|\partial_x u_k\|_{L^\infty_{xy}}\|J_x^s\omega\|_{L^2_{xy}}+(\|\omega\|_{L^\infty_{xy}}+\|\partial_x\omega\|_{L^\infty_{xy}})\|J_x^{s-1}\partial_x u_k\|_{L^2_{xy}}\Big] \\
&\lesssim \|J_x^s\omega\|_{L^2_{xy}}^2\|\omega\|_{L^\infty_{xy}}\|\partial_x u_k\|_{L^\infty_{xy}}+\|J_x^s\omega\|_{L^2_{xy}}\|J_x^s u_k\|_{L^2_{xy}}\|\omega\|_{L^\infty_{xy}}(\|\omega\|_{L^\infty_{xy}}+\|\partial_x\omega\|_{L^\infty_{xy}}).
\end{aligned}
$$

Also, $(V)_2 \lesssim \|J_x^s\omega\|_{L^2_{xy}}\|J_x^{s+1}u_k\|_{L^2_{xy}}\|\omega\|_{L^\infty_{xy}}^2$ and so therefore

$$
\begin{aligned}
(V) &\lesssim \|J_x^s\omega\|_{L^2_{xy}}^2(\beta_{u_k}(t)+\beta_\omega(t)) \\
&\quad + \|J_x^s\omega\|_{L^2_{xy}}\Big(\|J_x^s u_k\|_{L^2_{xy}}\|\omega\|_{L^\infty_{xy}}(\|\omega\|_{L^\infty_{xy}}+\|\partial_x\omega\|_{L^\infty_{xy}}) + \|J_x^{s+1}u_k\|_{L^2_{xy}}\|\omega\|_{L^\infty_{xy}}^2\Big).
\end{aligned}
$$

Now, putting together all the terms we get that

$$
\begin{aligned}
\frac{d}{dt}\|J_x^s\omega\|_{L^2_{xy}}^2 &\lesssim (\|J_x^s\omega\|_{L^2_{xy}}^2)(\beta_\omega(t)+\beta_{u_k}(t)) \\
&\quad + \|J_x^s\omega\|_{L^2_{xy}}\|J_x^{s+1}u_k\|_{L^2_{xy}}(\|\omega\|_{L^\infty_{xy}}^2+\|\omega\|_{L^\infty_{xy}}\|u_k\|_{L^\infty_{xy}}) \\
&\quad + \|J_x^s\omega\|_{L^2_{xy}}\|J_x^s u_k\|_{L^2_{xy}}(\|\omega\|_{L^\infty_{xy}}\|u_k\|_{L^\infty_{xy}}+\|\partial_x\omega\|_{L^\infty_{xy}}\|u_k\|_{L^\infty_{xy}}) \\
&\quad + \|J_x^s\omega\|_{L^2_{xy}}\|J_x^s u_k\|_{L^2_{xy}}(\|\omega\|_{L^\infty_{xy}}\|\partial_x u_k\|_{L^\infty_{xy}}+\|\omega\|_{L^\infty_{xy}}\|\partial_x\omega\|_{L^\infty_{xy}}+\|\omega\|_{L^\infty_{xy}}^2)
\end{aligned}
\tag{4.19}
$$

We are using the following variant of Grönwall's inequality:

Lemma 17. *If $\alpha(t), \beta(t)$ are two non-negative functions, and $\frac{d}{dt}u(t) \leq u(t)\beta(t) + \alpha(t)$ for all $t \in [0,T]$ then*

$$u(t) \leq e^{\int_0^t \beta(s)ds}\Big(u(0) + \int_0^t \alpha(s)ds\Big).$$

By putting $u(t) = \|J_x^s\omega\|_{L^2_{xy}}$, $\beta(t) = \beta_\omega(t) + \beta_{u_k}(t) \geq 0$ and

$$\begin{aligned}\alpha(t) = {} & \|J_x^{s+1}u_k\|_{L^2_{xy}}(\|\omega\|^2_{L^\infty_{xy}} + \|\omega\|_{L^\infty_{xy}}\|u_k\|_{L^\infty_{xy}}) \\ & + \|J_x^s u_k\|_{L^2_{xy}}(\|\omega\|_{L^\infty_{xy}}\|u_k\|_{L^\infty_{xy}} + \|\partial_x\omega\|_{L^\infty_{xy}}\|u_k\|_{L^\infty_{xy}}) \\ & + \|J_x^s u_k\|_{L^2_{xy}}(\|\omega\|_{L^\infty_{xy}}\|\partial_x u_k\|_{L^\infty_{xy}} + \|\omega\|_{L^\infty_{xy}}\|\partial_x\omega\|_{L^\infty_{xy}} + \|\omega\|^2_{L^\infty_{xy}}) \geq 0\end{aligned}$$

by applying the lemma to (4.19) together with Cauchy-Schwarz we get

$$\begin{aligned}\|J_x^s\omega\|_{L^\infty_T L^2_{xy}} \lesssim {} & \exp(\frac{1}{2}f_\omega(T)^2 + \frac{1}{2}f_{u_k}(T)^2)\Big[\|J_x^s\omega(0)\|_{L^2_{xy}} \\ & + \|J_x^s\omega\|_{L^\infty_T L^2_{xy}}\|J_x^{s+1}u_k\|_{L^\infty_T L^2_{xy}}(\|\omega\|^2_{L^2_T L^\infty_{xy}} + \|\omega\|_{L^2_T L^\infty_{xy}}\|u_k\|_{L^2_T L^\infty_{xy}}) \\ & + \|J_x^s\omega\|_{L^\infty_T L^2_{xy}}\|J_x^s u_k\|_{L^\infty_T L^2_{xy}}(\|\omega\|_{L^2_T L^\infty_{xy}} + \|\partial_x\omega\|_{L^2_T L^\infty_{xy}})\|u_k\|_{L^2_T L^\infty_{xy}} \\ & + \|J_x^s\omega\|_{L^\infty_T L^2_{xy}}\|J_x^s u_k\|_{L^\infty_T L^2_{xy}}\|\omega\|_{L^2_T L^\infty_{xy}}(\|\partial_x u_k\|_{L^2_T L^\infty_{xy}} + \|\partial_x\omega\|_{L^2_T L^\infty_{xy}}) \\ & + \|J_x^s\omega\|_{L^\infty_T L^2_{xy}}\|J_x^s u_k\|_{L^\infty_T L^2_{xy}}\|\omega\|^2_{L^2_T L^\infty_{xy}}\Big].\end{aligned}$$

b) We apply J_y^s to (4.18) and then we multiply by $J_y^s\omega$, in order to get

$$\begin{aligned}\frac{d}{dt}\|J_y^s\omega\|^2_{L^2} = {} & \int J_y^s(\omega^2\partial_x\omega)J_y^s\omega + 3\int J_y^s(u_k^2\partial_x\omega)J_y^s\omega + 3\int J_y^s(u_k\omega\partial_x u_k)J_y^s\omega \\ & - 3\int J_y^s(u_k\omega\partial_x\omega)J_y^s\omega - 3\int J_y^s(\omega^2\partial_x u_k)J_y^s\omega\end{aligned}$$

and we will analyze each term in the sum.

We have $(I) = \int J_y^s(\omega^2\partial_x\omega)J_y^s\omega$, $(II) = \int J_y^s(u_k^2\partial_x\omega)J_y^s\omega$, $(III) = \int J_y^s(u_k\omega\partial_x u_k)J_y^s\omega$,

$(IV) = \int J_y^s(u_k\omega\partial_x\omega)J_y^s\omega$ and $(V) = \int J_y^s(\omega^2\partial_x u_k)J_y^s\omega$.

For $(I) = \int J_y^s(\omega^2\partial_x\omega)J_y^s\omega = \int[J_y^s(\omega^2\partial_x\omega) - \omega^2 J_y^s\partial_x\omega]J_y^s\omega + \int \omega^2 J_y^s\partial_x\omega J_y^s\omega$, and we will denote $(I)_1 = \int[J_y^s(\omega^2\partial_x\omega) - \omega^2 J_y^s\partial_x\omega]J_y^s\omega$ and $(I)_2 = \int \omega^2 J_y^s\partial_x\omega J_y^s\omega$. For the first one, we have by the Kato-Ponce commutator estimate

$$
\begin{aligned}
(I)_1 &\leq \|J_y^s\omega\|_{L^2_{xy}}\|J_y^s(\omega^2\partial_x\omega) - \omega^2 J_y^s\partial_x\omega\|_{L^2_{xy}} \\
&\leq \|J_y^s\omega\|_{L^2_{xy}}\|\omega\|_{L^\infty_{xy}}\Big[\|\partial_x\omega\|_{L^\infty_{xy}}\cdot\|J_y^s\omega\|_{L^2_{xy}}+(\|\omega\|_{L^\infty_{xy}}+\|\partial_y\omega\|_{L^\infty_{xy}})\|J_y^{s-1}\partial_x\omega\|_{L^2_{xy}}\Big] \\
&\leq \|J_y^s\omega\|_{L^2_{xy}}\Big[\|\partial_x\omega\|_{L^\infty_{xy}}\cdot\|J_y^s\omega\|_{L^2_{xy}}\|\omega\|_{L^\infty_{xy}} \\
&+ (\|\omega\|^2_{L^\infty_{xy}}+\|\omega\|_{L^\infty}\|\partial_y\omega\|_{L^\infty_{xy}})(\|J_y^s\omega\|_{L^2_{xy}}+\|J_x^s\omega\|_{L^2_{xy}})\Big] \\
&\leq (\|J_y^s\omega\|^2_{L^2_{xy}}+\|J_y^s\omega\|_{L^2_{xy}}\|J_x^s\omega\|_{L^2_{xy}})\cdot\beta_\omega(t)
\end{aligned}
$$

and

$$
(I)_2 \leq \|J_y^s\omega\|^2_{L^2_{xy}}\|\omega\|_{L^\infty_{xy}}\|\partial_x\omega\|_{L^\infty_{xy}} \leq \|J_y^s\omega\|^2_{L^2_{xy}}\beta_\omega(t)
$$

so $(I) \lesssim (\|J_y^s\omega\|^2_{L^2_{xy}}+\|J_y^s\omega\|_{L^2_{xy}}\|J_x^s\omega\|_{L^2_{xy}})\beta_\omega(t)$.

Now, $(II) = \int J_y^s(u_k^2\partial_x\omega)J_y^s\omega = \int[J_y^s(u_k^2\partial_x\omega) - u_k^2 J_y^s\partial_x\omega]J_y^s\omega + \int u_k^2 J_y^s\partial_x\omega J_y^s\omega$ and we denote $(II)_1 = \int[J_y^s(u_k^2\partial_x\omega) - u_k^2 J_y^s\partial_x\omega]J_y^s\omega$ and $(II)_2 = \int u_k^2 J_y^s\partial_x\omega J_y^s\omega$. For the first term we have by the Kato-Ponce commutator estimate

$$
\begin{aligned}
(II)_1 &\lesssim \|J_y^s\omega\|_{L^2_{xy}}\cdot\|J_y^s(u_k^2\partial_x\omega) - u_k^2 J_y^s\partial_x\omega\|_{L^2_{xy}} \\
&\lesssim \|J_y^s\omega\|_{L^2_{xy}}\|u_k\|_{L^\infty_{xy}}\Big[\|\partial_x\omega\|_{L^\infty_{xy}}\|J_y^s u_k\|_{L^2_{xy}}+(\|u_k\|_{L^\infty_{xy}}+\|\partial_y u_k\|_{L^\infty})\|J_x^{s-1}\partial_x\omega\|_{L^2_{xy}}\Big] \\
&\lesssim \|J_y^s\omega\|^2_{L^2_{xy}}\beta_{u_k}(t) + \|J_y^s\omega\|_{L^2_{xy}}\|J_y^s u_k\|_{L^2_{xy}}\|\partial_y\omega\|_{L^\infty_{xy}}\|u_k\|_{L^\infty_{xy}} \\
&+ \|J_x^s\omega\|_{L^2_{xy}}\|J_y^s\omega\|_{L^2_{xy}}\|\partial_y u_k\|_{L^\infty_{xy}}\|u_k\|_{L^\infty_{xy}}
\end{aligned}
$$

where here we used that $\|J_x^{s-1}\partial_x\omega\|_{L^2_{xy}}\leq |J_y^s\omega\|_{L^2_{xy}}+|J_x^s\omega\|_{L^2_{xy}}$. Also, we have

$$(II)_2 \lesssim \|J_y^s\omega\|^2_{L^2_{xy}}\|u_k\|_{L^\infty}\|\partial_x u_k\|_{L^\infty}\lesssim \|J_y^s\omega\|^2_{L^2_{xy}}\beta_{u_k}(t)$$

Therefore,

$$\begin{aligned}(II) &\lesssim \|J_y^s\omega\|^2_{L^2_{xy}}\beta_{u_k}(t) + \|J_y^s\omega\|_{L^2_{xy}}\|J_y^s u_k\|_{L^2_{xy}}\|\partial_y\omega\|_{L^\infty_{xy}}\|u_k\|_{L^\infty_{xy}}\\ &\quad + \|J_x^s\omega\|_{L^2_{xy}}\|J_y^s\omega\|_{L^2_{xy}}\|\partial_y u_k\|_{L^\infty_{xy}}\|u_k\|_{L^\infty_{xy}}\end{aligned}$$

Let $(III)=\int J_y^s(u_k\omega\partial_x u_k)J_y^s\omega=\int[J_y^s(u_k\omega\partial_x u_k)-u_k\omega J_y^s\partial_x u_k]J_y^s\omega+\int u_k\omega J_y^s\partial_x u_k J_y^s\omega$ and denote $(III)_1=\int[J_y^s(u_k\omega\partial_x u_k)-u_k\omega J_y^s\partial_x u_k]J_y^s\omega$ and $(III)_2=\int u_k\omega J_y^s\partial_x u_k J_y^s\omega$. We have by the Kato-Ponce commutator estimate

$$\begin{aligned}(III)_1 &\lesssim \|J_y^s\omega\|_{L^2_{xy}}\|J_y^s(u_k\omega\partial_x u_k)-u_k\omega J_y^s\partial_x u_k\|_{L^2_{xy}}\\ &\lesssim \|J_y^s\omega\|_{L^2_{xy}}\Big[\|J_y^s u_k\|_{L^2_{xy}}\|\omega\|_{L^\infty_{xy}}\|\partial_x u_k\|_{L^\infty_{xy}}\\ &\quad + \|J_y^{s-1}\omega\|_{L^2_{xy}}(\|\partial_y u_k\|_{L^\infty_{xy}}\|\partial_x u_k\|_{L^\infty_{xy}}+\|u_k\|_{L^\infty_{xy}}\|\partial_x u_k\|_{L^\infty_{xy}})\\ &\quad + \|J_y^{s-1}\partial_x u_k\|_{L^2_{xy}}(\|\omega\|_{L^\infty_{xy}}\|\partial_y u_k\|_{L^\infty_{xy}}+\|u_k\|_{L^\infty_{xy}}\|\partial_y\omega\|_{L^\infty_{xy}})\\ &\quad + \|J_y^s\omega\|_{L^2_{xy}}\|u_k\|_{L^\infty_{xy}}\|\partial_x u_k\|_{L^\infty_{xy}}\Big]\\ &\lesssim \|J_y^s\omega\|^2_{L^2_{xy}}\beta_{u_k}(t)+\|J_y^s\omega\|_{L^2_{xy}}\|J_x^s u_k\|_{L^2_{xy}}(\|\omega\|_{L^\infty_{xy}}\|\partial_y u_k\|_{L^\infty_{xy}}+\|u_k\|_{L^\infty_{xy}}\|\partial_y\omega\|_{L^\infty_{xy}})\\ &\quad + \|J_y^s\omega\|_{L^2_{xy}}\|J_y^s u_k\|_{L^2_{xy}}(\|\omega\|_{L^\infty_{xy}}\|\partial_x u_k\|_{L^\infty_{xy}}+\|\omega\|_{L^\infty_{xy}}\|\partial_y u_k\|_{L^\infty_{xy}}+\|u_k\|_{L^\infty_{xy}}\|\partial_y\omega\|_{L^\infty_{xy}}).\end{aligned}$$

Also,

$$(III)_2 \lesssim \|J_y^s\omega\|_{L^2_{xy}}\|J_y^{s+1}u_k\|_{L^2_{xy}}\|\omega\|_{L^\infty_{xy}}\|u_k\|_{L^\infty_{xy}}+\|J_y^s\omega\|_{L^2_{xy}}\|J_x^s u_k\|_{L^2_{xy}}\|\omega\|_{L^\infty_{xy}}\|u_k\|_{L^\infty_{xy}}$$

and so therefore

$$
\begin{aligned}
(III) \lesssim\ & \|J_y^s\omega\|_{L^2_{xy}}^2\beta_{u_k}(t) + \|J_y^s\omega\|_{L^2_{xy}}\|J_y^{s+1}u_k\|_{L^2_{xy}}\|\omega\|_{L^\infty_{xy}}\|u_k\|_{L^\infty_{xy}} \\
& + \|J_y^s\omega\|_{L^2_{xy}}\|J_y^s u_k\|_{L^2_{xy}}(\|\omega\|_{L^\infty_{xy}}\|\partial_x u_k\|_{L^\infty_{xy}}+\|\omega\|_{L^\infty_{xy}}\|\partial_y u_k\|_{L^\infty_{xy}}+\|u_k\|_{L^\infty_{xy}}\|\partial_y\omega\|_{L^\infty_{xy}}) \\
& + \|J_y^s\omega\|_{L^2_{xy}}\|J_x^s u_k\|_{L^2_{xy}}(\|\omega\|_{L^\infty_{xy}}\|\partial_y u_k\|_{L^\infty_{xy}}+\|u_k\|_{L^\infty_{xy}}\|\partial_y\omega\|_{L^\infty_{xy}}+\|\omega\|_{L^\infty_{xy}}\|u_k\|_{L^\infty_{xy}}).
\end{aligned}
$$

Again, $(IV) = \int J_y^s(u_k\omega\partial_x\omega)J_y^s\omega = \int [J_y^s(u_k\omega\partial_x\omega) - u_k\omega J_y^s\partial_x\omega]J_y^s\omega + \int u_k\omega J_y^s\partial_x\omega J_y^s\omega$ and we denote $(IV)_1 = \int [J_y^s(u_k\omega\partial_x\omega) - u_k\omega J_y^s\partial_x\omega]J_y^s\omega$ and $(IV)_2 = \int u_k\omega J_y^s\partial_x\omega J_y^s\omega$. We have by the Kato-Ponce commutator estimate

$$
\begin{aligned}
(IV)_1 \lesssim\ & \|J_y^s\omega\|_{L^2_{xy}}\|J_y^s(u_k\omega\partial_x\omega) - u_k\omega J_y^s\partial_x\omega\|_{L^2_{xy}} \\
\lesssim\ & \|J_y^s\omega\|_{L^2_{xy}}\Big[\|\partial_x\omega\|_{L^\infty_{xy}}\|J_y^{s-1}\omega\|_{L^2_{xy}}(\|u_k\|_{L^\infty_{xy}}+\|\partial_y u_k\|_{L^\infty_{xy}}) \\
& + (\|u_k\|_{L^\infty_{xy}}\|\omega\|_{L^\infty_{xy}}+\|u_k\|_{L^\infty_{xy}}\|\partial_y\omega\|_{L^\infty_{xy}}+\|\partial_y u_k\|_{L^\infty_{xy}}\|\omega\|_{L^\infty_{xy}})\|J_y^{s-1}\partial_x\omega\|_{L^2_{xy}} \\
& + \|\partial_x\omega\|_{L^\infty_{xy}}\|J_y^s u_k\|_{L^2_{xy}}\|\omega\|_{L^\infty_{xy}}\Big] \\
\lesssim\ & \|J_y^s\omega\|_{L^2_{xy}}^2(\beta_{u_k}(t)+\beta_\omega(t)) + \|J_y^s\omega\|_{L^2_{xy}}\|J_y^s u_k\|_{L^2_{xy}}\|\omega\|_{L^\infty_{xy}}\|\partial_x\omega\|_{L^\infty_{xy}} \\
& + \|J_y^s\omega\|_{L^2_{xy}}\|J_x^s\omega\|_{L^2_{xy}}(\|u_k\|_{L^\infty_{xy}}\|\omega\|_{L^\infty_{xy}}+\|u_k\|_{L^\infty_{xy}}\|\partial_y\omega\|_{L^\infty_{xy}}+\|\partial_y u_k\|_{L^\infty_{xy}}\|\omega\|_{L^\infty_{xy}}).
\end{aligned}
$$

Also, $(IV)_2 \lesssim \|J_y^s\omega\|_{L^2_{xy}}^2(\|\partial_x u_k\|_{L^\infty_{xy}}\|\omega\|_{L^\infty_{xy}}+\|\partial_x\omega\|_{L^\infty_{xy}}\|u_k\|_{L^\infty_{xy}}) \lesssim \|J_y^s\omega\|_{L^2_{xy}}^2(\beta_{u_k}(t) + \beta_\omega(t))$ and so therefore

$$
\begin{aligned}
(IV) \lesssim\ & \|J_y^s\omega\|_{L^2_{xy}}^2(\beta_{u_k}(t)+\beta_\omega(t)) + \|J_y^s\omega\|_{L^2_{xy}}\|J_y^s u_k\|_{L^2_{xy}}\|\omega\|_{L^\infty_{xy}}\|\partial_x\omega\|_{L^\infty_{xy}} \\
& + \|J_y^s\omega\|_{L^2_{xy}}\|J_x^s\omega\|_{L^2_{xy}}(\|u_k\|_{L^\infty_{xy}}\|\omega\|_{L^\infty_{xy}}+\|u_k\|_{L^\infty_{xy}}\|\partial_y\omega\|_{L^\infty_{xy}}+\|\partial_y u_k\|_{L^\infty_{xy}}\|\omega\|_{L^\infty_{xy}}).
\end{aligned}
$$

Again, $(V) = \int J_y^s(\omega^2\partial_x u_k)J_y^s\omega = \int [J_y^s(\omega^2\partial_x u_k) - \omega^2 J_y^s\partial_x u_k]J_y^s\omega + \int \omega^2 J_y^s\partial_x u_k J_y^s\omega$ and we denote $(V)_1 = \int [J_y^s(\omega^2\partial_x u_k) - \omega^2 J_y^s\partial_x u_k]J_y^s\omega$ and $(V)_2 = \int \omega^2 J_y^s\partial_x u_k J_y^s\omega$.

We have by the Kato-Ponce commutator estimate

$$\begin{aligned}(V)_1 &\lesssim \|J_y^s\omega\|_{L^2_{xy}}\|J_y^s(\omega^2\partial_x u_k) - \omega^2 J_y^s\partial_x u_k\|_{L^2_{xy}} \\ &\lesssim \|J_y^s\omega\|_{L^2_{xy}}\|\omega\|_{L^\infty_{xy}}\Big[\|\partial_x u_k\|_{L^\infty_{xy}}\|J_y^s\omega\|_{L^2_{xy}} + (\|\omega\|_{L^\infty_{xy}} + \|\partial_y\omega\|_{L^\infty_{xy}})\|J_y^{s-1}\partial_x u_k\|_{L^2_{xy}}\Big] \\ &\lesssim \|J_y^s\omega\|^2_{L^2_{xy}}(\beta_{u_k}(t) + \beta_\omega(t)) + \|J_y^s\omega\|_{L^2_{xy}}(\|J_y^s u_k\|_{L^2_{xy}} + \|J_x^s u_k\|_{L^2_{xy}})\beta_\omega(t).\end{aligned}$$

Also,

$$(V)_2 \lesssim \|J_y^s\omega\|_{L^2_{xy}}\|J_y^{s+1}u_k\|_{L^2_{xy}}\|\omega\|^2_{L^\infty_{xy}} + \|J_y^s\omega\|_{L^2_{xy}}\|J_x^s u_k\|_{L^2_{xy}}\|\omega\|^2_{L^\infty_{xy}}$$

and so therefore

$$\begin{aligned}(V) &\lesssim \|J_y^s\omega\|^2_{L^2_{xy}}(\beta_{u_k}(t) + \beta_\omega(t)) + \|J_y^s\omega\|_{L^2_{xy}}\|J_y^s u_k\|_{L^2_{xy}}\beta_\omega(t) \\ &\quad + \|J_y^s\omega\|_{L^2_{xy}}\|J_x^s u_k\|_{L^2_{xy}}\beta_\omega(t) + \|J_y^s\omega\|_{L^2_{xy}}\|J_x^{s+1}u_k\|_{L^2_{xy}}\|\omega\|^2_{L^\infty_{xy}}.\end{aligned}$$

We make the following notation:

$$a(\omega, u_k) = \|\omega\|_{L^\infty_{xy}}(\|\partial_x u_k\|_{L^\infty_{xy}} + \|\partial_y u_k\|_{L^\infty_{xy}}) + \|u_k\|_{L^\infty_{xy}}\|\partial_y\omega\|_{L^\infty_{xy}} + \beta_\omega(t),$$

$$b(\omega, u_k) = \|\omega\|_{L^\infty_{xy}}(\|\partial_y u_k\|_{L^\infty_{xy}} + \|u_k\|_{L^\infty_{xy}}) + \|u_k\|_{L^\infty_{xy}}\|\partial_y\omega\|_{L^\infty_{xy}} + \beta_\omega(t),$$

$$c(\omega, u_k) = \|\omega\|_{L^\infty_{xy}}(\|\partial_y u_k\|_{L^\infty_{xy}} + \|u_k\|_{L^\infty_{xy}}) + \|u_k\|_{L^\infty_{xy}}\|\partial_y\omega\|_{L^\infty_{xy}} + \|u_k\|_{L^\infty_{xy}}\|\partial_y u_k\|_{L^\infty_{xy}} + \beta_\omega(t).$$

Now, putting together all the terms we get that

$$
\begin{aligned}
\frac{d}{dt}\|J_y^s\omega\|_{L^2_{xy}}^2 \lesssim \|J_y^s\omega\|_{L^2_{xy}}^2(\beta_\omega(t)+\beta_{u_k}(t)) \\
&+ \|J_y^s\omega\|_{L^2_{xy}}\|J_y^s u_k\|_{L^2_{xy}}a(\omega,u_k) \\
&+ \|J_y^s\omega\|_{L^2_{xy}}\|J_x^s u_k\|_{L^2_{xy}}b(\omega,u_k) \\
&+ \|J_y^s\omega\|_{L^2_{xy}}\|J_x^s \omega\|_{L^2_{xy}}c(\omega,u_k) \\
&+ \|J_y^s\omega\|_{L^2_{xy}}\|J_x^{s+1} u_k\|_{L^2_{xy}}(\|\omega\|_{L^\infty_{xy}}^2+\|\omega\|_{L^\infty_{xy}}\|u_k\|_{L^\infty_{xy}}).
\end{aligned}
\tag{4.20}
$$

Using the variant of Grönwall's inequality from part a) and applying it to (4.20) with $u(t)=\|J_y^s\omega\|_{L^2_{xy}}$, $\beta(t)=\beta_\omega(t)+\beta_{u_k}(t)\geq 0$ and

$$
\begin{aligned}
\alpha(t) = \|J_y^s\omega\|_{L^2_{xy}}\|J_y^s u_k\|_{L^2_{xy}}a(\omega,u_k) \\
&+ \|J_y^s\omega\|_{L^2_{xy}}\|J_x^s u_k\|_{L^2_{xy}}b(\omega,u_k) \\
&+ \|J_y^s\omega\|_{L^2_{xy}}\|J_x^s \omega\|_{L^2_{xy}}c(\omega,u_k) \\
&+ \|J_y^s\omega\|_{L^2_{xy}}\|J_x^{s+1} u_k\|_{L^2_{xy}}(\|\omega\|_{L^\infty_{xy}}^2+\|\omega\|_{L^\infty_{xy}}\|u_k\|_{L^\infty_{xy}}).
\end{aligned}
$$

we obtain

$$
\begin{aligned}
\|J_y^s\omega\|_{L_T^\infty L_{xy}^2} \lesssim \exp(\frac{1}{2}f_\omega(T)^2+\frac{1}{2}f_{u_k}(T)^2)\Big[&\|J_y^s\omega(0)\|_{L_{xy}^2}\\
&+\|J_y^s\omega\|_{L_T^\infty L_{xy}^2}\|J_y^s u_k\|_{L_{xy}^2}\|\omega\|_{L_T^2L_{xy}^\infty}(\|\partial_x u_k\|_{L_T^2L_{xy}^\infty}+\|\partial_y u_k\|_{L_T^2L_{xy}^\infty})\\
&+\|J_y^s\omega\|_{L_T^\infty L_{xy}^2}\|J_y^s u_k\|_{L_{xy}^2}\Big(\|u_k\|_{L_T^2L_{xy}^\infty}\|\partial_y\omega\|_{L_T^2L_{xy}^\infty}+f_\omega(T)^2\Big)\\
&+\|J_y^s\omega\|_{L_T^\infty L_{xy}^2}\|J_x^s u_k\|_{L_{xy}^2}\|\omega\|_{L_T^2L_{xy}^\infty}(\|u_k\|_{L_T^2L_{xy}^\infty}+\|\partial_y u_k\|_{L_T^2L_{xy}^\infty})\\
&+\|J_y^s\omega\|_{L_T^\infty L_{xy}^2}\|J_x^s u_k\|_{L_{xy}^2}\Big(\|u_k\|_{L_T^2L_{xy}^\infty}\|\partial_y\omega\|_{L_T^2L_{xy}^\infty}+f_\omega(T)^2\Big)\\
&+\|J_y^s\omega\|_{L_T^\infty L_{xy}^2}\|J_x^s \omega\|_{L_{xy}^2}\|\omega\|_{L_T^2L_{xy}^\infty}(\|u_k\|_{L_T^2L_{xy}^\infty}+\|\partial_y u_k\|_{L_T^2L_{xy}^\infty})\\
&+\|J_y^s\omega\|_{L_T^\infty L_{xy}^2}\|J_x^s \omega\|_{L_{xy}^2}\Big(\|u_k\|_{L_T^2L_{xy}^\infty}\|\partial_y\omega\|_{L_T^2L_{xy}^\infty}+f_\omega(T)^2\Big)\\
&+\|J_y^s\omega\|_{L_T^\infty L_{xy}^2}\|J_x^s \omega\|_{L_{xy}^2}\|u_k\|_{L_T^2L_{xy}^\infty}\|\partial_y u_k\|_{L_T^2L_{xy}^\infty}\\
&+\|J_y^{s+1}u_k\|_{L_T^\infty L_{xy}^2}(\|\omega\|_{L_T^\infty L_{xy}^2}^2+\|\omega\|_{L_T^\infty L_{xy}^2}\|u_k\|_{L_T^\infty L_{xy}^2})\Big].
\end{aligned}
$$

□

Lemma 18. *For $p \le s$, we have the following estimates:*

(a)

$$
\begin{aligned}
\|J_x^p[\omega(u_k^2+u_ku_{k'}+u_{k'}^2)]\|_{L_T^1L_{xy}^2} \lesssim\ &\|J_x^p\omega\|_{L_T^\infty L_{xy}^2}(\|u_k\|_{L_T^2L_{xy}^\infty}+\|u_{k'}\|_{L_T^2L_{xy}^\infty})\\
&+\|\omega\|_{L_T^2L_{xy}^\infty}\|\phi\|_{H^{s,s}}(\|u_k\|_{L_T^2L_{xy}^\infty}+\|u_{k'}\|_{L_T^2L_{xy}^\infty}).
\end{aligned}
$$

(b)

$$
\begin{aligned}
\|J_y^p[\omega(u_k^2+u_ku_{k'}+u_{k'}^2)]\|_{L_T^1L_{xy}^2} \lesssim\ &\|J_y^p\omega\|_{L_T^\infty L_{xy}^2}(\|u_k\|_{L_T^2L_{xy}^\infty}+\|u_{k'}\|_{L_T^2L_{xy}^\infty})\\
&+\|\omega\|_{L_T^2L_{xy}^\infty}\|\phi\|_{H^{s,s}}(\|u_k\|_{L_T^2L_{xy}^\infty}+\|u_{k'}\|_{L_T^2L_{xy}^\infty}).
\end{aligned}
$$

Proof. By using 7 part (c), we get that

$$\|J_x^p[\omega(u_k^2+u_ku_{k'}+u_{k'}^2)]\|_{L^2_{xy}} \lesssim \|J_x^p\omega\|_{L^\infty_T L^2_{xy}}(\|u_k^2+u_ku_{k'}+u_{k'}^2\|_{L^\infty_{xy}} \\ \|\omega\|_{L^\infty_{xy}}\|J_x^p(u_k^2+u_ku_{k'}+u_{k'}^2)\|_{L^2_{xy}}. \tag{4.21}$$

Observe that $\|u_k^2+u_ku_{k'}+u_{k'}^2\|_{L^\infty_{xy}}\lesssim \|u_k\|^2_{L^\infty_{xy}}+\|u_{k'}\|^2_{L^\infty_{xy}}$. Also, by 7 part (c) again, we have

$$\|J_x^p(u_k^2+u_ku_{k'}+u_{k'}^2)\|_{L^2_{xy}}\lesssim (\|J_x^pu_k\|_{L^2_{xy}}+\|J_x^pu_{k'}\|_{L^2_{xy}})(\|u_k\|_{L^\infty}+\|u_{k'}^2\|_{L^\infty_{xy}}).$$

By 4.12, we get $\|J_x^pu_k\|_{L^2_{xy}}+\|J_x^pu_{k'}\|_{L^2_{xy}}\lesssim \|J_x^p\phi_k\|_{L^2_{xy}}+\|J_x^p\phi_{k'}\|_{L^2_{xy}}\lesssim \|\phi\|_{H^{s,s}}$. Combining all the above observation together with 4.21, we get

$$\|J_x^p[\omega(u_k^2+u_ku_{k'}+u_{k'}^2)]\|_{L^2_{xy}} \lesssim \|J_x^p\omega\|_{L^2_{xy}}(\|u_k\|_{L^\infty_{xy}}+\|u_{k'}\|_{L^\infty_{xy}}) \\ +\|\omega\|_{L^2_TL^\infty_{xy}}\|\phi\|_{H^{s,s}}(\|u_k\|_{L^\infty_{xy}}+\|u_{k'}\|_{L^\infty_{xy}}).$$

Integrating both sides from 0 to T and applying Cauchy-Schwarz inequality, we obtain the conclusion of the lemma for J_x. The proof for J_y goes the same way. □

Lemma 19. *Suppose u_k satisfies the IVP (1.11) with initial data $\phi_k = P^k_{(5)}\phi$. We have $\|\omega\|_{L^2_TL^\infty_{xy}}\lesssim k^{(-1)-}$, $\|\partial_x\omega\|_{L^2_TL^\infty_{xy}}\lesssim k^{0-}$ and $\|\partial_y\omega\|_{L^2_TL^\infty_{xy}}\lesssim k^{0-}$ as $k\to\infty$. In particular, $f_\omega(T)\lesssim k^{0-}$ as $k\to\infty$.*

Proof. Take $\delta<\frac{s-\frac{5}{2}}{2}$. By the linear estimate in Proposition 4.2 applied to 4.18,

$$\|\omega\|_{L^2_TL^\infty_{xy}}\lesssim \|J_x^{\frac{3}{2}+\delta}\omega\|_{L^\infty_TL^2_{xy}}+\|J_x^{-\frac{3}{2}+\delta}J_y^{1+\delta}\omega\|_{L^\infty_TL^2_{xy}}+\|J_x^{\frac{1}{2}+\delta}J_y^{\delta}[\omega(u_k^2+u_ku_{k'}+u_{k'}^2)]\|_{L^1_TL^2_{xy}}.$$

From 4.16 and 4.17 we have $\|J_x^{\frac{3}{2}+\delta}\omega\|_{L^\infty_TL^2_{xy}}\lesssim k^{\frac{3}{2}+\delta-s}h_\phi^{(5)}(k)^{1-\frac{\frac{3}{2}+\delta}{s}}$, together with

$$\|J_x^{-\frac{3}{2}+\delta}J_y^{1+\delta}\omega\|_{L^\infty_TL^2_{xy}}\lesssim \|J_y^{1+\delta}\omega\|_{L^\infty_TL^2_{xy}}\lesssim k^{1+\delta-s}h_\phi^{(5)}(k)^{1-\frac{1+\delta}{s}}.$$

For the last term, we observe

$$\|J_x^{\frac{1}{2}+\delta}J_y^{\delta}[\omega(u_k^2+u_ku_{k'}+u_{k'}^2)]\|_{L_T^1L_{xy}^2}\lesssim \|J_x^{\frac{1}{2}+2\delta}[\omega(u_k^2+u_ku_{k'}+u_{k'}^2)]\|_{L_T^1L_{xy}^2} + \|J_y^{\frac{1}{2}+2\delta}[\omega(u_k^2+u_ku_{k'}+u_{k'}^2)]\|_{L_T^1L_{xy}^2}$$

By Lemma 18 we get that

$$\|J_x^{\frac{1}{2}+2\delta}[\omega(u_k^2+u_ku_{k'}+u_{k'}^2)]\|_{L_T^1L_{xy}^2}\lesssim \|J_x^{\frac{1}{2}+2\delta}\omega\|_{L_T^\infty L_{xy}^2}(\|u_k\|_{L_T^2L_{xy}^\infty}+\|u_{k'}\|_{L_T^2L_{xy}^\infty}) + \|\omega\|_{L_T^2L_{xy}^\infty}\|\phi\|_{H^{s,s}}(\|u_k\|_{L_T^2L_{xy}^\infty}+\|u_{k'}\|_{L_T^2L_{xy}^\infty})$$

and

$$\|J_y^{\frac{1}{2}+2\delta}[\omega(u_k^2+u_ku_{k'}+u_{k'}^2)]\|_{L_T^1L_{xy}^2}\lesssim \|J_y^{\frac{1}{2}+2\delta}\omega\|_{L_T^\infty L_{xy}^2}(\|u_k\|_{L_T^2L_{xy}^\infty}+\|u_{k'}\|_{L_T^2L_{xy}^\infty}) + \|\omega\|_{L_T^2L_{xy}^\infty}\|\phi\|_{H^{s,s}}(\|u_k\|_{L_T^2L_{xy}^\infty}+\|u_{k'}\|_{L_T^2L_{xy}^\infty}).$$

By 4.16 and 4.17 we have $\|J_x^{\frac{1}{2}+2\delta}\omega\|_{L_T^\infty L_{xy}^2}\lesssim k^{\frac{1}{2}+2\delta-s}h_\phi^{(5)}(k)^{1-\frac{\frac{1}{2}+2\delta}{s}}$ and $\|J_y^{\frac{1}{2}+2\delta}\omega\|_{L_T^\infty L_{xy}^2}\lesssim k^{\frac{1}{2}+2\delta-s}h_\phi^{(5)}(k)^{1-\frac{\frac{1}{2}+2\delta}{s}}$. By combining the previous observations, we obtain

$$\|\omega\|_{L_T^2L_{xy}^\infty}\lesssim k^{\frac{3}{2}+\delta-s}h_\phi^{(3)}(k)^{1-\frac{\frac{1}{2}+2\delta}{s}}\max(1,h_\phi^{(3)}(k)^{\frac{\delta-1}{s}}) + \|\omega\|_{L_T^2L_{xy}^\infty}\|\phi\|_{H^{s,s}}(\|u_k\|_{L_T^2L_{xy}^\infty}+\|u_{k'}\|_{L_T^2L_{xy}^\infty})$$

Since we consider that $\|\phi\|_{H^{s,s}}$ is small enough, such that $\|\phi\|_{H^{s,s}}(\|u_k\|_{L_T^2L_{xy}^\infty}+\|u_{k'}\|_{L_T^2L_{xy}^\infty})\leq \frac{1}{2}$, we get that

$$\|\omega\|_{L_T^2L_{xy}^\infty}\lesssim k^{\frac{3}{2}+2\delta-s}h_\phi^{(5)}(k)^{1-\frac{\frac{1}{2}+2\delta}{s}}\max(1,h_\phi^{(5)}(k)^{\frac{\delta-1}{s}})\to 0$$

as $k\to\infty$ since $\frac{3}{2}+2\delta<s$.

The linear estimate 4.2 applied to $\partial_x\omega$ results in

$$\|\partial_x\omega\|_{L^2_TL^\infty_{xy}}\lesssim \|J_x^{\frac{5}{2}+\delta}\omega\|_{L^\infty_TL^2_{xy}}+\|J_x^{-\frac{1}{2}+\delta}J_y^{1+\delta}\omega\|_{L^\infty_TL^2_{xy}}+\|J_x^{\frac{3}{2}+\delta}J_y^{\delta}[\omega(u_k^2+u_ku_{k'}+u_{k'}^2)]\|_{L^1_TL^2_{xy}}.$$

and by the same reasoning as above

$$\begin{aligned}\|\partial_x\omega\|_{L^2_TL^\infty_{xy}} &\lesssim k^{\frac{5}{2}+2\delta-s}h_\phi^{(5)}(k)^{1-\frac{\frac{3}{2}+2\delta}{s}}\max(1,h_\phi^{(3)}(k)^{\frac{\delta-1}{s}})\\ &\quad+\|\omega\|_{L^2_TL^\infty_{xy}}\|\phi\|_{H^{s,s}}(\|u_k\|_{L^2_TL^\infty_{xy}}+\|u_{k'}\|_{L^2_TL^\infty_{xy}})\end{aligned}$$

which, combined with the above fact that

$$\|\omega\|_{L^2_TL^\infty_{xy}}\lesssim k^{\frac{3}{2}+2\delta-s}h_\phi^{(5)}(k)^{1-\frac{\frac{3}{2}+2\delta}{s}}\max(1,h_\phi^{(5)}(k)^{\frac{\delta-1}{s}}),$$

for k large enough, it gives us $\|\partial_x\omega\|_{L^2_TL^\infty_{xy}}\lesssim k^{\frac{5}{2}+2\delta-s}\to 0$ as $k\to\infty$ since $\frac{5}{2}+2\delta<s$.

Lastly, the linear estimate 4.2 applied to $\partial_y\omega$ results in

$$\|\partial_y\omega\|_{L^2_TL^\infty_{xy}}\lesssim \|J_x^{\frac{3}{2}+\delta}J_y^{1}\omega\|_{L^\infty_TL^2_{xy}}+\|J_y^{2+\delta}\omega\|_{L^\infty_TL^2_{xy}}+\|J_x^{\frac{1}{2}+\delta}J_y^{\delta}[\omega(u_k^2+u_ku_{k'}+u_{k'}^2)]\|_{L^1_TL^2_{xy}}.$$

and by the same reasoning as above

$$\begin{aligned}\|\partial_y\omega\|_{L^2_TL^\infty_{xy}} &\lesssim k^{\frac{5}{2}+\delta-s}h_\phi^{(5)}(k)^{1-\frac{\frac{5}{2}+\delta}{s}}\max(1,h_\phi^{(5)}(k)^{\frac{\delta-1}{s}})\\ &\quad+\|\omega\|_{L^2_TL^\infty_{xy}}\|\phi\|_{H^{s,s}}(\|u_k\|_{L^2_TL^\infty_{xy}}+\|u_{k'}\|_{L^2_TL^\infty_{xy}})\end{aligned}$$

which, combined with the above fact that

$$\|\omega\|_{L^2_TL^\infty_{xy}}\lesssim k^{\frac{5}{2}+\delta-s}h_\phi^{(5)}(k)^{1-\frac{1+2\delta}{s}}\max(1,h_\phi^{(5)}(k)^{\frac{\delta-1}{s}}),$$

for k large enough, it gives us $\|\partial_x\omega\|_{L^2_TL^\infty_{xy}}\lesssim k^{\frac{5}{2}+2\delta-s}\to 0$ as $k\to\infty$ since $\frac{5}{2}+2\delta<s$.

$\square$

Corollary. *We have* $\|\omega\|_{H^{s,s}} \to 0$ *as* $k \to \infty$*, where* $s > \frac{5}{2}$ *for the initial value problem (1.11).*

Proof. From (4.14) and Lemma 19 we get $\|J_x^{s+1}u_k\|_{L_T^\infty L_{xy}^2}\|\omega\|_{L_T^2 L_{xy}^\infty} \lesssim k^1 \cdot k^{(-1)-} = k^{0-}$ and $k^{0-} \to 0$ as $k \to \infty$. From the Lemma 19 used in Lemma 16 we obtain

$$\|J_x^s\omega\|_{L_T^\infty L_{xy}^2} \lesssim \exp(\frac{1}{2}f_{u_k}(T)^2 + \frac{1}{2}f_\omega(T)^2)(\|J_x^s\omega(0)\|_{L_T^\infty L_{xy}^2} + Ck^{0-}) \to 0$$

as $k \to \infty$, where we used that $\|J_x^s\omega(0)\|_{L_T^\infty L_{xy}^2} \to 0$ as $k \to \infty$ and the boundedness of $f_{u_k}(T)$ and $f_\omega(T)$ by 7.

From (4.15) and Lemma 19 we get $\|J_y^{s+1}u_k\|_{L_T^\infty L_{xy}^2}\|\omega\|_{L_T^2 L_{xy}^\infty} \lesssim k^1 \cdot k^{(-1)-} = k^{0-}$ and $k^{0-} \to 0$ as $k \to \infty$. From the Lemma 19 used in Lemma 16 together with the fact we just proved, $\|J_x^s\omega\|_{L_T^\infty L_{xy}^2} \to 0$, we obtain

$$\|J_y^s\omega\|_{L_T^\infty L_{xy}^2} \lesssim \exp(\frac{1}{2}f_{u_k}(T)^2 + \frac{1}{2}f_\omega(T)^2)(\|J_y^s\omega(0)\|_{L_T^\infty L_{xy}^2} + Ck^{0-}) \to 0$$

as $k \to \infty$ and the boundedness of $f_{u_k}(T)$ and $f_\omega(T)$ by 7.

Therefore, as $\|J_x^s\omega\|_{L_T^\infty L_{xy}^2} + \|J_y^s\omega\|_{L_T^\infty L_{xy}^2} \to 0$ as $k \to \infty$, it means that $u \in C([0,T] : H^{s,s})$. $\square$

4.6 Continuity of the flow map

We assume that $T \in [0,\infty)$ and $\phi^l \to \phi$ in $H^{s,s}(\mathbb{R} \times \mathbb{T})$ as $l \to \infty$. We are going to prove that $u^l \to u$ in $C([-T,T] : H^{s,s}(\mathbb{R} \times \mathbb{T}))$ as $l \to \infty$, where u^l and u are solutions of the the initial value problem $\partial_t u - \partial_x^5 u - \partial_x^{-1}\partial_y^2 u + u^2\partial_x u = 0$ corresponding to initial data ϕ^l and ϕ, for $s > \frac{5}{2}$.

For $k \geq 1$, let as before, $\phi_k^l = P^k\phi^l$ and $u_k^l \in C([-T,T] : H^\infty)$ the corresponding solutions. Denote by $\omega_k = u_k - u$. By the same estimates from Lemma 16 and Lemma 19 applied to ω_k we get

$$\|u_k - u\|_{H^{s,s}} \lesssim \exp(\frac{1}{2}f_{\omega_k}(T)^2 + \frac{1}{2}f_{u_k}(T)^2)(\|\phi_k - \phi\|_{H^{s,s}} + C(T, \|\phi_k\|_{H^{s,s}}, \|\phi\|_{H^{s,s}})k^{0-}).$$

By the same reasoning, we have that

$$\|u_k^l - u^l\|_{H^{s,s}} \lesssim \exp(\frac{1}{2}f_{\omega_k^l}(T)^2 + \frac{1}{2}f_{u_k^l}(T)^2)(\|\phi_k^l - \phi^l\|_{H^{s,s}} + C(T, \|\phi_k^l\|_{H^{s,s}}, \|\phi^l\|_{H^{s,s}})k^{0-}).$$

Now, denote $\omega_k^l = u_k^l - u_k$. By the same estimates from Lemma 10 and Lemma 19 applied to ω_k^l

$$\|u_k^l - u_k\|_{H^{s,s}} \lesssim \exp(\frac{1}{2}f_{\omega_k^l}(T)^2 + \frac{1}{2}f_{u_k^l}(T)^2)(\|\phi_k^l - \phi_k\|_{H^{s,s}} + C(T, \|\phi_k^l\|_{H^{s,s}}, \|\phi_k\|_{H^{s,s}})k^{0-}).$$

By the boundedness of $f_{u_k}(T), f_{u_k^l}(T), f_{\omega_k}(T)$ and $f_{\omega_k^l}(T)$ by 7 and the triangle inequality, we get

$$\begin{aligned}
\|u^l - u\|_{H^{s,s}} &\leq \|u_k - u\|_{H^{s,s}} + \|u_k^l - u_k\|_{H^{s,s}} + \|u_k^l - u^l\|_{H^{s,s}} \\
&\lesssim \|\phi_k - \phi\|_{H^{s,s}} + \|\phi_k^l - \phi_k\|_{H^{s,s}} + \|\phi_k^l - \phi^l\|_{H^{s,s}} \\
&\quad + C(T, \|\phi\|_{H^{s,s}}, \|\phi_k\|_{H^{s,s}}, \|\phi^l\|_{H^{s,s}}, \|\phi_k^l\|_{H^{s,s}})k^{0-}
\end{aligned}$$

which, by letting $k \to \infty$, we get $\|u^l - u\|_{H^{s,s}} \lesssim \|\phi^l - \phi\|_{H^{s,s}}$ and proves the continuity of the flow map.

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