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MODULUS OF CONTINUITY METHOD FOR THE MUSKAT PROBLEM AND FRACTIONAL MEAN CURVATURE FLOW

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ABSTRACT

In this work, we consider two nonlinear, nonlocal, parabolic equations and show that under appropriate growth assumptions, solutions regularize. The first is the 2d Muskat problem, which models the evolution of the interface between oil and water in a tar sand. In the stable regime, the equation was known to be well-posed for smooth, flat initial data, but the slope can blow up in finite time for some large initial data [CCF^{+}12]. It was conjectured that slope 1 was the cutoff for global existence of graphical solutions. We resolve half of this conjecture, proving that the equation is globally well-posed whenever the initial data has slope less than 1. This work was originally published in [Cam19].

The second equation we consider is fractional mean curvature flow, which is a nonlocal fractional order analogue of the usual mean curvature flow. As with the local case, the regularity of general solutions of fractional mean curvature flow is very difficult to study. Without some form of star convexity or graphical assumption, solutions are known to pinch and develop singularities [CSV18, CDNV18]. Despite this, we prove that if our initial set E_0 is bounded between two Lipschitz subgraphs, then the minimal viscosity solution becomes a Lipschitz subgraph itself in finite time. This is a purely nonlocal phenomena, as the corresponding theorem is false for classical mean curvature flow.

For both equations we prove our results by showing that under our assumptions, evolving a solution over time quantitatively improves its modulus of continuity. This general method of proof was developed independently by Ishii-Lions in [IL90] for fully nonlinear elliptic equations and Kiselev, Nazarov, and Volberg in [AKV07] for active scalar equations.

CHAPTER 1

INTRODUCTION

In this work, we prove that two nonlinear nonlocal parabolic equations are regularizing under appropriate assumptions. While there will be many technical details and calculations throughout the proofs, the essential idea underlying our method in both cases is a very simple parabolic estimate:

Theorem 1.0.1. *(Propagation of uniform continuity) Let $u : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}$ solve the heat equation*

$$\partial_t u(t,x) - \Delta_x u(t,x) = 0, \qquad (t,x) \in (0,\infty) \times \mathbb{R}^d. \tag{1.1}$$

If $u(0,\cdot)$ has modulus of continuity $\omega : [0,\infty) \to [0,\infty)$, then so does $u(t,\cdot)$ for all $t \geq 0$.

Proof. Let $h \in \mathbb{R}^d$. Then as $u(0,\cdot)$ has modulus of continuity ω, we have that

$$u(0, x+h) \leq u(0,x) + \omega(|h|) \qquad x \in \mathbb{R}^d. \tag{1.2}$$

As both $(t,x) \to u(t,x+h)$ and $(t,x) \to u(t,x) + \omega(|h|)$ are solutions to the heat equation, we then have by the comparison principle that

$$u(t, x+h) \leq u(t,x) + \omega(|h|), \qquad (t,x) \in [0,\infty) \times \mathbb{R}^d. \tag{1.3}$$

As h was arbitrary, we then have that $u(t,\cdot)$ has modulus of continuity ω for all times $t \geq 0$. □

The key thing to note is that we only rely on two properties of the heat equation to prove the propagation of continuity. Namely,

1. Comparison principle

2. Translation invariance.

Any equation which has those two properties will automatically propagate the uniform continuity of initial data. Note though that the comparison principle and translation invariance don't give any improvement of regularity by themselves, as can be seen by considering solutions to the transport equation

$$\partial_t u(t,x) + b \cdot \nabla_x u(t,x) = 0, \tag{1.4}$$

where $b \in \mathbb{R}^d$ is a constant. Then $u(t,\cdot)$ has precisely the regularity of the initial data u_0.

However, if our parabolic equation has some uniform ellipticity, then the proof of propagation of regularity can easily be altered to instead give generation of regularity.

Theorem 1.0.2. *(Generation of Hölder continuity) Let $u : [0,\infty) \times \mathbb{R}^d \to \mathbb{R}$ solve the heat equation with $u(0,\cdot) = u_0 \in L^\infty(\mathbb{R}^d)$. Then for any $\alpha \in (0,1)$ and $t > 0$,*

$$\sup_{x \neq y} \frac{u(t,x) - u(t,y)}{|x-y|^\alpha} \le 2||u_0||_{L^\infty} \left(\frac{1}{4t(1-\alpha)} \right)^{\alpha/2} \tag{1.5}$$

Proof. To begin, let $C > 2||u_0||_{L^\infty} \left(\frac{1}{4(1-\alpha)} \right)^{\alpha/2}$ and define $\omega : (0,\infty) \times [0,\infty) \to \mathbb{R}$ by

$$\omega(t,r) = C \frac{r^\alpha}{t^{\alpha/2}}. \tag{1.6}$$

Note that by this choice of constant C, it follows that whenever $\omega(t,r) \le 2||u_0||_{L^\infty}$ we have the inequality

$$-2\alpha(1-\alpha)C\frac{r^{\alpha-2}}{t^{\alpha/2}} = 2\partial_r^2\omega(t,r) < \partial_t\omega(t,r) = -\frac{\alpha}{2}C\frac{r^\alpha}{t^{1+\alpha/2}}. \tag{1.7}$$

We will show that for all $t > 0$ and $x \neq y \in \mathbb{R}^d$,

$$u(t,x) - u(t,y) < \omega(t,|x-y|). \tag{1.8}$$

Suppose not. Then we claim that there must be a first crossing point. I.e., there exists some $t_0 \in (0, \infty)$ and $x_0 \neq y_0$ such that

$$\begin{cases} u(t,x) - u(t,y) \leq \omega(t, |x-y|), & t \in (0, t_0], x, y \in \mathbb{R}^d, \\ u(t_0, x_0) - u(t_0, y_0) = \omega(t_0, |x_0 - y_0|). \end{cases} \tag{1.9}$$

We will finish the proof assuming the existence of this crossing point. Details about how to guarantee the existence of a first crossing point can be seen in the break through arguments of Chapter 2.2.

At the crossing point, we have a number of differential inequalities. In particular, we get immediately the bound on the time derivatives

$$\partial_t \omega(t_0, |x_0 - y_0|) \leq \frac{d}{dt}\bigg|_{t=t_0} \left(u(t, x_0) - u(t, y_0)\right). \tag{1.10}$$

We can also bound the second derivatives in space. Note that for any $l \in S^{d-1}$ with $l \perp (x_0 - y_0)$, we have that the map

$$s \to u(t_0, x_0 + sl) - u(t_0, y_0 + sl), \tag{1.11}$$

has a global max at $s = 0$. Hence,

$$\partial_l^2 u(t_0, x_0) - \partial_l^2 u(t_0, y_0) \leq 0. \tag{1.12}$$

For any $x \in \mathbb{R}^d$, we have that

$$u(t_0, x) \leq \omega(t_0, |x - y_0|) + u(t_0, y_0), \tag{1.13}$$

with equality at $x = x_0$. Letting $e = \dfrac{x_0 - y_0}{|x_0 - y_0|}$, it then follows that

$$\partial_e^2 u(t_0, x_0) \leq \partial_e^2\bigg|_{x=x_0} (\omega(t_0, |x - y_0|)) = \partial_r^2 \omega(t_0, |x_0 - y_0|). \tag{1.14}$$

The same argument also gives that

$$\partial_e^2 u(t_0, y_0) \geq -\partial_r^2 \omega(t_0, |x_0 - y_0|). \tag{1.15}$$

Hence,

$$\Delta u(t_0, x_0) - \Delta u(t_0, y_0) \leq 2\partial_r^2 \omega(t_0, |x_0 - y_0|). \tag{1.16}$$

As at the touching point we have that $\omega(t_0, |x_0 - y_0|) = u(t_0, x_0) - u(t_0, y_0) \leq 2||u_0||_{L^\infty}$, it follows by (1.7) and the heat equation that

$$\frac{d}{dt}\bigg|_{t=t_0} (u(t, x_0) - u(t, y_0)) = \Delta u(t_0, x_0) - \Delta u(t_0, y_0) \leq 2\partial_r^2 \omega(t_0, |x_0 - y_0|) < \partial_t \omega(t_0, |x_0 - y_0|), \tag{1.17}$$

contradicting (1.10). Thus (1.8) holds whenever $C > 2||u_0||_{L^\infty} \left(\dfrac{1}{4(1-\alpha)}\right)^{\alpha/2}$, so the result follows.

□

1.1 Discussion of the basic estimate

In the later chapters of this work, we will be applying the essential proof scheme we used in the proof Theorem 1.0.2 in more complicated, nonlinear, nonlocal settings. As such, we first take some time to go over the qualities of the proof in detail.

The first thing to note is that this proof at its core is generalizing the comparison principle and as such is a proof for viscosity solutions. Indeed, this proof scheme seems to have first appeared for elliptic equations in the foundational work of Ishii-Lions [IL90]. In the proof, essentially what we do is double the spatial variables and consider $\delta u(t, x, y) = u(t, x) - u(t, y)$

as a viscosity solution to

$$\partial_t \delta u(t,x,y) - \Delta_{x,y}\delta u(t,x,y) = 0. \tag{1.18}$$

We then use that $\omega(t,r)$ is a supersolution to

$$\partial_t \omega(t,r) - 2\partial_r^2 \omega(t,r) > 0, \qquad (t,r) \text{ s.t. } \omega(t,r) \le 2||u_0||_{L^\infty}, \tag{1.19}$$

in order to guarantee that there can be no crossing point of $\omega(t,|x-y|)$ and $\delta u(t,x,y)$. The fact that $\omega(0,r) \equiv \infty$ and the maximum principle then guarantees that we can apply this comparison for any bounded initial data.

A priori it is surprising that $\omega(t,r)$ satisfying (1.19) was enough to make the argument of Theorem 1.0.2 go through. Naively one would expect to need instead the much stronger condition that $\omega(t,|x-y|)$ was a supersolution to (1.18), when in fact for dimension $d > 1$ $\omega(t,|x-y|)$ is a strict subsolution of (1.18). But by taking full advantage of the fact that $\omega(t,\cdot)$ is a function of $|x-y|$ and we were evaluating at a touching point, we were able to refine the calculation and show that a simpler equation (1.19) holding in a much smaller region sufficed.

Another point we would like to stress is that Theorem 1.0.2 is about how a growth condition like $u_0 \in L^\infty$ transfers to a regularity statement $u(t,\cdot) \in C^\alpha$ for positive time. Given any uniform sublinear growth condition

$$\sup_x u_0(x+h) - u_0(x) \le C(1+|h|)^\beta, \qquad 0 \le \beta < 1, \tag{1.20}$$

we could prove a corresponding uniform Hölder regularity estimate for $t > 0$. The comparison principle and translation invariance imply that for all later times t,

$$\sup_x u(t,x+h) - u(t,x) \le C(1+|h|)^\beta. \tag{1.21}$$

For $\alpha > \beta$ its then straightforward to construct a modulus $\omega(t,r) = C(t)r^\alpha$ that will be a supersolution to (1.19) whenever $\omega(t,r) < C(1+|r|)^\beta$. So the modulus of continuity method allows us to turn our fixed long range growth bounds into short range regularity.

Finally, as mentioned in the previous section, the three essential elements to get the gain in regularity using the modulus of continuity method are

1. Comparison principle,
2. Translation invariance,
3. Uniform ellipticity.

These three elements combined allow us to prove a quantitative gain in regularity like Theorem 1.0.2. However, once we have a quantitative estimate for a base equation, we can make quantitative alterations to that equation and still preserve the essential estimate. For example, given a $b \in L^\infty((0,\infty) \times \mathbb{R}^d; \mathbb{R}^d)$, we can easily alter the argument of Theorem 1.0.2 to work for the drift diffusion equation

$$\partial_t u(t,x) + b(t,x) \cdot \nabla u(t,x) - \Delta u(t,x) = 0, \tag{1.22}$$

which is no longer translation invariant. We simply treat the term $b(t,x) \cdot \nabla u(t,x)$ as an error term, and then take care to alter the definition of ω so that it instead satisfies

$$\partial_t \omega(t,r) - 2||b||_{L^\infty} \partial_r \omega(t,r) - 2\partial_r^2 \omega(t,r) > 0, \qquad (t,r) : \omega(t,r) \leq 2||u_0||_{L^\infty}. \tag{1.23}$$

One particular feature of this proof scheme is that it pairs very well with certain kinds of nonlinearities, such as those from active scalar equations. This is covered very well in [Kis11] and we shall talk about it in more detail in the next section. But now for illustrative purposes, consider Burger's equation with an added diffusive term,

$$\partial_t u(t,x) + u(t,x)\partial_x u(t,x) - \partial_x^2 u(t,x) = 0, \qquad (t,x) \in (0,\infty) \times \mathbb{R}. \tag{1.24}$$

While this is of the same form as (1.22) for initial data $u_0 \in L^\infty(\mathbb{R})$, we can take advantage of the fact that our drift $b(t,\cdot) = u(t,\cdot)$ now has modulus of continuity $\omega(t,\cdot)$. This essentially allows us to do an implicit bootstrapping argument, reducing the requirement on our modulus ω to

$$\partial_t \omega(t,r) - \omega(t,r)\partial_r \omega(t,r) - 2\partial_r^2 \omega(t,r) > 0, \qquad (t,r) : \omega(t,r) \leq 2||u_0||_{L^\infty}. \tag{1.25}$$

As $\omega(t,r) \leq 2||u_0||_{L^\infty} = 2||b||_{L^\infty}$, this significantly weakens the requirements on ω.

1.2 History of the proof technique

The general strategy of showing an equation generates a modulus of continuity seems to have originated in the work of Ishii-Lions [IL90]. There, the authors considered viscosity solutions u to the fully nonlinear elliptic equations

$$F(x, u(x), Du(x), D^2u(x)) = 0, \qquad x \in \Omega \subseteq \mathbb{R}^d. \tag{1.26}$$

Under uniform ellipticity and mild regularity assumptions on F, they prove Hölder regularity of u in the interior of Ω with Hölder constant depending on $||u||_{L^\infty(\Omega)}$, and uniform up to the boundary when $u\big|_{\partial\Omega} \in C^\alpha(\partial\Omega)$.

The method of Ishii-Lions has since been used to establish basic Hölder and Lipschitz bounds for a number of general nonlinear problems. It was used to give Lipschitz bounds on the prescribed mean curvature problem in [Bar91], give $C^{1,\alpha}$ bounds under natural assumptions in [Che93], give Hölder regularity for the Neumann problem in [BS06], Lipschitz bounds with superquadratic gradient growth in [LN17], and nonlocal equations in [GBI10, GBI11].

With the exception of [Bar91], the rest of the results above are all for rather general nonlinear elliptic/parabolic equations. The essential assumptions in each are some form of uniform ellipticity, and continuity/growth bounds on F in x. Reducing the proof to just a few requirements on your equation F of course allows this estimate to be used in a large variety of settings. However, this very broad point of view misses how this proof technique synergizes very well with certain kinds of quasilinear equations.

Independently of Ishii-Lions, this general proof strategy was rediscovered in 2006 by Kiselev, Nazarov, and Volberg in [AKV07]. There, the authors were trying to show global wellposedness for the 2d critical surface quasi-geostraphic equation

$$\partial_t \theta(t,x) - u(t,x)\cdot\nabla\theta(t,x) + (-\Delta)^{1/2}\theta(t,x) = 0, \qquad (t,x)\in(0,\infty)\times\mathbb{R}^2, \tag{1.27}$$

with the drift $u = u[\theta]$ given by

$$u = (u_1, u_2) = (-R_2\theta, R_1\theta), \tag{1.28}$$

where R_1, R_2 are the usual Riesz transforms in $\mathbb{R}^2$. The main ingredient to show global well-posedness is some kind of Hölder a priori estimate on θ, as $||u(t)||_{C^\alpha} \leq C_\alpha ||\theta(t)||_{C^\alpha}$. However, the only bounds available at the time were $||\theta(t)||_{L^p} \leq ||\theta_0||_{L^p}$ for all $1 \leq p \leq \infty$ due to the divergence free structure of u, and thus correspondingly $u \in BMO$.

Caffarelli and Vasseur were able to show in [CV10] that these a priori estimates were in fact sufficient, proving that for any solution θ to a drift diffusion equation is Hölder continuous for positive time whenever (1.27) with u divergence free and $u \in L^\infty_t(BMO_x)$. However, Kiselev, Nazarov, and Volberg were able to give a much more elementary proof in [AKV07] by using the modulus of continuity method and the structure of u (1.28).

In [AKV07], they show that if $\theta(t,\cdot)$ has uniform modulus of continuity $\omega(r)$, then u has modulus of continuity

$$|u(t,x)-u(t,y)| \leq C\int_0^{|x-y|}\frac{\omega(\eta)}{\eta}d\eta + C|x-y|\int_{|x-y|}^{\infty}\frac{\omega(\eta)}{\eta^2}d\eta. \tag{1.29}$$

They then show that at a point where we have equality $\theta(t_0, x_0) - \theta(t_0, x_0) = \omega(|x_0 - y_0|)$, we have a bound on the difference in half-Laplacians

$$\begin{aligned}(-\Delta)^{1/2}\theta(t_0, x_0) - (-\Delta)^{1/2}\theta(t_0, y_0)) \leq \int_0^r \frac{\omega(r+\eta) + \omega(r-\eta) - 2\omega(r)}{\eta^2} d\eta \\ + \int_r^\infty \frac{\omega(\eta+r) - \omega(\eta-r) - 2\omega(r)}{\eta^2} d\eta < 0,\end{aligned} \tag{1.30}$$

so long as ω is concave. Combining (1.29) and (1.30), it then follows that any modulus of continuity $\omega : [0, \infty) \to [0, \infty)$ satisfying

$$\begin{aligned}\left(C \int_0^r \frac{\omega(\eta)}{\eta} d\eta + Cr \int_r^\infty \frac{\omega(\eta)}{\eta^2} d\eta \right) \partial_r \omega(r) + \int_0^r \frac{\omega(r+\eta) + \omega(r-\eta) - 2\omega(r)}{\eta^2} d\eta \\ + \int_r^\infty \frac{\omega(\eta+r) - \omega(\eta-r) - 2\omega(r)}{\eta^2} d\eta < 0,\end{aligned} \tag{1.31}$$

will be propagated by the equation. They then show the Lipschitz modulus of continuity

$$\begin{cases} \omega(r) = r - r^{3/2}, & r \leq \delta, \\ \partial_r \omega(r) = \frac{\gamma}{r(4+\log(r/\delta))}, & r \geq \delta \end{cases} \tag{1.32}$$

and any rescaling of satisfy (1.31) for appropriate constants δ, γ. Any smooth initial data $\theta \in C^\infty(\mathbb{T}^2)$ satisfies some rescaling of ω, hence giving an a priori Lipschitz estimate on $\theta(t, \cdot)$ and thus proving global well-posedness.

Since that first paper, this proof strategy has been applied by Kiselev et al in a number of active scalar equations with fractional diffusions. That is, equations of the form

$$\partial_t \theta - u \cdot \nabla \theta - (-\Delta)^\alpha \theta = 0, \tag{1.33}$$

where $\alpha \in [0, 1]$ and $u = u[\theta]$ is some function of θ. The classic examples are the SQG equation where u satisfies (1.28), Burger's equation where $u(x) = \theta(x)$, and 2d Euler where θ is the vorticity and $u = \nabla^{\perp}(-\Delta)^{-1}\theta$.

For Burger's equation, it was proven in [AKS08] that solutions to (1.33) are smooth for $1/2 \leq \alpha \leq 1$ in the subcritical-critical case, but that shocks develop when $\alpha < 1/2$ in the supercritical regime. Regularity for logarithmically supercritical Burgers, SQG, and Euler's was proven in [MDV14].

Another fascinating result is the finite time Hölder regularization for the supercritical Burger's and SQG equations proved in [Kis11]. There Kiselev shows that in the supercritical regime $0 < \alpha < 1/2$, there exists a time $T = T(\alpha, \beta, ||\theta_0||_{L^\infty})$ such that for any solution θ of (1.33) has $||\theta(t)||_{C^\beta} \leq C$ for $t \geq T$. This is particular surprising for Burger's equation as we know there is finite time blow up for these norms from [AKS08]. For the proof, he showed that the equation propagated a family of moduli of continuity

$$\omega(t, r) \approx \delta(t) + Cr^\beta, \tag{1.34}$$

where $\delta(0) > 2||\theta_0||_{L^\infty}$ and $\delta(T) = 0$. Thus the moduli of continuity gives no new information at time $t = 0$, controls the size of shocks and large scale growth for $0 < t < T$, and then forces the solution to become β-Hölder continuous at time $t = T$.

1.3 Our contributions

In this work, we apply the modulus of continuity method of Ishii-Lions and Kiselev et al on two nonlinear, nonlocal parabolic equations.

The first is the 2d Muskat problem in the stable regime,

$$\partial_t f(t, x) = \int_{\mathbb{R}} \frac{f(t, x+h) - f(t, x) - \partial_x f(t, x)h}{(f(t, x+h) - f(t, x))^2 + h^2} dh. \tag{1.35}$$

The Muskat problem linearizes around a flat solution to the fractional heat equation

$$\partial_t f(t,x) = -(-\Delta)^{1/2} f(t,x), \tag{1.36}$$

showing the parabolicity of the problem for small sloped data. The problem has a maximum principle for the slope $\partial_x f$ when $||\partial_x f_0||_{L^\infty} < 1$ [CCGS13], however the equation is known to have finite time blow up for the slope for some large initial data [CCF$^+$12]. These results lead to a conjecture in the Muskat community:

Conjecture 1.3.1. *The 2d Muskat problem is globally well-posed for initial data $f_0 \in W^{1,\infty}(\mathbb{R})$ with $||\partial_x f_0||_{L^\infty} < 1$. Conversely, for any $\epsilon > 0$, there is initial data f_0 with $||\partial_x f_0||_{L^\infty} < 1 + \epsilon$ but $\partial_x f$ blows up in finite time.*

We will resolve half of Conjecture 1.3.1. Namely, we will show in Chapter 2 that

Theorem 1.3.1. *The Muskat equation* (1.35) *is globally well-posed for initial data $f_0 \in W^{1,\infty}$ with $||\partial_x f_0||_{L^\infty} < 1$.*

This work was originally published in [Cam19]. We prove Theorem 1.3.1 by using the modulus of continuity method on $\partial_x f$, proving the instantaneous generation Lipschitz bounds for the slope $\partial_x f(t,\cdot)$. By combining the local well posedness of the Muskat problem [CG07] with the continuation criteria of [CGSV17], the instantaneous generation of continuity for the slope $\partial_x f$ then gives global well-posednesss.

One way to see the importance of the continuity of the derivative is to examine the nonlinearity in (1.35). The function

$$\mathfrak{N}(x,h) := h^2 \left(\frac{1}{(f(t,x+h) - f(t,x))^2 + h^2} \right) = \frac{1}{\left(\frac{f(t,x+h)-f(t,x)}{|h|}\right)^2 + 1}, \tag{1.37}$$

is nonegative and bounded from below so long as $\partial_x f$ is bounded. And it will be continuous in x with a modulus of continuity depending on the modulus ω of $\partial_x f$. Thus with control over both of these properties, we would expect our equation to be well behaved. The assumption $||\partial_x f_0|| < 1$ gives us the control over the size of $\mathfrak{N}$, and the modulus of continuity method allows us to assume that $\mathfrak{N}(x, h)$ is continuous in x while we prove that same continuity estimate, essentially giving us an invaluable bootstrapping argument.

The second equation we shall study is fractional mean curvature flow in Chapter 3. Given an $s \in (0, 1)$ and sufficiently nice open set $E \subseteq \mathbb{R}^d$, we define the s-fractional mean curvature at a point $X \in \partial E$ by

$$H_s(X, E) = -s(1-s) \int_{\mathbb{R}^d} \frac{\mathbb{1}_E(X+Z) - \mathbb{1}_{\mathcal{C}E}(X+Z)}{|Z|^{d+s}} dZ. \tag{1.38}$$

where $\mathbb{1}_E, \mathbb{1}_{\mathcal{C}E}$ are the indicator functions of E and its compliment of $\mathcal{C}E$. We can then define the s-fractional mean curvature flow $t \to E_t$ by

$$\partial_t X(t) = -H_s(X(t), E_t)\nu(X(t)), \qquad X(t) \in \partial E_t. \tag{1.39}$$

As $s \to 1$, fractional mean curvature converges to the classical local mean curvature [CV11]. Thus s-mean curvature can be thought of as a nonlocal, fractional order analogue of the normal mean curvature.

In the case that your initial set E_0 is the subgraph of a smooth Lipschitz function $u_0 : \mathbb{R}^{d-1} \to \mathbb{R}$, then fractional mean curvature flow exists for all time[SV15] and can be equivalently described by

$$\partial_t u(t, x) = s(1-s)\sqrt{1 + |\nabla u(t, x)|^2} \int_{\mathbb{R}^{d-1}} \frac{u(t, x+z) - u(t, x)}{|z|^{d+s}} \Lambda\left(\frac{u(t, x+z) - u(t, x)}{|z|}\right) dz, \tag{1.40}$$

where the nonlinearity $\Lambda(L) = \frac{1}{L}\int_{-L}^{L} \frac{1}{(1+z_d^2)^{(d+s)/2}} dz_d$. Thus

$$\Lambda(||\nabla_x u||_{L^\infty}) \leq \Lambda\left(\frac{u(t,x+z)-u(t,x)}{|z|}\right) \leq 2, \tag{1.41}$$

so this is a nonlinear parabolic equation of order $1+s$, with the ellipticity constant depending on the Lipschitz constant of u. From this parabolicity, we see that fractional mean curvature flow is regularizing on Lipschitz subgraphs. And in Chapter 3, we shall show that the same is true for sets bounded between Lipschitz subgraphs.

Theorem 1.3.2. *Let $E_0 \subseteq \mathbb{R}^d$ be an open set and $u_0 \in \dot{W}^{1,\infty}(\mathbb{R}^{d-1})$ be a Lipschtiz function with $||\nabla_x u_0||_{L^\infty} = L$ and*

$$\left\{(x,x_d)\middle| x \in \mathbb{R}^{d-1}, x_d < u_0(x) - \frac{R}{2}\right\} \subseteq E_0 \subseteq \left\{(x,x_d)\middle| x \in \mathbb{R}^{d-1}, x_d < u_0(x) + \frac{R}{2}\right\}, \tag{1.42}$$

for some $R \geq 0$. Let E_t be the minimal viscosity supersolution of the flow (3.4)*, in the sense of Definition B.0.2. Then for all $t \geq R^{1+s}T(d,s,L)$, E_t will be a $(1+L)$-Lipschitz subgraph. The time $T(d,s,L)$ can be bounded explicitly, with*

$$T(d,s,L) \leq \frac{C(d)(1+L)^{d+s}}{s^2(1-s)}, \tag{1.43}$$

for some dimensional constant $C(d)$.

We stress that there is no initial regularity assumption on E_0 in Theorem 3.1.1. The boundary ∂E_0 can have positive measure, and the set E_0 does not even need to be connected. Our only assumption is that E_0 is a finite distance in the Hausdorff metric from a Lipschitz subgraph, which amounts to assuming some uniform long range growth bounds

$$E_0 \subseteq E_0 + (z, R+L|z|), \qquad \forall z \in \mathbb{R}^{d-1}. \tag{1.44}$$

Our proof of Theorem 1.3.2 is inspired by Kiselev's proof of long finite time Hölder regularization for supercritical Burger's equation [Kis11], which we discussed in the previous section. We show that the fractional mean curvature flow propagates a time dependent modulus of continuity $\omega(t, r)$. We discuss what it means for a set to have a modulus of continuity more in Chapter 3. But essentially, at time $t = 0$ our modulus $\omega(0, r) = R + Lr$ matches our long range growth bounds (1.44). For times $0 < t < T$, our modulus $\omega(t, \cdot)$ controls our distance from a Lipschitz subgraph, and then at time $t = T$ it forces our set to becomes a Lipschitz subgraph itself. Thus modulus of continuity argument allows us to transform these long range bounds into short distance regularity, giving us a Lipschitz subgraph in finite time.

Finally, the last thing we would like to note is that Theorem 1.3.2 is an inherently nonlocal in nature, with the time $T(d, s, L) \to \infty$ as $s \to 1$. This is a necessity, as the theorem is false for classical mean curvature flow. The set

$$E_0 = \{(x, x_d) | x_d < -1 \text{ or } 0 < x_d < 1\}, \tag{1.45}$$

is fixed by local mean curvature flow and hence never becomes graphical. In this case, it's clear that the barrier to regularity is multiplicity, as the problem is that ∂E_0 is a disjoint union of hyperplanes. But because of the nonlocal nature of fractional mean curvature, the points on the disjoint hyperplaces $\{x_d = \pm 1\}$ can still sense each other, and are no longer fixed. Direct calculation shows that flowing under fractional mean curvature flow, $E_t \to \{(x, x_d) | x_d < 0\}$ in finite time $T \sim \dfrac{1}{s(1-s)}$ for any $0 < s < 1$.

CHAPTER 2

THE MUSKAT PROBLEM

2.1 Introduction

The material in this chapter was first published in *Analysis & PDE* in vol. 12 (4) 2019, published by Mathematical Sciences Publishers [Cam19].

The Muskat problem was originally introduced by Muskat in [Mus34] in order to model the interface between water and oil in tar sands. In general, it describes the interface between two incompressible, immiscible fluids of different constant densities in a porous media. The fluids evolve according to Darcy's law, giving an evolution of the interface (see [CG07] for derivation of equations), and in 2D is analogous to the two phase Hele-Shaw cell (see [ST58]). In the case that the two fluids are of equal viscosity and the interface is given by the graph $y = f(t,x)$ with the denser fluid on bottom (i.e. the stable regime), the function f satisfies

$$f_t(t,x) = \int_{\mathbb{R}} \frac{(f_x(t,y) - f_x(t,x))(y-x)}{(f(t,y) - f(t,x))^2 + (y-x)^2} dy, \tag{2.1}$$

after the appropriate renormalization. By making a change of variables, (see the proof of Lemma 5.1 of [CG09]) we get the equivalent system

$$f_t(t,x) = \int_{\mathbb{R}} \frac{f(t,y) - f(t,x) - (y-x)f_x(t,x)}{(f(t,y) - f(t,x))^2 + (y-x)^2} dy, \tag{2.2}$$

which will be more useful for our purposes. Since the function f is Lipschitz, the above integral can be viewed as a nonlinear perturbation of the half Laplacian. In fact, it is easy to see that linearizing around a flat solution gives

$$f_t(t,x) = -c(-\Delta)^{1/2} f(t,x), \tag{2.3}$$

demonstrating the natural parabolicity of the problem.

The Muskat problem is known to be locally well-posed in H^k for $k \geq 3$ with solutions satisfying L^∞ and L^2 maximum principles, but neither imply any gain of derivatives (see [CG09], [CCGS13]).

Under the assumption $||f_0'||_{L^\infty} < 1$, there have been a number of positive results. In [CCGS13] the authors prove an L^∞ maximal principle for the slope f_x along with the existence of global weak Lipschitz solutions using a regularized system. Recently, [Gan17] improved the L^2 energy estimate of [CCGS13] (which holds for any solution) to one analogous with the energy estimate from the linear equation under this assumption on the slope. When the initial data $f_0 \in H^2(\mathbb{R})$ with $||f_0||_1 = || \, |\xi|\hat{f}_0(\xi)||_{L^1_\xi}$ less than some explicit constant $\approx 1/3$ (which implies slope less than 1), [CCGS16] proves that a unique global strong solution exists. In this case [PS17] proves optimal decay estimates on the norms $||f(t,\cdot)||_s = || \, |\xi|^s \hat{f}(t,\xi)||_{L^1_\xi}$, matching the estimates for the linear equation.

Recently, [DLL17] was also able to prove the existence of global weak solutions for arbitrarily large monotonic initial data. They did this using the regularized system from [CCGS13] to prove that both f and f_x still obey the maximum principle under this monotonicity assumption.

Because solutions to (2.2) have the natural scaling $\frac{1}{r}f(rt, rx)$, we see that L^∞ or sign bounds on the slope f_x are scale invariant properties. We fit these two types of assumptions into the same framework by showing that the critical quantity is in fact the product of the maximal and minimal slopes,

$$\beta(f_0') := (\sup_x f_0'(x))(\sup_y -f_0'(y)). \tag{2.4}$$

As we shall see in section 3, the derivative f_x obeys the equation

$$(f_x)_t(t,x) = f_{xx}(t,x) \int_{\mathbb{R}} \frac{-h}{\delta_h f(t,x)^2 + h^2} dh + \int_{\mathbb{R}} \delta_h f_x(t,x) K(t,x,h) dh. \tag{2.5}$$

where $\delta_h f(t,x) := f(t,x+h) - f(t,x)$ and the kernel K is uniformly elliptic of order 1 whenever $\beta(f_0') < 1$. Thus we naturally get regularizing effects from the equation whenever the initial data satisfies this bound. It's clear that $||f_0'||_{L^\infty} < 1$ implies $\beta(f_0') < 1$, and for bounded monotonic data we get that $\beta(f_0') = 0$ since either $\sup f_0' = 0$ or $\inf f_0' = 0$. Thus this $\beta(f_0') < 1$ provides a natural interpolation between these two types of assumptions.

In contrast to the positive results, [CCF+12] shows that there is an open subset of initial data in H^4 such that the Rayleigh-Taylor condition breaks down in finite time. That is, $\lim_{t\to t_0-} ||f_x(t,\cdot)||_{L^\infty} = \infty$ for some time t_0, after which the interface between the fluids can no longer be described by a graph.

The authors of [CGSV17] made great progress towards proving global regularity. They proved that if the initial data $f_0 \in H^k$, then the solution f will exist and remain in H^k so long as the slope $f_x(t,\cdot)$ remains bounded and uniformly continuous. Thus the natural next step is to prove the generation of a modulus of continuity for f_x, hence

Theorem 2.1.1. *Let $f_0 \in W^{1,\infty}(\mathbb{R})$ with*

$$\beta(f_0') := (\sup_x f_0'(x))(\sup_y -f_0'(y)) < 1. \tag{2.6}$$

Then there exists a classical solution

$$f \in C([0,\infty) \times \mathbb{R}) \cap C^{1,\alpha}_{loc}((0\,\infty) \times \mathbb{R}) \cap L^\infty_{loc}((0,\infty); C^{1,1}), \tag{2.7}$$

to (2.2) with f_x satisfying both the maximum principle and

$$f_x(t,x) - f_x(t,y) \le \rho\left(\frac{|x-y|}{t}\right), \quad t > 0, x \neq y \in \mathbb{R}, \tag{2.8}$$

for some Lipschitz modulus of continuity ρ depending solely on $\beta(f_0')$,$||f_0'||_{L^\infty}$.

In the case that $f_0 \in C^{1,\epsilon}(\mathbb{R})$ for some $\epsilon > 0$, then the solution f is unique with $f \in L^\infty([0,\infty); C^{1,\epsilon})$.

The uniqueness statement follows essentially from the uniqueness theorem of [CGSV17]. We note in the appendix the few small changes needed to their proof in order to apply it here.

The most vital part of Theorem 2.1.1 is the spontaneous generation of the modulus $\rho(\cdot/t)$, as everything else will follow from that. The spontaneous generation/propogation of a general modulus of continuity has old roots as classical Holder estimates, but its only recently that the idea to tailor make moduli for specific equations emerged. The technique first appeared in [AKV07], where the authors used it to prove global well-posedness for the surface quasi-geostraphic equation. It has had great success at proving regularity for a number of active scalar equations, that is equations of the form

$$\theta_t + (u \cdot \nabla)\theta + \mathcal{L}\theta = 0, \tag{2.9}$$

where u is a flow depending on θ and $\mathcal{L}$ is some diffusive operator. See [Kis11], [MDV14] for a good overview of results using this method.

To date, these tailor made moduli have only been applied to cases where all the nonlinearity has been in the flow velocity u, and the diffusive term $\mathcal{L}$ has been rather nice (typically $(-\Delta)^\alpha$, or at least a Fourier multiplier). We will be applying this method to f_x, which solves the active scalar equation (2.5). Note that in this equation, the kernel K defined in (2.25) is a highly nonlinear function of f, f_x. Thus this is the first time the method has been applied in a fully nonlinear equation.

We prove Theorem 2.1.1 by deriving a priori estimates for smooth solutions to (2.2) with initial data $f_0 \in C_c^\infty(\mathbb{R})$ depending primarily on $\beta(f_0'), ||f_0'||_{L^\infty}$. We prove enough estimates that by approximating in $W^{1,\infty}_{loc}$ with smooth compactly supported initial data, we get solutions f^ϵ which will converge along subsequences in C^1_{loc} to a solution f solving (2.2) for arbitrary initial data $f_0 \in W^{1,\infty}(\mathbb{R})$ with $\beta(f_0') < 1$.

The rest of the paper is organized as follows. We begin by repeating the breakthrough argument of [AKV07] in Section 2. In Section 3, we differentiate (2.2) to derive the equation for f_x, showing that it satisfies the maximum principle when $\beta(f_0') < 1$. In Section 4, we state how a modulus of continuity ω interacts with the equation in our main technical lemma. In Sections 5 and 6 we then derive the bounds on the drift and diffusion terms necessary to prove that lemma. In Section 7, we apply our main technical lemma to a specific modulus of continuity, and finally in Section 8 we complete the proof of (2.8) by choosing the correct modulus ρ. In Section 9, we then use (2.8) to prove a few estimates on regularity in time, guaranteeing enough compactness to prove that there are classical solutions for rough initial data. Finally in the appendix, we give a quick outline for how to modify the uniqueness proof of [CGSV17] to work for initial data $f_0 \in C^{1,\epsilon}(\mathbb{R})$ with $\beta(f_0') < 1$.

2.2 Breakthrough Scenario

Assume that $f_0 \in C_c^\infty(\mathbb{R})$ with $\beta(f_0') < 1$, so that there exists a solution $f \in C^1((0, T_+); H^k)$ for k arbitrarily large and some $T_+ > 0$ by [CG09]. Note that under the assumption that $\beta(f_0') < 1$, we will show that the maximum principle holds (see Section 3 Proposition 2.3.1) and hence $||f_x||_{L^\infty([0,T_+)\times\mathbb{R})} \leq ||f_0'||_{L^\infty}$ is uniformly bounded. Fix a Lipschitz modulus ρ which we will define later. For sufficiently small times, $f_x(t, \cdot)$ will have modulus $\rho(\cdot/t)$ since it is smooth and bounded. It then follows by the main theorem of [CGSV17] that as long as $f_x(t, \cdot)$ continues to have modulus $\rho(\cdot/t)$, the solution f will exist with $T_+ > t$.

So, we proceed as in [AKV07]'s proof for quasi-geostraphic equation. Suppose that $f_x(t, \cdot)$ satisfies (2.8) for all $t < T$. Then by continuity,

$$f_x(T, x) - f_x(T, y) \leq \rho\left(\frac{|x - y|}{T}\right), \quad \forall x \neq y \in \mathbb{R}. \tag{2.10}$$

We first prove that if we have the strict inequality $f_x(T, x) - f_x(T, y) < \rho(|x - y|/T)$, then $f_x(t, \cdot)$ will have modulus $\rho(\cdot/t)$ for $t \leq T + \epsilon$.

Lemma 2.2.1. *Let $f \in C([0,T_+);C^3_0(\mathbb{R}))$, and $T \in (0,T_+)$. Suppose that $f(T,\cdot)$ satisfies*

$$f_x(T,x) - f_x(T,y) < \rho\left(|x-y|/T\right), \forall x \neq y \in \mathbb{R}, \tag{2.11}$$

for some Lipschitz modulus of continuity ρ with $\rho''(0) = -\infty$. Then

$$f_x(T+\epsilon,x) - f_x(T+\epsilon,y) < \rho(|x-y|/(T+\epsilon)), \forall x \neq y \in \mathbb{R}, \tag{2.12}$$

for all $\epsilon > 0$ sufficiently small.

Proof. To begin, note that for any compact compact subset $K \subset \mathbb{R}^2 \setminus \{(x,x)|x \in \mathbb{R}\}$,

$$\begin{aligned} &f_x(T,x) - f_x(T,y) < \rho(|x-y|/T) \quad \forall (x,y) \in K \\ &\qquad \Rightarrow \quad f_x(T+\epsilon,x) - f_x(T+\epsilon,y) < \rho(|x-y|/(T+\epsilon)) \qquad \forall (x,y) \in K, \end{aligned} \tag{2.13}$$

for $\epsilon > 0$ sufficiently small by uniform continuity. So, we only need to focus on pairs (x,y) that are either close to the diagonal, or that are large.

To handle (x,y) near the diagonal, we start by noting that $f(T,\cdot) \in C^3(\mathbb{R})$ and $\rho''(0) = -\infty$. Thus for every x we get that

$$|f_{xx}(T,x)| < \frac{\rho'(0)}{T}. \tag{2.14}$$

Since $f \in C([0,T_+);C^3_0(\mathbb{R}))$, $f_{xx}(T,x) \to 0$ as $x \to \infty$. Thus we can take the point where $\max_x |f_{xx}(T,x)|$ is achieved to get that

$$||f_{xx}(T,\cdot)||_{L^\infty} < \frac{\rho'(0)}{T}. \tag{2.15}$$

By continuity of f_{xx}, we thus have $||f_{xx}(T+\epsilon,\cdot)||_{L^\infty} < \dfrac{\rho'(0)}{T+\epsilon}$ for $\epsilon > 0$ sufficiently small.

Hence,

$$f_x(T+\epsilon,x) - f_x(T+\epsilon,y) < \rho\left(\frac{|x-y|}{T+\epsilon}\right), \quad |x-y| < \delta, \tag{2.16}$$

for ϵ, δ sufficiently small.

Now let $R_1, R_2 > 0$ be such that

$$\rho(R_1/(T+\epsilon)) > \mathrm{osc}_{\mathbb{R}} f_x(T+\epsilon,\cdot), \tag{2.17}$$

and that $|x| > R_2$ implies

$$|f_x(T+\epsilon,x)| < \frac{\rho(\delta/(T+\epsilon))}{2}, \tag{2.18}$$

for $\epsilon > 0$ sufficiently small. Taking $R = R_1 + R_2$, it's easy to check that $|x| > R$ implies that

$$|f_x(T+\epsilon,x) - f_x(T+\epsilon,y)| < \rho(|x-y|/(T+\epsilon)), \quad \forall y \neq x. \tag{2.19}$$

Finally, taking $K = \{(x,y) \in \mathbb{R}^2 : |x-y| \geq \delta, x, y \in \overline{B_R}\}$, we're done.

$\square$

Thus by the lemma, if f_x was to lose its modulus after time T, we must have that there exist $x \neq y \in \mathbb{R}$ with

$$f_x(T,x) - f_x(T,y) = \rho\left(\frac{|x-y|}{T}\right). \tag{2.20}$$

We will show for a smooth solution f of (2.2) and the correct choice of ρ that in this case

$$\frac{d}{dt}\left(f_x(t,x) - f_x(t,y)\right)\bigg|_{t=T} < \frac{d}{dt}\left(\rho\left(\frac{|x-y|}{t}\right)\right)\bigg|_{t=T}, \tag{2.21}$$

contradicting the fact that f_x had modulus $\rho(\cdot/t)$ for time $t < T$.

Thus we just need to prove (2.21) to complete the proof of the generation of modulus of continuity (2.8) of Theorem 2.1.1.

2.3 Equation for f_x

So, we just need to prove (2.21). To begin, we need to examine the equation that f_x solves. Since everything we will be doing is for some fixed time $T > 0$, we will suppress the time variable from now on. Differentiating (2.2), we see that f_x solves

$$\begin{aligned}(f_x)_t(x) = f_{xx}(x) \int_{\mathbb{R}} \frac{x-y}{(f(y)-f(x))^2+(y-x)^2} dy \\ + \int_{\mathbb{R}} (f(y)-f(x)-(y-x)f_x(x)) \frac{2\left((f(y)-f(x))f_x(x)+(y-x)\right)}{\left((f(y)-f(x))^2+(y-x)^2\right)^2} dy.\end{aligned} \tag{2.22}$$

To simplify notation, we reparametrize (2.22) by taking $y = x + h$, and letting

$$\delta_h f(x) := f(x+h) - f(x),$$

we get

$$\begin{aligned}(f_x)_t(x) = f_{xx}(x) \int_{\mathbb{R}} \frac{-h}{(\delta_h f(x))^2 + h^2} dh \\ + \int_{\mathbb{R}} (\delta_h f(x) - h f_x(x)) \frac{2\left(\delta_h f(x) f_x(x) + h\right)}{\left(\delta_h f(x)^2 + h^2\right)^2} dh.\end{aligned} \tag{2.23}$$

Note that

$$\delta_h f(x) - h f_x(x) = \int_0^h \delta_s f_x(x) ds,$$

for $h > 0$, and

$$\delta_h f(x) - h f_x(x) = -\int_h^0 \delta_s f_x(x) ds,$$

for $h < 0$.

With that in mind, define

$$k(x,s) = \frac{2\left(\delta_s f(x) f_x(x) + s\right)}{\left(\delta_s f(x)^2 + s^2\right)^2}, \tag{2.24}$$

and

$$K(x,h) = \begin{cases} \int\limits_h^\infty k(x,s)ds, & h > 0 \\ \int\limits_{-\infty}^h -k(x,s)ds, & h < 0 \end{cases}. \tag{2.25}$$

Then integrating (2.23) by parts, we have that f_x solves the equation

$$(f_x)_t(x) = f_{xx}(x) \int\limits_{\mathbb{R}} \frac{-h}{\delta_h f(x)^2 + h^2} dh + \int\limits_{\mathbb{R}} \delta_h f_x(x) K(x,h) dh. \tag{2.26}$$

As

$$\frac{-\beta(f_x)}{s} \leq \frac{f_x(x)\delta_s f(x)}{s} \leq \frac{||f_x||_{L^\infty}^2}{s}, \tag{2.27}$$

we see that

$$\frac{2(1-\beta(f_x))}{(1+||f_x||_{L^\infty}^2)^2} \frac{1}{|s|^3} \leq \operatorname{sgn}(s) k(x,s) \leq \frac{2(1+||f_x||_{L^\infty}^2)}{|s|^3},$$

and hence

$$\frac{1-\beta(f_x)}{(1+||f_x||_{L^\infty}^2)^2} \frac{1}{h^2} \leq K(x,h) \leq \frac{1+||f_x||_{L^\infty}^2}{h^2}. \tag{2.28}$$

Thus in the case that $\beta(f_x) \leq 1$, we then have that the kernel K is a nonnegative, from which we get immediately

Proposition 2.3.1. *(Maximum Principle)*

Let f_x be a sufficiently smooth solution to (2.26) *with* $\beta(f_0') \leq 1$*. Then for any* $0 \leq s \leq t$*, we have that*

$$\inf_y f_x(s,y) \leq \inf_y f_x(t,y) \leq \sup_y f_x(t,y) \leq \sup_y f_x(s,y). \tag{2.29}$$

In particular, since $\beta(f_0') < 1$ the maximum principle tells us that

$$\beta(f_x) \leq \beta(f_0') < 1, \qquad ||f_x||_{L^\infty} \leq ||f_0'||_{L^\infty} < \infty. \tag{2.30}$$

Thus we get that

$$0 < \frac{\lambda}{h^2} \leq K(x,h) \leq \frac{\Lambda}{h^2}, \tag{2.31}$$

where

$$\lambda = \frac{1 - \beta(f_0')}{(1 + ||f_0'||_{L^\infty}^2)^2}, \quad \Lambda = 1 + ||f_0'||_{L^\infty}^2. \tag{2.32}$$

Thus K is comparable to the kernel for $(-\Delta)^{1/2}$, so f_x solves the uniformly elliptic equation (2.26). Note that the sole reason we require $\beta(f_0') < 1$ is to ensure this ellipticity of K.

2.4 Moduli Estimates

Our goal is to show that if $f_x(T, \cdot)$ has modulus $\rho(\cdot/T)$ and equality is achieved at two points (2.20), then (2.21) must hold, contradicting the assumptions of the breakthrough argument (see section 2). To that end, we first need to understand how a modulus of continuity interacts with the equation for f_x (2.26). Hence,

Lemma 2.4.1. *Let $f : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ be a bounded smooth solution to (2.2) with $\beta(f_0') < 1$, and $\omega : [0,\infty) \to [0,\infty)$ be some fixed modulus of continuity. Assume that at some fixed time T that*

$$\begin{aligned} \delta_h f_x(T,x) &\leq \omega(|h|), \\ f_x(T,\xi/2) - f_x(T,-\xi/2) &= \omega(\xi), \end{aligned} \tag{2.33}$$

for all $h \in \mathbb{R}$, and for some $\xi > 0$. Then

$$\frac{d}{dt}(f_x(t,\xi/2) - f_x(t,-\xi/2))\Big|_{t=T} \le A\omega'(\xi)\left(\int_0^{\xi}\frac{\omega(h)}{h}dh + \xi\int_{\xi}^{\infty}\frac{\omega(h)}{h^2}dh + \ln(M+1)\omega(\xi)\right)$$
$$+ A\omega(\xi)\int_{M\xi}^{\infty}\frac{\omega(h)}{h^2}dh + 2(\Lambda-\lambda)\int_{\xi}^{M\xi}\frac{(\omega(h-\xi)-\omega(\xi))_+}{h^2}dh$$
$$+ 2\lambda\int_0^{\xi}\frac{\delta_h\omega(\xi)+\delta_{-h}\omega(\xi)}{h^2}dh + 2\lambda\int_{\xi}^{\infty}\frac{\omega(h+\xi)-\omega(h)-\omega(\xi)}{h^2}dh, \tag{2.34}$$

for any $M \ge 1$, where A depends only on $||f_0'||_{L^\infty}$ and λ, Λ are as in (2.32).

This is the main technical lemma that we need. Since solutions to (2.2) are closed under translation and sign change, it suffices to consider the above situation for our proof of (2.21).

Note that (4.2) holds for any value of the parameter $M \ge 1$. Later in Lemma 6.1, we will essentially use two different values of M depending on the size of ξ. In the small ξ regime we can simply take $M = 1$, but in the large ξ regime we will need to take M to be a sufficiently large constant depending only on initial data (but not on exact size of ξ) in order to control the size of the error term $\omega(\xi)\int_{M\xi}^{\infty}\frac{\omega(h)}{h^2}dh$.

The proof for Lemma 2.4.1 is essentially a nondivergence form argument; our function f_x is touched from above at $\xi/2$ by our modulus ω, and its touched from below at $-\xi/2$ by $-\omega$. Specifically,

$$\begin{aligned} \delta_h f_x(\xi/2) &\le \delta_h\omega(\xi), \qquad \forall h > -\xi, \\ \delta_h f_x(-\xi/2) &\ge -\delta_{-h}\omega(\xi), \qquad \forall h < \xi. \end{aligned} \tag{2.35}$$

From (2.35), we want to derive as much information as we can and bound $\frac{d}{dt}(f_x(\xi/2) - f_x(-\xi/2))$. To that end, by dividing (2.35) through by h and taking the limit as $h \to 0$, we then get that

$$f_{xx}(\xi/2) = f_{xx}(-\xi/2) = \omega'(\xi). \tag{2.36}$$

Hence by our equation for f_x (2.26), we have that

$$\begin{aligned}
\frac{d}{dt}(f_x(\xi/2) - f_x(-\xi/2)) = \omega'(\xi) \int_{\mathbb{R}} \left(\frac{-h}{\delta_h f(\xi/2)^2 + h^2} - \frac{-h}{\delta_h f(-\xi/2)^2 + h^2} \right) dh \\
+ \int_{\mathbb{R}} \delta_h f_x(\xi/2) K(\xi/2, h) - \delta_h f_x(-\xi/2) K(-\xi/2, h) dh \\
= \omega'(\xi) \int_{\mathbb{R}} \left(\frac{-h}{\delta_h f(\xi/2)^2 + h^2} - \frac{-h}{\delta_h f(-\xi/2)^2 + h^2} \right) dh \\
+ \omega'(\xi) \int_{-M\xi}^{M\xi} (hK(\xi/2, h) - hK(-\xi/2, h))\, dh \\
+ \int_{-M\xi}^{M\xi} (\delta_h f_x(\xi/2) - h\omega'(\xi)) K(\xi/2, h) - (\delta_h f_x(-\xi/2) - h\omega'(\xi)) K(-\xi/2, h) dh \\
+ \int_{|h| > M\xi} \delta_h f_x(\xi/2) K(\xi/2, h) - \delta_h f_x(-\xi/2) K(-\xi/2, h) dh,
\end{aligned} \tag{2.37}$$

for any $M \geq 1$. The first two terms of the RHS of (2.37) act as a drift, giving rise to the first two error terms of (2.34). The latter two terms of (2.37) act as a diffusion, giving rise to both the helpful (negative) terms in (2.34), as well as additional error terms (the middle terms of (2.34)) arising from the difference in the kernels, $|K(\xi/2, h) - K(-\xi/2, h)|$.

2.5 Bounds on Drift terms

We begin proving Lemma 2.4.1 by bounding the drift terms of (2.37), starting with

Lemma 2.5.1. *Under the assumptions of Lemma 2.4.1,*

$$\omega'(\xi)\left|\int_{\mathbb{R}} \frac{-h}{\delta_h f(\xi/2)^2+h^2} - \frac{-h}{\delta_h f(-\xi/2)^2+h^2} dh\right| \lesssim \omega'(\xi)\left(\int_0^\xi \frac{\omega(h)}{h} dh + \xi\int_\xi^\infty \frac{\omega(h)}{h^2} dh\right). \tag{2.38}$$

Proof. We want to bound (2.38) by symmetrizing the kernels for $|h| < \xi$, and and then using the continuity in the first variable for $|h| > \xi$. To that end,

$$\begin{aligned}
&\omega'(\xi)\int_{\mathbb{R}} \left(\frac{-h}{\delta_h f(\xi/2)^2+h^2} - \frac{-h}{\delta_h f(-\xi/2)^2+h^2}\right) dh \\
&\quad\leq \omega'(\xi)\int_0^\xi h\left|\frac{\delta_h f(\xi/2)^2-\delta_{-h} f(\xi/2)^2}{(\delta_h f(\xi/2)^2+h^2)(\delta_{-h} f(\xi/2)^2+h^2)} + \frac{\delta_h f(-\xi/2)^2-\delta_{-h} f(-\xi/2)^2}{(\delta_h f(-\xi/2)^2+h^2)(\delta_{-h} f(-\xi/2)^2+h^2)}\right| dh \\
&\qquad + \omega'(\xi)\int_{|h|>\xi} |h|\left|\frac{\delta_h f(\xi/2)^2-\delta_h f(-\xi/2)^2}{(\delta_h f(\xi/2)^2+h^2)(\delta_h f(-\xi/2)^2+h^2)}\right| dh.
\end{aligned} \tag{2.39}$$

We bound the first integral using

$$\begin{aligned}
|\delta_h f(x)| &\lesssim |h|, \\
|\delta_h f(x)+\delta_{-h} f(x)| &= \left|\int_0^h f_x(x+s)-f_x(x+s-h)ds\right| \leq \omega(h)h,
\end{aligned} \tag{2.40}$$

Thus get that for $0 \leq h < \xi$,

$$\left|\frac{\delta_h f(x)^2-\delta_{-h} f(x)^2}{(\delta_h f(x)^2+h^2)(\delta_{-h} f(x)^2+h^2)}\right| \lesssim \frac{\omega(h)}{h^2}, \tag{2.41}$$

and hence

$$\int_0^\xi h\left|\frac{\delta_h f(\xi/2)^2-\delta_{-h} f(\xi/2)^2}{(\delta_h f(\xi/2)^2+h^2)(\delta_{-h} f(\xi/2)^2+h^2)} dh\right| \lesssim \int_0^\xi \frac{\omega(h)}{h} dh. \tag{2.42}$$

For $|h| \geq \xi$, we bound $|\delta_h f(\xi/2) + \delta_h f(-\xi/2)| \lesssim |h|$ and

$$\begin{aligned} \left|\delta_h f(\xi/2) - \delta_h f(-\xi/2)\right| &= \left|\int_0^h f_x(\xi/2 + s) - f_x(-\xi/2 + s) ds\right| \\ &= \left|\int_0^\xi f_x(h - \xi/2 + s) - f_x(-\xi/2 + s) ds\right| \leq \xi\omega(|h|), \end{aligned} \tag{2.43}$$

in order to get

$$\int_{|h|>\xi} |h| \left|\frac{\delta_h f(\xi/2)^2 - \delta_h f(-\xi/2)^2}{(\delta_h f(\xi/2)^2 + h^2)(\delta_h f(-\xi/2)^2 + h^2)}\right| dh \lesssim \xi \int_\xi^\infty \frac{\omega(h)}{h^2} dh. \tag{2.44}$$

Putting (2.42) and (2.44) together, we thus have

$$\omega'(\xi) \int_{\mathbb{R}} \left(\frac{-h}{\delta_h f(\xi/2)^2 + h^2} - \frac{-h}{\delta_h f(-\xi/2)^2 + h^2}\right) dh \lesssim \omega'(\xi) \left(\int_0^\xi \frac{\omega(h)}{h} dh + \xi \int_\xi^\infty \frac{\omega(h)}{h^2} dh\right). \tag{2.45}$$

□

That leaves us with the second drift term of (2.37),

Lemma 2.5.2. *Under the assumptions of Lemma 2.4.1, for any $M \geq 1$*

$$\omega'(\xi)\left|\int_{-M\xi}^{M\xi} hK(\xi/2, h) - hK(-\xi/2, h) dh\right| \lesssim \omega'(\xi) \left(\int_0^\xi \frac{\omega(h)}{h} dh + \xi \int_\xi^\infty \frac{\omega(h)}{h^2} dh + \ln(M+1)\omega(\xi)\right). \tag{2.46}$$

Proof. To begin, we note

$$\omega'(\xi)\Big|\int_{-M\xi}^{M\xi} hK(\xi/2,h) - hK(-\xi/2,h)dh\Big|$$
$$\leq \omega'(\xi)\int_{0}^{M\xi} h\Big|K(\xi/2,h) - K(\xi/2,-h) - K(-\xi/2,h) + K(-\xi/2,-h)\Big|dh. \tag{2.47}$$

Recall the definition of K, (2.25),

$$K(x,h) = \begin{cases} \int_h^\infty k(x,s)ds, & h > 0 \\ \int_{-\infty}^h -k(x,s)ds, & h < 0 \end{cases}, \tag{2.48}$$
$$k(x,s) = \frac{2\left(\delta_s f(x) f_x(x) + s\right)}{\left(\delta_s f(x)^2 + s^2\right)^2}.$$

So, to control (2.47) we first need to bound $|k(x,s)+k(x,-s)|$ for $0 \leq s < \xi$, and $|k(\xi/2,s) - k(-\xi/2,s)|$ for $|s| > \xi$. For the first, using the bounds (2.40) we see that

$$\begin{aligned} |k(x,s) + k(x,-s)| &= \left|\frac{2\left(\delta_s f(x) f_x(x) + s\right)}{\left(\delta_s f(x)^2 + s^2\right)^2} + \frac{2\left(\delta_{-s} f(x) f_x(x) - s\right)}{\left(\delta_{-s} f(x)^2 + s^2\right)^2}\right| \\ &\leq \frac{2|\delta_s f(x) + \delta_{-s} f(x)| \cdot |f_x(x)|}{\left(\delta_{-s} f(x)^2 + s^2\right)^2} \\ &\quad + 2|\delta_s f(x) f_x(x) + s| \left|\frac{\left(\delta_s f(x)^2 + s^2\right)^2 - \left(\delta_{-s} f(x)^2 + s^2\right)^2}{\left(\delta_s f(x)^2 + s^2\right)^2\left(\delta_{-s} f(x)^2 + s^2\right)^2}\right| \\ &\lesssim \frac{\omega(s)}{s^3} + s\left|\frac{\delta_s f(x)^4 - \delta_{-s} f(x)^4 + 2s^2(\delta_s f(x)^2 - \delta_{-s} f(x)^2)}{s^8}\right| \\ &\lesssim \frac{\omega(s)}{s^3}. \end{aligned} \tag{2.49}$$

For the second, using (2.40), (2.43), and (2.33) we get that

$$\begin{aligned}
|k(\xi/2,s)-k(-\xi/2,s)| &= \left|\frac{2\left(\delta_s f(\xi/2) f_x(\xi/2)+s\right)}{\left(\delta_s f(\xi/2)^2+s^2\right)^2}-\frac{2\left(\delta_s f(-\xi/2) f_x(-\xi/2)+s\right)}{\left(\delta_s f(-\xi/2)^2+s^2\right)^2}\right| \\
&\leq 2\frac{|\delta_s f(\xi/2) f_x(\xi/2)-\delta_s f(-\xi/2) f_x(-\xi/2)|}{\left(\delta_s f(-\xi/2)^2+s^2\right)^2} \\
&\quad + 2|\delta_s f(\xi/2) f_x(\xi/2)+s|\left|\frac{\left(\delta_s f(\xi/2)^2+s^2\right)^2-\left(\delta_s f(-\xi/2)^2+s^2\right)^2}{\left(\delta_s f(\xi/2)^2+s^2\right)^2\left(\delta_s f(-\xi/2)^2+s^2\right)^2}\right| \\
&\lesssim \frac{|\delta_s f(\xi/2)-\delta_s f(-\xi/2)|\cdot|f_x(\xi/2)|}{s^4}+\frac{|\delta_s f(-\xi/2)|\cdot|f_x(\xi/2)-f_x(-\xi/2)|}{s^4} \\
&\quad + |s|\left|\frac{\delta_s f(\xi/2)^4-\delta_s f(-\xi/2)^4+s^2\left(\delta_s f(\xi/2)^2-\delta_s f(-\xi/2)^2\right)}{s^8}\right| \\
&\lesssim \frac{\xi\omega(s)}{s^4}+\frac{\omega(\xi)}{s^3}.
\end{aligned} \tag{2.50}$$

So using (2.49) and (2.50), we can first bound

$$\begin{aligned}
&\int_0^\xi h\Big|K(\xi/2,h)-K(\xi/2,-h)-K(-\xi/2,h)+K(-\xi/2,-h)\Big|dh \\
&\lesssim \int_0^\xi h\int_h^\xi \frac{\omega(s)}{s^3}ds dh+\int_0^\xi h\int_\xi^\infty \frac{\xi\omega(s)}{s^4}+\frac{\omega(\xi)}{s^3}ds dh \\
&\lesssim \int_0^\xi \frac{\omega(s)}{s^3}\int_0^s h dh ds+\int_\xi^\infty \frac{\xi^3\omega(s)}{s^4}+\frac{\xi^2\omega(\xi)}{s^3}ds \\
&\lesssim \int_0^\xi \frac{\omega(s)}{s}ds+\xi\int_\xi^\infty \frac{\omega(s)}{s^2}ds+\omega(\xi).
\end{aligned} \tag{2.51}$$

For the rest of (2.47), we use (2.50) again to also bound

$$\begin{aligned}\int_{M\xi>|h|>\xi} |h| \left|K(\xi/2,h) - K(-\xi/2,h)\right| dh &\lesssim \int_{\xi}^{M\xi} h \int_{h}^{\infty} \frac{\omega(\xi)}{s^3} + \frac{\xi\omega(s)}{s^4} ds \\ &\lesssim \omega(\xi) \int_{\xi}^{M\xi} \frac{1}{h} dh + \xi \int_{\xi}^{M\xi} \frac{\omega(h)}{h^2} dh \\ &\lesssim \ln(M)\omega(\xi) + \xi \int_{\xi}^{\infty} \frac{\omega(h)}{h^2} dh.\end{aligned} \tag{2.52}$$

□

2.6 Bounds on Diffusive Terms

Now we move on to proving an upper bound for the diffusive terms of (2.37). We can rewrite them as

$$\begin{aligned}&\int_{-M\xi}^{M\xi} (\delta_h f_x(\xi/2) - h\omega'(\xi))K(\xi/2,h) - (\delta_h f_x(-\xi/2) - h\omega'(\xi))K(-\xi/2,h) dh \\ &\qquad + \int_{|h|>M\xi} \delta_h f_x(\xi/2) K(\xi/2,h) - \delta_h f_x(-\xi/2) K(-\xi/2,h) dh \\ &= \int_{-M\xi}^{M\xi} (\delta_h f_x(\xi/2) - h\omega'(\xi))K(\xi/2,h) - (\delta_h f_x(-\xi/2) - h\omega'(\xi))K(-\xi/2,h) dh \\ &\quad + \int_{|h|>M\xi} \left[\delta_h f_x(\xi/2) - \delta_h f_x(-\xi/2)\right] K(\xi/2,h) dh \\ &\quad + \int_{|h|>M\xi} \delta_h f_x(-\xi/2) \left[K(\xi/2,h) - K(-\xi/2,h)\right] dh.\end{aligned} \tag{2.53}$$

We begin by bounding the last term, which is an error term.

Lemma 2.6.1. *Under the assumptions of Lemma 2.4.1,*

$$\Bigg|\int\limits_{|h|>M\xi} \delta_h f_x(-\xi/2)\left[K(\xi/2,h)-K(-\xi/2,h)\right]\Bigg| dh \lesssim \omega(\xi)\int\limits_{M\xi}^{\infty}\frac{\omega(h)}{h^2}dh + \omega'(\xi)\xi\int\limits_{\xi}^{\infty}\frac{\omega(h)}{h^2}dh. \tag{2.54}$$

Proof. Using the fact that f_x has modulus ω and the bounds 2.50, it follows that

$$\begin{aligned}
\int\limits_{|h|>M\xi} \delta_h f_x(-\xi/2)[K(\xi/2,h)-K(-\xi/2,h)]dh &\lesssim \int\limits_{M\xi}^{\infty}\omega(h)\int\limits_{h}^{\infty}\frac{\omega(\xi)}{s^3}+\frac{\xi\omega(s)}{s^4}dsdh \\
&\lesssim \omega(\xi)\int\limits_{M\xi}^{\infty}\frac{\omega(h)}{h^2}dh + \int\limits_{M\xi}^{\infty}\omega(h)\int\limits_{h}^{\infty}\frac{\xi\omega(\xi)+\xi\omega'(\xi)(s-\xi)}{s^4}dsdh \\
&\lesssim \omega(\xi)\int\limits_{M\xi}^{\infty}\frac{\omega(h)}{h^2}dh + \omega(\xi)\int\limits_{M\xi}^{\infty}\frac{\xi\omega(h)}{h^3}dh + \omega'(\xi)\xi\int\limits_{M\xi}^{\infty}\frac{\omega(h)}{h^2}dh \\
&\lesssim \omega(\xi)\int\limits_{M\xi}^{\infty}\frac{\omega(h)}{h^2}dh. + \omega'(\xi)\xi\int\limits_{\xi}^{\infty}\frac{\omega(h)}{h^2}dh.
\end{aligned} \tag{2.55}$$

□

For the other two terms in (2.53), we bound them in two stages.

Lemma 2.6.2. *Under the assumptions of Lemma 2.4.1,*

$$\begin{aligned}\int_{-M\xi}^{M\xi} &(\delta_h f_x(\xi/2) - h\omega'(\xi))K(\xi/2,h) - (\delta_h f_x(-\xi/2) - h\omega'(\xi))K(-\xi/2,h)dh \\ &+ \int_{|h|>M\xi} [\delta_h f_x(\xi/2) - \delta_h f_x(-\xi/2)]\, K(\xi/2,h)dh \\ &\qquad \le \lambda \int_{\mathbb{R}} \frac{\delta_h f_x(\xi/2) - \delta_h f_x(-\xi/2)}{h^2} dh + 2(\Lambda - \lambda) \int_{\xi}^{M\xi} \frac{(\omega(h-\xi) - \omega(\xi))_+}{h^2} dh \\ &\qquad\qquad + \omega'(\xi) \int_{\xi<|h|<M\xi} \Big| h\,[K(\xi/2,h) - K(-\xi/2,h)] \Big| dh.\end{aligned} \tag{2.56}$$

Proof. We can bound the second term of (2.56) rather easily. Since

$$\delta_h f_x(\xi/2) - \delta_h f_x(-\xi/2) = (f_x(h+\xi/2) - f_x(h-\xi/2)) - \omega(\xi) \le 0, \tag{2.57}$$

by the uniform ellipticity of K,

$$\int_{|h|>M\xi} [\delta_h f_x(\xi/2) - \delta_h f_x(-\xi/2)]\, K(\xi/2,h)dh \le \lambda \int_{|h|>M\xi} \frac{\delta_h f_x(\xi/2) - \delta_h f_x(-\xi/2)}{h^2} dh. \tag{2.58}$$

To bound the first term, we first define

$$G(\xi,h) = (\delta_h f_x(\xi/2) - h\omega'(\xi))K(\xi/2,h) - (\delta_h f_x(-\xi/2) - h\omega'(\xi))K(-\xi/2,h). \tag{2.59}$$

Note that since ω is concave and touches f_x from above (see (2.35)), it follows that

$$\begin{aligned}\delta_h f_x(\xi/2) - \omega'(\xi)h \le \delta_h \omega(\xi) - \omega'(\xi)h \le 0, \quad h \ge -\xi \\ \delta_h f_x(-\xi/2) - \omega'(\xi)h \ge -\delta_{-h}\omega(\xi) - h\omega'(\xi) \ge 0, \quad h \le \xi\end{aligned} \tag{2.60}$$

Thus for $|h| \leq \xi$, by the uniform ellipticity of K we have the bound

$$G(\xi, h) \leq \lambda \frac{\delta_h f_x(\xi/2) - \delta_h f_x(-\xi/2)}{h^2}. \tag{2.61}$$

That just leaves us with the case $\xi \leq |h| \leq M\xi$ to analyze. Note that we can write G in two distinct ways:

$$\begin{aligned} G(\xi, h) &= (\delta_h f_x(\xi/2) - \delta_h f_x(-\xi/2))K(\frac{\xi}{2}, h) + (\delta_h f_x(-\xi/2) - h\omega'(\xi))(K(\frac{\xi}{2}, h) - K(\frac{-\xi}{2}, h)) \\ &= (\delta_h f_x(\xi/2) - \delta_h f_x(-\xi/2))K(\frac{-\xi}{2}, h) + (\delta_h f_x(\xi/2) - h\omega'(\xi))(K(\frac{\xi}{2}, h) - K(\frac{-\xi}{2}, h)). \end{aligned} \tag{2.62}$$

By (2.60), $\delta_h f_x(\xi/2) - h\omega'(\xi) \leq 0$ for all $h > \xi$. Thus if $K(\xi/2, h) - K(-\xi/2, h) \geq 0$, then

$$G(\xi, h) \leq \lambda \frac{\delta_h f_x(\xi/2) - \delta_h f_x(-\xi/2)}{h^2}, \qquad \text{if } K(\xi/2, h) - K(-\xi/2, h) \geq 0 \tag{2.63}$$

On the other hand, since

$$\delta_h f_x(-\xi/2) = \delta_{h-\xi} f(\xi/2) + \omega(\xi) \geq -\omega(h - \xi) + \omega(\xi) \tag{2.64}$$

for $h \geq \xi$, we see that

$$\begin{aligned} G(\xi, h) &\leq \lambda \frac{\delta_h f_x(\xi/2) - \delta_h f_x(-\xi/2)}{h^2} + (\delta_h f_x(-\xi/2) - h\omega'(\xi))(K(\xi/2, h) - K(-\xi/2, h)) \\ &\leq \lambda \frac{\delta_h f_x(\xi/2) - \delta_h f_x(-\xi/2)}{h^2} + (\Lambda - \lambda)\frac{(\omega(h - \xi) - \omega(\xi))_+}{h^2} \\ &\quad + h\omega'(\xi)|K(\xi/2, h) - K(-\xi/2, h)|, \\ &\text{if } K(\xi/2, h) - K(-\xi/2, h) \leq 0. \end{aligned} \tag{2.65}$$

Putting these two together, we get that

$$G(\xi,h) \le \lambda\frac{\delta_h f_x(\xi/2)-\delta_h f_x(-\xi/2)}{h^2}+(\Lambda-\lambda)\frac{(\omega(h-\xi)-\omega(\xi))_+}{h^2}+h\omega'(\xi)|K(\frac{\xi}{2},h)-K(\frac{-\xi}{2},h)|. \tag{2.66}$$

for $h \ge \xi$. A similar argument can be made in the case that $h \le -\xi$.

Putting this all together,

$$\int_{-M\xi}^{M\xi} G(\xi,h)dh + \int_{|h|>M\xi} \left[\delta_h f_x(\xi/2)-\delta_h f_x(-\xi/2)\right] K(\xi/2,h)dh$$
$$\le \lambda\int_{\mathbb{R}} \frac{\delta_h f_x(\xi/2)-\delta_h f_x(-\xi/2)}{h^2}dh + 2(\Lambda-\lambda)\int_{\xi}^{M\xi}\frac{(\omega(h-\xi)-\omega(\xi))_+}{h^2}dh$$
$$+\,\omega'(\xi)\int_{\xi<|h|<M\xi}\left|h\left[K(\xi/2,h)-K(-\xi/2,h)\right]\right|dh. \tag{2.67}$$

□

It's clear that we can bound $\int_{\xi<|h|<M\xi}\left|h\left[K(\xi/2,h)-K(-\xi/2,h)\right]\right|dh$ as in (2.52). Thus the only thing remaining to prove (2.34) is

Lemma 2.6.3. *Under the assumptions of Lemma 2.4.1,*

$$\lambda\int_{\mathbb{R}}\frac{\delta_h f_x(\xi/2)-\delta_h f_x(-\xi/2)}{h^2}dh \le 2\lambda\int_0^{\xi}\frac{\delta_h\omega(\xi)+\delta_{-h}\omega(\xi)}{h^2}dh+2\lambda\int_{\xi}^{\infty}\frac{\omega(\xi+h)-\omega(h)-\omega(\xi)}{h^2}dh. \tag{2.68}$$

Proof. To see this, note that formally we should have

$$\int_{\mathbb{R}}\frac{\delta_h f_x(\xi/2)-\delta_h f_x(-\xi/2)}{h^2}dh = \int_{\mathbb{R}} f_x(y)\left(\frac{1}{(y-\xi/2)^2}-\frac{1}{(y+\xi/2)^2}\right)-\frac{\omega(\xi)}{y^2}dy. \tag{2.69}$$

Thus in order to get an upper bound on (2.69), we should be taking an upper bound on $f_x(y)$ when $y > 0$ and a lower bound when $y < 0$. Note by (2.35) that

$$\begin{aligned} f_x(y) \leq f_x(\xi/2) + \omega(y+\xi/2) - \omega(\xi) = f_x(-\xi/2) + \omega(y+\xi/2), \quad y > -\xi/2, \\ f_x(y) \geq f_x(-\xi/2) - \omega(-y+\xi/2) + \omega(\xi) = f_x(\xi/2) - \omega(-y+\xi/2), \quad y < \xi/2. \end{aligned} \tag{2.70}$$

In particular, using the upper bounds bounds on $\delta_h f_x(\pm\xi/2)$ for $h > 0$ and the lower bounds for $\delta_h f_x(\pm\xi/2)$ for $h < 0$ give the result. To rigorously justify this though, we will bound

$$\int_{\epsilon}^{\infty} \frac{\delta_h f_x(\xi/2) - \delta_h f_x(-\xi/2)}{h^2} dh$$

from above. Taking $\epsilon \to 0$, we'll get

$$\int_0^{\infty} \frac{\delta_h f_x(\xi/2) - \delta_h f_x(-\xi/2)}{h^2} dh \leq \int_0^{\xi} \frac{\delta_h \omega(\xi) + \delta_{-h}\omega(\xi)}{h^2} dh + \int_{\xi}^{\infty} \frac{\omega(\xi+h) - \omega(h) - \omega(\xi)}{h^2} dh. \tag{2.71}$$

The bound for $\int\limits_{-\infty}^{0}$ follows from identical arguments.

So, fix some $\epsilon << \xi$. By splitting the integral into a several pieces and reparameterizing, we get that

$$\begin{aligned} \int_{\epsilon}^{\infty} \frac{\delta_h f_x(\xi/2) - \delta_h f_x(-\xi/2)}{h^2} dh &= \int_{\epsilon+\xi/2}^{\infty} \frac{f_x(y)}{(y-\xi/2)^2} dy - \int_{\epsilon-\xi/2}^{\infty} \frac{f_x(y)}{(y+\xi/2)^2} dy - \int_{\epsilon}^{\infty} \frac{\omega(\xi)}{y^2} dy \\ &= \int_{\epsilon+\xi/2}^{\infty} f_x(y) \left(\frac{1}{(y-\xi/2)^2} - \frac{1}{(y+\xi/2)^2} \right) dy - \int_{\epsilon}^{\infty} \frac{\omega(\xi)}{y^2} dy - \int_{\epsilon-\xi/2}^{\epsilon+\xi/2} \frac{f_x(y)}{(y+\xi/2)^2} dy. \end{aligned} \tag{2.72}$$

In the first integral of the second line, since $y > \xi/2$ we have that $(y-\xi/2)^{-2} > (y+\xi/2)^{-2}$.

So applying the upper bound in (2.70) gives an upper bound on the integral,

$$\begin{aligned}
&\int\limits_{\epsilon+\xi/2}^{\infty} f_x(y)\left(\frac{1}{(y-\xi/2)^2}-\frac{1}{(y+\xi/2)^2}\right)dy \\
&\qquad \leq \int\limits_{\epsilon+\xi/2}^{\infty} \left(f_x(\xi/2)+\omega(y+\xi/2)-\omega(\xi)\right)\left(\frac{1}{(y-\xi/2)^2}-\frac{1}{(y+\xi/2)^2}\right)dy \\
&\qquad = \int\limits_{\epsilon+\xi/2}^{\infty} \frac{f_x(\xi/2)+\omega(y+\xi/2)-\omega(\xi)}{(y-\xi/2)^2}dy - \int\limits_{\epsilon+\xi/2}^{\infty} \frac{f_x(\xi/2)+\omega(y+\xi/2)-\omega(\xi)}{(y+\xi/2)^2}dy
\end{aligned} \tag{2.73}$$

By reparametrizing back, we get that

$$\begin{aligned}
&\int\limits_{\epsilon+3\xi/2}^{\infty} \frac{f_x(\xi/2)+\omega(y+\xi/2)-\omega(\xi)}{(y-\xi/2)^2}dy - \int\limits_{\epsilon+\xi/2}^{\infty} \frac{f_x(\xi/2)+\omega(y+\xi/2)-\omega(\xi)}{(y+\xi/2)^2}dy - \int\limits_{\epsilon+\xi}^{\infty} \frac{\omega(\xi)}{y^2}dy \\
&\qquad = \int\limits_{\epsilon+\xi}^{\infty} \frac{\omega(\xi+h)-\omega(h)-\omega(\xi)}{h^2}dh
\end{aligned} \tag{2.74}$$

Hence combining (2.72),(2.73), and (2.74) gives us

$$\begin{aligned}\int_{\epsilon}^{\infty} \frac{\delta_h f_x(\xi/2) - \delta_h f_x(-\xi/2)}{h^2} dh &\leq \int_{\epsilon+\xi}^{\infty} \frac{\omega(\xi+h) - \omega(h) - \omega(\xi)}{h^2} dh \\ &\quad + \int_{\epsilon}^{\epsilon+\xi} \frac{f_x(\xi/2) + \omega(\xi+h) - 2\omega(\xi)}{h^2} dh - \int_{\epsilon}^{\epsilon+\xi} \frac{f_x(h-\xi/2)}{h^2} dh \\ &= \int_{\epsilon+\xi}^{\infty} \frac{\omega(\xi+h) - \omega(h) - \omega(\xi)}{h^2} dh \\ &\quad + \int_{\epsilon}^{\epsilon+\xi} \frac{\delta_h \omega(\xi) + f_x(\xi/2) - f_x(h-\xi/2) - \omega(\xi)}{h^2} dh. \end{aligned} \tag{2.75}$$

Now for $h < \xi$, we have that $f_x(\xi/2) - f_x(h-\xi/2) \leq \omega(\xi - h)$, and thus

$$\int_{\epsilon}^{\xi} \frac{\delta_h \omega(\xi) + f_x(\xi/2) - f_x(h-\xi/2) - \omega(\xi)}{h^2} dh \leq \int_{\epsilon}^{\xi} \frac{\delta_h \omega(\xi) + \delta_{-h}\omega(\xi)}{h^2} dh. \tag{2.76}$$

Taking the limit as $\epsilon \to 0$, we then get

$$\int_{0}^{\infty} \frac{\delta_h f_x(\xi/2) - \delta_h f_x(-\xi/2)}{h^2} dh \leq \int_{0}^{\xi} \frac{\delta_h \omega(\xi) + \delta_{-h}\omega(\xi)}{h^2} dh + \int_{\xi}^{\infty} \frac{\omega(\xi+h) - \omega(h) - \omega(\xi)}{h^2} dh. \tag{2.77}$$

□

2.7 Modulus Inequality

Combining all the estimates from the previous two sections, we get a proof of Lemma 2.4.1. Thus under the assumptions (2.33), we have that

$$\begin{aligned}\frac{d}{dt}(f_x(\xi/2) - f_x(-\xi/2)) \leq & A\omega'(\xi)\left(\int_0^\xi \frac{\omega(h)}{h}dh + \xi\int_\xi^\infty \frac{\omega(h)}{h^2}dh + \ln(M+1)\omega(\xi)\right) \\ & + A\omega(\xi)\int_{M\xi}^\infty \frac{\omega(h)}{h^2}dh + 2(\Lambda - \lambda)\int_\xi^{M\xi} \frac{(\omega(h-\xi) - \omega(\xi))_+}{h^2}dh \\ & + 2\lambda\int_0^\xi \frac{\delta_h\omega(\xi) + \delta_{-h}\omega(\xi)}{h^2}dh + 2\lambda\int_\xi^\infty \frac{\omega(h+\xi) - \omega(h) - \omega(\xi)}{h^2}dh,\end{aligned} \tag{2.78}$$

for any $M \geq 1$, where A is a constant depending only on $||f_0'||_{L^\infty}$.

In [AKV07], the authors showed that the modulus

$$\begin{cases} \omega(\xi) = \xi - \xi^{3/2}, & 0 \leq \xi \leq \delta \\ \omega'(\xi) = \dfrac{\gamma}{\xi(4 + \log(\xi/\delta))}, & \xi \geq \delta \end{cases}, \tag{2.79}$$

satisfies

$$\begin{aligned} & A\omega'(\xi)\left(\int_0^\xi \frac{\omega(h)}{h}dh + \xi\int_\xi^\infty \frac{\omega(h)}{h^2}dh\right) \\ & + \lambda\int_0^\xi \frac{\delta_h\omega(\xi) + \delta_{-h}\omega(\xi)}{h^2}dh + \lambda\int_\xi^\infty \frac{\omega(h+\xi) - \omega(h) - \omega(\xi)}{h^2}dh < 0,\end{aligned} \tag{2.80}$$

for all $\xi \in \mathbb{R}$ so long as δ, γ are sufficiently small.

With that in mind, we will show that

Lemma 2.7.1. *Under the assumptions of Lemma 2.4.1 for the modulus* ω *defined in* (2.79)*,*

$$\frac{d}{dt}(f_x(\xi/2) - f_x(-\xi/2)) < -\omega'(\xi)\omega(\xi), \tag{2.81}$$

as long as δ, γ *are taken sufficiently small depending on* $\beta(f_0'), ||f_0'||_{L^\infty}$.

Proof. By the Lemma 2.4.1 and (2.80) which was proven in [AKV07], it suffices to show

$$\begin{aligned} A\omega'(\xi)\ln(M+1)\omega(\xi) + A\omega(\xi)\int_{M\xi}^{\infty}\frac{\omega(h)}{h^2}dh + 2(\Lambda-\lambda)\int_{\xi}^{M\xi}\frac{(\omega(h-\xi)-\omega(\xi))_+}{h^2}dh \\ + \lambda\int_0^{\xi}\frac{\delta_h\omega(\xi)+\delta_{-h}\omega(\xi)}{h^2}dh + \lambda\int_{\xi}^{\infty}\frac{\omega(h+\xi)-\omega(h)-\omega(\xi)}{h^2}dh \\ \le -\omega'(\xi)\omega(\xi) \end{aligned} \tag{2.82}$$

for the correct choices of M, and δ, γ sufficiently small.

We proceed very similarly to [AKV07]. To begin, for $\xi \le \delta$ we take $M = 1$. Then we just need to show that

$$\begin{aligned} A\omega'(\xi)\omega(\xi) + A\omega(\xi)\int_{\xi}^{\infty}\frac{\omega(h)}{h^2}dh + \lambda\int_0^{\xi}\frac{\delta_h\omega(\xi)+\delta_{-h}\omega(\xi)}{h^2}dh + \lambda\int_{\xi}^{\infty}\frac{\omega(h+\xi)-\omega(h)-\omega(\xi)}{h^2}dh \\ \le -\omega'(\xi)\omega(\xi). \end{aligned} \tag{2.83}$$

In this regime, note that we have the bounds

$$\begin{cases} \int\limits_{\xi}^{\delta} \frac{\omega(h)}{h^2} dh \leq \log(\delta/\xi), \\ \int\limits_{\delta}^{\infty} \frac{\omega(h)}{h^2} dh = \frac{\omega(\delta)}{\delta} + \gamma \int\limits_{\delta}^{\infty} \frac{1}{h^2(4+\log(h/\delta))} dh \leq 1 + \frac{\gamma}{4\delta} \leq 2 \text{ if you take } \gamma < 4\delta, \\ \omega'(\xi) \leq 1, \\ \omega(\xi) \leq \xi, \\ \int\limits_{0}^{\xi} \frac{\omega(\xi+h)+\omega(\xi-h)-2\omega(\xi)}{h^2} \leq \xi\omega''(\xi) = -\frac{3}{2}\xi\xi^{-1/2}. \end{cases} \tag{2.84}$$

Putting this all together, we get that

$$\begin{aligned} &(A+1)\omega'(\xi)\omega(\xi) + A\omega(\xi) \int\limits_{\xi}^{\infty} \frac{\omega(h)}{h^2} dh + \lambda \int\limits_{0}^{\xi} \frac{\omega(\xi+h)+\omega(\xi-h)-2\omega(\xi)}{h^2} dh \\ &\quad + \lambda \int\limits_{\xi}^{\infty} \frac{\omega(\xi+h)-\omega(h)-\omega(\xi)}{h^2} dh \leq \xi \left((A+1)(3+\log(\delta/\xi)) - \frac{3}{2}\lambda\xi^{-1/2} \right) < 0, \end{aligned} \tag{2.85}$$

assuming that δ is sufficiently small.

Now assume that $\xi \geq \delta$. Then what we need to show is

$$\begin{aligned} &A\omega'(\xi)\ln(M+1)\omega(\xi) + A\omega(\xi) \int\limits_{M\xi}^{\infty} \frac{\omega(h)}{h^2} dh + 2(\Lambda-\lambda) \int\limits_{\xi}^{M\xi} \frac{(\omega(h-\xi)-\omega(\xi))_+}{h^2} dh \\ &\quad + \lambda \int\limits_{0}^{\xi} \frac{\delta_h\omega(\xi)+\delta_{-h}\omega(\xi)}{h^2} dh + \lambda \int\limits_{\xi}^{\infty} \frac{\omega(h+\xi)-\omega(h)-\omega(\xi)}{h^2} dh \leq -\omega'(\xi)\omega(\xi). \end{aligned} \tag{2.86}$$

We first bound our new error terms. Using the definition of ω and integrating by parts, we see that

$$
\begin{aligned}
2(\Lambda-\lambda)\int_{2\xi}^{M\xi}\frac{\omega(h-\xi)-\omega(\xi)}{h^2}dh &\le 2(\Lambda-\lambda)\int_{\xi}^{\infty}\frac{\omega(h)-\omega(\xi)}{h^2}dh \\
&\le 2(\Lambda-\lambda)\int_{\xi}^{\infty}\frac{\gamma}{h^2(4+\log(h/\delta))}dh \\
&\le \frac{2(\Lambda-\lambda)\gamma}{\xi} \le \frac{\lambda}{4}\frac{\omega(\delta)}{\xi} \le \frac{\lambda}{4}\frac{\omega(\xi)}{\xi},
\end{aligned} \tag{2.87}
$$

assuming $\gamma \le \frac{\lambda}{8(\Lambda-\lambda)}\omega(\delta)$.

In order to bound our other new error term, we will be taking M sufficiently large and then γ sufficiently small depending on M, δ. Noting that $\omega(\xi) \le 2||f_0'||_{L^\infty}$, we can bound our other new error term by integrating by parts

$$
\begin{aligned}
A\omega(\xi)\int_{M\xi}^{\infty}\frac{\omega(h)}{h^2}dh &\le \frac{2A||f_0'||_{L^\infty}}{M}\frac{\omega(M\xi)}{\xi} + 2A||f_0'||_{L^\infty}\int_{M\xi}^{\infty}\frac{\gamma}{h^2(4+\log(h/\delta))}dh \\
&\le \frac{2A||f_0'||_{L^\infty}}{M}\frac{\omega(M\xi)}{\xi} + \frac{2A||f_0'||_{L^\infty}}{M}\frac{\gamma}{\xi} \\
&\le \frac{\lambda}{16}\frac{\omega(M\xi)}{\xi} + \frac{\lambda}{8}\frac{\omega(\xi)}{\xi},
\end{aligned} \tag{2.88}
$$

assuming that

$$
M \ge \frac{32A||f_0'||_{L^\infty}}{\lambda},
$$

and then γ is sufficiently small so that

$$
\frac{2||f_0'||_{L^\infty}A}{M}\gamma \le \frac{\lambda}{8}\omega(\delta) \le \frac{\lambda}{8}\omega(\xi).
$$

Note that this is where we set a value for M, and that γ is taken sufficiently small depending on M. Now that the value for M is fixed, we can also control the value $\omega(M\xi)$ by taking γ

sufficiently small that

$$\omega(M\xi) = \omega(\xi) + \int_{\xi}^{M\xi} \frac{\gamma}{h(4+\log(h/\delta))} dh \leq \omega(\xi) + \gamma \ln(M) \leq \omega(\xi) + \omega(\delta)$$
$$\leq 2\omega(\xi). \tag{2.89}$$

Hence,

$$A\omega(\xi) \int_{M\xi}^{\infty} \frac{\omega(h)}{h^2} dh \leq \frac{\lambda}{16} \frac{\omega(M\xi)}{\xi} + \frac{\lambda}{8} \frac{\omega(\xi)}{\xi} \leq \frac{\lambda}{4} \frac{\omega(\xi)}{\xi}. \tag{2.90}$$

Using the same integration by parts tricks, we can also show

$$\lambda \int_{\xi}^{\infty} \frac{\omega(h+\xi) - \omega(h) - \omega(\xi)}{h^2} dh \leq -\frac{3}{4} \lambda \frac{\omega(\xi)}{\xi}. \tag{2.91}$$

for γ sufficiently small.

So combining these together, we get that

$$A\omega(\xi) \int_{M\xi}^{\infty} \frac{\omega(h)}{h^2} dh + 2(\Lambda - \lambda) \int_{2\xi}^{M\xi} \frac{\omega(h-\xi) - \omega(\xi)}{h^2} dh + \lambda \int_{\xi}^{\infty} \frac{\omega(h+\xi) - \omega(h) - \omega(\xi)}{h^2} dh \leq \frac{-\lambda}{4} \frac{\omega(\xi)}{\xi}. \tag{2.92}$$

Since $\omega'(\xi)\omega(\xi) \leq \dfrac{\gamma \omega(\xi)}{\xi}$, we finally get that

$$(A \ln(M+1) + 1)\omega'(\xi)\omega(\xi) - \frac{\lambda}{4} \frac{\omega(\xi)}{\xi} \leq \frac{\omega(\xi)}{\xi} \left((A \ln(M+1) + 1)\gamma - \lambda/4 \right) < 0, \tag{2.93}$$

if γ is taken sufficiently small. □

2.8 Our choice for the modulus ρ

We've now shown that for the modulus defined in (2.79) that if the assumptions (2.33) hold that

$$\frac{d}{dt}\left(f_x(t,\xi/2)-f_x(t,-\xi/2)\right)\Big|_{t=T} < -\omega'(\xi)\omega(\xi). \tag{2.94}$$

We claim that in fact (2.94) will hold for any rescaling $\omega_r(h)=\omega(rh)$ as well. To see this, fix some $r>0$, and suppose that $f(t,x)$ satisfies the conditions of Lemma 2.4.1 for ω_r at time T and distance ξ. Take $\tilde{f}(t,x)=rf(t/r,x/r)$, which is also a solution of (2.2). Then $\tilde{f}_x$ is a solution of (2.26) with $\beta(\tilde{f}_0')=\beta(f_0')$, $||\tilde{f}_0'||_{L^\infty}=||f_0'||_{L^\infty}$, and satisfying the conditions of Lemma 2.4.1 for ω at time rT and distance $r\xi$. Hence by Lemma 2.7.1

$$\begin{aligned}\frac{d}{dt}\left(f_x(t,\xi/2)-f_x(t,-\xi/2)\right)\Big|_{t=T} &= r\frac{d}{dt}\left(\tilde{f}_x(t,r\xi/2)-\tilde{f}_x(t,-r\xi/2)\right)\Big|_{t=rT} \\ &< -r\omega'(r\xi)\omega(r\xi)=-\omega_r'(\xi)\omega_r(\xi).\end{aligned} \tag{2.95}$$

So, (2.94) will hold for any rescaling ω_r. Also note that for $f_x(T,\xi/2)-f_x(T,-\xi/2)=\omega(\xi)$ to hold, we must necessarily have $\omega(\xi)\le 2||f_x(T,\cdot)||_{L^\infty}<2||f_0'||_{L^\infty}$. Thus taking

$$C=\sup_{0<h<\omega^{-1}(2||f_0'||_{L^\infty})}\frac{h}{\omega(h)}=\frac{\omega^{-1}(2||f_0'||_{L^\infty})}{2||f_0'||_{L^\infty}}, \tag{2.96}$$

we see that

$$\omega(h)\ge\frac{h}{C}. \tag{2.97}$$

for all relevant h. Define

$$\rho(h):=\omega(Ch), \tag{2.98}$$

so that

$$\rho(h)\ge h, \tag{2.99}$$

for all $h\in[0,\rho^{-1}(2||f_0'||_{L^\infty})]$.

Now, suppose that at time T, f satisfies the assumptions (2.33) for $\rho(\cdot/T)$. Then since $\rho(\cdot/T)$ is a rescaling of ω, we have that

$$\begin{aligned}\frac{d}{dt}\left(f_x(T,\xi/2)-f_x(T,-\xi/2)\right) &< -\frac{d}{dh}\rho(h/T)\Big|_{h=\xi}\rho(\xi/T)=\frac{-1}{T}\rho'(\xi/T)\rho(\xi/T)\\ &\le \frac{-\xi}{T^2}\rho'(\xi/T)=\frac{d}{dt}\rho(\xi/t)\Big|_{t=T}.\end{aligned} \tag{2.100}$$

Thus we've constructed a modulus ρ which satisfies (2.21), completing the proof of the generation of a Lipschitz modulus of continuity (2.8) in our main theorem.

2.9 Regularity in Time

With the construction of the modulus ρ, we get universal Lipschitz bounds in space for $f_x(t,\cdot)$. By the structure of (2.2), we also get regularity in space for f_t.

Proposition 2.9.1. *Let $f:(0,T)\times\mathbb{R}\to\mathbb{R}$ be a classical solution to* (2.2) *with $||f(t,\cdot)||_{W^{1,\infty}}$ bounded and $||f_{xx}(t,\cdot)||_{L^\infty}\lesssim 1/t.$. Then $f_t(t,\cdot)$ is Log-Lipschitz in space with*

$$|f_t(t,\cdot)|\lesssim \max\{-\log(t),1\},\qquad |f_t(t,x)-f_t(t,y)|\lesssim -\log(|x-y|)|x-y|\left(1+\frac{1}{t}\right)$$
$$0<|x-y|<1/2. \tag{2.101}$$

Proof. For $t<1$, we have that

$$\begin{aligned}|f_t(t,x)| &= \left|\int_{\mathbb{R}}\frac{\delta_h f(t,x)-hf_x(t,x)}{\delta_h f(t,x)^2+h^2}dh\right| \le \left|\int_0^\infty \frac{\delta_h f(t,x)+\delta_{-h}f(t,x)}{\delta_{-h}f(t,x)^2+h^2}dh\right|\\ &\quad+\left|\int_0^\infty \frac{(\delta_h f(t,x)-hf_x(t,x))(\delta_h f(t,x)^2-\delta_{-h}f(t,x)^2)}{(\delta_h f(t,x)^2+h^2)(\delta_{-h}f(t,x)^2+h^2)}dh\right|\\ &\lesssim \int_0^t \frac{1}{t}dh+\int_t^1\frac{1}{h}dh+\int_1^\infty \frac{1}{h^2}+\frac{1}{h^3}dh\lesssim -\log(t)+1.\end{aligned} \tag{2.102}$$

For $t > 1$, you can similarly show $|f_t(t,x)| \lesssim 1$, proving the first bound.

For regularity in space, we see that

$$\begin{aligned} f_t(t,x) - f_t(t,y) &= \int_{\mathbb{R}} \frac{\delta_h f(t,x) - h f_x(t,x)}{\delta_h f(t,x)^2 + h^2} - \frac{\delta_h f(t,y) - h f_x(t,y)}{\delta_h f(t,y)^2 + h^2} dh \\ &= \int_{\mathbb{R}} \frac{\delta_h f(t,x) - h f_x(t,x) - (\delta_h f(t,y) - h f_x(t,y))}{\delta_h f(t,y)^2 + h^2} dh \\ &\quad + \int_{\mathbb{R}} \frac{(\delta_h f(t,x) - h f_x(t,x))(\delta_h f(t,x)^2 - \delta_h f(t,y)^2)}{(\delta_h f(t,x)^2 + h^2)(\delta_h f(t,y)^2 + h^2)} dh \\ &\leq \Bigg| \int_{|h|<|x-y|} \Bigg| + \Bigg| \int_{|x-y|<|h|<1} \Bigg| + \Bigg| \int_{|h|>1} \Bigg| \end{aligned} \tag{2.103}$$

For $|h| < |x-y|$, we can bound similarly to before to get that

$$\Bigg| \int_{|h|<|x-y|} \Bigg| \lesssim \int_0^{|x-y|} \frac{1}{t} dh = \frac{|x-y|}{t}. \tag{2.104}$$

For midsize $|x-y| < |h| < 1$, we have that

$$\begin{aligned} \Big| \delta_h f(t,x) - h f_x(t,x) - (\delta_h f(t,y) - h f_x(t,y)) \Big| &= \Big| \int_0^h \delta_s f_x(t,x) - \delta_s f_x(t,y) ds \Big| \lesssim \frac{|x-y|h}{t}, \\ \Big| \delta_h f(t,x) - \delta_h f(t,y) \Big| &= \Big| \int_0^h f_x(t,x+s) - f_x(t,y+s) ds \Big| \lesssim \frac{|x-y|h}{t}. \end{aligned} \tag{2.105}$$

Thus

$$\Bigg| \int_{|x-y|<|h|<1} \Bigg| \lesssim \frac{|x-y|}{t} \int_{|x-y|}^{1} \frac{1}{h} dh = \frac{-\ln(|x-y|)|x-y|}{t}. \tag{2.106}$$

Finally, we use L^∞ bounds on f to get that

$$\left|\int\limits_{|h|>1}\right| \leq \left|\int\limits_{|h|>1} \frac{\delta_h f(t,x)-\delta_h f(t,y)}{\delta_h f(t,y)^2+h^2} + \frac{(\delta_h f(t,x)-hf_x(t,x))(\delta_h f(t,x)^2-\delta_h f(t,y)^2)}{(\delta_h f(t,x)^2+h^2)(\delta_h f(t,y)^2+h^2)}dh\right|$$
$$+ |f_x(t,x)-f_x(t,y)|\left|\int\limits_{|h|>1} \frac{-h}{\delta_h f(t,y)^2+h^2}dh\right|$$
$$\lesssim |x-y|\int\limits_1^\infty \frac{1}{h^2}+\frac{1}{h^3}dh + \frac{|x-y|}{t}\int\limits_1^\infty \frac{1}{h^3}dh \lesssim \left(1+\frac{1}{t}\right)|x-y|. \tag{2.107}$$

Putting this all together, we thus have that

$$|f_t(t,x)-f_t(t,y)| \lesssim -\ln(|x-y|)|x-y|\left(1+\frac{1}{t}\right). \tag{2.108}$$

□

Recall that in section 2, we assumed that our initial data $f_0 \in C_c^\infty(\mathbb{R})$ so that by the local existence results of [CG09], there is a unique solution $f \in C^1((0,T_+);H^k)$ for k arbitrarily large and some $T_+ > 0$. We were then able to prove the existence of the modulus ρ as in Theorem 2.1.1 depending only on $\beta(f_0'), ||f_0'||_{L^\infty}$, and hence with the solution f existing for all time by the main theorem of [CGSV17]. For an arbitrary $f_0 \in W^{1,\infty}(\mathbb{R})$ with $\beta(f_0') < 1$, the same result holds true by compactness. Let $\eta \in C_c^\infty(\mathbb{R})$ be a smooth mollifier, and $\phi \in C_c^\infty(\mathbb{R})$ be a smooth cutoff function. For $f_0 \in W^{1,\infty}(\mathbb{R})$ with $\beta(f_0') < 1$, take $f_0^{(\epsilon)}(x) := (f_0 * \eta_\epsilon)(x)\phi(\epsilon x)$. Then $f_0^{(\epsilon)} \to f_0$ in $W^{1,\infty}_{loc}$, with $\beta(f_0^{(\epsilon)\prime}), ||f_0^{(\epsilon)}||_{W^{1,\infty}(\mathbb{R})} \to \beta(f_0'), ||f_0||_{W^{1,\infty}(\mathbb{R})}$ respectively as $\epsilon \to 0$. Thus for ϵ sufficiently small, $\beta(f_0^{(\epsilon)\prime}) < 1$ and the results of the previous section hold for the solution to the mollified problem $f^{(\epsilon)}$. The L^∞ bound on $f_t^{(\epsilon)}$ proven above along with the maximum principle for $f_x^{(\epsilon)}$ is enough to ensure that there a subsequence $f^{(\epsilon_k)}$ converging in $C_{loc}([0,\infty)\times\mathbb{R})$ to a Lipschitz (weak) solution f to the original problem. In order to get a classical C^1 solution, we need regularity estimates for $f_x^{(\epsilon)}, f_t^{(\epsilon)}$ in both time and space. The modulus ρ and Proposition 2.9.1 give the regularity in space that we need for f_x, f_t. All that leaves is to prove regularity in time.

Proposition 2.9.2. *Let f be a sufficiently smooth solution to* (2.2) *with* $\beta(f_0') < 1$. *Then* $f_x, f_t \in C^\alpha_{loc}((0,\infty)\times\mathbb{R})$ *with*

$$||f_x||_{C^\alpha(Q_{t/4}(t,x))}, ||f_t||_{C^\alpha(Q_{t/4}(t,x))} \leq C(\beta(f_0'), ||f||_{L_t^\infty((t/2,3t/2);W_x^{2,\infty}(\mathbb{R}))}) \max\{t^{-\alpha}, 1\}, \tag{2.109}$$

where $Q_r(s,y) = (s-r, s] \times B_r(y)$, *and* $\alpha > 0$ *depends only on* $\beta(f_0'), ||f_0'||_{L^\infty}$.

Proof. We have that f_x solves

$$(f_x)_t(t,x) = f_{xx}(t,x)\int_{\mathbb{R}} \frac{-h}{\delta_h f(t,x)^2 + h^2} dh + \int_{\mathbb{R}} \delta_h f_x(t,x) K(t,x,h) dh, \tag{2.110}$$

where $\dfrac{\lambda}{h^2} \leq K(t,x,h) \leq \dfrac{\Lambda}{h^2}$ is uniformly elliptic with ellipticity constants λ, Λ depending on $\beta(f_0'), ||f_0'||_{L^\infty}$. Rewriting this, we have that f_x satisfies

$$\begin{aligned}(f_x)_t - \int_{\mathbb{R}} \delta_h f_x(t,x)\left(\frac{K(t,x,h)+K(t,x,-h)}{2}\right) dh &= f_{xx}(t,x)\int_{\mathbb{R}} \frac{-h}{\delta_h f(t,x)^2 + h^2} dh \\ &+ \int_{\mathbb{R}} \delta_h f_x(t,x)\left(\frac{K(t,x,h)-K(t,x,-h)}{2}\right) dh.\end{aligned} \tag{2.111}$$

Let $F(t,x)$ denote the righthand side of (2.111). Then $F(t,x)$ is locally bounded with $|F(t,x)|$ controlled by $||f(t,\cdot)||_{W^{2,\infty}}$. Then since $(K(t,x,h)+K(t,x,-h))/2$ is a symmetric uniformly elliptic kernel, it follows that we have local C^α bounds for $\alpha \leq \alpha_0$ for some α_0 depending on ellipticity constants (see [Sil11]).

So, all we have to do is give bounds on $F(t,x)$ depending only on $||f(t,\cdot)||_{W^{2,\infty}}$. Similar to proof of Lemma 2.5.1,

$$\int_{\mathbb{R}} \frac{-h}{\delta_h f(t,x)^2 + h^2} dh = \int_0^\infty h \frac{\delta_h f(t,x)^2 - \delta_{-h} f(t,x)^2}{(\delta_h f(t,x)^2 + h^2)(\delta_{-h} f(t,x)^2 + h^2)} dh \lesssim \int_0^1 1 dh + \int_1^\infty \frac{1}{h^3} dh \lesssim 1. \tag{2.112}$$

Also similar to the proof of Lemma 2.5.2 (specifically (2.49)), we have that

$$|K(t,x,h) - K(t,x,-h)| \lesssim \min\{\frac{1}{h}, \frac{1}{h^3}\}, \tag{2.113}$$

so

$$\left| \int_{\mathbb{R}} \delta_h f_x(t,x) \left(\frac{K(t,x,h) - K(t,x,-h)}{2} \right) dh \right| \lesssim \int_0^1 1 dh + \int_1^\infty \frac{1}{h^3} dh \lesssim 1. \tag{2.114}$$

Thus since we've bounded the right hand side of (2.111) depending only on $||f(t,\cdot)||_{W^{2,\infty}}$, we have our local C^α bounds for f_x for all α sufficiently small. A C^α bound that is uniform in x for f_x then gives a log C^α estimate for f_t, similar to the proof for regularity in space in Proposition 2.9.1. Thus we have C^α estimates for both f_x, f_t.

□

CHAPTER 3

FRACTIONAL MEAN CURVATURE FLOW

3.1 Introduction

For sufficiently regular set $E \subseteq \mathbb{R}^d$ and $s \in (0,1)$, we define the s-fractional perimeter of E

$$\begin{aligned} P_s(E) := s(1-s)[\mathbb{1}_E]_{\dot{W}^{s,1}} &= s(1-s) \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{|\mathbb{1}_E(X) - \mathbb{1}_E(Y)|}{|X-Y|^{d+s}} dX dY \\ &= 2s(1-s) \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{\mathbb{1}_E(X)\mathbb{1}_{\mathcal{C}E}(Y)}{|X-Y|^{d+s}} dX dY, \end{aligned} \tag{3.1}$$

where $\mathbb{1}_E, \mathbb{1}_{\mathcal{C}E}$ are the characteristic functions of E and its compliment $\mathcal{C}E$. The s-fractional perimeter interpolates between our usual notions of perimeter and Lebesgue measure, with

$$\lim_{s\to 1} P_s(E) = C_d P(E), \qquad \lim_{s\to 0} P_s(E) = C_d' \mathcal{L}^d(E), \tag{3.2}$$

for bounded regular sets (see [CV11], [SDV13]). The s-fractional perimeter was first introduced in [CRS10] where the authors studied the regularity of minimizers, known as nonlocal minimal surfaces. Minimizers satisfy the Euler-Lagrange equation

$$H_s(X,E) := -s(1-s) P.V. \int_{\mathbb{R}^d} \frac{\mathbb{1}_E^{\pm}(X+Z)}{|Z|^{d+s}} dZ = 0, \tag{3.3}$$

for all points $X \in \partial E$, where $\mathbb{1}_E^{\pm} = \mathbb{1}_E - \mathbb{1}_{\mathcal{C}E}$ is the signed characteristic function. The quantity H_s is called the s-fractional mean curvature, and it converges to the classical mean curvature as $s \to 1$ [CV11], so fractional mean curvature can be thought of as a nonlocal, fractional order analogue of local mean curvature.

We are interested in studying the regularizing effects of the flow

$$\partial_t X(t) = -H_s(X, E_t)\nu(X), \quad X(t) \in \partial E_t. \tag{3.4}$$

Solutions to the flow (3.4) are translation invariant, satisfy the comparison principle, and have natural rescaling $t \to \frac{1}{R} E_{R^{1+s}t}$ for any $R > 0$.

In the case that E_0 is the subgraph of a smooth Lipschitz function $u_0 : \mathbb{R}^{d-1} \to \mathbb{R}$), the flow exists for all time [SV15] and can be equivalently described by

$$\partial_t u(t,x) = s(1-s)\sqrt{1+|\nabla u(t,x)|^2} \int\limits_{\mathbb{R}^{d-1}} \frac{u(t,x+z)-u(t,x)}{|z|^{d+s}} \Lambda\left(\frac{u(t,x+z)-u(t,x)}{|z|}\right) dz, \tag{3.5}$$

where the nonlinearity $\Lambda(L) = \frac{1}{L}\int\limits_{-L}^{L} \frac{1}{(1+z_d^2)^{(d+s)/2}} dz_d$. Thus

$$\Lambda(||\nabla_x u||_{L^\infty}) \leq \Lambda\left(\frac{u(t,x+z)-u(t,x)}{|z|}\right) \leq 2, \tag{3.6}$$

so this is a nonlinear parabolic equation of order $1+s$, with the ellipticity constant depending on the Lipschitz constant of u. From this parabolicity, we see that fractional mean curvature flow is regularizing on Lipschitz subgraphs.

While in general smooth solutions $t \to E_t$ of (3.4) do not exist for all times t for non-graphical initial data, it's possible to define weak viscosity solutions via the level set method which will exist for all time. See [Imb09, CS10, CMP15, ACP19] or the appendix for details. In this article, we shall show that for any initial data E_0 that is bounded between two Lipschitz subgraphs, the minimal viscosity supersolution will itself become a Lipschitz subgraph in finite time.

Theorem 3.1.1. *Let $E_0 \subseteq \mathbb{R}^d$ be an open set and $u_0 \in \dot{W}^{1,\infty}(\mathbb{R}^{d-1})$ be a Lipschtiz function with $||\nabla_x u_0||_{L^\infty} = L$ and*

$$\left\{(x,x_d)\middle| x \in \mathbb{R}^{d-1}, x_d < u_0(x) - \frac{R}{2}\right\} \subseteq E_0 \subseteq \left\{(x,x_d)\middle| x \in \mathbb{R}^{d-1}, x_d < u_0(x) + \frac{R}{2}\right\}, \tag{3.7}$$

for some $R \geq 0$. Let E_t be the minimal viscosity supersolution of the flow (3.4)*, in the sense* *of Definition B.0.2. Then for all* $t \geq R^{1+s}T(d,s,L)$*,* E_t *will be a* $(1+L)$*-Lipschitz subgraph. The time* T *can be calculated explicitly, with*

$$T(d,s,L) \leq \frac{C(d)(1+L)^{d+s}}{s^2(1-s)}, \tag{3.8}$$

for some dimensional constant $C(d)$*.*

We stress that there is no initial regularity assumption on E_0 in Theorem 3.1.1. The boundary ∂E_0 can have positive measure, and the set E_0 does not even need to be connected. Our only assumption is that E_0 is a finite distance in the Hausdorff metric from a Lipschitz subgraph. This finite distance allows us to recover the regularizing effects of fractional mean curvature flow on Lipschitz functions (3.5).

Taking $u_0 \equiv 0$ and $R = 2$, we get the immediate corollary

Corollary 3.1.1. *Let* $E_0 \subseteq \mathbb{R}^d$ *be an open set with*

$$\left\{(x,x_d)\middle| x_d < -1\right\} \subseteq E_0 \subseteq \left\{(x,x_d)\middle| x_d < 1\right\}. \tag{3.9}$$

Let E_t *be the minimal viscosity supersolution of the flow* (3.4)*, in the sense of Definition B.0.2. Then for all* $t \geq T(d,s)$*,* E_t *will be a* 1*-Lipschitz subgraph.*

From Corollary 3.1.1 it's clear that Theorem 3.1.1 can be viewed as a parabolic version of the "flat implies smooth" result of [CRS10] for nonlocal minimal surfaces. One key difference between these results is that the proof of flat implies smooth in [CRS10] is by compactness, with non explicit constants. Conversely, our proof is constructive, giving a explicit modulus of continuity for the set E_t. See Subsection 3.1.2 for a more detailed discussion of our approach.

Lastly, we wish to stress that the result Theorem 3.1.1 is inherently nonlocal in nature, with the time $T(d,s,L) \to \infty$ as $s \to 1$. This is a necessity, as the theorem is false for classical mean curvature flow. The set

$$E_0 = \{(x, x_d) | x_d < -1 \text{ or } 0 < x_d < 1\}, \tag{3.10}$$

is fixed by local mean curvature flow and hence never becomes graphical. In this case, it's clear that the barrier to regularity is multiplicity, as the problem is that ∂E_0 is a disjoint union of hyperplanes. But because of the nonlocal nature of fractional mean curvature, the points on the disjoint hyperplaces $\{x_d = \pm 1\}$ can still sense each other, and are no longer fixed. Direct calculation shows that flowing under fractional mean curvature flow, $E_t \to \{(x, x_d) | x_d < 0\}$ in finite time $T \sim \dfrac{1}{s(1-s)}$ for any $0 < s < 1$.

3.1.1 Background

Nonlocal perimeters arises naturally in the context of phase transition problems with very long range interactions. In [SV12], the authors consider the energies

$$\mathcal{E}_s(U) := \epsilon^s ||U||_{H^{s/2}(\Omega)} + \int_\Omega W(U(X))dX, \tag{3.11}$$

where $s \in (0, 2)$ and W is a standard double well potential. They show that after appropriate rescaling, the functionals Γ-converge as $\epsilon \to 0$ to classical perimeter functional for $s \in [1, 2)$, but converge to the s-fractional perimeter in the more nonlocal case $s \in (0, 1)$.

The s-fractional mean curvature flow was first defined in [CS10], where the authors were investigating the convergence of the threshold dynamics for the fractional heat equation $U_t + (-\Delta_X)^{s/2}U = 0$. In the case that $s \in [1, 2)$, the evolution of the interface $\{U(t, \cdot) = 0\}$ converged to classical mean curvature flow, but for $s \in (0, 1)$ it instead converges to s-fractional mean curvature flow.

Motivated by these applications and the parallels to classical minimal surfaces, there has been a sustained effort over the past 10 years to study the regularity of local minimizers of fractional perimeter, s-minimal surfaces. [CRS10] began the study, recovering a number of the tools from classical minimal surfaces such as density estimates, monotonicity formula, and the improvement of flatness argument. Nonlocal minimal surfaces are known to be smooth whenever they are Lipschitz [FV17], smooth outside of a set of codimension 2 for any $s \in (0,1)$ [CRS10], and for s sufficiently close to 1 the singularity set in fact has codimension at least 8 [CV13], matching the regularity theory for the local case.

There are however key differences between the regularity theory for the nonlocal and the local cases. Stable nonlocal minimal surfaces satisfy a universal BV estimate [ECV19], which is false without additional assumptions for classical minimal surfaces, and an important unsolved problem with the appropriate additional assumptions when $d > 3$. There is also an example of a nontrivial stable s-minimal cone in $\mathbb{R}^7$ for small s [DdPW18], showing the regularity of nonlocal minimal surfaces is different than the classical case for s bounded away from 1. It is still an open problem though if this is the case for minimizing nonlocal minimal surfaces.

Since nonlocal mean curvature flow's introduction in [CS10], properties of smooth solutions have been studied in [SV15] and radial self-shrinkers in [CN18]. Most work however on fractional mean curvature flow has focused on the study of weak solutions via the level set method and the singularities they develop.

The level set method was popularized in [OS88], where it was used as a numerical tool to study the evolution of classical mean curvature flow past the point of singularities. This was made analytically rigorous by [ES91], and is an invaluable tool in the study of mean curvature flow. The key insight of the level set method is to replace the evolution of the boundary $t \to \partial E_t$ with the evolutions of the zero level set of a function $t \to \{U(t,\cdot) = 0\}$, where U now solves a degenerate parabolic equation based on the original flow.

The existence, uniqueness, and comparison principal for global viscosity solutions defined via the level set method for fractional mean curvature flow was first shown by [Imb09], and then later expanded to more general nonlocal and even crystalline flows in [CMP15, ACP19].

We review the definitions and essential results in the appendix.

One type of singularity particular to level set flows is "fattening." It refers to when the level set $\{U(t,\cdot)=0\}$ which represent our "boundary" develops a nonempty interior which corresponds to a lack of uniqueness in the geometric flow. As at most countably many level sets $\{U(t,\cdot)=\gamma\}$ can fatten, it is in some sense a rare phenomena. However there has a been an intense study to see what kind of properties of the initial set E_0 rule out the possibility. For fractional mean curvature flow, it was shown in [CSV18] that a smooth simple closed curve can fatten, in contrast to Grayson's theorem for classical mean curvature flow. The preprint [CDNV18] goes through a number of illustrative examples of when fattening does or does not occur for nonlocal flows, proving smooth strictly star convex sets don't fatten. The "strictness" on the strictly star convex assumption is necessary though, as [CDNV18] also gives an example of a star convex set which does fatten.

In this paper, we circumvent the issue of fattening by instead showing that the two sets $\partial\{U(t,\cdot)<0\}$ and $\partial\{U(t,\cdot)>0\}$ each independently regularize.

3.1.2 Argument Outline

Our proof of Theorem 3.1.1 is inspired by Kiselev's proof of eventual Hölder regularization for solutions to the supercritical Burger's equation in [Kis11]. There, Kiselev shows that solutions to

$$\begin{aligned}\partial_t u(t,x)+u(t,x)\partial_x u(t,x)+(-\Delta)^{\alpha}u(t,x)=0,\\ \alpha\in(0,1/2).\end{aligned}\tag{3.12}$$

becomes Hölder continuous in finite time. A priori, this is surprising as solutions to the equation are known to develop shocks [AKS08]. For the proof, he showed that the equation propagated a family of moduli of continuity

$$\omega(t,r)\approx\delta(t)+Cr^{\beta},$$

where $\delta(0) > 2||u_0||_{L^\infty}$ and $\delta(T_{\alpha,\beta}) = 0$. Thus the moduli of continuity gives no new information at time $t = 0$, controls the size of shocks for $0 < t < T_{\alpha,\beta}$, and then forces the solution to become β-Holder continuous at time $t = T_{\alpha,\beta}$.

Allowing $\omega(t, 0) > 0$ lets us apply the concept of a modulus of continuity to a discontinuous function. But it makes just as much sense to apply it in this case to a multivalued function like the boundary of a set which can fold over itself. Then at time $t = T$, satisfying the modulus forces the boundary to be graphical. Our goal is to construct an explicit time dependent family of moduli of continuity and show that exactly this occurs for fractional mean curvature flow.

In order to make our proof the most clear and understandable, we first prove Theorem 3.1.1 in the case that the flow $t \to E_t$ is smooth. We begin by defining what it means for a set to have a modulus of continuity in Section 2, and showing that our assumption (3.9) is equivalent to assuming our initial set has a Lipschitz modulus of continuity. In Section 3, we repeat the breakthrough argument of [Kis11] to set up an eventual proof by contradiction. In section 4, we make a number of curvature estimates and reduce the proof by contradiction to the construction of a modulus of continuity satisfying an integral inequality. In section 5, we construct that modulus of continuity and finish the proof by contradiction, completing the proof in the smooth case.

In sections 6 and 7, we extend the smooth proof to work in the viscosity solution framework. In section 6, we prove a number of technical lemmas in order to formally justify the break through argument of section 3 and estimates of section 4 for almost every level set $\{U(t, \cdot) = \gamma\}$ under the additional assumption that our initial set E_0 is asymptotically flat. In section 7, we then apply limiting arguments to apply the result to the boundary of every level set without the flatness assumption, completing the general proof.

3.2 Moduli of Continuity for Sets

Our first step is to extend the idea of a modulus of continuity to a nongraphical set E. With that in mind, we define what we call the upper and lower boundaries of a set by

Definition 3.2.1. *(Upper and lower boundaries) Let $E \subseteq \mathbb{R}^d$. Assume that for any $x \in \mathbb{R}^{d-1}$, the sets*

$$\{x_d | (x, x_d) \in \overline{E}\}, \qquad \{x_d | (x, x_d) \in \overline{\mathcal{C}E}\}, \tag{3.13}$$

are bounded and nonempty. Then we define the upper and lower boundaries of E in the x_d-direction $\overline{u}, \underline{u} : \mathbb{R}^{d-1} \to \mathbb{R}$ to be

$$\overline{u}(x) := \sup\{x_d | (x, x_d) \in \overline{E}\}, \qquad \underline{u}(x) := \inf\{x_d | (x, x_d) \in \overline{\mathcal{C}E}\}. \tag{3.14}$$

Upper and lower boundaries could analogously be defined for any direction $e \in S^{d-1}$. Without loss of generality, we restrict ourselves to the positive x_d-direction, which corresponds to thinking of our set E as close to a subgraph.

Note that equivalently

$$\overline{u}(x) = \max\{x_d | (x, x_d) \in \partial E\}, \qquad \underline{u}(x) = \min\{x_d | (x, x_d) \in \partial E\}, \tag{3.15}$$

once we know that our set E both contains and is contained by a subgraph.

Definition 3.2.2. *Let $E \subseteq \mathbb{R}^d$ be a set with upper and lower boundaries in the x_d-direction, and $\omega : [0, \infty) \to [0, \infty)$ be a continuous function. Then we say that E has modulus ω in the x_d-direction if for all $x, y \in \mathbb{R}^{d-1}$,*

$$\overline{u}(x) - \underline{u}(y) \leq \omega(|x - y|). \tag{3.16}$$

Note that we don't force $\omega(0) = 0$ in our definition of a modulus of continuity, which allows for this definition to make sense when ∂E is not graphical. Indeed, if E has modulus

ω with $\omega(0) = 0$, then necessarily we have that

$$\overline{u}(x) = \underline{u}(x)\ \forall x \in \mathbb{R}^{d-1} \quad \Rightarrow \quad \partial E = \text{graph}(u : \mathbb{R}^{d-1} \to \mathbb{R}). \tag{3.17}$$

To begin, we first note that $\overline{u}, \underline{u}$ always have some underlying continuity:

Proposition 3.2.1. *Let $E \subseteq \mathbb{R}^d$ be a set with upper and lower boundaries $\overline{u}, \underline{u}$. Then $\overline{u}, \underline{u}$ are upper/lower semicontinuous respectively.*

Proof. We show that $\overline{u}$ is upper semicontinuous. Fix $x_0 \in \mathbb{R}^{d-1}$, and let $x_n \to x_0$. Without loss of generality, by passing to a subsequence we may suppose that $\lim\limits_{n\to\infty} \overline{u}(x_n) = L$. Thus as $(x_n, \overline{u}(x_n)) \in \overline{E}$ for all n, we have that $(x_0, L) \in \overline{E}$ as well. But then by the definition of $\overline{u}$, $\overline{u}(x_0) \geq L$. Thus $\overline{u}(x_0) \geq \limsup\limits_{x \to x_0} \overline{u}(x)$, so it is upper semicontinuous. □

A modulus of continuity can also be equivalently described on the level of sets as

Definition 3.2.3. *Let $\omega : [0, \infty) \to [0, \infty)$ be a continuous function. Then we say that an open (or closed) set $E \subseteq \mathbb{R}^d$ has modulus of continuity ω in the x_d-direction if for all $(z, z_d) \in \mathbb{R}^d$ with $z_d \geq \omega(|z|)$,*

$$E - (z, z_d) \subseteq E. \tag{3.18}$$

Proposition 3.2.2. *Let $E \subseteq \mathbb{R}^d$ be an open set with upper and lower boundaries in the x_d-direction. Then definitions 3.2.2 and 3.2.3 are equivalent.*

Proof. Assume $\overline{u}(x) - \underline{u}(y) \leq \omega(|x - y|)$ for all x, y, and fix some $Z \in \mathbb{R}^d$ with $z_d \geq \omega(|z|)$ and point $(x, x_d) \in E$. Then by our assumption and the definition of $\overline{u}$ we have that

$$x_d < \overline{u}(x) \leq \underline{u}(x - z) + \omega(|z|) \leq \underline{u}(x - z) + z_d. \tag{3.19}$$

As $x_d - z_d < \underline{u}(x - z)$, we thus have by the definition of $\underline{u}$ that $(x - z, x_d - z_d) \in E$. As $(x, x_d) \in E$ was arbitrary, we have that $E - (z, z_d) \subseteq E$. Thus E has modulus ω in the sense of Definition 3.2.3.

Conversely, suppose that $\overline{u}(x_0) - \underline{u}(y_0) = \omega(|x_0 - y_0|) + \epsilon$ for some $x_0, y_0 \in \mathbb{R}^{d-1}$ and $\epsilon > 0$. As $(x_0, \overline{u}(x_0)) \in \partial E$ and ω is continuous, we can find a point $(x, x_d) \in E$ such that $|\overline{u}(x_0) - x_d| < \epsilon/2$ and $|\omega(|x - y_0|) - \omega(|x_0 - y_0|)| < \epsilon/2$. Taking $Z = (x - y_0, x_d - \underline{u}(y_0))$, we have that

$$x_d - \underline{u}(y_0) > \overline{u}(x_0) - \underline{u}(y_0) - \epsilon/2 = \omega(|x_0 - y_0|) + \epsilon/2 > \omega(|x - y_0|), \tag{3.20}$$

but that $(x, x_d) - (x - y_0, x_d - \underline{u}(y_0)) = (y_0, \underline{u}(y_0)) \notin E$. Thus E does not have modulus ω in the sense of Definition 3.2.3.

□

For most of our purposes, we'll be thinking about moduli of continuity in terms of the upper and lower boundaries, but the set definition works particularly will with the comparison principle and provides the easiest way to rigorously prove propagation of moduli for viscosity solutions.

Proposition 3.2.3. *Let $t \to E_t$ be the minimal viscosity supersolution . Then E_t has modulus ω for all time t.*

Proof. The proof follows by translation invariance and the comparison principle for the minimal viscosity supersolution, which we prove in Proposition B.0.1 in the appendix.

Fix $Z = (z, z_d) \in \mathbb{R}^d$ with $z_d \geq \omega(|z|)$, and let $E_t(Z) = E_t + Z$. Then by translation invariance, $t \to E_t(Z)$ is the minimal viscosity supersolution of fractional mean curvature flow for the initial data $E_0(Z)$.

As $E_0 \subseteq E_0(Z)$ by Definition 3.2.3, it follows by the comparison principle that $E_t \subseteq E_t(Z)$ for all $t \geq 0$. Since $Z \in \mathbb{R}^d$ with $z_d \geq \omega(|z|)$ was arbitrary, we thus have that E_t has modulus ω for all times t. □

Thus as in the graphical case, comparison principle and translation invariance imply that any modulus of continuity is propagated. Rather than just propagation though, our goal is to show an *improvement* in our modulus of continuity.

We now turn our attention to Theorem 3.1.1. There we assume that our initial set E_0 is bounded between two Lipschitz subgraphs. Since our plan is to describe an improvement in the modulus of continuity of E_t, it will be more convenient for us to translate this assumption into one about the modulus of continuity of E_0 directly. Luckily, these are equivalent notions.

Proposition 3.2.4. *Let $E \subseteq \mathbb{R}^d$ be an open set. Then E has modulus of continuity $\omega(r) = R + Lr$ in the x_d-direction if and only if there exists a Lipschitz function $u \in \dot{W}^{1,\infty}(\mathbb{R}^{d-1})$ with $||\nabla u||_{L^\infty} \leq L$ and*

$$\left\{(x, x_d)\middle| x \in \mathbb{R}^{d-1}, x_d < u(x) - \frac{R}{2}\right\} \subseteq E \subseteq \left\{(x, x_d)\middle| x \in \mathbb{R}^{d-1}, x_d < u(x) + \frac{R}{2}\right\}, \quad (3.21)$$

Proof. To begin, suppose that E has modulus $\omega(r) = R + Lr$. Then define $u : \mathbb{R}^{d-1} \to \mathbb{R}$ by

$$u(x) = \frac{R}{2} + \inf_{y \in \mathbb{R}^{d-1}} \underline{u}(y) + L|x - y|. \quad (3.22)$$

Note that u is a well defined function, as for any $y \in \mathbb{R}^{d-1}$

$$\underline{u}(y) + L|x - y| \geq \overline{u}(x) - \omega(|x - y|) + L|x - y| \geq \overline{u}(x) - R. \quad (3.23)$$

Hence $u(x)$ exists with

$$u(x) + \frac{R}{2} \geq \overline{u}(x), \quad (3.24)$$

so

$$E \subseteq \{(x, x_d)|x_d < \overline{u}(x)\} \subseteq \left\{(x, x_d)\middle| x_d < u(x) + \frac{R}{2}\right\}. \quad (3.25)$$

As we also have by construction that $u(x) - \frac{R}{2} \leq \underline{u}(x)$, we also get the reverse inclusion

$$\left\{(x, x_d)\middle| x_d < u(x) - \frac{R}{2}\right\} \subseteq E. \quad (3.26)$$

Finally as u is the infimum of L-Lipschitz functions, it follows that u is L-Lipschitz itself.

Thus we've shown the first direction.

For the converse, suppose that $u \in \dot{W}^{1,\infty}(\mathbb{R}^{d-1})$ with $||\nabla u||_{L^\infty} \leq L$ satisfies (3.21). Then since u is continuous, it follows that

$$\overline{E} \subseteq \left\{(x, x_d)\middle| x_d \leq u(x) + \frac{R}{2}\right\}, \qquad \left\{(x, x_d)\middle| x_d \geq u(x) - \frac{R}{2}\right\} \subseteq \mathcal{C}E. \tag{3.27}$$

Hence, $\overline{u}, \underline{u}$ are well defined with $\overline{u} \leq u + \frac{R}{2}$ and $\underline{u} \geq u - \frac{R}{2}$. Thus for any $x, y \in \mathbb{R}^{d-1}$,

$$\overline{u}(x) - \underline{u}(y) \leq u(x) - u(y) + R \leq R + L|x - y|. \tag{3.28}$$

Thus E has modulus of continuity $\omega(r) = R + L|x - y|$. □

To make our strategy for the proof of Theorem 3.1.1 more clear and understandable, let's first consider the case that we have a smooth, open initial data E_0 which has modulus

$$\omega(r) = R + Lr. \tag{3.29}$$

and the flow $t \to E_t$ is smooth and exists for all time t. By rescaling the flow $t \to \frac{1}{R}E_{R^{1+s}t}$, we can assume without loss of generality that $R = 1$. Our goal then is to find a time dependent family of moduli of continuity $\omega : [0, \infty) \times [0, \infty) \to [0, \infty)$ satisfying

$$\begin{cases} 1). \quad \omega(0, r) > 1 + Lr \text{ for all } r \text{ and } \omega(t, r) > 1 + Lr \text{ for all } r > 2 \text{ and times } t \in [0, T) \\ 2). \quad \omega(t, \cdot) \text{ is } C^{1,1}, 0 \leq \partial_r \omega(t, \cdot) \leq 1 + L, \omega(t, 0) > 0, \text{ and } \partial_r \omega(t, 0) = 0 \text{ for all } t \in [0, T), \\ 3). \quad \omega(T, \cdot) \text{ is } (1 + L)\text{-Lipschitz with } \omega(T, 0) = 0, \end{cases} \tag{3.30}$$

for some time T. Then proving E_t has modulus $\omega(t, \cdot)$ for all $t \in [0, T]$ would prove the Theorem 3.1.1 for smooth flows.

3.3 Breakthrough Argument

Let $\omega(t,r)$ be a time dependent family of moduli of continuity satisfying out assumptions (3.30).

Let $E_0 \subseteq \mathbb{R}^d$ be a smooth open set satisfying the modulus $1+Lr$, and assume the flow $t \to E_t$ is smooth and exists for all time t. Assume additionally that E_0 is flat at infinity, so

$$\partial E_0 \setminus (B_M^{d-1} \times \mathbb{R}) = \{(x,0) : |x| \geq M\}, \tag{3.31}$$

for some $M > 0$. Letting $\overline{u}(t,\cdot), \underline{u}(t,\cdot)$ denote the upper and lower boundaries of E_t, it then follows by Proposition 3.6.2 that

$$\lim_{|x|\to\infty} \overline{u}(t,x) = \lim_{|x|\to\infty} \underline{u}(t,x) = 0, \qquad \text{uniformly for } t \in [0,T]. \tag{3.32}$$

Furthermore by the Proposition 3.2.3, E_t will have modulus $1+Lr$ for all times t.

By assumption 1). in (3.30), we know automatically that E_0 has modulus $\omega(0,\cdot)$. Since the flow is smooth, it follows by continuity that E_t will have modulus $\omega(t,\cdot)$ for sufficiently small times t.

Suppose that E_t loses the modulus $\omega(t,\cdot)$ before time T. Let

$$t_0 = \sup\{t \in [0,T] : E_t \text{ has modulus } \omega(t,\cdot)\}. \tag{3.33}$$

Then since have the modulus of continuity $\omega(t,\cdot)$ is a closed condition, we know that E_{t_0} has modulus $\omega(t_0,\cdot)$.

Suppose that E_{t_0} strictly had the modulus ω. That is, for any $x,y \in \mathbb{R}^{d-1}$

$$\overline{u}(t_0,x) - \underline{u}(t_0,y) < \omega(t_0,|x-y|). \tag{3.34}$$

We will show that in this case that $E_{t_0+\epsilon}$ has the modulus $\omega(t_0+\epsilon,\cdot)$ for ϵ sufficiently small, contradicting the definition of t_0.

Let $\delta = \min\{\omega(t,0) : 0 \le t \le \frac{t_0+T}{2}\}$. By (3.32) we have that there is some $R > 0$ such that for $|x|, |y| > R$,

$$\overline{u}(t_0+\epsilon, x) - \delta/3 < 0 < \underline{u}(t_0+\epsilon, y) + \delta/3, \tag{3.35}$$

for any $0 \le \epsilon \le \frac{T-t_0}{2}$.

Now suppose that $|x| > R+2$. Then for any $y \in \mathbb{R}^{d-1}$, we have that either $|y| > R$ or $|x-y| > 2$. If $|x-y| > 2$, then by assumption 1). of (3.30)

$$\overline{u}(t_0+\epsilon, x) - \underline{u}(t_0+\epsilon, y) \le 1 + L|x-y| < \omega(t_0+\epsilon, |x-y|), \tag{3.36}$$

for any ϵ. If $|y| > R$, then similarly we have for any $0 \le \epsilon \le \frac{T-t_0}{2}$

$$\overline{u}(t_0+\epsilon, x) - \underline{u}(t_0+\epsilon, y) < 2\delta/3 < \omega(t_0+\epsilon, 0) \le \omega(t_0+\epsilon, |x-y|). \tag{3.37}$$

A symmetric argument clearly works in the case that $|y| > R+2$. Thus the only thing that remains to show that $E_{t_0+\epsilon}$ has modulus $\omega(t_0+\epsilon, \cdot)$ is to consider the case when both $|x|, |y| \le R+2$.

We know from Proposition (3.2.1) that $\overline{u}(t,\cdot), \underline{u}(t,\cdot)$ are upper/lower semicontinuous in space respectively. Since by assumption the flow $t \to E_t$ is smooth, they will be semicontinuous in time as well. Thus by uniform semicontinuity,

$$\overline{u}(t_0+\epsilon, x) - \underline{u}(t_0+\epsilon, y) < \omega(t_0+\epsilon, |x-y|), \tag{3.38}$$

for ϵ sufficiently small when $|x|, |y| < R+2$. Hence, $E_{t_0+\epsilon}$ has modulus $\omega(t_0+\epsilon, \cdot)$ for ϵ sufficiently small, violating the definition of t_0 (3.33).

Thus if the set E_t was to lose the modulus $\omega(t,\cdot)$ before time T, then necessarily there must be two points $x, y \in \mathbb{R}^{d-1}$ such that

$$\overline{u}(t_0, x) - \underline{u}(t_0, y) = \omega(t_0, |x-y|). \tag{3.39}$$

We will show in sections 3 and 4 that in this case for the right choice of family $\omega(t,r)$,

$$\partial_t \overline{u}(t_0, x) - \partial_t \underline{u}(t_0, y) < \partial_t \omega(t_0, |x-y|), \tag{3.40}$$

contradicting the fact that E_t had the modulus before time t_0.

3.4 Curvature Estimates

Everything from now on will be at a fixed time $t_0 \in (0, T)$, so we will simply suppress the time variable. Our standing assumption is that the open set $E = E_{t_0}$ has some modulus $\omega(\cdot)$ satisfying

$$\begin{cases} 1). \ \ \omega(0) > 0 \text{ and } \omega(r) > 1 + Lr \text{ for } r > 2, \\ 2). \ \ \omega(\cdot) \text{ is } C^{1,1} \text{ with } 0 \leq \omega'(\cdot) \leq (1+L) \text{ and } \omega'(0) = 0, \\ 3). \ \ \overline{u}(x) - \underline{u}(y) \leq \omega(|x-y|), \quad \forall x, y \in \mathbb{R}^{d-1}, \\ 4). \ \ \overline{u}(\frac{\xi}{2}) - \underline{u}(\frac{-\xi}{2}) = \omega(|\xi|), \end{cases} \tag{3.41}$$

for some $\xi \in \mathbb{R}^{d-1}$ with $|\xi| \leq 2$.

Our goal is to use our assumptions (3.41) and the equation (3.4) to bound the difference between $\partial_t \overline{u}(\xi/2) - \partial_t \underline{u}(-\xi/2)$ from above in terms of ω.

To begin, we're first going to derive the proper equation for $\overline{u}, \underline{u}$. Note that they are respectively upper and lower semicontinuous, and locally smooth (since E is smooth) whenever the outward unit normal doesn't lie in the $\mathbb{R}^{d-1}$ plane. As $\overline{u}$ is touched from above by

ω and $\underline{u}$ is touched from below by $-\omega$, this is necessarily the case so we thus have that

$$\nabla \overline{u}\left(\frac{\xi}{2}\right) = \nabla \underline{u}\left(\frac{-\xi}{2}\right) = \omega'(|\xi|)\frac{\xi}{|\xi|}. \tag{3.42}$$

Note that as $\omega'(0) = 0$ by 2) in (3.41), this is still valid when $\xi = 0$.

The point $(\frac{\xi}{2}, \overline{u}(\frac{\xi}{2})) \in \partial E$ thus has outward unit normal $\dfrac{(-\omega'(|\xi|)\frac{\xi}{|\xi|}, 1)}{\sqrt{1+\omega'(|\xi|)^2}}$, so from the fractional mean curvature flow equation (3.4) we get that

$$\partial_t \overline{u}\left(\frac{\xi}{2}\right) = s(1-s)\sqrt{1+\omega'(|\xi|)^2}\,P.V. \int\limits_{\mathbb{R}^{d-1}} \int\limits_{\mathbb{R}} \frac{\mathbb{1}_E^{\pm}\left(\frac{\xi}{2}+z, \overline{u}\left(\frac{\xi}{2}\right)+z_d\right)}{(|z|^2+z_d^2)^{(d+s)/2}} dz_d dz, \tag{3.43}$$

where $\mathbb{1}_E^{\pm}(X) = \mathbb{1}_E(X) - \mathbb{1}_{\mathcal{C}E}(X)$ is the signed characteristic function. Similarly,

$$\partial_t \underline{u}\left(\frac{-\xi}{2}\right) = s(1-s)\sqrt{1+\omega'(|\xi|)^2}\,P.V. \int\limits_{\mathbb{R}^{d-1}} \int\limits_{\mathbb{R}} \frac{\mathbb{1}_E^{\pm}\left(\frac{-\xi}{2}+z, \underline{u}\left(\frac{-\xi}{2}\right)+z_d\right)}{(|z|^2+z_d^2)^{(d+s)/2}} dz_d dz. \tag{3.44}$$

Taking the difference and moving like constants to the other side, we thus have that

$$\frac{\partial_t \overline{u}\left(\frac{\xi}{2}\right) - \partial_t \underline{u}\left(\frac{-\xi}{2}\right)}{s(1-s)\sqrt{1+\omega'(|\xi|)^2}} = P.V. \int\limits_{\mathbb{R}^d} \frac{\mathbb{1}_E^{\pm}\left(\frac{\xi}{2}+z, \overline{u}\left(\frac{\xi}{2}\right)+z_d\right) - \mathbb{1}_E^{\pm}\left(\frac{-\xi}{2}+z, \underline{u}\left(\frac{-\xi}{2}\right)+z_d\right)}{(|z|^2+z_d^2)^{(d+s)/2}} dZ. \tag{3.45}$$

Remark. *Its important to note that here we are implicitly taking advantage of the fact that in the smooth case, $H_s(X, E) = H_s(X, \overline{E})$. This will no longer be true in general for viscosity solutions, and that difference is the source of most of the extra technical difficulties in that regime.*

Lemma 3.4.1. *Let $E \subseteq \mathbb{R}^d$ be an open set with modulus ω satisfying* (3.41). *Then*

$$\mathbb{1}_E^{\pm}\left(\frac{\xi}{2}+z, \overline{u}\left(\frac{\xi}{2}\right)+z_d\right) - \mathbb{1}_E^{\pm}\left(\frac{-\xi}{2}+z, \underline{u}\left(\frac{-\xi}{2}\right)+z_d\right) \le 0. \tag{3.46}$$

Furthermore, for any $0 \neq z \in \mathbb{R}^{d-1}$,

$$\int_{\mathbb{R}} \frac{\mathbb{1}_E^{\pm}\left(\frac{\xi}{2}+z, \overline{u}\left(\frac{\xi}{2}\right)+z_d\right) - \mathbb{1}_E^{\pm}\left(\frac{-\xi}{2}+z, \underline{u}\left(\frac{-\xi}{2}\right)+z_d\right)}{(|z|^2+z_d^2)^{(d+s)/2}} dz_d$$
$$\leq \int_{\underline{u}\left(\frac{\xi}{2}+z\right)-\overline{u}\left(\frac{\xi}{2}\right)}^{\overline{u}\left(\frac{-\xi}{2}+z\right)-\underline{u}\left(\frac{-\xi}{2}\right)} \frac{\mathbb{1}_E^{\pm}\left(\frac{\xi}{2}+z, \overline{u}\left(\frac{\xi}{2}\right)+z_d\right) - \mathbb{1}_E^{\pm}\left(\frac{-\xi}{2}+z, \underline{u}\left(\frac{-\xi}{2}\right)+z_d\right)}{(|z|^2+\omega(|z|)^2)^{(d+s)/2}} dz_d \tag{3.47}$$

Proof. To begin, note that the fact that E has modulus ω and 4). in (3.41) immediately implies

$$E - \left(\frac{\xi}{2}, \overline{u}\left(\frac{\xi}{2}\right)\right) \subseteq E + \left(\xi - \frac{\xi}{2}, \omega(|\xi|) - \overline{u}\left(\frac{\xi}{2}\right)\right) = E - \left(\frac{-\xi}{2}, \underline{u}\left(\frac{-\xi}{2}\right)\right), \tag{3.48}$$

and hence (3.46).

This is just a reflection of Proposition 3.2.3. In order to turn this into a quantitative statement, we'll have to rely on the definitions of $\overline{u}, \underline{u}$ (see Definition 3.2.1) and assumption 3). of (3.41).

Now fix some $z \neq 0$. By (3.46), we get immediately that if $\mathbb{1}_E^{\pm}\left(\frac{\xi}{2}+z, \overline{u}\left(\frac{\xi}{2}\right)+z_d\right) = 1$ then $\mathbb{1}_E^{\pm}\left(\frac{-\xi}{2}+z, \underline{u}\left(\frac{-\xi}{2}\right)+z_d\right) = 1$, and if $\mathbb{1}_E^{\pm}\left(\frac{-\xi}{2}+z, \underline{u}\left(\frac{-\xi}{2}\right)+z_d\right) = -1$ then $\mathbb{1}_E^{\pm}\left(\frac{\xi}{2}+z, \overline{u}\left(\frac{\xi}{2}\right)+z_d\right) = -1$. In particular, by the definition of $\underline{u}\left(\frac{\xi}{2}+z\right), \overline{u}\left(\frac{-\xi}{2}+z\right)$ this implies that

$$\begin{gathered}\mathbb{1}_E^{\pm}\left(\frac{\xi}{2}+z, \overline{u}\left(\frac{\xi}{2}\right)+z_d\right) - \mathbb{1}_E^{\pm}\left(\frac{-\xi}{2}+z, \underline{u}\left(\frac{-\xi}{2}\right)+z_d\right) = 0 \\ \text{for } z_d < \underline{u}\left(\frac{\xi}{2}+z\right) - \overline{u}\left(\frac{\xi}{2}\right) \text{ or } z_d \geq \overline{u}\left(\frac{-\xi}{2}+z\right) - \underline{u}\left(\frac{-\xi}{2}\right).\end{gathered} \tag{3.49}$$

As $\underline{u}\left(\frac{\xi}{2}+z\right) - \overline{u}\left(\frac{\xi}{2}\right) \geq -\omega(|z|)$ and $\overline{u}\left(\frac{-\xi}{2}+z\right) - \underline{u}\left(\frac{-\xi}{2}\right) \leq \omega(|z|)$ by 3). in (3.41), combining (3.46) and (3.49) gives us

$$\begin{aligned}
&\int_{\mathbb{R}} \frac{\mathbb{1}_E^{\pm}\left(\frac{\xi}{2}+z,\overline{u}\left(\frac{\xi}{2}\right)+z_d\right)-\mathbb{1}_E^{\pm}\left(\frac{-\xi}{2}+z,\underline{u}\left(\frac{-\xi}{2}\right)+z_d\right)}{(|z|^2+z_d^2)^{(d+s)/2}}dz_d \\
&= \int_{\underline{u}\left(\frac{\xi}{2}+z\right)-\overline{u}\left(\frac{\xi}{2}\right)}^{\overline{u}\left(\frac{-\xi}{2}+z\right)-\underline{u}\left(\frac{-\xi}{2}\right)} \frac{\mathbb{1}_E^{\pm}\left(\frac{\xi}{2}+z,\overline{u}\left(\frac{\xi}{2}\right)+z_d\right)-\mathbb{1}_E^{\pm}\left(\frac{-\xi}{2}+z,\underline{u}\left(\frac{-\xi}{2}\right)+z_d\right)}{(|z|^2+z_d^2)^{(d+s)/2}}dz_d \qquad (3.50)\\
&\le \int_{\underline{u}\left(\frac{\xi}{2}+z\right)-\overline{u}\left(\frac{\xi}{2}\right)}^{\overline{u}\left(\frac{-\xi}{2}+z\right)-\underline{u}\left(\frac{-\xi}{2}\right)} \frac{\mathbb{1}_E^{\pm}\left(\frac{\xi}{2}+z,\overline{u}\left(\frac{\xi}{2}\right)+z_d\right)-\mathbb{1}_E^{\pm}\left(\frac{-\xi}{2}+z,\underline{u}\left(\frac{-\xi}{2}\right)+z_d\right)}{(|z|^2+\omega(|z|)^2)^{(d+s)/2}}dz_d.
\end{aligned}$$

□

Lemma 3.4.2. *Let $E \subseteq \mathbb{R}^d$ be an open set with modulus ω satisfying* (3.41). *Then*

$$\begin{aligned}
&P.V. \int_{\mathbb{R}^{d-1}} \int_{\mathbb{R}} \frac{\mathbb{1}_E^{\pm}\left(\frac{\xi}{2}+z,\overline{u}\left(\frac{\xi}{2}\right)+z_d\right)-\mathbb{1}_E^{\pm}\left(\frac{-\xi}{2}+z,\underline{u}\left(\frac{-\xi}{2}\right)+z_d\right)}{(|z|^2+z_d^2)^{(d+s)/2}}dz dz_d \\
&\qquad \le \frac{-2}{(3(1+L))^{d+s}} \int_{\mathbb{R}^{d-1}} \frac{\omega(|\xi|)-\left(\overline{u}\left(\frac{\xi}{2}+z\right)-\underline{u}\left(\frac{-\xi}{2}+z\right)\right)}{(|z|^2+\omega(0)^2)^{(d+s)/2}}dz \qquad (3.51)\\
&\qquad + \frac{-2}{(3(1+L))^{d+s}} \int_{\mathbb{R}^{d-1}} \frac{\overline{u}(z)-\underline{u}(z)}{\max_{\pm}(|z \pm \frac{\xi}{2}|^2+\omega(0)^2)^{(d+s)/2}}dz.
\end{aligned}$$

Proof. By Lemma 3.4.1 it suffices to show that

$$\int_{\mathbb{R}^{d-1}} \left(\int_{\underline{u}\left(\frac{\xi}{2}+z\right)-\overline{u}\left(\frac{\xi}{2}\right)}^{\overline{u}\left(\frac{-\xi}{2}+z\right)-\underline{u}\left(\frac{-\xi}{2}\right)} \frac{\mathbb{1}_E^{\pm}\left(\frac{\xi}{2}+z,\overline{u}\left(\frac{\xi}{2}\right)+z_d\right)-\mathbb{1}_E^{\pm}\left(\frac{-\xi}{2}+z,\underline{u}\left(\frac{-\xi}{2}\right)+z_d\right)}{(|z|^2+\omega(|z|)^2)^{(d+s)/2}} dz_d \right) dz$$
$$\leq \frac{-2}{(3(1+L))^{d+s}} \int_{\mathbb{R}^{d-1}} \frac{\omega(|\xi|)-\left(\overline{u}\left(\frac{\xi}{2}+z\right)-\underline{u}\left(\frac{-\xi}{2}+z\right)\right)}{(|z|^2+\omega(0)^2)^{(d+s)/2}} dz$$
$$+ \frac{-2}{(3(1+L))^{d+s}} \int_{\mathbb{R}^{d-1}} \frac{\overline{u}(z)-\underline{u}(z)}{\max_{\pm}(|z\pm\frac{\xi}{2}|^2+\omega(0)^2)^{(d+s)/2}} dz. \tag{3.52}$$

Fix some $z \neq 0$. Using the definitions of $\overline{u}, \underline{u}$, we can more precisely bound

$$\int_{\underline{u}\left(\frac{\xi}{2}+z\right)-\overline{u}\left(\frac{\xi}{2}\right)}^{\overline{u}\left(\frac{-\xi}{2}+z\right)-\underline{u}\left(\frac{-\xi}{2}\right)} \frac{\mathbb{1}_E^{\pm}\left(\frac{\xi}{2}+z,\overline{u}\left(\frac{\xi}{2}\right)+z_d\right)-\mathbb{1}_E^{\pm}\left(\frac{-\xi}{2}+z,\underline{u}\left(\frac{-\xi}{2}\right)+z_d\right)}{(|z|^2+\omega(|z|)^2)^{(d+s)/2}} dz_d, \tag{3.53}$$

using that

$$\overline{u}\left(\frac{\xi}{2}+z\right)-\overline{u}\left(\frac{\xi}{2}\right) \leq z_d < \underline{u}\left(\frac{-\xi}{2}+z\right)-\underline{u}\left(\frac{-\xi}{2}\right)$$
$$\Rightarrow \mathbb{1}_E^{\pm}\left(\frac{\xi}{2}+z,\overline{u}\left(\frac{\xi}{2}\right)+z_d\right)-\mathbb{1}_E^{\pm}\left(\frac{-\xi}{2}+z,\underline{u}\left(\frac{-\xi}{2}\right)+z_d\right) = -2. \tag{3.54}$$

Thus we can bound (3.53) by

$$\int\limits_{\underline{u}\left(\frac{\xi}{2}+z\right)-\overline{u}\left(\frac{\xi}{2}\right)}^{\overline{u}\left(\frac{-\xi}{2}+z\right)-\underline{u}\left(\frac{-\xi}{2}\right)} \frac{\mathbb{1}_E^{\pm}\left(\frac{\xi}{2}+z,\overline{u}\left(\frac{\xi}{2}\right)+z_d\right)-\mathbb{1}_E^{\pm}\left(\frac{-\xi}{2}+z,\underline{u}\left(\frac{-\xi}{2}\right)+z_d\right)}{(|z|^2+\omega(|z|)^2)^{(d+s)/2}}dz_d$$

$$\le \int\limits_{\overline{u}\left(\frac{\xi}{2}+z\right)-\overline{u}\left(\frac{\xi}{2}\right)}^{\underline{u}\left(\frac{-\xi}{2}+z\right)-\underline{u}\left(\frac{-\xi}{2}\right)} \frac{-2}{(|z|^2+\omega(|z|)^2)^{(d+s)/2}}dz_d$$

$$+ \int\limits_{\underline{u}\left(\frac{\xi}{2}+z\right)-\overline{u}\left(\frac{\xi}{2}\right)}^{\overline{u}\left(\frac{\xi}{2}+z\right)-\overline{u}\left(\frac{\xi}{2}\right)} \frac{\mathbb{1}_E^{\pm}\left(\frac{\xi}{2}+z,\overline{u}\left(\frac{\xi}{2}\right)+z_d\right)-\mathbb{1}_E^{\pm}\left(\frac{-\xi}{2}+z,\underline{u}\left(\frac{-\xi}{2}\right)+z_d\right)}{(|z|^2+\omega(|z|)^2)^{(d+s)/2}}dz_d$$

$$+ \int\limits_{\underline{u}\left(\frac{-\xi}{2}+z\right)-\underline{u}\left(\frac{-\xi}{2}\right)}^{\overline{u}\left(\frac{-\xi}{2}+z\right)-\underline{u}\left(\frac{-\xi}{2}\right)} \frac{\mathbb{1}_E^{\pm}\left(\frac{\xi}{2}+z,\overline{u}\left(\frac{\xi}{2}\right)+z_d\right)-\mathbb{1}_E^{\pm}\left(\frac{-\xi}{2}+z,\underline{u}\left(\frac{-\xi}{2}\right)+z_d\right)}{(|z|^2+\omega(|z|)^2)^{(d+s)/2}}dz_d. \tag{3.55}$$

Using 4). in (3.41) we can rewrite the first integral on the right hand side of (3.55) as

$$\int\limits_{\overline{u}\left(\frac{\xi}{2}+z\right)-\overline{u}\left(\frac{\xi}{2}\right)}^{\underline{u}\left(\frac{-\xi}{2}+z\right)-\underline{u}\left(\frac{-\xi}{2}\right)} \frac{-2}{(|z|^2+\omega(|z|)^2)^{(d+s)/2}}dz_d = (-2)\frac{\omega(|\xi|)-\left(\overline{u}\left(\frac{\xi}{2}+z\right)-\underline{u}\left(\frac{-\xi}{2}+z\right)\right)}{(|z|^2+\omega(|z|)^2)^{(d+s)/2}}. \tag{3.56}$$

As for the other two integrals on the right hand side of (3.55), by translating our z_d

bounds and using that $\overline{u}\left(\frac{\xi}{2}\right) - \underline{u}\left(\frac{-\xi}{2}\right) = \omega(|\xi|)$ we get that

$$\int_{\underline{u}\left(\frac{\xi}{2}+z\right)-\overline{u}\left(\frac{\xi}{2}\right)}^{\overline{u}\left(\frac{\xi}{2}+z\right)-\overline{u}\left(\frac{\xi}{2}\right)} \mathbb{1}_E^{\pm}\left(\frac{\xi}{2}+z, \overline{u}\left(\frac{\xi}{2}\right)+z_d\right) - \mathbb{1}_E^{\pm}\left(\frac{-\xi}{2}+z, \underline{u}\left(\frac{-\xi}{2}\right)+z_d\right) dz_d$$
$$= \int_{\underline{u}\left(\frac{\xi}{2}+z\right)}^{\overline{u}\left(\frac{\xi}{2}+z\right)} \mathbb{1}_E^{\pm}\left(\frac{\xi}{2}+z, z_d\right) - \mathbb{1}_E^{\pm}\left(\frac{-\xi}{2}+z, z_d-\omega(|\xi|)\right) dz_d \tag{3.57}$$
$$= \int_{\underline{u}\left(\frac{\xi}{2}+z\right)}^{\overline{u}\left(\frac{\xi}{2}+z\right)} \mathbb{1}_E^{\pm}\left(\frac{\xi}{2}+z, z_d\right) - 1 dz_d.$$

To go from line 2 to line 3, note that $z_d < \overline{u}\left(\frac{\xi}{2}+z\right) \le \underline{u}\left(\frac{-\xi}{2}+z\right) + \omega(|\xi|)$. Hence, $\mathbb{1}_E^{\pm}\left(\frac{-\xi}{2}+z, z_d-\omega(|\xi|)\right) = 1$.

By a symmetric argument,

$$\int_{\underline{u}\left(\frac{-\xi}{2}+z\right)-\overline{u}\left(\frac{-\xi}{2}\right)}^{\overline{u}\left(\frac{-\xi}{2}+z\right)-\overline{u}\left(\frac{-\xi}{2}\right)} \mathbb{1}_E^{\pm}\left(\frac{\xi}{2}+z, \overline{u}\left(\frac{\xi}{2}\right)+z_d\right) - \mathbb{1}_E^{\pm}\left(\frac{-\xi}{2}+z, \underline{u}\left(\frac{-\xi}{2}\right)+z_d\right) dz_d$$
$$= \int_{\underline{u}\left(\frac{-\xi}{2}+z\right)}^{\overline{u}\left(\frac{-\xi}{2}+z\right)} -1 - \mathbb{1}_E^{\pm}\left(\frac{-\xi}{2}+z, z_d\right) dz_d. \tag{3.58}$$

Shifting the value of z in (3.57) and (3.58) by $\mp\xi/2$ and adding them together thus gives that

$$\int_{\underline{u}(z)}^{\overline{u}(z)} \frac{\mathbb{1}_E^{\pm}(z,z_d)-1}{(|z-\frac{\xi}{2}|^2+\omega(|z-\frac{\xi}{2}|)^2)^{(d+s)/2}} + \frac{-1-\mathbb{1}_E^{\pm}(z,z_d)}{(|z+\frac{\xi}{2}|^2+\omega(|z+\frac{\xi}{2}|)^2)^{(d+s)/2}} dz_d$$
$$\leq \int_{\underline{u}(z)}^{\overline{u}(z)} \frac{-2}{\max_{\pm}(|z\pm\frac{\xi}{2}|^2+\omega(|z\pm\frac{\xi}{2}|)^2)^{(d+s)/2}} = (-2)\frac{\overline{u}(z)-\underline{u}(z)}{\max_{\pm}(|z\pm\frac{\xi}{2}|^2+\omega(|z\pm\frac{\xi}{2}|)^2)^{(d+s)/2}} \tag{3.59}$$

Plugging in (3.59) and (3.56) into (3.55) and integrating in z thus gives us that

$$P.V. \int_{\mathbb{R}^{d-1}} \int_{\mathbb{R}} \frac{\mathbb{1}_E^{\pm}\left(\frac{\xi}{2}+z, \overline{u}\left(\frac{\xi}{2}\right)+z_d\right) - \mathbb{1}_E^{\pm}\left(\frac{-\xi}{2}+z, \underline{u}\left(\frac{-\xi}{2}\right)+z_d\right)}{(|z|^2+z_d^2)^{(d+s)/2}} dz dz_d$$
$$\leq -2\int_{\mathbb{R}^{d-1}} \frac{\omega(|\xi|)-\left(\overline{u}\left(\frac{\xi}{2}+z\right)-\underline{u}\left(\frac{-\xi}{2}+z\right)\right)}{(|z|^2+\omega(|z|)^2)^{(d+s)/2}} dz \tag{3.60}$$
$$-2\int_{\mathbb{R}^{d-1}} \frac{\overline{u}(z)-\underline{u}(z)}{\max_{\pm}(|z\pm\frac{\xi}{2}|^2+\omega(|z\pm\frac{\xi}{2}|)^2)^{(d+s)/2}} dz$$

Finally, the proof is complete using that ω is $(1+L)$-Lipschtiz by assumption 3). in (3.41), so

$$\frac{1}{(|z|^2+\omega(|z|)^2)^{(d+s)/2}} \geq \frac{1}{(|z|^2+((1+L)|z|+\omega(0))^2)^{(d+s)/2}}$$
$$\geq \frac{1}{((2(1+L)^2+1)|z|^2+2\omega(0)^2)^{(d+s)/2}} \tag{3.61}$$
$$\geq \frac{1}{(3(1+L))^{d+s}} \frac{1}{(|z|^2+\omega(0)^2)^{(d+s)/2}}.$$

□

Combining (3.45) with Lemma 3.4.2 thus gives us that

$$\partial_t\overline{u}\left(\frac{\xi}{2}\right)-\partial_t\underline{u}\left(\frac{-\xi}{2}\right)$$
$$\lesssim_{d,s,L}-\int\limits_{\mathbb{R}^{d-1}}\frac{\omega(|\xi|)-\left(\overline{u}\left(\frac{\xi}{2}+z\right)-\underline{u}\left(\frac{-\xi}{2}+z\right)\right)}{(|z|^2+\omega(0)^2)^{(d+s)/2}}dz \tag{3.62}$$
$$-\int\limits_{\mathbb{R}^{d-1}}\frac{\overline{u}(z)-\underline{u}(z)}{\max\limits_{\pm}(|z\pm\frac{\xi}{2}|^2+\omega(0)^2)^{(d+s)/2}}dz.$$

In order to remove the dependence on $\overline{u},\underline{u}$ from (3.62) and get an upper bound depending only on ω, we now alter a useful integral rearrangement argument of [MDV14] to get

Lemma 3.4.3. *Let E be an open set with modulus ω satisfying assumptions* (3.41)*. Then*

$$\int\limits_{\mathbb{R}^{d-1}}\frac{\omega(|\xi|)-\left(\overline{u}\left(\frac{\xi}{2}+z\right)-\underline{u}\left(\frac{-\xi}{2}+z\right)\right)}{(|z|^2+\omega(0)^2)^{(d+s)/2}}dz+\int\limits_{\mathbb{R}^{d-1}}\frac{\overline{u}(z)-\underline{u}(z)}{\max\limits_{\pm}(|z\pm\frac{\xi}{2}|^2+\omega(0)^2)^{(d+s)/2}}dz$$
$$\geq C(d,s)\int\limits_0^{\frac{|\xi|}{2}}\frac{2\omega(|\xi|)-\omega(|\xi|+2\eta)-\omega(|\xi|-2\eta)}{(\eta^2+\omega(0)^2)^{(2+s)/2}}d\eta \tag{3.63}$$
$$+C(d,s)\int\limits_{\frac{|\xi|}{2}}^{\infty}\frac{2\omega(|\xi|)+\omega(2\eta-|\xi|)-\omega(2\eta+|\xi|)}{(\eta^2+\omega(0)^2)^{(2+s)/2}}d\eta.$$

where $C(d,s)=\dfrac{\Gamma\left(\frac{d-2}{2}\right)\Gamma\left(\frac{2+s}{2}\right)}{2\Gamma\left(\frac{d+s}{2}\right)}$ *for $d\geq 3$, and* $C(2,s)=1$.

Proof. We prove Lemma 3.4.3 in the case that $d\geq 3$. The case when $d=2$ follows from a clear simplification.

We write $z\in\mathbb{R}^{d-1}$ as $z=(\eta,\nu)\in\mathbb{R}\times\mathbb{R}^{d-2}$, and without loss of generality assume that $\xi=(|\xi|,0)$.

To begin, let

$$K(\eta,\nu)=\frac{1}{(\eta^2+|\nu|^2+\omega(0)^2)^{(d+s)/2}}. \tag{3.64}$$

As $K(\eta,\nu)$ is radial, we have that

$$\begin{aligned}\int\limits_{\mathbb{R}^{d-2}}\int\limits_{\mathbb{R}} &\left[\omega(|\xi|)-\overline{u}\left(\frac{|\xi|}{2}+\eta,\nu\right)+\underline{u}\left(-\frac{|\xi|}{2}+\eta,\nu\right)\right]K(\eta,\nu)d\eta d\nu \\ &= \int\limits_{\mathbb{R}^{d-2}}\int\limits_{\mathbb{R}}\left[\omega(|\xi|)-\overline{u}\left(\frac{|\xi|}{2}+\eta,\nu\right)+\underline{u}\left(-\frac{|\xi|}{2}-\eta,\nu\right)\right]K(\eta,\nu)d\eta d\nu.\end{aligned} \tag{3.65}$$

Fix some $\nu \in \mathbb{R}^{d-1}$. Then breaking up the η integral, rearranging, and adding terms gives

$$\begin{aligned}\int\limits_{\mathbb{R}} &\left[\omega(|\xi|)-\overline{u}\left(\frac{|\xi|}{2}+\eta,\nu\right)+\underline{u}\left(-\frac{|\xi|}{2}-\eta,\nu\right)\right]K(\eta,\nu)d\eta \\ &= \int\limits_{-\frac{|\xi|}{2}}^{\infty}\left[\omega(|\xi|)-\overline{u}\left(\frac{|\xi|}{2}+\eta,\nu\right)+\underline{u}\left(-\frac{|\xi|}{2}-\eta,\nu\right)\right]K(\eta,\nu)d\eta \\ &\quad+\int\limits_{-\frac{|\xi|}{2}}^{\infty}\left[\omega(|\xi|)-\overline{u}\left(-\frac{|\xi|}{2}-\eta,\nu\right)+\underline{u}\left(\frac{|\xi|}{2}+\eta,\nu\right)\right]K(-|\xi|-\eta,\nu)d\eta \\ &= \int\limits_{-\frac{|\xi|}{2}}^{\infty}\omega(|\xi|)(K(\eta,\nu)+K(|\xi|+\eta,\nu)) \\ &\qquad\qquad -\left[\overline{u}\left(\frac{|\xi|}{2}+\eta,\nu\right)-\underline{u}\left(-\frac{|\xi|}{2}-\eta,\nu\right)\right](K(\eta,\nu)-K(|\xi|+\eta,\nu))d\eta \\ &\quad-\int\limits_{-\frac{|\xi|}{2}}^{\infty}\left[\overline{u}\left(\frac{|\xi|}{2}+\eta\right)-\underline{u}\left(\frac{|\xi|}{2}+\eta\right)+\overline{u}\left(-\frac{|\xi|}{2}-\eta\right)-\underline{u}\left(\frac{-|\xi|}{2}-\eta\right)\right]K(|\xi|+\eta)d\eta \\ &= \int\limits_{-\frac{|\xi|}{2}}^{\infty}\omega(|\xi|)(K(\eta,\nu)+K(|\xi|+\eta,\nu)) \\ &\qquad\qquad -\left[\overline{u}\left(\frac{|\xi|}{2}+\eta,\nu\right)-\underline{u}\left(-\frac{|\xi|}{2}-\eta,\nu\right)\right](K(\eta,\nu)-K(|\xi|+\eta,\nu))d\eta \\ &\quad-\int\limits_{\mathbb{R}}[\overline{u}(\eta,\nu)-\underline{u}(\eta,\nu)]\min\{K\left(\eta\pm\frac{|\xi|}{2},\nu\right)\}d\eta.\end{aligned} \tag{3.66}$$

The last integral is the only real difference between the argument we give and the one [MDV14] gives, and it arises naturally out of the fact that ∂E is not the graph of a function. However, noting that

$$\min_{\pm} K\left(z \pm \frac{\xi}{2}\right) = \frac{1}{\max_{\pm}(|z \pm \frac{\xi}{2}|^2 + \omega(0)^2)^{(d+s)/2}}, \tag{3.67}$$

we see this term is precisely

$$\int_{\mathbb{R}^{d-2}} \int_{\mathbb{R}} [\overline{u}(\eta,\nu) - \underline{u}(\eta,\nu)] \min\{K\left(\eta \pm \frac{|\xi|}{2}, \nu\right)\} d\eta d\nu = \int_{\mathbb{R}^{d-1}} \frac{\overline{u}(z) - \underline{u}(z)}{\max_{\pm}(|z \pm \frac{\xi}{2}|^2 + \omega(0)^2)^{(d+s)/2}} dz \tag{3.68}$$

Integrating in ν, taking into account this cancelation, adding/subtracting $\omega(|\xi| + 2\eta)$, and using that K is nonincreasing, we get that

$$\begin{aligned}
&\int_{\mathbb{R}^{d-2}} \int_{-\frac{|\xi|}{2}}^{\infty} \left[\omega(|\xi|) - \overline{u}\left(\frac{|\xi|}{2} + \eta, \nu\right) + \underline{u}\left(-\frac{|\xi|}{2} + \eta, \nu\right)\right] K(\eta,\nu) d\eta d\nu \\
&+ \int_{\mathbb{R}^{d-2}} \int_{\mathbb{R}} [\overline{u}(\eta,\nu) - \underline{u}(\eta,\nu)] \min\{K\left(\eta \pm \frac{|\xi|}{2}, \nu\right)\} d\eta d\nu. \\
&= \int_{\mathbb{R}^{d-2}} \int_{-\frac{|\xi|}{2}}^{\infty} \omega(|\xi|)(K(\eta,\nu) + K(|\xi| + \eta, \nu)) - \omega(|\xi| + 2\eta)(K(\eta,\nu) - K(|\xi| + \eta,\nu)) d\eta d\nu \\
&+ \int_{\mathbb{R}^{d-2}} \int_{-\frac{|\xi|}{2}}^{\infty} \left[\omega(|\xi| + 2\eta) - \overline{u}\left(\frac{|\xi|}{2} + \eta, \nu\right) + \underline{u}\left(-\frac{|\xi|}{2} - \eta, \nu\right)\right] (K(\eta,\nu) - K(|\xi| + \eta, \nu)) d\eta d\nu \\
&\geq \int_{\mathbb{R}^{d-2}} \int_{-\frac{|\xi|}{2}}^{\infty} \omega(|\xi|)[K(\eta,\nu) + K(|\xi| + \eta,\nu)] - \omega(|\xi| + 2\eta)[K(\eta,\nu) - K(|\xi| + \eta, \nu)] d\eta d\nu
\end{aligned} \tag{3.69}$$

It then follows identically to the argument in the appendix of [MDV14] that

$$\int_{\mathbb{R}^{d-2}} \int_{-\frac{|\xi|}{2}}^{\infty} \omega(|\xi|)[K(\eta,\nu)+K(|\xi|+\eta,\nu)] - \omega(|\xi|+2\eta)[K(\eta,\nu)-K(|\xi|+\eta,\nu)]d\eta d\nu$$

$$= \int_0^{\frac{|\xi|}{2}} (2\omega(|\xi|) - \omega(|\xi|+2\eta) - \omega(|\xi|-2\eta))\,\tilde{K}(\eta)d\eta \tag{3.70}$$

$$+ \int_{\frac{|\xi|}{2}}^{\infty} (2\omega(|\xi|) + \omega(2\eta-|\xi|) - \omega(2\eta+|\xi|))\,\tilde{K}(\eta)d\eta,$$

where

$$\tilde{K}(\eta) = \int_{\mathbb{R}^{d-2}} K(\eta,\nu)d\nu = \frac{1}{(\eta^2+\omega(0)^2)^{(2+s)/2}} \left(\int_0^{\infty} \frac{r^{d-3}}{(1+r^2)^{(d+s)/2}} dr \right). \tag{3.71}$$

Note that

$$\int_0^{\infty} \frac{r^{d-3}}{(1+r^2)^{(d+s)/2}} dr = \frac{1}{2} B\left(\frac{d-2}{2}, \frac{2+s}{2} \right) = \frac{\Gamma\left(\frac{d-2}{2}\right)\Gamma\left(\frac{2+s}{2}\right)}{2\Gamma\left(\frac{d+s}{2}\right)}, \tag{3.72}$$

where $B(x,y)$ is the Beta function.

Thus

$$\int_{\mathbb{R}^{d-1}} \frac{\omega(|\xi|) - \left(\overline{u}\left(\frac{\xi}{2}+z\right) - \underline{u}\left(\frac{-\xi}{2}+z\right)\right)}{(|z|^2+\omega(0)^2)^{(d+s)/2}} dz + \int_{\mathbb{R}^{d-1}} \frac{\overline{u}(z) - \underline{u}(z)}{\max_{\pm}(|z \pm \frac{\xi}{2}|^2 + \omega(0)^2)^{(d+s)/2}} dz$$

$$\geq C(d,s) \int_0^{\frac{|\xi|}{2}} \frac{2\omega(|\xi|) - \omega(|\xi|+2\eta) - \omega(|\xi|-2\eta)}{(\eta^2+\omega(0)^2)^{(2+s)/2}} d\eta \tag{3.73}$$

$$+ C(d,s) \int_{\frac{|\xi|}{2}}^{\infty} \frac{2\omega(|\xi|) + \omega(2\eta-|\xi|) - \omega(2\eta+|\xi|)}{(\eta^2+\omega(0)^2)^{(2+s)/2}} d\eta.$$

□

3.5 Construction of modulus and completion of break through argument

Combining (3.45) with Lemmas 3.4.2 and 3.4.3, we have under the assumptions of (3.41) that

$$\begin{aligned}\partial_t \overline{u}\left(t_0, \frac{\xi}{2}\right) - \partial_t \underline{u}\left(t_0, \frac{-\xi}{2}\right) \lesssim_{d,s,L} & \int_0^{\frac{|\xi|}{2}} \frac{\omega(|\xi| + 2\eta) + \omega(|\xi| - 2\eta) - 2\omega(|\xi|)}{(\eta^2 + \omega(0)^2)^{(2+s)/2}} d\eta \\ & + \int_{\frac{|\xi|}{2}}^{\infty} \frac{\omega(2\eta + |\xi|) - \omega(2\eta - |\xi|) - 2\omega(|\xi|)}{(\eta^2 + \omega(0)^2)^{(2+s)/2}} d\eta.\end{aligned} \tag{3.74}$$

As the right hand side of (3.74) only depends on ω, we can now make our choice of a family of moduli of continuity $\omega : [0, \infty) \times [0, \infty) \to [0, \infty)$ to complete the breakthrough argument. With that in mind, define

$$\omega(t, r) = \delta(t) + \begin{cases} 1 + Lr, & r \geq 2 \\ (1 + L)r - \frac{r^{1+s}}{2^{1+s}}, & c\delta(t)^2 \leq r \leq 2, \\ A(\delta)r^2 + B(\delta), & 0 \leq r \leq c\delta(t)^2 \end{cases} \tag{3.75}$$

where $0 < c << 1$, and

$$\begin{aligned} A(\delta) &= \frac{1 + L}{2c\delta^2} - \frac{1 + s}{2^{2+s}} (c\delta^2)^{s-1}, \\ B(\delta) &= \frac{(1 + L)c\delta^2}{2} - \frac{1 - s}{2^{2+s}} (c\delta^2)^{1+s}. \end{aligned} \tag{3.76}$$

are chosen so that $\omega(t, \cdot)$ is C^1, and $\delta : [0, T] \to [0, 1]$ is a to be determined non increasing function with $\delta(0) = 1$ and $\delta(T) = 0$. The function $\delta(t) \approx \omega(t, 0)$ and essentially represents how far the boundary of our set E_t is from being a graph.

Lemma 3.5.1. *For $\omega(t,r)$ as defined in* (3.75)*, $\partial_t\omega(t,r) > 2\delta'(t)$ for $0 < c < \frac{2}{5(1+L)}$.*

Proof. Examining the formula in (3.75) and the fact that $\delta'(t) \leq 0$, its clear that it suffices to show

$$A'(\delta(t))r^2 + B'(\delta(t)) < 1, \qquad 0 \leq r \leq c\delta(t)^2,\ 0 \leq c \leq \frac{2}{5(1+L)}. \tag{3.77}$$

Differentiating B and using that $0 \leq \delta \leq 1$, we have that

$$B'(\delta) = (1+L)c\delta - \frac{(1-s)(1+s)}{2^{1+s}}c^{1+s}\delta^{1+2s} \leq (1+L)c < \frac{2}{5}. \tag{3.78}$$

Similarly, as

$$A'(\delta) = \frac{-3(1+L)}{2c\delta^3} + \frac{(1+s)(3-2s)}{2^{2+s}c^{1-s}\delta^{3-2s}} \tag{3.79}$$

and $0 \leq r \leq c\delta^2$, we have that

$$A'(\delta)r^2 \leq \frac{(1+s)(3-2s)}{2^{2+s}c^{1-s}\delta^{3-2s}}r^2 \leq \frac{3}{2}c^{1+s}\delta^{1+2s} \leq \frac{3}{2}c < \frac{3}{5}. \tag{3.80}$$

□

Thus $\partial_t\omega(t,\cdot)$ will always be comparable to $\delta'(t)$. Our goal now is to bound

$$\int_0^{\frac{|\xi|}{2}} \frac{\omega(t,|\xi|+2\eta)+\omega(t,|\xi|-2\eta)-2\omega(t,|\xi|)}{(\eta^2+\omega(t,0)^2)^{(2+s)/2}}d\eta + \int_{\frac{|\xi|}{2}}^{\infty} \frac{\omega(t,2\eta+|\xi|)-\omega(t,2\eta-|\xi|)-2\omega(t,|\xi|)}{(\eta^2+\omega(t,0)^2)^{(2+s)/2}}d\eta, \tag{3.81}$$

in terms of $\delta(t) \approx \omega(t,0)$.

If $\omega(t,\cdot)$ was a concave function, then both of the integrals in (3.81) would be nonpositive. However because we needed $\partial_r\omega(t,0) = 0$ in our construction of ω in case the touching point $\xi = 0$, $\omega(t,\cdot)$ will be convex near 0. Thus the first integral of (3.81) can be positive. However, we will show that as long as c is taken small, it will be under control.

Lemma 3.5.2. *Let* $\omega(t,r)$ *be as defined in* (3.75) *and* $|\xi| \le 2$*. Then*

$$\int_0^{\frac{|\xi|}{2}} \frac{\omega(t,|\xi|+2\eta)+\omega(t,|\xi|-2\eta)-2\omega(t,|\xi|)}{(\eta^2+\omega(t,0)^2)^{(2+s)/2}} d\eta \le (1+L)c^2. \tag{3.82}$$

Proof. We claim that

$$\omega(t,|\xi|+2\eta)+\omega(t,|\xi|-2\eta)-2\omega(t,|\xi|) \le \begin{cases} 0, & 2\eta-|\xi| \ge c\delta^2, \\ (1+L)c\delta^2, & \text{otherwise} \end{cases} \tag{3.83}$$

Given (3.83), it follows immediately that for $|\xi| \le c\delta^2$

$$\begin{aligned} \int_0^{\frac{|\xi|}{2}} \frac{\omega(t,|\xi|+2\eta)+\omega(t,|\xi|-2\eta)-2\omega(t,|\xi|)}{(\eta^2+\omega(t,0)^2)^{(2+s)/2}} d\eta &\le \int_0^{c\delta(t)^2} \frac{(1+L)c\delta(t)^2}{\delta(t)^{2+s}} d\eta \\ &= (1+L)c^2\delta(t)^{2-s}. \end{aligned} \tag{3.84}$$

For $|\xi| \ge c\delta(t)^2$, we similarly have that

$$\begin{aligned} \int_0^{\frac{|\xi|}{2}} \frac{\omega(t,|\xi|+2\eta)+\omega(t,|\xi|-2\eta)-2\omega(t,|\xi|)}{(\eta^2+\omega(t,0)^2)^{(2+s)/2}} d\eta &\le \int_{\frac{|\xi|-c\delta(t)^2}{2}}^{\frac{|\xi|}{2}} \frac{(1+L)c\delta(t)^2}{\delta(t)^{2+s}} d\eta \\ &= (1+L)c^2\delta(t)^{2-s}. \end{aligned} \tag{3.85}$$

As $\delta \le 1$ always, we thus have that

$$\int_0^{\frac{|\xi|}{2}} \frac{\omega(t,|\xi|+2\eta)+\omega(t,|\xi|-2\eta)-2\omega(t,|\xi|)}{(\eta^2+\omega(t,0)^2)^{(2+s)/2}} d\eta \le (1+L)c^2, \tag{3.86}$$

for all $|\xi| \le 2$.

All that remains is to prove the claim (3.83). Consider the function

$$\tilde{\omega}(t,r) = \delta(t) + \begin{cases} 1 + Lr, & r \geq 2 \\ (1+L)r - \frac{r^{1+s}}{2^{1+s}}, & 0 \leq r \leq 2, \end{cases} . \tag{3.87}$$

Then for fixed t, $\tilde{\omega}(t,\cdot)$ is a concave function of r. Thus whenever $|\xi| - 2\eta \geq c\delta(t)^2$, we have that

$$\omega(t,|\xi|+2\eta)+\omega(t,|\xi|-2\eta)-2\omega(t,|\xi|) = \tilde{\omega}(t,|\xi|+2\eta)+\tilde{\omega}(t,|\xi|-2\eta)-2\tilde{\omega}(t,|\xi|) \leq 0. \tag{3.88}$$

As we also have that

$$0 \leq \omega(t,r) - \tilde{\omega}(t,r) \leq \omega(t,0) - \tilde{\omega}(t,0) = B(\delta(t)) \leq \frac{(1+L)c\delta(t)^2}{2}, \tag{3.89}$$

it follows that when $|\xi| - 2\eta \leq c\delta(t)^2$

$$\begin{aligned} &\omega(t,|\xi|+2\eta) + \omega(t,|\xi|-2\eta) - 2\omega(t,|\xi|) \\ &\qquad \leq \tilde{\omega}(t,|\xi|+2\eta) + \tilde{\omega}(t,|\xi|-2\eta) - 2\tilde{\omega}(t,|\xi|) + (1+L)c\delta(t)^2 \\ &\qquad \leq (1+L)c\delta(t)^2. \end{aligned} \tag{3.90}$$

□

With Lemma 3.5.2, we can bound the first integral in (3.81) by an arbitrarily small constant as $c \to 0$. All that remains now is to get a good, negative upper bound on the second integral.

Lemma 3.5.3. *Let $\omega(t,r)$ be as defined in* (3.75) *and $|\xi| \leq 2$. Then*

$$\int_{\frac{|\xi|}{2}}^{\infty} \frac{\omega(t,2\eta+|\xi|) - \omega(t,2\eta-|\xi|) - 2\omega(t,|\xi|)}{(\eta^2+\omega(t,0)^2)^{(2+s)/2}} d\eta \lesssim -1. \tag{3.91}$$

Proof. Again, take $\tilde{\omega}(t,r)$ to be

$$\tilde{\omega}(t,r) = \delta(t) + \begin{cases} 1 + Lr, & r \geq 2 \\ (1+L)r - \frac{r^{1+s}}{2^{1+s}}, & 0 \leq r \leq 2, \end{cases} \tag{3.92}$$

Then we claim that for $\eta \geq \dfrac{|\xi|}{2}$,

$$\omega(t, 2\eta+|\xi|) - \omega(t, 2\eta-|\xi|) - 2\omega(t,|\xi|) \leq \begin{cases} \tilde{\omega}(2\eta+|\xi|) - \tilde{\omega}(2\eta-|\xi|) - 2\tilde{\omega}(|\xi|), & |\xi| \geq \frac{\delta(t)}{2(1+L)}, , \\ -\delta(t), & |\xi| \leq \frac{\delta(t)}{2(1+L)}, \end{cases} \tag{3.93}$$

To see this, note that $\tilde{\omega}(t,r) \leq \omega(t,r)$ with equality for $r \geq c\delta(t)^2$ by the definition of $\tilde{\omega}$ (3.87). Thus in the case that $|\xi| \geq \dfrac{\delta(t)}{2(1+L)}, \eta \geq \dfrac{|\xi|}{2}$, we have that $2\eta + |\xi| \geq \dfrac{\delta(t)}{1+L} \geq c\delta(t)^2$ so (3.93) follows immediately. And when $|\xi| \leq \dfrac{\delta(t)}{2(1+L)}$, we then have that

$$\omega(t, 2\eta+|\xi|) - \omega(t, 2\eta-|\xi|) - 2\omega(t,|\xi|) \leq 2|\xi|(1+L) - 2\omega(t,0) \leq -\delta(t). \tag{3.94}$$

Thus we've proven (3.93). Note that as

$$\tilde{\omega}(t, 2\eta+|\xi|) - \tilde{\omega}(t, 2\eta-|\xi|) - 2\tilde{\omega}(t,|\xi|) \leq \tilde{\omega}(t,2\eta) - \tilde{\omega}(t,2\eta) - 2\tilde{\omega}(t,0) \leq -2\delta(t), \tag{3.95}$$

we always have that

$$\omega(t, 2\eta+|\xi|) - \omega(t, 2\eta-|\xi|) - 2\omega(t,|\xi|) \leq -\delta(t). \tag{3.96}$$

Now to prove (3.91), we will consider three cases. First consider small ξ, where $|\xi| \leq \delta(t)$.

Then using (3.96) we get that

$$\int_{\frac{|\xi|}{2}}^{\infty} \frac{\omega(t,2\eta+|\xi|)-\omega(t,2\eta-|\xi|)-2\omega(t,|\xi|)}{(\eta^2+\omega(t,0)^2)^{(2+s)/2}} d\eta \lesssim \int_{\frac{\delta(t)}{2}}^{\delta(t)} \frac{-\delta(t)}{\delta(t)^{2+s}} d\eta \\ \lesssim -\delta(t)^{-s} \leq -1. \tag{3.97}$$

as $\delta \leq 1$.

For the second case, we consider midsize ξ where $\delta(t) \leq |\xi| \leq \frac{1}{2}$. Then using (3.93), we have that

$$\begin{aligned} \int_{\frac{|\xi|}{2}}^{\infty} \frac{\omega(t,2\eta+|\xi|)-\omega(t,2\eta-|\xi|)-2\omega(t,|\xi|)}{(\eta^2+\omega(t,0)^2)^{(2+s)/2}} d\eta &\lesssim \int_{\frac{|\xi|}{2}}^{\frac{1}{2}} \frac{-(2\eta+|\xi|)^{1+s}+(2\eta-|\xi|)^{1+s}+2|\xi|^{1+s}}{\eta^{2+s}} d\eta \\ &= \int_{\frac{1}{2}}^{\frac{1}{2|\xi|}} \frac{-(2\tilde{\eta}+1)^{1+s}+(2\tilde{\eta}-1)^{1+s}+2}{\tilde{\eta}^{2+s}} d\tilde{\eta} \\ &\leq \int_{\frac{1}{2}}^{1} \frac{-(2\tilde{\eta}+1)^{1+s}+(2\tilde{\eta}-1)^{1+s}+2}{\tilde{\eta}^{2+s}} d\tilde{\eta} \\ &\leq \int_{\frac{1}{2}}^{1} \frac{-2^{1+s}+2}{\tilde{\eta}^{2+s}} d\tilde{\eta} \leq \int_{\frac{1}{2}}^{1} \frac{-2\ln(2)s}{\tilde{\eta}^{2+s}} d\tilde{\eta} \\ &\lesssim -s. \end{aligned} \tag{3.98}$$

Finally, suppose that ξ is large so $2 \geq |\xi| \geq \frac{1}{2}$. Then $\omega(t,|\xi|) \geq \frac{1}{4} + L|\xi|$, so

$$\int_{\frac{|\xi|}{2}}^{\infty} \frac{\omega(t,2\eta+|\xi|)-\omega(t,2\eta-|\xi|)-2\omega(t,|\xi|)}{(\eta^2+\omega(t,0)^2)^{(2+s)/2}} d\eta \lesssim \int_{2}^{\infty} \frac{-2\omega(t,|\xi|)}{\eta^{2+s}} d\eta \\ \lesssim -1 \tag{3.99}$$

□

Take $c << \dfrac{s}{1+L}$ and $\delta(t) = \dfrac{T-t}{T}$ in (3.75) for some

$$T \gtrsim \frac{(3(1+L))^{d+s}\Gamma\left(\frac{d+s}{2}\right)}{s^2(1-s)\Gamma\left(\frac{d-2}{2}\right)\Gamma\left(\frac{2+s}{2}\right)}, \quad d \geq 3, \qquad \left(T \gtrsim \frac{(1+L)^{2+s}}{s^2(1-s)}, \quad d=2\right) \tag{3.100}$$

Then combining Lemmas 3.4.2 through 3.5.3, we have under the assumptions of the break through argument at the end of Section 2 that

$$\begin{aligned}
&\partial_t \overline{u}\left(t_0, \frac{\xi}{2}\right) - \partial_t \underline{u}\left(t_0, \frac{-\xi}{2}\right) \\
&\quad = s(1-s)\sqrt{1+\partial_r\omega^2} \int\limits_{\mathbb{R}^{d-1}} \int\limits_{\mathbb{R}} \frac{\mathbb{1}_E^{\pm}\left(\frac{\xi}{2}+z, \overline{u}\left(\frac{\xi}{2}\right)+z_d\right) - \mathbb{1}_E^{\pm}\left(\frac{-\xi}{2}+z, \underline{u}\left(\frac{-\xi}{2}\right)+z_d\right)}{(|z|^2+z_d^2)^{(d+s)/2}} dz dz_d \\
&\quad \leq C(d,s,L) \int\limits_0^{\frac{|\xi|}{2}} \frac{\omega(|\xi|+2\eta)+\omega(|\xi|-2\eta)-2\omega(|\xi|)}{(\eta^2+\omega(0)^2)^{(2+s)/2}} d\eta \\
&\quad\quad + C(d,s,L) \int\limits_{\frac{|\xi|}{2}}^{\infty} \frac{\omega(2\eta+|\xi|)-\omega(2\eta-|\xi|)-2\omega(|\xi|)}{(\eta^2+\omega(0)^2)^{(2+s)/2}} d\eta \\
&\quad \leq \frac{-2}{T} = 2\delta'(t_0) < \partial_t\omega(t,|\xi|),
\end{aligned} \tag{3.101}$$

a contradiction. Thus for any smooth flow $t \to E_t$ with with initial data E_0 satisfying the modulus $1+Lr$ and flat at infinity, we have that E_t has modulus $\omega(t,\cdot)$ for all $t \in [0,T]$. In particular, ∂E_T is a $(1+L)$-Lipschitz graph.

3.6 Viscosity Solutions and Technicalities

Sections 3 through 5 gave the proof of Theorem 3.1.1 in the case that we have a smooth flow. But even for smooth initial data, there's no guarantee a unique, smooth solution of fractional mean curvature flow exists. So instead we work with viscosity solutions. See the appendix or [Imb09, CMP15] for appropriate definitions and details.

Fix a tuple (E_0^-, Γ_0, E_0^+) with $E_0^\pm \subseteq \mathbb{R}^d$ open, $\Gamma_0 \subseteq \mathbb{R}^d$ closed, all are disjoint and $E_0^- \cup \Gamma_0 \cup E_0^+ = \mathbb{R}^d$. We then have that there is a unique viscosity solutions (E_t^-, Γ_t, E_t^+) of (3.4) in the sense of Definition B.0.2 for all times t.

If we knew a priori that $\mathcal{L}^d(\Gamma_t) = 0$ for all times t, then we could repeat the same argument as in the smooth case with only minor alterations. However that is not the case in general, so we must adjust.

Our first goal is to prove Theorem 3.1.1 under the assumptions that

$$\begin{cases} 1).E_0^\pm \text{ have modulus } (1-\eta) + Lr, \\ 2).0 \in \Gamma_0 \\ 3).\Gamma_0 \setminus (B_M^{d-1} \times \mathbb{R}) = \{(x,0) : x \in \mathbb{R}^{d-1}, |x| \geq M\}, \end{cases} \tag{3.102}$$

for some $0 < \eta << 1$ and $1 << M < \infty$. We will later be able to remove the last condition and let $\eta \to 0$. But for now these are convenient assumptions.

Let $U_0 : \mathbb{R}^d \to \mathbb{R}$ be the signed distance function

$$U_0(X) = \begin{cases} \min\{d(X,\Gamma_0), 1\}, & X \in E_0^+, \\ \max\{-d(X,\Gamma_0), -1\}, & X \in E_0^- \end{cases}, \tag{3.103}$$

and let $U(t,X)$ be the unique viscosity solution to the level set equation (B.5) for the initial data U_0. Then for any $\gamma \in [-1,1]$, we can define the tuple $(E_t^{\gamma-}, \Gamma_t^\gamma, E_t^{\gamma+})$ by

$$E_t^{\gamma-} = \{U(t,\cdot) < \gamma\}, \quad \Gamma_t^\gamma = \{U(t,\cdot) = \gamma\}, \quad E_t^{\gamma+} = \{U(t,\cdot) > \gamma\}. \tag{3.104}$$

Note that $(E_t^{0-}, \Gamma_t^0, E_t^{0+}) = (E_t^-, \Gamma_t, E_t^+)$ is our original viscosity solution triple.

Lemma 3.6.1. *Let (E_0^-, Γ_0, E_0^+) satisfy (3.102) and $(E_t^{\gamma-}, \Gamma_t^\gamma, E_t^{\gamma+})$ be as in (3.104). Then $E_t^{\gamma\pm}$ have modulus of continuity $1 + Lr$ for all times t and $|\gamma| < \dfrac{\eta}{\sqrt{1+L^2}}$.*

Proof. Let $A = \{(x, x_d) : x_d \leq 1 - \eta + L|x|\}$ and $B = \{(y, y_d) : y_d \geq 1 + L|y|\}$. Direct calculation then gives that

$$d(A, B) = \frac{\eta}{\sqrt{1 + L^2}}. \tag{3.105}$$

Using the set formulation of a modulus of continuity (Definition 3.2.3), checking cases then gives you that $E_0^{\gamma\pm}$ have modulus $1 + Lr$ for $|\gamma| \leq \frac{\eta}{\sqrt{1+L^2}}$. Proposition 3.2.3 then implies that this remains true for all $t \geq 0$. □

Proposition 3.6.1. *Let* $(E_t^{\gamma-}, \Gamma_t^\gamma, E_t^{\gamma+})$ *be as in* (3.104). *Then for almost every* $\gamma \in [-1, 1]$,

$$\mathcal{L}^d(\Gamma_t^\gamma) = 0 \textit{ for almost every time } t \geq 0. \tag{3.106}$$

Proof. This follows easily from the fact that there are at most countably many $\gamma \in \mathbb{R}$ such that

$$\mathcal{L}^{d+1}(\{U(\cdot, \cdot) = \gamma\}) = \mathcal{L}^{d+1}\left(\bigcup_{t\geq 0}\{t\} \times \Gamma_t^\gamma\right) \neq 0 \tag{3.107}$$

□

As a slight abuse of notation, we define $\overline{u}^\gamma, \underline{u}^\gamma : [0, \infty) \times \mathbb{R}^{d-1} \to \mathbb{R}$ by

$$\overline{u}^\gamma(t, x) = \max\{x_d | (x, x_d) \in \Gamma_t^\gamma\}, \quad \underline{u}^\gamma(t, x) = \min\{x_d | (x, x_d) \in \Gamma_t^\gamma\}. \tag{3.108}$$

Our goal now is to show that for any γ such that Lemma 3.6.1 and (3.106) hold, Γ_t^γ becomes a $(1 + L)$-Lipschitz graph in finite time. Explicitly,

Lemma 3.6.2. *Assume* (E_0^-, Γ_0, E_0^+) *satisfy* (3.102). *Then for any* $|\gamma| < \frac{\eta}{\sqrt{1+L^2}}$ *with*

$$\mathcal{L}^d(\Gamma_t^\gamma) = 0 \textit{ for almost every time } t \geq 0, \tag{3.109}$$

Γ_t^γ has modulus $\omega\left(\frac{t}{2}, \cdot\right)$ for times $t \in [0, 2T]$. That is,

$$\overline{u}^\gamma(t,x) - \underline{u}^\gamma(t,y) \leq \omega\left(\frac{t}{2}, |x-y|\right), \tag{3.110}$$

for all $(t,x,y) \in [0,2T] \times \mathbb{R}^{d-1} \times \mathbb{R}^{d-1}$ where ω is as in (3.75) and T is as in (3.100).

Remark. *For simplicity we prove that $E_t^{\gamma\pm}$ have modulus $\omega(\frac{t}{2}, \cdot)$, but similar arguments can be made to show that $E_t^{\gamma\pm}$ has modulus $\omega((1-\epsilon)t, \cdot)$ for any $\epsilon > 0$, and hence modulus $\omega(t, \cdot)$ by continuity.*

Note that by continuity, it suffices to prove that Γ_t^γ has modulus $\omega(\frac{t}{2}, \cdot)$ at times $t \leq t_0$, for some arbitrary $t_0 < 2T$. The key advantage to proving this is that

$$\inf\left\{\omega\left(\frac{t}{2}, r\right) : r \geq 0, 0 \leq t \leq t_0\right\} = \omega\left(\frac{t_0}{2}, 0\right) \geq \delta\left(\frac{t_0}{2}\right) > 0. \tag{3.111}$$

The ω bound from below and Proposition 3.6.2 then allows us to rule out any crossing points at infinity.

Proposition 3.6.2. *Suppose that our initial data $U_0 : \mathbb{R}^d \to [-1,1]$ is as in (3.103) for some tuple (E_0^-, Γ_0, E_0^+) satisfying (3.102) Then for any $\delta > 0$ and $t_0 < \infty$, there exists an $r = r(M, \delta, t_0) < \infty$ such that*

$$|U(t,x,x_d) - U_0(x,x_d)| < \delta/2, \qquad |x| \geq r(M,\delta,t_0),\ t \in [0,t_0]. \tag{3.112}$$

Proof. Let $\phi \in C_c^\infty(\mathbb{R})$ with $\phi \geq 0$, $\phi \equiv 1$ on $[-1/2, 1/2]$, and $\text{supp}(\phi) \subseteq [-1,1]$. Let

$$V_r(t,x,x_d) = \frac{\delta}{2t_0}t + 2\phi\left(\frac{|x|}{r}\right) + \min\{(x_d+1)_+ - 1, 1\} \tag{3.113}$$

Then for $r \geq 2M$, we have that $V_r(0,x,z) \geq U_0(x,z)$. Since $\phi \in C_c^\infty(\mathbb{R})$, we have that if we take r sufficiently large depending on $\dfrac{\delta}{t_0}$, then V_r will be a supersolution to (B.3). Thus by the comparison principle Theorem B.0.1, for any $t \in [0, t_0]$ and $|x| \geq r$ and $x_d \in \mathbb{R}$

$$U(t,X) \leq V_r(t,X) = \frac{\delta}{2t_0}t + \min\{(x_d+1)_+ - 1, 1\} \leq \frac{\delta}{2} + \min\{(x_d+1)_+ - 1, 1\} = \frac{\delta}{2} + U_0(X). \tag{3.114}$$

A symmetric proof works for the opposite inequality. □

The fact that we are restricting ourselves to times $0 \leq t \leq t_0 < 2T$ and Proposition 3.6.2 effectively allows us to redo the breakthrough argument of section 2. However, when we try to redo the estimates from section 3, we run into a problem because our "boundary" Γ_t^γ might have positive measure.

Again, if we knew in fact that $\mathcal{L}^d(\Gamma_t^\gamma) = 0$ for all times t, then the argument from the smooth case would work with minor alterations. The main problem when we only have $\mathcal{L}^d(\Gamma_t^\gamma) = 0$ for a.e. time t is that at the breakthrough argument relies on having an open interval of times where we can run it. Else at the crossing time t_0 we have no guarantee that $\mathcal{L}^d(\Gamma_{t_0}^\gamma) = 0$.

Thus in order to deal with this, we're going need to adjust the modulus estimates in Section 3 to work when we only have that $\mathcal{L}^d(\Gamma_t^\gamma)$ is *small*. Luckily, that will be true on an open interval of times.

Proposition 3.6.3. *Let $\gamma \in \mathbb{R}$ and $R > 0$. Then the function*

$$t \to \mathcal{L}^d(\Gamma_t^\gamma \cap B_R^d) \tag{3.115}$$

is upper semicontinuous. In particular, for any $\epsilon > 0$ the set of times

$$\{t \in (0,T) : \mathcal{L}^d(\Gamma_t^\gamma \cap B_R^d) < \epsilon\} \tag{3.116}$$

is open.

Proof. Let $\epsilon > 0$ and $t \in [0, \infty)$. Then it suffices to show that there is some $\delta > 0$ such that if $|t-t'|<\delta$ then

$$\mathcal{L}^d(\Gamma^\gamma_{t'}\cap B^d_R) < \mathcal{L}^d(\Gamma^\gamma_t\cap B^d_R)+\epsilon. \tag{3.117}$$

Let $N_r(\Gamma^\gamma_t)=\{X\in\mathbb{R}^d : d(X,\Gamma^\gamma_t)<r\}$. Then for r sufficiently small,

$$\mathcal{L}^d(N_r(\Gamma^\gamma_t)\cap B^d_R) < \mathcal{L}^d(\Gamma^\gamma_t\cap B^d_R)+\epsilon. \tag{3.118}$$

Let $K=\{t\}\times(\overline{B^d_R}\setminus N_r(\Gamma^\gamma_t))$. Then K is compact with $K\cap U^{-1}(\gamma)=\emptyset$. Hence, $d(K,U^{-1}(\gamma))=\delta>0$ for some δ. In particular, $|t-t'|<\delta$ implies that $(\overline{B^d_R}\setminus N_r(\Gamma^\gamma_t))\cap\Gamma^\gamma_{t'}=\emptyset$. Thus

$$\mathcal{L}^d(\Gamma^\gamma_{t'}\cap B^d_R)\le\mathcal{L}^d(N_r(\Gamma^\gamma_t)\cap B^d_R) < \mathcal{L}^d(\Gamma^\gamma_t\cap B^d_R)+\epsilon. \tag{3.119}$$

□

Now with Proposition 3.6.3, we first make our choice of $R=R(t_0)$ as

$$R(t_0)=r\left(M,\delta\left(\frac{t_0}{2}\right),t_0\right)+3+LM+\left(\frac{1}{8(2+L)(1-s)T}\right)^{-1/s}. \tag{3.120}$$

where $r\left(M,\delta\left(\frac{t_0}{2}\right),t_0\right)$ is as in Proposition 3.6.2. With this choice of $R(t_0)$, by Proposition 3.6.3 we have the set of times

$$\mathcal{T}(t_0)=\left\{t\in(0,t_0):\mathcal{L}^d(\Gamma^\gamma_t\cap B^d_{R(t_0)})<\frac{\epsilon(t_0)^{d+s}}{8(2+L)s(1-s)T}\right\}=\bigcup_i(a_i,b_i), \tag{3.121}$$

is an open set of full measure on $(0,t_0)$, where

$$\epsilon(t_0)=\left(\frac{c\delta(\frac{t_0}{2})^2}{8(2+L)sT}\right)^{\frac{1}{(1-s)}}. \tag{3.122}$$

The breakthrough argument of section 2 is designed to work on an open interval of times. However, a finite union of open intervals works just as well. For $N\in\mathbb{N}$, define

$\alpha_N : [0, t_0] \to [0, t_0]$ by $\alpha_N(0) = 0$ and

$$\alpha'_N(t) = \begin{cases} \frac{1}{2}, & t \in (a_i, b_i) \text{ for } i = 1, \ldots, N, \\ 0, & \text{otherwise} \end{cases}. \tag{3.123}$$

Note that $\lim_{N\to\infty} \alpha_N(t) = \dfrac{t}{2}$ for $t \in [0, t_0]$ as $\mathcal{T}(t_0)$ has full measure. Thus it would suffice to show that Γ_t^γ has modulus $\omega(\alpha_N(t), \cdot)$ for every $t \in [0, t_0]$ to prove Lemma 3.6.2.

With all of this set up, we can now define our test function

$$\Phi : [0, t_0] \times \mathbb{R}^{d-1} \times \mathbb{R} \times \mathbb{R}^{d-1} \times \mathbb{R} \to \mathbb{R}$$

by

$$\Phi(t, x, x_d, y, y_d) = \left(\omega(\alpha_N(t), |x - y|) - (x_d - y_d) + \frac{1}{2}\right)_+ + \frac{1}{2}. \tag{3.124}$$

The function Φ encodes that Γ_t^γ has the modulus $\omega(\alpha_N(t), \cdot)$, in the sense that

Lemma 3.6.3. *Γ_t^γ has modulus $\omega(\alpha_N(t), \cdot)$ if and only if for all $X, Y \in \mathbb{R}^d$,*

$$\mathbb{1}_{E_t^{\gamma-} \cup \Gamma_t^\gamma}(X) - \mathbb{1}_{E_t^{\gamma-}}(Y) \le \Phi(t, X, Y). \tag{3.125}$$

Similarly, Γ_t^γ only has the modulus strictly if the inequality above is strict.

Proof. We simply show the first statement, as the second follows similarly.

One direction is straightforward, as if we take $X = (x, \overline{u}^\gamma(t, x))$ and $Y = (y, \underline{u}^\gamma(t, y))$ then

$$\begin{gathered} 1 = \mathbb{1}_{E_t^{\gamma-} \cup \Gamma_t^\gamma}(X) - \mathbb{1}_{E_t^{\gamma-}}(Y) \le \Phi(t, X, Y) = \omega(\alpha_N(t), |x - y|) - (\overline{u}^\gamma(t, x) - \underline{u}^\gamma(t, y)) + 1, \\ \Rightarrow \overline{u}^\gamma(t, x) - \underline{u}^\gamma(t, y) \le \omega(\alpha_N(t), |x - y|). \end{gathered} \tag{3.126}$$

As for the converse, suppose that $\overline{u}^\gamma(t, x) - \underline{u}^\gamma(t, y) \le \omega(\alpha_N(t), |x - y|)$ for all $x, y \in \mathbb{R}^{d-1}$. Then since indicator functions can only take the values 0 or 1 and $\Phi > 0$, it follows immediately that if $\mathbb{1}_{E_t^{\gamma-}\cup\Gamma_t^{\gamma}}(X) - \mathbb{1}_{E_t^{\gamma-}}(Y) \neq 1$ then

$$\mathbb{1}_{E_t^{\gamma-}\cup\Gamma_t^{\gamma}}(X) - \mathbb{1}_{E_t^{\gamma-}}(Y) \leq 0 < \Phi(t, X, Y). \tag{3.127}$$

In the case that $\mathbb{1}_{E_t^{\gamma-}\cup\Gamma_t^{\gamma}}(X) - \mathbb{1}_{E_t^{\gamma-}}(Y) = 1$, we have that $X \in E_t^{\gamma-} \cup \Gamma_t^{\gamma}$ and $Y \notin E_t^{\gamma-}$. Using the monotonicity of Φ in the x_d, y_d variables we have that

$$\begin{aligned}\mathbb{1}_{E_t^{\gamma-}\cup\Gamma_t^{\gamma}}(X) - \mathbb{1}_{E_t^{\gamma-}}(Y) = 1 &\leq 1 + \omega(\alpha_N(t), |x-y|) - (\overline{u}^{\gamma}(t,x) - \underline{u}^{\gamma}(t,y)) \\ &\leq \omega(\alpha_N(t), |x-y|) - (x_d - y_d) + 1 = \Phi(t, X, Y).\end{aligned} \tag{3.128}$$

□

Thus in order to prove Lemma 3.6.2 it suffices to show that

$$\mathbb{1}_{E_t^{\gamma-}\cup\Gamma_t^{\gamma}}(X) - \mathbb{1}_{E_t^{\gamma-}}(Y) \leq \Phi(t, X, Y), \quad \text{for all } (t, X, Y) \in [0, t_0] \times \mathbb{R}^d \times \mathbb{R}^d. \tag{3.129}$$

With the help of our assumptions (3.102), Proposition 3.6.2 and the definition (3.75) of ω, we can now formally justify a large portion of the breakthrough argument by showing that

Lemma 3.6.4. *Let Φ be as in* (3.124)*. Then for* $|\gamma| < \dfrac{\eta}{\sqrt{1+L^2}}$,

$$\begin{cases} \Phi(0, X, Y) > \mathbb{1}_{E_0^{\gamma-}\cup\Gamma_0^{\gamma}}(X) - \mathbb{1}_{E_0^{\gamma-}}(Y), & X, Y \in \mathbb{R}^d, \\ \Phi(t, X, Y) > \mathbb{1}_{E_t^{\gamma-}\cup\Gamma_t^{\gamma}}(X) - \mathbb{1}_{E_t^{\gamma-}}(Y), & t \in [0, t_0], x, y \in \mathbb{R}^{d-1}, |x_d| \text{ or } |y_d| > 1 + LM, \\ \Phi(t, X, Y) > \mathbb{1}_{E_t^{\gamma-}\cup\Gamma_t^{\gamma}}(X) - \mathbb{1}_{E_t^{\gamma-}}(Y), & t \in [0, t_0], |x| \text{ or } |y| > r(M, \delta(t_0/2), t_0) + 2. \end{cases} \tag{3.130}$$

Proof. When $t = 0$, by Lemma 3.6.1 and the definition (3.75) we have that $E_t^{\gamma\pm}$ strictly has the modulus $\omega(0, r) > 1 + Lr$. Thus by Lemma 3.6.3

$$\Phi(0, X, Y) > \mathbb{1}_{E_0^{\gamma-}\cup\Gamma_0^{\gamma}}(X) - \mathbb{1}_{E_t^{\gamma-}}(Y). \tag{3.131}$$

Note that our assumptions (3.102) on (E_0^-, Γ_0, E_0^+) imply that

$$\{\pm x_d \geq 1 - \eta + LM\} \subseteq E_0^{\pm} \tag{3.132}$$

Thus for $|\gamma| < \dfrac{\eta}{\sqrt{1+L^2}}$, we have that

$$\{\pm x_d \geq 1 + LM\} \subseteq E_0^{\gamma\pm}. \tag{3.133}$$

By the comparison principle, this remains true for all later times. So if $x_d > 1 + LM$ or $y_d < -1 - LM$, then

$$\Phi(t, x, x_d, y, y_d) > 0 \geq \mathbb{1}_{E_t^{\gamma-} \cup \Gamma_t^{\gamma}}(x, x_d) - \mathbb{1}_{E_t^{\gamma-}}(y, y_d). \tag{3.134}$$

On the other hand if $x_d < -1 - LM$ and $y_d \geq -1 - LM$, then $x_d \leq y_d$ and hence

$$\Phi(t, x, x_d, y, y_d) > \omega(\alpha_N(t), |x-y|) + 1 > \mathbb{1}_{E_t^{\gamma-} \cup \Gamma_t^{\gamma}}(x, x_d) - \mathbb{1}_{E_t^{\gamma-}}(y, y_d). \tag{3.135}$$

The same of course holds if $y_d > 1 + LM$ and $x_d \leq 1 + LM$.

Finally, suppose that $|x| > r(M, \delta(t_0/2), t_0) + 2$. Then by Proposition 3.6.2, we have that for any $t \in [0, t_0]$ and $x_d \in \mathbb{R}$ that

$$|U(t, x, x_d) - U_0(x, x_d)| < \frac{\delta(t_0/2)}{2}. \tag{3.136}$$

In particular,

$$\mathbb{1}_{E_t^{\gamma-} \cup \Gamma_t^{\gamma}}(x, x_d) = 1 \quad \Rightarrow \quad x_d < \gamma + \frac{\delta(t_0/2)}{2}. \tag{3.137}$$

Let $Y = (y, y_d) \in \mathbb{R}^d$. Then either $|y| > r(M, \delta(t_0/2), t_0)$ or $|x - y| > 2$. If $|y| > r(M, \delta(t_0/2), t_0)$, then by the same argument $\mathbb{1}_{E_t^{\gamma-}}(y, y_d) = 0$ implies $y_d > \gamma - \dfrac{\delta(t_0/2)}{2}$.

Hence,

$$\mathbb{1}_{E_t^{\gamma-}\cup\Gamma_t^{\gamma}}(x,x_d) - \mathbb{1}_{E_t^{\gamma-}}(y,y_d) = 1 < 1 + \delta(t_0/2) - (x_d - y_d) < \Phi(t,x,x_d,y,y_d). \quad (3.138)$$

On the other hand, if $|x-y| > 2$, then $\omega(\alpha_N(t), |x-y|) > 1 + L|x-y|$. Thus if $x_d - y_d \leq 1 + L|x-y|$, then by comparison principle and the definition of ω

$$\mathbb{1}_{E_t^{\gamma-}\cup\Gamma_t^{\gamma}}(x,x_d) - \mathbb{1}_{E_t^{\gamma-}}(y,y_d) \leq 1 < (\omega(\alpha_N(t), |x-y|) - (x_d - y_d)) + 1 \leq \Phi(t,x,x_d,y,y_d). \quad (3.139)$$

And if $x_d - y_d > 1 + L|x-y|$, then by Lemma 3.6.1

$$\mathbb{1}_{E_t^{\gamma-}\cup\Gamma_t^{\gamma}}(x,x_d) - \mathbb{1}_{E_t^{\gamma-}}(y,y_d) = 0 < \Phi(t,x,x_d,y,y_d). \quad (3.140)$$

Thus $|x| > r(M, \delta(t_0/2), t_0) + 2$ implies that

$$\mathbb{1}_{E_t^{\gamma-}\cup\Gamma_t^{\gamma}}(x,x_d) - \mathbb{1}_{E_t^{\gamma-}}(y,y_d) < \Phi(t,x,x_d,y,y_d). \quad (3.141)$$

A symmetric argument clearly works in the case that $|y| > r(M, \delta(t_0/2), t_0) + 2$. □

Combining Lemmas 3.6.3 and 3.6.4, we now just have to show that

$$\begin{aligned} &t \in [0,t_0],\ |x|,|y| \leq r(M,\delta(t_0/2),t_0),\ |x_d|,|y_d| \leq 1 + LM \\ &\Rightarrow \quad \Phi(t,x,x_d,y,y_d) \geq \mathbb{1}_{E_t^{\gamma-}\cup\Gamma_t^{\gamma}}(x,x_d) - \mathbb{1}_{E_t^{\gamma-}}(y,y_d). \end{aligned} \quad (3.142)$$

To do this, we need to use our equations. By Theorem B.0.2 that $\mathbb{1}_{E_t^{\gamma-}\cup\Gamma_t^{\gamma}}(X)$ is a viscosity subsolution to (B.1) and that $\mathbb{1}_{E_t^{\gamma-}}(Y)$ is a viscosity supersolution to (B.3). By standard viscosity solution arguments, it then follows that

Lemma 3.6.5. *The function* $\mathbb{1}_{E_t^{\gamma -}\cup\Gamma_t^\gamma}(X) - \mathbb{1}_{E_t^{\gamma -}}(Y)$ *is a viscosity subsolution to*

$$\begin{aligned}\partial_t V(t,X,Y) \leq & - H_s(X, \{V(t,\cdot,Y) \geq V(t,X,Y)\})|\nabla_X V(t,X,Y)| \\ & + H_s(Y, \{V(t,X,\cdot) > V(t,X,Y)\})|\nabla_Y V(t,X,Y)|.\end{aligned} \tag{3.143}$$

In particular, if Ψ *crosses* $\mathbb{1}_{E_t^{\gamma -}\cup\Gamma_t^\gamma} - \mathbb{1}_{E_t^{\gamma -}}$ *at* (t,X,Y) *with* $\nabla_X\Psi, \nabla_Y\Psi \neq 0$*, then for any* $\epsilon > 0$

$$\begin{aligned}&\partial_t\Psi(t,X,Y) \leq \\ &\quad s(1-s)|\nabla_X\Psi|\left(\int\limits_{|Z|>\epsilon} \frac{\mathbb{1}^{\pm}_{E_t^{\gamma -}\cup\Gamma_t^\gamma}(X+Z)}{|Z|^{d+s}}dZ + P.V.\int\limits_{|Z|<\epsilon} \frac{\mathbb{1}^{\pm}_{\{\Psi(t,\cdot,Y)\geq\Psi(t,X,Y)\}}(X+Z)}{|Z|^{d+s}}dZ\right) \\ &\quad - s(1-s)|\nabla_Y\Psi|\left(\int\limits_{|Z|>\epsilon} \frac{\mathbb{1}^{\pm}_{E_t^{\gamma -}}(Y+Z)}{|Z|^{d+s}}dZ + P.V.\int\limits_{|Z|<\epsilon} \frac{\mathbb{1}^{\pm}_{\{\Psi(t,X,\cdot)>\Psi(t,X,Y)\}}(Y+Z)}{|Z|^{d+s}}dZ\right).\end{aligned} \tag{3.144}$$

We leave the proof of Lemma 3.6.5 to the appendix.

Our plan now is to show that (3.154) can never hold.

3.6.1 Proof of Lemma 3.6.2

We now prove that for any initial data (E_0^-, Γ_0, E_0^+) satisfying (3.102) and any $\gamma \in \left(\frac{-\eta}{\sqrt{1+L^2}}, \frac{\eta}{\sqrt{1+L^2}}\right)$ with

$$\mathcal{L}^d(\Gamma_t^\gamma) = 0 \text{ for almost every time } t \geq 0, \tag{3.145}$$

that Γ_t^γ has modulus $\omega(\alpha_N(t),\cdot)$ for times $t \in [0, t_0]$ where $t_0 < 2T$ is arbitrary. By taking $N \to \infty$ and then $t_0 \to 2T$, it then follows that Γ_{2T}^γ will have modulus $\omega(T,\cdot)$ and thus be $(1+L)$-Lipschitz.

By Lemma 3.6.3, it suffices to show that

$$\begin{aligned}\mathbb{1}_{E_t^{\gamma-}\cup\Gamma_t^\gamma}(x,x_d) - \mathbb{1}_{E_t^{\gamma-}}(y,y_d) &\le \Phi(t,x,x_d,y,y_d) \\ &= \left(\omega(\alpha_N(t),|x-y|) - (x_d-y_d) + \frac{1}{2}\right)_+ + \frac{1}{2}\end{aligned} \tag{3.146}$$

for all $(t,X,Y) \in [0,t_0] \times \mathbb{R}^d \times \mathbb{R}^d$. By Lemma 3.6.4, we need only consider the case that

$$t \in [0,t_0], \quad |x|,|y| \le r(M,\delta(t_0/2),t_0) + 2, \quad |x_d|,|y_d| \le 1 + LM. \tag{3.147}$$

Recall that $\alpha_N'(t) \neq 0$ only for $t \in \bigcup_{i=1}^N (a_i,b_i)$, where

$$\bigcup_i (a_i,b_i) = \mathcal{T}(t_0) = \left\{ t \in (0,t_0) : \mathcal{L}^d(\Gamma_t^\gamma \cap B^d_{R(t_0)}) < \frac{\epsilon(t_0)^{d+s}}{8(2+L)s(1-s)T} \right\}, \tag{3.148}$$

and

$$\epsilon(t_0) := \left(\frac{c\delta(t_0/2)^2}{8(2+L)sT}\right)^{1/(1-s)} \quad R(t_0) := r\left(M,\delta(t_0/2),t_0\right) + 3 + LM + \left(\frac{1}{8(2+L)(1-s)T}\right)^{-1/s} \tag{3.149}$$

Without loss of generality, by reindexing we may assume that $0 \le a_1 < b_1 \le a_2 < \ldots \le b_N \le t_0$. It follows by the comparison principle and Lemma 3.6.4 that

$$\begin{aligned} &\mathbb{1}_{E_0^{\gamma-}\cup\Gamma_0^\gamma}(X) - \mathbb{1}_{E_0^{\gamma-}}(Y) < \Phi(0,X,Y) \\ \Rightarrow \quad &\mathbb{1}_{E_t^{\gamma-}\cup\Gamma_t^\gamma}(X) - \mathbb{1}_{E_t^{\gamma-}}(Y) < \Phi(0,X,Y) = \Phi(t,X,Y), \quad t \in [0,a_1]. \end{aligned} \tag{3.150}$$

By induction, suppose that $\mathbb{1}_{E_t^{\gamma-}\cup\Gamma_t^\gamma}(X) - \mathbb{1}_{E_t^{\gamma-}}(Y) < \Phi(t,X,Y)$ for $t \in [0,a_i]$. As $\mathbb{1}_{E_t^{\gamma-}\cup\Gamma_t^\gamma}(X) - \mathbb{1}_{E_t^{\gamma-}}(Y)$ is upper semicontinuous and Φ is continuous, by Lemma 3.6.4 we get that

$$\mathbb{1}_{E_t^{\gamma-}\cup\Gamma_t^\gamma}(X) - \mathbb{1}_{E_t^{\gamma-}}(Y) < \Phi(t,X,Y) \text{ for } t \in (a_i, a_i+\epsilon), \tag{3.151}$$

for some $\epsilon > 0$. Thus there are two cases. If $\mathbb{1}_{E_t^{\gamma-}\cup\Gamma_t^{\gamma}}(X) - \mathbb{1}_{E_t^{\gamma-}}(Y) < \Phi(t, X, Y)$ for $t \in (a_i, b_i]$, then by the same argument as above

$$\begin{aligned} &\mathbb{1}_{E_{b_i}^{\gamma-}\cup\Gamma_{b_i}^{\gamma}}(X) - \mathbb{1}_{E_{b_i}^{\gamma-}}(Y) < \Phi(b_i, X, Y) \\ \Rightarrow \quad &\mathbb{1}_{E_t^{\gamma-}\cup\Gamma_t^{\gamma}}(X) - \mathbb{1}_{E_t^{\gamma-}}(Y) < \Phi(b_i, X, Y) = \Phi(t, X, Y), \quad t \in [b_i, a_{i+1}]. \end{aligned} \tag{3.152}$$

Else, we must that that Φ crosses $\mathbb{1}_{E_t^{\gamma-}\cup\Gamma_t^{\gamma}} - \mathbb{1}_{E_t^{\gamma-}}$ for some $t \in (a_i, b_i]$. If the crossing point was at time b_i, then by replacing $\Phi(t)$ with $\Phi(t-\epsilon) > \Phi(t)$ for some arbitrary $\epsilon > 0$, we can regain the strict inequality. Following the rest of the argument, we then get that at time t_0, $\Gamma_{t_0}^{\gamma}$ has modulus $\omega(\alpha_N(t_0 - N\epsilon), \cdot)$ for an arbitrary $\epsilon > 0$. Hence, $\Gamma_{t_0}^{\gamma}$ has modulus $\omega(\alpha_N(t_0), \cdot)$.

So without loss of generality, we may assume that any crossing point happens in the open interval $(a_i, b_i) \subset \mathcal{T}(t_0)$. We will show that this is not possible, proving that $\mathbb{1}_{E_t^{\gamma-}\cup\Gamma_t^{\gamma}} - \mathbb{1}_{E_t^{\gamma-}} < \Phi$ for all times $t \in [0, t_0]$ and thus completing the proof of Lemma 3.6.2 by Lemma 3.6.3.

So now assume that Φ crosses $\mathbb{1}_{E_t^{\gamma-}\cup\Gamma_t^{\gamma}} - \mathbb{1}_{E_t^{\gamma-}}$ for some time $t \in (a_i, b_i)$. By Lemma 3.6.4 we can thus assume our crossing point (t, x, x_d, y, y_d) satisfies

$$t \in (a_i, b_i), \quad |x|, |y| \leq r(M, \delta(t_0/2), t_0) + 2, \quad |x_d|, |y_d| \leq 1 + LM. \tag{3.153}$$

At the crossing point, we necessarily have that $\mathbb{1}_{E_t^{\gamma-}\cup\Gamma_t^{\gamma}}(x, x_d) - \mathbb{1}_{E_t^{\gamma-}}(y, y_d) = 1$. The strict monotonicity of Φ in x_d, y_d then implies that $x_d = \overline{u}^{\gamma}(t, x)$ and $y_d = \underline{u}^{\gamma}(t, y)$.

By the definition of Φ (3.124) we have that Φ is C^1 in time and $C^{1,1}$ in X, Y on a neighborhood of the set $\Phi^{-1}(1) \subseteq [0, t_0] \times \mathbb{R}^d \times \mathbb{R}^d$. Thus Φ is a valid test function for our purposes. Taking $X = (x, \overline{u}^{\gamma}(t, x))$ and $Y = (y, \underline{u}^{\gamma}(t, y))$, applying Lemma 3.6.5 gives us that

$$\partial_t\Phi(t,X,Y) \le$$

$$s(1-s)|\nabla_X\Phi|\left(\int\limits_{|Z|>\epsilon(t_0)} \frac{\mathbb{1}^{\pm}_{E_t^{\gamma-}\cup\Gamma_t^{\gamma}}(X+Z)}{|Z|^{d+s}}dZ + \int\limits_{|Z|<\epsilon(t_0)} \frac{\mathbb{1}^{\pm}_{\{\Phi(t,\cdot,Y)\ge\Phi(t,X,Y)\}}(X+Z)}{|Z|^{d+s}}dZ\right)$$
$$- s(1-s)|\nabla_Y\Phi|\left(\int\limits_{|Z|>\epsilon(t_0)} \frac{\mathbb{1}^{\pm}_{E_t^{\gamma-}}(Y+Z)}{|Z|^{d+s}}dZ + \int\limits_{|Z|<\epsilon(t_0)} \frac{\mathbb{1}^{\pm}_{\{\Phi(t,X,\cdot)>\Phi(t,X,Y)\}}(Y+Z)}{|Z|^{d+s}}dZ\right) \tag{3.154}$$

We will show that (3.154) is not possible, thus ruling out any crossing points.

From the definition of Φ (3.124), we can immediately calculate that

$$\partial_t\Phi(t,X,Y) = \frac{1}{2}\partial_t\omega(\alpha_N(t),|x-y|) > \delta'(t/2) = \frac{-1}{T}, \tag{3.155}$$

and

$$\begin{aligned} &|\nabla_X\Phi(t,X,Y)| = |\nabla_Y\Phi(t,X,Y)| = \sqrt{1+\partial_r\omega(\alpha_N(t),|x-y|)^2}, \\ \Rightarrow \quad &1 \le |\nabla_X\Phi(t,X,Y)| = |\nabla_Y\Phi(t,X,Y)| \le 2+L \end{aligned} \tag{3.156}$$

As $||\Phi(t,\cdot,Y)||_{\dot{W}^{2,\infty}} \le ||\omega(\alpha_N(t),\cdot)||_{\dot{W}^{2,\infty}} \le (c\delta(t_0/2)^2)^{-1}$, we have that

$$\begin{aligned} &\left|P.V.\int\limits_{|Z|<\epsilon(t_0)} \frac{\mathbb{1}^{\pm}_{\{\Phi(t,\cdot,Y)\ge\Phi(t,X,Y)\}}(X+Z)}{|Z|^{d+s}}dZ\right| \\ &\qquad \le \int\limits_{|z|<\epsilon(t_0)} \int\limits_{-(c\delta(t_0/2)^2)^{-1}|z|^2/2}^{(c\delta(t_0/2)^2)^{-1}|z|^2/2} \frac{1}{(|z|^2+z_d^2)^{(d+s)/2}}dz_d dz \\ &\qquad \le \int\limits_{|z|<\epsilon(t_0)} \frac{(c\delta(t_0/2)^2)^{-1}}{|z|^{d+s-2}}dz \le \frac{\epsilon(t_0)^{1-s}}{(1-s)c\delta(t_0/2)^2} \\ &\qquad = \frac{1}{8(2+L)s(1-s)T}. \end{aligned} \tag{3.157}$$

The same holds for the Y integral as well, so combining (3.157) with (3.156) gives

$$s(1-s)|\nabla_X \Phi(t,X,Y)| \left| P.V. \int_{|Z|<\epsilon(t_0)} \frac{\mathbb{1}^{\pm}_{\{\Phi(t,\cdot,Y)\geq \Phi(t,X,Y)\}}(X+Z)}{|Z|^{d+s}} dZ \right| + s(1-s)|\nabla_Y \Phi(t,X,Y)| \left| P.V. \int_{|Z|<\epsilon(t_0)} \frac{\mathbb{1}^{\pm}_{\{\Phi(t,X,\cdot)>\Phi(t,X,Y)\}}(Y+Z)}{|Z|^{d+s}} dZ \right| \leq \frac{1}{4T}. \tag{3.158}$$

Plugging in (3.158) and (3.156) into (3.154) we get that

$$\frac{\partial_t \Phi(t,X,Y) - \frac{1}{4T}}{s(1-s)\sqrt{1+\partial_r \omega(\alpha_N(t),|x-y|)^2}} \leq \int_{|Z|>\epsilon(t_0)} \frac{\mathbb{1}^{\pm}_{E_t^{\gamma-}\cup \Gamma_t^{\gamma}}(X+Z) - \mathbb{1}^{\pm}_{E_t^{\gamma-}}(Y+Z)}{|Z|^{d+s}} dZ \tag{3.159}$$

As

$$\mathbb{1}^{\pm}_{E_t^{\gamma-}\cup \Gamma_t^{\gamma}}(X+Z) - \mathbb{1}^{\pm}_{E_t^{\gamma-}}(Y+Z) = 2\mathbb{1}_{\Gamma_t^{\gamma}}(X+Z) + \mathbb{1}^{\pm}_{E_t^{\gamma-}}(X+Z) - \mathbb{1}^{\pm}_{E_t^{\gamma-}}(Y+Z) \tag{3.160}$$

we then have that

$$\partial_t \Phi(t,X,Y) \leq \frac{1}{4T} + 2s(1-s)\sqrt{1+\partial_r \omega(\alpha_N(t),|x-y|)^2} \int_{|Z|>\epsilon(t_0)} \frac{\mathbb{1}_{\Gamma_t^{\gamma}}(X+Z)}{|Z|^{d+s}} dZ + s(1-s)\sqrt{1+\partial_r \omega(\alpha_N(t),|x-y|)^2} \int_{|Z|>\epsilon(t_0)} \frac{\mathbb{1}^{\pm}_{E_t^{\gamma-}}(X+Z) - \mathbb{1}^{\pm}_{E_t^{\gamma-}}(Y+Z)}{|Z|^{d+s}} dZ. \tag{3.161}$$

Letting $r_s = \left(\frac{1}{8(2+L)(1-s)T} \right)^{-1/s}$ and noting that

$$|X| + r_s \leq r(M, \delta(t_0/2), t_0) + 3 + LM + r_s = R(t_0), \tag{3.162}$$

by Lemma 3.6.4, we can bound the integral of the boundary term using (3.148) as

$$\begin{aligned}
\int\limits_{|Z|>\epsilon(t_0)} \frac{\mathbb{1}_{\Gamma_t^\gamma}(X+Z)}{|Z|^{d+s}} dZ &\leq \int\limits_{|Z|>r_s} \frac{1}{|Z|^{d+s}} dZ + \epsilon^{-(d+s)} \int\limits_{|Z|<r_s} \mathbb{1}_{\Gamma_t^\gamma}(X+Z) dZ \\
&\leq \int\limits_{r_s}^{\infty} \frac{1}{r^{1+s}} dr + \epsilon(t_0)^{-(d+s)} \mathcal{L}^d(B_{r_s}^d(X) \cap \Gamma_t^\gamma) \\
&\leq \frac{1}{s r_s^s} + \epsilon(t_0)^{-(d+s)} \mathcal{L}^d(B_{R(t_0)}^d \cap \Gamma_t^\gamma) \\
&\leq \frac{1}{8(2+L)s(1-s)T} + \frac{1}{8(2+L)s(1-s)T} \\
&= \frac{1}{4(2+L)s(1-s)T}.
\end{aligned} \tag{3.163}$$

Plugging (3.163) back into (3.161) then gives us

$$\frac{\partial_t \Phi(t,X,Y) - \frac{3}{4T}}{s(1-s)\sqrt{1+\partial_r \omega(\alpha_N(t), |x-y|)^2}} \leq \int\limits_{|Z|>\epsilon(t_0)} \frac{\mathbb{1}^{\pm}_{E_t^{\gamma-}}(X+Z) - \mathbb{1}^{\pm}_{E_t^{\gamma-}}(Y+Z)}{|Z|^{d+s}} dZ. \tag{3.164}$$

Now the only thing that remains is to bound

$$\int\limits_{|Z|>\epsilon(t_0)} \frac{\mathbb{1}^{\pm}_{E_t^{\gamma-}}(X+Z) - \mathbb{1}^{\pm}_{E_t^{\gamma-}}(Y+Z)}{|Z|^{d+s}} dZ. \tag{3.165}$$

Applying Lemma 3.4.1, we get

$$\int\limits_{|Z|>\epsilon(t_0)} \frac{\mathbb{1}^{\pm}_{E_t^{\gamma-}}(X+Z) - \mathbb{1}^{\pm}_{E_t^{\gamma-}}(Y+Z)}{|Z|^{d+s}} dZ$$

$$\leq \int\limits_{|z|>\epsilon(t_0)} \int\limits_{\underline{u}^\gamma(t,x+z)-\overline{u}^\gamma(t,x)}^{\overline{u}^\gamma(t,y+z)-\underline{u}^\gamma(t,y)} \frac{\mathbb{1}^{\pm}_{E_t^{\gamma-}}(x+z,\overline{u}^\gamma(t,x)+z_d) - \mathbb{1}^{\pm}_{E_t^{\gamma-}}(y+z,\underline{u}^\gamma(t,y)+z_d)}{(|z|^2+\omega(\alpha_N(t),|z|)^2)^{(d+s)/2}} dz_d dz$$

$$= \int\limits_{\mathbb{R}^{d-1}} \int\limits_{\underline{u}^\gamma(t,x+z)-\overline{u}^\gamma(t,x)}^{\overline{u}^\gamma(t,y+z)-\underline{u}^\gamma(t,y)} \frac{\mathbb{1}^{\pm}_{E_t^{\gamma-}}(x+z,\overline{u}^\gamma(t,x)+z_d) - \mathbb{1}^{\pm}_{E_t^{\gamma-}}(y+z,\underline{u}^\gamma(t,y)+z_d)}{(|z|^2+\omega(\alpha_N(t),|z|)^2)^{(d+s)/2}} dz_d dz$$

$$- \int\limits_{|z|<\epsilon(t_0)} \int\limits_{\underline{u}^\gamma(t,x+z)-\overline{u}^\gamma(t,x)}^{\overline{u}^\gamma(t,y+z)-\underline{u}^\gamma(t,y)} \frac{\mathbb{1}^{\pm}_{E_t^{\gamma-}}(x+z,\overline{u}^\gamma(t,x)+z_d) - \mathbb{1}^{\pm}_{E_t^{\gamma-}}(y+z,\underline{u}^\gamma(t,y)+z_d)}{(|z|^2+\omega(\alpha_N(t),|z|)^2)^{(d+s)/2}} dz_d dz \tag{3.166}$$

We can bound that last error term rather easily by

$$- \int\limits_{|z|<\epsilon(t_0)} \int\limits_{\underline{u}^\gamma(t,x+z)-\overline{u}^\gamma(t,x)}^{\overline{u}^\gamma(t,y+z)-\underline{u}^\gamma(t,y)} \frac{\mathbb{1}^{\pm}_{E_t^{\gamma-}}(x+z,\overline{u}^\gamma(t,x)+z_d) - \mathbb{1}^{\pm}_{E_t^{\gamma-}}(y+z,\underline{u}^\gamma(t,y)+z_d)}{(|z|^2+\omega(\alpha_N(t),|z|)^2)^{(d+s)/2}} dz_d dz$$

$$\begin{aligned}
&\leq \int\limits_{|z|<\epsilon(t_0)} \int\limits_{-\omega(\alpha_N(t),|z|)}^{\omega(\alpha_N(t),|z|)} \frac{2}{\omega(\alpha_N(t),0)^{d+s}} dz_d dz \\
&\leq \frac{4\omega(\alpha_N(t),\epsilon(t_0))\mathcal{L}^{d-1}(B_1^{d-1})}{\delta(t_0/2)^{d+s}} \epsilon(t_0)^{d-1} \\
&= \frac{4\omega(\alpha_N(t),\epsilon(t_0))\mathcal{L}^{d-1}(B_1^{d-1})}{\delta(t_0/2)^{d+s}} \left(\frac{c\delta(t_0/2)^2}{16sT}\right)^{(d-1)/(1-s)} \\
&\lesssim_{d,s,L,M} \delta(t_0/2)^{d-1-s}
\end{aligned} \tag{3.167}$$

Thus as long as we take t_0 sufficiently close to $2T$ and thus $\delta(t_0/2)$ sufficiently small, we can guarantee that

$$-\int\limits_{|z|<\epsilon(t_0)}\ \int\limits_{\underline{u}^\gamma(t,x+z)-\overline{u}^\gamma(t,x)}^{\overline{u}^\gamma(t,y+z)-\underline{u}^\gamma(t,y)} \frac{\mathbb{1}^{\pm}_{E_t^{\gamma-}}(x+z,\overline{u}^\gamma(t,x)+z_d)-\mathbb{1}^{\pm}_{E_t^{\gamma-}}(y+z,\underline{u}^\gamma(t,y)+z_d)}{(|z|^2+\omega(\alpha_N(t),|z|)^2)^{(d+s)/2}}dz_d dz \leq \frac{1}{4(2+L)s(1-s)T}. \tag{3.168}$$

Since we care about that limit as $t_0 \to 2T$, this isn't an issue.

Plugging (3.166) and (3.168) into (3.164) we get that

$$\frac{\partial_t\Phi(t,X,Y)-\frac{1}{T}}{s(1-s)\sqrt{1+\partial_r\omega(\alpha_N(t),|x-y|)^2}} \leq \int\limits_{\mathbb{R}^{d-1}}\ \int\limits_{\underline{u}^\gamma(t,x+z)-\overline{u}^\gamma(t,x)}^{\overline{u}^\gamma(t,y+z)-\underline{u}^\gamma(t,y)} \frac{\mathbb{1}^{\pm}_{E_t^{\gamma-}}(x+z,\overline{u}^\gamma(t,x)+z_d)-\mathbb{1}^{\pm}_{E_t^{\gamma-}}(y+z,\underline{u}^\gamma(t,y)+z_d)}{(|z|^2+\omega(\alpha_N(t),|z|)^2)^{(d+s)/2}}dz_d dz \tag{3.169}$$

Now after all of this setup, we have essentially returned to smooth case. Indeed, all of our integral bounds in sections 3 and 4 never used that the flow $t \to E_t$ was smooth. So we can apply Lemmas 3.4.2 through 3.5.3 to get that just as in the smooth case,

$$s(1-s)\sqrt{1+\partial_r\omega^2}\int\limits_{\mathbb{R}^{d-1}}\int\limits_{\mathbb{R}} \frac{\mathbb{1}^{\pm}_{E_t^{\gamma-}}(x+z,\overline{u}^\gamma(t,x)+z_d)-\mathbb{1}^{\pm}_{E_t^{\gamma-}}(y+z,\underline{u}^\gamma(t,y)+z_d)}{(|z|^2+\omega(\alpha_N(t),|z|)^2)^{(d+s)/2}}dz_d dz \leq \frac{-2}{T}. \tag{3.170}$$

Recalling (3.155), we thus have that

$$\partial_t\Phi(t,X,Y) \leq \frac{1}{T}-\frac{2}{T}=\frac{-1}{T}=\delta'(t/2)<\alpha_N'(t)\partial_t\omega(\alpha_N(t),|x-y|)=\partial_t\Phi(t,X,Y), \tag{3.171}$$

a contradiction. Thus we could not have a crossing point for times $t \in (a_i, b_i)$. By induction, it then follows that there is no crossing point for any time $t \in [0, t_0]$, so

$$\mathbb{1}_{E_t^{\gamma-} \cup \Gamma_t^{\gamma}}(X) - \mathbb{1}_{E_t^{\gamma-}}(Y) \leq \Phi(t, X, Y), \qquad \text{for all } (t, X, Y) \in [0, t_0] \times \mathbb{R}^d \times \mathbb{R}^d. \quad (3.172)$$

Hence, Γ_t^{γ} has modulus $\omega(\alpha_N(t), \cdot)$ for all times $t \in [0, t_0]$. Letting $N \to \infty$ and $t_0 \to 2T$, we then have that Γ_t^{γ} has modulus $\omega(t/2, \cdot)$ for all times $t \in [0, 2T]$. Thus Γ_{2T}^{γ} is $(1 + L)$-Lipschitz.

3.7 Proofs by Approximations

Corollary 3.7.1. *Let* (E_0^-, Γ_0, E_0^+) *satisfy* (3.102). *Then for every* $\gamma \in \left(\frac{-\eta}{\sqrt{1+L^2}}, \frac{\eta}{\sqrt{1+L^2}}\right)$, $\partial E_{2T}^{\gamma\pm}$ *is* $(1 + L)$*-Lipschitz.*

Proof. Without loss of generality, we will simply prove in the case that $\gamma = 0$ that ∂E_{2T}^- is $(1 + L)$-Lipschitz. Let $X_0 \in \partial E_{2T}^-$ be arbitrary, and consider a sequence $(\gamma_n)_{n=1}^{\infty}$ such that $\frac{-\eta}{\sqrt{1+L^2}} < \gamma_1 < \gamma_2 < \ldots < \gamma_n \to 0$ and γ_n satisfies

$$\mathcal{L}^d(\Gamma_t^{\gamma_n}) = 0, \qquad \text{for almoset every } t. \quad (3.173)$$

For each $n \in N$, let $X_n \in \Gamma_{2T}^{\gamma_n}$ be a point closest to X_0. Then

$$|X_n - X_0| \geq |X_{n+1} - X_0| \to 0. \quad (3.174)$$

By Lemma 3.6.2, $\Gamma_{2T}^{\gamma_n} = \mathrm{graph}(u^{\gamma_n} : \mathbb{R}^{d-1} \to \mathbb{R})$ is a $(1 + L)$-Lipschitz graph with $\overline{E_{2T}^{\gamma_n-}} = \{(x, x_d) | x_d \leq u^{\gamma_n}(x)\}$, it follows that

$$\{X_n - (z, z_d) : z \in \mathbb{R}^{d-1}, z_d \geq (1 + L)|z|\} \subseteq \overline{E_{2T}^{\gamma_n-}} \subseteq E_{2T}^-. \quad (3.175)$$

Thus taking the union we get

$$\{X_0-(z,z_d) : z \in \mathbb{R}^{d-1}, z_d > (1+L)|z|\} \subseteq \bigcup_{n=1}^{\infty} \{X_n-(z,z_d) : z \in \mathbb{R}^{d-1}, z_d \geq (1+L)|z|\} \subseteq E_{2T}^-. \tag{3.176}$$

As $X_0 \in \partial E_{2T}^-$ was arbitrary, we thus have that ∂E_{2T}^- is a $(1+L)$-Lipschitz graph. □

We've now proven Theorem 3.1.1 under the assumptions that

$$\begin{cases} 1).E_0^{\pm} \text{ have modulus } (1-\eta)+Lr, \\ 2).0 \in \Gamma_0 \\ 3).\Gamma_0 \setminus (B_M^{d-1} \times \mathbb{R}) = \{(x,0) : x \in \mathbb{R}^{d-1}, |x| \geq M\}, \end{cases} \tag{3.177}$$

for some $0 < \eta << 1$ and $1 << M < \infty$. Our next goal is to remove the flatness assumption and allow more arbitrary behavior at infinity by letting $M \to \infty$, but in order to justify that we need some compactness.

Lemma 3.7.1. *(Compactness) Let $U_0 : \mathbb{R}^d \to \mathbb{R}$ be 1-Lipschitz. Then the unique viscosity solution $U : [0,\infty) \times \mathbb{R}^d \to \mathbb{R}$ is 1-Lipschitz in space with*

$$\sup_{X \in \mathbb{R}^d} \sup_{t,t'>0} \frac{|U(t,X) - U(t',X)|}{|t-t'|^{1/(1+s)}} \leq \left[(1+s)H_s(B_1^d)\right]^{1/(1+s)}. \tag{3.178}$$

Proof. By Theorem B.0.1 we have that the viscosity solution $U(t,\cdot)$ will be 1-Lipschitz for all times t, so we only need to prove the $C^{1/(1+s)}$ estimate in time. Without loss of generality, assume that $t < t'$,$U(t',X) = 0$, and $U(t,X) > 0$. As $U(t,\cdot)$ is 1-Lipschitz, we have that

$$d(X,\Gamma_t) \geq U(t,X). \tag{3.179}$$

Thus $B^d(X,U(t,X)) \subseteq E_t^+$. Let $H_s(B_1^d)$ be the s-fractional mean curvature of B_1^d. Then

$$\tau \to B^d(X, \left[U(t,X)^{1+s} - (1+s)H_s(B_1^d)\tau\right]^{1/(1+s)}) \tag{3.180}$$

is smooth fractional mean curvature flow for $0 \leq \tau < \frac{U(t,X)^{1+s}}{(1+s)H_s(B_1^d)}$. Thus by comparison principle (Proposition B.0.1), we then have that $X \in E^+_{t+\tau}$ for all such τ. As $t' > t$ and $U(t', X) = 0$, we thus have that $|t'-t| \geq \frac{U(t,X)^{1+s}}{(1+s)H_s(B_1^d)}$. Thus

$$\sup_{t'>t} \frac{|U(t,X) - U(t',X)|}{|t-t'|^{1/(1+s)}} \leq \Big[(1+s)H_s(B_1^d)\Big]^{1/(1+s)}. \tag{3.181}$$

□

3.7.1 Proof of Theorem 3.1.1

Proof. It suffices to show that if (E_0^-, Γ_0, E_0^+) is such that $E_0^\pm$ have modulus of continuity $1 - \eta + Lr$ for some $\eta > 0$, then $\partial E_t^\pm$ are $(1+L)$-Lipschtiz graphs for $t \geq 2T(d,s,L)$.

Without loss of generality, we may assume that $0 \in \Gamma_0$. To begin, let U be the viscosity solution of the level set equation (B.5) for the initial data U_0 as in (3.103), the signed distance function of Γ_0.

For a set $A \subseteq \mathbb{R}^d$ and $x \in \mathbb{R}^{d-1}$, let $A(x) = \{x_d | (x, x_d) \in A\}$. Then for $M >> 1$, we define the sets $E_0^{\pm(M)}$ by

$$E_0^\pm(x) = \begin{cases} E_0^\pm(x), & |x| \leq M \\ E_0^\pm(M\hat{x}), & M \leq |x| \leq M^2/2, \\ \frac{2(M^2-|x|)}{M^2} E_0^\pm(M\hat{x}), & M^2/2 \leq |x| \leq M^2, \\ \{\pm x_d > 0\}, & |x| \geq M^2 \end{cases} \tag{3.182}$$

where $\hat{x} = \dfrac{x}{|x|}$. The sets $E_0^\pm$ are open and disjoint, and have modulus of continuity $1 - \frac{\eta}{2} + Lr$ for M sufficiently large. Taking $\Gamma_0^{(M)} = \mathbb{R}^d \setminus (E_0^{-(M)} \cup E_0^{+(M)})$ and $U_0^{(M)}$ to be the signed distance function of $\Gamma_0^{(M)}$, we then have a unique viscosity solution $U^{(M)}(t,X)$ with $U^{(M)}(0,0) = 0$.

By Lemma 3.7.1 we have that $(U^{(M)}(\cdot,\cdot))_M$ is precompact in $C_{loc}([0,\infty) \times \mathbb{R}^d)$. By classical viscosity solution arguments, we have that any limit will also be a viscosity solution. As $\lim_{M\to\infty} U_0^{(M)}(X) = U_0(X)$, we have by the uniqueness of viscosity solutions that there is a sequence $M_k \to \infty$ such that $U^{(M_k)}(t,X) \to U(t,X)$ in $C_{loc}([0,\infty)\times\mathbb{R}^d)$.

Now fix some time $t \geq 2T(d,s,L)$ and point $X_0 \in \partial E_t^- = \partial\{U(t,\cdot) < 0\}$. Let $(\gamma_n)_{n=1}^\infty$ be such that $\frac{-\eta}{2\sqrt{1+L^2}} < \gamma_1 < \gamma_2 < \ldots < \gamma_n \to 0$, and let $X_n \in \Gamma_t^{\gamma_n} = \{U(t,\cdot) = \gamma_n\}$ be a point closest to X_0. Without loss of generality, we can assume that $|X_1 - X_0| \leq 1$, and hence

$$1 \geq |X_n - X_0| \geq |X_{n+1} - X_0| \to 0. \tag{3.183}$$

As $U^{(M_k)}(t,\cdot) \to U(t,\cdot)$ in C_{loc}, we can find a $k(n) \in \mathbb{N}$ such that

$$|U^{(M_{k(n)})}(t,X) - U(t,X)| < -\gamma_n/2,\ |X - X_0| \leq n \quad \Rightarrow \quad -\eta < 2\gamma_n < U^{(M_{k(n)})}(t,X_n) < \gamma_n/2. \tag{3.184}$$

By Corollary 3.7.1 applied to level sets of $U^{(M_k)}$, we have that

$$\{X_n - (z,z_d) : z \in \mathbb{R}^{d-1}, z_d \geq (1+L)|z|\} \cap B_n^d(X_0) \subseteq \{U^{(M_{k(n)})}(t,\cdot) \leq \gamma_n/2\} \cap B_n^d(X_0) \subseteq E_t^-. \tag{3.185}$$

Thus taking the union it follows that

$$\begin{aligned}\{X_0 - (z,z_d) :& z \in \mathbb{R}^{d-1}, z_d > (1+L)|z|\} \\ &\subseteq \bigcup_{n=1}^\infty \Big(\{X_n - (z,z_d) : z \in \mathbb{R}^{d-1}, z_d \geq (1+L)|z|\} \cap B_n^d(X_0)\Big) \subseteq E_t^-.\end{aligned} \tag{3.186}$$

As $X_0 \in \partial E_t^-$ was arbitrary, we thus have that ∂E_t^- is a $(1+L)$-Lipschitz graph. $\square$

Appendices

APPENDIX A

APPENDIX A: UNIQUENESS FOR THE 2D MUSKAT PROBLEM

We now prove that if our initial data $f_0 \in C^{1,\epsilon}(\mathbb{R})$ with $\beta(f_0') < 1$, then the solution f given by Theorem 2.1.1 is unique with $f \in L^\infty([0,\infty); C^{1,\epsilon})$. As mentioned before, this essentially follows from the uniqueness theorem given in [CGSV17], which under our assumptions simplifies to

Theorem A.0.1. *(Constantin et al) Let $f \in L^\infty([0,T]; W^{1,\infty})$ be a classical, C^1 solution to (2.2) with initial data $f(0,x) = f_0(x)$. Assume that $\lim_{x\to\infty} f(t,x) = 0$, and that there is some modulus of continuity $\tilde{\rho}$ such that*

$$f_x(t,x) - f_x(t,y) \le \tilde{\rho}(|x-y|), \quad \forall 0 \le t \le T,\ x \neq y \in \mathbb{R}. \tag{A.1}$$

Then the solution f is unique.

The authors of [CGSV17] note that the uniform continuity assumption should be the only real assumption; the decay is assumed for convenience in their proof. So, we start by proving that if $f_0 \in C^{1,\epsilon}(\mathbb{R})$, then the solution $f \in L^\infty([0,\infty); C^{1,\epsilon})$. To begin, suppose that $f_0 \in C^{1,1}(\mathbb{R})$. Then necessarily f_0' has modulus $\rho(\cdot/\delta)$ for some $\delta > 0$ sufficiently small. The same proof for the instantaneous generation of the modulus ρ will give that $f_x(t,\cdot)$ has modulus $\rho(\cdot/t+\delta)$. Hence $f_x(t,\cdot)$ has modulus $\rho(\cdot/\delta)$ for all $t \ge 0$.

If $f_0 \in C^{1,\epsilon}(\mathbb{R})$, we can make the same essential argument by changing the definition of ρ , ω. You can repeat the arguments of section 7 and 8 for the modulus

$$\begin{cases} \omega^{(\epsilon)}(\xi) = \xi^\epsilon, & 0 \le \xi \le \delta \\ \omega^{(\epsilon)\,\prime}(\xi) = \dfrac{\gamma}{\xi(4+\log(\xi/\delta))}, & \xi \ge \delta \end{cases}. \tag{A.2}$$

All the error terms for $\xi \le \delta$ are of order $\xi^{2\epsilon-1}$, while the diffusion term is of the order

$\xi^{\epsilon-1}$, so there are no problems as long as δ is sufficiently small. The argument for $\xi \geq \delta$ is identical to the original. Taking $\rho^{(\epsilon)}$ to be some suitable rescaling of $\omega^{(\epsilon)}$, we then have that if f_0' has modulus $\rho^{(\epsilon)}(\cdot/\delta)$, then $f_x(t,\cdot)$ will have modulus $\rho^{(\epsilon)}(\cdot/t+\delta)$.

Thus if $f_0 \in C^{1,\epsilon}(\mathbb{R})$, then the solution f given by Theorem 3.1.1 will satisfy the main uniform continuity assumption of Theorem A.0.1. Our solution f will not decay as $x \to \infty$, but that assumption isn't truly necessary.

Let f_1, f_2 be two uniformly continuous, classical solutions to (2.2) with the same initial data, and let $M(t) = ||f_1(t,\cdot) - f_2(t,\cdot)||_{L^\infty}$. With the decay assumption, the authors of [CGSV17] are able to assume that for almost every t, there is a point $x(t) \in \mathbb{R}$ such that

$$M(t) = |f_1(t,x(t)) - f_2(t,x(t))|, \quad \frac{d}{dt}M(t) = \left(\frac{d}{dt}|f_1 - f_2|\right)(t,x(t)). \tag{A.3}$$

They then bound $\frac{d}{dt}|f_1(t,x(t)) - f_2(t,x(t))|$ using equation (2.2), $\tilde{\rho}$, and $W^{1,\infty}$ bounds.

Without the decay assumption, you instead use that

$$\frac{d}{dt}M(t) \leq \sup\{\frac{d}{dt}|f_1(t,x) - f_2(t,x)| : |f_1(t,x) - f_2(t,x)| \geq M(t) - \delta\}, \tag{A.4}$$

where $\delta > 0$ is arbitrary. When you go to bound $\frac{d}{dt}|f_1(t,x) - f_2(t,x)|$, you then get new error terms which can be bounded by

$$C(\tilde{\rho}, \max_i ||f_i(t,\cdot)||_{W^{1,\infty}}, M(t)) \left(\delta + |f_{1,x}(t,x) - f_{2,x}(t,x)|\right). \tag{A.5}$$

Since $f_{i,x}(t,x)$ is bounded and has modulus $\tilde{\rho}$, it then follows that

$$|f_{1,x}(t,x) - f_{2,x}(t,x)| = o_\delta(1). \tag{A.6}$$

Thus by taking δ sufficiently small depending on $\tilde{\rho}, \max_i ||f_i(t,\cdot)||_{W^{1,\infty}}, M(t)$, we can guarantee that the new error terms $\lesssim M(t)$. Then the original proof of [CGSV17] goes through.

APPENDIX B

APPENDIX B: VISCOSITY SOLUTIONS FOR FRACTIONAL MEAN CURVATURE FLOW

In this appendix, we review the necessary definitions and essential existence/uniqueness results for defining weak solutions via the level set method. For more details, we refer the reader to [Imb09, CMP15, ACP19], or [ES91] for details about level set method for classical mean curvature flow.

Definition B.0.1.

1). An upper semicontinuous function $U : (0,T) \times \mathbb{R}^d \to \mathbb{R}$ *is a viscosity subsolution to the level set equation*

$$\partial_t U(t,X) \leq -H_s(X, \{U(t,\cdot) \geq U(t,X)\})|\nabla_X U(t,X)| \tag{B.1}$$

if whenever Ψ *is a smooth test function such that crosses* U *from above at* (t,X)*, then* $\partial_t \Psi(t,X) \leq 0$ *if* $\nabla_X \Psi(t,X) = 0$ *or else*

$$\partial_t \Psi(t,X) \leq \left(\int\limits_{|Z|>\epsilon} \frac{\mathbb{1}^{\pm}_{\{U(t,\cdot)\geq U(t,X)\}}(X+Z)}{|Z|^{d+s}} dZ + \int\limits_{|Z|<\epsilon} \frac{\mathbb{1}^{\pm}_{\{\Psi(t,\cdot)\geq \Psi(t,X)\}}(X+Z)}{|Z|^{d+s}} dZ \right) |\nabla_X \Psi(t,X)|, \tag{B.2}$$

for any $\epsilon > 0$.

2). A lower semicontinuous function $U : (0,T) \times \mathbb{R}^d \to \mathbb{R}$ *is a viscosity supersolution to the level set equation*

$$\partial_t U(t,X) \geq -H_s(X, \{U(t,\cdot) > U(t,X)\})|\nabla_X U(t,X)|, \tag{B.3}$$

if whenever Ψ is a smooth test function such that crosses U from below at (t,X), then $\partial_t\Psi(t,X)\geq 0$ if $\nabla_X\Psi(t,X)=0$ or else

$$\partial_t\Psi(t,X)\geq \left(\int\limits_{|Z|>\epsilon}\frac{\mathbb{1}^{\pm}_{\{U(t,\cdot)>U(t,X)\}}(X+Z)}{|Z|^{d+s}}dZ+\int\limits_{|Z|<\epsilon}\frac{\mathbb{1}^{\pm}_{\{\Psi(t,\cdot)>\Psi(t,X)\}}(X+Z)}{|Z|^{d+s}}dZ\right)|\nabla_X\Psi(t,X)|, \tag{B.4}$$

for any $\epsilon>0$.

3). We say that a continuous function $U:(0,T)\times\mathbb{R}^d\to\mathbb{R}$ is a viscosity solution to the level set equation

$$\partial_t U(t,X)=-H_s(X,\{U(t,\cdot)\geq U(t,X)\})|\nabla_X U(t,X)|, \tag{B.5}$$

if U is both a subsolution and a supersolution.

Theorem B.0.1. *[Imb09] Let $U_0\in\dot{W}^{1,\infty}(\mathbb{R}^d)$, and suppose that $U,V:[0,T]\times\mathbb{R}^d\to\mathbb{R}$ are sub/super solutions respectively to* (B.1),(B.3) *with $U(0,X)\leq U_0(X)\leq V(0,X)$. Then $U(t,X)\leq V(t,X)$ for all $(t,X)\in[0,T]\times\mathbb{R}^d$.*

In particular, there is a unique viscosity solution $U:[0,\infty)\times\mathbb{R}^d\to\mathbb{R}$ to (B.5) *with $U(0,X)=U_0(X)$. Furthermore,*

$$||\nabla_X U||_{L^\infty([0,\infty)\times\mathbb{R}^d)}=||\nabla_X U_0||_{L^\infty(\mathbb{R}^d)}. \tag{B.6}$$

The sets

$$\{U(t,\cdot)<0\},\quad \{U(t,\cdot)=0\},\quad \{U(t,\cdot)>0\} \tag{B.7}$$

depend only on the initial sets

$$\{U_0<0\},\qquad \{U_0=0\},\quad \{U_0>0\}. \tag{B.8}$$

Definition B.0.2. *Let* (E_0^-, Γ_0, E_0^+) *be such that* $E_0^{\pm}$ *are open,* Γ_0 *is closed, all are mutually disjoint with* $E_0^- \cup \Gamma_0 \cup E_0^+ = \mathbb{R}^d$*. Then we define the viscosity solution to fractional mean curvature flow* (3.4) *as follows. Let* $U_0 \in \dot{W}^{1,\infty}(\mathbb{R}^d)$ *be such that that*

$$E_0^- = \{U_0 < 0\}, \quad \Gamma_0 = \{U_0 = 0\}, \quad E_0^+ = \{U_0 > 0\}, \tag{B.9}$$

and $U(t,X)$ *be the unique viscosity solution to* (B.5) *with initial data* U_0*. Then the sets*

$$E_t^- = \{U(t,\cdot) < 0\}, \quad \Gamma_t = \{U(t,\cdot) = 0\}, \quad E_t^+ = \{U(t,\cdot) > 0\}, \tag{B.10}$$

are independent of the choice of U_0*. We call* $t \to (E_t^-, \Gamma_t, E_t^+)$ *the viscosity solution to fractional mean curvature flow. The flows* $t \to E_t^- \cup \Gamma_t$ *and* $t \to E_t^-$ *are called the maximal subsolution and minimal superoslution of the flow* (3.4) *respectively.*

The names maximal subsolution and minimal supersolution are quite nature for $E_t^- \cup \Gamma_t$ and E_t^-, as in fact

Theorem B.0.2. *[Imb09] Let* (E_t^-, Γ_t, E_t^+) *be the viscosity solution of fractional mean curvature flow for the initial data* (E_0^-, Γ_0, E_0^+)*. Then the indicator functions* $\mathbb{1}_{E_t^- \cup \Gamma_t}, \mathbb{1}_{E_t^-}$ *are the maximal subsolution and minimal supersolution to* (B.1),(B.3) *for the initial data* $U_0 = \mathbb{1}_{E_0^- \cup \Gamma_0}$ *and* $U_0 = \mathbb{1}_{E_0^-}$ *respectively.*

While the comparison principle on the level of sets is proven in [Imb09], for our purposes it is more useful to apply it to just the minimal superosolutions. Hence,

Proposition B.0.1. *(Comparison Principle) Let* $E_0, F_0 \subseteq \mathbb{R}^d$ *be open sets with* $E_0 \subseteq F_0$*, and* E_t, F_t *be the minimal viscosity supersolutions of the flow. Then* $E_t \subseteq F_t$ *for all times* $t \geq 0$*.*

Proof. Note that as $E_0 \subseteq F_0$,

$$\mathbb{1}_{E_0}(X) \leq \mathbb{1}_{F_0}(X). \tag{B.11}$$

Thus as $\mathbb{1}_{E_t}$ is the minimal supersolution of (B.3) with respect to the initial data $U_0 = \mathbb{1}_{E_0}$ and $\mathbb{1}_{F_t}$ is a supersolution, we have by minimality that

$$\mathbb{1}_{E_t}(X) \leq \mathbb{1}_{F_t}(X), \tag{B.12}$$

for all t, and hence $E_t \subseteq F_t$. □

Proposition B.0.2. *Basic properties of fractional mean curvature:*

Let $E \subseteq \mathbb{R}^d$ with $0 \in \partial E$ and $H_s(0, E)$ well defined.

1. *Translation invariance: for any $Y \in \mathbb{R}^d$, $H_s(Y, E + Y) = H_s(0, E)$.*

2. *Symmetry: $H_s(0, -E) = H_s(0, E)$*

3. *Scaling: for any $r > 0$, $r^s H_s(0, rE) = H_s(0, E)$.*

4. *Monotonicty: if $E \subseteq F$ is with $0 \in \partial F$, then $H_s(0, F) \leq H_s(0, E)$.*

Proof. 1-3. follow by making a simple change of variables to

$$P.V. \int_{\mathbb{R}^d} \frac{\mathbb{1}^{\pm}_{E}(Z)}{|Z|^{d+s}} dZ. \tag{B.13}$$

4. follows from noting that $\mathbb{1}^{\pm}_{E}(Z) \leq \mathbb{1}^{\pm}_{F}(Z)$ for all $Z \in \mathbb{R}^d$. □

Corollary B.0.1. *Let $U(t, X)$ be a viscosity subsolution to* (B.1). *Then so is*

1. *$U(t, X + Y)$ for fixed $Y \in \mathbb{R}^d$,*

2. *$U(t, -X)$,*

3. *$U(r^{1+s}t, rX)$ for $r > 0$.*

Lemma B.0.1. *Suppose that $F_n \in C_{loc}((0,T); W^{2,\infty}(\mathbb{R}^d))$ and $F_n \to F$ in $C_{loc}((0,T)\times\mathbb{R}^d)$. Then for any $(t,X) \in (0,T) \times \mathbb{R}^d$,*

$$\limsup_{n\to\infty} -H_s(X, \{F_n(t,\cdot) \geq F_n(t,X)\}) \leq -H_s(X, \{F(t,\cdot) \geq F(t,X)\}) \tag{B.14}$$

In particular, for fixed $F \in C((0,T); W^{2,\infty}(\mathbb{R}^d))$ the function

$$(t,X) \to -H_s(X, \{F(t,\cdot) \geq F(t,X)\}), \tag{B.15}$$

is upper semicontinuous.

Proof. Fix some point $(t,X) \in (0,T) \times \mathbb{R}^d$ and let $\epsilon > 0$. As $(F_n)_{n=1}^{\infty}$ is uniformly $C^{1,1}$ in space for times $t' \in (\frac{t}{2}, \frac{t+T}{2})$, we have that

$$\left| s(1-s) \int\limits_{|Z|<r} \frac{\mathbb{1}^{\pm}_{\{F_n(t,\cdot)\geq F_n(t,X)\}}(X+Z)}{|Z|^{d+s}} dZ \right| \lesssim r. \tag{B.16}$$

uniformly in n. Taking $r_0 \lesssim \epsilon$, we thus have that

$$\left| s(1-s) \int\limits_{|Z|<r_0} \frac{\mathbb{1}^{\pm}_{\{F_n(t,\cdot)\geq F_n(t,X)\}}(X+Z)}{|Z|^{d+s}} dZ \right| \leq \frac{\epsilon}{5}, \tag{B.17}$$

for all n.

Now take $R_0 > 0$ large enough so that

$$\left| s(1-s) \int\limits_{\mathbb{R}^d \setminus B^d_{R_0}} \frac{\mathbb{1}^{\pm}_{\{F_n(t,\cdot)\geq F_n(t,X)\}}(X+Z)}{|Z|^{d+s}} dZ \right| \leq s(1-s) \int\limits_{\mathbb{R}^d \setminus B^d_{R_0}} \frac{1}{|Z|^{d+s}} dZ \leq \frac{\epsilon}{5} \tag{B.18}$$

Combining (B.17) and (B.18) gives us that

$$-H_s(X, \{F_n(t, \cdot \geq F_n(t, X)\}) + H_s(X, \{F(t, \cdot) \geq F(t, X)\})$$
$$\leq \frac{4}{5}\epsilon + s(1-s) \int\limits_{B^d_{R_0} \backslash B^d_{r_0}} \frac{\mathbb{1}^{\pm}_{\{F_n(t,\cdot) \geq F_n(t,X)\}}(X+Z) - \mathbb{1}^{\pm}_{\{F(t,\cdot) \geq F(t,X)\}}(X+Z)}{|Z|^{d+s}} dZ. \tag{B.19}$$

Note that as $\eta \to 0+$, we have the convergence

$$\mathbb{1}^{\pm}_{\{F(t,\cdot) \geq F(t,X) - \eta\}} \xrightarrow{L^1_{loc}} \mathbb{1}^{\pm}_{\{F(t,\cdot) \geq F(t,X)\}}. \tag{B.20}$$

Hence for some $\eta_0 > 0$ sufficiently small,

$$s(1-s) \int\limits_{B^d_{R_0} \backslash B^d_{r_0}} \frac{\mathbb{1}^{\pm}_{\{F(t,\cdot) \geq F(t,X) - \eta_0\}} - \mathbb{1}^{\pm}_{\{F(t,\cdot) \geq F(t,X)\}}(X+Z)}{|Z|^{d+s}} dZ < \frac{\epsilon}{5}. \tag{B.21}$$

As $F_n \to F$ in C_{loc}, we have that for n sufficiently large that

$$\{Z : F_n(t, X+Z) \geq F_n(t, X)\} \cap B^d_{R_0} \subseteq \{Z : F(t, X+Z) \geq F(t, X) - \eta_0\} \cap B^d_{R_0}. \tag{B.22}$$

In particular, we then have that

$$\mathbb{1}^{\pm}_{\{F_n(t,\cdot) \geq F(t,X)\}}(X+Z) \leq \mathbb{1}^{\pm}_{\{F(t,\cdot) \geq F(t,X) - \eta_0\}}(X_0+Z), \qquad |Z| \leq R_0. \tag{B.23}$$

Plugging in (B.23) and (B.21) into (B.19) then gives us that

$$-H_s(X, \{F(t, \cdot \geq F(t, X)\}) + H_s(X_0, \{F(t_0, \cdot) \geq F(t_0, X_0)\}) < \epsilon, \tag{B.24}$$

whenever n is sufficiently large. $\square$

Lemma B.0.2. *Let* $U_1, U_2 \in L^\infty((0,T); W^{1,\infty}(\mathbb{R}^d)) \cap L^\infty(\mathbb{R}^d; C^{1/(1+s)}(0,T))$ *with* U_1 *a subsolution to* (B.1) *and* U_2 *a viscosity supersoltuion to* (B.3). *Then* $U_1(t,X) - U_2(t,Y)$ *is a viscosity subsolution to* (3.143).

Proof. Our goal is show that $V(t,X,Y) = U_1(t,X) - U_2(t,Y)$ is a subsolution of (3.143).

Suppose $\Psi : (0,T) \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ satisfies

$$\begin{cases} \Psi(t_0, X_0, Y_0) = V(t_0, X_0, Y_0), & \\ \Psi(t,X,Y) > V(t,X,Y), & (t,X,Y) \neq (t_0, X_0, Y_0) \end{cases}. \tag{B.25}$$

In fact, without loss of generality we may assume that

$$\Psi(t,X,Y) - (U_1(t,X) - U_2(t,Y)) \geq \alpha \min\{|t-t_0|^2 + |X-X_0|^2 + |Y-Y_0|^2, 1\}, \tag{B.26}$$

for some $\alpha > 0$ arbitrarily small.

Consider $\Psi_\epsilon(t,s,X,Y) = \Psi(\frac{t+s}{2}, X, Y) + \frac{|t-s|^2}{2\epsilon}$. Then

$$\begin{aligned} &\Psi_\epsilon(t,s,X,Y) - (U_1(t,X) - U_2(s,Y)) \\ &\quad \geq \Psi(\frac{t+s}{2}, X, Y) + \frac{|t-s|^2}{2\epsilon} - (U_1(\frac{t+s}{2}, X) - U_2((\frac{t+s}{2}, Y)) \\ &\qquad - \frac{|t-s|^{\frac{1}{1+s}}}{2}(||U_1||_{L^\infty_X C_t^{\frac{1}{1+s}}} + ||U_2||_{L^\infty_Y C_t^{\frac{1}{1+s}}}) \\ &\quad \geq \alpha \min\{|\frac{t+s}{2} - t_0|^2 + |X-X_0|^2 + |Y-Y_0|^2, 1\} + \frac{|t-s|^2}{2\epsilon} - C|t-s|^{\frac{1}{1+s}} \end{aligned} \tag{B.27}$$

Taking $\epsilon << \alpha \min\{\sqrt{t_0}, \sqrt{T-t_0}\}$, we can guarantee then that

$$\Psi_\epsilon(t,s,X,Y) - (U_1(t,X) - U_2(s,Y)) > 0, \qquad t = 0, T \text{ or } s = 0, T. \tag{B.28}$$

As the RHS of (B.27) $\to \infty$ as $|(X,Y)| \to \infty$, we thus have that a global minimum

$(t_\epsilon, s_\epsilon, X_\epsilon, Y_\epsilon)$ exists. Since

$$\begin{aligned}\Psi_\epsilon(t,s,X,Y) - (U_1(t,X) - U_2(s,Y)) \geq \frac{|t-s|^2}{2\epsilon} - C|t-s| \geq -\tilde{C}\epsilon^{\frac{1+s}{1+2s}} \\ \Psi_\epsilon(t_0,t_0,X_0,Y_0) - (U_1(t_0,X_0) - U_2(t_0,Y_0)) = 0,\end{aligned} \tag{B.29}$$

it then follows that at the minimum

$$|X_\epsilon - X_0|^2, |Y_\epsilon - Y_0|^2 \leq \frac{\tilde{C}\epsilon^{\frac{1+s}{1+2s}}}{2\alpha}. \tag{B.30}$$

Thus the sequence $(t_\epsilon, s_\epsilon, X_\epsilon, Y_\epsilon)$ is bounded. Passing to a convergent subsequence, the uniqueness of the minimum for $\Psi(t,X,Y) - (U_1(t,X) - U_2(t,Y))$ then implies that

$$(t_\epsilon, s_\epsilon, X_\epsilon, Y_\epsilon) \to (t_0, t_0, X_0, Y_0). \tag{B.31}$$

Now, assume that $\nabla_X \Psi(t_0, X_0, Y_0), \nabla_Y \Psi(t_0.X_0, Y_0) \neq 0$. Then by continuity, for ϵ sufficiently small

$$\nabla_X \Psi_\epsilon(t_\epsilon, s_\epsilon, X_\epsilon, Y_\epsilon), \nabla_Y \Psi_\epsilon(t_\epsilon, s_\epsilon, X_\epsilon, Y_\epsilon) \neq 0. \tag{B.32}$$

As U_1 is a subsolution to (B.1) and U_2 a viscosity supersoltuion to (B.3), it follows that

$$\begin{aligned}\partial_t\Psi_\epsilon(t_\epsilon, s_\epsilon, X_\epsilon, Y_\epsilon) + \partial_s\Psi_\epsilon(t_\epsilon, s_\epsilon, X_\epsilon, Y_\epsilon) \leq \\ - H_s(X_\epsilon, \{\Psi_\epsilon(t_\epsilon, s_\epsilon, \cdot, Y_\epsilon) \geq \Psi_\epsilon(t_\epsilon, s_\epsilon, X_\epsilon, Y_\epsilon)\})|\nabla_X \Psi_\epsilon(t_\epsilon, s_\epsilon, X_\epsilon, Y_\epsilon)| \\ + H_s(Y_\epsilon, \{\Psi_\epsilon(t_\epsilon, s_\epsilon, X_\epsilon, \cdot) > \Psi_\epsilon(t_\epsilon, s_\epsilon, X_\epsilon, Y_\epsilon)\})|\nabla_Y \Psi_\epsilon(t_\epsilon, s_\epsilon, X_\epsilon, Y_\epsilon)|\end{aligned} \tag{B.33}$$

We have by continuity that

$$\partial_t\Psi_\epsilon(t_\epsilon,s_\epsilon,X_\epsilon,Y_\epsilon)+\partial_s\Psi_\epsilon(t_\epsilon,s_\epsilon,X_\epsilon,Y_\epsilon)=\frac{\partial_t\Psi(t_\epsilon,X_\epsilon,Y_\epsilon)+\partial_t\Psi(s_\epsilon,X_\epsilon,Y_\epsilon)}{2}\to\partial_t\Psi(t_0,X_0,Y_0),$$
$$|\nabla_X\Psi_\epsilon(t_\epsilon,s_\epsilon,X_\epsilon,Y_\epsilon)|\to|\nabla_X\Psi(t_0,X_0,Y_0)|,$$
$$|\nabla_Y\Psi_\epsilon(t_\epsilon,s_\epsilon,X_\epsilon,Y_\epsilon)|\to|\nabla_Y\Psi(t_0,X_0,Y_0)|, \tag{B.34}$$

As the s-mean curvature of closed superlevel sets $H_s(X,\{F(\cdot)\geq F(X)\})$ is upper semicontinuous by Lemma B.0.1, it then follows that

$$\begin{aligned}\partial_t\Psi(t_0,X_0,Y_0)&=\lim_{\epsilon\to 0}\partial_t\Psi_\epsilon(t_\epsilon,s_\epsilon,X_\epsilon,Y_\epsilon)+\partial_s\Psi_\epsilon(t_\epsilon,s_\epsilon,X_\epsilon,Y_\epsilon)\\&\leq\limsup_{\epsilon\to 0}-H_s(X_\epsilon,\{\Psi_\epsilon(t_\epsilon,s_\epsilon,\cdot,Y_\epsilon)\geq\Psi_\epsilon(t_\epsilon,s_\epsilon,X_\epsilon,Y_\epsilon)\})|\nabla_X\Psi_\epsilon(t_\epsilon,s_\epsilon,X_\epsilon,Y_\epsilon)|\\&\qquad+H_s(Y_\epsilon,\{\Psi_\epsilon(t_\epsilon,s_\epsilon,X_\epsilon,\cdot)>\Psi_\epsilon(t_\epsilon,s_\epsilon,X_\epsilon,Y_\epsilon)\})|\nabla_Y\Psi_\epsilon(t_\epsilon,s_\epsilon,X_\epsilon,Y_\epsilon)|\\&\leq -H_s(X_0,\{\Psi(t_0\cdot,Y_0)\geq\Psi(t,X_0,Y_0)\})|\nabla_X\Psi(t_0,X_0,Y_0)|\\&\qquad+H_s(Y_0,\{\Psi(t_0,X_0,\cdot)>\Psi(t_0,X_0,Y_0)\})|\nabla_Y\Psi(t_0,X_0,Y_0)|,\end{aligned} \tag{B.35}$$

thus proving the lemma. □

Proof of Lemma 3.6.5

Proof. Let (E_t^-,Γ_t,E_t^+) be our viscosity solution. For $n\in\mathbb{N}$, let $U^{(n\pm)}(t,X)$ be the viscosity solution of (B.5) for the initial data

$$U_0^{(n+)}(X)=\begin{cases}0, & X\in\overline{E_0^+},\\ \min\{1,nd(X,\Gamma_0)\}, & X\in E_0^-\cup\Gamma_0,\end{cases}, \tag{B.36}$$
$$U_0^{(n-)}(Y)=-U_0^{(n+)}(Y)$$

By Lemmas B.0.2 and 3.7.1, we have that $U^{(n+)}(t,X)-U^{(n-)}(t,Y)$ is a viscosity subsolution of the doubled equation (3.143).

$$\mathbb{1}_{E_t^- \cup \Gamma_t}(X) - \mathbb{1}_{E_t^-}(Y) = \limsup_{(t_n, X_n, Y_n) \to (t, X, Y)} U^{(n+)}(t_n, X_n) - U^{(n-)}(t_n, Y_n), \tag{B.37}$$

we have that $\mathbb{1}_{E_t^- \cup \Gamma_t}(X) - \mathbb{1}_{E_t^-}(Y)$ is the upper semicontinuous envelope of subsolutions, and thus is a subsolution itself by standard viscosity solution theory. See Theorem 1 (Discontinuous Stability) of [Imb09] for the essential details. □

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