Relative Entropy of Random States and Black Holes: Supplemental material

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I. CHAOTIC EIGENSTATES

We provide a numerical study of relative entropy between mid-spectrum eigenstates of integrable and chaotic spin chains of length N with Hamiltonian

$$H = -\sum_{i=1}^{N} \left(Z_i Z_{i+1} + h_x X_i + h_z Z_i \right), \tag{1}$$

where X and Z are Pauli spin operators. We take $h_x = 1$, $h_z = 0$ for the integrable limit and $h_x = -1.05$, $h_z = 0.5$ for the chaotic regime as in Ref. [1]. We also numerically study the Sachdev-Ye-Kitaev model [2] with Hamiltonian

$$H = \sum_{j<k<l<m}^{N} J_{ijkl} \chi_j \chi_k \chi_l \chi_m, \quad \overline{J_{ijkl}^2} = \frac{6}{(N-3)(N-2)(N-1)} J^2, \tag{2}$$

where the χ_i's are Majorana fermions and J_{ijkl} is a Gaussian random variable. The comparison between numerical data and (23) from the main text is shown in Fig. 1. The eigenstates are chosen randomly from the middle of the spectrum. The SYK model matches very well with (23). This may be expected because the Hamiltonian is a random matrix and the SYK model is known to be closely related to low-dimensional gravitational systems. The chaotic spin chain eigenstates have relative entropy close to, but noticeably larger than, random mixed states. This is reasonable because these eigenstates are not truly random and therefore should be more easily distinguishable. It would be interesting to understand whether this is a finite size bug or a feature that holds in the thermodynamic limit. Meanwhile, the integrable eigenstates are even more distinguishable, which is consistent with their violation of the eigenstate thermalization hypothesis. Moreover, the variance in relative entropy from eigenstate to eigenstate is much larger for the integrable spin chain.

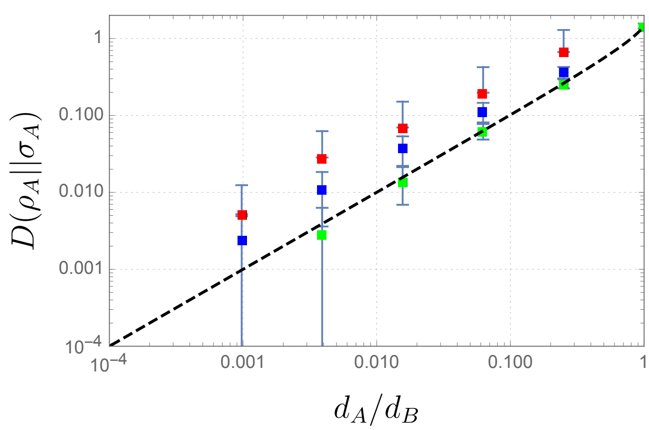

FIG. 1. The relative entropy between 10^3 random pairs of mid-spectrum eigenstates. The blue (red) data points are for the chaotic (integrable) spin chain with 12 spins and the dashed line is (23) from the main text. The green data points are for the SYK model with 20 Majorana fermions. We have omitted the lower error bars for the red data points for clarity, as they are very large and get in the way of the other data.

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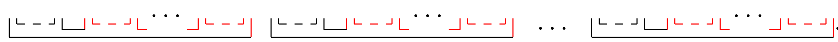
II. COMMUTING OF THE LOGARITHM AND ENSEMBLE AVERAGE

In the main text, we computed the ensemble average of $\mathrm{Tr}\left[\rho_A \sigma_A^{n-1}\right]$ and then took the logarithm to find the relative entropy. In general, the logarithm and ensemble average do not commute. However, for large Hilbert space dimensions, they approximately commute as we now show. The average of a logarithm can be computed from a further replica trick

$$\overline{\log \mathrm{Tr}\left[\rho_A \sigma_A^{n-1}\right]} = \lim_{m\to 0} \frac{\overline{\left(\mathrm{Tr}\left[\rho_A \sigma_A^{n-1}\right]\right)^m} - 1}{m}. \tag{3}$$

Therefore, we must compute the moments $\overline{\left(\mathrm{Tr}\left[\rho_A \sigma_A^{n-1}\right]\right)^m}$. This consists of m copies of the diagrams we previously considered where the ensemble averaging is allowed to connect any of the m copies

$$\left(\mathrm{Tr}\left[\rho_A \sigma_A^{n-1}\right]\right)^m = \tag{4}$$



A straightforward generalization of the analysis in the main text gives

$$\overline{\left(\mathrm{Tr}\left[\rho_A \sigma_A^{n-1}\right]\right)^m} = \frac{1}{(d_A d_B)^{mn}} \sum_{\tau \in S_m \times S_{m(n-1)}} d_A^{D((\eta^{-1})^{\times m} \circ \tau)} d_B^{D(\tau)}, \tag{5}$$

where the S_m factor of τ acts on the copies of ρ_A (black lines) and the $S_{m(n-1)}$ factor acts on the copies of σ_A (red lines). $(\eta^{-1})^{\times m}$ is the permutation element that implements the trace structure. In cycle notation,

$$(\eta^{-1})^{\times m} = \prod_{i=0}^{m-1} (ni+1, ni+2, \ldots, ni+n). \tag{6}$$

For large Hilbert space dimensions, we need to sum over the permutations that maximize $D((\eta^{-1})^{\times m} \circ \tau) + D(\tau)$. These are non-crossing within each block of n density matrices and have $D((\eta^{-1})^{\times m} \circ \tau) + D(\tau) = m(n+1)$ [3]. Therefore, at leading order, the sum factorizes as

$$\overline{\left(\mathrm{Tr}\left[\rho_A \sigma_A^{n-1}\right]\right)^m} \simeq \left(\frac{1}{(d_A d_B)^n} \sum_{\tau \in \mathbb{1} \times S_{n-1}} d_A^{D(\eta^{-1} \circ \tau)} d_B^{D(\tau)} \right)^m = \overline{\left(\mathrm{Tr}\left[\rho_A \sigma_A^{n-1}\right]\right)}^m. \tag{7}$$

This immediately implies that for large Hilbert space dimensions,

$$\overline{\log \mathrm{Tr}\left[\rho_A \sigma_A^{n-1}\right]} \simeq \log \overline{\mathrm{Tr}\left[\rho_A \sigma_A^{n-1}\right]}. \tag{8}$$

The corrections to this formula come from the subleading permutations in $S_m \times S_{m(n-1)}$ that are not of the form $(\mathbb{1} \times S_{n-1})^{\times m}$. An identical argument can be made for $\overline{\log \mathrm{Tr}\left[\rho_A^n\right]}$.

Furthermore, we note that the same argument for $m = 2$ implies that the variance of $\mathrm{Tr}\left[\rho_A \sigma_A^{n-1}\right]$ and $\mathrm{Tr}\left[\rho_A^n\right]$ vanish in the limit of large Hilbert space dimensions. This explains the numerical results for finite dimensional systems in the main text.

III. HYPERGEOMETRIC REPRESENTATION

In this section, we show (in reverse order) how the sums involving the Narayana numbers lead to standard hypergeoemtric functions. The hypergeometric functions are defines as a power series

$${}_2F_1(a, b, c; z) := \sum_{m=0}^{\infty} \frac{(a)_m (b)_m}{(c)_m} \frac{z^m}{m!}, \tag{9}$$

where $(q)_n$ is the Pochhammer symbol defined as

$$(q)_m = \begin{cases} 1, & m = 0 \\ \prod_{i=0}^{m-1} (q+i), & m > 1 \end{cases}. \tag{10}$$

The hypergeometric series terminates when either a or b is zero or a negative integer, in which case

$$ {}_2F_1(a,b,c;z) := \sum_{k=0}^{-a} (-1)^k \binom{-a}{k} \frac{(b)_k}{(c)_k} z^k. \tag{11}$$

This is relevant for us because the hypergeometric functions we are interested in always satisfy this condition. In particular, plugging in the arguments for the Rényi entropies, we have

$$ {}_2F_1\left(1-n,-n;2;\frac{d_A}{d_B}\right) = \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} \frac{(-n)_k}{(2)_k} \left(\frac{d_A}{d_B}\right)^k \tag{12}$$

Note that

$$ (-q)_m = \begin{cases} 1, & m=0 \\ (-1)^m \prod_{i=0}^{m-1}(q-i), & m>1 \end{cases}, \tag{13}$$

so

$$ {}_2F_1\left(1-n,-n;2;\frac{d_A}{d_B}\right) = \sum_{k=0}^{n-1} \binom{n-1}{k} \frac{n!}{(n-k)!(k+1)!} \left(\frac{d_A}{d_B}\right)^k. \tag{14}$$

Redefining $k \to k-1$ and converting the factorials into binomial coefficients, we get

$$ {}_2F_1\left(1-n,-n;2;\frac{d_A}{d_B}\right) = \sum_{k=1}^{n} \frac{1}{k} \binom{n-1}{k-1} \binom{n}{k-1} \left(\frac{d_A}{d_B}\right)^{k-1} \tag{15}$$

The coefficients in this series are precisely the Narayana numbers, $N_{n,k}$. An analogous analysis can be made for ${}_2F_1\left(1-n,2-n;2;\frac{d_A}{d_B}\right)$ which was the relevant hypergeometric function for $\mathrm{Tr}\left[\rho_A \sigma_A^{n-1}\right]$.

Next, we perform the analytic continuation needed to compute the relative entropy. For $\mathrm{Tr}\left[\rho_A^n\right]$,

$$ \lim_{n\to 1} \frac{1}{1-n} \log\left[d_A^{1-n}\, {}_2F_1\left(1-n,-n;2;\frac{d_A}{d_B}\right)\right] = \log d_A - \partial_n \sum_{k=1}^{n} N_{n,k} \left(\frac{d_A}{d_B}\right)^{k-1} \Bigg|_{n=1}. \tag{16}$$

The key point is that all Narayana numbers with $k>2$ are proportional to $(n-1)^2$ so their derivative evaluated at $n=1$ is trivial. Only the $k=2$ term is nontrivial. Therefore,

$$ \lim_{n\to 1} \frac{1}{1-n} \log\left[d_A^{1-n}\, {}_2F_1\left(1-n,-n;2;\frac{d_A}{d_B}\right)\right] = \log d_A - \frac{d_A}{2d_B}. \tag{17}$$

This is Page's formula [4]. For the other hypergeometric function, we have

$$ \lim_{n\to 1} \frac{1}{1-n} \log\left[d_A^{1-n}\, {}_2F_1\left(1-n,2-n;2;\frac{d_A}{d_B}\right)\right] = \log d_A - \partial_n \sum_{k=1}^{\infty} N_{n-1,k} \left(\frac{d_A}{d_B}\right)^{k-1} \Bigg|_{n=1}. \tag{18}$$

We take the sum to run to infinity for the purpose of analytic continuation even though the Narayana numbers are trivial for $k>n$. We can Taylor expand the Narayana number around $n=1$ for integer k

$$ N_{n-1,k} = \begin{cases} 0, & k=1 \\ -\frac{n-1}{k(k-1)} + O(n-1)^2, & k>1 \end{cases}. \tag{19}$$

Therefore,

$$ \begin{aligned} \lim_{n\to 1} \frac{1}{1-n} \log\left[d_A^{1-n}\, {}_2F_1\left(1-n,2-n;2;\frac{d_A}{d_B}\right)\right] &= \log d_A + \sum_{k=2}^{\infty} \frac{1}{k(k-1)} \left(\frac{d_A}{d_B}\right)^{k-1} \\ &= \log d_A + 1 + \left(\frac{d_B}{d_A} - 1\right) \log\left(1 - \frac{d_A}{d_B}\right). \end{aligned} \tag{20}$$

Taking the difference between this and (17) gives the relative entropy stated in the main text.

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