



An algebraic variant of the Fischer–Grauert Theorem

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Abstract

A well-known theorem of W. Fischer and H. Grauert states that analytic fiber spaces with all fibers isomorphic to a fixed compact connected complex manifold are locally trivial. Motivated by this result, we show that if k is an algebraically closed field of infinite transcendence degree over its prime field, then every smooth projective family over a reduced k -scheme of finite type with isomorphic fibers having reduced automorphism group schemes is locally trivial in the étale topology. We do so by reducing the problem to the case when the base is a smooth integral curve, and then, using the vanishing of the Kodaira–Spencer map, we prove formal triviality of such families at every geometric point of the base. We also provide examples of smooth projective fibrewise trivial families in positive characteristic whose Kodaira–Spencer map are nowhere vanishing.

1 Introduction

Suppose $f: \mathcal{X} \rightarrow S$ is a morphism of complex manifolds such that for every $s \in S$, the fibre $\mathcal{X}_s := f^{-1}(s)$ is analytically isomorphic to a fixed complex manifold X . It is natural to investigate to what extent the family f is, locally on S , isomorphic to the trivial family. In 1965, Wolfgang Fischer and Hans Grauert gave the following answer to this question.

Theorem 1.1 [7] *Let $f: \mathcal{X} \rightarrow S$ be a proper submersion of complex manifolds. Suppose that the fibres \mathcal{X}_s are analytically isomorphic to a fixed complex manifold X . Then, f is locally trivial, i.e. for every $s \in S$ there exists an open neighborhood $U_s \subset S$ such that $\mathcal{X}_{U_s} := f^{-1}(U_s)$ and $X \times U_s$ are isomorphic as analytic fibre bundles over U_s .*

Generalizations of this result are known in the context of complex spaces (in the sense of Grauert). An analogous statement when $f: \mathcal{X} \rightarrow S$ is a flat proper morphism of complex spaces with S reduced was proved Schuster [26, Satz 4.9]. Other results may also be found e.g. in a work of Donin [4, 3.1, 3.2] and in a work of Hauser–Müller [14].

One may expect that a theorem of this kind should have an analog in algebraic geometry. In this paper, we show that it is indeed the case. Before we state our main result, let us introduce some terminology that we use throughout this work.

Definition 1.2 Let k be an algebraically closed field and let X be a k -scheme. A flat morphism of finite type k -schemes $f: \mathcal{X} \rightarrow S$ is **fibrewise trivial** (with fibre X), if there exists an

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isomorphism $\mathcal{X}_s = \mathcal{X} \times_S s \simeq X$ for every $s \in S(k)$. A fibrewise trivial family f with fibre X is **trivial** if \mathcal{X} and $X \times_k S$ are isomorphic as S -schemes. A family f is **Zariski-locally trivial** if for every $s \in S(k)$ there exists an open neighborhood $s \in U \subset S$ such that $f_U: \mathcal{X}_U \rightarrow U$ is trivial. We say that a morphism of finite type k -schemes $f: \mathcal{X} \rightarrow S$ is **isotrivial at $s \in S(k)$** if there exists an étale neighborhood $V \rightarrow S$ of s such that $f_V: \mathcal{X} \times_S V \rightarrow V$ is trivial. We call f **isotrivial** if it is isotrivial at every $s \in S(k)$.

It is well-known that an analog of Theorem 1.1 does not hold if we interpret the concept of local triviality in the naïve manner, i.e. a fibrewise trivial family $f: \mathcal{X} \rightarrow S$ may not be Zariski-locally trivial. A standard example is to take $S = \text{Spec } k[t, t^{-1}]$ and consider an elliptic S -scheme $p: \mathcal{E} \subset \mathbb{P}_S^2 = \text{Proj } \mathcal{O}_S[X, Y, Z] \rightarrow S$ cut out by the equation $Y^2Z - X^3 - tZ^3 = 0$. Clearly \mathcal{E}/S is fibrewise trivial. The generic fibre \mathcal{E}_η of \mathcal{E} is an elliptic curve over $\mathbb{C}(t)$ defined by the equation $Y^2Z = X^3 - tZ^3$ and it is not isomorphic to the base change $\mathcal{E}_1 \times_{\mathbb{C}} \mathbb{C}(t)$ of the fibre of p [23, Exercise 4.D],¹ so p isn't Zariski-locally trivial.

Although the Zariski topology is too coarse for our purposes, the above example suggests that we could obtain more satisfying results when working in finer Grothendieck topologies, in particular in the étale topology of the base of considered family. From Definition 1.2, it follows that when S is connected, an isotrivial family $f: \mathcal{X} \rightarrow S$ is fibrewise trivial. In this paper, we investigate whether the converse statement holds under some reasonable assumptions about the family f . This topic appears to be well-known to experts. In [17, Remark 1.4], the author mentions briefly the fact that projective families over \mathbb{C} with isomorphic fibers are *generically* locally trivial with respect to the étale topology.² In [2, Section 1], authors claim that an algebraic variant of Theorem 1.1 should follow from [27, Proposition 2.6.10] which states that isotriviality is equivalent to formal triviality (6.1) for flat projective families.

That being said, none of the sources listed above provides a precise statement of a variant of Theorem 1.1. Nor is it immediately clear what assumptions about a fibrewise trivial family $f: \mathcal{X} \rightarrow S$ are sufficient to deduce formal triviality of f at every closed points of its base. To the author's knowledge, both an explicit statement and a proof of such a result appear to be missing from the literature. In what follows, we attempt to fill the gap by providing a precise statement of an algebraic variant of Theorem 1.1 (Theorem 1.3) for fibrewise trivial families of projective k -varieties in the case when k is a *sufficiently large* algebraically closed field, and all varieties and morphisms are algebraic, together with a detailed proof of this assertion. Namely, our main result is the following theorem.

Theorem 1.3 *Let k be an algebraically closed field of infinite transcendence degree over its prime field. Suppose that $f: \mathcal{X} \rightarrow S$ is a smooth, projective, fibrewise trivial morphism of finite type k -schemes with fibre X . Suppose furthermore that S is reduced, and that the automorphism group scheme (Proposition 2.5) Aut_k^X of the fibre is reduced. Then f is isotrivial.*

Note that the assumption about Aut_k^X being reduced is automatic when $\text{char } k = 0$ [21, Theorem 3.23].

Let us present a brief outline of the proof of Theorem 1.3 when $\text{char } k = 0$. First, we prove generic isotriviality (Theorem 3.2), a minor generalization of the result mentioned in [17], where the author considers only schemes over complex numbers. This is the only step in the proof where we make use of the condition $\text{tr. deg}(k/\mathbb{F}) = \infty$ for \mathbb{F} being the prime field of k .

¹ But they become isomorphic after passing to the extension $\mathbb{C}(t^{\frac{1}{6}})$ of $\mathbb{C}(t)$.

² Note, however, that the notion of isotriviality in op.cit. differs from the one being used throughout this paper: compare [17, Definition 1.2] with Definition 1.2.

We use this result to show that the relative Kodaira–Spencer map $\text{KS}_f : \mathcal{T}_{S/k} \rightarrow R^1 f_* \mathcal{T}_{\mathcal{X}/S}$ associated to f vanishes on S (Proposition 4.2). Next, we reduce Theorem 1.3 to the situation when S is a smooth integral curve of finite type over k , using a variant of the valuative criterion of flatness (Proposition 5.1). Finally (Sect. 6), we prove formal triviality at every closed point of the base with an explicit calculation, using the vanishing of KS_f .

The strategy behind the proof of Theorem 1.3 in positive characteristic (Sect. 7) is the same except for a few caveats—the final calculation done in the proof of the char $k = 0$ case relies heavily on this assumption. Hence, assuming $\text{char } k = p > 0$, we prove formal triviality using the fact that now our schemes are equipped with Frobenius morphism. We apply a variant of Ogus’ lemma (Proposition 7.2), which originally states that the vanishing of the Kodaira–Spencer map guarantees that a smooth proper family $\mathcal{X} \rightarrow S = \text{Spec } k[[t]]$ descends along Frobenius $F_S : S \rightarrow S$ (Lemma 7.1). In our setting, when S is a smooth curve over k , we show that the descent can be done in an appropriate étale neighborhood of any given point $s \in S(k)$. We deduce this fact from the original result of Ogus by approximating the scheme $\text{Spec } \widehat{\mathcal{O}}_{S,s}$ with smooth S -schemes [28, Tag 07GC].

Lastly, we show how to construct a fibrewise trivial family with nowhere vanishing Kodaira–Spencer map (Sect. 8), thereby proving that \mathcal{X}/S may not even be generically isotrivial in the case when Aut_k^X is non-reduced.

We think that an interesting question that we didn’t find an answer to is whether we can remove the assumptions about the scheme Aut_k^X by working with fppf topology.

Question 1.4 Let k be an algebraically closed field of infinite transcendence degree over its prime field. Suppose that $f : \mathcal{X} \rightarrow S$ is a smooth, projective, fibrewise trivial morphism of finite type k -schemes with S reduced. Does it imply that f is fppf-locally trivial?

2 Preliminaries

2.1 Notation

Throughout, k always denotes an algebraically closed field. Given a scheme S , we denote the category of S -schemes by Sch_S . When $S = \text{Spec } A$ is affine, we write Sch_A for $\text{Sch}_{\text{Spec } A}$. An opposite category to category \mathcal{C} is denoted by \mathcal{C}^{op} . We often write \mathcal{X}/S for an S -scheme $\mathcal{X} \rightarrow S$, provided that the structure morphism is clear from the context. Given \mathcal{X}/S and $\phi : T \rightarrow S$, we write $\phi^* \mathcal{X}$ for a choice of pullback of \mathcal{X} along ϕ . When the structure morphism $T \rightarrow S$ is clear, we also denote $\phi^* \mathcal{X}$ with \mathcal{X}_T .

Given a scheme S , we denote with $\text{Coh}(S)$ the category of coherent sheaves on S . When $\mathcal{F} \in \text{Coh}(S)$ and $s \in S$ is a point, we denote with \mathcal{F}_s the stalk of \mathcal{F} at s and by $\mathcal{F}(s)$ the fibre of \mathcal{F} at s , i.e. $\mathcal{F}(s) = \mathcal{F}_s / \mathfrak{m}_s \mathcal{F}_s$, where $\mathfrak{m}_s \subset \mathcal{O}_{S,s}$ is the maximal ideal.

Given a ring A and $r \in A$, we denote r -torsion of A by $A[r]$. We also write $A[r^\infty] := \bigcup_{n \geq 0} A[r^n]$.

A few times, e.g. in the proof of generic isotriviality (Theorem 3.2) and of étale local Frobenius descent (Proposition 7.2) for families with vanishing Kodaira–Spencer map, we will make use of the fact that a finite type morphism $f : \mathcal{X} \rightarrow \text{Spec } A$ can be described with finitely many equations only, hence can be defined over a finitely generated \mathbb{Z} -algebra $B \subset A$, see [16, II.5.10].

Definition 2.2 Suppose $S = \text{Spec } A$ is affine, let $B \subset A$ be a subring, and let $\mathcal{X} \in \text{Sch}_S$. We say that \mathcal{X} is **defined over** B if there exists a **model of \mathcal{X} over B** , i.e. $\mathcal{X}^{(B)} \in \text{Sch}_B$ of finite type such that \mathcal{X} is isomorphic to $\mathcal{X}^{(B)} \times_{\text{Spec } B} S$ as S -schemes.

2.3 Hom, Isom and Aut schemes

From now on, let S be a finite type k -scheme and \mathcal{X}, \mathcal{Y} be two S -schemes. We define the functors $\mathbf{Hom}_S^{\mathcal{X}, \mathcal{Y}}$ of morphisms and $\mathbf{Isom}_S^{\mathcal{X}, \mathcal{Y}}$ of isomorphisms between \mathcal{X} and \mathcal{Y} . Using the theory of Hilbert scheme [10], one can show that these functors are representable by a locally finite type S -schemes $\text{Hom}_S^{\mathcal{X}, \mathcal{Y}}$ and $\text{Isom}_S^{\mathcal{X}, \mathcal{Y}}$ respectively. The existence of $\text{Isom}_S^{\mathcal{X}, \mathcal{Y}}$ will be the main tool in our proof of Theorem 1.3 and we will use the existence of $\text{Hom}_S^{\mathcal{X}, \mathcal{Y}}$ in Sect. 8.

Definition 2.4 Suppose that \mathcal{X} and \mathcal{Y} are S -schemes. We define the functor of morphisms $\mathbf{Hom}_S^{\mathcal{X}, \mathcal{Y}} : \text{Sch}_S^{\text{op}} \rightarrow \text{Set}$ as follows: given $T \in \text{Sch}_S$ we set

$$\mathbf{Hom}_S^{\mathcal{X}, \mathcal{Y}}(T) = \left\{ \phi : \mathcal{X}_T \longrightarrow \mathcal{Y}_T \mid \begin{array}{l} \phi \text{ is a morphism} \\ \text{of } T\text{-schemes} \end{array} \right\}.$$

We also define the functor of isomorphisms $\mathbf{Isom}_S^{\mathcal{X}, \mathcal{Y}} : \text{Sch}_S^{\text{op}} \rightarrow \text{Set}$ as follows: given $T \in \text{Sch}_S$ we set

$$\mathbf{Isom}_S^{\mathcal{X}, \mathcal{Y}}(T) = \left\{ \phi : \mathcal{X}_T \longrightarrow \mathcal{Y}_T \mid \begin{array}{l} \phi \text{ is an isomorphism} \\ \text{of } T\text{-schemes} \end{array} \right\}.$$

Also, we define $\mathbf{Aut}_S^{\mathcal{X}} := \mathbf{Isom}_S^{\mathcal{X}, \mathcal{X}}$.

Let $\mathcal{X} \in \text{Sch}_S$. Then we define the Hilbert functor $\mathbf{Hilb}_S^{\mathcal{X}} : \text{Sch}_S^{\text{op}} \rightarrow \text{Set}$ associating to $S' \in \text{Sch}_S$ the set of S' -flat closed subschemes of $\mathcal{X} \times_S S'$ [10, p. 265]. When \mathcal{X}/S is projective, then $\mathbf{Hilb}_S^{\mathcal{X}}$ is representable by an S -scheme, denoted $\text{Hilb}_S^{\mathcal{X}}$ (op.cit.).

Proposition 2.5 [10, 4.c] Suppose \mathcal{X}/S and \mathcal{Y}/S are projective. Then $\mathbf{Hom}_S^{\mathcal{X}, \mathcal{Y}}$ and $\mathbf{Isom}_S^{\mathcal{X}, \mathcal{Y}}$ are representable by schemes, locally of finite type over S . We denote these scheme with $\text{Hom}_S^{\mathcal{X}, \mathcal{Y}}$ and $\text{Isom}_S^{\mathcal{X}, \mathcal{Y}}$ respectively. What's more, the obvious functor $\mathbf{Isom}_S^{\mathcal{X}, \mathcal{Y}} \rightarrow \mathbf{Hom}_S^{\mathcal{X}, \mathcal{Y}}$ induces an open embedding $\text{Isom}_S^{\mathcal{X}, \mathcal{Y}} \rightarrow \text{Hom}_S^{\mathcal{X}, \mathcal{Y}}$ of S -schemes and similarly the functor $\mathbf{Hom}_S^{\mathcal{X}, \mathcal{Y}} \rightarrow \mathbf{Hilb}_S^{\mathcal{X} \times_S \mathcal{Y}}$ associating morphisms to their graphs [27, 4.6.6] induces an open embedding $\text{Hom}_S^{\mathcal{X}, \mathcal{Y}} \rightarrow \text{Hilb}_S^{\mathcal{X} \times_S \mathcal{Y}}$ of S -schemes.

Corollary 2.6 Suppose that \mathcal{X}/S is projective. Then the functor $\mathbf{Aut}_S^{\mathcal{X}}$ is representable by a locally of finite type group S -scheme, denoted $\text{Aut}_S^{\mathcal{X}}$.

Now suppose $\mathcal{X} \rightarrow S$ is projective and fix a relatively very ample line bundle $\mathcal{O}_{\mathcal{X}}(1)$ on \mathcal{X} . Let $\mathcal{F} \in \text{Coh}(\mathcal{X})$. Given a point $s \in S$, write \mathcal{F}_s for $\mathcal{F}_{\mathcal{X}_s}$. Then, $\mathcal{O}_{\mathcal{X}}(1)$ restricts to a very ample line bundle $\mathcal{O}_{\mathcal{X}_s}(1)$ on the projective $k(s)$ -scheme \mathcal{X}_s . Thus, for every point $s \in S$, we obtain a Hilbert polynomial $p_{\mathcal{F}_s}(m)$. In case $\mathcal{F} = \mathcal{O}_{\mathcal{Z}}$ for some subscheme $\mathcal{Z} \subset \mathcal{X}$, we denote $p_{\mathcal{O}_{\mathcal{Z}_s}}$ with $p_{\mathcal{Z}_s}$.

Consider the function $s \mapsto p_{\mathcal{F}_s}(m)$ for $s \in S(k)$. In case \mathcal{F} is S -flat, this function is locally constant on S . Given $P \in \mathbb{Q}[y]$, we define the subfunctor $\mathbf{Hilb}_S^{\mathcal{X}, P}$ of $\mathbf{Hilb}_S^{\mathcal{X}}$ as follows

$$\mathbf{Hilb}_S^{\mathcal{X}, P}(T) = \left\{ \mathcal{Z} \subset \mathcal{X} \times_S T \mid \begin{array}{l} \mathcal{Z} \text{ is a } T\text{-flat closed subscheme of } \mathcal{X} \times_S T, \\ \text{and } p_{\mathcal{Z}_t} = P \text{ for all } t \in T \end{array} \right\}.$$

Then

$$\mathbf{Hilb}_S^{\mathcal{X}} = \bigsqcup_{P \in \mathbb{Q}[y]} \mathbf{Hilb}_S^{\mathcal{X}, P} \tag{2.6.1}$$

and when \mathcal{X}/S is projective, each $\mathbf{Hilb}_S^{\mathcal{X}, P}$ is representable by a *projective* S -scheme.

2.7. As a result, when \mathcal{X}/S and \mathcal{Y}/S are projective, by Proposition 2.5 and (2.6.1), we find that $\mathbf{Isom}_S^{\mathcal{X}, \mathcal{Y}} = \bigsqcup_{n \in \mathbb{N}} I_n$ is a countable disjoint union with each I_n quasi-projective over S . In particular, for $p : \mathbf{Isom}_S^{\mathcal{X}, \mathcal{Y}} \rightarrow S$ denoting the structure morphism, we deduce from the Chevalley Theorem that each $p(I_n)$ is constructible in S .

2.8 Exponential law

Let $\mathbf{Hom}_S^{\mathcal{X}, \mathcal{Y}}$ be the functor as in Definition 2.4. Then, treating schemes as functors of points, we find that we have an isomorphism

$$\mathbf{Hom}_S(T, \mathbf{Hom}_S^{\mathcal{X}, \mathcal{Y}}) \xrightarrow{\sim} \mathbf{Hom}_S(T \times_S \mathcal{X}, \mathcal{Y}) = \mathbf{Hom}_T(T \times_S \mathcal{X}, T \times_S \mathcal{Y}) \tag{2.8.1}$$

which is natural in $T \in \mathbf{Sch}_S$. In (2.8.1), \mathbf{Hom}_S denotes morphisms in the category of sheaves on \mathbf{Sch}_S with fppf topology. When \mathcal{X} and \mathcal{Y} are projective over S , then $\mathbf{Hom}_S^{\mathcal{X}, \mathcal{Y}}$ is representable by an S -scheme so (2.8.1) induces a bijection

$$\mathbf{Hom}_S(T, \mathbf{Hom}_S^{\mathcal{X}, \mathcal{Y}}) \xrightarrow{\sim} \mathbf{Hom}_S(T \times_S \mathcal{X}, \mathcal{Y}) \tag{2.8.2}$$

between sets of morphisms of S -schemes. We call it the **exponential law**.

2.9 Deformation theory

Let k be an algebraically closed field. Denote \mathbf{Art}_k the full subcategory of the category of k -algebras, consisting of local Artinian k -algebras with residue field equal to k .

Definition 2.10 (see [27, 1.2.1]) Let $X \in \mathbf{Sch}_k$. A **deformation of X over $A \in \mathbf{Art}_k$** is a Cartesian diagram

$$\begin{array}{ccc} X & \xrightarrow{j} & \mathcal{X} \\ \downarrow & & \downarrow \pi \\ \mathrm{Spec} k & \longrightarrow & \mathrm{Spec} A \end{array}$$

with π flat.

We denote the set of isomorphism classes of deformations of X over A by $\mathrm{Def}_X(A)$. The association $A \mapsto \mathrm{Def}_X(A)$ defines a functor $\mathrm{Def}_X : \mathbf{Art}_k \rightarrow \mathbf{Set}$. In particular, notice that here, a deformation is the data of an A -scheme \mathcal{X} and an isomorphism $X \xrightarrow{\sim} \mathcal{X} \times_A k$.

Definition 2.11 [27, p. 44] Suppose $X \in \mathbf{Sch}_k$. We say that Def_X is **pro-representable** if $\mathrm{Def}_X(\cdot) = \mathbf{Hom}_{k\text{-alg}}(R, \cdot)$ for some complete local noetherian k -algebra R with residue field k . We say $X \in \mathbf{Sch}_k$ has **pro-representable deformations** when Def_X is pro-representable.

The common tool for determining whether X has pro-representable deformations is *Schlessinger’s criterion* [25, Theorem 2.11]. Standard examples of schemes with pro-representable deformations include smooth projective connected curves over k [27, Corollary 2.6.6] and abelian varieties [24, Theorem 2.2.1].

3 Generic isotriviality

The first step in our proof of Theorem 1.3 is showing that our claim holds generically on (i.e., over an open dense subset of) S . The proof of this part relies on the following well-known lemma from the theory of constructible sets.

Proposition 3.1 *Let k be an algebraically closed field of infinite transcendence degree over its prime field \mathbb{F} , and let S be an irreducible k -scheme of finite type. Suppose that $\mathcal{C} = \{C_i\}_{i \in \mathbb{N}}$ is a countable family of constructible subsets of S , satisfying $S(k) \subset \bigcup_{i \in \mathbb{N}} C_i(k)$. Additionally, assume that there exists an intermediate extension $\mathbb{F} \subset \ell \subset k$ with $\text{tr. deg}(\ell/\mathbb{F}) < \infty$ such that S admits a model S^ℓ over ℓ and all $C_i \in \mathcal{C}$ admit models C_i^ℓ which are constructible subsets of S^ℓ . Then there exists $C \in \mathcal{C}$ dense in S .*

Proof Let $\eta \in S$ be the generic point of S . Every constructible set is a union of finitely many constructible irreducible sets. Therefore, by replacing each C_i with the collection of its irreducible components, we may assume that \mathcal{C} is a countable family of constructible irreducible subsets of S . If $\dim S = 0$ then $|S|$ is a single point so there is nothing to prove. Hence we may suppose $\dim S \geq 1$. We prove the assertion in three steps.

Step 1: Suppose that $S = \mathbb{A}_k^1 = \text{Spec } k[T]$. Replacing ℓ with $\bar{\ell}$, we may assume that ℓ is algebraically closed (preserving the condition $\text{tr. deg}(\ell/\mathbb{F}) < \infty$). By assumption there exist constructible subsets $C_i^\ell \subset \mathbb{A}_\ell^1$ such that $C_i = C_i^\ell \otimes_\ell k$. If none of C_i is dense in S , then each C_i is a finite union of closed points, thus so is each C_i^ℓ . For $\lambda \in k \setminus \ell$, let $m = (T - \lambda)$ and suppose $p = [m] \in \mathbb{A}_k^1$ belongs to some C_i . Then the image of p in \mathbb{A}_ℓ^1 is the generic point of \mathbb{A}_ℓ^1 , hence C_i^ℓ is dense, a contradiction.

Step 2: $S = \mathbb{A}_k^d$. We proceed by induction on d . The base of induction $d = 1$ is done in **Step 1**. Let $d > 1$ and suppose \mathbb{A}_k^r satisfies the assertion of the proposition for all $1 \leq r < d$. Suppose that none of the C_i 's contain η . Then $\dim C_i \leq d - 1$ for all $i \in \mathbb{N}$. Since every C_i is assumed to be irreducible, there exists at most one hyperplane $H_i \subset S$ contained in $\overline{C_i}$. Let $H \subset \mathbb{A}_k^d$ be a hyperplane different from H_i 's (such an H exists because each H_i is defined over ℓ hence it suffices to choose H which is not defined over ℓ) with generic point ξ . By our assumptions, $\overline{C_i} \cap H \subsetneq H$ is a proper closed subset of H , so $\xi \notin C_i \cap H$. The family $\mathcal{C}|_H := \{C_i \cap H\}_{i \in \mathbb{N}}$ consists of countably many constructible subsets of H , with none of $C_i \cap H$ dense in H . By the inductive assumption, $H(k) \not\subset \bigcup_{i \in \mathbb{N}} (C_i \cap H)$. Then, for every $p \in H(k) \setminus (\bigcup_{i \in \mathbb{N}} (C_i \cap H))$ we have $p \in S(k) \setminus (\bigcup_{i \in \mathbb{N}} C_i)$, a contradiction.

Step 3: Let S be an arbitrary irreducible k -scheme of dimension $d \geq 1$. We may assume that S is affine. By Noether's Normalization Lemma [28, Tag 0CBI] there exists a finite surjective map $\pi : S \rightarrow \mathbb{A}_k^d$. For the generic point $\mu \in \mathbb{A}_k^d$, we have $\eta = \pi^{-1}(\mu)$. Write $Z_i = \pi(C_i)$. Each Z_i is irreducible and by Chevalley's Theorem it is also constructible. Surjectivity of π implies $\mathbb{A}_k^d(k) \subseteq \bigcup_{i \in \mathbb{N}} Z_i$. By **Step 2**, there exists $i_0 \in \mathbb{N}$ such that Z_{i_0} is dense in \mathbb{A}_k^d , meaning that $\mu \in Z_{i_0}$. Then $\eta \in C_{i_0}$, proving that C_{i_0} is dense in S . \square

We apply this result to the scheme of isomorphisms $\text{Isom}_S^{\mathcal{X}, X \times S}$ to show that one of its components dominates S .

Theorem 3.2 *Suppose that k is algebraically closed, of infinite transcendence degree over its prime field \mathbb{F} . Let $f : \mathcal{X} \rightarrow S$ be a smooth, projective, fibrewise trivial morphism of finite type k -schemes with S reduced and with fibre X . Then there exists an open dense subset U of S such that $f_U : \mathcal{X}_U \rightarrow U$ is fppf-locally trivial. If we assume furthermore that the group scheme $\text{Aut}_k^{\mathcal{X}} := \text{Isom}_k^{\mathcal{X}, \mathcal{X}}$ is reduced, then \mathcal{X}_U/U is isotrivial.*

Proof It is enough that we restrict ourselves to the case when S is affine. Let $\mathbb{F} \subset \ell \subset k$ be an algebraically closed subfield of k satisfying $\text{tr. deg}(\ell/\mathbb{F}) < \infty$, such that there exist models (Definition 2.2) $S^\ell, \mathcal{X}^\ell, X^\ell$ and $f^\ell : \mathcal{X}^\ell \rightarrow S^\ell$ over ℓ of S, \mathcal{X}, X and f respectively (see [16, II.5.10]), with the property that f^ℓ is fibrewise trivial with fiber X^ℓ . Since $\text{tr. deg}(\ell/\mathbb{F}) < \infty$, we find that $\text{tr. deg}(k/\ell)$ is infinite.

By Proposition 2.5, $\mathcal{I} = \text{Isom}_S^{\mathcal{X}, X \times_k S}$ and $\mathcal{I}^\ell = \text{Isom}_{S^\ell}^{\mathcal{X}^\ell, X^\ell \times_k S^\ell}$ are schemes locally of finite type over S and S^ℓ respectively. Moreover, we have $\mathcal{I}^\ell \otimes_\ell k \simeq \mathcal{I}$ and the structural morphisms $p : \mathcal{I} \rightarrow S$ and $p^\ell : \mathcal{I}^\ell \rightarrow S^\ell$ satisfy $p^\ell \times_{S^\ell} S = p$. By 2.7, $\mathcal{I}^\ell = \bigsqcup_{i \in \mathbb{N}} \mathcal{I}_i^\ell$ with each \mathcal{I}_i^ℓ locally of finite type over S^ℓ . Since f is fibrewise trivial, $\mathcal{I}_s \simeq \text{Isom}_k^{\mathcal{X}_s, X} \simeq \text{Aut}_k^X$ is nonempty at every $s \in S(k)$. Thus $S(k) \subseteq p(\mathcal{I}) = \bigcup_{i \in \mathbb{N}} p(\mathcal{I}_i)$ with $\mathcal{I}_i = \mathcal{I}_i^\ell \times_{S^\ell} S$. Again by 2.7, each $p(\mathcal{I}_i)$ is constructible in S , and since each \mathcal{I}_i is defined over ℓ , it follows that each $p(\mathcal{I}_i)$ is defined over ℓ as well. Hence by Proposition 3.1 for some $j \in \mathbb{N}$ and $I := \mathcal{I}_j$, the image $p(I)$ contains an open dense subset of S . Since S is reduced, by [28, Tag 052B] there exists $U \subseteq p(I)$ nonempty and open in S such that $p : I_U \rightarrow U$ is faithfully flat. Then $(\text{id}_{I_U} : I_U \rightarrow I_U) \in \mathbf{Isom}_U^{\mathcal{X}_U, X \times_k U}(I_U)$ viewed as a morphism of U -schemes gives an isomorphism $\mathcal{X}_{I_U} \simeq X \times_k I_U$ of I_U -schemes, proving that \mathcal{X}_U/U is fppf-locally trivial.

Suppose furthermore that $I_s = \text{Aut}_k^X$ is reduced. Then I_s is smooth over k by [21, Proposition 1.28], hence $p|_{I_U} : I_U \rightarrow U$ is smooth. By [13, 17.16.3], there exists an étale surjective morphism $V \rightarrow U$ such that the restriction $I_V \rightarrow V$ admits a section $\sigma : V \rightarrow I_V$. Then $\sigma \in \text{Isom}_V^{\mathcal{X}_V, X \times_k V}(V)$ proving that $\mathcal{X}_V \simeq X \times_k V$. Therefore $f_U : \mathcal{X}_U \rightarrow U$ is isotrivial. \square

Remark 3.3 The above step is the only one in the proof of Theorem 1.3 where the assumption about k being *large* is used. Once the generic isotriviality holds for fibrewise trivial families over \mathbb{F} -varieties for \mathbb{F} being a prime field, so does Theorem 1.3, but the author does not know how to get rid of the assumption $\text{tr. deg}(k/\mathbb{F}) = \infty$.

4 Global vanishing of the Kodaira–Spencer map

In this short section we review the construction of the Kodaira–Spencer map and prove that the Kodaira–Spencer map for fibrewise trivial families $f : \mathcal{X} \rightarrow S$ satisfying assumptions as in Theorem 1.3 vanishes.

4.1 Review of the Kodaira–Spencer map

Let $f : \mathcal{X} \rightarrow S$ be a smooth fibrewise trivial morphism of schemes with S reduced. Recall that the cotangent sequence

$$0 \longrightarrow f^* \Omega_{S/k}^1 \longrightarrow \Omega_{\mathcal{X}/k}^1 \longrightarrow \Omega_{\mathcal{X}/S}^1 \longrightarrow 0 \tag{4.1.1}$$

is exact [13, 17.2.3]. The cotangent sheaf $\Omega_{\mathcal{X}/S}^1$ is locally free, which implies that $\mathcal{E}xt^1(\Omega_{\mathcal{X}/S}^1, \mathcal{O}_{\mathcal{X}})$ is trivial, hence the dualization of (4.1.1)

$$0 \longrightarrow \mathcal{T}_{\mathcal{X}/S} \longrightarrow \mathcal{T}_{\mathcal{X}/k} \longrightarrow f^* \mathcal{T}_{S/k} \longrightarrow 0$$

is also exact. After applying the left-exact functor f_* , we obtain a long exact sequence of \mathcal{O}_S -modules

$$0 \longrightarrow f_*\mathcal{T}_{\mathcal{X}/S} \longrightarrow f_*\mathcal{T}_{\mathcal{X}/k} \longrightarrow f_*f^*\mathcal{T}_{S/k} \xrightarrow{\delta_{\mathcal{X}/S}} R^1f_*\mathcal{T}_{\mathcal{X}/S} \longrightarrow \dots \tag{4.1.2}$$

Denote the composition of $\delta_{\mathcal{X}/S}$ with the adjunction unit

$$\mathcal{T}_{S/k} \longrightarrow f_*f^*\mathcal{T}_{S/k} \xrightarrow{\delta_{\mathcal{X}/S}} R^1f_*\mathcal{T}_{\mathcal{X}/S} \tag{4.1.3}$$

by KS_f . Note that fibrewise triviality of \mathcal{X}/S implies $s \mapsto h^1(\mathcal{X}_s, \mathcal{T}_{\mathcal{X}_s/k})$ is a constant function on $S(k)$. It follows from Grauert’s Semicontinuity Theorem [11, 7.8.4] that $R^1f_*\mathcal{T}_{\mathcal{X}/S}$ is locally free and that the natural morphism

$$R^1f_*\mathcal{T}_{\mathcal{X}/S}(s) \longrightarrow H^1(\mathcal{X}_s, \mathcal{T}_{\mathcal{X}_s/k})$$

is an isomorphism. Therefore, the map KS_f specializes at a point $s \in S(k)$ under the above isomorphism to $\text{KS}_f(s) : \mathcal{T}_{S/k}(s) \longrightarrow H^1(\mathcal{X}_s, \mathcal{T}_{\mathcal{X}_s/k})$.

Proposition 4.2 *Assume $f : \mathcal{X} \rightarrow S$ satisfies conditions of Theorem 1.3. The Kodaira–Spencer morphism KS_f vanishes on S .*

Proof By Proposition 3.2, there exists an open dense subset $U \subset S$ such that f_U is isotrivial. By [27, Corollary 2.6.11], $\text{KS}_f(s) = 0$ for all $s \in U(k)$. Since KS_f is a morphism of locally free sheaves and S (hence U) is reduced, it follows that $\text{KS}_f|_U \equiv 0$. Hence, the image of KS_f is a torsion subsheaf of $R^1f_*\mathcal{T}_{\mathcal{X}/S}$. Since $R^1f_*\mathcal{T}_{\mathcal{X}/S}$ is locally free, it follows that the image must be trivial. \square

Remark 4.3 In fact, the assumption that Aut_k^X is reduced is crucial for $\text{KS}_f = 0$. Later, in Sect. 8, we show the existence of fibrewise trivial families with nowhere vanishing Kodaira–Spencer maps.

5 A variant of the valuative criterion of flatness

Our intermediate step in the proof of Theorems 1.3 is the reduction to the case when S is a smooth curve. As we show in Proposition 6.4, isotriviality of a family $f : \mathcal{X} \rightarrow S$ is equivalent to flatness of $\text{Isom}_S^{\mathcal{X}, X \times S}$. It turns out that it is sufficient to check the latter condition only on smooth curves mapping to S . This is captured in the main result of this section, Proposition 5.1. This proposition is inspired by Grothendieck’s valuative criterion of flatness [12, 11.8.1], as well as notes of McKernan [20, Lemma 4.10], but it appears that one cannot simply apply this result here, since not all valuation rings arise as stalks of structure sheaves of smooth k -curves.

Proposition 5.1 *Let $p : \mathcal{I} \rightarrow S$ be a finite type morphism with geometrically reduced fibers with S reduced and of finite type over k . Suppose furthermore that for every smooth integral curve C over k and k -morphism $C \rightarrow S$, the base change $p_C : \mathcal{I}_C \rightarrow C$ is flat. Then p is flat.*

While for simplicity of the argument we prove this claim as stated, we also want to announce that the assumption about geometrically reduced fibers is in fact redundant. An

alternative proof of the variant of Proposition 5.1 where condition on fibers is dropped will appear elsewhere.

The idea behind our proof below is as follows: first, by blowing up singularities of S we show that it is enough to restrict ourselves to morphisms with smooth targets. Then, working via induction on $\dim S$ we reduce the problem of flatness to the case when S itself is a smooth integral curve. The key tool in the inductive step is the slicing criterion of flatness [5, Cor. 6.9] and the assumption on geometrically reduced fibers allows us to apply it without complications.

Another key ingredient in the proof of Proposition 5.1 is the following fact.

Proposition 5.2 [12, 11.6.1] *Let A be a local, integral, geometrically unibranch [28, Tag 06DT] ring. Let $S = \text{Spec } A$ and $f : X \rightarrow S$ be a finitely presented morphism. Let A' be a local ring and $S' = \text{Spec } A'$. Suppose $u^\# : A \rightarrow A'$ is a local injective homomorphism and let $u : S' \rightarrow S$ be the associated morphism of schemes. Let $x \in X$ with $s = f(x)$ closed in S . Suppose $x' \in X' = X \times_S S'$ is a point whose image in X is equal to x and whose image in S' is the closed point $s' \in S'$. If X' is S' -flat at x' , then X is S -flat at x .*

Our proof of Proposition 5.1 consists of the following two lemmas.

Lemma 5.3 *Suppose Proposition 5.1 holds when S is smooth over k . Then Proposition 5.1 is true for an arbitrary reduced S of finite type over k .*

Proof Let $\{S_i\}_{i=1}^m$ be the set of irreducible reduced components of S , let $S' = \bigsqcup_{i=1}^m S_i$ and write $\mathcal{I}_i = \mathcal{I} \times_S S_i$. The canonical map $g : S' \rightarrow S$ is finite (hence in particular proper), surjective, and since S is reduced, it follows that for every $s \in S$, the associated morphism $\mathcal{O}_{S,s} \rightarrow (g_* \mathcal{O}_{S'})_s$ is injective. Therefore, by applying [12, 11.5.3], it suffices to prove that $\mathcal{I} \otimes_S S'$ is S' -flat, i.e. that each \mathcal{I}_i is flat over S_i . In other words, we may assume that S is integral.

Let $\nu : S' \rightarrow S$ be the normalization of S in the function field $K(S)$. By Noether’s theorem [5, Corollary 13.13] ν is a finite surjective morphism. By [12, 11.5.3], it is enough to show that the base change $p' : \mathcal{I}' = \mathcal{I} \times_S S' \rightarrow S'$ of p is flat. Hence, we may additionally assume that S is normal.

Let $y \in \mathcal{I}(k)$ and $p(y) = s \in S$. Since S is normal, $\mathcal{O}_{S,s}$ is geometrically unibranch. Let $\rho : \tilde{S} \rightarrow S$ be a normalization in $K(S)$ of a blow-up of S at s . Then $\rho^{-1}(s)$ is a codimension 1 subset in \tilde{S} . Denote the smooth locus of \tilde{S} by \tilde{S}° . Since \tilde{S} is normal, $\tilde{S} \setminus \tilde{S}^\circ$ is a codimension ≥ 2 subset. Hence there exists $\tilde{s} \in \tilde{S}^\circ$ satisfying $\rho(\tilde{s}) = s$. By assumption $p_{\tilde{S}^\circ} : \mathcal{I}_{\tilde{S}^\circ} \rightarrow \tilde{S}^\circ$ is flat. Since ρ is birational, $\mathcal{O}_{S,s} \rightarrow \mathcal{O}_{\tilde{S}^\circ, \tilde{s}}$ is a local injective ring homomorphism. By Proposition 5.2, since $\mathcal{I}_{\tilde{S}^\circ}$ is \tilde{S}° -flat at \tilde{y} for any \tilde{y} lying above y such that $p_{\tilde{S}^\circ}(\tilde{y}) = \tilde{s}$, it follows that \mathcal{I} is S -flat at y . □

Lemma 5.4 *Suppose that Proposition 5.1 is true in case when S is smooth of pure dimension 1 over k . Then Proposition 5.1 is true for every smooth S of finite type over k .*

Proof We induct on $\dim S$. The case $\dim S = 1$ is automatic. Suppose the claim holds for all smooth S' of finite type over k satisfying $\dim S' < d$. Let S be as in the lemma’s statement and such that $\dim S = d$. Since S is smooth, it is a disjoint union of smooth integral schemes, hence we can reduce to the case when S is smooth of dimension d and integral.

We show first that p is an open morphism. Since p is of finite type, by Chevalley’s Theorem the image $p(\mathcal{I})$ is constructible in S . Since S is integral, in order to show that it is open, it suffices to prove that $\dim p(\mathcal{I}) = \dim S$. If $\dim p(\mathcal{I}) < \dim S$, we can cut S down

to a smooth curve C such that $\mathcal{I} \times_S C$ is non-empty and supported at only finitely many points of C , which contradicts flatness of p_C . This proves that p is an open morphism. As a consequence, we may further assume that p is surjective onto S .

Flatness of a morphism is a local condition the target – write $S_{(s)} := \text{Spec } \mathcal{O}_{S,s}$; in order to prove that p is flat, it suffices to show that for all $s \in S$, the base change $p_{(s)} : \mathcal{I} \times_S S_{(s)} \rightarrow S_{(s)}$ is flat at every $x \in \mathcal{I} \times_S S_{(s)}$ mapping to s , so we may further reduce to the case when p is surjective and S is local, i.e. $S = \text{Spec } R$ for R a regular local ring. Denote by \mathfrak{m} the maximal ideal of R and let $x \in \mathcal{I}$ be such that $p(x) = s$. Since flatness is also Zariski-local on the source, we may replace \mathcal{I} by an affine connected open neighborhood x , and write $A := \Gamma(\mathcal{I}, \mathcal{O}_{\mathcal{I}})$, which a finite type R -algebra. Then $\mathcal{I} \rightarrow S$ remains surjective (the image of $p(\mathcal{I})$ is open and contains the only closed point of S) and since R is an integral domain, the corresponding ring homomorphism $R \rightarrow A$ is injective.

Let $r \in \mathfrak{m} \setminus \mathfrak{m}^2$, so that R/r is regular. Since p is assumed to be surjective, by inductive hypothesis, A/r is faithfully flat over R/r . Therefore, by the slicing criterium of flatness [5, Corollary 6.9], in order to conclude that A is R -flat, it is enough to show that A is r -torsion-free. Observe that since geometric fibers of p are reduced, by [9, Prop. 14.57] we know that A/r is reduced.

To prove that $A[r] = 0$, it is sufficient to show the inclusion $A[r^\infty] \subset rA$. Indeed this inclusion implies that $A[r^\infty]$ is r -divisible, so in particular $A[r^\infty] \subset \bigcap_{n \in \mathbb{N}} r^n A$. Since $\text{Spec } A$ is connected, we are done by the Krull’s Intersection Theorem [28, Tag 00IP].

We prove the inclusion $A[r^\infty] \subset rA$. Let $a \in A[r^\infty]$ and $n \in \mathbb{N}$ be such that $ar^n = 0$. Let $r' \in \mathfrak{m} \setminus \mathfrak{m}^2$ be such that r, r' form a regular sequence. By inductive hypothesis A/r' is faithfully flat over R/r' . The latter is a domain, $R/r' \rightarrow A/r'$ is injective and $r^n \neq 0$ in A/r' . Thus A/r' is r^n -torsion-free, hence the equality $ar^n = 0$ in A/r' must imply $a \in r'A$. Since this holds for infinitely many such r' , we deduce that a must be nilpotent in A . Indeed, we have $V(a) \supset \bigcup_{r' \nmid r} V(r'A) = \bigcup_{r' \nmid r} p^{-1}(V(r'))$. To show nilpotence of a , it suffices to show that $\bigcup_{r' \nmid r} V(r'A)$ is dense in \mathcal{I} . This holds because $\bigcup_{r' \nmid r} V(r')$ is dense in S and p is an open morphism. Therefore $V(a) = \mathcal{I}$ as topological spaces, which implies our claim. Finally, since A/r is reduced and a is nilpotent in A , it follows that $a \in rA$ proving that $A[r^\infty] \subset rA$ and completing the proof of the lemma. \square

Proof of 5.1 Clearly the result holds in the case when S is a smooth, integral curve. Now, combine Lemma 5.3 with Lemma 5.4. \square

6 Proof of Theorem 1.3 when char $k = 0$

In this section we prove Theorem 1.3. Using global vanishing of Kodaira–Spencer map KS_f , we show formal triviality of f at every geometric point of S . We do so by reducing the problem to the case when S is a smooth curve. Then we prove the desired result by lifting id_X inductively to S_n -isomorphisms $\mathcal{X}_{S_n} \rightarrow X \times S_n$ for all $n \geq 1$.

6.1 Formal triviality

First, let us recall the notion of formal triviality and how formal triviality implies isotriviality in the case of projective families. Let $f : \mathcal{X} \rightarrow S$ be a morphism of schemes and let $s \in S(k)$. Denote the maximal ideal of $\mathcal{O}_{S,s}$ with \mathfrak{m}_s and write $S_n = \text{Spec } \mathcal{O}_{S,s}/\mathfrak{m}_s^{n+1}$. Take $\mathcal{X}_n := \mathcal{X}_{S_n}$, in particular $\mathcal{X}_0 = \mathcal{X}_s$. Then, each $f_n : \mathcal{X}_n \rightarrow S_n$ is a local deformation of $f_0 : \mathcal{X}_0 \rightarrow S_0$.

These mappings are compatible, i.e. the following diagram

$$\begin{array}{ccccccc}
 \cdots & \longrightarrow & \mathcal{X}_{n-1} & \longrightarrow & \mathcal{X}_n & \longrightarrow & \mathcal{X}_{n+1} & \longrightarrow & \cdots \\
 & & \downarrow f_{n-1} & & \downarrow f_n & & \downarrow f_{n+1} & & \\
 \cdots & \longrightarrow & S_{n-1} & \longrightarrow & S_n & \longrightarrow & S_{n+1} & \longrightarrow & \cdots
 \end{array}$$

commutes. Passing to the colimit, it gives an adic morphism of formal schemes $\mathcal{X} \rightarrow \widehat{S}_y$.

Definition 6.2 We say that \mathcal{X}/S is **formally trivial** at $s \in S(k)$ if for all n , $f_n : \mathcal{X}_n \rightarrow S_n$ is a trivial deformation of f_0 .

Clearly, if a family \mathcal{X}/S is isotrivial at $s \in S(k)$, then it is formally trivial. The converse holds as well, provided the family in question is projective. Our proof of this fact is a variant of [27, Corollary 2.6.10].

Proposition 6.3 *Suppose f is a flat projective family which is formally trivial at $s \in S(k)$. Then f is isotrivial at s .*

Proof The functor $F = \mathbf{Isom}_S^{\mathcal{X}, X \times S}$ is locally of finite presentation. Since f is formally trivial at $s \in S$, the formal completions of \mathcal{X} and $X \times S$ along s are isomorphic as formal schemes over $\text{Spf } \widehat{\mathcal{O}}_{S,s}$. By Grothendieck’s Existence Theorem ([6, Corollary 8.4.7], see also [27, Theorem 2.5.11]), this isomorphism is induced by a unique isomorphism $\xi : \mathcal{X}_{\widehat{\mathcal{O}}_{S,s}} \rightarrow X \times \text{Spec } \widehat{\mathcal{O}}_{S,s}$ of $\widehat{\mathcal{O}}_{S,s}$ -schemes. Hence $\xi \in F(\widehat{\mathcal{O}}_{S,s})$. By Artin’s Approximation Theorem [1, Theorem 1.12], there exists $\tilde{\xi} \in F(\widehat{\mathcal{O}}_{S,s})$ with $\xi \equiv \tilde{\xi} \pmod{\mathfrak{m}}$ for $\widehat{\mathcal{O}}_{S,s}$ the henselization of $\mathcal{O}_{S,s}$. Let $\varphi : S' \rightarrow S$ be an étale neighborhood of s such that $\tilde{\xi}$ is the image of an element $\xi' \in F(S')$. Then $\mathcal{X}_{S'}$ and $X \times S'$ are isomorphic as S' -schemes. The result follows. \square

As announced in Sect. 5, we show that under assumptions of Theorem 1.3, isotriviality is equivalent to flatness of $\mathbf{Isom}_S^{\mathcal{X}, X \times_k S}$.

Proposition 6.4 *Let $f : \mathcal{X} \rightarrow S$ be a smooth, projective, fibrewise trivial morphism with fiber X . Suppose Aut_k^X is reduced. Then f is isotrivial if and only if $\mathbf{Isom}_S^{\mathcal{X}, X \times S}$ is flat over S .*

Proof By our assumptions Aut_k^X is k -smooth [21, Proposition 1.28], thus $\mathbf{Isom}_S^{\mathcal{X}, X \times S}$ is flat over S if and only if it is smooth over S .

If $\mathcal{I} = \mathbf{Isom}_S^{\mathcal{X}, X \times S}$ is smooth over S , then by [13, 17.16.3 (ii)] there exists an étale surjective morphism $S' \rightarrow S$ and an S -morphism $\sigma : S' \rightarrow \mathcal{I}$, proving that $\mathcal{X}_{S'} \simeq X \times S'$.

Conversely, suppose that f is isotrivial. Let $\phi : S' \rightarrow S$ be an étale surjective morphism such that $\mathcal{X}_{S'} \simeq X \times_S S'$ as S' -schemes. Then $\mathcal{I}_{S'} \simeq \mathbf{Isom}_S^{\mathcal{X}, X \times S} \times_S S' \simeq \text{Aut}_k^X \times_k S'$ is obviously S' -flat. We have a Cartesian diagram

$$\begin{array}{ccc}
 \mathcal{I}_{S'} & \xrightarrow{\psi} & \mathcal{I} \\
 p' \downarrow & & \downarrow p \\
 S' & \xrightarrow{\phi} & S
 \end{array}$$

Clearly ψ is flat and so is the composition $\phi \circ p' = p \circ \psi$. The result follows from [28, Tag 02JZ] applied to $X = \mathcal{I}$, $Y = \mathcal{I}_{S'}$ and $\mathcal{G} = \mathcal{O}_{\mathcal{I}}$. \square

Now we are ready to prove our main result, under the assumption $\text{char } k = 0$.

Proof of Theorem 1.3 in the case $\text{char } k = 0$ By Proposition 6.4 it suffices to show that $\text{Isom}_S^{\mathcal{X}, \mathcal{X} \times S}$ is flat over S . By Proposition 5.1 it is enough to prove it under the assumption that S is a smooth integral curve. Fix $s \in S(k)$. By Propositions 6.3 and 6.4, it suffices to show that \mathcal{X}/S is formally trivial at s . Write $\widehat{\mathcal{O}}_{S,s} = k[[t]]$, let $S_n = \text{Spec } k[[t]]/(t^{n+1})$, $\mathcal{X}_n = \mathcal{X}_{S_n}$ and $f_n : \mathcal{X}_n \rightarrow S_n$.

We proceed by induction on n . When $n = 0$, there is nothing to prove. Suppose that $n \geq 1$ and that we are given an isomorphism $\phi : \mathcal{X}_{n-1} \rightarrow X \times S_{n-1}$ of S_{n-1} -schemes. Let $\mathcal{U} = \{\mathcal{U}_i\}$ be an affine open cover of \mathcal{X}_n , $U_i = \mathcal{U}_i \times_{S_n} S_0$. Then $\{U_i\}$ is an affine open cover of X_0 and each $\mathcal{U}_i \rightarrow S_n$ is a deformation of $U_i \rightarrow S_0$. Thus there exist isomorphisms $\phi_i : \mathcal{U}_i \rightarrow U_i \times S_n$ [27, Theorem 1.2.4], each being a lift of the restriction of ϕ to $\mathcal{U}_i \times_{S_n} S_{n-1}$ for every $i \in I$.

Denote $\mathcal{U}_{ij} = \mathcal{U}_i \cap \mathcal{U}_j$ and $U_{ij} = U_i \cap U_j$. Since all \mathcal{X}_m are separated, U_{ij} and \mathcal{U}_{ij} are also affine. Write $\phi_{ij} = \phi_i^{-1}|_{\mathcal{U}_{ij}} \circ \phi_j|_{\mathcal{U}_{ij}}$ and let ψ_{ij} be the automorphism of $U_{ij} \times S_n$ making the following diagram of isomorphisms

$$\begin{array}{ccc} \mathcal{U}_{ij} & \xrightarrow{\phi_j} & U_{ij} \times S_n \\ \phi_{ij} \downarrow & & \downarrow \psi_{ij} \\ \mathcal{U}_{ij} & \xrightarrow{\phi_j} & U_{ij} \times S_n \end{array}$$

commute. An explicit calculation shows that $\psi_{ij} = (\phi_j|_{\mathcal{U}_{ij}}) \circ (\phi_i|_{\mathcal{U}_{ij}})^{-1}$. By our inductive assumption each ψ_{ij} descends to identity on $U_{ij} \times S_{n-1}$ under the base change $S_{n-1} \rightarrow S_n$. For $\pi : \text{Spec } B \rightarrow \text{Spec } A$, let $\pi^\sharp : A \rightarrow B$ denote the associated homomorphism of rings. By inductive assumption ψ_{ij}^\sharp is the identity modulo t^n we obtain $\psi_{ij}^\sharp = \text{id}_{U_{ij} \times S_n}^\sharp + t^n d_{ij}$ for some function $d_{ij} : \Gamma(U_{ij} \times S_n, \mathcal{O}_{U_{ij} \times S_n}) \rightarrow \Gamma(U_{ij} \times S_n, \mathcal{O}_{U_{ij} \times S_n})$. Explicit calculation (cf. [27, Lemma 1.2.6]) shows that for ψ_{ij}^\sharp to be a ring homomorphism, d_{ij} must be a k -derivation, hence, slightly abusing notation, we can associate it with a section $d_{ij} \in \Gamma(U_{ij}, \mathcal{T}_{U_{ij} \times S_n})$ of the tangent bundle, and $\phi_{ij}^\sharp = \text{id}_{\mathcal{U}_{ij}}^\sharp + t^n \phi_j^* d_{ij}$ for $\tau = f_n^* t$, the image of t under the homomorphism $\Gamma(S_n, \mathcal{O}_{S_n}) \rightarrow \Gamma(\mathcal{X}_n, \mathcal{O}_{\mathcal{X}_n})$, and $\phi_j^* d_{ij} := \phi_j^{-1} \circ d_{ij} \circ \phi_j$. Also, observe that the equality $\phi_{ij} = \phi_{ji}^{-1}$ implies that $d_{ij} = -d_{ji}$ for all i, j .

Since f is smooth, we have a short exact sequence

$$0 \rightarrow \mathcal{T}_{\mathcal{X}_n/S_n} \rightarrow \mathcal{T}_{\mathcal{X}_n/k} \rightarrow f_n^* \mathcal{T}_{S_n/k} \rightarrow 0. \tag{6.4.1}$$

The existence of isomorphisms ϕ_i implies that the above short exact sequence is locally split. For each \mathcal{U}_i , consider the section $\sigma_i : f_n^* \mathcal{T}_{S_n/k}|_{\mathcal{U}_i} \rightarrow \mathcal{T}_{\mathcal{X}_n/k}|_{\mathcal{U}_i}$ given by the composition

$$f_n^* \mathcal{T}_{S_n/k}|_{\mathcal{U}_i} \hookrightarrow \phi_i^*(\text{pr}_{1i}^* \mathcal{T}_{U_i/k} \oplus \text{pr}_{2i}^* \mathcal{T}_{S_n/k}) \simeq \mathcal{T}_{\mathcal{X}_n/k}|_{\mathcal{U}_i}. \tag{6.4.2}$$

with $\text{pr}_{1i} : U_i \times S_n \rightarrow U_i$ and $\text{pr}_{2i} : U_i \times S_n \rightarrow S_n$ projections and the first arrow in (6.4.2) being the inclusion of the second direct summand. In particular, $f_n|_{\mathcal{U}_i} = \text{pr}_{2i} \circ \phi_i$. Consider the Čech complex associated to (6.4.1) and the covering \mathcal{U} :

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & \bigoplus_i \Gamma(\mathcal{U}_i, \mathcal{T}_{\mathcal{X}_n/S_n}) & \longrightarrow & \bigoplus_i \Gamma(\mathcal{U}_i, \mathcal{T}_{\mathcal{X}_n/k}) & \longrightarrow & \bigoplus_i \Gamma(\mathcal{U}_i, f_n^* \mathcal{T}_{S_n/k}) \longrightarrow 0 \\
 & & \downarrow & & \downarrow \delta & & \downarrow \\
 0 & \longrightarrow & \bigoplus_{i,j} \Gamma(\mathcal{U}_{ij}, \mathcal{T}_{\mathcal{X}_n/S_n}) & \longrightarrow & \bigoplus_{i,j} \Gamma(\mathcal{U}_{ij}, \mathcal{T}_{\mathcal{X}_n/k}) & \longrightarrow & \bigoplus_{i,j} \Gamma(\mathcal{U}_{ij}, f_n^* \mathcal{T}_{S_n/k}) \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & \vdots & & \vdots & & \vdots
 \end{array}$$

Let $[(\tau \frac{d}{dt})|_{\mathcal{U}_i}]_i \in \bigoplus_i \Gamma(\mathcal{U}_i, f_n^* \mathcal{T}_{S_n/k})$ be the product of restrictions of the global section $\tau \frac{d}{dt} = f_n^*(t \frac{d}{dt}) \in \Gamma(\mathcal{X}_n, f_n^* \mathcal{T}_{S_n/k})$ to each of $\mathcal{U}_i \in \mathfrak{U}$. By the above observations, the considered 0-cocycle lifts via chosen sections σ_i 's to the element $[\phi_i^* \text{pr}_{2i}^*(t \frac{d}{dt})]_i \in \bigoplus_i \Gamma(\mathcal{U}_i, \mathcal{T}_{\mathcal{X}_n/k})$. Write $t_i = \text{pr}_{2i}^* t \in \Gamma(U_i \times S_n, \mathcal{O}_{U_i \times S_n})$. We have $\mathcal{O}_{U_i \times S_n} \simeq \mathcal{O}_{U_i}[t_i]/(t_i^{n+1})$. It follows that $\text{pr}_{2i}^*(\frac{d}{dt}) = \partial_i := \frac{\partial}{\partial t_i}$, which is the \mathcal{O}_{U_i} -derivation with respect to the variable t_i .

The map δ sends $[\phi_i^* \text{pr}_{2i}^*(t \frac{d}{dt})]_i$ to the element of $\bigoplus_{i,j} \Gamma(\mathcal{U}_{ij}, \mathcal{T}_{\mathcal{X}_n/k})$ given by

$$\delta \left[\phi_i^* \text{pr}_{2i}^* \left(t \frac{d}{dt} \right) \right]_{ij} = \phi_i^* \text{pr}_{2i}^* \left(t \frac{d}{dt} \right) \Big|_{\mathcal{U}_{ij}} - \phi_j^* \text{pr}_{2j}^* \left(t \frac{d}{dt} \right) \Big|_{\mathcal{U}_{ij}} = \phi_i^* (t_i \partial_i) \Big|_{\mathcal{U}_{ij}} - \phi_j^* (t_j \partial_j) \Big|_{\mathcal{U}_{ij}}.$$

Now, observe that

$$\begin{aligned}
 \phi_i^* (t_i \partial_i) \Big|_{\mathcal{U}_{ij}} - \phi_j^* (t_j \partial_j) \Big|_{\mathcal{U}_{ij}} &= \phi_i^* (t_i \partial_i - (\phi_j \phi_i^{-1})^* (t_j \partial_j)) \Big|_{\mathcal{U}_{ij}} \\
 &= \phi_i^* (t_i \partial_i - \psi_{ij}^* (t_j \partial_j)).
 \end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
 \psi_{ij}^* \left(t_j \frac{\partial}{\partial t_j} \right) &= \psi_{ij}^\# \circ \left(t_j \frac{\partial}{\partial t_j} \right) \circ \psi_{ji}^\# \\
 &= \psi_{ij}^\# \left(t_j \frac{\partial}{\partial t_j} (\text{id} - t_j^n \mathbf{d}_{ij}) \right) = \psi_{ij}^\# \left(t_j \frac{\partial}{\partial t_j} - nt_j^n \mathbf{d}_{ij} - \underbrace{t_j^{n+1} \mathbf{d}_{ij} \circ \frac{\partial}{\partial t_j}}_{=0} \right) \\
 &= t_j \frac{\partial}{\partial t_j} - nt_j^n \mathbf{d}_{ij} = t_j \frac{\partial}{\partial t_j} + nt_j^n \mathbf{d}_{ji}.
 \end{aligned}$$

Since $f_n|_{\mathcal{U}_{ij}} = \text{pr}_{2i} \circ (\phi_i|_{\mathcal{U}_{ij}}) = \text{pr}_{2j} \circ (\phi_j|_{\mathcal{U}_{ij}})$, we get

$$\phi_i^* \left(t_i \frac{\partial}{\partial t_i} \right) \Big|_{\mathcal{U}_{ij}} - \phi_j^* \left(t_j \frac{\partial}{\partial t_j} \right) \Big|_{\mathcal{U}_{ij}} = \phi_i^* \left[(nt_j^n \mathbf{d}_{ji})|_{\mathcal{U}_{ij}} \right] = n\tau^n \phi_i^* \mathbf{d}_{ji}.$$

As a result, we obtain a 1-cocycle $(n\tau^n \phi_i^* \mathbf{d}_{ji})_{i,j} \in \bigoplus_{i,j} \Gamma(\mathcal{U}_{ij}, \mathcal{T}_{\mathcal{X}_n/S_n})$. It is the image of $\tau \frac{d}{dt} \in H^0(\mathcal{X}_n, f_n^* \mathcal{T}_{S_n/k})$ under the map $H^0(\mathcal{X}_n, f_n^* \mathcal{T}_{S_n/k}) \rightarrow H^1(\mathcal{X}_n, \mathcal{T}_{\mathcal{X}_n/S_n})$ arising

from the long exact sequence. This is the morphism of $\Gamma(S_n, \mathcal{O}_{S_n})$ -modules that makes the diagram

$$\begin{array}{ccc} H^0(\mathcal{X}_n, \widetilde{f_n^* \mathcal{T}_{S_n/k}}) & \longrightarrow & H^1(\mathcal{X}_n, \widetilde{\mathcal{T}_{\mathcal{X}_n/S_n}}) \\ \downarrow \simeq & & \downarrow \simeq \\ (f_n)_* f_n^* \mathcal{T}_{S_n/k} & \xrightarrow{\delta_{\mathcal{X}/S}} & \mathbf{R}^1 f_* \mathcal{T}_{\mathcal{X}_n/S_n} \end{array}$$

commute. Furthermore, $\tau \frac{d}{dt} \in \Gamma(\mathcal{X}_n, f_n^* \mathcal{T}_{S_n/k}) = \Gamma(S_n, (f_n)_* f_n^* \mathcal{T}_{S_n/k})$ is clearly in the image of $\mathcal{T}_{S_n/k} \rightarrow (f_n)_* f_n^* \mathcal{T}_{S_n/k}$. Therefore, since S is smooth, $\delta_{\mathcal{X}_n/S_n} = \delta_{\mathcal{X}/S}|_{S_n}$ and by Proposition 4.2, $\delta_{\mathcal{X}_n/S_n}(t \frac{d}{dt}) = 0$. As a result, the cohomology class of $(n\tau^n \phi_i^* d_{ji})_{i,j} \in \bigoplus_{i,j} \Gamma(U_{ij}, \mathcal{T}_{\mathcal{X}_n/S_n})$ is trivial. Thus, since $\text{char } k = 0$ and n is invertible, the class $[\phi_i^* d_{ij}]_{i,j} = -[\phi_i^* d_{ji}]_{i,j}$ is also a 1-coboundary. However, by the construction of derivations d_{ij} , this implies that f_n is trivial as a deformation of f_{n-1} . In other words ϕ lifts to a morphism $\tilde{\phi} : \mathcal{X}_n \rightarrow X \times S_n$ which completes the inductive step and proves that \mathcal{X}/S is formally trivial at s . \square

Remark 6.5 Observe that the last part of the proof of Theorem 1.3 relies on the assumption that $\text{char } k = 0$.

7 Proof of Theorem 1.3 when $\text{char } k = p > 0$

In this section, we prove Theorem 1.3 in the case when $\text{char } k = p > 0$. Our main idea is the same as in the proof of characteristic zero case, i.e. we deduce the result from vanishing of $\text{KS}_f : \mathcal{T}_S \rightarrow \mathbf{R}^1 f_* \mathcal{T}_{\mathcal{X}/S}$. However, in order to prove formal triviality of \mathcal{X}/S , we observe that étale locally, our family can be descended along the absolute Frobenius $F_S : S \rightarrow S^{(1)}$. This is inspired by the following result appearing in the work of Ogus.

Lemma 7.1 [22, Lemma 3.5] *Suppose $f : \mathcal{X} \rightarrow S = \text{Spec } k[[t_1, t_2, \dots, t_r]]$ is smooth and proper, let $F_S : S \rightarrow S^{(1)}$ be the Frobenius map. Suppose that the Kodaira–Spencer class $\xi \in H^1(\mathcal{X}, \mathcal{T}_{\mathcal{X}/S}) \otimes \Omega_{S/k}^1$ is trivial. Then \mathcal{X}/S can be descended to $S^{(1)}$, meaning that there exists $\mathcal{X}^{(1)}/S^{(1)}$ and a Cartesian square*

$$\begin{array}{ccc} \mathcal{X} & \longrightarrow & \mathcal{X}^{(1)} \\ f \downarrow & & \downarrow f^{(1)} \\ S & \xrightarrow{F_S} & S^{(1)}. \end{array}$$

Yet, the above cannot be applied directly to \mathcal{X}/S with S (locally) of finite type over k . For the purpose of isotriviality, however, it is enough to descend along Frobenius only étale-locally on S . Also, in order to apply this technique inductively, we need to ensure that there exists an étale neighborhood of any given point $s \in S(k)$, such that we can force our local Frobenius descent of \mathcal{X} to be smooth projective and fibrewise trivial. Combining Lemma 7.1 with Popescu’s Theorem, we propose the following workaround of this issue.

Proposition 7.2 *Let S be a smooth curve, $s \in S(k)$ and $f : \mathcal{X} \rightarrow S$ be a smooth projective morphism. Suppose the Kodaira–Spencer map $\text{KS}_f : \mathcal{T}_S \rightarrow \mathbf{R}^1 f_* \mathcal{T}_{\mathcal{X}/S}$ vanishes. Then there exists an étale neighborhood $s \rightarrow U \rightarrow S$ such that $f_U : \mathcal{X}_U \rightarrow U$ descends to a smooth, projective, fibrewise trivial family $f^{(1)} : \mathcal{X}^{(1)} \rightarrow U^{(1)}$ along the Frobenius $F_U : U \rightarrow U^{(1)}$.*

Proof We may assume that S is affine, say $S = \text{Spec } A$. By Popescu’s Theorem [28, Tag 07GC] the ring homomorphism

$$\Gamma(S, \mathcal{O}_S) =: A \rightarrow \widehat{\mathcal{O}}_{S,s}$$

is a colimit of a filtered diagram \mathcal{D} of smooth A -algebras. By [12, 8.9.1.(iii)] there exists $R_0 \in \mathcal{D}$ and a model $\widehat{f}_0^{(1)} : \mathcal{X}_0 \rightarrow \text{Spec } R_0^{(1)}$ of $\widehat{f}^{(1)}$. Then $\widehat{\mathcal{O}}_{S,s}$ is a filtered colimit of R_0 -algebras R_α which are smooth A -algebras. Let

$$\mathcal{X}_\alpha := \mathcal{X}_0 \otimes_{R_0^{(1)}} R_\alpha^{(1)} \xrightarrow{\widehat{f}_\alpha^{(1)}} \text{Spec } R_\alpha^{(1)}.$$

Then, by [12, 8.10.5.(xiii)] and [13, 17.7.8.(ii)] there exists α such that $\widehat{f}_\alpha^{(1)} : \mathcal{X}_\alpha \rightarrow \text{Spec } R_\alpha^{(1)}$ is smooth and projective with $\widehat{f}_\alpha^{(1)} \otimes_{R_\alpha^{(1)}} \widehat{\mathcal{O}}_{S,s}^{(1)} = \widehat{f}^{(1)}$. Denote $R = R_\alpha$ and $\mathcal{X}^{(R)} = \mathcal{X}_\alpha$.

Next, let \mathcal{D}_R be the coslice category corresponding to $R \in \mathcal{D}$. Since \mathcal{D} is filtered, we have $\text{colim } \mathcal{D} = \text{colim } \mathcal{D}_R$. Consider the functor $\mathcal{I} = \mathbf{Isom}_{\mathcal{X}_R, F_R^* \mathcal{X}^{(R)}}$ which is locally finitely presented. Recall that we are given an isomorphism

$$\iota = \left(\mathcal{X}_R \otimes_R \widehat{\mathcal{O}}_{S,s} \xrightarrow{\sim} \widehat{\mathcal{X}} \xrightarrow{\sim} F_S^* \mathcal{X} \xrightarrow{\sim} (F_R^* \mathcal{X}^{(R)}) \otimes_R \widehat{\mathcal{O}}_{S,s} \right) \in \mathcal{I}(\widehat{\mathcal{O}}_{S,s}) = \mathcal{I}(\text{colim } \mathcal{D}_R).$$

Since \mathcal{I} is locally of finite presentation, we have $\mathcal{I}(\text{colim } \mathcal{D}_R) = \text{colim } \mathcal{I}(\mathcal{D}_R)$, so ι is the image of an element $\iota_C \in \mathcal{I}(C)$ for some $C \in \mathcal{D}_R$. In other words, after pulling back $\mathcal{X}^{(R)}$ along $R \rightarrow C$, we obtain an isomorphism

$$\iota_C : \mathcal{X}_C \xrightarrow{\sim} \mathcal{X}_R \otimes_R C \rightarrow (F_R^* \mathcal{X}^{(R)}) \otimes_R C \xrightarrow{\sim} F_C^*(\mathcal{X}^{(R)} \otimes_{R^{(1)}} C^{(1)})$$

satisfying $\iota_C \otimes_C \widehat{\mathcal{O}}_{S,s} = \iota$. Denote $\mathcal{X}^{(C)} := \mathcal{X}^{(R)} \otimes_R C$. Finally, take $U \rightarrow S$ to be an étale neighborhood of s which admits a section $U \rightarrow \text{Spec } C$ over S . Then $\mathcal{X}^{(1)} := \mathcal{X}^{(C)} \times_{C^{(1)}} U^{(1)}$ and $\iota_U := \iota_C \otimes_C U : \mathcal{X}_U \xrightarrow{\sim} F_U^* \mathcal{X}^{(1)}$ satisfy the desired properties. \square

Now we can finish the proof of Theorem 1.3.

Proof of Theorem 1.3 in the case $\text{char } k = p > 0$ Again, proving isotrivality is equivalent with proving that $\text{Isom}_{\mathcal{X}, \mathcal{X} \times_S}$ is S -smooth and by Proposition 5.1, it is enough to prove that when S is a smooth curve, so let us reduce to this case. It suffices to show that \mathcal{X}/S is formally trivial at every point $s \in S(k)$. Fix such a point s , let \mathfrak{m}_s be the maximal ideal in $\mathcal{O}_{S,s}$, and let $S_n = \text{Spec } \mathcal{O}_{S,s}/\mathfrak{m}_s^{n+1}$. By Theorem 3.2, the family \mathcal{X}/S is generically isotrivial, hence the Kodaira–Spencer map $\text{KS}_f : \mathcal{T}_S \rightarrow \mathbb{R}^1 f_* \mathcal{T}_{\mathcal{X}/S}$ vanishes (Proposition 4.2). By Proposition 7.2 we find that there exists an étale neighborhood $U \rightarrow S$ of s such that \mathcal{X}_U descends to some smooth, projective and fibrewise trivial $f^{(1)} : \mathcal{X}^{(1)} \rightarrow U^{(1)}$ with fibre X along Frobenius $F_U : U \rightarrow U^{(1)}$. Repeating the above argument, we find a sequence of étale neighborhoods

$$\cdots \rightarrow U_{m+1} \rightarrow U_m \rightarrow U_{m-1} \rightarrow \cdots \rightarrow U_1 \rightarrow S$$

of s such that \mathcal{X}_{U_m} descends to some $f^{(m)} : \mathcal{X}^{(m)} \rightarrow U_m^{(m)}$ along m^{th} -power of Frobenius $F_{U_m}^m : U_m \rightarrow U_m^{(m)}$. This in particular shows that $\mathcal{X} \times_S S_{p^m-1} \simeq X \times_k S_{p^m-1}$ for all $m \in \mathbb{N}$. Therefore \mathcal{X} is formally trivial at s . \square

In particular, we obtain the following result.

Corollary 7.3 *Suppose $k = \bar{k}$, $\text{tr. deg}(k/\mathbb{F}) = \infty$ and $f : \mathcal{X} \rightarrow S$ is smooth, projective, fibrewise trivial family with fibre X . Suppose furthermore that Def_X is pro-representable. Then \mathcal{X}/S is isotrivial.*

This follows easily from the following lemma.

Lemma 7.4 *Suppose X is smooth projective k -scheme such that the functor $\text{Def}_X : \text{Art}_k \rightarrow \text{Set}$ is pro-representable. Then the group scheme Aut_k^X is smooth.*

Proof Let Def_X be prorepresentable by $\text{Spf } R$ for R a complete local k -algebra R and $\hat{u} : \mathcal{X} \rightarrow \text{Spf } R$ be the universal formal family. By [27, Prop 2.6.2] it follows that the functor of automorphisms of \hat{u} is prorepresentable by an affine formal scheme $\text{Aut}_{\hat{u}}$ such that the obvious morphism $\text{Aut}_{\hat{u}} \rightarrow \text{Def}_X$ is formally smooth. Let $\text{Spec } k \rightarrow \text{Def}_X$ be the inclusion of the closed point which corresponds to the trivial deformation $X \rightarrow \text{Spec } k$. Then the fiber product $\text{Aut}_{\hat{u}} \times_{\text{Def}_X} \text{Spec } k$ is formally smooth over $\text{Spec } k$, both treated as formal schemes over k in the sense of [3, Chapter I.6], and it coincides with the formal completion (in the sense of *loc.cit.*) of Aut_k^X . Since Aut_k^X is also locally of finite presentation over k (Corollary 2.6), it follows that Aut_k^X is smooth [28, Tag 02H6]. \square

Proof of Lemma 7.3 It is a formal consequence of Lemma 7.4 and Theorem 1.3. \square

8 Fibrewise trivial family with non-vanishing Kodaira–Spencer map.

In this section, we employ quotients of trivial families by a group action in order to construct fibrewise trivial families $f : \mathcal{X} \rightarrow S$ over a smooth curve such that the Kodaira–Spencer map $\text{KS}_f(s) : \mathcal{T}_S(s) \rightarrow H^1(\mathcal{X}_s, \mathcal{T}_{\mathcal{X}_s})$ is nonzero at every $s \in S$. In particular, such a family is not isotrivial, thus showing that the assumption that Aut_k^X is reduced in Theorem 1.3 is necessary.

8.1 Quotients of trivial families by diagonal group actions

Let S and X be two k -schemes such that X is smooth and projective. Then the group scheme Aut_k^X exists. Let G be a group scheme with a group scheme homomorphism $\iota : G \rightarrow \text{Aut}_k^X$, which in turn [by the exponential law (2.8.2)] induces a left group action $\mu : G \times X \rightarrow X$. Given $g : T \rightarrow G$, we write $\iota(g) : T \times X \rightarrow T \times X$ the T -automorphism associated to $\iota \circ g : T \rightarrow \text{Aut}_k^X$. Then, by the adjunction (2.8.2) we have that the square

$$\begin{CD} T \times X @>\iota(g)>> T \times X \\ @Vg \times \text{id}_XVV @VV\text{pr}_2V \\ G \times X @>\mu>> X \end{CD}$$

commutes. Let $S' \rightarrow S$ be an fppf G -torsor, $m : S' \times G \rightarrow S'$ be the associated right G -action on S' . Then, we equip $S' \times X$ with **diagonal (right) action** of G : $((s', x), g) \mapsto (s'g, \iota(g)^{-1}x)$. Suppose furthermore that the fppf quotient $(S' \times X)/G$ is representable by a scheme \mathcal{X} . Observe that we have the following commutative square

$$\begin{CD} S' \times G \times X @>\text{id}_{S'} \times \mu>> S' \times X \\ @V m \times \text{id}_X VV @VV\pi V \\ S' \times X @>\pi>> \mathcal{X}. \end{CD} \tag{8.1.1}$$

Denote by $f : \mathcal{X} \rightarrow S$ the induced map on quotients so that the following square

$$\begin{CD} S' \times X @>\pi>> \mathcal{X} \\ @V\text{pr}_1VV @VVfV \\ S' @>q>> S \end{CD}$$

commutes with the horizontal arrows being quotient maps. Then, denoting $q^*\mathcal{X}$ the pullback of \mathcal{X} along q , we find that the induced canonical morphism $\phi : S' \times_k X \rightarrow q^*\mathcal{X}$ is an (G -equivariant) isomorphism, i.e. the above square is cartesian.

Similarly, it is clear that the morphism $q^*f : q^*\mathcal{X} \rightarrow S'$ is G -equivariant as well. The isomorphism ϕ induces a morphism $[\phi] : S' \rightarrow \mathcal{I} := \text{Isom}_S^{S \times X, \mathcal{X}}$ of S -schemes. We know that S' is equipped with right G -action and at the same time, $\mathcal{H} := \text{Hom}_S^{S \times X, \mathcal{X}}$ is equipped with right Aut_k^X -action $\nu : \mathcal{H} \times_k \text{Aut}_k^X \rightarrow \mathcal{H}$ defined as follows: for $\sigma \in \text{Aut}_k^X(T)$ and $(\alpha : T \times X \rightarrow T \times_S \mathcal{X}) \in \mathcal{H}(T)$ we take

$$\nu(\alpha, \sigma) : T \times X \xrightarrow{\sigma} T \times X \xrightarrow{\alpha} T \times_S \mathcal{X}. \tag{8.1.2}$$

This action obviously restricts to $\mathcal{I} = \text{Isom}_S^{S \times X, \mathcal{X}}$ which is an open subscheme of \mathcal{H} . This Aut_k^X -action on \mathcal{I} makes \mathcal{I} into Aut_k^X -torsor, as it is trivialised by the fppf cover $S' \rightarrow S$. With a slight abuse of notation, we will denote this action by ν as well. We claim that the two group actions on S' and \mathcal{I} are compatible with respect to $[\phi] : S' \rightarrow \mathcal{I}$ and ι .

Proposition 8.2 *Under the assumptions made in 8.1, the morphism $[\phi] : S' \rightarrow \mathcal{I}$ is equivariant with respect to $\iota : G \rightarrow \text{Aut}_k^X$, i.e. the following square*

$$\begin{CD} S' \times G @>([\phi], \iota)>> \mathcal{I} \times_k \text{Aut}_k^X \\ @V m VV @VV \nu V \\ S' @>[\phi]>> \mathcal{I} \end{CD} \tag{8.2.1}$$

is commutative.

Proof Since $\mathcal{I} \subset \mathcal{H}$ is a monomorphism, to prove the desired claim it suffices to show that the composition of the two squares below

$$\begin{CD} S' \times G @>([\phi], \iota)>> \mathcal{I} \times_k \text{Aut}_k^X @<< \hookrightarrow \mathcal{H} \times_k \text{Aut}_k^X \\ @V m VV @VV \nu V @VV \nu V \\ S' @>[\phi]>> \mathcal{I} @<< \hookrightarrow \mathcal{H} \end{CD}$$

is commutative. This in turn, by Yoneda’s lemma, is equivalent with proving that

$$\begin{CD} \text{Hom}_S(T, S') \times \text{Hom}_S(T, G) @>([\phi], \iota)>> \text{Hom}_S(T, \mathcal{I}) \times \text{Hom}_S(T, \text{Aut}_k^X) \\ @V m VV @VV \nu V \\ \text{Hom}_S(T, S') @>[\phi]>> \text{Hom}_S(T, \mathcal{I}) \end{CD}$$

8.4 Fibrewise trivial families with nowhere vanishing Kodaira–Spencer map

Suppose that $\text{char } k = p > 0$ and let $X \in \text{Sch}_k$ be a smooth projective scheme such that the connected component of Aut_k^X containing 1 is isomorphic to $\mu_p =: G$. For instance, one can set $\text{char } k = 2$ and choose X to be a generic supersingular Enriques surface [19, Example 5.6]. Take $S = \mathbb{G}_m$, $S' = \mathbb{G}_m$, and let $S' \rightarrow S$ be the p^{th} power map, which makes S' into a nontrivial μ_p -torsor over S . Then the quotient $(S' \times_k X)/G$ by the diagonal action is representable by a scheme \mathcal{X} [29, Section 4]. Again, let $f : \mathcal{X} \rightarrow S$ be as in 8.1.

Proposition 8.5 *Consider the setup as in 8.4. Then the Kodaira–Spencer map of $f : \mathcal{X} \rightarrow S$ is nowhere vanishing. In particular, f is not isotrivial.*

Proof Pick $s \in S(k)$. To show that KS_f does not vanish at s , it suffices to show that $\text{KS}_{(\times s)^* f}$ does not vanish at $s = 1$ for $(\times s)^* f : (\times s)^* \mathcal{X} \rightarrow S$ being the pullback of f along $S \xrightarrow{\times s} S$, so we may simply assume that $s = 1$.

Let $v : \text{Spec } k[\varepsilon]/(\varepsilon^2) =: D \rightarrow \mathbb{G}_m = \text{Spec } k[t, t^{-1}]$ be given by $t \mapsto 1 + \varepsilon$. It is enough to show that $v^* \mathcal{X} \rightarrow D$ is a non-trivial deformation of X/k . For this to hold, it suffices to prove that $\text{Isom}_D^{v^* \mathcal{X}, D \times_k X} = v^* \mathcal{I} \rightarrow D$ does not admit a section, where $\mathcal{I} = \text{Isom}_S^{\mathcal{X}, S \times X} \rightarrow S$. If it did have a section then $v^* \mathcal{I} \simeq \text{Aut}_k^X \times_k D$ and its cohomology class $[v^* \mathcal{I}]$ would be zero in $H_{\text{fppf}}^1(D, \text{Aut}_k^X)$. However, we have the commutative diagram

$$\begin{CD}
 H_{\text{fppf}}^1(S, G) @>H_{\text{fppf}}^1(\iota)>> H_{\text{fppf}}^1(S, \text{Aut}_k^X) \\
 @VVv^*V @VVv^*V \\
 H_{\text{fppf}}^1(D, G) @>H_{\text{fppf}}^1(\iota)>> H_{\text{fppf}}^1(D, \text{Aut}_k^X)
 \end{CD} \tag{8.5.1}$$

and by Corollary 8.3 we see that $[v^* \mathcal{I}] = H_{\text{fppf}}^1(\iota)(v^*[S'])$. Clearly the image $v^*[S'] \in H_{\text{fppf}}^1(D, G)$ corresponds to the morphism $\text{Spec } k[\varepsilon']/((\varepsilon')^{2p}) \rightarrow \text{Spec } k[\varepsilon]/(\varepsilon^2)$ given by $\varepsilon \mapsto (\varepsilon')^p$ which is a nontrivial G -torsor. It remains to show that the bottom map in (8.5.1) has trivial kernel—note that since $G = (\text{Aut}_k^X)^0$ and G_{red} is trivial, it follows that $(\text{Aut}_k^X)_{\text{red}} \simeq \pi_0(\text{Aut}_k^X)$. This is because the composition $(\text{Aut}_k^X)_{\text{red}} \subset \text{Aut}_k^X \rightarrow \text{Aut}_k^X/G = \pi_0(\text{Aut}_k^X)$ [21, 2.37] is bijection on topological spaces, both sides are disjoint unions of k -points, so it must be an isomorphism of schemes. Since both sides carry a k -group scheme structure and the isomorphism respects the group law, it is a k -group scheme isomorphism. Thus the epimorphism in the exact sequence from *loc.cit.*

$$0 \rightarrow G \xrightarrow{\iota} \text{Aut}_k^X \xrightarrow{q} \text{Aut}_k^X/G \rightarrow 0 \tag{8.5.2}$$

admits a section. For every $T \in \text{Sch}_k$, we have the induced exact sequence of pointed sets [8, Ch. III, Prop 3.2.2]

$$0 \rightarrow H^0(T, G) \rightarrow H^0(T, \text{Aut}_k^X) \rightarrow H^0(T, \text{Aut}_k^X/G) \xrightarrow{d} H_{\text{fppf}}^1(T, G) \rightarrow H_{\text{fppf}}^1(T, \text{Aut}_k^X). \tag{8.5.3}$$

Suppose that $T'/T \in H_{\text{fppf}}^1(T, G)$ induces the trivial Aut_k^X -torsor. By (8.5.3) there exists a section $s : T \rightarrow \text{Aut}_k^X/G$ such that $d(s) = [T']$. By (8.5.2) s lifts to $\tilde{s} : T \rightarrow \text{Aut}_k^X$, so that $q\tilde{s} = s$. Therefore $[T'] = d(q_*\tilde{s}) = *$ by exactness of (8.5.3), proving the triviality of T' . Apply this to $T = D$. □

Remark 8.6 We briefly remark on a slightly different example of a fibrewise trivial family which is not isotrivial. Let X be a surface with ample canonical divisor K_X , and such that $H^0(X, \mathcal{F}_X) \neq 0$. Then the scheme of automorphisms Aut_k^X is finite and non-reduced. Suppose furthermore that Aut_k^X contains $\mu_p = \text{Spec } k[t]/(t^p - 1)$ as a subgroup. Here, one can take for instance $p = 5$ and a hypersurface in \mathbb{P}^3 of degree 5 (which is automatically of general type), invariant under the Godeaux action on \mathbb{P}^3 [18, 1], and on which the restriction of μ_5 -action is free. Then for the μ_p -torsor $S' \rightarrow S$ as in 8.4 and the quotient $\mathcal{X} := (X \times S')/\mu_p$ by diagonal action of μ_p , the induced fibrewise trivial family $f: \mathcal{X} \rightarrow S$ is nowhere isotrivial - one uses [15, Prop. 6] to show that $\mathcal{I} = \text{Isom}^{S \times X, \mathcal{X}}$ is finite over S and, using the same technique, one can prove that isotriviality would in fact imply that $S \times X \simeq \mathcal{X}$. However, we don't know if its Kodaira–Spencer map is nowhere vanishing.

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Data availability Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

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References

1. Artin, M.: Algebraic approximation of structures over complete local rings. *Publications Mathématiques de l'Institut des Hautes Études Scientifiques* **36**(1), 23–58 (1969)
2. Bogomolov, F.A., Böhning, C., Graf von Bothmer, H.-C.: Birationally isotrivial fiber spaces. *Eur. J. Math.* **2**(1), 45–54 (2016)
3. Demazure, M.: Lectures on p -Divisible Groups. *Lecture Notes in Mathematics*, vol. 302. Springer, Berlin (1986). (reprint of the 1972 original)
4. Donin, I.F.: Conditions for the triviality of deformations of complex structures. *Mat. Sb. (N.S.)* **81**(123), 610–621 (1970)
5. Eisenbud, D.: *Commutative Algebra, Graduate Texts in Mathematics*, vol. 150. Springer, New York (1995). (with a view toward algebraic geometry)
6. Fantechi, B., Göttsche, L., Illusie, L., Kleiman, S.L., Nitsure, N., Vistoli, A.: *Fundamental Algebraic Geometry, Mathematical Surveys and Monographs*, vol. 123. American Mathematical Society, Providence (2005). (Grothendieck's FGA explained)
7. Fischer, W., Grauert, H.: Lokal-triviale Familien kompakter komplexer Mannigfaltigkeiten. *Nachr. Akad. Wiss. Göttingen Math. Phys. Kl. II* **89–94**, 1965 (1965)
8. Giraud, J.: *Cohomologie non abélienne. Die Grundlehren der mathematischen Wissenschaften, Band 179*. Springer, Berlin (1971)
9. Görtz, U., Wedhorn, T.: *Algebraic geometry I. Schemes—with Examples and Exercises. Springer Studium Mathematik–Master. Springer Spektrum, Wiesbaden* (2020). (second edition [of 2675155])
10. Grothendieck, A.: Techniques de construction et théorèmes d'existence en géométrie algébrique iv : les schémas de hilbert. In: *Séminaire Bourbaki : années 1960/61, exposés, number 6 in Séminaire Bourbaki. Société mathématique de France, pp. 205–222* (1961) (talk:221)

11. Grothendieck, A.: éléments de géométrie algébrique: III. étude cohomologique des faisceaux cohérents. Seconde partie. Publications Mathématiques de l’IHÉS **17**, 5–91 (1963)
12. Grothendieck, A.: Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. III. Inst. Hautes Études Sci. Publ. Math. **28**, 255 (1966)
13. Grothendieck, A.: Éléments de géométrie algébrique: IV. étude locale des schémas et des morphismes de schémas, Quatrième partie. Publications Mathématiques de l’IHÉS **32**, 5–361 (1967)
14. Hauser, H., Müller, G.: The trivial locus of an analytic map germ. Ann. Inst. Fourier (Grenoble) **39**(4), 831–844 (1989)
15. Kollár, J.: Moduli of varieties of general type. In: Handbook of Moduli. Vol. II. Advanced Lectures in Mathematics (ALM), vol. 25, pp. 131–157. International Press, Somerville, MA (2013)
16. Kollár, J.: Rational Curves on Algebraic varieties, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 32. Springer, Berlin (1996)
17. Kovács, S.: Strong non-isotriviality and rigidity. Recent progress in arithmetic and algebraic geometry, 47–55, Contemp. Math., **386**, Amer. Math. Soc., Providence, RI. <http://www.ams.org/mathscinet-getitem?mr=2182770> (2005)
18. Lang, W.E.: Classical Godeaux surface in characteristic P . Math. Ann. **256**(4), 419–427 (1981)
19. Martin, G.: Infinitesimal automorphisms of algebraic varieties and vector fields on elliptic surfaces. Alg. Num. Theor. **16**(7), 1655–1704 (2022)
20. McKernan, J.: Lecture notes at MIT, course 18.726, Lecture 14. http://math.mit.edu/~mckernan/Teaching/07-08/Spring/18.726/l_14.pdf (2008). Accessed May 2020
21. Milne, J.S.: Algebraic Groups. Cambridge Studies in Advanced Mathematics, vol. 170. Cambridge University Press, Cambridge (2017)
22. Ogus, A.: F -crystals and Griffiths transversality. In: Proceedings of the International Symposium on Algebraic Geometry (Kyoto Univ., Kyoto, 1977). Kinokuniya Book Store, Tokyo, pp. 15–44 (1978)
23. Olsson, M.: Algebraic Spaces and Stacks, American Mathematical Society Colloquium Publications, vol. 62. American Mathematical Society, Providence (2016)
24. Oort, F.: Finite group schemes, local moduli for abelian varieties, and lifting problems. Compos. Math. **23**, 265–296 (1971)
25. Schlessinger, M.: Functors of Artin rings. Trans. Am. Math. Soc. **130**, 208–222 (1968)
26. Schuster, H.W.: Zur Theorie der Deformationen kompakter komplexer Räume. Invent. Math. **9**, 284–294 (1969/70)
27. Sernesi, E.: Deformations of Algebraic Schemes. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], No. 334. Springer, Berlin (2006)
28. Stacks Project Authors. Stacks Project. <https://stacks.math.columbia.edu> (2020). Accessed May 2020
29. Tziolas, N.: Quotients of schemes by α_p or μ_p actions in characteristic $p > 0$. Manuscr. Math. **152**(1–2), 247–279 (2017)

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