Supplementary Materials for: Learning Increases Growth and Reduces Inequality in Shared Noisy Environments

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I. SUPPLEMENTARY MATERIAL

A. Information quantities

1. *Kelly growth rate*

Multiplying and dividing by $P(e|s)$ in the logarithm of Eq. 2 yields

$$\begin{aligned}\gamma &= \sum_{e,s} P(e,s)\log\left[w_e P(e|s)\frac{X(e|s)}{P(e|s)}\right] \\ &= \sum_{e,s} P(e,s)\log\frac{P(e|s)}{P(e)} - P(s)P(e|s)\log\frac{P(e|s)}{X(e|s)} \\ &= I(E;S) - \mathrm{E}_s\big(D_{KL}\big[P(E|s)||X(E|s)\big]\big),\end{aligned} \tag{1}$$

where E_s is an expectation value over all signal states.

B. Simplified growth model

Consider a conditional probability that is degenerate off-diagonal,

$$P(e|s) = f(p,l) = \begin{cases} p & \text{if } s = e, \\ \frac{1-p}{l-1} & \text{if } s \neq e. \end{cases} \tag{2}$$

The "correct" outcome corresponding to the sampled event occurs with conditional probability $0 < p \leq 1$, and all other "incorrect" guesses occur with some uniform probability normalized to

$$\sum_{e}^{l-1} P(e|s) = 1 - p; \quad s \neq e. \tag{3}$$

We describe the agent's posterior for all agents with the same form, with the "correct" binomial coefficient x. Thus, we calculate the growth rate by taking the expectation value of the posterior over the set of signals, summing over diagonal and off-diagonal components separately.

The mutual information separates into a term of only $l = 1/P(e)$, an on-diagonal, and off-diagonal term

$$\begin{aligned}I(E;S) &= \sum_{e,s}^{l} P(e,s)\big[\log l + \log P(e|s)\big] \\ &= \log l + p\log p + (1-p)\log\frac{1-p}{l-1} \\ &= H(E) - H(E|S),\end{aligned} \tag{4}$$

with the entropy of the outcome given by $H(E) = \log l$ and the reduction in entropy by the signal given by $H(E|S) = -p\log p - (1-p)\log\frac{1-p}{l-1}$. The information maximizes as $p \to 1$ and increases with l, and vanishes at $p \to 1/l$. The divergence is

$$\begin{aligned}\mathrm{E}_s\big[D_{KL}(P||X)\big] &= \sum_{e,s} P(e,s)\log\frac{P(e|s)}{X(e|s)} \\ &= p\log\frac{p}{x} + (1-p)\log\frac{1-p}{1-x},\end{aligned} \tag{5}$$

which is always non-negative and vanishes when $x \to p$. We can write the growth rate as the difference between these two terms as

$$\gamma = \mathrm{E}\big[\log l f(x,l)\big] = \log l + p\log x + (1-p)\log\frac{1-x}{l-1}. \tag{6}$$

C. Variance of growth model

The volatility can be calculated via the second moment of the stochastic growth rate as

$$\sigma = \sqrt{E\big[\log\big(lf(x,l)\big)^2\big] - E\big[\log lf(x,l)\big]^2}. \tag{7}$$

$\mathrm{E}\big[\log lf(x,l)\big]$ is simply γ, and the second term is

$$\begin{aligned}\mathrm{E}\big[\log lf(x,l)\big]^2 &= \left(\log l + p\log x + (1-p)\log\frac{1-x}{l-1}\right)^2 \\ &= \log^2 l + p^2\log^2 x + (1-p)^2\log^2\frac{1-x}{l-1} \\ &\quad + 2p\log l\log x + 2(1-p)\log l\log\frac{1-x}{l-1} \\ &\quad + 2p(1-p)\log x\log\frac{1-x}{l-1}.\end{aligned} \tag{8}$$

The first term expands to

$$\begin{aligned}\mathrm{E}\big[\log\big(lf(x,l)\big)^2\big] &= \mathrm{E}_{e,s}\big[(\log P(e|s)+\log l)^2\big]\\ &= \log^2 l + p\log^2 x + (1-p)\log^2\frac{1-x}{l-1}\\ &+ 2p\log l\log x + 2(1-p)\log l\log\frac{1-x}{l-1}.\end{aligned} \tag{9}$$

Combining these two quantities yields the volatility, where $(1-p)-(1-p)^2 = p(1-p)$,

$$\begin{aligned}\sigma_n &= \sqrt{p(1-p)\left[\log^2 x + \log^2\frac{1-x}{l-1} - 2\log x\log\frac{1-x}{l-1}\right]}\\ &= \sqrt{p(1-p)}\log\frac{x(l-1)}{1-x}.\end{aligned} \tag{10}$$

The variance of investment clusters of size $1/\omega$ scales as

$$\sigma_t^2 = \frac{1}{\gamma}\sigma_n^2, \tag{11}$$

where the subscript t denotes the temporal variance.

D. Latent Dirichlet Allocation

In this section, we derive the Latent Dirichlet Allocation (LDA) mode for the degenerate multinomial environment. The Bayesian update equation is given by

$$X(e|s) \propto \frac{\big(m_{(-s)}^{(-e)} + \tilde{\beta}_s^e\big)}{\big(M^{(-s)} + \tilde{B}^s\big)}(n_{(-e)} + \tilde{\alpha}_e), \tag{12}$$

for $m_{(-s)}^{(-e)}$ number of samples of outcome s conditional on e excluding the current, $n_{(-e)}$ the number of samples of e excluding the current in a batch of $n = \sum_e n_{(-e)}$ trials, where $M^{(-s)} = \sum_e m_{(-s)}^{(-e)}$. We set $\alpha_e = 1$, as every event is equally likely. For $s = e$, $\tilde{\beta}_s^e = x_e$, and for $s \neq e$, $\beta_{es} = \frac{(l-1)}{1-x}$ to impose degenerate off-diagonal conditions on $s|e$. We introduce $\tilde{B}^s = \sum_e \tilde{\beta}_s^e$, whereby symmetry, $\tilde{B}^s \equiv \tilde{B} = 1$, and we count over the diagonals, $n_{e=s}$, and off diagonals, $n_{e\neq s}$. Therefore the diagonal environmental posterior is

$$P(e|s) \propto \frac{\big(m_{(-s=e)}^{(-e)} + x_e\big)}{\big(M^{(-s)} + 1\big)}(n_{(-e)} + 1), \tag{13}$$

and the off-diagonal is

$$P(e|s) \propto \frac{\big(m_{(-s\neq e)}^{(-e)} + \frac{1-x_e}{l-1}\big)}{\big(M^{(-s)} + 1\big)}(n_{(-e)} + 1). \tag{14}$$

E. Asymptotic, temporal behavior

We introduce the temporal behavior, with two constants. We multiply the number of observations by the observation rate ω, with units *samples/time* and the inference rate k, with unit *time/update*. The inference rate counts the number of samples per Bayesian update, and the observation rate counts the updates per unit time. We introduce the inference time, k, a hyperprior magnitude that weighs the evidence versus the prior, leaving

$$P(e|s) \propto \frac{(m_{(-s)}^{(-e)}/\omega + \tilde{\beta}_s^e k)}{(M^{(-s)}/\omega + 1k)}(n_{(-e)}/\omega + \tilde{1}/k). \tag{15}$$

Over many observations, the law of large numbers argues that each outcome count converges to the environmental posterior with some noise, ξ_i as

$$\begin{aligned}M^{(-s)}/\omega &\to P(s)Nt + \xi_s\\ n_{(-e)}/\omega &\to P(e)Nt + \xi_e\\ m_{(-s)}^{(-e)}/\omega &\to P(s|e)Nt + \xi_{s|e},\end{aligned} \tag{16}$$

where the ξ's are fluctuation terms representing deviations from the mean. Over many i.i.d observations of events, $\xi \to 0$. The marginal terms converge to uniform over all states, and the agent posterior converges to the dynamical distribution

$$X(e,\lambda|s) = \frac{P(s|e)\lambda + X(s|e)}{1+\lambda}, \tag{17}$$

where we have converted to the time domain $t = N/\omega$, and substituted the dimensionless inference sample size $\lambda = t/kl$. Over long times, the distribution converges to the environmental posterior by

$$\begin{aligned}X(e,\lambda|s) &\propto \frac{P(s|e)\lambda + X(s|e)}{P(s)\lambda + 1}\big(P(e)\lambda + \alpha_e\big)\\ &\to \left(P(s|e) + \frac{X(s|e,0)}{\lambda}\right)\frac{P(e)}{P(s)} = P(e|s),\end{aligned} \tag{18}$$

yielding power law time-averaged behavior. At early times, as $t \to 0$ the posterior is proportional to the agent's initial agent posterior, $X(E|S)$, and converges to $P(E|S)$ as $kl \ll t \to \infty$. If agents are initialized with the same diagonal posterior value such that $X(s|e) = X(e', s')$ for all $e = e, s' = s'$, we can assume that the diagonals of an agent uniformly converge to p in time such that $X(s|e) \propto x(t)$ for all $s = e$,

F. Growth rate population variance

The mean growth rate is computed, where for brevity, the expected divergence for agent i with signals $s_i \in S_i$

is given as $\mathrm{E}_{s_i}\big(D_{KL}\big[P(E|s_i)||X(E|s_i]\big) \equiv D_i$, and the mutual information between individual signals and the environment, $I(E;S_i) \equiv I_i$

$$
\begin{aligned}
\langle\gamma_i\rangle &= \frac{1}{N}\sum_i I(E;S_i) - \mathrm{E}_{s_i}\big(D_{KL}\big[P(E|s_i)||X(E|s_i]\big) \\
&= \big\langle I_i\big\rangle - \langle D_i\rangle,
\end{aligned}
\tag{19}
$$

where angle brackets denote population arithmetic means. The variance in growth rates is calculated

$$
\begin{aligned}
\mathrm{Var}_N[\gamma_i] &= \big\langle(\gamma_i - \langle\gamma_i\rangle)^2\big\rangle, \\
&= \big\langle\gamma_i^2 + \langle\gamma_i\rangle^2 - 2\gamma_i\langle\gamma_i\rangle\big\rangle \\
&= \langle I_i^2\rangle - \langle I_i\rangle^2 + \langle D_i^2\rangle - \langle D_i\rangle^2 \\
&- 2\big(\langle I_i D_i\rangle - \langle I_i\rangle\langle D_i\rangle\big) \\
&= \mathrm{Var}_N[I_i] + \mathrm{Var}_N[D_i] - 2\mathrm{Covar}_N\big[I_i D_i\big].
\end{aligned}
\tag{20}
$$

When all agents are exposed to the same environment and share the same likelihood, the first and third terms vanish, leaving

$$
\mathrm{Var}_N[\gamma_i] = \mathrm{Var}_N\Bigg[\mathrm{E}_{s_i}\Bigg(D_{KL}\big[P(S|s_i)||X(E|s_i)\big]\Bigg)\Bigg].
\tag{21}
$$

G. Binomial parameter variance

The binomial variance can be computed exactly as

$$
\begin{aligned}
\mathrm{Var}_N\big[x_i(\lambda)\big] &= \frac{1}{N}\sum_i^N\left[\frac{p\lambda + x_i}{1+\lambda}\right]^2 - \left[\frac{p\lambda + \langle x_j\rangle}{1+\lambda}\right]^2 \\
&= \frac{1}{N}\sum_i^N\left[2\frac{x_i p\lambda + x_i^2}{(1+\lambda)^2} - 2\frac{\langle x_j\rangle p\lambda - \langle x_j\rangle^2}{(1+\lambda)^2}\right] \\
&= \frac{\langle x^2\rangle - \langle x_j\rangle^2}{(1+\lambda)^2} = \frac{\sigma_x^2}{(1+\lambda)^2}.
\end{aligned}
\tag{22}
$$

H. Multinomial growth rate variance

The variance of a function, $\gamma(x)$, of a random variable, x, is given generally by the Taylor expansion of that function [62]. It is written as

$$
\begin{aligned}
\mathrm{Var}_N\big(\gamma[x_i(\lambda)]\big) &= \gamma'\big[\big\langle x_i(\lambda)\big\rangle\big]\mathrm{Var}_N\big[x_i(\lambda)\big] \\
&- \frac{\gamma''\big[\big\langle x_i(\lambda)\big\rangle\big]^2}{4}\mathrm{Var}_N^2\big[x_i(\lambda)\big] + \bar{T}^3,
\end{aligned}
\tag{23}
$$

where primes denote differentiation with respect to x, and $\bar{T}^3$ are higher order terms that are only relevant at small times. The first and second-order derivatives of γ are given by

$$
\begin{aligned}
\gamma'(x) &= \frac{p}{x} - \frac{1-p}{1-x} \\
\gamma''(x) &= -\left[\frac{p}{x^2} + \frac{1-p}{(1-x)^2}\right],
\end{aligned}
\tag{24}
$$

and the variance term is given by

$$
\mathrm{Var}_N\big[x_i(\lambda)\big] = \frac{\sigma_x^2}{(1+\lambda)^2}.
\tag{25}
$$

The growth rate variance after small times is given by

$$
\begin{aligned}
\mathrm{Var}_N\big(\gamma[x_i(\lambda)]\big) &= \left[\frac{p}{\bar{x}} - \frac{1-p}{1-\bar{x}}\right]\frac{\sigma_x^2}{(1+\lambda)^2} \\
&+ \left[\frac{p}{\bar{x}^2} + \frac{1-p}{(1-\bar{x})^2}\right]\left[\frac{\sigma_x^2}{(1+\lambda)^2}\right]^2,
\end{aligned}
\tag{26}
$$

were for brevity, $\bar{x} \equiv \big\langle x_i(\lambda)\big\rangle$.