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ON THE SPECTRUM OF SINGULAR VALUES OF MULTI-Z-SHAPED GRAPH MATRICES

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ABSTRACT

This thesis studies graph matrices. Graph matrices are a type of random matrices that were invented as a powerful tool for analyzing large and complicated moment matrices which often arise in the analysis of the Sum of Square Hierarchy. They are also useful for other methods involving higher moments. Formally, a graph matrix is defined as a function mapping an $n \times n$ random input matrix to an output matrix which is generally larger and more complicated. This function is described by a small underlying shape. Previous studies on graph matrices mainly focused on their norm bounds. In this thesis, we further investigate their spectrum.

We start with determining the spectrum of singular values of Z-shaped and multi-Z-shaped graph matrices when the entries of the random input matrix have distribution ± 1 and the dimension parameter $n \to \infty$. This result can be seen as an analog of Wigner's Semicircle Law in the special case of ± 1 distribution, instead of any arbitrary distribution with mean 0 and variance 1.

We then generalize our result to multi-Z-shaped graph matrices with arbitrary input distributions with variance 1 and 0 odd moments. We achieve this using the $\circ_R$ operation, which mixes two distributions Ω and Ω' via a random orthogonal matrix.

This $\circ_R$ operation is closely connected to free probability theory, more precisely the free multiplicative convolution in the random matrix setting. As a part of our analysis, we prove a new direct formula for the moments of the product of two freely independent variables. We also prove some new results on non-crossing partitions which is an essential part of free probability theory.

CHAPTER 1

INTRODUCTION

The main topic of this thesis is graph matrices, their spectrum, and their connection to random matrix theory and free probability theory. We start by introducing graph matrices.

1.1 Introducing Graph Matrices

A graph matrix is a random matrix associated with an underlying small graph α, which we call a *shape* (see Definition 2.1.3). The entries of a graph matrix depend on its shape α and a random input. To give a flavor of graph matrices without giving the formal definition (We formally define graph matrices in Definition 2.1.5 and Definition 2.1.10), we give two examples below.

1. A simple example of a graph matrix is the $n \times n$ ± 1 adjacency matrix of a random graph $G \sim G\left(n, 1/2\right)$, with 0's on the diagonal. More precisely, $M(i,i) = 0$ for all $i \in [n]$, and for $i \neq j$, $M(i,j) = 1$ if $\{i,j\}$ is an edge in G and -1 otherwise.

   We visualize the matrix with the line shape α in the following way: the row index i corresponds to a vertex u, the column index corresponds to a vertex v, and the entry $M(i,j)$ corresponds to the edge (u,v).

2. For a more complicated example, which is the central object of study for the first half of this thesis, consider a $n(n-1) \times n(n-1)$ random matrix M with a random input graph $G \sim G\left(n, 1/2\right)$ where

$$M\left((i_1,i_2),(j_1,j_2)\right) = \begin{cases} x(i_1,j_1) \cdot x(j_1,i_2) \cdot x(i_2,j_2) & \text{if } i_1,i_2,j_1,j_2 \text{ are all distinct} \\ 0 & \text{otherwise} \end{cases}$$

   and $x(i,j) = 1$ if $\{i,j\}$ is an edge in G and -1 otherwise.

Again we can visualize this matrix by making the following correspondence: the row index (i_1, i_2) and column index (j_1, j_2) correspond to pairs of vertices (u_1, u_2) and (v_1, v_2) respectively, and the three edge variables $x(i_1, j_1)$, $x(j_1, i_2)$ and $x(i_2, j_2)$ correspond to the edges $\{u_1, v_1\}$, $\{v_1, u_2\}$ and $\{u_2, v_2\}$. The resulting shape is called the Z-shape (see Definition 2.1.15) and the matrix is thus called the Z-shaped graph matrix. See Figure 1.1 for an illustration.

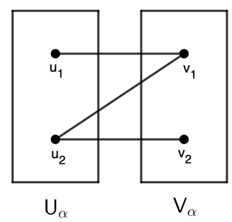


Figure 1.1: Z-shape

Graph matrices were first introduced and studied by Medarametla and Potechin in their paper "Bounds on the norms of uniform low degree graph matrices" (Medarametla and Potechin [2016], later updated in Ahn et al. [2020]) as a tool for analyzing large and complicated moment matrices which often appear in the sum of squares hierarchy analysis. These moment matrices can frequently be decomposed into linear combinations of graph matrices, thus understanding properties of graph matrices gives a systematic way of handling moment matrices. Ahn et al. [2020] proved rough norm bounds which are tight up to a polylog(n) factor on all graph matrices. Barak et al. [2019] was the first paper that used graph matrices to analyze the performance of SOS, in this case the planted clique problem. Graph matrices and its norm bound were then used as a useful tool to analyze many SOS problems later on (Ghosh et al. [2020], Potechin and Rajendran [2020], Kivva and Potechin [2020], Jones et al. [2022], Mohanty et al. [2020]), and also other methods involving higher moments (Venkat et al. [2022], Bafna et al. [2022], Rajendran and Tulsiani [2022]).

1.2 Background in Random Matrix Theory

Despite its somewhat complicated definition, graph matrices are random matrices, so to study them we are in the realm of random matrix theory (RMT). Some well-known results from RMT are Wigner's semicircle law (Wigner [1958, 1993]), Girko's circular law (Girko [1984]), Marchenko–Pastur law (Marčenko and Pastur [1967]), the Tracy-Widom distribution (Tracy and Widom [1994, 2002]), and the Matrix Bernstein Inequality (Oliveira [2009], Tropp [2012]). The first four are results about the distribution of eigenvalues of random matrices, whereas the last one gives a probabilistic norm bound on the sum of independent random matrices.

We will list these theorems formally.

Definition 1.2.1 (Wigner matrix)**.** A *Wigner matrix M* is a random Hermitian matrix where entries M_{ij} for $i < j$ are i.i.d. complex random variables with mean 0 and variance 1, $M_{ij} = \overline{M_{ji}}$ for $i > j$, and the diagonal entries are i.i.d. real random variables with bounded mean and variance.

Theorem 1.2.2 (Wigner's semicircle law)**.** *Let M_n be a sequence of $n \times n$ Wigner matrices. Let $\overline{M}_n = \frac{1}{\sqrt{n}} M_n$ be its normalization. Then the empirical distribution of eigenvalues (ESD) of $\overline{M}_n$ converges to the* semi-circle law $F_{sc}(x)$ *almost surely as $n \to \infty$, where $F_{sc}(x)$ is the distribution function whose density function $f_{sc}(x)$ is*

$$f_{sc}(x) = \begin{cases} \frac{1}{2\pi}\sqrt{4-x^2} & \textit{if } |x| \leq 2 \\ 0 & \textit{otherwise} \end{cases} \tag{1.1}$$

Girko's circular law, which arose as a natural counterpart of Wigner's semi-circle law, concerns the spectrum of the *non-Hermitian* random matrices.

Theorem 1.2.3 (Girko's circular law)**.** *Let M_n be a sequence of $n \times n$ matrices whose entries are i.i.d. complex random variables with mean* 0 *and variance* 1*. Let $\overline{M}_n = \frac{1}{\sqrt{n}} M_n$*

be its normalization. Then the empirical distribution of eigenvalues of $\overline{M}_n$ converges to the uniform distribution over the unit disk in the complex plane (Circular law) as $n \to \infty$ almost surely.

Remark 1.2.4. *Girko's circular law was originally posed as a conjecture in 1950. It was then partially proved by Mehta [1967], Edelman [1988], Girko [1984, 2004a,a], Bai [1997], Bai and Silverstein [2010], Götze and Tikhomirov [2010], Pan and Zhou [2010], Tao and Vu [2008], and finally proved in full generality by Tao et al. [2010].*

Theorem 1.2.5 (Marchenko–Pastur law)**.** *Let X_n be a sequence of $n \times m$ matrices whose entries are i.i.d. random variables with mean 0 and bounded variance σ^2. If $m/n \to c$ where $0 < c \leq 1$ as $n \to \infty$, then the ESD of $\frac{1}{n}XX^T$ converges to the* Marchenko–Pastur law *$F_c(x)$ almost surely as $n \to \infty$, where $F_c(x)$ is the distribution function whose density function $f_c(x)$ is*

$$f_c(x) = \frac{1}{2\pi\sigma^2} \cdot \frac{\sqrt{(c_+ - x)(x - c_-)}}{cx} \cdot \mathbb{1}_{x \in [c_-, c_+]} \tag{1.2}$$

where $c_\pm = \sigma^2 \left(1 \pm \sqrt{c}\right)^2$.

In contrast to the first three which are results on the limiting distribution of eigenvalues of random matrices in different settings, Tracy-Widom concerns the distribution of the *largest* eigenvalue.

Theorem 1.2.6 (Tracy-Widom distribution)**.** *Let M be an $n \times n$ Gaussian Unitary Ensemble (random Hermitian matrix where real and imaginary parts of off-diagonal entries are i.i.d. from $\mathcal{N}(0, 1/2)$ and diagonals are i.i.d from $\mathcal{N}(0,1)$). Let $\lambda_{\max}$ be the maximum eigenvalue of M. Consider $F(x)$, the distribution function of centered and scaled $\lambda_{\max}$ as $n \to \infty$,*

$$F(x) = \lim_{n\to\infty} \mathbb{P}\left(\left(\lambda_{\max} - \sqrt{2n}\right)\sqrt{2}n^{1/6} \leq x\right). \tag{1.3}$$

Then

$$F(s) = \det(I - A_s) = \exp\left(-\int_s^\infty (x-s)q^2(x)\,dx\right) \tag{1.4}$$

where A_s is an operator acting on $L^2(s,\infty)$ with kernel

$$\frac{Ai(x)Ai'(y) - Ai'(x)Ai(y)}{x-y}, \tag{1.5}$$

where the Airy function $Ai(x)$ is the solution to the Airy equation $y'' - xy = 0$ subject to the condition $y \to 0$ as $x \to \infty$, and $q(x)$ is the unique solution to the differential equation

$$q''(x) = xq(x) + 2q(x)^3 \tag{1.6}$$

satisfying $q(x) \sim Ai(x)$ as $x \to \infty$.

On the norm bound side, Matrix Bernstein gives a probabilistic bound on the upper tail of the spectral norm of a sum of independent zero-centered random matrices.

Theorem 1.2.7 (Matrix Bernstein)**.** *Let $A_1, \ldots, A_n$ be independent $d_1 \times d_2$ random matrices where for each $i \in [n]$, $\mathbb{E}[A_i] = \mathbf{0}$ and $\|A_i\| \leq L$ for some L. Let $A = A_1 + \cdots + A_n$ and $v(A)$ be its matrix variance defined as*

$$v(A) = \max\left\{\left\|\mathbb{E}\, AA^*\right\|, \left\|\mathbb{E}\, A^*A\right\|\right\}. \tag{1.7}$$

Then for any $\epsilon \geq 0$,

$$\mathbb{P}(\|A\| \geq \epsilon) \leq (d_1 + d_2) \cdot \exp\left(\frac{-\epsilon^2/2}{v(A) + \epsilon L/3}\right). \tag{1.8}$$

One key observation we would like to point out is that the graph matrices share similarity with classical random matrix theory, but are also distinct from it in certain aspects. In particular, when it comes to the study of limiting spectrum distributions, the entries are often independent and identical random variables in the classical RMT setting, whereas for graph matrices, every entry has dependency on a small set of other entries which share common edges (eg. the Z-shaped graph matrix example earlier in Section 1.1).

1.2.1 Methodologies

Since this thesis exclusively studies the limiting spectrums of graph matrices, we will focus on the ESD aspect of the RMT. Two methods are often used to analyze the limiting spectrum of a sequence of random matrices M_n, the moment method (more details in Section 3.1) and the Stieltjes transform method. These methods can be found in Bai and Silverstein [2010].

Moment method Consider the setup where M_n is a sequence of random matrices and $f_n(x)$ is the distribution of eigenvalues of M_n. The moment method is relatively straightforward: it considers all k^{th} moments $\beta_{k,n} := \int x^k f_n(x)\,dx$ and then computes the limits β_k as $n \to \infty$ for each k. The goal is then to find a distribution function $f(x)$ whose moments match with all β_k, together with confirming the conditions that guarantee the convergence of $f_n(x)$. In the setup of random matrices, these moments $\beta_{k,n}$ of eigenvalues turn out to be $\frac{1}{n}\operatorname{tr}\left({M_n}^k\right)$, thus we also refer to the moment method as the "Trace Power Method".

Stieltjes transform The Stieltjes transform, on the other hand, is a complex-analytic method. The Stieltjes transform of a measure μ on $\mathbb{R}$ with density function ρ is defined as $s_\rho(z) = \int \frac{1}{x-z} \cdot \rho(x)\,dx$ for $z \in \mathbb{C}$ not in the support of $\rho(x)$. The method uses the identity $s_n(z) = \int \frac{1}{x-z} \cdot f_n(x)\,dx = \frac{1}{n}\operatorname{tr}\left((M_n - zI)^{-1}\right)$, and then finds the limit $s(z)$ of $s_n(z)$ as $n \to \infty$ and the corresponding density function $f(x)$ whose Stieltjes transform is $s(z)$.

1.2.2 Wigner's Semicircle Law

As an example, we will consider the setup of Wigner's semicircle law. Recall Definition 1.2.1 and the Semicircle law.

Definition 1.2.8 (Semicircle Law)**.** The *semi-circle law* $F_{sc}(x)$ is a distribution function whose density function $f_{sc}(x)$ is

$$f_{sc}(x) = \begin{cases} \frac{1}{2\pi}\sqrt{4-x^2} & \text{if } |x| \le 2 \\ 0 & \text{otherwise} \end{cases} \tag{1.9}$$

For the Stieltjes transform method, we first consider the Stieltjes transform of the semi-circle law.

Lemma 1.2.9 (Lemma 2.11 from Bai and Silverstein [2010])**.** *Let $s_{sc}(z)$ be the Stieltjes transform of the Semicircle law $F_{sc}(x)$. i.e.*

$$s_{sc}(z) = \frac{1}{2\pi}\int_{-2}^{2} \frac{1}{x-z} \cdot \sqrt{4-x^2}\, dx\,. \tag{1.10}$$

Then

$$s_{sc}(z) = -\frac{1}{2}\left(z - \sqrt{z^2-4}\right). \tag{1.11}$$

Remark 1.2.10. *We can also compute the semicircle law $f_{sc}(x)$ backward from its Stieltjes transform $s_{sc}(z) = -\frac{1}{2}\left(z - \sqrt{z^2-4}\right)$ which is obtained as a limit of $s_n(z)$, by applying the inverse formula $f_{sc}(x) = \lim_{\varepsilon\to 0^+} \frac{s_{sc}(x-i\varepsilon) - s_{sc}(x+i\varepsilon)}{2\pi i}$.*

To prove Theorem 1.2.2, it suffices to prove that the Stieltjes transform $s_n(z)$ of normalized Wigner matrices, which is $\frac{1}{n}\operatorname{tr}\left(\left(\overline{M}_n - zI\right)^{-1}\right)$, converges almost surely to $s_{sc}(z) = -\frac{1}{2}\left(z - \sqrt{z^2-4}\right)$ for every z in the upper half plane. We do so by proving $\mathbb{E}\left[s_n(z)\right]$ converges to $s_{sc}(z)$ and then handling $s_n(z) - \mathbb{E}\left[s_n(z)\right]$. We need the inverse matrix formula to compute $\operatorname{tr}\left(\left(\overline{M}_n - zI\right)^{-1}\right)$.

Lemma 1.2.11. *Let A be an $n \times n$ invertible matrix. Then*

$$\left(A^{-1}\right)_{kk} = \frac{1}{a_{kk} - {w_k}' {A_k}^{-1} v_k} \tag{1.12}$$

where A_k is the $(n-1) \times (n-1)$ matrix obtained from A by deleting its k^{th} row and column, w_k' (resp., v_k) is the k^{th} row (resp. column) of A with the k^{th} entry deleted.

In particular, if A is a Hermitian matrix, then

$$\left(A^{-1}\right)_{kk} = \frac{1}{a_{kk} - {v_k}^* {A_k}^{-1} v_k} \tag{1.13}$$

Corollary 1.2.12.

$$\operatorname{tr}\left((M - zI)^{-1}\right) = \sum_{k=1}^{n} \frac{1}{m_{kk} - z - {v_k}^* \left(M_k - zI\right)^{-1} v_k} \tag{1.14}$$

where v_k is the k^{th} column of M_k with the k^{th} entry deleted.

With some reduction steps, we may assume that the diagonal elements are zero (removing diagonal elements does not change the limiting spectrum). The important identity we want to obtain here is that

$${v_k}^* \left(\left(\overline{M}_n\right)_k - zI\right)^{-1} v_k = \mathbb{E}\left[s_n(z)\right] + o(1) \tag{1.15}$$

since then we can conclude that (recall we assumed that $m_{kk} = 0$ for all k)

$$\begin{aligned}
\mathbb{E}\left[s_n(z)\right] &= \frac{1}{n}\mathbb{E}\left[\operatorname{tr}\left(\left(\overline{M}_n - zI\right)^{-1}\right)\right] \\
&= \mathbb{E}\left[\frac{1}{-z - {v_k}^*\left(M_k - zI\right)^{-1} v_k}\right] = -\frac{1}{z + \mathbb{E}\left[s_n(z)\right]} + o(1) \\
\Longrightarrow \mathbb{E}\left[s_n(z)\right]^2 &+ z\mathbb{E}[s_n(z)] + 1 = 0 \text{ as } n \to \infty \\
\Longrightarrow \mathbb{E}[s_n(z)] &= \frac{-z + \sqrt{4 - z^2}}{2} = s_{sc}(z) \text{ as } n \to \infty
\end{aligned}$$

Finally, to obtain Equation (1.15),

$$\begin{aligned}
&\mathbb{E}\left[{v_k}^*\left(\left(\overline{M}_n\right)_k - zI\right)^{-1} v_k\right] = \mathbb{E}\left[\sum_{i,j}\left(\left(\overline{M}_n\right)_k - zI\right)^{-1}_{ij} (\bar{v}_k)_i\,(v_k)_j\right] \\
&= \sum_i \frac{1}{n}\left(\left(\overline{M}_n\right)_k - zI\right)^{-1}_{ii} \text{ since } \mathbb{E}\,(v_k)_i = 0, \mathbb{E}\,|(v_k)_i|^2 = \mathbb{E}\left|\frac{1}{\sqrt{n}}M_{ik}\right|^2 = \frac{1}{n} \\
&= \frac{1}{n}\operatorname{tr}\left(\left(\frac{1}{\sqrt{n}}M_{n-1} - zI\right)^{-1}\right) \approx s_{n-1}(z) = s_n(z) + o(1) \text{ (stability of } s_n(z)) \\
\Longrightarrow\ &{v_k}^*\left(\left(\overline{M}_n\right)_k - zI\right)^{-1} v_k = \mathbb{E}\left[s_n(z)\right] + o(1) \text{ w.h.p.}
\end{aligned}$$

Now we turn to the moment method. One of the most important steps of analyzing the limiting spectrum of M_n is to compute $\lim_{n\to\infty} \frac{1}{n} \operatorname{tr}\left({M_n}^k\right)$. In our case when M_n are random matrices, we need to first consider $\lim_{n\to\infty} \frac{1}{n}\mathbb{E}\left[\operatorname{tr}\left({M_n}^k\right)\right]$ for easier analysis.

For simplicity, we assume that the input distribution of the Wigner matrix is even. Then $\mathbb{E}\left[\operatorname{tr}\left({M_n}^{2k+1}\right)\right] = 0$ since it expands to a summation of products of an odd number of entries, and each random entry has zero odd moments. Note that in the general case, we are still able to bound the magnitude of $\mathbb{E}\left[\operatorname{tr}\left({M_n}^{2k+1}\right)\right]$.

With some combinatorial arguments, one can deduce that the expected even trace powers of normalized M_n converge to the Catalan numbers.

Theorem 1.2.13. *Let M_n be a sequence of $n \times n$ Wigner matrices and $\overline{M}_n = \frac{1}{\sqrt{n}} M_n$ be its normalization. Then*

$$\lim_{n\to\infty} \frac{1}{n}\mathbb{E}\left[\operatorname{tr}\left(\overline{M}_n{}^{2k}\right)\right] = C_k \tag{1.16}$$

where $C_k = \frac{1}{k+1}\binom{2k}{k}$ is the k^{th} Catalan number, and

$$\lim_{n\to\infty} \frac{1}{n}\mathbb{E}\left[\operatorname{tr}\left(\overline{M}_n{}^{2k+1}\right)\right] = 0. \tag{1.17}$$

We can then verify that the moments of the semicircle distribution are indeed the Catalan numbers.

Proposition 1.2.14. *Recall the semicircle law $F_{sc}(x)$ from Definition 1.2.8. For all $k \geq 0$,*

$$\int_{-\infty}^{\infty} f_{sc}(x)\cdot x^{2k}\,dx = \int_{-2}^{2} \frac{1}{2\pi}\sqrt{4-x^2}\cdot x^{2k}\,dx = C_k \tag{1.18}$$

and

$$\int_{-\infty}^{\infty} f_{sc}(x)\cdot x^{2k+1}\,dx = \int_{-2}^{2} \frac{1}{2\pi}\sqrt{4-x^2}\cdot x^{2k+1}\,dx = 0. \tag{1.19}$$

Thus the expected moments of normalized Wigner matrices converge to the moments of the semicircle law. With some more careful analysis on the variance of the trace powers and verifying the Carleman condition (see Lemma 3.1.1), we can ensure the convergence of $\lim_{n\to\infty} \frac{1}{n} \operatorname{tr}\left(\overline{M}_n{}^{k}\right)$ and thus prove Theorem 1.2.2.

1.3 Contribution of this Thesis

1.3.1 Previous Results on Graph Matrices

Ahn et al. [2020] established the following norm bounds on all graph matrices.

Definition 1.3.1 (Vertex Separator)**.** Given a graph $G = (V(G), E(G))$ and $U, V \subseteq V(G)$, we say that $S \subseteq V(G)$ is a *vertex separator between U and V* if all paths from U to V intersect S.

Theorem 1.3.2 (Theorem 2.24 in Ahn et al. [2020])**.** *Let α be a shape with parts U_α and V_α without isolated middle vertices. Let $s_{\min}$ be the minimum size of a vertex separator between U_α and V_α. Then w.h.p.*

$$\|M_\alpha\| = \tilde{O}\left(n^{(|V(\alpha)|-s_{\min})/2)}\right) \tag{1.20}$$

Moreover, if α has u isolated middle vertices, then $\|M_\alpha\| = \tilde{O}\left(n^{(|V(\alpha)|+u-s_{\min})/2)}\right)$.

The above theorem works for graph matrices with one type of vertex. There are generalizations of this result for graph matrices where there are multiple types of vertices (Ahn et al. [2020]) and where the input is sparse (Jones et al. [2022], Rajendran and Tulsiani [2022]).

There were a few open problems on graph matrices after Ahn et al. [2020].

1. The norm bound given in Ahn et al. [2020] is tight up to a polylog(n) factor. Is it possible to perform a more careful analysis to get the exact norm bound?

2. Can we analyze the limiting distribution of singular values of graph matrices (equivalently, eigenvalue distribution of the symmetric matrix $M_\alpha M_\alpha^T$) and obtain an analog of the Wigner's semicircle Law?

3. Can we analyze the limiting distribution of eigenvalues of graph matrices and obtain an analog of the Girko's circular law?

In this thesis, we aim to take a step forward and answer the second question. More precisely, we are interested in analyzing the limiting behaviour of singular values for a sequence of $a(n) \times b(n)$ graph matrices as $n \to \infty$. As a first step, we consider the previous introduced Z-shaped graph matrices M_{α_Z}, and its generalization, multi-Z-shaped graph matrices $M_{\alpha_{Z(m)}}$ (formally defined in Definition 2.1.15 and Definition 2.2.17). As the name suggests,

$\alpha_{Z(m)}$ is formed by stacking $m-1$ Z shapes together ($2m$ vertices in total). Analyzing the limiting distribution of singular values of normalized $M_{\alpha_{Z(m)}}$ can be seen as an analog of the Wigner's semicircle law, in the sense that the base case of $M_{\alpha_{Z(m)}}$, M_{α_0} (see Example 2.1.7, 3 and Figure 2.1c), is a special Winger matrix where the entries are ± 1 with equal probability.

In the first part of this thesis, we analyze Z-shaped and multi-Z shaped graph matrices with ± 1 random inputs. We use the moment method for our analysis. There are two main steps:

Step 1. Obtain the expected trace powers of normalized Z-shaped and multi-Z shaped graph matrices. Indeed there is a similarity with the Wigner's semicircle law. As seen in the previous section, $\lim_{n \to \infty} \frac{1}{n} \mathbb{E}\left[\operatorname{tr}\left(\overline{M}_n{}^{2k}\right)\right] = C_k$ where $\overline{M}_n = \frac{1}{\sqrt{n}} M_n$ is the normalized Wigner matrix and C_k is the k^{th} Catalan number. For a multi-Z-shaped matrix $M_{\alpha_{Z(m)}}$ of dimension $r_{m,n} \sim n^m$, the expected trace power of $\frac{1}{n^{m/2}} M_{\alpha_{Z(m)}}$ as $n \to \infty$ turns out to be $C(m,k) = \frac{1}{mk+1}\binom{(m+1)k}{k}$, the generalized level-$m$ Catalan number. The high level idea for obtaining this result is to prove the trace power satisfies the same recurrence relation (see Theorem 5.1.7) for the (generalized) Catalan numbers through repeatedly splitting the *constraint graph* (see Definition 3.2.9) which represents the trace power.

Step 2. Solve for $f_{\alpha_{Z(m)}}(x)$, the limiting spectrum of normalized (multi) Z-shaped graph matrices, knowing its moments: $\int x^{2k} f_{\alpha_{Z(m)}}(x)\,dx = C(m,k)$. We do this by obtaining a differential equation for $f_{\alpha_{Z(m)}}(x)$ through performing integration by parts repeatedly on the moment formula and using the recurrence relation between $C(m,k)$ and $C(m,k+1)$. We successfully find an explicit expression for $f_{\alpha_Z}(x)$ (when $m=2$, see Theorem 2.1.17), and only obtained the differential equations for $m = 2, 3$ (see (4.13) and (5.7)), but the techniques work for higher m in general.

However, there are limitations to our results. Unlike Wigner's Semicircle Law, which holds as long as the entries of the symmetric random matrix have mean 0 and variance 1, our results on $M_{\alpha_{Z(m)}}$ only hold if the input distribution for the slanted edges [1] is $\{-1, 1\}$. We would like to know if we can push our analysis further to graph matrices with other input distributions (eg. Gaussian), or optimally, any arbitrary distribution with mean 0 and variance 1.

In the second part of this thesis, we resolve this question using an operation $\circ_R$ (see Definition 2.2.12) which mixes two distributions Ω and Ω' via a random orthogonal matrix. We show that if we take Ω_Z to be the limiting distribution of the singular values of $\frac{1}{n}M_{\alpha_Z}$ with the $\{-1, 1\}$ input distribution, then for any distribution Ω with variance 1 and zero odd moments, the limiting singular values distribution of $\frac{1}{n}M_{\alpha_Z}$ with the Ω input distribution is $\Omega_Z \circ_R \Omega$. We further show that this result generalizes nicely for multi-Z shaped graph matrices mixed with multiple distributions $\Omega^{(i)}$, namely $\Omega_{Z(m)} \circ_R \Omega^{(1)} \circ_R \ldots \circ_R \Omega^{(s)}$. The proof idea is similar to our first result on Z-shaped graph matrices: we use the moment method and show the equality of the moments by showing they satisfy the same recurrence relation. A nice byproduct of our analysis is that $\Omega_{Z(m)} \circ_R \Omega_{Z(n)}$ is the same as $\Omega_{Z(m+n)}$, thus allowing us to "stack" together multi-Z shapes through the $\circ_R$ operator.

This $\circ_R$ operation turns out to to be closely connected to free probability theory. Free probability theory, in contrast with the classical probability theory, studies non-commutative random variables that are *freely independent* (see Definition 2.2.3). It was first introduced by Voiculescu around 1986 to solve longstanding problems in the area of operator algebra, and in 1991 its connection with the random matrix theory was discovered (Voiculescu [1995, 1997]). Free probability theory then found its application in fields like Large deviations theory and Quantum information theory. Vaguely speaking, the "freeness" in free probability theory corresponds to the eigenspaces of random matrices being in "generic" positions, and random matrices with such property are free random variables. Similarly, our $\circ_R$ operation handles

1. the input distributions for the other two edges can be arbitrary as long as they have mean 0 and variance 1

matrices with the singular vectors in generic positions (where this freeness is ensured through the random orthogonal matrix).

An important piece in free probability theory is to study products of freely independent variables. With some reduction steps, computing moments of $\Omega \circ_R \Omega'$ becomes equivalent to computing moments of ab where a, b are freely independent. The existing formula for moments of ab (Theorem 3.4.11) is expressed through summation of free cumulants (see Definition 3.4.9) under non-crossing partitions. We managed to derive a new direct formula (see Theorem 2.2.8) for the moment cumulant formula by analyzing the combinatorics of these non-crossing partitions.

CHAPTER 2

DEFINITIONS AND RESULTS

2.1 Graph Matrices

In this section we will give formal definitions of graph matrices and state our results on the Z-shaped graph matrix.

2.1.1 Definitions

In order to state our results, we need a few definitions.

Definition 2.1.1 (Fourier Characters)**.** Given a graph $G = (V(G), E(G))$ and a multi-set of possible edges $E \subseteq \binom{V(G)}{2}$, define $\chi_E(G) = \prod_{e \in E} e(G)$ where the *edge variable* $e(G)$ is $e(G) = 1$ if $e \in E(G)$ and -1 otherwise.

Remark 2.1.2. *Notice that* $\chi_E(G) = (-1)^{\#(\textit{edges in } E \textit{ that are not in } G)} = (-1)^{E \setminus E(G)}$.

Definition 2.1.3 (Shapes)**.** We define a *shape* α to be a graph with vertices $V(\alpha)$, edges $E(\alpha)$, and distinguished tuples of vertices $U_\alpha = \left(u_1, \ldots, u_{|U_\alpha|}\right)$ and $V_\alpha = \left(v_1, \ldots, v_{|V_\alpha|}\right)$.

Definition 2.1.4 (Bipartite Shapes)**.** We say that a shape α is *bipartite* if $U_\alpha \cap V_\alpha = \emptyset$, $V(\alpha) = U_\alpha \cup V_\alpha$, and all edges in $E(\alpha)$ are between U_α and V_α.

Definition 2.1.5 (Graph Matrices)**.** Given a shape α, we define the *graph matrix* M_α (which depends on the input graph G) to be the $\frac{n!}{(n-|U_\alpha|)!} \times \frac{n!}{(n-|V_\alpha|)!}$ matrix with rows indexed by tuples of $|U_\alpha|$ distinct vertices, columns indexed by tuples of $|V_\alpha|$ distinct vertices, and entries

$$M_\alpha(A, B) = \sum_{\substack{\sigma: V(\alpha) \to V(G):\, \sigma \text{ is injective}, \\ \sigma(U_\alpha) = A, \sigma(V_\alpha) = B}} \chi_{\sigma(E(\alpha))}(G).$$

Now we give some basic examples of shapes and the corresponding graph matrices.

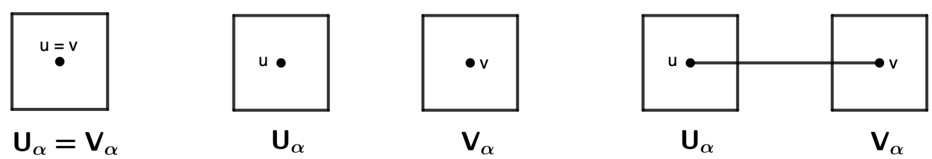
Definition 2.1.6. Let α_0 be the bipartite shape with vertices $V(\alpha_0) = \{u, v\}$ and a single edge $\{u, v\}$ with distinguished tuples of vertices $U_{\alpha_0} = (u)$ and $V_{\alpha_0} = (v)$. We call α_0 the *line shape.* See Figure 2.1c for an illustration.

Example 2.1.7. *Let $\alpha = (V(\alpha), E(\alpha))$, U_α, V_α be distinguished tuples of $V(\alpha)$. Let G be an input graph with n vertices. Here are some important examples:*

1. *when $U_\alpha = V_\alpha = V(\alpha) = \{v\}$ and $E(\alpha) = \emptyset$, $M_\alpha(G)$ is the $n \times n$ identity matrix.*

2. *when $V(\alpha) = \{u, v\}, U_\alpha = \{u\}, V_\alpha = \{v\}$, and $E(\alpha) = \emptyset$, $M_\alpha(G)$ is the all one matrix with zeros on the diagonal.*

3. *when $\alpha = \alpha_0$ the line shape, $M_\alpha(G)$ is a ± 1 matrix with zeros on diagonal.*



(a) example 1 (b) example 2 (c) example 3, the line shape α_0

Figure 2.1: Examples of graph matrices

We can also consider the linear combination of the graph matrices M_α's for different α's:

4. *Let $\alpha_E = (U_\alpha \sqcup V_\alpha, E)$ where $U_\alpha = \{u_1, u_2\}, V_\alpha = \{v_1, v_2\}$ and E is a subset of $E' :=$ set of all possible edges between $\{u_1, u_2, v_1, v_2\}$. Then for any A, B tuples of $V(G)$ of size two,*

$$\sum_{E \subset E'} M_{\alpha_E}(G)(A, B) = \begin{cases} \sum_{E \subset E'} 1 = 2^{|E'|} = 2^6 & \text{if } A, B \text{ form a 4-clique} \\ 0 & \text{otherwise} \end{cases} \tag{2.1}$$

Thus the graph matrix $M = \frac{1}{2^6} \sum_{E \subset E'} M_{\alpha_E}$ *is such that* $M(G)(A, B) = 1$ *if* A, B *forms a* 4*-clique in* G *and* 0 *otherwise.*

5. *Example 4 can be generalized to detecting* k*-cliques in graph* G *for any* k*. For a given* k*, let* $\alpha_E = (U_\alpha \sqcup V_\alpha, E)$ *where* $|U_\alpha| = \left\lfloor \frac{k}{2} \right\rfloor$, $|V_\alpha| = \left\lceil \frac{k}{2} \right\rceil$ *and* $E \subset E'$ *for* $E' =$ *all possible edges on* $U_\alpha \sqcup V_\alpha$*. Then*

$$M = \frac{1}{2^{\binom{k}{2}}} \cdot \sum_{E \subset E'} M_{\alpha_E}$$

is such that $M(G)(A, B) = 1$ *if* A, B *forms a* k*-clique in* G *and* 0 *otherwise.*

We now introduce a more general definition of graph matrices, by associating each shape α with a set of distributions $\Omega_{E(\alpha)} = \{\Omega_e : e \in E(\alpha)\}$, one for each edge of α.

Instead of having a single random input matrix, we have a random input matrix M^e for each edge e of our shape.

Definition 2.1.8 (Random Input Matrix M^e)**.** Let α be a shape associated with distributions $\Omega_{E(\alpha)}$. For each $e \in E(\alpha)$, we define the *random input matrix* M^e to be a symmetric random matrix where each entry $M^e(i, j)$ is drawn independently from Ω_e.

Definition 2.1.9 (Fourier Characters $\chi_{\sigma(E(\alpha))}$)**.** Let α be a shape associated with distributions $\Omega_{E(\alpha)}$. Given an injective map $\sigma : V(\alpha) \to [n]$, we define

$$\chi_{\sigma(E(\alpha))} = \prod_{e=(u,v) \in E(\alpha)} M^e(\sigma(u), \sigma(v)). \tag{2.2}$$

Definition 2.1.10 (Graph Matrices with input distributions $\Omega_{E(\alpha)}$)**.** Given a shape α associated with distributions $\Omega_{E(\alpha)}$, we define $M_{\alpha, \Omega_{E(\alpha)}}$, the *graph matrix with input distributions* $\Omega_{E(\alpha)}$, to be the $\frac{n!}{(n-|U_\alpha|)!} \times \frac{n!}{(n-|V_\alpha|)!}$ matrix with rows indexed by tuples of $|U_\alpha|$ distinct vertices, columns indexed by tuples of $|V_\alpha|$ distinct vertices, and entries

$$M_{\alpha,\Omega_{E(\alpha)}}(A,B) = \sum_{\substack{\sigma:V(\alpha)\to[n]:\,\sigma\text{ is injective},\\ \sigma(U_\alpha)=A,\sigma(V_\alpha)=B}} \chi_{\sigma(E(\alpha))}\,. \tag{2.3}$$

Remark 2.1.11. *In this thesis, we only consider bipartite shapes so for all A and B, we will either have that $A \cap B = \emptyset$ and there is exactly one injective map σ such that $\sigma(U_\alpha) = A$ and $\sigma(V_\alpha) = B$ or $A \cap B \neq \emptyset$ and $M_{\alpha,\Omega_{E(\alpha)}}(A,B) = 0$ as there are no such injective maps σ.*

Definition 2.1.12 (Graph Matrices with a single input distribution Ω)**.** If $\Omega_e = \Omega$ for all $e \in E(\alpha)$ then we denote the corresponding graph matrix $M_{\alpha,\Omega_{E(\alpha)}}$ as $M_{\alpha,\Omega}$. Note that even though all Ω_e are the same, there is an independent input matrix M^e with the same underlying distribution Ω for each edge e.

Definition 2.1.13 (± 1 distribution)**.** We define the ± 1 *distribution* $\Omega_{\pm 1}$ to be the distribution which is 1 with probability 1/2 and -1 with probability 1/2.

Definition 2.1.14. Given a shape α, if $\Omega_{E(\alpha)}$ consists of only $\Omega_{\pm 1}$, then we denote $M_{\alpha,\Omega_{E(\alpha)}}$ as just M_α. In other words, we take graph matrices to have input distributions $\Omega_{\pm 1}$ by default.

2.1.2 Our Results

Our main result is determining the spectrum of the singular values of the Z-shaped graph matrix.

Definition 2.1.15. Let α_Z be the bipartite shape with vertices $V(\alpha_Z) = \{u_1, u_2, v_1, v_2\}$ and edges $E(\alpha_Z) = \{\{u_1, v_1\}, \{u_2, v_1\}, \{u_2, v_2\}\}$ with distinguished tuples of vertices $U_{\alpha_Z} = (u_1, u_2)$ and $V_{\alpha_Z} = (v_1, v_2)$. See Figure 2.2 for an illustration. We call α_Z the *Z-shape*.

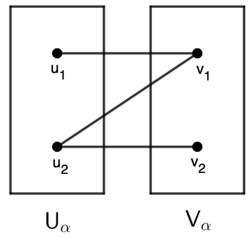


Figure 2.2: Z-shape α_Z

Definition 2.1.16. Let $a = \dfrac{3\sqrt{3}}{2}$ and define $g_{\alpha_Z} : (0, \infty) \to \mathbb{R}$ be the function such that

$$g_{\alpha_Z}(x) = \frac{i}{\pi}\left(\sqrt{3}\sin\left(\frac{1}{3}\arctan\left(\frac{3}{\sqrt{4x^2/3 - 9}}\right)\right) + \cos\left(\frac{1}{3}\arctan\left(\frac{3}{\sqrt{4x^2/3 - 9}}\right)\right)\right) \tag{2.4}$$

if $x \in (0, a]$ and $g_{\alpha_Z}(x) = 0$ if $x > a$. See Figure 2.3 for an illustration.

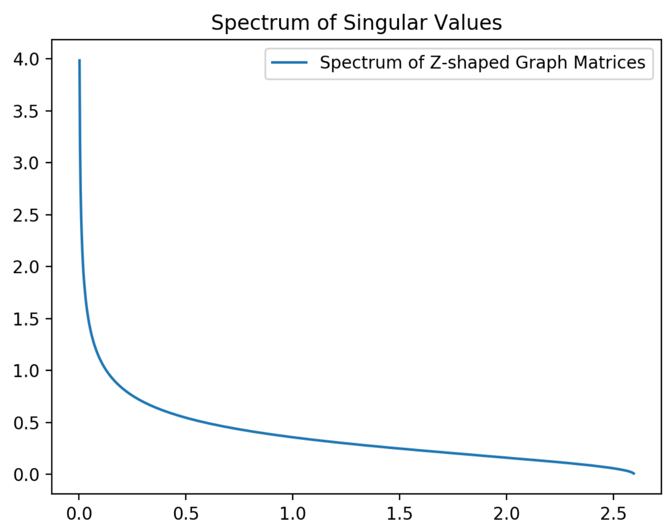


Figure 2.3: The limiting distribution of the singular values of $\frac{1}{n}M_{\alpha_Z}$ as $n \to \infty$

Theorem 2.1.17. *As $n \to \infty$, the spectrum of the singular values of $\frac{1}{n}M_{\alpha_Z}$ approaches g_{α_Z}.*

After proving this result, we apply our techniques to give a partial analysis of the spectrum of the singular values of multi-Z-shaped graph matrices.

2.2 Free Probability and the $\circ_R$ Operation

In this section we will list some necessary definitions from free probability theory to state our result of the moment formula. After that we will define the $\circ_R$ operation formally and state some results on its connection to the graph matrix.

2.2.1 Free Probability

Definition 2.2.1. We say that $(\mathcal{A}, \varphi)$ is a *non-commutative probability space* if $\mathcal{A}$ is a unital algebra over $\mathbb{C}$ and $\varphi : \mathcal{A} \to \mathbb{C}$ is a unital linear functional.

In this paper, we focus on the following type of non-commutative probability space. For further examples of non-commutative probability spaces, see Nica and Speicher [2006].

Example 2.2.2. $\left(M_n(\mathbb{C}), \bar{\operatorname{tr}}\right)$ *is a non-commutative probability space where* $M_n(\mathbb{C})$ *is the algebra of* $n \times n$ *complex matrices and* $\bar{\operatorname{tr}}$ *is the normalized trace* $\bar{\operatorname{tr}}(A) = \frac{1}{n} \cdot \sum_{j=1}^{n} A_{jj}$.

Definition 2.2.3 (Free Independence)**.** Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space. Let $\{\mathcal{A}_i : i \in \mathcal{I}\}$ be a collection of unital subalgebras of $\mathcal{A}$. We say that $\{\mathcal{A}_i : i \in \mathcal{I}\}$ are *freely independent* if for any $k \in \mathbb{Z}^+$, $\varphi(a_1 \ldots a_k) = 0$ whenever

1. $a_j \in \mathcal{A}_{i_j}$ for all $j \in [k]$
2. $\varphi(a_j) = 0$ for all $j \in [k]$
3. $i_j \neq i_{j+1}$ for any $j \in [k-1]$

Let $X_i \subset \mathcal{A}$ for $i \in \mathcal{I}$. We say that $\{X_i\}_{i \in \mathcal{I}}$ are *freely independent* if $\{\mathcal{A}_i\}_{i \in \mathcal{I}}$ are freely independent where $\mathcal{A}_i := \operatorname{alg}(1, X_i)$ is the unital algebra generated by the set X_i.

Let $a_i \in \mathcal{A}$ for $i \in \mathcal{I}$. We say that $\{a_i\}_{i \in \mathcal{I}}$ are *freely independent* if $\{\mathcal{A}_i\}_{i \in \mathcal{I}}$ are freely independent where $\mathcal{A}_i := \operatorname{alg}(1, a_i)$ is the unital algebra generated by a_i.

Given freely independent random variables a and b, the moments of the product of a and b (i.e., $\{\varphi((ab)^k) : k \in \mathbb{N}\}$) are completely determined by the moments of a and b. These moments can be computed directly using the definition of free independence but this quickly gets complicated.

Example 2.2.4. *If $(\mathcal{A}, \varphi)$ is a non-commutative probability space and $a, b \in \mathcal{A}$ are freely independent then:*

1. $\varphi(ab) = \varphi(a)\varphi(b)$ *since*

$$\begin{aligned} 0 &= \varphi\Big((a - \varphi(a))\,(b - \varphi(b))\Big) \\ &= \varphi\left(ab - \varphi(a)b - a\varphi(b) + \varphi(a)\varphi(b)\right) = \varphi(ab) - \varphi(a)\varphi(b) \end{aligned}$$

   More generally, $\varphi(a^m b^n) = \varphi(a^m)\varphi(b^n)$.

2. $\varphi(aba) = \varphi(a^2)\varphi(b)$ *since*

$$\begin{aligned} 0 &= \varphi\Big((a - \varphi(a))\,(b - \varphi(b))\,(a - \varphi(a))\Big) \\ &= \varphi\Big(a\,(b - \varphi(b))\,(a - \varphi(a))\Big) - \varphi(a)\varphi\Big((b - \varphi(b))\,(a - \varphi(a))\Big) \\ &= \varphi(aba) - \varphi(a)\varphi(ba) - \varphi(b)\varphi(a^2) + \varphi(a)\varphi(b)\varphi(a) - 0 \\ &= \varphi(aba) - \varphi(a^2)\varphi(b) \end{aligned}$$

3. $\varphi(abab) = -\varphi(a)^2\varphi(b)^2 + \varphi(a^2)\varphi(b)^2 + \varphi(a)^2\varphi(b^2)$ *since*

$$\begin{aligned}
0 =& \varphi\left((a-\varphi(a))\,(b-\varphi(b))\,(a-\varphi(a))\,(b-\varphi(b))\right) \\
=& \varphi\left(a\,(b-\varphi(b))\,(a-\varphi(a))\,(b-\varphi(b))\right) - \varphi(a)\varphi\left((b-\varphi(b))\,(a-\varphi(a))\,(b-\varphi(b))\right) \\
=& \varphi\,(abab) - \varphi(b)\varphi\,(aba) - \varphi(a)\varphi\left(ab^2\right) + \varphi(a)\varphi(b)\varphi\,(ab) - \varphi(b)\varphi\left(a^2b\right) + \varphi(b)^2\varphi\left(a^2\right) \\
& + \varphi(a)\varphi(b)\varphi\,(ab) - \varphi(a)^2\varphi(b)^2 - 0 \\
=& \varphi\,(abab) - \varphi(b)^2\varphi\left(a^2\right) - \varphi(a)^2\varphi\left(b^2\right) + \varphi(a)^2\varphi(b)^2 - \varphi(b)^2\varphi\left(a^2\right) + \varphi(b)^2\varphi\left(a^2\right) \\
& + \varphi(a)^2\varphi(b)^2 - \varphi(a)^2\varphi(b)^2 \\
=& \varphi(abab) + \varphi(a)^2\varphi(b)^2 - \varphi(a)^2\varphi(b^2) - \varphi(a^2)\varphi(b)^2
\end{aligned}$$

We now give our direct formula for the moments of the product of two freely independent random variables.

Definition 2.2.5. Let $P_k = \left\{ \vec{\alpha} = (\alpha_1, \ldots, \alpha_k) \subseteq [k]^k : \sum_{i=1}^{k} i \cdot \alpha_i = k \right\}$.

Definition 2.2.6. Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space. Given $\vec{\alpha} \in P_k$ and $a \in \mathcal{A}$, we denote $\vec{\varphi_a}^{\vec{\alpha}}$ to be

$$\vec{\varphi_a}^{\vec{\alpha}} = \prod_{i=1}^{k} \varphi(a^i)^{\alpha_i} = \varphi(a)^{\alpha_1}\varphi(a^2)^{\alpha_2}\ldots\varphi(a^k)^{\alpha_k} \tag{2.5}$$

Definition 2.2.7. Given $\vec{\alpha}, \vec{\beta} \in P_k$, we denote $C\left(\vec{\alpha}, \vec{\beta}\right)$ to be

$$C\left(\vec{\alpha}, \vec{\beta}\right) = (-1)^{A+B-k-1} \cdot k \cdot \binom{A+B-2}{k-1} \cdot \frac{(A-1)!}{\alpha_1!\ldots\alpha_k!} \cdot \frac{(B-1)!}{\beta_1!\ldots\beta_k!}\ . \tag{2.6}$$

where $A = \alpha_1 + \cdots + \alpha_k$ and $B = \beta_1 + \cdots + \beta_k$.

Note that when $A + B \leq k$, $C\left(\vec{\alpha}, \vec{\beta}\right) = 0$.

Theorem 2.2.8. *If* $(\mathcal{A}, \varphi)$ *is a non-commutative probability space and* $a, b \in \mathcal{A}$ *are freely* *independent then*

$$\varphi\left((ab)^k\right) = \sum_{\vec{\alpha},\vec{\beta}\in P_k} C\left(\vec{\alpha},\vec{\beta}\right)\cdot \vec{\varphi_a}^{\vec{\alpha}} \cdot \vec{\varphi_b}^{\vec{\beta}} \tag{2.7}$$

Example 2.2.9. *We list below the first few cases when $k = 1, 2, 3, 4$.*

1. $\varphi(ab) = \varphi(a)\varphi(b)$.

2. $\varphi(abab) = -\varphi(a)^2\varphi(b)^2 + \varphi(a)^2\varphi(b^2) + \varphi(a^2)\varphi(b)^2$.

3. $\varphi(ababab) = 2\varphi(a)^3\varphi(b)^3 - 3\varphi(a)^3\varphi(b)\varphi(b^2) - 3\varphi(a)\varphi(a^2)\varphi(b)^3 + 3\varphi(a)\varphi(a^2)\varphi(b)\varphi(b^2) + \varphi(a)^3\varphi(b^3) + \varphi(a^3)\varphi(b)^3$.

4. $\varphi(abababab) = -5\varphi(a)^4\varphi(b)^4 + 10\varphi(a)^4\varphi(b)^2\varphi(b^2) + 10\varphi(a)^2\varphi(a^2)\varphi(b)^4 - 4\varphi(a)^4\varphi(b)\varphi(b^3) - 4\varphi(a)\varphi(a^3)\varphi(b)^4 - 2\varphi(a)^4\varphi(b^2)^2 - 2\varphi(a^2)^2\varphi(b)^4 + \varphi(a)^4\varphi(b^4) + \varphi(a^4)\varphi(b)^4 - 16\varphi(a)^2\varphi(a^2)\varphi(b)^2\varphi(b^2) + 4\varphi(a)^2\varphi(a^2)\varphi(b)\varphi(b^3) + 4\varphi(a)\varphi(a^3)\varphi(b)^2\varphi(b^2) + 2\varphi(a)^2\varphi(a^2)\varphi(b^2)^2 + 2\varphi(a^2)^2\varphi(b)^2\varphi(b^2)$.

Note that we computed the first two results directly in Example 2.2.4.

2.2.2 $\circ_R$ Operation

We now formally define the operation $\circ_R$ and describe its connection with the moments of the product of two freely independent random variables.

Definition 2.2.10. We say D is an Ω-*distribution diagonal matrix* if D is diagonal and each diagonal entry of D is drawn independently from the distribution Ω.

Definition 2.2.11. Given $n \in \mathbb{N}$, we choose a random orthogonal matrix R as follows:

1. Choose the first row $\vec{r}_1$ to be a random vector in $S^{n-1} = \{\vec{x} \in \mathbb{R}^n : \|\vec{x}\| = 1\}$. One way to do this is to choose the entries of $\vec{r}_1$ to be random Gaussian variables and then rescale $\vec{r}_1$ so that $\|\vec{r}_1\| = 1$.

2. For each $j \in \{2, \ldots, n\}$, choose the jth row $\vec{r}_j$ to be a random vector in the $(n-j)$-dimensional sphere $\{\vec{x} \in \mathbb{R}^n : \|\vec{x}\| = 1, \vec{x} \cdot \vec{r}_i = 0 \text{ for all } i \in [j-1]\}$. One way to do this is to choose the entries of $\vec{r}_j$ to be random Gaussian variables, remove the components of $\vec{r}_j$ which are parallel to $\vec{r}_1, \ldots, \vec{r}_{j-1}$, and then rescale $\vec{r}_j$ so that $\left\|\vec{r}_j\right\| = 1$.

Definition 2.2.12. Given two distributions Ω and Ω', we define $\Omega \circ_R \Omega'$ to be the limiting distribution as $n \to \infty$ of the singular values of the random matrix $M = DRD'$ where D and D' are $n \times n$ Ω and Ω'-distribution diagonal matrices, respectively, and R is a random $n \times n$ orthogonal matrix.

Proposition 2.2.13. *The operation $\circ_R$ is commutative and associative.*

Proof.

1. $\Omega \circ_R \Omega' = \Omega' \circ_R \Omega$: Let D and D' be Ω and Ω'-distribution diagonal matrices and R be a random orthognal matrix. $\left(DRD'\right)^T = D'R^T D$ where R^T is also a random orthogonal matrix. Since M and M^T have the same singular values, $\{DRD' : R \text{ is a random orthogonal matrix}\}$ and $\{D'RD : R \text{ is a random orthogonal matrix}\}$ have the same singular value distributions.

2. $\left(\Omega \circ_R \Omega'\right) \circ_R \Omega'' = \Omega \circ_R \left(\Omega' \circ_R \Omega''\right)$: Let D, D' and D'' be Ω, Ω' and Ω''-distribution diagonal matrices. Let $D_{\Omega \circ_R \Omega'}$ and $D_{\Omega' \circ_R \Omega''}$ be $\Omega \circ_R \Omega'$ and $\Omega' \circ_R \Omega''$-distribution diagonal matrices. Since $D_{\Omega \circ_R \Omega'}$ and DRD' have the same singular values distributions in the limit as $n \to \infty$, $\left\{R_1 D_{\Omega \circ_R \Omega'} R_2 : R_1, R_2 \text{ orthogonal}\right\} =$
$\left\{\hat{R}_1 \left(DRD'\right) \hat{R}_2 : \hat{R}_1, \hat{R}_2 \text{ orthogonal}\right\}$.

We have that in the limit as $n \to \infty$,

$$
\begin{aligned}
&\left\{R_1\left(D_{\Omega\circ_R\Omega'}R_2D''\right)R_3 : R_1, R_2, R_3 \text{ orthogonal}\right\}\\
=&\left\{\left(R_1D_{\Omega\circ_R\Omega'}R_2\right)D''R_3 : R_1, R_2, R_3 \text{ orthogonal}\right\}\\
=&\left\{\left(\hat{R}_1\left(DRD'\right)\hat{R}_2\right)D''R_3 : R, \hat{R}_1, \hat{R}_2, R_3 \text{ orthogonal}\right\}\\
=&\left\{R_1DR_2D'R_3D''R_4 : R_1, R_2, R_3, R_4 \text{ orthogonal}\right\}\\
=&\left\{R_1D\left(R_2\left(D'R_3D''\right)R_4\right) : R_1, R_2, R_3, R_4 \text{ orthogonal}\right\}\\
=&\left\{R_1D\left(\hat{R}_2D_{\Omega'\circ_R\Omega''}\hat{R}_4\right) : R_1, \hat{R}_2, \hat{R}_4 \text{ orthogonal}\right\}\\
=&\left\{R_1\left(DR_2D_{\Omega'\circ_R\Omega''}\right)R_4 : R_1, R_2, R_4 \text{ orthogonal}\right\}
\end{aligned}
$$

For any matrix M, M and R_1MR_2 have the same singular values if R_1 and R_2 are orthogonal. Thus, in the limit as $n \to \infty$, $\left\{D_{\Omega\circ_R\Omega''}RD'' : R \text{ orthogonal}\right\}$ and $\left\{DRD_{\Omega'\circ_R\Omega''} : R \text{ orthogonal}\right\}$ have the same singular value distributions.

□

To see why the $\circ_R$ operation is connected with free probability theory, we make the following observation.

Proposition 2.2.14. *Let R_n be an $n \times n$ random orthogonal matrix, D_n and D'_n be $n \times n$ random diagonal matrices where the diagonal elements are drawn independently from distributions Ω and Ω', respectively. Let $M_n = D_nR_nD'_n$, $A_n = {D_n}^2$, $B_n = R_n{D'_n}^2R_n^T$, and $\varphi = \bar{\text{tr}} \otimes \mathbb{E}$. Then*

$$\frac{1}{n} \cdot \mathbb{E}\left[\text{tr}\left(\left(M_nM_n^T\right)^k\right)\right] = \varphi\left(\left(A_nB_n\right)^k\right) \tag{2.8}$$

Proof. Observe that

$$\begin{aligned}\frac{1}{n}\cdot\mathbb{E}\left[\operatorname{tr}\left(\left(MM^T\right)^k\right)\right] =&\mathbb{E}\left[\frac{1}{n}\operatorname{tr}\left(DRD'^2R^TD^2RD'^2R^T\dots D^2RD'^2R^TD\right)\right]\\ =&\mathbb{E}\left[\frac{1}{n}\operatorname{tr}\left(D^2\left(RD'^2R^T\right)D^2\left(RD'^2R^T\right)\dots D^2\left(RD'^2R^T\right)\right)\right]\\ =&\mathbb{E}\left[\frac{1}{n}\operatorname{tr}\left(A_nB_nA_nB_n\dots A_nB_n\right)\right]\\ =&\varphi\left((A_nB_n)^k\right)\end{aligned}$$

where the second equality uses the fact that $\operatorname{tr}(AB)=\operatorname{tr}(BA)$. □

As we discuss in Section 3.4, an important result in free probability theory is that in the limit as $n\to\infty$, the variables A_n and B_n are freely independent. Together with Theorem 2.2.8, this gives the following corollary.

Definition 2.2.15. Given a distribution Ω and $\vec{\alpha}\in P_k$, we denote

1. $\Omega_{2m}=\mathbb{E}_{x\sim\Omega}\left[\,x^{2m}\right]$, and
2. $\vec{\Omega}^{\vec{\alpha}}=\Omega_2^{\alpha_1}\dots\Omega_{2k}^{\alpha_k}$.

Below is a corollary from Theorem 2.2.8.

Corollary 2.2.16. *Let Ω and Ω' be two distributions. Then for all $k\in\mathbb{N}$,*

$$\left(\Omega\circ_R\Omega'\right)_{2k}=\sum_{\vec{\alpha},\vec{\beta}\in P_k}C\left(\vec{\alpha},\vec{\beta}\right)\cdot\vec{\Omega}^{\vec{\alpha}}\cdot\vec{\Omega'}^{\vec{\beta}} \tag{2.9}$$

where if we let $a=a_1+\cdots+a_k$ and $b=b_1+\cdots+b_k$, then

$$C\left(\vec{\alpha},\vec{\beta}\right)=(-1)^{k+a+b-1}\cdot k\cdot\binom{a+b-2}{k-1}\cdot\frac{(a-1)!}{\alpha_1!\dots\alpha_k!}\cdot\frac{(b-1)!}{\beta_1!\dots\beta_k!}\,. \tag{2.10}$$

2.2.3 Application of the $\circ_R$ Operation to Graph Matrices

We now state our main results on the limiting distribution of the singular values of multi-Z-shaped graph matrices.

Definition 2.2.17. Let $\alpha_{Z(m)}$ be the bipartite shape with vertices $V\left(\alpha_{Z(m)}\right) = \{u_1, \ldots, u_m, v_1, \ldots, v_m\}$ and edges $E\left(\alpha_{Z(m)}\right) = \Big\{\{u_i, v_i\} : i \in [m]\Big\} \cup \Big\{\{u_{i+1}, v_i\} : i \in [m-1]\Big\}$ with distinguished tuples of vertices $U_{\alpha_{Z(m)}} = (u_1, \ldots, u_m)$ and $V_{\alpha_{Z(m)}} = (v_1, \ldots, v_m)$. We call the slanted edges $\Big\{\{u_{i+1}, v_i\} : i \in [m-1]\Big\}$ *spokes* and call $\{u_{i+1}, v_i\}$ the i^{th} spoke. See Figure 2.4 for an illustration.

We refer to $\alpha_{Z(m)}$ as the *m−layer Z-shape* or the *Z(m)-shape*. When $m = 2$, the shape is exactly a Z shape.

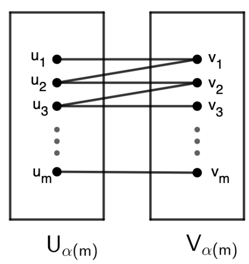


Figure 2.4: The $m-$layer Z-shape $\alpha_{Z(m)}$

Definition 2.2.18. Given $\Omega^{(1)}, \ldots, \Omega^{(s)}$ and $m \geq s+1$, we define

1. $\Omega_{\alpha_{Z(m)},s}$ to be the set of distributions associated with $\alpha_{Z(m)}$ where the i^{th} spoke has distribution $\Omega^{(i)}$ for $i = 1, \ldots, s$ and all other edges have distribution $\Omega_{\pm 1}$, and

2. $M_{\alpha_{Z(m)},s}$ to be the graph matrix with shape $\alpha_{Z(m)}$ and input distributions $\Omega_{\alpha_{Z(m)},s}$, and

3. $M^{(G)}_{Z(m),s} = \dfrac{1}{n^{m/2}} \cdot M_{\alpha_{Z(m)},s}$ to be the normalized graph matrix.

Definition 2.2.19. Given $\Omega^{(1)}, \ldots, \Omega^{(s)}$ and $m \geq s+1$, we define

1. $\Omega_{Z(m)}$ to be the limiting distribution of the singular values of $\frac{1}{n^{m/2}} \cdot M_{\alpha_{Z(m)}}$, or equivalently, $M^{(G)}_{Z(m),0}$.

2. $\Omega_{Z(m),\Omega^{(1)},\ldots,\Omega^{(s)}}$ to be the limiting distribution of the singular values of $M^{(G)}_{Z(m),s}$.

With these definitions, we can now state our main results which show that the spectrum of the singular values of multi-Z-shaped graph matrices can be easily described using the $\circ_R$ operation.

Definition 2.2.20. We define $\mathbf{\Omega}$ to be the set of all random distributions and $\mathbf{\Omega_{0,1}}$ to be the set of distributions Ω with 0 odd moments and variance 1.

Definition 2.2.21. A distribution Ω satisfies Carleman's condition if $\sum_{k=1}^{\infty} \beta_{2k}^{-\frac{1}{2k}} = \infty$ where $\beta_{2k} = \mathbb{E}_{\Omega}\left[x^{2k}\right]$ is the 2k-th moment of Ω.

Theorem 2.2.22. *For all $\Omega^{(1)}, \ldots, \Omega^{(s)} \in \mathbf{\Omega_{0,1}}$ which satisfy Carleman's condition,*

$$\Omega_{Z(m)} \circ_R \Omega^{(1)} \circ_R \ldots \circ_R \Omega^{(s)} = \Omega_{Z(m),\Omega^{(1)},\ldots,\Omega^{(s)}} \tag{2.11}$$

Remark 2.2.23. *Carleman's condition is needed as it ensures that the resulting distribution can be recovered from its moments.*

Theorem 2.2.24. *For any $m, m' \in \mathbb{N}$,*

$$\Omega_{Z(m)} \circ_R \Omega_{Z(m')} = \Omega_{Z(m+m')} \tag{2.12}$$

CHAPTER 3

PRELIMINARIES

3.1 The Trace Power Method

In this section, we will give a more detailed explanation of the moment method introduced in Section 1.2.1.

Lemma 3.1.1 (Carleman Condition, Akhiezer [2020])**.** *Let μ be a measure on $\mathbb{R}$ and let $\beta_k = \int x^k \, d\mu(x)$ be its moments. If $\sum_k {\beta_{2k}}^{-1/2k} = \infty$, then μ is the unique measure on $\mathbb{R}$ with (β_k) as its moments.*

We call the condition $\sum_k {\beta_{2k}}^{-1/2k} = \infty$ the Carleman Condition*.*

Lemma 3.1.2 (Moment Convergence Theorem, Lemma B.1 and B.3 in Bai and Silverstein [2010])**.** *Let $\{F_n(x) : n \geq 1\}$ be a sequence of distribution functions and $F(x)$ be a distribution function. $\{F_n(x)\}$ converges weakly to $F(x)$ if the following are true:*

1. *F_n has finite moments of all orders for all $n \geq 1$. i.e. For all $n \geq 1$, $k \geq 0$, $\int x^k \, dF_n(x) < \infty$.*
2. *For each $k \geq 0$, $\beta_k := \lim_{n\to\infty} \int x^k \, dF_n(x)$ exists and is finite. Moreover, $\int x^k \, dF(x) = \beta_k$ for each $k \geq 0$.*
3. *(Carleman Condition) $\sum {\beta_{2k}}^{-1/2k} = \infty$.*

Definition 3.1.3. Let M be a random symmetric matrix with dimension m. We define $F^M(x)$*, the empirical spectral distribution (ESD) of* M to be

$$F^M(x) := \frac{1}{m} \left| \{i \in [m] : \lambda_i \leq x\} \right| \tag{3.1}$$

where $\lambda_1, \ldots, \lambda_m$ are the eigenvalues of M.

We also consider the notation

$$f^M(x) := \frac{1}{m} \sum_{i=1}^{m} \delta_{\lambda_j}(x) \tag{3.2}$$

for its corresponding probability density function.

Let $\{M_n : n \in \mathbb{N}\}$ be a family of random symmetric $r(n) \times r(n)$ matrices. For each $n \in \mathbb{N}$, we denote $F_n(x) := F^{M_n}(x)$.

Proposition 3.1.4. *Let M be a random symmetric matrix with dimension m. Then*

$$\int_{-\infty}^{\infty} x^k \, dF^M(x) = \frac{1}{m} \operatorname{tr}\left(M^k\right) \tag{3.3}$$

Corollary 3.1.5. *Let $\{A_n : n \in \mathbb{N}\}$ be a family of random symmetric $r(n) \times r(n)$ matrices. If $F(x)$ is some distribution function such that the following are true:*

1. $\frac{1}{r(n)} \operatorname{tr}\left({A_n}^k\right) < \infty$ *for all* $n \geq 1, k \geq 0$.
2. *For each* $k \geq 0$, $\frac{1}{r(n)} \operatorname{tr}\left({A_n}^k\right)$ *converges to* β_k *in probability (respectively, almost surely). Moreover,* $\int x^k \, dF(x) = \beta_k$ *for all* $k \geq 0$.
3. $\sum {\beta_{2k}}^{-1/2k} = \infty$.

Then $F_n(x)$, the distribution of eigenvalues of A_n, converges to $F(x)$ in probability (respectively, almost surely).

To ensure the convergence of $\frac{1}{n} \operatorname{tr}\left({M_n}^k\right)$, we further separate into two steps: compute the limit of the expected value of the trace power $\frac{1}{n}\mathbb{E}\left[\operatorname{tr}\left({M_n}^k\right)\right]$ as $n \to \infty$, and bound its variance.

Lemma 3.1.6. *Let x_n be a sequence of random variables and μ a constant. Assume that* $\lim_{n\to\infty} \mathbb{E}[x_n] = \mu$. *Then*

1. x_n *converges to* μ *in probability if* $\mathrm{Var}(x_n) = o(1)$.

2. x_n *converges to* μ *almost surely if* $\sum_n \mathrm{Var}(x_n) < \infty$.

Proof.

1. We want to show that $\lim_{n\to\infty} \mathbb{P}(|x_n - \mu| > \epsilon) = 0$ for any $\epsilon > 0$. Equivalently, $\lim_{n\to\infty} \mathbb{P}(|x_n - \mu| < \epsilon) = 1$ for any $\epsilon > 0$.

   By the triangle inequality, $|x_n - x| \leq |x_n - \mu_n| + |\mu_n - \mu|$ where $\mu_n = \mathbb{E}[x_n]$.

   Since $\lim_{n\to\infty} \mu_n = \mu$, there exists N_1 such that $|\mu_n - \mu| < \epsilon/2$ for any $n \geq N_1$.

   By Chebyshev's inequality, $\mathbb{P}(|x_n - \mu_n| \geq \delta) \leq \sigma_n^2/\delta^2$ where $\sigma_n^2 = \mathrm{Var}(x_n)$. Thus $\mathbb{P}(|x_n - \mu_n| < \delta) > 1 - \sigma_n^2/\delta^2$.

   For any fixed $\epsilon > 0$ and $\epsilon' > 0$, since $\sigma_n^2 = o(1)$, there exists N_2 such that $\sigma_n^2 \geq \frac{\epsilon^2 \epsilon'}{4}$ for any $n \geq N_2$. Let $N = \max(N_1, N_2)$. Then for all $n \geq N$,

$$\begin{aligned}
\mathbb{P}(|x_n - x| < \epsilon) &\geq \mathbb{P}(|x_n - \mu_n| + |\mu_n - \mu| < \epsilon) \\
&\geq \mathbb{P}(|x_n - \mu_n| < \epsilon/2)\, \mathbb{P}(|\mu_n - \mu| < \epsilon/2) \\
&\geq 1 - 4\sigma_n^2/\epsilon^2 \geq 1 - \epsilon' \\
\implies \lim_{n\to\infty} \mathbb{P}(|x_n - \mu| < \epsilon) = 1
\end{aligned}$$

2. We want to show that $\mathbb{P}\left(\lim_{n\to\infty} x_n = x\right) = 1$. Equivalently, $\mathbb{P}\left(\lim_{n\to\infty} x_n \neq x\right) = 0$.

$$\begin{aligned}
\mathbb{P}\left(\lim_{n\to\infty} x_n \neq x\right) &= \mathbb{P}(\forall N, \exists\, n \geq N \text{ such that } |x_n - x| \geq \epsilon) \\
&\leq \mathbb{P}(|x_n - x| \geq \epsilon \;\forall n \geq N) \text{ for some fixed } N \\
&\leq \sum_{n=N}^{\infty} \mathbb{P}(|x_n - x| \geq \epsilon\,) \\
&\leq \sum_{n=N}^{\infty} \sigma_n^2/\epsilon^2 \to 0 \text{ as } N \to \infty \text{ since } \sum_n \sigma_n^2 < \infty
\end{aligned}$$

□

With Lemma 3.1.6, Corollary 3.1.5 can be rewritten as the following.

Corollary 3.1.7. *Let $\{A_n : n \in \mathbb{N}\}$ be a family of random symmetric $r(n) \times r(n)$ matrices. If $F(x)$ is some distribution function such that the following are true:*

1. $\frac{1}{r(n)} \operatorname{tr}\left(A_n{}^k\right) < \infty$ *for all* $n \geq 1, k \geq 0$.
2. *For each* $k \geq 0$, $\beta_k = \lim_{n\to\infty} \frac{1}{r(n)} \mathbb{E}\left[\operatorname{tr}\left(A_n{}^k\right)\right]$ *exists and is finite. Moreover,* $\int x^k \, dF(x) = \beta_k$ *for all* $k \geq 0$.
3. $\lim_{n\to\infty} \operatorname{Var}\left(\frac{1}{r(n)} \operatorname{tr}\left(A_n{}^k\right)\right) = 0$ *(respectively,* $\sum_n \operatorname{Var}\left(\frac{1}{r(n)} \operatorname{tr}\left(A_n{}^k\right)\right) < \infty$*).*
4. *(Carleman Condition)* $\sum \beta_{2k}{}^{-1/2k} = \infty$.

Then $F_n(x)$, the distribution of eigenvalues of A_n, converges to $F(x)$ in probability (respectively, almost surely).

Since graph matrices are not symmetric in general, we want to consider the limiting distribution of singular values of $\{M_n : n \geq 1\}$ instead of eigenvalues. We can do so by applying Corollary 3.1.5 to $A_n = M_n M_n^T$.

Remark 3.1.8. *Let M be a $a \times b$ matrix. We define the distribution function of its singular values by replacing λ_i with σ_i and the dimension m with $r = \min\{a, b\}$ in Definition 3.1.3 where $\sigma_1, \ldots, \sigma_r$ are the singular values of M.*

Corollary 3.1.9. *Let $\{M_n : n \in \mathbb{N}\}$ be a family of random $a(n) \times b(n)$ matrices. Let $r(n) = \min\{a(n), b(n)\}$. If $F(x)$ is some distribution function such that the following are true:*

1. $\frac{1}{r(n)} \mathbb{E}\left[\operatorname{tr}\left(\left(M_n M_n{}^T\right)^k\right)\right] < \infty$ *for all* $n \geq 1, k \geq 0$.

2. For each $k \geq 0$, $\beta_k = \lim_{n\to\infty} \frac{1}{r(n)} \mathbb{E}\left[\operatorname{tr}\left(\left(M_n {M_n}^T\right)^k\right)\right]$ *exists and is finite. Moreover,* $\int x^{2k}\, dF(x) = \beta_k$ *for all* $k \geq 0$.

3. $\lim_{n\to\infty} \operatorname{Var}\left(\frac{1}{r(n)} \operatorname{tr}\left(\left(M_n {M_n}^T\right)^k\right)\right) = 0$ *(respectively,* $\sum_n \operatorname{Var}\left(\frac{1}{r(n)} \operatorname{tr}\left(\left(M_n {M_n}^T\right)^k\right)\right) < \infty$*).*

4. (Carleman Condition) $\sum {\beta_{2k}}^{-1/2k} = \infty$.

Then $F_n(x)$*, the distribution of singular values of* M_n*, converges to* $F(x)$ *in probability (respectively, almost surely).*

3.2 Constraint Graphs

To use the trace power method to analyze M_α, we use several definitions and results from Section 3 of Ahn et al. [2020].

Definition 3.2.1 (Definition 3.2 of Ahn et al. [2020])**.** Given a shape α and a $q \in \mathbb{N}$, we define $H(\alpha, 2q)$ to be the multi-graph which is formed as follows:

1. Take q copies $\alpha_1, \ldots, \alpha_q$ of α and take q copies $\alpha_1^T, \ldots, \alpha_q^T$ of α^T, where α^T is the shape obtained from α by switching the role of U_α and V_α.

2. For all $i \in [q]$, we glue them together by setting $V_{\alpha_i} = U_{\alpha_i^T}$ and $V_{\alpha_i^T} = U_{\alpha_{i+1}}$ (where $\alpha_{q+1} = \alpha_1$).

We define $V(\alpha, 2q) = V(H(\alpha, 2q))$ and we define $E(\alpha, 2q) = E(H(\alpha, 2q))$. See Figure 3.1 for an illustration.

Remark 3.2.2. *$H(\alpha, 2q)$ is defined as a multi-graph because edges will be duplicated if U_α or V_α contains one or more edges. That said, in this paper we only consider α such that U_α and V_α do not contain any edges, so here $H(\alpha, 2q)$ will always be a graph.*

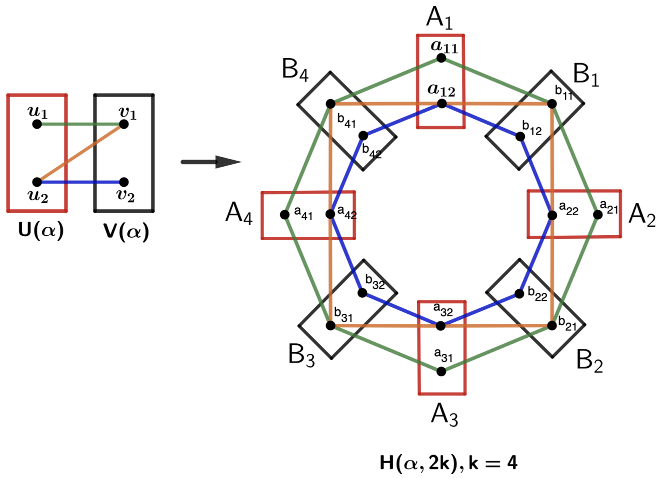


Figure 3.1: On the left is a shape α, on the right is $H(\alpha, 2q)$ where $q = 4$.

Definition 3.2.3 (Definition 3.4 of Ahn et al. [2020]: Piecewise injectivity)**.** We say that a map $\phi : V(\alpha, 2q) \to [n]$ is *piecewise injective* if ϕ is injective on each piece $V(\alpha_i)$ and each piece $V\left(\alpha_i^T\right)$ for all $i \in [q]$. In other words, $\phi(u) \neq \phi(v)$ whenever $u, v \in V(\alpha_i)$ for some $i \in [q]$ or $u, v \in V\left(\alpha_i^T\right)$ for some $i \in [q]$.

As observed in Ahn et al. [2020], with these definitions $\mathbb{E}\left[\operatorname{tr}\left(\left(M_\alpha M_\alpha^T\right)^q\right)\right]$ can be re-expressed as follows.

Proposition 3.2.4 (Proposition 3.5 of Ahn et al. [2020])**.** *For all shapes α and all $q \in \mathbb{N}$,*

$$\mathbb{E}\left[\operatorname{tr}\left(\left(M_\alpha M_\alpha^T\right)^q\right)\right] = \sum_{\substack{\phi: V(\alpha,2q) \to [n]: \\ \phi \text{ is piecewise injective}}} \mathbb{E}\left[\chi_{\phi(E(\alpha,2q))}(G)\right].$$

To analyze this expression, we use constraint graphs.

Definition 3.2.5. We define a relation $\equiv$ on the set of acyclic graphs where $G \equiv G'$ if

1. G and G' have the same vertex set V.

2. For all $u, v \in V$, there is a path from u to v in G if and only if there is a path from u to v in G'.

Proposition 3.2.6. $\equiv$ *is an equivalence relation.*

Proposition 3.2.7. *If $G \equiv G'$ and V is their vertex set, then $v \in V$ is isolated in G if and only if v is isolated in G'.*

Proposition 3.2.8. *If $G \equiv G'$, then* $|E(G)| = \left|E(G')\right|$.

Proof. Let V be the vertex set for G, G'. Since $G \equiv G'$, they have the same vertex sets of the connected components, $V = V_1 \sqcup \cdots \sqcup V_k$. Let $T_1, \ldots, T_k$ and $T'_1, \ldots, T'_k$ be the connected components of G and G', respectively, where T_i, T'_i are induced by V_i. Since G and G' are acyclic graphs, T_i, T'_i are trees for all $i \in [k]$. Thus $|E(G)| = \sum_{i=1}^{k} |E(T_i)| = \sum_{i=1}^{k} |V_i| - 1 = \sum_{i=1}^{k} \left|E(T'_i)\right| = \left|E(G')\right|$. □

Definition 3.2.9. Given a set of vertices V, a *constraint graph C on V (represented by G)* is the equivalence class of an acyclic graph G on V. i.e. $C = [G] = \left\{G' \text{ acyclic} : G' \equiv G\right\}$.

We define $V(C)$, the *vertices of C*, to be V. We say two vertices u, v in C are *constrained together* if for some representative graph $G \in C$, there is a path between u and v in G. Denote this as $u \longleftrightarrow v$ in C.

We define $|E(C)|$, the *number of edges of C* to be $|E(G)|$ for any $G \in C$. By Proposition 3.2.8, this is well-defined.

Given a representative graph $G \in C$, we call the edges of G *constraint edges*.

Proposition 3.2.10. *Given a set of vertices, let C be a constraint graph on V. If $u \longleftrightarrow v$ in C, then for all $G \in C$, there is a path between u and v in G.*

Definition 3.2.11. Given a set of vertices V and a map $\phi : V \to [n]$, we construct an acyclic graph $G(\phi)$ as follows:

1. We take $V(G(\phi)) = V$.

2. For each pair of vertices $u, v \in V$ such that $\phi(u) = \phi(v)$, we add an edge between u and v.

3. As long as there is a cycle, we delete one edge of this cycle (this choice is arbitrary). We do this until there are no cycles left.

We define the *constraint graph* $C(\phi)$ *on* V *associated to* ϕ to be the equivalence class of $G(\phi)$ under $\equiv$.

Proposition 3.2.12. *Let* $C(\phi)$ *be a constraint graph on* V *associated to* $\phi : V \to [n]$*, then two vertices* u, v *in* $C(\phi)$ *are constrained together if and only if* $\phi(u) = \phi(v)$*.*

Definition 3.2.13 (Definition 3.8 of Ahn et al. [2020]: Constraint graphs on $H(\alpha, 2q)$)**.** We define $\mathcal{C}_{(\alpha,2q)} = \{C(\phi) : \phi : V(\alpha, 2q) \to [n]$ is piecewise injective$\}$ to be the set of all possible constraint graphs on $V(\alpha, 2q)$ which come from a piecewise injective map $\phi : V(\alpha, 2q) \to [n]$.

Given a constraint graph $C \in \mathcal{C}_{(\alpha,2q)}$, we make the following definitions:

1. We define $N(C) = \left|\{\phi : V(\alpha, 2q) \to [n] : \phi \text{ is piecewise injective}, C(\phi) = C\}\right|$.
2. We define $\text{val}(C) = \mathbb{E}\left[\chi_{\phi(E(\alpha,2q))}(G)\right]$ where $\phi : V(\alpha, 2q) \to [n]$ is any piecewise injective map such that $C(\phi) = C$.

We say that a constraint graph C on $H(\alpha, 2q)$ is *nonzero-valued* if $\text{val}(C) \neq 0$.

As observed in Ahn et al. [2020], with these definitions $\mathbb{E}\left[\text{tr}\left((M_\alpha M_\alpha^T)^q\right)\right]$ can be re-expressed as follows.

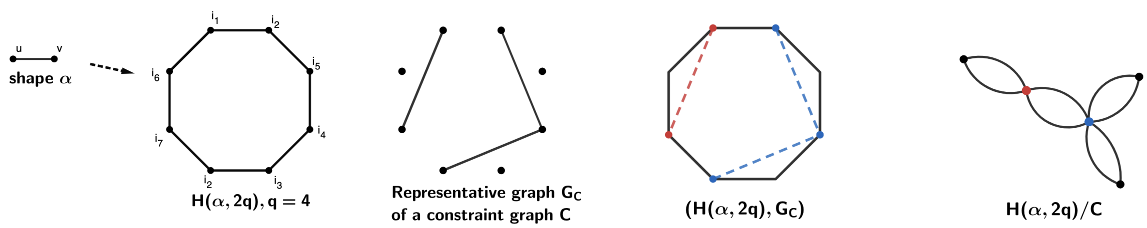
Proposition 3.2.14 (Proposition 3.9 of Ahn et al. [2020])**.** *For all shapes* α *and all* $q \in \mathbb{N}$*,*

$$\mathbb{E}\left[\text{tr}\left((M_\alpha M_\alpha^T)^q\right)\right] = \sum_{C \in \mathcal{C}_{(\alpha,2q)}} N(C)\,\text{val}(C).$$

Definition 3.2.15. Let H be a multi-graph and C a constraint graph on $V(H)$. For e, e' two edges in H, we say that *e and e' are made equal by C* if $\phi(e) = \phi(e')$ where $\phi : |V(H)| \to [n]$ is any map such that $C(\phi) = C$. We denote this as $e \longleftrightarrow e'$ by C.

Remark 3.2.16. *Given a multi-graph H and a constraint graph C on $V(H)$, it is convenient to take a representative graph G_C for C and overlay H and G_C for analysis. See Figure 3.2b for an illustration. We draw $E(G_C)$ with different colors/patterns to distinguish it from $E(H)$.*

Definition 3.2.17. Given a multi-graph H and a constraint graph C on $V(H)$, we pick a canonical $\phi : V(H) \to [n]$ such that $C(\phi) = C$. We define H/C to be the multi-graph with vertices $V(H/C) = \{\phi(v) : v \in H\}$ and edges $E(H/C) = \{\phi(e) : e \in E(H)\}$ (note that this is a multi-set). The idea is that H/C is obtained by starting with the graph H and contracting along the constraint edges in C. See Figure 3.2b for an illustration.



(a) $H(\alpha, 2q)$ and a representative graph G_C of a constraint graph $C \in \mathcal{C}_{(\alpha,2q)}$.

(b) Left: Overlay of $H(\alpha, 2q)$ and G_C Right: $H(\alpha, 2q)/C$

Figure 3.2: Illustration of Definition 3.2.9 and Definition 3.2.17.

Definition 3.2.18 (Induced constraint graphs)**.** Given a multi-graph H, a constraint graph C on $V(H)$, and a set of vertices $V \subseteq V(H)$, we define the *induced constraint graph C' on V* to be the constraint graph such that $V(C') = V$ and for all $u, v \in V$, $u \longleftrightarrow v$ in C' if and only if $u \longleftrightarrow v$ in C.

Proposition 3.2.19 (Proposition 3.10 of Ahn et al. [2020])**.** *For every constraint graph $C \in \mathcal{C}_{(\alpha,2q)}$, $\mathrm{val}(C) = 1$ if every edge in $\phi(E(\alpha, 2q))$ appears an even number of times and $\mathrm{val}(C) = 0$ otherwise (where $\phi : V(\alpha, 2q) \to [n]$ is any piecewise injective map such that $C(\phi) = C$). Alternatively, we can say that $\mathrm{val}(C) = 1$ if every edge in $H(\alpha, 2q)/C$ appears an even number of times and $\mathrm{val}(C) = 0$ otherwise.*

Proposition 3.2.20. *For every constraint graph $C \in \mathcal{C}_{(\alpha,2q)}$,*

$$N(C) = \frac{n!}{\Big(n - |V(\alpha,2q)| + |E(C)|\Big)!}\,. \tag{3.4}$$

Proof. Observe that choosing a piecewise injective map ϕ such that $C(\phi) = C$ is equivalent to choosing a distinct element of $[n]$ for each of the $n - |V(\alpha,2q)| + |E(C)|$ connected components of C. □

Since the number of constraint graphs in $\mathcal{C}_{(\alpha,2q)}$ depends on q but not on n, as $n \to \infty$ we only care about the nonzero-valued constraint graphs in $\mathcal{C}_{(\alpha,2q)}$ which have the minimum possible number of edges. We call such constraint graphs dominant.

Definition 3.2.21 (Dominant Constraint Graphs)**.** we say a constraint graph $C \in \mathcal{C}_{(\alpha,2q)}$ is a *dominant constraint graph* if

1. $\mathrm{val}(C) \neq 0$
2. $|E(C)| = \min\left\{\left|E(C')\right| : C' \in \mathcal{C}_{(\alpha,2q)}, \mathrm{val}(C') \neq 0\right\}$

We now state the number of edges in dominant constraint graphs.

Definition 3.2.22 (Vertex Separators)**.** We say that $S \subseteq V(\alpha)$ is a *vertex separator* of α if every path from a vertex $u \in U_\alpha$ to a vertex $v \in V_\alpha$ contains at least one vertex in S.

Definition 3.2.23. Given a shape α, define s_α to be the minimum size of a vertex separator of α.

Lemma 3.2.24 (Follows from Lemma 6.4 of Ahn et al. [2020])**.** *For any bipartite shape α, for any nonzero-valued $C \in \mathcal{C}_{(\alpha,2q)}$, $|E(C)| \geq (q-1)s_\alpha$. Moreover, the bound is tight. i.e. There exists a nonzero-valued $C \in \mathcal{C}_{(\alpha,2q)}$ such that $|E(C)| = (q-1)s_\alpha$.*

Remark 3.2.25. *In Ahn et al. [2020], this result was only proved for well-behaved constraint graphs (see Definition 4.1.21). That said, using the ideas in Appendix B of Ahn et al. [2020], it can be shown for all constraint graphs $C \in \mathcal{C}_{(\alpha,2q)}$. For details, see the appendix.*

Corollary 3.2.26. *For all bipartite shapes* α*, for all dominant constraint graphs* $C \in \mathcal{C}_{(\alpha,2q)}$*,* $|E(C)| = (q-1)s_\alpha$*.*

The following Corollary follows from Proposition 3.2.14, Proposition 3.2.20 and Corollary 3.2.26.

Corollary 3.2.27. *For all bipartite shapes* α*, taking* $r_{approx}(n) = \dfrac{n!}{(n-s_\alpha)!}$*,*

$$\lim_{n\to\infty} \frac{1}{r_{approx}(n)}\mathbb{E}\left[\operatorname{tr}\left(\left(\frac{M_\alpha M_\alpha^T}{n^{|V(\alpha)|-s_\alpha}}\right)^q\right)\right] = \left|\left\{C \in \mathcal{C}_{(\alpha,2q)} : C \textit{ is dominant}\right\}\right|.$$

Thus, to determine the spectrum of the singular values of M_α for a bipartite shape α, we need to count the number of constraint graphs $C \in \mathcal{C}_{(\alpha,2q)}$ such that C is dominant.

Remark 3.2.28. *We write* r_{approx} *rather than* r *here because if* $s_\alpha \le \min\{|U_\alpha|, |V_\alpha|\}$ *then the rank of* M_α *will generally be* $\dfrac{n!}{\left(n-\min\{|U_\alpha|,|V_\alpha|\}\right)!}$ *rather than* $\dfrac{n!}{(n-s_\alpha)!}$*.*

Remark 3.2.29. *The same statement is true for general* α *except that the number of edges in a dominant constraint graph* $C \in \mathcal{C}_{(\alpha,2q)}$ *is* $q|V(\alpha) \setminus (U_\alpha \cup V_\alpha)| + (q-1)(s_\alpha - |U_\alpha \cap V_\alpha|)$ *rather than* $(q-1)s_\alpha$*.*

3.3 Non-crossing Partitions

Non-crossing partition plays a key role in our analysis of the $\circ_R$ operation and free probability theory in general (see Nica and Speicher [2006]). It was first studied systematically by Kreweras [1972] and Poupard [1972]. Many of the following definitions and theorems are from the lecture notes of Nica and Speicher [2006].

Definition 3.3.1 (Non-crossing Partitions)**.**

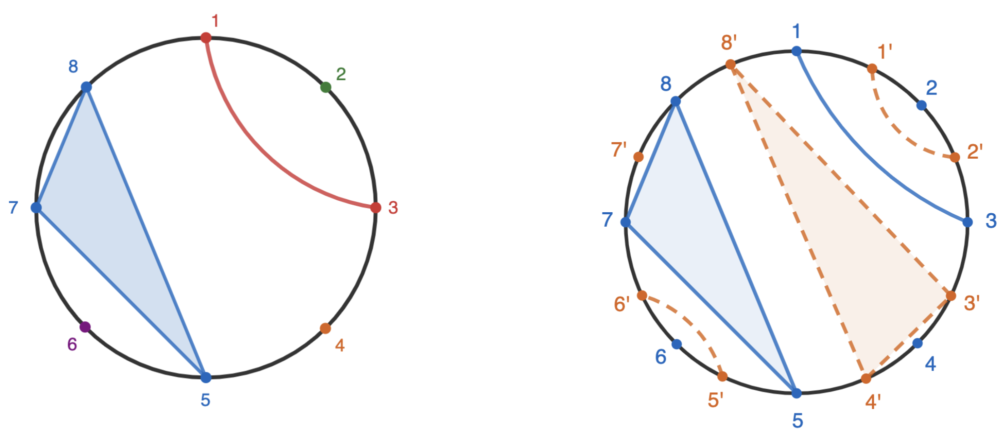
A *partition* π *of* $[n]$ is $\pi = \{P_1, \ldots, P_m\}$ where $\bigsqcup_{i=1}^{m} P_i = [n]$ for each $i \in [m]$.

We say a partition $\pi = \{P_1, \ldots, P_m\}$ is *non-crossing* if there do not exist $a < b < c < d$ such that $a, c \in P_i$ and $b, d \in P_j$ for some $i \neq j \in [m]$.

For $n \in \mathbb{N}$, we denote $P(n)$ to be the set of partitions of $[n]$ and $NC(n)$ to be the non-crossing ones.

Remark 3.3.2. *Given a non-crossing partition π of $[n]$, we can visualize the partition $\pi = \{P_1, \ldots, P_m\}$ as placing polygons of size $|P_i|$ within a cycle of length n such that no two polygons intersect each other. When $|P_i| = 1$, the corresponding polygon is simply a point.*

Example 3.3.3. *Consider the partition $\pi = \{\{2\}, \{4\}, \{6\}, \{1, 3\}, \{5, 7, 8\}\}$ of $[8]$. Then π can be represented as placing the line $\{4, 6\}$ and the triangle $\{5, 7, 8\}$ on the cycle $\mathcal{C}_8$ as shown in Figure 3.3a.*



(a) $\{1, 3\}$ represents a line, and $\{5, 7, 8\}$ represents a triangle and all others are points.

(b) The blue solid parts are π, and the orange dashed parts are $K(\pi)$.

Figure 3.3: Illustration of π and $K(\pi)$. Here $\pi = \{\{2\}, \{4\}, \{6\}, \{1, 3\}, \{5, 7, 8\}\}$.

Definition 3.3.4. We say a partition $\pi = \{P_1, \ldots, P_m\}$ is *of parts with sizes* $(n_1, \ldots, n_m)$ if $|P_i| = n_i$ for all $i \in [m]$.

Let $\vec{\alpha} = (\alpha_1, \ldots, \alpha_n) \in P_n$. We denote $\mathcal{NP}(\vec{\alpha})$ to be the set of non-crossing partitions of parts with sizes $\left(\underbrace{1,\ldots,1}_{\alpha_1 \text{ times}}, \underbrace{2,\ldots,2}_{\alpha_2 \text{ times}}, \ldots, \underbrace{n,\ldots,n}_{\alpha_n \text{ times}}\right)$. In other words, π has α_i number of parts with size i for each $i \in [n]$.

Definition 3.3.5. Let $\pi = \{V_1, \ldots, V_r\}, \sigma = \{W_1, \ldots, W_s\} \in NC(n)$. We define $\pi \leq \sigma$ if for each $i \in [r]$, $V_i \subset W_j$ for some $j \in [s]$.

For $\pi, \sigma \in NC(n)$, we denote $[\pi, \sigma] = \{\tau \in NC(n) : \pi \leq \tau \leq \sigma\}$

Proposition 3.3.6. *$(NC(n), \leq)$ is a poset (partially ordered set). In particular, $1_n = \{\{1, 2, \ldots, n\}\}$ is the maximal element and $0_n = \{\{1\}, \ldots, \{n\}\}$ is the minimal element.*

Definition 3.3.7 (Kreweras Complement Map, Definition 9.21 from Nica and Speicher [2006])**.** We define the *Kreweras complement map* $K : NC(n) \to NC(n)$ as the following:

1. take $\pi \in NC(n)$.

2. expand the vertex set $[n]$ into $\{1, \bar{1}, 2, \bar{2}, \ldots, n, \bar{n}\}$.

3. $K(\pi) = \sigma$ where σ is the biggest element in $NC(\bar{1}, \ldots, \bar{n}) \cong NC(n)$ where $\pi \cup \sigma \in NC(1, \bar{1}, \ldots, n, \bar{n})$.

Example 3.3.8. *Let $\pi = \{\{2\}, \{4\}, \{6\}, \{1, 3\}, \{5, 7, 8\}\}$ of $[8]$ as in the previous example. Then $K(\pi) = \{\{1, 2\}, \{3, 4, 8\}, \{5, 6\}, \{7\}\}$.*

See Figure 3.3b for an illustration.

Proposition 3.3.9. *Let $\pi \leq \sigma \in NC(n)$. Then $[\pi, \sigma] \cong [K(\sigma), K(\pi)]$.*

Theorem 3.3.10 (Canonical Factorization, Theorem 9.29 from Nica and Speicher [2006])**.** *Let $\pi \leq \sigma \in NC(n)$. Then there exists a canonical choice of $(k_1, \ldots, k_n) \in \left(\mathbb{Z}_{\geq 0}\right)^n$ such that*

$$[\pi, \sigma] \cong NC(1)^{k_1} \times \cdots \times NC(n)^{k_n} \tag{3.5}$$

Following the proof for Theorem 3.3.10, the canonical factorization for a special interval $[0, \pi]$ is as follows.

Corollary 3.3.11. *Let $\pi \in NC(n)$. Assume $\pi \in \mathcal{NP}(\vec{\alpha})$ for some $\vec{\alpha} \in P_n$. Then the canonical factorization gives*

$$[0, \pi] \cong NC(1)^{\alpha_1} \times \cdots \times NC(n)^{\alpha_n} = \prod_{V \in \pi} NC(|V|) \tag{3.6}$$

and

$$[\pi, 1_n] \cong [0_n, K(\pi)] \cong \prod_{V \in K(\pi)} NC(|V|) \tag{3.7}$$

Example 3.3.12. *Let $\pi = \{\{2\}, \{4\}, \{6\}, \{1, 3\}, \{5, 7, 8\}\}$ be as in previous example. Then $[0, \pi] \cong NC(1)^3 \times NC(2) \times NC(3)$.*

Kreweras [1972] proved the following result on the number of non-crossing partitions. Later Liaw et al. [1998] gave a shorter and more direct proof.

Theorem 3.3.13. *Let $\vec{\alpha} \in P_k$. Then*

$$|\mathcal{NP}(\vec{\alpha})| = \binom{k}{a-1} \cdot \frac{(a-1)!}{\alpha_1! \dots \alpha_k!}. \tag{3.8}$$

We now give an extended definition of $\mathcal{NP}(\vec{\alpha})$.

Definition 3.3.14. Given $\vec{\alpha} \in P_k$, we define $\mathcal{NP}(m\vec{\alpha})$ to be the set of non-crossing partitions of $[mk]$ corresponding to augmented $\vec{\alpha}$: there are α_i parts of size mi in a partition $P \in \mathcal{NP}(m\vec{\alpha})$.

A direct corollary is the following.

Corollary 3.3.15. *Let $\vec{\alpha} \in P_k$. Then*

$$|\mathcal{NP}(m\vec{\alpha})| = \binom{mk}{a-1} \cdot \frac{(a-1)!}{\alpha_1! \dots \alpha_k!}. \tag{3.9}$$

Proof. This is an application of Theorem 3.3.13. Since $\mathcal{NP}(m\vec{\alpha})$ is the set of non-crossing partitions of $[mk]$ with a parts, and there are α_i number of parts with size mi, $|\mathcal{NP}(m\vec{\alpha})| = \binom{mk}{a-1} \cdot \frac{(a-1)!}{\alpha_1! \ldots \alpha_k!}$, as needed. □

Using Corollary 3.3.15, the generalized Catalan number can be interpreted as the number of non-crossing partitions in the following way.

Definition 3.3.16. Let $C(k,m) = \frac{1}{mk+1}\binom{(m+1)k}{k}$ denote the level-m k^{th} Catalan number. In particular, when $m=1$, $C(k,1) = C_k$ is the k^{th} Catalan number.

Corollary 3.3.17. *Let* $\vec{\alpha} = (k, 0, \ldots, 0) \in P_k$. *Then*

$$C(k,m) = \left|\mathcal{NP}\left((m+1)\vec{\alpha}\right)\right| \tag{3.10}$$

i.e. $C(k,m)$ *is the number of non-crossing partitions of* $[(m+1)k]$ *where all the parts are of size* $(m+1)$.

Proof. By Corollary 3.3.15,

$$\begin{aligned}\left|\mathcal{NP}\left((m+1)\vec{\alpha}\right)\right| &= \binom{(m+1)k}{k-1} \cdot \frac{(k-1)!}{k!} = \frac{((m+1)k)!}{(k-1)!(mk+1)!} \cdot \frac{1}{k} = \frac{((m+1)k)!}{k!(mk)!} \cdot \frac{1}{mk+1} \\ &= \binom{(m+1)k}{k} \cdot \frac{1}{mk+1} = C(k,m).\end{aligned}$$

as needed. □

3.4 Results from Free Probability

In this section we give several results from free probability theory which we need for our analysis. The definitions and theorems are from the lecture notes of Nica and Speicher Nica and Speicher [2006].

Definition 3.4.1. Let P be a finite poset and denote $P^{(2)} = \{(\pi, \sigma) \in P \times P : \pi \leq \sigma\}$. Let $F, G : P^{(2)} \to \mathbb{C}$. We define the *convolution of F and G*, $F * G : P^{(2)} \to \mathbb{C}$ to be

$$(F * G)(\pi, \sigma) = \sum_{\tau \in P:\, \pi \leq \tau \leq \sigma} F(\pi, \tau) G(\tau, \sigma) \tag{3.11}$$

Definition 3.4.2. Let P be a finite poset. Define $\delta : P^{(2)} \to \mathbb{C}$ to be

$$\delta(\pi, \sigma) = \begin{cases} 1 & \text{if } \pi = \sigma \\ 0 & \text{if } \pi < \sigma \end{cases} \tag{3.12}$$

Proposition 3.4.3. *δ is a unit element for the convolution operation. i.e. $\delta * F = F * \delta = F$.*

Definition 3.4.4. Let P be a finite poset. We define

1. the *Zeta function* $\zeta : P^{(2)} \to \mathbb{C}$ to be $\zeta(\pi, \sigma) = 1$ for any $(\pi, \sigma) \in P^{(2)}$
2. the *Möbius function* $\mu = \zeta^{-1}$ under the convolution operation. i.e. $\mu * \zeta = \zeta * \mu = \delta$.

Proposition 3.4.5.

1. *Let P, Q be finite posets and $\Phi : P \to Q$ be an isomorphism. Then $\mu_P(\pi, \sigma) = \mu_Q(\Phi(\pi), \Phi(\sigma))$.*
2. *Let $P_1, \ldots, P_n$ be finite posets and let $P = P_1 \times \cdots \times P_n$. Then*

$$\mu_P((\pi_1, \ldots, \pi_n), (\sigma_1, \ldots, \sigma_n)) = \mu_{P_1}(\pi_1, \sigma_1) \ldots \mu_{P_n}(\pi_n, \sigma_n). \tag{3.13}$$

Proposition 3.4.6. *Let $n \in \mathbb{N}$ and let μ_n be the Möbius function for $NC(n)$. Then $\mu_n(0_n, 1_n) = (-1)^{n-1} C_{n-1}$ where $C_k = \frac{1}{k+1}\binom{2k}{k}$ is the k^{th} Catalan number.*

Corollary 3.4.7. *Let $\pi \leq \sigma \in NC(n)$. Then $\mu_n(\pi, \sigma) = \prod_{i=1}^{n} \left((-1)^{i-1} C_{i-1}\right)^{k_i}$ where $[\pi, \sigma] \cong NC(1)^{k_1} \times \cdots \times NC(n)^{k_n}$ is the canonical factorization.*

Definition 3.4.8. Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space. For $n \in \mathbb{N}$ and $V \subset [n]$, we define $\varphi(V)[a_1, \ldots, a_n]$ to be

$$\varphi(V)[a_1, \ldots, a_n] = \varphi(a_{i_1} \ldots a_{i_m}) \text{ where } V = \{i_1, \ldots, i_m\} \subset [n] \tag{3.14}$$

Let $\pi = \{V_1, \ldots, V_r\} \in NC(n)$, we further define $\varphi_\pi[a_1, \ldots, a_n]$ to be

$$\varphi_\pi[a_1, \ldots, a_n] = \prod_{i=1}^{r} \varphi(V_i)[a_1, \ldots, a_n] \tag{3.15}$$

When $\pi = 1_n$, we denote $\varphi_{1_n}[a_1, \ldots, a_n]$ as $\varphi_n(a_1 \ldots a_n)$ which is the same as $\varphi(a_1 \ldots a_n)$.

Definition 3.4.9 (Free Cumulants)**.** Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space. For $n \in \mathbb{N}$, and $\pi \in NC(n)$, the *free cumulant* κ_π is a multilinear functional $\kappa_\pi : \mathcal{A}^n \to \mathbb{C}$ where

$$\kappa_\pi[a_1, \ldots, a_n] = \sum_{\sigma \in NC(n) \colon \sigma \leq \pi} \varphi_\sigma[a_1, \ldots, a_n] \cdot \mu(\sigma, \pi) \tag{3.16}$$

In particular, when $\pi = 1_n$, we denote $\kappa_{1_n}[a_1, \ldots, a_n]$ as $\kappa_n(a_1, \ldots, a_n)$. We have that

$$\kappa_n(a_1, \ldots, a_n) = \sum_{\sigma \in NC(n)} \varphi_\sigma[a_1, \ldots, a_n] \cdot \mu(\sigma, 1_n) \tag{3.17}$$

Let $V = \{i_1, \ldots, i_m\} \subset [n]$. We define $\kappa(V)[a_1, \ldots, a_n] = \kappa_m(a_{i_1}, \ldots, a_{i_m})$.

Proposition 3.4.10 (Proposition 11.4 of Nica and Speicher [2006])**.** *Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space. Then*

1. $\kappa_\pi[a_1, \ldots, a_n] = \prod_{V \in \pi} \kappa(V)[a_1, \ldots, a_n]$
2. *(Möbius Inversion)* $\varphi(a_1 \ldots a_n) = \sum_{\pi \in NC(n)} \kappa_\pi[a_1, \ldots, a_n]$

Theorem 3.4.11 (Theorem 14.4 of Nica and Speicher [2006])**.** *Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space. Let $a_1, \ldots, a_n, b_1, \ldots, b_n \in \mathcal{A}$ where $\{a_1, \ldots, a_n\}$ and $\{b_1, \ldots, b_n\}$ are freely* *independent. Then*

$$\varphi(a_1b_1\dots a_nb_n) = \sum_{\pi\in NC(n)} \kappa_\pi[a_1,\dots,a_n]\cdot\varphi_{K(\pi)}[b_1,\dots,b_n] \tag{3.18}$$

In particular, we can apply this to $a_i = a$ and $b_i = b$ for all $i \in [n]$, then

$$\varphi\left((ab)^n\right) = \varphi(ab\dots ab) = \sum_{\pi\in NC(n)} \kappa_\pi[a,\dots,a]\cdot\varphi_{K(\pi)}[b,\dots,b] \tag{3.19}$$

Definition 3.4.12. Let μ and ν be compactly supported probability measure on $\mathbb{R}^+$. The *multiplicative free convolution* $\mu \boxtimes \nu$ is the distribution of $\sqrt{x}y\sqrt{x}$ where x, y are positive elements in some C^*-probability space, x and y are free and $x \sim \mu$, $y \sim \nu$.

Theorem 3.4.13 (Theorem 23.14 of Nica and Speicher [2006])**.** *Let $(A_n)_{n\in\mathbb{N}}$ and $(B_n)_{n\in\mathbb{N}}$ be sequences of $n\times n$ matrices such that A_n converges in distribution (with respect to tr) for $n\to\infty$ and B_n converges in distribution (with respect to tr) for $n\to\infty$. Furthermore, let $(U_n)_{n\in\mathbb{N}}$ be a sequence of Haar unitary $n\times n$ random matrices. Then, $U_nA_nU_n^*$ and B_n are asymptotically free for $n\to\infty$.*

CHAPTER 4

THE Z-SHAPE GRAPH MATRIX

Chapter 4 and 5 are from the paper Cai and Potechin [2020].

4.1 The Trace Power of Z-shaped Graph Matrices

Recall that α_Z is the bipartite shape with vertices $V(\alpha_Z) = \{u_1, u_2, v_1, v_2\}$, distinguished tuples of vertices $U_{\alpha_Z} = (u_1, u_2)$ and $V_{\alpha_Z} = (v_1, v_2)$, and edges $E(\alpha_Z) = \Big\{\{u_1, v_1\}, \{u_2, v_1\}, \{u_2, v_2\}\Big\}$ (see Definition 2.1.15 and Figure 2.2). M_{α_Z} is a graph matrix with dimension $r(n) = n(n-1)$ (see Definition 2.1.5). In this section, we determine

$$\lim_{n\to\infty} \frac{1}{r(n)} \mathbb{E}_{G\sim G(n,1/2)} \left[\operatorname{tr} \left(\left(\frac{M_{\alpha_Z} M_{\alpha_Z}^T}{n^2} \right)^q \right) \right]$$

by counting the number of dominant constraint graphs in $\mathcal{C}_{(\alpha_Z,2q)}$.

Remark 4.1.1. *For α_Z, the size of the minimum separator is $s_{\alpha_Z} = 2$. By Corollary 3.2.26, dominant constraint graphs $C \in \mathcal{C}_{(\alpha_Z,2q)}$ have $2(q-1)$ edges.*

Definition 4.1.2.

$$D_n = \frac{1}{2n+1}\binom{3n}{n}. \tag{4.1}$$

Remark 4.1.3. *D_n is a special case of the generalized Catalan number, which is defined as $A_n(k,r) = \frac{r}{nk+r}\binom{nk+r}{n}$. Note that $A_n(2,1) = \frac{1}{2n+1}\binom{2n+1}{n} = \frac{1}{n+1}\binom{2n}{n}$ is the Catalan number we know. $A_n(3,1) = \frac{1}{3n+1}\binom{3n+1}{n} = \frac{1}{2n+1}\binom{3n}{n}$ is the D_n defined above.*

Below is the main result of this section.

Theorem 4.1.4. *For all $q \in \mathbb{N}$, the number of dominant constraint graphs $C \in \mathcal{C}_{(\alpha_Z,2q)}$ is D_q.*

As a direct result of Theorem 4.1.4 and Corollary 3.2.27, we get the following corollary.

Corollary 4.1.5. *Let* $M_n = \frac{1}{n} M_{\alpha_Z}(G)$ *where* $G \sim G(n, 1/2)$ *and let* $r(n) = n(n-1)$ *be the dimension of* M_{α_Z}*. Recall that* $D_q = \dfrac{1}{2q+1}\dbinom{3q}{q}$*. Then*

$$\lim_{n \to \infty} \frac{1}{r(n)} \mathbb{E}_{G \sim G(n,1/2)} \left[\operatorname{tr} \left(\left(M_n(G) M_n(G)^T \right)^q \right) \right] = D_q. \tag{4.2}$$

Proof. By Corollary 3.2.27,

$$\lim_{n \to \infty} \frac{1}{r_{approx}(n)} \mathbb{E} \left[\operatorname{tr} \left(\left(\frac{M_{\alpha_Z} M_{\alpha_Z}^T}{n^{|V(\alpha_Z)| - s_{\alpha_Z}}} \right)^q \right) \right] = \left| \left\{ C \in \mathcal{C}_{(\alpha_Z, 2q)} : C \text{ is dominant} \right\} \right|.$$

Since $s_{\alpha_Z} = 2$ and $|V(\alpha_Z)| = 4$, $r_{approx}(n) = \dfrac{n!}{\left(n - s_{\alpha_Z}\right)!} = n(n-1) = r(n)$ and $\dfrac{M_{\alpha_Z} M_{\alpha_Z}^T}{n^{|V(\alpha_Z)| - s_{\alpha_Z}}} = M_n {M_n}^T$. By Theorem 4.1.4, $\left| \left\{ C \in \mathcal{C}_{(\alpha_Z, 2q)} : C \text{ is dominant} \right\} \right| = D_q$ and the result follows. □

4.1.1 Recurrence Relation for D_n

One of the key ingredients for proving Theorem 4.1.4 is the following recurrence relation on D_n.

Theorem 4.1.6. *Let* D_n *be defined as in Definition 4.1.2. Then*

$$D_{n+1} = \sum_{i,j,k \geq 0:\, i+j+k=n} D_i D_j D_k = \sum_{i=0}^{n} D_i \left(\sum_{j=0}^{n-i} D_j D_{n-i-j} \right). \tag{4.3}$$

To prove this recurrence relation, we consider walks on grids. This proof is a generalization of the third proof in the Wikipedia article on Catalan numbers.

Definition 4.1.7 (Grid Walk)**.** Let m, n be two positive integers. A *grid walk from* $(0,0)$ *to* (m,n) is a sequence of $(m+n)$ coordinates $(z_0, z_1, z_2, \ldots, z_{m+n})$ where

1. $z_i = (x_i, y_i)$ where $x_i \in [m]$, $y_i \in [n]$ for each $i \in [m+n]$,

2. $z_0 = (0,0)$ and $z_{m+n} = (m,n)$,

3. $z_{i+1} - z_i = (1,0)$ or $(0,1)$ for any $i \in [m+n]$.

Pictorially, a grid is a walk from $(0,0)$ to (m,n) that steps on integer coordinates and only moves straight up or straight right.

A *grid walk from* $(0,0)$ *to* (m,n) *weakly below the diagonal* is a grid walk $(z_1, \ldots, z_{m+n})$ where $z_i = (x_i, y_i)$ and for all i, $y_i/x_i \leq m/n$.

Proof of Theorem 4.1.6. Let W_n be the set of all grid walks from $(0,0)$ to $(n,2n)$ weakly below the diagonal and let $d_n = |W_n|$. We will prove that d_n satisfies the recurrence relation in Theorem 4.1.6:

$$d_{n+1} = \sum_{i,j,k \geq 0 : i+j+k=n} d_i d_j d_k = \sum_{i=0}^{n} d_i \left(\sum_{j=0}^{n-i} d_j d_{n-i-j} \right). \tag{4.4}$$

1. $d_n = \sum_{\substack{i,j,k \geq 0: \\ i+j+k=n-1}} d_i d_j d_k = \sum_{i=0}^{n-1} d_i \left(\sum_{j=0}^{n-1-i} d_j d_{n-i-j} \right)$:

   We will establish a bijection between W_n and $W'_n := \bigcup_{\substack{i,j,k \geq 0: \\ i+j+k=n-1}} W_i \times W_j \times W_k$.

   - Let $w = (z_1, \ldots, z_{3n})$ be a grid walk from $(0,0)$ to $(n,2n)$ weakly below the diagonal. Consider the first point that w touches the diagonal i.e. let $a \in [n]$ be smallest such that $z_i = (a, 2a)$ for some $i \in [3n]$. Then $w_1 = (z_i, z_{i+1}, \ldots, z_{3n})$ is a grid walk from $(a, 2a)$ to $(n, 2n)$ weakly below the diagonal. After translation $w_1 \in W_{n-a}$.

     Let d' be the line parallel to the diagonal which passes $(a, 2a-1)$. Since z_i is the first point touching the diagonal, $(z_1, \ldots, z_{i-1})$ is weakly below d'. Let $z_j = (b, 2b-1)$ be the first point touching d'. Then $w_2 = (z_j, \ldots, z_{i-1})$ is a grid walk from $(b, 2b-1)$ to $(a, 2a-1)$ weakly below the diagonal. After translation $w_2 \in W_{a-b}$.

Let d'' be the line parallel to the diagonal which passes $(b, 2b-2)$. Since z_j is the first point touching d', $(z_2, \ldots, z_{j-1})$ is weakly below d''. Then $w_3 = (z_2, \ldots, z_{j-1})$ is a grid walk from $(1,0)$ to $(b, 2b-2)$ weakly below the diagonal d''. After translation $w_3 \in W_{b-1}$.

Thus from $w \in W_n$ we get a tuple $(w_1, w_2, w_3) \in W_{n-a} \times W_{a-b} \times W_{b-1}$ where a, b are uniquely determined by w. Note $(n-a)+(a-b)+(b-1) = n-1$, thus $(w_1, w_2, w_3) \in W'_n$.

- Conversely, given a $(w_1, w_2, w_3) \in W'_n$, let $(a_i, 2a_i)$ be the last coordinate point of w_i. Let

  $$w = \big((0,0), w_1 + (1,0), w_2 + (a_1+1, 2a_1+1), w_3 + (a_1+a_2+1, 2(a_1+a_2+1))\big)$$

  where if $w = (z_1, \ldots, z_k)$ is a grid walk then $w + (s,t)$ means translate every coordinate point z_i in w by (s,t). We can easily check that $w \in W_n$.

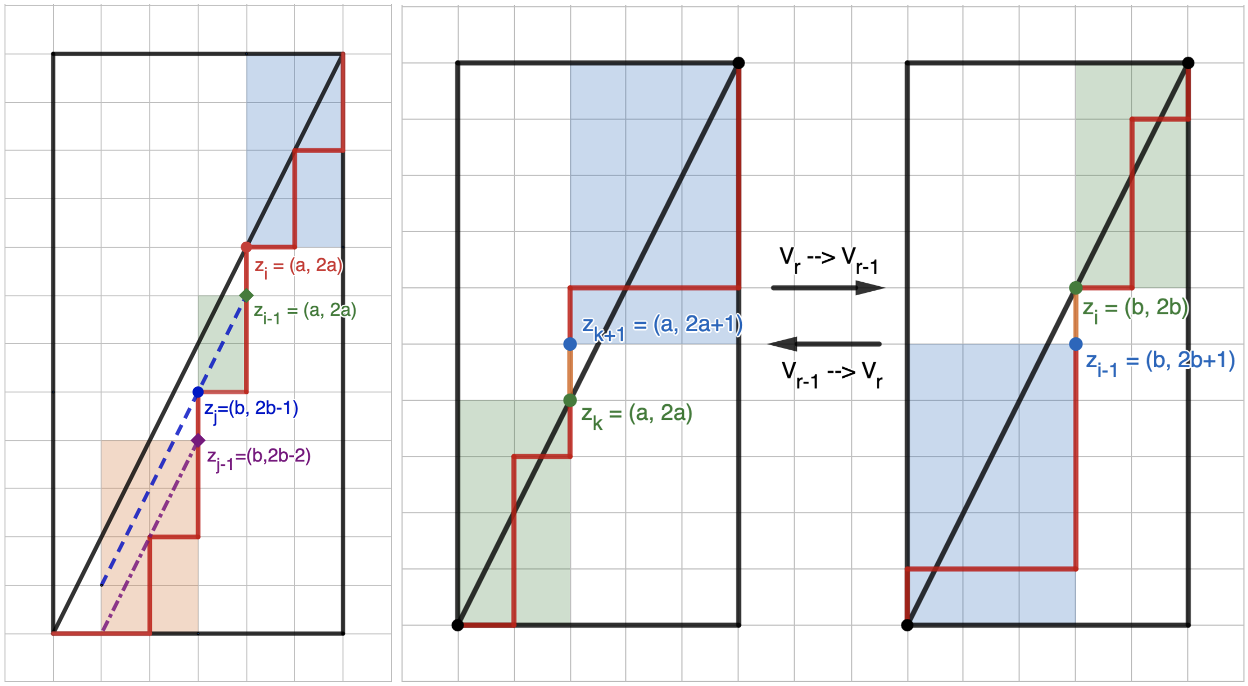
- It is not hard to check this is a bijection.

2. $D_n = d_n = \dfrac{1}{2n+1}\dbinom{3n}{n}$:

For $i \in \{0, 1, \ldots, 2n\}$, let V_r = the set of grid walks from $(0,0)$ to $(n, 2n)$ that has r vertical steps above the diagonal. i.e. for $w = (z_1, \ldots, z_{3n}) \in V_r$, there are r $z_j = (x_j, y_j)$'s such that $y_j/x_j > 2$. Let G_n = the set of all grid walk from $(0,0)$ to $(n, 2n)$. We have that $|G_n| = \dbinom{3n}{n}$. Note that $V_0 = W_n$ and $\bigcup_{r=0}^{2n} V_r = G_n$. We will prove that $|V_r| = |V_{r-1}|$ for all $r \in [2n]$, then $|V_0| = d_n = \dfrac{1}{2n+1}\dbinom{3n}{n}$ as needed.

Claim 4.1.8. $|V_{r-1}| = |V_r|$ *for all* $r \in [2n]$.

Proof. We will find a bijection between V_r and V_{r-1} for each $r \in [2n]$.



(a) proof part 1

(b) proof part 2

Figure 4.1: Illustration of Theorem 4.1.6

- Let $w = (z_0, \ldots, z_{3n}) \in V_r$ and let z_k be the **last** point where the walk is on the diagonal and then takes a step upwards. i.e. z_k is the last point such that $z_k = (a, 2a)$ for some $a \in [n]$ and $z_{k+1} - z_k = (0, 1)$. Let $w_1 = (z_0, \ldots, z_k)$ and $w_2 = (z_{k+1}, \ldots, z_{3n})$. Let $w' = (w'_1, w'_2)$ where $w'_1 = w_2 - (a, 2a + 1)$ and $w'_2 = w_1 + (n - a, 2n - 2a))$ (see Figure 4.1, w' exchanges the green and blue part of w). Then w'_2 has the same number of steps above the diagonal as w_1 does. Moreover, since z_k is the last point such that w passes the diagonal vertically through it, w'_1 has exactly one less vertical step above the diagonal than (z_k, w_2) does. Thus $w' \in V_{r-1}$.
- Let $w = (z_0, \ldots, z_{3n}) \in V_{r-1}$. Let z_i be the first point such that w touches the diagonal from below. i.e. z_i is the first such that $z_i = (b, 2b)$ for some $b \in [n]$ and $z_i - z_{i-1} = (0, 1)$. Let $w_1 = (z_0, \ldots, z_{i-1})$ and $w_2 = (z_i, \ldots, z_{3n})$. Let $w' = (w'_1, w'_2)$ where $w'_1 = w_2 - (b, 2b)$ and $w'_2 = (n - b, 2n - 2b + 1) + w_1$. Then w'_1 has the same number of steps above the diagonal as w_2 does. Moreover, since z_i is the first such that w touches the diagonal from below, $((n - b, 2n - 2b), w'_2)$ has exactly one more step above the diagonal than w_1 does. Thus $w' \in V_r$.
- It is not hard to check that this gives a bijection.

□

To conclude, we proved that $D_n = \frac{1}{2n+1}\binom{3n}{n} = d_n =$ the number of grid walks from $(0,0)$ to $(n, 2n)$ that are weakly below the diagonal $= \sum_{i,j,k \geq 0:\, i+j+k=n-1} D_i D_j D_k$. □

4.1.2 Properties of Dominant Constraint Graphs on a Cycle

In order to count the number of dominant constraint graphs in $\mathcal{C}_{(\alpha_Z, 2q)}$, we need a few properties of these constraint graphs. As a warm-up, we first consider dominant constraint graphs on a cycle of length $2q$. The first part of this analysis is essentially the same as Lemma 4.4 of Ahn et al. [2020], but we will need a few additional properties.

Definition 4.1.9. Let α_0 be the line shape as in Definition 2.1.6. Let $H(\alpha_0, 2q)$ be the multi-graph as in Definition 3.2.1. We label the vertices of $H(\alpha_0, 2q)$ as $\left\{i_j : j \in [2q]\right\}$.

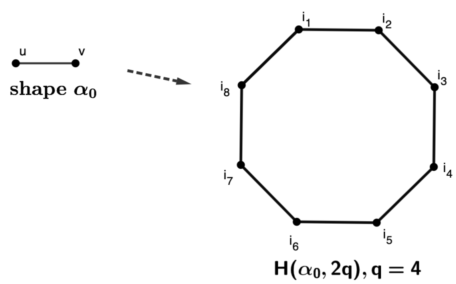


Figure 4.2: α_0 is the line shape. $H(\alpha_0, 2q)$ is a cycle of length $2q$.

We say a representative graph G of a constraint graph $C \in \mathcal{C}_{(\alpha_0, 2q)}$ is *explicitly non-crossing* if no two constraint edges of G cross. Note: constraint edges $\{i_x, i_y\}$ and $\{i_s, i_t\}$ where $x < y$ and $s < t$ cross if $x < s < y < t$ or $s < x < t < y$. We say G is *crossing* if it is not explicitly non-crossing.

We say a constraint graph $C \in \mathcal{C}_{(\alpha_0, 2q)}$ is *non-crossing* if there is a representative graph $G \in C$ that is explicitly non-crossing. We say C is *crossing* if it is not non-crossing. See Figure 4.3 for an illustration.

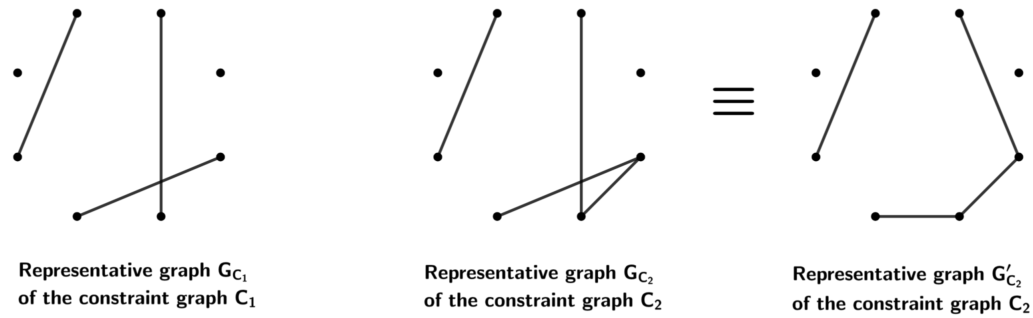


Figure 4.3: $C_1, C_2 \in \mathcal{C}_{(\alpha_0,2q)}$. C_1 is crossing; C_2 is non-crossing since G'_{C_2} is explicitly non-crossing even though G_{C_2} is crossing.

Definition 4.1.10. Let α_0 be the line shape. We say a constraint graph $C \in \mathcal{C}_{(\alpha_0,2q)}$ is *parity preserving* if for all $i_x, i_y \in V(\alpha_0, 2q)$ such that $i_x \longleftrightarrow i_y$, $|x-y|$ is even.

Lemma 4.1.11. *All dominant constraint graphs in* $\mathcal{C}_{(\alpha_0,2q)}$ *are non-crossing and parity-preserving.*

To prove this lemma, we need the following observation about isolated vertices.

Definition 4.1.12. Given a multi-graph H and a constraint graph C on H, we say that a vertex $v \in V(C) = V(H)$ is *isolated* if for any $G \in C$, v is not incident to any constraint edges in G. Note that by Proposition 3.2.7, this is well-defined.

Lemma 4.1.13. *If C is a nonzero-valued constraint graph on $H(\alpha_0, 2q)$ and C has an isolated vertex i_j, then $i_{j-1} \longleftrightarrow i_{j+1}$. In the cases when $j = 1$ or $j = 2q$, $i_0 = i_{2q}$ and $i_{2q+1} = i_1$ respectively.*

Proof. Recall that by Proposition 3.2.19, C is nonzero-valued if and only if each edge in $H(\alpha_0, 2q)/C$ appears an even number of times. Since i_j is isolated, the only way this can happen is if $i_{j-1} \longleftrightarrow i_{j+1}$. □

With this observation, we can now prove Lemma 4.1.11.

Proof of Lemma 4.1.11. Since C is dominant, each edge appears even number of times in $H(\alpha_0, 2q)/C$ and there are exactly $(q-1)$ constraint edges in C. We prove the lemma by induction on q.

- When $q = 1$, there are no constraint edge so the lemma trivially holds. For $q = 2$, $|E(C)| = 1$. In order for each edge to appear even number of times in $H(\alpha_0, 2q)/C$, either $i_1 \longleftrightarrow i_3$ or $i_2 \longleftrightarrow i_4$, which implies that C is parity preserving. If $i_1 \longleftrightarrow i_3$ in C, we choose $G_C \in C$ to have a single constraint edge $\{i_1, i_3\}$. If $i_2 \longleftrightarrow i_4$ in C, we choose $G_C \in C$ to have a single constraint edge $\{i_2, i_4\}$. In either case G_C is explicitly non-crossing, thus C is non-crossing.

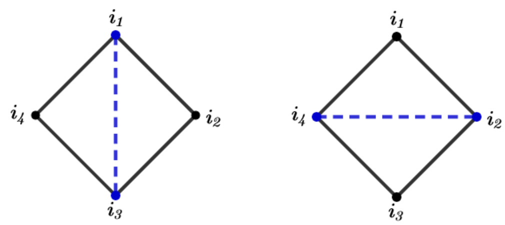


Figure 4.4: Illustration of base case of the proof: $H(\alpha_0, 2)$ overlay with $G_C \in C$ where G_C consists of a single constraint edge, either $\{i_1, i_3\}$ or $\{i_2, i_4\}$.

- $q \implies (q+1)$: Consider a constraint graph C on $H(\alpha_0, 2q+2)$ with vertices $\{i_1, \ldots, i_{2q+2}\}$.

  Since C is dominant by assumption, there are only q constraint edges in C, so C must have an isolated vertex. Without loss of generality assume this vertex is i_{2q+2}. Then by Lemma 4.1.13, $i_1 \longleftrightarrow i_{2q+1}$ and there exists $G \in C$ such that $\{i_1, i_{2q+1}\}$ is an constraint edge in G. Note that $(2q+1) - 1 = 2q$ is even. Contracting the constraint edge $\{i_{2q+1}, i_1\}$ (identifying i_1 with i_{2q+1}) results in $H(\alpha_0, 2q)$ with vertices $\{i_1, \ldots, i_{2q}\}$ and two edges $\{i_{2q+1}, i_{2q}\} = \{i_1, i_{2q}\}$ attached to $H(\alpha_0, 2q)$. See Figure 4.5 for an illustration.

  Let G' be the induced subgraph of G on $H(\alpha_0, 2q)$. Since G' has one less edge than G, and edges in $H(\alpha_0, 2q)$ are only made equal to edges in $H(\alpha_0, 2q)$ by C, the constraint graph $C' = \left[G'\right] \in C_{(\alpha_0,2q)}$ represented by G' is dominant. By the inductive hypothesis, C' is non-crossing and parity preserving, which implies that C is parity preserving. Choosing a representative graph of C' that is explicitly non-crossing and adding in the constraint edge $\{i_1, i_{2q-1}\}$, we get an explicitly non-crossing representative graph of C, which implies that C is non-crossing, as needed.

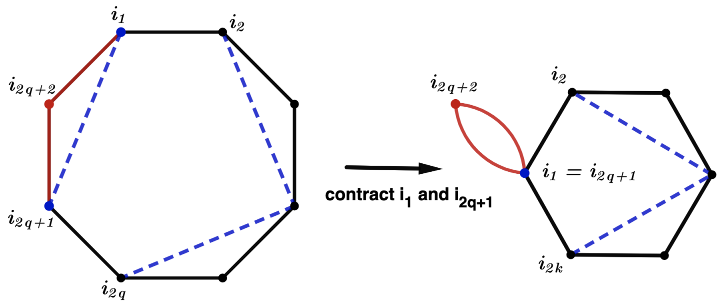


Figure 4.5: Illustration of the inductive step of the proof for Lemma 4.1.11: i_{2q+2} is isolated and $\{i_1, i_{2q+1}\}$ is a constraint edge in a representative graph $G \in C \in \mathcal{C}_{(\alpha_0,2q)}$.

□

We now show a few additional properties of dominant constraint graphs in $\mathcal{C}_{(\alpha_0,2q)}$.

Corollary 4.1.14. *Let $C \in \mathcal{C}_{(\alpha_0,2q)}$ be a dominant constraint graph. If $i_x \longleftrightarrow i_y$ and $i_v \longleftrightarrow i_w$ for some $x \le v \le y \le w$, then $i_x \longleftrightarrow i_y \longleftrightarrow i_v \longleftrightarrow i_w$.*

Proof. If $x = v$, $v = y$, or $y = w$ then $i_x \longleftrightarrow i_y \longleftrightarrow i_v \longleftrightarrow i_w$ so we can assume that $x < v < y < w$. By Lemma 4.1.11, C is non-crossing, so there exists $G \in C$ that is explicitly non-crossing. We think of $H(\alpha_0, 2q)$ as a circle, vertices of G as points on the circle and edges of G as chords. Since $i_x \longleftrightarrow i_y$ and $i_v \longleftrightarrow i_w$, there is a path from x to y and a path from v to w G. These paths do not leave the circle, so they must intersect. Since there are no crossings, they must intersect at an index which implies that $i_v \longleftrightarrow i_x \longleftrightarrow i_y \longleftrightarrow i_w$, as needed. □

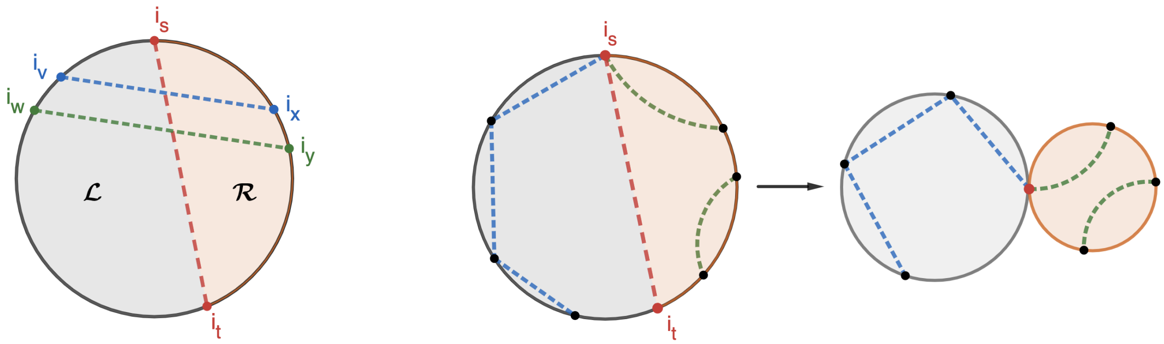
Corollary 4.1.15. *Let $C \in \mathcal{C}_{(\alpha_0,2q)}$ be a dominant constraint graph. If $i_s \longleftrightarrow i_t$ for some $1 \le s < t \le 2q$, then there exists an explicitly non-crossing $G \in C$ such that $\{i_s, i_t\}$ is an edge in G. Moreover, if we let $\mathcal{R} = \{\{i_x, i_{x+1}\} : s \le x < t\}$ and $\mathcal{L} = E(\alpha_0, 2q) \setminus \mathcal{R}$ then edges in $\mathcal{R}$ can only be made equal to edges in $\mathcal{R}$ by C and edges in $\mathcal{L}$ can only be made equal to edges in $\mathcal{L}$ by C.*

Proof. For the first part, let G be an explicitly non-crossing representative graph of C, we will adjust G as follows. Let V be the connected component of G which contains i_s. Delete all edges between vertices in V and then add an edge from each vertex in $V \setminus \{i_s\}$ to i_s. We claim that the adjusted G is still explicitly non-crossing. Assume not. Then there is an edge $\{i_x, i_y\}$ which crosses one of these new edges $\{i_s, i_v\}$. Since $i_x \longleftrightarrow i_y$, $i_s \longleftrightarrow i_v$ and these edges cross, by Corollary 4.1.14, $i_x \longleftrightarrow i_y \longleftrightarrow i_s \longleftrightarrow i_v$. But then $x, y \in V$ so we would have deleted the edge $\{i_x, i_y\}$, which is a contradiction.

For the second part, assume not and let $e_1 = \{i_x, i_y\} \in \mathcal{R}$ and $e_2 = \{i_v, i_w\} \in \mathcal{L}$ be edges such that $e_1 \longleftrightarrow e_2$. Since e_1, e_2 are edges, $|x - y| = |v - w| = 1$. Without loss of generality, assume x, v are even and y, w are odd. Since C is parity preserving, $i_x \longleftrightarrow i_v$ and $i_y \longleftrightarrow i_w$. Since $e_1 \in \mathcal{R}$ and $e_2 \in \mathcal{L}$, $s \le x \le t \le v$ or $v \le s \le x \le t$. By Corollary 4.1.14, $i_s \longleftrightarrow i_x \longleftrightarrow i_t \longleftrightarrow i_v$. Following similar logic, $i_s \longleftrightarrow i_y \longleftrightarrow i_t \longleftrightarrow i_w$. Thus $i_s \longleftrightarrow i_t \longleftrightarrow i_x \longleftrightarrow i_y \longleftrightarrow i_v \longleftrightarrow i_w$, contradicting that C is parity preserving. ☐

Corollary 4.1.16. *Let $C \in \mathcal{C}_{(\alpha_0,2q)}$ be a dominant constraint graph. If $i_s \longleftrightarrow i_t$ for some $1 \le s < t \le 2q$, contracting i_s and i_t splits $H(\alpha_0, 2q)$ into $H(\alpha_0, t - s)$ and $H(\alpha_0, 2q - (t - s))$. Letting C' and C'' be the induced constraint graphs on $H(\alpha_0, t - s)$ and $H(\alpha_0, 2q - (t - s))$ respectively, C' and C'' are dominant. See Figure 4.6b for an illustration.*

Proof. By Corollary 4.1.15, no edge in $H(\alpha_0, t - s)$ can be made equal to an edge in $H(\alpha_0, 2q - (t - s))$, so C' and C'' are nonzero-valued constraint graphs in $\mathcal{C}_{(\alpha_0,t-s)}$ and $\mathcal{C}_{(\alpha_0,2q-(t-s))}$ respectively. This implies that $\left|E(C')\right| \ge (t - s)/2 - 1$ and $\left|E(C')\right| \ge q - (t -$



(a) Illustration of Corollary 4.1.15: $i_s \longleftrightarrow i_t$.

(b) Illustration of Corollary 4.1.16: contract i_s and i_t.

Figure 4.6: Illustration of Corollary 4.1.15 and Corollary 4.1.16: $i_s \longleftrightarrow i_t$ in a dominant constraint graph C, contracting i_s and i_t splits C into two dominant constraint graphs.

$s)/2 - 1$. Since $|E(C)| = q - 1$ as C is dominant and $|E(C)| = |E(C')| + |E(C'')| + 1$ (here the additional edge is $\{i_s, i_t\}$), we must have that $|E(C')| = (t - s)/2 - 1$ and $|E(C')| = q - (t - s)/2 - 1$, so C' and C'' are dominant, as needed. □

Lemma 4.1.17. *Consider a dominant constraint graph C on $H(\alpha_0, 2q)$. If i_j is the first vertex i_1 is constrained to (i.e. if j is the smallest index such that $i_1 \longleftrightarrow i_j$), then $i_2 \longleftrightarrow i_{j-1}$.*

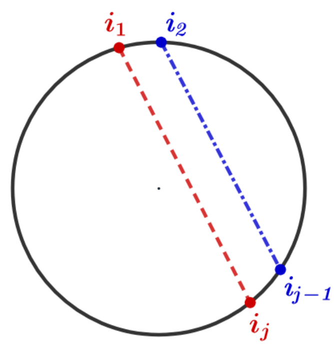


Figure 4.7: Illustration of Lemma 4.1.17: i_j is the first vertex i_1 is constrained to.

Proof. Contract the edge $\{i_1, i_j\}$, splitting $H(\alpha_0, 2q)$ into $H(\alpha_0, j-1)$ and $H(\alpha_0, 2q-j+1)$. Let C' be the induced constraint graph on $H(\alpha_0, j-1)$. Since i_j is the first vertex i_1 is constrained to, $i_1 = i_j$ is isolated in $H(\alpha_0, j-1)$. By Lemma 4.1.13, $i_2 \longleftrightarrow i_{j-1}$ in C' and thus $i_2 \longleftrightarrow i_{j-1}$ in C, as needed. $\square$

List of Properties of Dominant Constraint Graphs on a Cycle

For convenience, here is a list of the properties we have shown. If $C \in \mathcal{C}_{(\alpha_0,2q)}$ is a dominant constraint graph then

1. $|E(C)| = q - 1$.
2. C is parity-preserving.
3. C is non-crossing.
4. If $i_x \longleftrightarrow i_y$ and $i_v \longleftrightarrow i_w$ for some $x \le v \le y \le w$, then $i_x \longleftrightarrow i_y \longleftrightarrow i_v \longleftrightarrow i_w$.
5. If $i_s \longleftrightarrow i_t$ for some $1 \le s < t \le 2q$, then there is an explicitly non-crossing representative graph $G_C \in C$ so that it includes the edge $\{i_s, i_t\}$. Moreover, if we let $\mathcal{R} = \{\{i_x, i_{x+1}\} : s \le x < t\}$ and $\mathcal{L} = E(\alpha_0, 2q) \setminus \mathcal{R}$ then edges in $\mathcal{R}$ can only be made equal to edges in $\mathcal{R}$ by C and edges in $\mathcal{L}$ can only be made equal to edges in $\mathcal{L}$ by C.
6. If $i_s \longleftrightarrow i_t$ for some $1 \le s < t \le 2q$, contracting i_s and i_t splits $H(\alpha_0, 2q)$ into $H(\alpha_0, t-s)$ and $H(\alpha_0, 2q-(t-s))$. Letting C' and C'' be the induced constraint graphs on $H(\alpha_0, t-s)$ and $H(\alpha_0, 2q-(t-s))$ respectively, C' and C'' are dominant.
7. If i_j is the first vertex i_1 is constrained to (i.e. if j is the smallest index such that $i_1 \longleftrightarrow i_j$), then $i_2 \longleftrightarrow i_{j-1}$.

4.1.3 Properties of Dominant Constraint Graphs on $H(\alpha_Z, 2q)$

Now that we have analyzed dominant constraint graphs in $\mathcal{C}_{(\alpha_0,2q)}$, we can analyze dominant constraint graphs in $\mathcal{C}_{(\alpha_Z,2q)}$.

Definition 4.1.18. Let α_Z be the Z-shape as defined in Definition 2.1.15 and let $H(\alpha_Z, 2q)$ be the multi-graph as defined in Definition 3.2.1. We label the vertices of $V_{(\alpha_Z)_i}$ as $\{a_{i1}, a_{i2}, b_{i1}, b_{i2}\}$ and the vertices of $V_{\left(\alpha_Z^T\right)_i}$ as $\{b_{i1}, b_{i2}, a_{(i+1)1}, a_{(i+1)2}\}$. We call the induced subgraph of $H(\alpha_Z, 2q)$ on vertices $\{a_{i1}, b_{i1} : i \in [q]\}$ the *outer wheel* W_1 and the induced subgraph on vertices $\{a_{i2}, b_{i2} : i \in [q]\}$ the *inner wheel* W_2. We denote the vertices of W_i as V_i and edges as E_i.

We label the "middle edges" of $H(\alpha, 2q)$ in the following way: let $e_{2i-1} = \{a_{i2}, b_{i1}\}$ and $e_{2i} = \{b_{i1}, a_{(i+1)2}\}$ for $i = 1, \dots, q$. We call the edges $\{e_i : i \in [2q]\}$ the *spokes* of $H(\alpha_Z, 2q)$. See Figure 4.8 for an illustration.

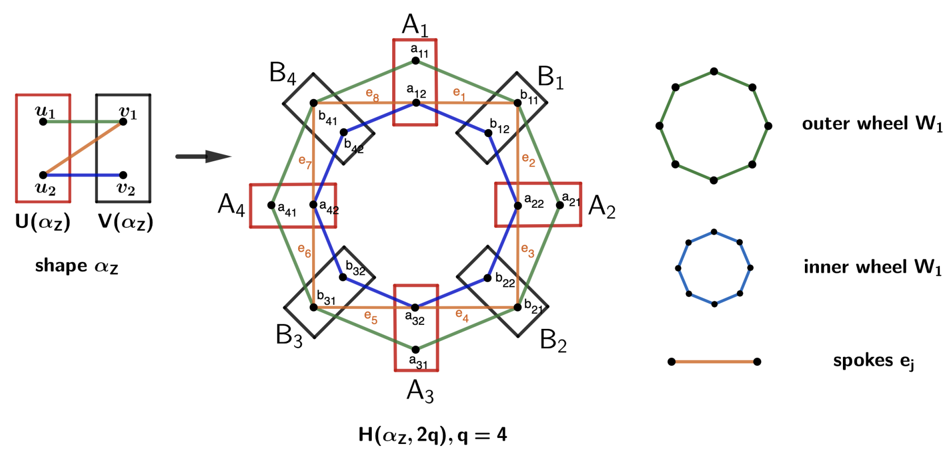


Figure 4.8: $H(\alpha_Z, 2q)$ where $q = 4$.

Definition 4.1.19. Let C be a constraint graph in $\mathcal{C}_{(\alpha_Z, 2q)}$. For $i = 1, 2$, we take C_i to be the induced constraint graph of C on the vertices V_i.

Remark 4.1.20. *Each wheel W_i can be viewed as $H(\alpha_0, 2q)$. The induced constraint graph C_i can be viewed as a constraint graph on $H(\alpha_0, 2q)$.*

The key property that we need about dominant constraint graphs in $\mathcal{C}_{(\alpha_Z, 2q)}$ is that they are well-behaved. This implies that the induced constraint graphs C_1, C_2 are dominant constraint graphs in $\mathcal{C}_{(\alpha_0, 2q)}$.

Definition 4.1.21. Given a shape α, we say that a constraint graph $C \in \mathcal{C}_{(\alpha,2q)}$ is *well-behaved* if whenever $u \longleftrightarrow v$ in C, u and v are copies of the same vertex in α or α^T.

Theorem 4.1.22. *All dominant constraint graphs in $\mathcal{C}_{(\alpha_Z,2q)}$ are well-behaved.*

This theorem is surprisingly tricky to prove, so we defer its proof to the appendix.

Remark 4.1.23. *This theorem is not true for all shapes α. In particular, this theorem is false for the bipartite shape α with $U_\alpha = (u_1, u_2)$, $V_\alpha = (v_1, v_2)$, $V(\alpha) = U_\alpha \cup V_\alpha$, and $E(\alpha) = \{\{u_1, v_1\}, \{u_1, v_2\}, \{u_2, v_1\}, \{u_2, v_2\}\}$.*

Definition 4.1.24. Let C be a constraint graph in $\mathcal{C}_{(\alpha_Z,2q)}$.

1. We say C is *wheel-respecting* if whenever $u \longleftrightarrow v$, $u, v \in V_i$ for some $i \in \{1, 2\}$ (i.e. no two vertices on different wheels are constrained together). Note that if C is wheel-respecting then if G_1 and G_2 are representatives of C_1 and C_2, the graph G with $V(G) = V_1 \cup V_2$ and $E(G) = E(G_1) \cup E(G_2)$ is a representative of C.

2. We say C is *parity-preserving* if for each $i \in \{1, 2\}$, the induced constraint graphs C_i of C on V_i is parity-preserving.

3. We say that C is *non-crossing* if the induced constraint graphs C_1 and C_2 are non-crossing.

Proposition 4.1.25. *Let C be a constraint graph in $\mathcal{C}_{(\alpha_Z,2q)}$. C is well-behaved if and only if C is wheel-respecting and parity-preserving.*

Corollary 4.1.26. *If C is a dominant constraint graph in $\mathcal{C}_{(\alpha_Z,2q)}$ then*

1. *C is wheel-respecting, parity-preserving, and non-crossing.*

2. *The induced constraint graphs C_1,C_2 are dominant constraint graphs in $\mathcal{C}_{(\alpha_0,2q)}$.*

Proof. Since dominant constraint graphs in $\mathcal{C}_{(\alpha_Z,2q)}$ are well-behaved, C is wheel-respecting and parity-preserving. Since C is wheel-respecting, C can only make edges in W_i equal to other edges in W_i, so C_1 and C_2 must be nonzero-valued constraint graphs in $\mathcal{C}_{(\alpha_0,2q)}$. Since $|E(C)| = 2q - 2 = |E(C_1)| + |E(C_2)|$, we must have that $|E(C_1)| = |E(C_2)| = q - 1$ and thus C_1 and C_2 are dominant. This implies that C_1 and C_2 are non-crossing, so C is non-crossing. □

We now show a few additional properties of dominant constraint graphs in $\mathcal{C}_{(\alpha_Z,2q)}$. We start with the following fact about the spokes of $H(\alpha_Z, 2q)$.

Lemma 4.1.27. *Let C be a dominant constraint graph in $\mathcal{C}_{(\alpha_Z,2q)}$ and consider the spokes $\{e_i : i \in [2q]\}$ of $H(\alpha_Z, 2q)$. If $e_i \longleftrightarrow e_j$ and $e_s \longleftrightarrow e_t$ for some $i < s < j < t$, then $e_i \longleftrightarrow e_s \longleftrightarrow e_j \longleftrightarrow e_t$.*

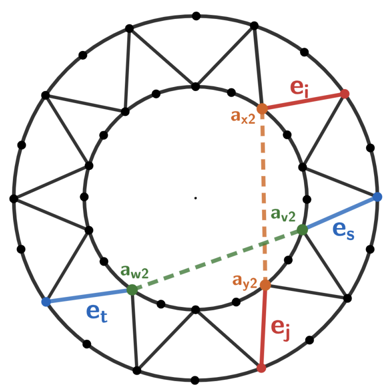


Figure 4.9: Illustration of Lemma 4.1.27: e_i, e_j and e_s, e_t "cross" each other.

Proof. By the definition of the spokes e_i's, one of the endpoints of e_i is a_{x2} where $x = \lfloor i/2 \rfloor + 1$. Since C is well-behaved and $e_i \longleftrightarrow e_j$, $a_{x2} \longleftrightarrow a_{y2}$ where $x = \lfloor i/2 \rfloor + 1$ and $y = \lfloor j/2 \rfloor + 1$. Similarly since $e_s \longleftrightarrow e_t$, $a_{v2} \longleftrightarrow a_{w2}$ where $v = \lfloor s/2 \rfloor + 1$ and $w = \lfloor t/2 \rfloor + 1$. Since $i < s < j < t$, we have $x \le v \le y \le w$. By Corollary 4.1.26, C_2 is a dominant constraint graph on W_2. By Corollary 4.1.14, $a_{x2} \longleftrightarrow a_{y2} \longleftrightarrow a_{v2} \longleftrightarrow a_{w2}$.

Similarly we can argue that $b_{x'1} \longleftrightarrow b_{y'1} \longleftrightarrow b_{v'1} \longleftrightarrow b_{w'1}$ where $b_{x'1}, b_{y'1}, b_{v'1}, b_{w'1}$ are the endpoints of e_i, e_s, e_j, e_t, respectively. Thus $e_i \longleftrightarrow e_s \longleftrightarrow e_j \longleftrightarrow e_t$. □

Combining this fact about the spokes of $H(\alpha_Z, 2q)$ with the following lemma, we can show that constraint edges between vertices which are not incident to any spokes split $H(\alpha_Z, 2q)$ into two parts, which is the main result needed to prove Theorem 4.1.4.

Lemma 4.1.28. *For all $m \in \mathbb{N}$, if M is a perfect matching on the indices $[2m]$ such that no two edges of M cross (i.e. there is no pair of edges $\{i,j\}, \{k,l\} \in M$ such that $i < k < j < l$) then either $\{1, 2m\} \in M$ or there is a sequence of indices $i_1 < \ldots < i_k$ such that*

1. *For all $j \in [k]$, i_j is even.*
2. *$\{1, i_1\} \in M$ and $\{i_k + 1, 2m\} \in M$.*
3. *For all $j \in [k-1]$, $\{i_j + 1, i_{j+1}\} \in M$.*

See Figure 4.10a for an illustration.

Proof. We prove this by induction on m. The base case $m = 1$ is trivial. For the inductive step, assume the result is true for m and consider a matching M on $[2m+2]$ such that no edges of M cross. If $\{1, 2m+2\} \in M$ then we are done, so we can assume that $\{1, 2m+2\} \notin M$.

Choose $s < t \in [2m+2]$ such that $\{s,t\} \in M$ and $t - s$ is minimized. We claim that $t = s+1$. To see this, assume that $t > s+1$. Since M is a perfect matching, $\{s+1, x\} \in M$ for some $x \in [2m+2]$. Since no two edges of M cross, we must have that $s+1 < x < t$. However, this implies that $x - (s+1) < t - s$, contradicting our choice of s and t.

Now consider the matching M' obtained from M by deleting the indices $s, s+1$ and decreasing all indices greater than $s+1$ by 2. By the inductive hypothesis, either $\{i'_1, i'_{2m}\} \in M$, or there is a sequence $i'_1 < \ldots < i'_k$ such that for all $j \in [k]$, i'_j is even, $\{1, i'_1\} \in M$ and $\{i'_k, 2m\} \in M$, and for all $j \in [k-1]$, $\{i'_j + 1, i'_{j+1}\} \in M$. In the later case we can modify this sequence as follows to obtain the desired sequence:

1. Increase all indices in this sequence which are greater than or equal to s by 2.

2. If $s-1$ is in this sequence, insert $s+1$ after it.

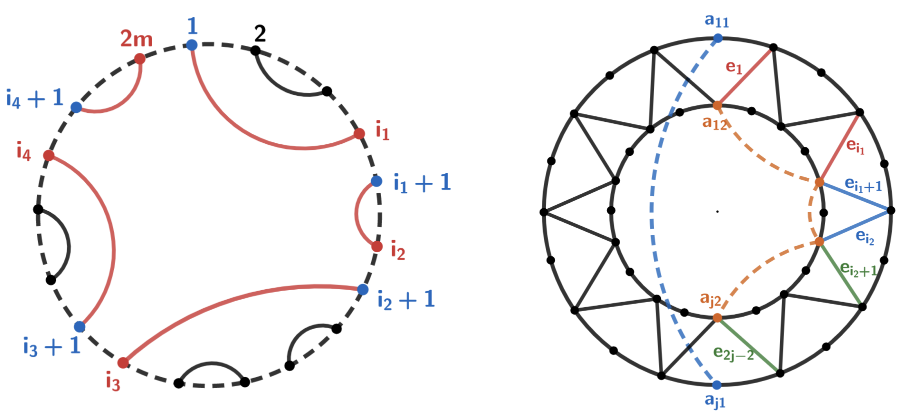
If $\{i'_1, i'_{2m}\} \in M$, then there are three cases:

1. When $1 < s < 2m+1$, $i'_1 = i_1$ and $i'_{2m} = i_{2m+2}$, then we are done.

2. When $s = 1$, $i'_1 = i_3$ and $i'_{2m} = i_{2m+2}$, then we have a sequence with single element $i_1 = 2$ such that $\{1, 2\} \in M$ and $\{3, 2m+2\} \in M$.

3. When $s = 2m+1$, $i'_1 = i_1$ and $i'_{2m} = i_{2m}$, then we have a sequence with single element $i_1 = 2m$ such that $\{1, 2m\} \in M$ and $\{2m+1, 2m+2\} \in M$.

□



(a) Illustration of Lemma 4.1.28: solid lines are matchings.

(b) Illustration of Lemma 4.1.29: $a_{11} \longleftrightarrow a_{j1}$ implies $a_{12} \longleftrightarrow a_{j2}$.

Figure 4.10: Illustration of Lemma 4.1.28 and Lemma 4.1.29.

Lemma 4.1.29. *Let C be a dominant constraint graph in $\mathcal{C}_{(\alpha_Z, 2q)}$. If $a_{i1} \longleftrightarrow a_{j1}$ for some $1 \leq i < j \leq q$, then $a_{i2} \longleftrightarrow a_{j2}$. Moreover, the spokes $\{e_x : x \in [2i-1, 2j-2]\}$ can only be made equal to each other. Similarly, if $b_{i2} \longleftrightarrow b_{j2}$ for some $1 \leq i < j \leq q$, then $b_{i1} \longleftrightarrow b_{j1}$ and the spokes $\{e_x : x \in [2i, 2j-1]\}$ can only be made equal to each other.*

Proof. We prove the first statement as the proof for the second statement is similar. Without loss of generality, assume $i = 1$. Observe that for all $x \in [j-1]$ and all $y \in [j, q]$, we cannot have that $b_{x1} \longleftrightarrow b_{y1}$. Otherwise, by Corollary 4.1.14 we would have that $a_{11} \longleftrightarrow a_{j1} \longleftrightarrow b_{x1} \longleftrightarrow b_{y1}$, contradicting the fact that C is well-behaved.

This implies that the spokes $\{e_x : x \in [2j-2]\}$ can only be made equal to each other. By Lemma 4.1.27, there must be a perfect matching M on the indices $[2j-2]$ such that

1. If $\{x, y\} \in M$ then $e_x \longleftrightarrow e_y$ (M describes how the spokes $\{e_x : x \in [2j-2]\}$ are paired up).

2. No two edges of M cross (there is no pair of edges $\{x, y\}, \{z, w\} \in M$ such that $x < z < y < w$).

By Lemma 4.1.28, either $\{1, 2j-2\} \in M$ or there is a sequence of indices $i_1 < \ldots < i_k$ such that

1. For all $l \in [k]$, i_l is even.

2. $\{1, i_1\} \in M$ and $\{i_k + 1, 2j-2\} \in M$.

3. For all $l \in [k-1]$, $\{i_l + 1, i_{l+1}\} \in M$.

If $\{1, 2j-2\} \in M$ then $a_{12} \longleftrightarrow a_{j2}$. Otherwise, we make the following observations (see Figure 4.10b for an illustration):

1. Since $\{1, i_l\} \in M$, $a_{12} \longleftrightarrow a_{(i_l/2+1)2}$.

2. For all $l \in [k-1]$, since $\{i_l + 1, i_{l+1}\} \in M$, $a_{(i_l/2+1)2} \longleftrightarrow a_{(i_{l+1}/2+1)2}$.

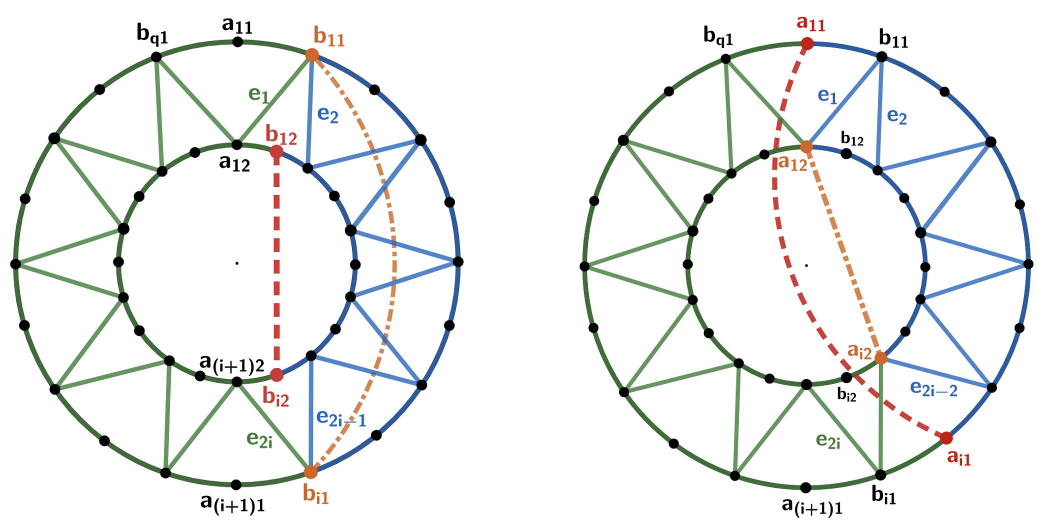
3. Since $\{i_k + 1, 2(j-1)\} \in M$, $a_{(i_k/2+1)2} \longleftrightarrow a_{j2}$.

Putting these observations together, $a_{12} \longleftrightarrow a_{j2}$, as needed. □

Corollary 4.1.30. *If C is a dominant constraint graph in $\mathcal{C}_{(\alpha_Z, 2q)}$ then the following statements are true:*

1. *If* $a_{i1} \longleftrightarrow a_{j1}$ *for some* $1 \leq i < j \leq q$ *then* $a_{i2} \longleftrightarrow a_{j2}$. *Moreover, contracting* a_{i1} *and* a_{j1} *together and contracting* a_{i2} *and* a_{j2} *together splits* $H(\alpha_Z, 2q)$ *into* $H(\alpha_Z, 2(j-i))$ *and* $H(\alpha_Z, 2(q-j+i))$, *and the induced constraint graphs* $C' \in \mathcal{C}_{(\alpha_Z, 2(j-i))}$ *and* $C'' \in \mathcal{C}_{(\alpha_Z, 2(q-j+i))}$ *are dominant.*

2. *Similarly, if* $b_{i2} \longleftrightarrow b_{j2}$ *for some* $1 \leq i < j \leq q$ *then* $b_{i1} \longleftrightarrow b_{j1}$. *Moreover, contracting* b_{i1} *and* a_{j1} *together and contracting* b_{i2} *and* b_{j2} *together splits* $H(\alpha_Z, 2q)$ *into* $H(\alpha_Z, 2(j-i))$ *and* $H(\alpha_Z, 2(q-j+i))$, *and the induced constraint graphs* $C' \in \mathcal{C}_{(\alpha_Z, 2(j-i))}$ *and* $C'' \in \mathcal{C}_{(\alpha_Z, 2(q-j+i))}$ *are dominant.*



(a) Illustration of Corollary 4.1.30:$b_{12} \longleftrightarrow b_{i2}$

(b) Illustration of Corollary 4.1.30:$a_{11} \longleftrightarrow a_{i1}$

Figure 4.11: An edge can only be made equal to the edges with the same color.

List of Properties of Dominant Constraint Graphs on $H(\alpha_Z, 2q)$

For convenience, here is a list of the properties we have shown. If $C \in \mathcal{C}_{(\alpha_Z, 2q)}$ is a dominant constraint graph then

1. $|E(C)| = 2q - 2$.

2. C is wheel-respecting, parity-preserving, and non-crossing.

3. The induced constraint graphs C' and C'' on the two wheels W_1 and W_2 are dominant constraint graphs in $\mathcal{C}_{(\alpha_0,2q)}$.

4. If $e_i \longleftrightarrow e_j$ and $e_s \longleftrightarrow e_t$ for some $i < s < j < t$, then $e_i \longleftrightarrow e_s \longleftrightarrow e_j \longleftrightarrow e_t$.

5. If $a_{i1} \longleftrightarrow a_{j1}$ for some $1 \leq j < i \leq q$ then $a_{i2} \longleftrightarrow a_{j2}$. Moreover, contracting a_{i1} and a_{j1} together and contracting a_{i2} and a_{j2} together splits $H(\alpha_Z, 2q)$ into $H(\alpha_Z, 2(j-i))$ and $H(\alpha_Z, 2(q-j+i))$ and the induced constraint graphs $C' \in \mathcal{C}_{(\alpha_Z,2(j-i))}$ and $C'' \in \mathcal{C}_{(\alpha_Z,2(q-j+i))}$ are dominant.

6. Similarly, if $b_{i2} \longleftrightarrow b_{j2}$ for some $1 \leq j < i \leq q$ then $b_{i1} \longleftrightarrow b_{j1}$. Moreover, contracting b_{i1} and a_{j1} together and contracting b_{i2} and b_{j2} together splits $H(\alpha_Z, 2q)$ into $H(\alpha_Z, 2(j-i))$ and $H(\alpha_Z, 2(q-j+i))$ and the induced constraint graphs $C' \in \mathcal{C}_{(\alpha_Z,2(j-i))}$ and $C'' \in \mathcal{C}_{(\alpha_Z,2(q-j+i))}$ are dominant.

4.1.4 Proof of Theorem 4.1.4

Now we are ready to prove the main result of this section, Theorem 4.1.4.

Definition 4.1.31. Define $D(q,m)$ to be the number of dominant constraint graphs C in $\mathcal{C}_{(\alpha_Z,2(q+m))}$ such that $a_{12} \longleftrightarrow a_{22} \longleftrightarrow \ldots \longleftrightarrow a_{(m+1)2}$.

Remark 4.1.32. *Notice that for the case when $m = 0$, $D(q,0) = \left|\left\{C \in \mathcal{C}_{(\alpha_Z,2q)} : C \text{ is dominant}\right\}\right|$. For the case when $q = 0$, we consider constraint graphs C on $H(\alpha_Z, 2m)$ where all the vertices a_{i2} on W_2 are constrained together by C. i.e. C can be viewed as a dominant constraint graph on W_1. Thus $D(0,m) = \left|\left\{C \in \mathcal{C}_{(\alpha_0,2m)} : C \text{ is dominant}\right\}\right|$, which is the Catalan numbers.*

Proof of Theorem 4.1.4. To prove Theorem 4.1.4, we prove the following two statements:

1. For all $q \in \mathbb{N}$, $D(q,0) = \sum_{i=1}^{q} D(q-i,1) \cdot D(i-1,0)$.

2. For all $q \in \mathbb{N} \cup \{0\}$, $D(q,1) = \sum_{i=0}^{q} D(i,0) \cdot D(q-i,0)$.

Combining these two statements, for all $n \in \mathbb{N} \cup \{0\}$,

$$D(n+1,0) = \sum_{i,j,k \geq 0: i+j+k=n} D(i,0)D(j,0)D(k,0).$$

This is the same recurrence relation as we have for D_n and we have that $D(0,0) = D_0 = 1$, so these two statements imply that $D(n,0) = D_n = \frac{1}{2n+1}\binom{3n}{n}$, as needed.

To prove the first statement, given a dominant constraint graph C in $\mathcal{C}_{(\alpha_Z,2q)}$, if a_{12} is not isolated then let $i \in [q-1]$ be the first index such that $a_{12} \longleftrightarrow a_{(i+1)2}$. By Lemma 4.1.17, $b_{12} \longleftrightarrow b_{i2}$. By Corollary 4.1.30, $b_{11} \longleftrightarrow b_{i1}$. Moreover, contracting b_{11} and b_{i1} together and contracting b_{12} and b_{i2} together splits $H(\alpha_Z, 2q)$ into $H(\alpha_Z, 2(i-1))$ and $H(\alpha_Z, 2(q-i+1))$ and the induced constraint graphs $C' \in \mathcal{C}_{(\alpha_Z,2(i-1))}$ and $C'' \in \mathcal{C}_{(\alpha_Z,2(q-i+1))}$ are dominant. Now observe that

1. Since $a_{12} \longleftrightarrow a_{(i+1)2}$ in C, $a_{12} \longleftrightarrow a_{22}$ in C''.

2. If we are given dominant constraint graphs $C' \in \mathcal{C}_{(\alpha_Z,2(i-1))}$ and $C'' \in \mathcal{C}_{(\alpha_Z,2(q-i+1))}$ such that $a_{12} \longleftrightarrow a_{22}$ in C'', we can recover C and i by reversing this process. Thus, this map is a bijection.

This implies that the number of dominant constraint graphs C in $\mathcal{C}_{(\alpha_Z,2q)}$ such that $i \in [q-1]$ is the first index such that $a_{12} \longleftrightarrow a_{(i+1)2}$ is $D(q-i,1) \cdot D(i-1,0)$. For an illustration of this argument, see Figure 4.12.

If a_{12} is isolated then we must have that $b_{12} \longleftrightarrow b_{q2}$. In this case, $b_{11} \longleftrightarrow b_{q1}$ and contracting along these edges gives us $H(\alpha_Z, 2(q-1))$. Thus, the number of dominant constraint graphs C in $\mathcal{C}_{(\alpha_Z,2q)}$ such that a_{12} is isolated is $D(q-1,0) = D(q-1,0)D(0,1)$

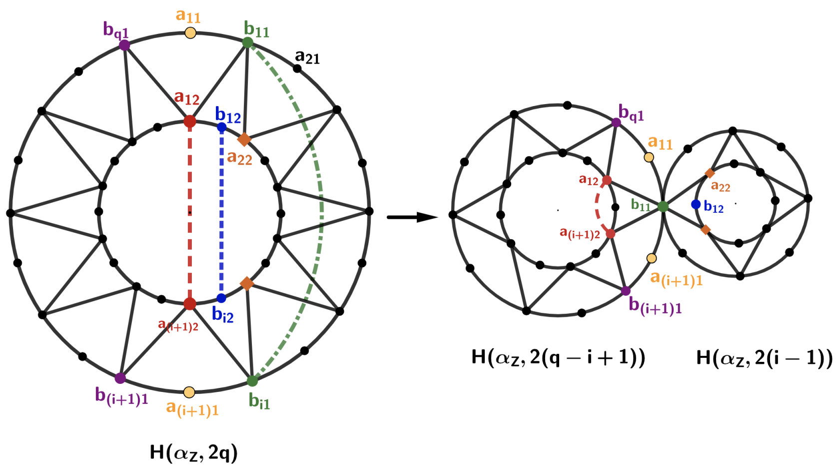


Figure 4.12: $a_{(i+1)2}$ is the first vertex that a_{12} is constrained to. As a result $H(\alpha_Z, 2q)$ is split into $H(\alpha_Z, 2(i-1))$ and $H(\alpha_Z, 2(q-i+1))$.

as $D(0,1) = 1$. Putting everything together,

$$D(q,0) = \sum_{i=1}^{q} D(q-i,1) \cdot D(i-1,0).$$

To prove the second statement, given a dominant constraint graph C in $\mathcal{C}_{(\alpha_Z,2(q+1))}$ such that $a_{12} \longleftrightarrow a_{22}$, consider the first index i such that $b_{11} \longleftrightarrow b_{(i+1)1}$. If b_{11} is isolated then we take $i = 0$. We have the following cases:

1. $i = 0$: if b_{11} is not constrained to any vertex, then it implies that $a_{11} \longleftrightarrow a_{21}$. Merging a_{11} and a_{21}, a_{12} and a_{22} and deleting spokes e_1 and e_2, we get $H(\alpha_Z, 2q)$. The induced constraint graph C' of C on $H(\alpha_Z, 2q)$ is dominant, so this gives $D(q,0) = D(q,0) \cdot D(0,0)$ possible constraint graphs.

2. $i \in [q]$: By Lemma 4.1.17, $a_{21} \longleftrightarrow a_{(i+1)1}$. By Corollary 4.1.30, $a_{22} \longleftrightarrow a_{(i+1)2}$. Moreover, contracting a_{21} and $a_{(i+1)1}$ together and contracting a_{22} and $a_{(i+1)2}$ together splits $H(\alpha_Z, 2(q+1))$ into $H(\alpha_Z, 2(i-1))$ and $H(\alpha_Z, 2(q-i+2))$ and the induced constraint graphs $C' \in \mathcal{C}_{(\alpha_Z,2(i-1))}$ and $C'' \in \mathcal{C}_{(\alpha_Z,2(q-i+2))}$ are dominant. Now observe that

(a) Since $a_{12} \longleftrightarrow a_{22}$ in C and $b_{11} \longleftrightarrow b_{(i+1)1}$ in C, $a_{12} \longleftrightarrow a_{22}$ in C'' and $b_{11} \longleftrightarrow b_{21}$ in C''. Contracting these edges gives us $H(\alpha_Z, 2(q-i+1))$, so C'' corresponds to a dominant constraint graph in $\mathcal{C}_{(\alpha_Z, 2(q-i+1))}$.

(b) If we are given dominant constraint graphs $C' \in \mathcal{C}_{(\alpha_Z, 2(i-1))}$ and $C'' \in \mathcal{C}_{(\alpha_Z, 2(q-i+1))}$, we can recover C and i by reversing this process. Thus, this map is a bijection.

This gives $D(i-1,0) \cdot D(q-i+1,0)$ dominant constraint graphs. For an illustration of this argument, see Figure 4.13.

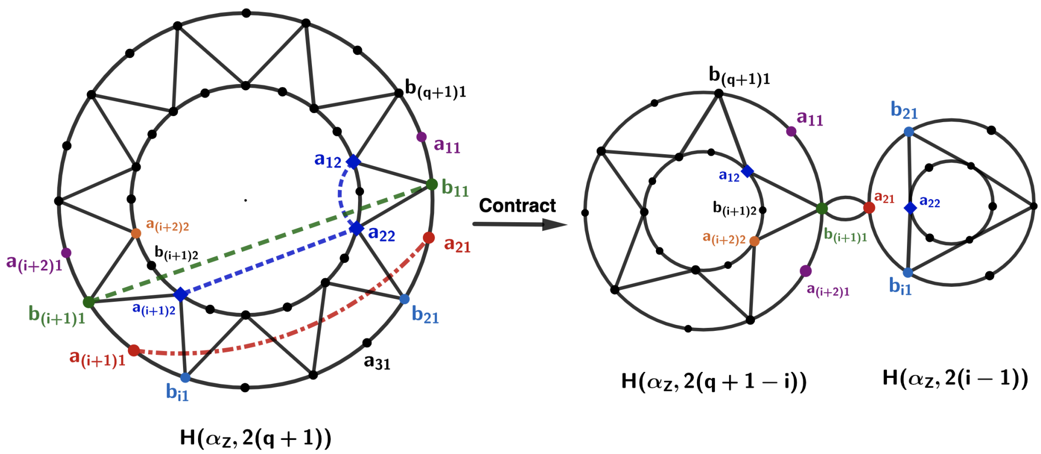


Figure 4.13: C is a dominant constraint graph in $\mathcal{C}_{(\alpha_Z, 2(q+1))}$ such that $\alpha_{12} \longleftrightarrow a_{22}$. $b_{(i+1)1}$ is the first vertex that b_{11} is constrained to.

Putting everything together,

$$D(q,1) = D(q,0) \cdot D(0,0) + \sum_{i=1}^{q} D(i-1,0) \cdot D(q-i+1,0) = \sum_{i=0}^{q} D(i,0) \cdot D(q-i,0)$$

as needed. □

4.2 The Spectrum of Z-shaped Graph Matrices

In this section, we will determine the limiting spectrum of singular values of the Z-shaped graph matrices. We will first show how to do so with the Wigner matrix as a warm-up, and then derive the spectrum for the Z-shaped graph matrix with the similar technique.

4.2.1 Warm-up: The Spectrum of Wigner's Matrix

Before we derive the limiting spectrum for the Z-shape graph matrices, we will show the simpler but similar analysis for deriving Wigner's Semicircle Law from the trace power moments.

Recall the Wigner matrix and the semicircle law defined in Definition 1.2.1 and Definition 1.2.8, and Theorem 1.2.13 states that the expected even trace power of normalized Wigner matrices are the Catalan numbers.

$$\lim_{n\to\infty} \frac{1}{n}\mathbb{E}\left[\operatorname{tr}\left(\overline{M}_n{}^{2k}\right)\right] = C_k \tag{4.5}$$

where $\overline{M}_n = \frac{1}{\sqrt{n}}M_n$ and M_n are $n \times n$ Wigner matrices.

One can verify that $\int_{-2}^{2}\left(\frac{1}{2\pi}\sqrt{4-x^2}\right)\cdot x^{2k}\,dx = C_k$ and conclude Theorem 1.2.2 after checking the other conditions in Corollary 3.1.7. Here we will show an alternative way to get $f_{sc}(x)$ by deriving a differential equation for $f_{sc}(x)$ from the recurrence relation for C_k.

Lemma 4.2.1. *Assume $f(x)$ is an function satisfying that for all $k \in \mathbb{N}$,*

$$\int_{-2}^{2} x^{2k}\cdot f(x)\,dx = C_k \tag{4.6}$$

and moreover,

1. *$f(x)$ is continuously differentiable on $(-2, 2)$.*
2. $\lim_{x\to 2^-} f(x) = \lim_{x\to -2^+} f(x) = 0.$

Then $f(x)$ satisfies the following differential equation on $(-2, 2)$:

$$\left(4-x^2\right)f'(x) + xf(x) = 0.. \tag{4.7}$$

Proof.

Proposition 4.2.2.

$$\frac{C_{k+1}}{C_k} = \frac{2(2k+1)}{k+2}. \tag{4.8}$$

Proof.

$$\frac{C_{k+1}}{C_k} = \frac{\dfrac{1}{k+2} \cdot \dfrac{(2k+2)!}{(k+1)!(k+1)!}}{\dfrac{1}{k+1} \cdot \dfrac{(2k)!}{k!k!}} = \frac{(k+1)\cdot(2k+2)(2k+1)}{(k+2)\cdot(k+1)^2} = \frac{2(2k+1)}{k+2}.$$

□

For any $k \geq 1$, using integration by parts and the boundary condition that $\lim_{x \to 2^-} f(x) = \lim_{x \to -2^+} f(x) = 0$,

$$\int_{-2}^{2} x^{2k+1} \cdot f'(x)\, dx = \left[x^{2k+1} \cdot f(x)\right]_{-2}^{2} - (2k+1) \cdot \int_{-2}^{2} f(x) \cdot x^{2k}\, dx = -(2k+1)C_k$$

We have the following relations together with Equation (4.8).

i. $\int_{-2}^{2} x^{2k} \cdot f(x)\, dx = C_k$

ii. $\int_{-2}^{2} x^{2k+2} \cdot f(x)\, dx = C_{k+1}$

iii. $\int_{-2}^{2} x^{2k+1} \cdot f'(x)\, dx = -(2k+1)C_k$

iv. $\int_{-2}^{2} x^{2k+3} \cdot f'(x)\, dx = -(2k+3)C_{k+1}$

v. $(k+2)C_{k+1} = 2(2k+1)C_k$

Thus multiplying v. by -2 and rewriting using i. to iv., we get

$$\begin{aligned}
&-2(k+2)C_{k+1} = -(2k+3)C_{k+1} - C_{k+1} = -4(2k+1)C_k \\
\implies &\int_{-2}^{2} x^{2k+3} \cdot f'(x)\,dx - \int_{-2}^{2} x^{2k+2} \cdot f(x)\,dx = 4\int_{-2}^{2} x^{2k+1} \cdot f'(x)\,dx \\
\implies &\int_{-2}^{2} x^{2k+1} \cdot \left(\left(4-x^2\right) f'(x) + xf(x)\right)\,dx = 0.
\end{aligned}$$

With similar argument as in Lemma 4.2.14, we can conclude that

$$\left(4-x^2\right) f'(x) + xf(x) = 0.$$

□

Finally we can solve the ODE for the solution of $f(x)$.

Proposition 4.2.3. *Assume $f(x)$ satisfies Equation (4.7) and $\int_{-2}^{2} f(x)\,dx = 1$. Then $f(x) = \frac{1}{2\pi}\sqrt{4-x^2}$ on $(-2,2)$.*

Proof.

$$\begin{aligned}
&\left(4-x^2\right) f'(x) + xf(x) = 0 \implies \frac{f'(x)}{f(x)} = -\frac{x}{4-x^2} \\
\implies &\int \frac{f'(x)}{f(x)}\,dx = \int -\frac{x}{4-x^2}\,dx \implies \ln|f(x)| = \frac{1}{2}\ln\left|4-x^2\right| + C \\
\implies &f(x) = A \cdot \sqrt{4-x^2} \text{ for some constant } A \\
\implies &f(x) = \frac{1}{2\pi}\sqrt{4-x^2} \text{ since } \int_{-2}^{2} f(x)\,dx = 1
\end{aligned}$$

□

4.2.2 The Spectrum of the Z-shaped Graph Matrix

We now find the limiting distribution of the spectrum of the singular values of $\frac{1}{n}M_{\alpha_Z}$ as $n \to \infty$.

Definition 4.2.4. Let $a = \dfrac{3\sqrt{3}}{2}$ and define $g_{\alpha_Z} : (0, \infty) \to \mathbb{R}$ be the function such that

$$f(x) = \frac{i}{\pi}\left(\sqrt{3}\sin\left(\frac{1}{3}\arctan\left(\frac{3}{\sqrt{4x^2/3-9}}\right)\right) + \cos\left(\frac{1}{3}\arctan\left(\frac{3}{\sqrt{4x^2/3-9}}\right)\right)\right) \quad (4.9)$$

if $x \in (0, a]$ and $g_{\alpha_Z}(x) = 0$ if $x > a$.

Theorem 4.2.5. *As $n \to \infty$, the spectrum of the singular values of $\frac{1}{n}M_{\alpha_Z}$ approaches g_{α_Z} almost surely.*

Proof. To prove this, we need to show that g_{α_Z} satisfies the conditions of Corollary 3.1.9. Denote $M_n = \frac{1}{n}M_{\alpha_Z}$. Here $\beta_k = \lim_{n\to\infty} \dfrac{1}{r(n)}\mathbb{E}\left[\operatorname{tr}\left(\left(M_n {M_n}^T\right)^k\right)\right]$ where $r(n) = n(n-1)$ is the dimension of the graph matrix M_{α_Z}. We proved in Section 4.1 that $\beta_k = D_k = \dfrac{1}{2k+1}\dbinom{3k}{k}$. We will verify the last two conditions first.

Lemma 4.2.6. $\sum_k {D_{2k}}^{-1/2k} = \infty$.

Proof.

$$D_k = \frac{1}{2k+1}\binom{3k}{k} \le \binom{3k}{k} \le 2^{3k} \text{ since } \sum_{k=0}^{n}\binom{n}{k} = 2^n \implies \binom{n}{k} \le 2^n \text{ for any } k$$

$$\implies D_k^{-1/k} > 2^{-3} \implies D_{2k}^{-1/2k} \not\to 0 \text{ as } k \to \infty$$

Alternative proof: Using Sterling's formula, we can estimate $D_k^{-1/k}$ asymptotically.

$$D_k = \frac{1}{2k+1} \cdot \binom{3k}{k} = \frac{1}{2k+1} \frac{(3k)!}{k!(2k)!} \sim \frac{1}{2k} \cdot \frac{\sqrt{2\pi \cdot 3k}\left(\frac{3k}{e}\right)^{3k}}{\sqrt{2\pi \cdot 2k}\left(\frac{2k}{e}\right)^{2k}\sqrt{2\pi k}\left(\frac{k}{e}\right)^{k}} \sim C \cdot k^{-3/2} \cdot \left(\frac{27}{4}\right)^k$$

$$\implies D_k^{-1/k} \sim C^{-1/k} \cdot k^{3/2k} \cdot \frac{4}{27} \to \frac{4}{27} \neq 0 \text{ as } k \to \infty$$

Thus $\sum_k D_{2k}{}^{-1/2k} = \infty$. ∎

Lemma 4.2.7. *For each k, $\sum_n \operatorname{Var}\left(\frac{1}{r(n)} \operatorname{tr}\left(\left(M_n {M_n}^T\right)^k\right)\right) < \infty$.*

Proof.

$$\begin{aligned}
\beta_{n,k} &:= \operatorname{Var}\left(\frac{1}{r(n)} \operatorname{tr}\left(\left(M_n {M_n}^T\right)^k\right)\right) = n^{-4} \cdot n^{-4k} \cdot \operatorname{Var}\left(\operatorname{tr}\left(\left(M_{\alpha_Z} M_{\alpha_Z}^T\right)^k\right)\right) \\
&= n^{-4(k+1)} \cdot \left(\mathbb{E}\left[\operatorname{tr}\left(\left(M_{\alpha_Z} M_{\alpha_Z}^T\right)^k\right)^2\right] - \mathbb{E}\left[\operatorname{tr}\left(\left(M_{\alpha_Z} M_{\alpha_Z}^T\right)^k\right)\right]^2\right) \\
&= n^{-4(k+1)} \cdot \left(\sum_{A_i, B_j, A'_i, B'_j \subseteq \binom{[n]}{2}} \mathbb{E}\left[\prod_{i=1}^k M_{\alpha_Z}(A_i, B_i) M_{\alpha_Z}^T(B_i, A_{i+1}) M_{\alpha_Z}\left(A'_i, B'_i\right) M_{\alpha_Z}^T\left(B'_i, A'_{i+1}\right)\right]\right. \\
&\qquad \left. - \mathbb{E}\left[\prod_{i=1}^k M_{\alpha_Z}(A_i, B_i) M_{\alpha_Z}^T(B_i, A_{i+1})\right] \cdot \mathbb{E}\left[\prod_{i=1}^k M_{\alpha_Z}\left(A'_i, B'_i\right) M_{\alpha_Z}^T\left(B'_i, A'_{i+1}\right)\right]\right)
\end{aligned}$$

To analyze the term in the summation, we consider two copies $H_1(\alpha_Z, 2k)$ and $H_2(\alpha_Z, 2k)$ with vertices A_i, B_i and A'_i, B'_i respectively, where the constraint edges of a constraint graph C can now go in between H_1 and H_2. Denote

$$\begin{aligned}
\operatorname{val}(C) =& \mathbb{E}\left[\prod_{i=1}^k M_{\alpha_Z}(A_i, B_i) M_{\alpha_Z}^T\left(B_i, A_{i+1}\right) M_{\alpha_Z}\left(A'_i, B'_i\right) M_{\alpha_Z}^T\left(B'_i, A'_{i+1}\right)\right] \\
&- \mathbb{E}\left[\prod_{i=1}^k M_{\alpha_Z}(A_i, B_i) M_{\alpha_Z}^T\left(B_i, A_{i+1}\right)\right] \mathbb{E}\left[\prod_{i=1}^k M_{\alpha_Z}\left(A'_i, B'_i\right) M_{\alpha_Z}^T\left(B'_i, A'_{i+1}\right)\right]
\end{aligned}$$

Observe that $\operatorname{val}(C) = 1$ if under C, each edge appears even number of times in $H_1 \cup H_2$, but there is at least one edge appears odd number of times when restricted to only H_1 or H_2 (call this special edge the "odd" edge). $\text{val}(C) = 0$ in all other situations.

It suffices to prove that there are at most $4k+2$ distinct vertices under the constraint graphs C where $\text{val}(C) = 1$. Because then $\sum_C N(C)\,\text{val}(C) = O_k\left(n^{4k+2}\right) \implies \beta_{n,k} = O_k\left(n^{-2}\right) \implies \sum_n \beta_{n,k} = \sum_n O_k\left(n^{-2}\right) < \infty$.

We will only prove the above claim for well-behaved constraint graphs for now. There are two cases:

1. The odd edge e is among one of the wheels of H_1 or H_2. Denote $W_{i,j}$ the j^{th} wheel of H_i. Observe that for each $j = 1, 2$, $W_{1,j} \cup W_{2,j}$ has $4k$ edges, thus $W_{1,j} \cup W_{2,j}/C$ has at most $2k$ multi-edges (since each edge has multiplicity at least 2). W.L.O.G. assume e is in $W_{1,1}$, the outer wheel of H_1. Then the following are true.

   i. $W_{1,1} \cup W_{2,1}/C$ is a connected graph: since e is an odd edge, it is constrained to some edge in $W_{2,1}$.

   ii. $W_{1,1} \cup W_{2,1}/C$ has at least one cycle: each vertex in $W_{1,1}/C$ has an even degree and the multi-edge containing e has an odd multiplicity, thus there must be one cycle.

   Therefore $W_{1,1} \cup W_{2,1}/C$ has at most $2k$ vertices.

   On the other hand, $W_{1,2} \cup W_{2,2}/C$ has at most $2k+2$ vertices because it has at most two connected components. Thus all together there are $\leq 2k + (2k+2) = 4k+2$ distinct vertices.

2. The odd edge e is among the spokes. W.L.O.G assume $e \in H_1$. Then e is constrained to some e' in H_2. This implies that for each $j = 1, 2$, $W_{1,j} \cup W_{2,j}/C$ is connected, and thus has $\leq 2k+1$ vertices. Thus all together there are $\leq 2(2k+1) = 4k+2$ distinct vertices.

□

The remaining condition we need to verify is that for all $k \in \mathbb{N}$,

$$\int_{x=0}^{a} x^{2k} g_{\alpha_Z}(x)\, dx = \beta_k = D_k$$

To prove that $\int_{x=0}^{a} x^{2k} g_{\alpha_Z}(x)\, dx = D_k$, we proceed as follows:

1. We derive a differential equation for g_{α_Z} based on a recurrence relation for D_k (see Theorem 4.2.9).

2. We prove that if g_{α_Z} satisfies this differential equation and some conditions at $x = 0$ and $x = a$ then $\int_{x=0}^{a} x^{2k} g_{\alpha_Z}(x) dx = D_k$ (see Theorem 4.2.16).

3. We verify that g_{α_Z} satisfies the required conditions (see Theorem 4.2.17).

Remark 4.2.8. *Technically, only steps 2 and 3 are needed. We include the first step because it gives better intuition for where the differential equation comes form.*

Theorem 4.2.9. *Let* $D_k = \frac{1}{2k+1}\binom{3k}{k}$ *and* $a = \lim_{k\to\infty} D_{k+1}/D_k = 3\sqrt{3}/2$. *Assume* $f(x)$ *is an function satisfying that for all* $k \in \mathbb{N}$,

$$\int_0^a x^{2k} \cdot f(x)\, \mathrm{dx} = D_k \tag{4.10}$$

and moreover,

1. $f(x)$ *is twice continuously differentiable on* $(0, a)$.

2. $\lim_{x\to 0^+} x f(x) = 0$ *and* $\lim_{x\to 0^+} x^2 f'(x) = 0$.

3. $\lim_{x\to a^-} f(x) = 0$ *and* $\lim_{x\to a^-} f'(x)(4x^2 - 27) = \lim_{x\to a^-} 8af'(x)(x - a) = 0$.

4. $\lim_{x\to 0^+} x^3 f''(x) = 0$ *and* $\lim_{x\to a^-}(a - x)^2 f''(x) = 0$.

Then $f(x)$ satisfies the following differential equation on $(0, a)$:

$$(4x^4 - 27x^2)f''(x) + (8x^3 - 27x)f'(x) + 3f(x) = 0.. \tag{4.11}$$

Proof. To prove this, we use the following recurrence relation for $D_k = \frac{1}{2k+1}\binom{3k}{k}$.

Proposition 4.2.10.

$$\frac{D_k}{D_{k-1}} = \frac{3(3k-1)(3k-2)}{2k(2k+1)}. \tag{4.12}$$

Proof. Observe that

$$\begin{aligned}\frac{D_k}{D_{k-1}} &= \frac{2k-1}{2k+1} \cdot \frac{(3k)!(2k-2)!(k-1)!}{(3k-3)!(2k)!k!} \\ &= \frac{2k-1}{2k+1} \cdot \frac{(3k)(3k-1)(3k-2)}{(2k)(2k-1)k} = \frac{3(3k-1)(3k-2)}{2k(2k+1)}.\end{aligned}$$

□

We also need the following relationship between the moments of f and the moments of its derivatives.

Definition 4.2.11. For all $j \in \{0, 1, 2\}$ and $k \in \mathbb{Z}$ such that $k \geq j$, we define $A(j, k)$ to be $A(j, k) := \int_0^a f^{(j)}(x) \cdot x^k \,\mathrm{dx}$ where $f^{(j)}(x)$ denotes the j^{th} derivation of f. Notice that $A(0, 2k) = D_k$.

Lemma 4.2.12. *For all $j \in \{1, 2\}$ and $k \in \mathbb{Z}$ such that $k \geq j$,*

$$A(j, k) = \left[f^{(j-1)}(x) \cdot x^k\right]_0^a - kA(j-1, k-1).$$

Proof. Using integration by parts, we have that

$$\begin{aligned} A(j,k) = \int_0^a f^{(j)}(x) \cdot x^k \,\mathrm{dx} &= \left[f^{(j-1)}(x) \cdot x^k \right]_0^a - \int_0^a k f^{(j-1)}(x) \cdot x^{k-1} \,\mathrm{dx} \\ &= \left[f^{(j-1)}(x) \cdot x^k \right]_0^a - kA(j-1,k-1). \end{aligned}$$

□

Corollary 4.2.13. *If* $\lim_{x \to 0^+} xf(x) = 0$ *and* $\lim_{x \to a^-} f(x) = 0$ *then*

1. *For all* $k \in \mathbb{N}$, $A(1,k) = kA(0,k-1)$

2. *For all* $k \in \mathbb{N}$ *such that* $k \geq 2$,

$$A(2,k) = \left[f'(x) \cdot x^k \right]_0^a - kA(1,k-1) = \left[f'(x) \cdot x^k \right]_0^a + k(k-1)A(0,k-2).$$

Using Corollary 4.2.13, Proposition 4.2.10, and the fact that $A(0,2k) = D_k$, we have that for all $k \in \mathbb{N}$,

$$\begin{aligned} A(2,2k+2) &= \left[f'(x) \cdot x^{2k+2} \right]_0^a - 2kA(1,2k+1) - 2A(1,2k+1) \\ &= \left[f'(x) \cdot x^{2k+2} \right]_0^a + (2k)(2k+1)A(0,2k) - 2A(1,2k+1) \\ &= \left[f'(x) \cdot x^{2k+2} \right]_0^a + 3(3k-1)(3k-2)A(0,2k-2) - 2A(1,2k+1). \end{aligned}$$

Multiplying both sides by 4 and repeatedly applying Corollary 4.2.13, we get

$$\begin{aligned}
&4A(2,2k+2)\\
=&4\Big[f'(x)\cdot x^{2k+2}\Big]_0^a + 27(2k)(2k-1)A(0,2k-2) + (-54k+24)A(0,2k-2) - 8A(1,2k+1)\\
=&4\Big[f'(x)\cdot x^{2k+2}\Big]_0^a + 27\cdot\left(-\Big[f'(x)\cdot x^{2k}\Big]_0^a + A(2,2k)\right) - 27(2k-1)A(0,2k-2)\\
&-3A(0,2k-2) - 8A(1,2k+1)\\
=&\left[x^{2k}f'(x)\cdot\left(4x^2-27\right)\right]_0^a + 27A(2,2k) + 27A(1,2k-1) - 3A(0,2k-2) - 8A(1,2k+1)\\
=&27A(2,2k) + 27A(1,2k-1) - 3A(0,2k-2) - 8A(1,2k+1).
\end{aligned}$$

where the last inequality holds because $\lim_{x\to a^-}(4x^2-27)f'(x)=0$ and $\lim_{x\to 0^+}x^2f'(x)=0$ by assumption.

Writing the $A(j,k)$'s above as integrals, we get that for all $k\in\mathbb{N}$

$$\int_0^a \left(4f''(x)\cdot x^4 - 27f''(x)\cdot x^2 - 27f'(x)\cdot x + 8f'(x)\cdot x^3 + 3f(x)\right)\cdot x^{2k-2}\,\mathrm{dx} = 0.$$

One way for this equation to be true is if $(4x^4-27x^2)f''(x)+(8x^3-27x)f'(x)+3f(x)=0$ on $(0,a)$. As shown by the following lemma and corollary, this is the only way for this equation to be true for all $k\in\mathbb{Z}$, which completes the proof of Theorem 4.2.9.

Lemma 4.2.14. *Let a be some positive constant. If f is continuous on $[0,a]$ and $\int_0^a f(x)x^{2k}\,\mathrm{dx}=0$ for all nonnegative integers k, then $f=0$ on $(0,a)$.*

Proof. Let $M>0$ be an upper bound of f on $[0,a]$. For an arbitrary $\epsilon>0$, let $p(x)$ be a polynomial such that $\left|p(x)-f(\sqrt{x})\right| < \dfrac{\epsilon}{M\cdot a}$ for all $x\in(0,a^2)$. Taking $p_\epsilon(x)=p(x^2)$, p_ϵ is a linear combination of monomials of even power and $|p_\epsilon(x)-f(x)| < \dfrac{\epsilon}{M\cdot a}$ for all $x\in(0,a)$. Thus $\int_0^a (f(x)-p_\epsilon(x))\cdot f(x)\,\mathrm{dx} < \epsilon$. On the other hand, since all even moments of $f(x)$ are zero,

$$\int_0^a (f(x) - p_\epsilon(x)) \cdot f(x)\,\mathrm{dx} = \int_0^a f(x)^2 - p_\epsilon(x) f(x)\,\mathrm{dx} = \int_0^a f(x)^2\,\mathrm{dx}\,.$$

Thus $\int_0^a f(x)^2\,\mathrm{dx} < \epsilon$ for all $\epsilon > 0$ and we conclude that $f(x) = 0$ on $(0, a)$. □

Corollary 4.2.15. *Let a be some positive constant. If f is continuous on $(0, a)$, $\lim_{x\to 0^+} x^2 f(x) = 0$, $\lim_{x\to a^-} (a-x)^2 f(x) = 0$, and $\int_0^a f(x) x^{2k}\,\mathrm{dx} = 0$ for all nonnegative integers k, then $f = 0$ on $(0, a)$.*

Proof. This follows by applying Lemma 4.2.14 to the function $f(x)x^2(a^2 - x^2)^2$. □

□

We now confirm that if f satisfies the differential equation $(4x^4 - 27x^2)f''(x) + (8x^3 - 27x)f'(x) + 3f(x) = 0$, the conditions of Theorem 4.2.9, and the condition that $\int_0^a f(x)\,\mathrm{dx} = 1$, then $\int_0^a x^{2k} \cdot f(x)\,\mathrm{dx} = D_k$.

Theorem 4.2.16. *Let $a = \lim_{k\to\infty} D_k/D_{k-1} = 3\sqrt{3}/2$. Let f be a function satisfying the following ODE on $(0, a)$*

$$(4x^4 - 27x^2)f''(x) + (8x^3 - 27x)f'(x) + 3f(x) = 0 \tag{4.13}$$

and the first three conditions in Theorem 4.2.9, i.e.

1. *$f(x)$ is twice continuously differentiable on $(0, a)$.*
2. *$\lim_{x\to 0^+} x f(x) = 0$ and $\lim_{x\to 0^+} x^2 f'(x) = 0$.*
3. *$\lim_{x\to a^-} f(x) = 0$ and $\lim_{x\to a^-} f'(x)(4x^2 - 27) = \lim_{x\to a^-} 8af'(x)(x - a) = 0$.*

Moreover, assume that $\int_0^a f(x)\,\mathrm{dx} = 1$. Then for all $k \in \mathbb{N} \cup \{0\}$,

$$\int_0^a x^{2k} \cdot f(x)\,\mathrm{dx} = D_k\,. \tag{4.14}$$

Proof. Notice that $\int_0^a f(x)\,\mathrm{dx} = 1 = D_0$ by assumption. We aim to prove that for all $k \in \mathbb{N} \cup \{0\}$,

$$(2k+3)(2k+2)\int_0^a x^{2k+2} \cdot f(x)\,\mathrm{dx} = 3(3k+2)(3k+1)\int_0^a x^{2k} \cdot f(x)\,\mathrm{dx}\,.$$

If so, then since $(2k+3)(2k+2)D_{k+1} = 3(3k+2)(3k+1)D_k$, we can prove Theorem 4.2.16 by induction on k.

We multiply Equation (4.13) by x^{2k} and integrate from 0 to a:

$$\begin{aligned}
0 &= \int_0^a (4x^4 - 27x^2)f''(x) \cdot x^{2k} + (8x^3 - 27x)f'(x) \cdot x^{2k} + 3x^{2k} f(x)\,\mathrm{dx} \\
&= \left(\left[f'(x)(4x^4 - 27x^2)x^{2k}\right]_0^a - \int_0^a f'(x)\left(4(2k+4)x^{2k+3} - 27(2k+2)x^{2k+1}\right)\mathrm{dx}\right) \\
&\qquad\qquad + \int_0^a (8x^{2k+3} - 27x^{2k+1})f'(x)\,\mathrm{dx} + \int_0^a 3x^{2k} f(x)\,\mathrm{dx} \\
&= -\int_0^a \left(8(k+1)x^{2k+3} - 27(2k+1)x^{2k+1}\right) f'(x)\,\mathrm{dx} + \int_0^a 3x^{2k} f(x)\,\mathrm{dx} \\
&= -\left[f(x)\left(8(k+1)x^{2k+3} - 27(2k+1)x^{2k+1}\right)\right]_0^a \\
&\qquad\qquad + \int_0^a \left(8(k+1)(2k+3)x^{2k+2} - 27(2k+1)^2 x^{2k}\right) f(x)\,\mathrm{dx} + \int_0^a 3x^{2k} f(x)\,\mathrm{dx} \\
&= \int_0^a \left(8(k+1)(2k+3)x^{2k+2} - 3(36k^2 + 36k + 8)x^{2k}\right) f(x)\,\mathrm{dx} \\
&= \int_0^a \left(8(k+1)(2k+3)x^{2k+2} - 12(3k+1)(3k+2)x^{2k}\right) f(x)\,\mathrm{dx}
\end{aligned}$$

as $\left[f'(x)(4x^4 - 27x^2)x^{2k}\right]_0^a$ and $\left[f(x)\left(8(k+1)x^{2k+3} - 27(2k+1)x^{2k+1}\right)\right]_0^a$ are zero by the assumed conditions on f.

Rearranging the last step we get

$$(2k+2)(2k+3)\int_0^a f(x)x^{2k+2}\,\mathrm{dx} = 3(3k+1)(3k+2)\int_0^a f(x)x^{2k}\,\mathrm{dx}$$

as needed. □

Using WolframAlpha to solve the above ODE and analyzing the constant coefficient by the imposed initial conditions, we can get an explicit solution for $f(x)$. We verify the solution below.

Theorem 4.2.17. *Let $a = 3\sqrt{3}/2$ and $f(x)$ be such that*

$$f(x) = \frac{i}{\pi}\left(\sqrt{3}\sin\left(\frac{1}{3}\arctan\left(\frac{3}{\sqrt{4x^2/3-9}}\right)\right) + \cos\left(\frac{1}{3}\arctan\left(\frac{3}{\sqrt{4x^2/3-9}}\right)\right)\right) \tag{4.15}$$

for $0 < x < a$. Then $f(x)$ is an solution to the ODE (4.11) on $(0, a)$. Moreover, f satisfies the conditions listed in Theorem 4.2.16.

Proof. We first verify that this $f(x)$ satisfies the ODE (4.11)

$$(4x^4 - 27x^2)f''(x) + (8x^3 - 27x)f'(x) + 3f(x) = 0.$$

on $(0, a)$.

For simplicity, we will denote $g(x) = \frac{1}{3} \cdot \arctan\left(\frac{3}{\sqrt{4x^2/3-9}}\right)$. Then

1. $f(x) = \frac{i}{\pi}\left(\sqrt{3}\sin(g(x)) + \cos(g(x))\right)$.
2. $f'(x) = \frac{i}{\pi}\left(\sqrt{3}\cos(g(x)) - \sin(g(x))\right) \cdot g'(x) = \frac{i}{\pi}\left(\sqrt{3}\cos(g(x)) - \sin(g(x))\right)\frac{-1}{x\sqrt{4x^2/3-9}}$.
3. $f''(x) = \frac{i}{\pi}\left(-\sqrt{3}\sin g(x) - \cos g(x)\right) \cdot (g'(x))^2 + \frac{i}{\pi}\left(\sqrt{3}\cos g(x) - \sin g(x)\right) \cdot g''(x)$
$= \frac{-i}{\pi}\left(\sqrt{3}\sin g(x) + \cos g(x)\right) \cdot \frac{1}{x^2(4x^2/3-9)} + \frac{i}{\pi}\left(\sqrt{3}\cos g(x) - \sin g(x)\right) \cdot \frac{8x^2/3-9}{x^2(4x^2/3-9)^{3/2}}$.

Plugging the above into the LHS of (4.11), one can verify that $(4x^4 - 27)f''(x) + (8x^3 - 27x)f'(x) + 3f(x) = 0$.

Now we check the conditions listed in Theorem 4.2.16. For this purpose, it is more convenient to re-write $f(x)$ as a function all of real terms.

We will use the following facts:

1. $\arctan(ix) = \frac{i}{2}\ln\left(\frac{1+x}{1-x}\right)$.

2. $\sin(ix) = i\cdot\sinh(x) = \frac{i}{2}\cdot\left(e^x - e^{-x}\right)$, $\cos(ix) = \cosh(x) = \frac{e^x+e^{-x}}{2}$.

Recall that the domain for $f(x)$ is $0 < x \le 3\sqrt{3}/2$. Let $y = \frac{3}{\sqrt{-4x^2/3+9}}$. Note that y is a real variable and $y \ge 1$. Also note that $\frac{3}{\sqrt{4x^2/3-9}} = -iy$ and $g(x) = \frac{1}{3}\cdot\arctan(-iy) = \frac{i}{6}\ln\left(\frac{1-y}{1+y}\right)$.

Let $z = \frac{y-1}{y+1} = \frac{27-2x^2-9\sqrt{9-4x^2/3}}{2x^2}$. Note that z is a real variable and $z \ge 0$. Now observe that

1. $g(x) = \frac{i}{6}\ln(-z)$

2. $\sin(g(x)) = \frac{i}{2}\left((-z)^{1/6} - (-z)^{-1/6}\right) = \frac{i}{2}\left(\left(\frac{\sqrt{3}+i}{2}\right)z^{1/6} - \left(\frac{\sqrt{3}-i}{2}\right)z^{-1/6}\right)$

3. $\cos(g(x)) = \frac{1}{2}\left((-z)^{1/6} + (-z)^{-1/6}\right) = \frac{1}{2}\left(\left(\frac{\sqrt{3}+i}{2}\right)z^{1/6} + \left(\frac{\sqrt{3}-i}{2}\right)z^{-1/6}\right)$.

Plugging in the above equations to $f(x)$ and simplifying, we get that

$$f(x) = \frac{i}{\pi}\left(\sqrt{3}\sin(g(x) + \cos(g(x))\right) = \frac{-1}{\pi}\cdot\left(z^{1/6} - z^{-1/6}\right),$$
$$f'(x) = \frac{i}{\pi}\left(\sqrt{3}\cos(g(x) - \sin(g(x))\right)\cdot\frac{-1}{x\sqrt{4x^2/3-9}} = \frac{-1}{\pi}\cdot\left(z^{1/6} + z^{-1/6}\right)\cdot\frac{1}{x\sqrt{9-4x^2/3}}.$$

Recall that $y = \frac{3}{\sqrt{-4x^2/3+9}}$ and $z = \frac{y-1}{y+1}$. Observe that

1. As $x \to 0^+$, $y \approx 1 + \frac{2x^2}{27}$. Thus, $\lim_{x\to 0^+}\frac{z}{x^2} = \frac{1}{27}$.

2. As $x \to a^-$, $y \to \infty$. Thus, $\lim_{x\to a^-} z = 1$.

Thus,

1. f is twice differentiable on $(0, a)$.

2. $\lim_{x\to 0^+} xf(x) = \lim_{x\to 0} x \cdot \left(\frac{z^{-1/6}}{\pi}\right) = 0.$

3. $\lim_{x\to 0^+} x^2 f'(x) = \lim_{x\to 0} x \cdot \left(\frac{-z^{-1/6}}{3\pi}\right) = 0.$

4. $\lim_{x\to a^-} f(x) = \frac{-1}{\pi}(1-1) = 0.$

5. $\lim_{x\to a^-}(4x^2-27)f'(x) = \lim_{x\to a^-} \frac{1}{\pi}(z^{1/6}+z^{-1/6}) \cdot \left(\frac{\sqrt{3(27-4x^2)}}{x}\right) = 0.$

Now we will prove the last piece of this Theorem: $\int_0^a f(x)\,\mathrm{dx} = 1.$

We have that $a = \frac{3\sqrt{3}}{2}$, $z = \frac{y-1}{y+1} = \frac{27-2x^2-9\left(9-4x^2/3\right)^{1/2}}{2x^2} = \frac{27-27\left(1-x^2/a^2\right)^{1/2}}{2x^2} - 1$, and $f(x) = -\frac{1}{\pi}\cdot\left(z^{1/6}-z^{-1/6}\right)$.

Let $x = a\sin\theta$. Then $z = \frac{27-27\cos\theta}{2a^2\sin^2\theta} - 1 = \frac{2(1-\cos\theta)}{\sin^2\theta} - 1 = \frac{1-\cos\theta}{1+\cos\theta}$. Expressing $\cos\theta$ in terms of z, we get $\cos\theta = \frac{1-z}{1+z}$, thus $\sin\theta = \frac{2\sqrt{z}}{1+z}$. Moreover,

$$\mathrm{dz} = \left(\frac{1-\cos\theta}{1+\cos\theta}\right)' d\theta = \frac{2\sin\theta}{(1+\cos\theta)^2}d\theta = \frac{2\sin\theta(1-\cos\theta)}{\sin^2\theta(1+\cos\theta)} = \frac{2z}{\sin\theta}d\theta \implies d\theta = \frac{\sqrt{z}}{z(1+z)}\,\mathrm{dz}\,.$$

Thus

$$\begin{aligned}\int_0^a f(x)\,\mathrm{dx} &= \frac{-1}{\pi}\int_0^{\pi/2}\left(\left(\frac{1-\cos\theta}{1+\cos\theta}\right)^{1/6} - \left(\frac{1-\cos\theta}{1+\cos\theta}\right)^{-1/6}\right) a\cos\theta d\theta \\ &= \frac{-a}{\pi}\int_0^1 \left(z^{1/6}-z^{-1/6}\right)\cdot\left(\frac{1-z}{1+z}\right)\cdot\frac{\sqrt{z}}{z(1+z)}\,\mathrm{dz} \\ &= \frac{-a}{\pi}\int_0^1 \frac{(1-z)(z^{2/3}-z^{1/3})}{z(1+z)^2}\,\mathrm{dz}\,.\end{aligned}$$

Let $w = z^3$, then

$$\int_0^a f(x)\,\mathrm{dx} = -\frac{a}{\pi}\int_0^1 \frac{(1-w^3)(w-1)}{(1+w^3)^2}\,\mathrm{dw}$$
$$= -\frac{a}{\pi}\int_0^1 \frac{-4/3}{(1+w)^2} + \frac{2w}{(w^2-w+1)^2} + \frac{-5/3}{w^2-w+1}\,\mathrm{dw}$$
$$= -\frac{a}{\pi}\left(\frac{4}{3}\left[\frac{1}{1+w}\right]_0^1 + \int_0^1 \frac{2w-1}{(w^2-w+1)^2}\,\mathrm{dw} + \int_0^1 \frac{1}{(w^2-w+1)^2}\,\mathrm{dw} + \int_0^1 \frac{-5/3}{w^2-w+1}\,\mathrm{dw}\right)$$
$$= -\frac{a}{\pi}\left(-\frac{2}{3} + \left[\frac{-1}{(w^2-w+1)}\right]_0^1 + \int_0^1 \frac{1}{(w^2-w+1)^2}\,\mathrm{dw} + \int_0^1 \frac{-5/3}{w^2-w+1}\,\mathrm{dw}\right)$$
$$= -\frac{a}{\pi}\left(-\frac{2}{3} + \int_0^1 \frac{1}{\left((w-\frac{1}{2})^2+\frac{3}{4}\right)^2}\,\mathrm{dw} + \int_0^1 \frac{-5/3}{w^2-w+1}\,\mathrm{dw}\right).$$

Lemma 4.2.18. *For any $b \neq 0$,*

$$\int \frac{1}{(x^2+b^2)^2}\,\mathrm{dx} = \frac{1}{2b^2}\left(\int \frac{1}{x^2+b^2}\,\mathrm{dx} - \frac{x}{x^2+b^2}\right). \tag{4.16}$$

Proof.

$$\int \frac{1}{(x^2+b^2)^2}\,\mathrm{dx} = \frac{1}{b^2}\int \frac{x^2+b^2}{(x^2+b^2)^2} + \frac{-x^2}{x^2+b^2}\,\mathrm{dx}$$
$$= \frac{1}{b^2}\left(\int \frac{1}{x^2+b^2}\,\mathrm{dx} + \int \frac{-x}{2}d\left(\frac{1}{x^2+b^2}\right)\right)$$
$$= \frac{1}{b^2}\left(\int \frac{1}{x^2+b^2}\,\mathrm{dx} - \frac{x}{2(x^2+b^2)} + \int \frac{-\frac{1}{2}}{x^2+b^2}\,\mathrm{dx}\right)$$
$$= \frac{1}{2b^2}\left(\int \frac{1}{x^2+b^2}\,\mathrm{dx} - \frac{x}{x^2+b^2}\right).$$

□

Apply the lemma to the $\int_0^1 \frac{1}{\left((w-1/2)^2+3/4\right)^2}\,\mathrm{dw}$ term, we get that

$$\begin{aligned}\int_0^a f(x)\,\mathrm{dx} = &-\frac{a}{\pi}\left(-\frac{2}{3}+\frac{2}{3}\left(\int_0^1 \frac{1}{(w-1/2)^2+3/4}\,\mathrm{dw} - \left[\frac{w-1/2}{w^2-w+1}\right]_0^1\right)\right. \\ &\left.+\int_0^1 \frac{-5/3}{(w-1/2)^2+3/4}\,\mathrm{dw}\right) \\ = &-\frac{a}{\pi}\left(-\frac{2}{3}+\frac{2}{3}+\int_0^1 \frac{-1}{(w-1/2)^2+3/4}\,\mathrm{dw}\right) \\ = &\frac{a}{\pi}\left[\frac{1}{\sqrt{3}/2}\arctan\left(\frac{w-1/2}{\sqrt{3}/2}\right)\right]_0^1 = \frac{3\sqrt{3}/2}{\pi}\cdot\frac{2\pi}{3\sqrt{3}} = 1.\end{aligned}$$

□

□

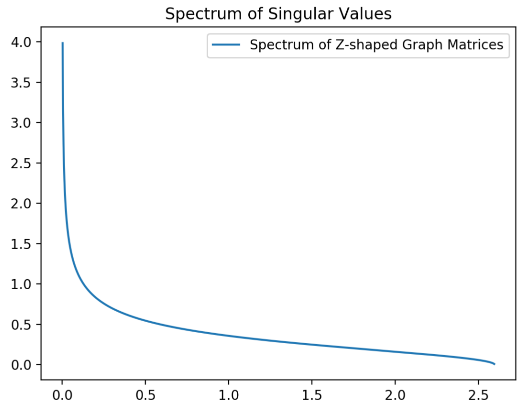
Figure 4.14 shows some graphs of $g_{\alpha_Z}(x)$ and samplings of singular values of M_{α_Z} for $n = 20, 30, 70$. We can see that the sampled distribution of the singular values of M_{α_Z} gets closer to $g_{\alpha_Z}(x)$ as n gets bigger.

4.2.3 Behavior near $x = 0$ and $x = a$ and numerically solving the differential equation

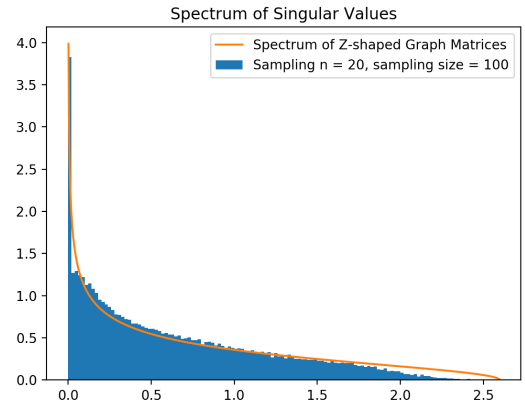
We now consider the behavior of the differential equation $(4x^4-27)f''(x)+(8x^3-27x)f'(x)+3f(x) = 0$ near $x = 0$ and near $x = a$. While this kind of analysis is not necessary for this differential equation as we were able to find an explicit solution, this kind of analysis is very useful for differential equations where we cannot find an explicit solution.

When x is very close to 0, the differential equation is approximately

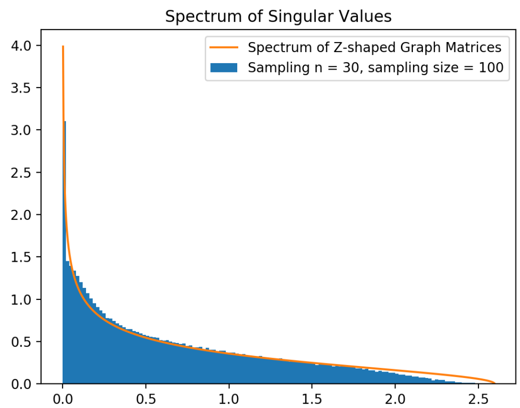
$$-27x^2 f''(x) - 27xf'(x) + 3f(x) \sim 0.$$



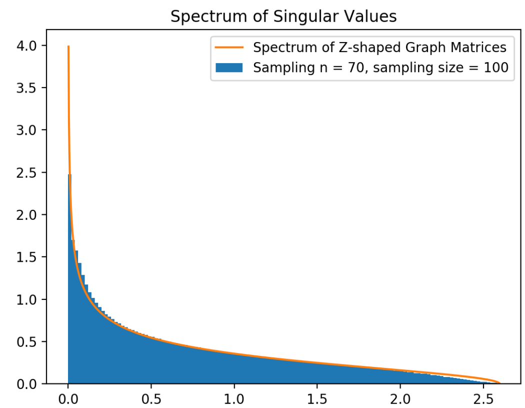
(a) The spectrum of singular values



(b) Sampling of singular values of M_{α_Z} where $n = 20$



(c) Sampling of singular values of M_{α_Z} where $n = 30$



(d) Sampling of singular values of M_{α_Z} where $n = 70$

Figure 4.14: The Spectrum of singular values of the Z-shape graph matrix and some samplings of the Z-shape graph matrices with random input graphs on n vertices, for $n = 20, 30, 70$.

Plugging in $f(x) = x^r$, we obtain that

$$(-27r(r-1) - 27r + 3)\, x^r = \left(-27r^2 + 3\right) x^r = 0$$

which is satisfied when $r = \pm 1/3$. Thus, near $x = 0$ the solution to the differential equation is approximately $c_1 x^{1/3} + c_2 x^{-1/3}$.

When x is very close to a, we observe that

1. $4x^4 - 27x^2 = 4x^2(x-a)(x+a) \approx 8a^3(x-a)$,

2. $8x^3 - 27x = 4x^3 + 4x^2(x-a)(x+a) \approx 4a^3$,

If we further assume that $\lim_{x\to a} f(x) = 0$ then when x is very close to a, the differential equation is approximately

$$-8a^3(a-x)f''(x) + 4a^3 f'(x) = 0.$$

Plugging in $f'(x) = (a-x)^r$, we obtain that

$$8ra^3(a-x)^r + 4a^3(a-x)^r = (8r+4)a^3(a-x)^r = 0$$

which is satisfied when $r = -\frac{1}{2}$. Thus, the solution to the differential equation near $x = a$ which is 0 at $x = a$ is approximately $c\sqrt{a-x}$.

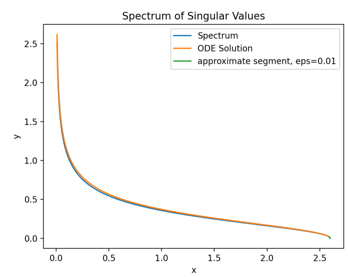
This analysis helps us solve this differential equation numerically. To solve this differential equation numerically, we need an initial point x_0 and the initial conditions $f(x_0)$ and $f'(x_0)$. However, we can't use $x_0 = 0$ because $\lim_{x\to 0^+} f(x) = \infty$ and we can't use $x_0 = a$ because $\lim_{x\to a^-} f'(x) = -\infty$. Instead, we proceed as follows:

1. Choose an $\epsilon > 0$ and approximate the solution by $\sqrt{a-x}$ on the interval $[a-\epsilon, a]$.

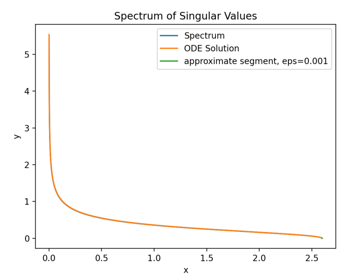
2. Numerically solve the differential equation on the interval $(0, a-\epsilon)$.

3. Scale the resulting function f so that $\int_{x=0}^{a} f(x)dx = 1$.

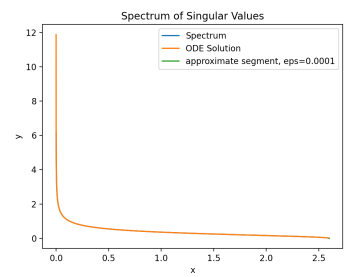
Figure 4.15 shows several plots of the explicit solution we get in Theorem 4.2.17 with the numerical solution we get for various ϵ's. One can see that as ϵ gets smaller, the approximated spectrum gets closer to the actual spectrum. When $\epsilon = 0.0001$, the two curves are almost identical.



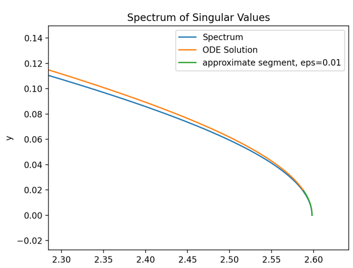
(a) $\epsilon = 0.01$



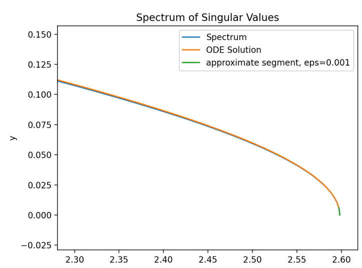
(b) $\epsilon = 0.001$



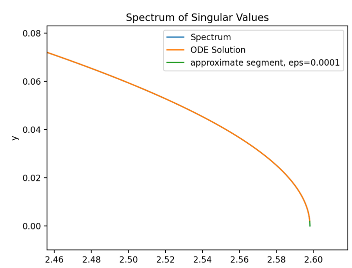
(c) $\epsilon = 0.0001$



(d) $\epsilon = 0.01$ tail zoom in



(e) $\epsilon = 0.001$ tail zoom in



(f) $\epsilon = 0.0001$ tail zoom in

Figure 4.15: Explicit Solution of the Spectrum of the singular values (blue curve) and the numerical ODE solution (orange curve) with approximated tail segment (green curve) with different approximating intervals (different ϵ's).

CHAPTER 5

THE MULTI-Z-SHAPE GRAPH MATRIX

5.1 The Trace Power of Multi-Z-shaped Graph Matrices

Now we consider a generalization of the Z-shape graph matrix discussed in Chapter 4.

Remark 5.1.1. *For the $m-$layer Z-shape $\alpha_{Z(m)}$, the size of the minimum separator is m. By Lemma 3.2.24, for any nonzero-valued constraint graph $C \in \mathcal{C}_{\left(\alpha_{Z(m)}, 2q\right)}$, $|E(C)| \geq m \cdot (q-1)$. By Corollary 3.2.26, dominant constraint graphs $C \in \mathcal{C}_{\left(\alpha_{Z(m)}, 2q\right)}$ have $m \cdot (q-1)$ edges.*

Definition 5.1.2. For m, n positive integers,

$$D(m,n) = \frac{1}{m \cdot n + 1} \binom{(m+1) \cdot n}{n}. \tag{5.1}$$

Remark 5.1.3. *The number $D_n = \frac{1}{2n+1}\binom{3n}{n}$ in Chapter 4 is $D(2,n)$ here. Also $D(m-1,n) = A_n(m,1)$ where the generalized Catalan number $A_n(k,r) = \frac{r}{nk+r}\binom{nk+r}{n}$ is defined in Remark 4.1.3.*

Below is the main result of this section.

Theorem 5.1.4. *Let $\alpha_{Z(m)}$ be the $m-$layer Z-shape as in Definition 2.2.17. Then the number of dominant constraint graphs $C \in \mathcal{C}_{\left(\alpha_{Z(m)}, 2q\right)}$ is $D(m,q)$.*

Remark 5.1.5. *When $m = 2$, $D(m,n) = D(2,n) = D_n$, $\alpha_{Z(m)} = \alpha_Z$, and this theorem is exactly Theorem 4.1.4.*

A direct corollary we get from the above theorem is the following.

Corollary 5.1.6. *Let $\alpha_{Z(m)}$ be the m-layer Z-shape as in Definition 2.2.17. Let $M_{n,m} = \frac{1}{n^{m/2}} M_{\alpha_{Z(m)}}(G)$ be the graph matrix where $G \sim G(n,1/2)$ and $r(n,m) = \frac{n!}{(n-m)!}$ be the* *dimension of* $M_{n,m}$. *Recall that* $D(m,q) = \dfrac{1}{mq+1}\dbinom{(m+1)q}{q}$. *Then*

$$\lim_{n\to\infty} \frac{1}{r(n,m)} \mathbb{E}\left[\mathrm{tr}\left(\left(M_{n,m}M_{n,m}^T\right)^q\right)\right] = D(m,q). \tag{5.2}$$

Proof. Recall that Corollary 3.2.27 says that for any bipartite shape α,

$$\lim_{n\to\infty} \frac{1}{r_{approx}(n)} \mathbb{E}\left[\mathrm{tr}\left(\left(\frac{M_\alpha M_\alpha^T}{n^{|V(\alpha)|-s_\alpha}}\right)^q\right)\right] = \left|\left\{C \in \mathcal{C}_{(\alpha,2q)} : C \text{ is dominant}\right\}\right|.$$

Since $s_{\alpha_{Z(m)}} = m$ and $\left|V(\alpha_{Z(m)})\right| = 2m$, $r_{approx}(n) = \dfrac{n!}{\left(n - s_{\alpha_{Z(m)}}\right)!} = \dfrac{n!}{(n-m)!} =$ $r(n,m)$ and $\dfrac{M_{\alpha_{Z(m)}}M_{\alpha_{Z(m)}}^T}{n^{\left|V(\alpha_{Z(m)})\right|-s_{\alpha_{Z(m)}}}} = \dfrac{M_{\alpha_{Z(m)}}M_{\alpha_{Z(m)}}^T}{n^m} = M_{n,m}M_{n,m}^T$.

By Theorem 5.1.4, $\left|\left\{C \in \mathcal{C}_{\left(\alpha_{Z(m)},2q\right)} : C \text{ is dominant}\right\}\right| = D(m,q)$ and the result follows.

□

5.1.1 Recurrence Relation for $D(m,n)$

To prove the main result for this section, We need the following crucial recurrence relation for $D(m,n)$.

Theorem 5.1.7.

$$D(m,n+1) = \sum_{\substack{i_0,\ldots,i_m \geq 0: \\ i_0+\cdots+i_m=n}} D(m,i_0)\ldots D(m,i_m)\,. \tag{5.3}$$

Proof. The proof is similar to the proof of Theorem 4.1.6. Let $W_{m,n} :=$ *the set of all grid walks from* $(0,0)$ *to* (n,mn) *that are weakly below the diagonal* and $d_{m,n} = \left|W_{m,n}\right|$. We will prove that $d_{m,n} = D(m,n)$ and that $d_{m,n}$ satisfies the recurrence relation in the theorem.

1. $d_{m,n} = D(m,n)$:

   For $r \in \{0, 1, \ldots, mn\}$, let $W_{m,n}(r) :=$ *the set of grid walks from* $(0,0)$ *to* (n, mn) *that are* r *steps above the diagonal.* Then $\bigcup_{r=0}^{mn} W_{m,n}(r)$ is the set of all grid walks from $(0,0)$ to (n, mn), which has cardinality $\binom{(m+1)n}{n}$. Also $|W_{m,n}(0)| = |W_{m,n}| = d_{m,n}$. By the same proof as in the Theorem 4.1.6, there is a bijection between $W_{m,n}(r)$ and $W_{m,n}(r-1)$ for each $r \in [mn]$. Thus $d_{m,n} = \frac{1}{mn+1}\binom{(m+1) \cdot n}{n}$.

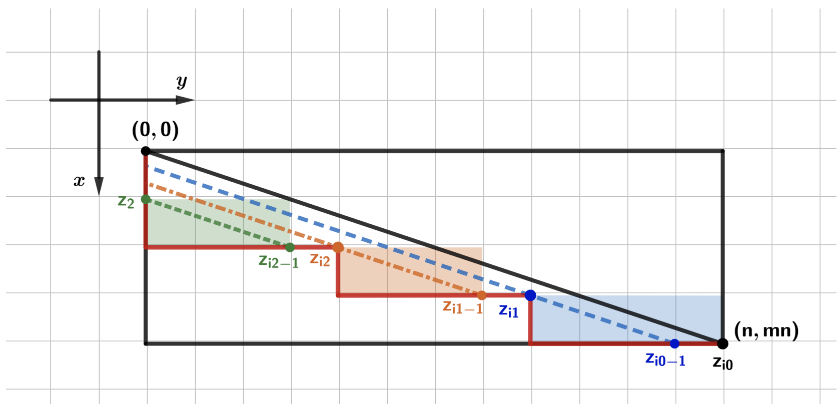


Figure 5.1: Illustration of the proof part 2 for Theorem 5.1.7.

2. $d_{m,n} = \sum_{\substack{i_0,\ldots,i_{m+1} \geq 0: \\ i_0+\cdots+i_{m+1}=n-1}} d_{m,i_1} \ldots d_{m,i_{m+1}}$:

   Similar to the proof of Theorem 4.1.6, we now will find a bijection between $W_{m,n}$ and

$$\bigcup_{\substack{i_0,\ldots,i_m \geq 0: \\ i_0+\cdots+i_m=n-1}} W_{m,i_0} \times \cdots \times W_{m,i_m}.$$

   Let d_k be the line that is shifted k vertical grids down from the diagonal. i.e. d_k is the line $y = m \cdot x - k$. Let $w = \left(z_1, \ldots, z_{n\cdot(m+1)}\right) \in W_{m,n}$.

   i. Let $z_{i_0} = (a_0, m \cdot a_0)$ be the first point where w touches the diagonal. Then $w_0 := \left(z_{i_0}, \ldots, z_{n\cdot(m+1)}\right)$ can be viewed as an element in $W_{m,n-a_0}$. Moreover, $w' := \left(z_2, \ldots, z_{i_0-1}\right)$ is strictly below the diagonal d_0, thus weakly below d_1.

ii. Let $z_{i_1} = (a_1, m \cdot a_1 - 1)$ be the first point where w' touches d_1. Then $w_1 := (z_{i_1}, \ldots, z_{i_0 - 1})$ can be viewed as an element in $W_{m, a_0 - a_1}$. Moreover, $w' := (z_2, \ldots, z_{i_1 - 1})$ is strictly below d_1, thus weakly below d_2.

iii. Continuing this way, we get a sequence of points $z_{i_0}, \ldots, z_{i_{m-1}}$ and walks $w_0, \ldots, w_{m-1}$ where each $z_{i_j} = (a_j, m \cdot a_j - j)$ is a point on d_j and each w_i can be viewed as an element in $W_{m, a_{i-1} - a_i}$.

iv. Since $z_{i_{m-1}}$ is the first point touching d_{m-1}, $w_m = (z_2, \ldots, z_{i_{m-1}-1})$ is strictly below d_{m-1} and thus weakly below d_m. Since d_m crosses $(1,0) = z_2$, w_m can be viewed as an element in $W_{m, a_{m-1} - 1}$.

Since $(n - a_0) + (a_0 - a_1) + \cdots + (a_{m-1} - 1) = n - 1$, we conclude that

$$(w_0, \ldots, w_m) \in \bigcup_{\substack{i_0, \ldots, i_m \geq 0: \\ i_0 + \cdots + i_m = n-1}} W_{m, i_0} \times \cdots \times W_{m, i_m}.$$

The other direction of the bijection can be constructed in a backward manner. It is not hard to prove this construction gives a bijection.

Combining 1 and 2, we conclude that $D(m, n)$ satisfies the recurrence relation.

□

5.1.2 *Properties of Dominant Constraint Graphs on* $H\left(\alpha_{Z(m)}, 2q\right)$

Definition 5.1.8. Let $\alpha_{Z(m)}$ be the multi-layer Z-shape as in Definition 2.2.17 and let $H\left(\alpha_{Z(m)}, 2q\right)$ be the multi-graph as in Definition 3.2.1. We label the vertices of $V_{(\alpha_Z)_i}$ as $\{a_{ij}, b_{ij} : j \in [m]\}$ and the vertices of $V^T_{(\alpha_Z)_i}$ as $\{b_{ij}, a_{(i+1)j}\}$. Let $V_i = \{a_{ij}, b_{ij} : i \in [q]\}$. For $j \in [m]$, we call the induced subgraph of $H\left(\alpha_{Z(m)}, 2q\right)$ on vertices V_i the j^{th} *wheel* W_j.

We label the "middle edges" of $H(\alpha, 2q)$ in the following way: let $e_{2i-1,j} = \{a_{i(j+1)}, b_{ij}\}$ and $e_{2i,j} = \{b_{ij}, a_{(i+1)(j+1)}\}$ for $i = 1, \ldots, q$. For a fixed $j \in [m]$, we call the edges $e_{i,j}$'s the *spokes between wheels* W_j *and* W_{j+1} of $H\left(\alpha_{Z(m)}, 2q\right)$. See Figure 5.2 for an illustration.

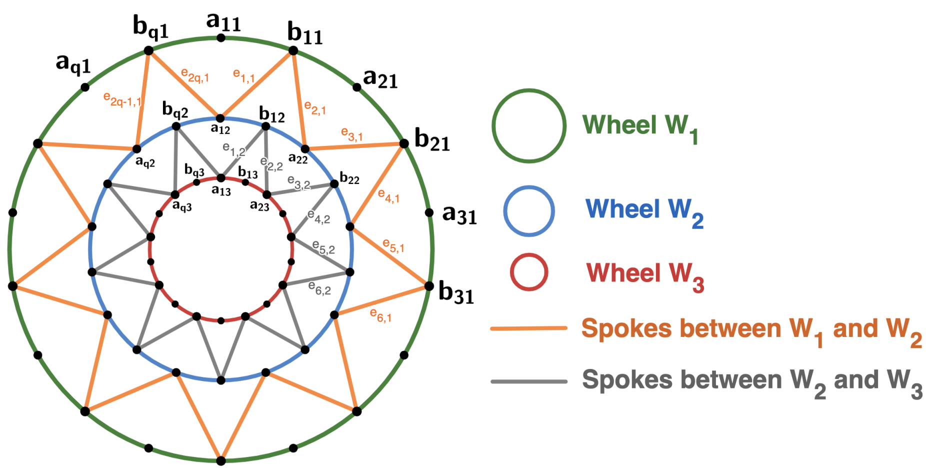


Figure 5.2: $H\left(\alpha_{Z(m)}, 2q\right)$, here $m = 3$

Definition 5.1.9. Let $\alpha_{Z(m)}$ be the multi-layer Z-shape. Let C be a constraint graph on $H(\alpha_Z, 2q)$. For $i = 1, 2$, We denote C_i the induced subgraph of C on vertices V_i. We call C_i the *induced constraint graph* of C on V_i.

Recall that a constraint graph $C \in \mathcal{C}_{(\alpha,2q)}$ is *well-behaved* if whenever $u \longleftrightarrow v$ in C, u and v are copies of the same vertex in α or α^T.

Theorem 5.1.10. *All dominant constraint graphs in* $\mathcal{C}_{\left(\alpha_{Z(m)},2q\right)}$ *are well-behaved.*

Proof. See appendix. □

We extend our definitions of wheel-respecting, parity-preserving, and non-crossing to $\mathcal{C}_{\left(\alpha_{Z(m)},2q\right)}$.

Definition 5.1.11. Let C be a constraint graph in $\mathcal{C}_{\left(\alpha_{Z(m)},2q\right)}$.

1. We say C *wheel-respecting* if for all $u \longleftrightarrow v$, $u, v \in V_i$ for some $i \in [m]$. (i.e. no two vertices on different wheels are constrained together by C).

2. We say C is *parity-preserving* if for each $i \in [m]$, the induced constraint graphs C_i of C on V_i is parity-preserving.

3. We say C is *non-crossing* if the induced constraint graphs C_i's are non-crossing.

Remark 5.1.12. *If $C \in \mathcal{C}_{\left(\alpha_{Z(m)},2q\right)}$ is wheel-respecting and G_i are representatives of C_i, then the graph G with $V(G) = V(\alpha_{Z(m)}, 2q)$ and $E(G) = E(G_1) \sqcup \cdots \sqcup E(G_m)$ is a representative of C.*

Proposition 5.1.13. *Let C be a constraint graph in $\mathcal{C}_{\left(\alpha_{Z(m)},2q\right)}$. C is well-behaved if and only if C is wheel-respecting and parity-preserving.*

Corollary 5.1.14. *If C is a dominant constraint graph in $\mathcal{C}_{\left(\alpha_{Z(m)},2q\right)}$, then*

1. *C is wheel-respecting, parity-preserving and non-crossing.*

2. *The induced constraint graphs C_i on wheels W_i are dominant constraint graphs in $\mathcal{C}_{(\alpha_0,2q)}$.*

The same proofs for Lemma 4.1.29 yields the following statement.

Lemma 5.1.15. *Let $\alpha_{Z(m)}$ be as in Definition 2.2.17 and let C be a dominant constraint graph in $\mathcal{C}_{\left(\alpha_{Z(m)},2q\right)}$. If $a_{sj} \longleftrightarrow a_{tj}$ for some $1 \le s < t \le q$ and $j \in [m-1]$, then $a_{s(j+1)} \longleftrightarrow a_{t(j+1)}$. Moreover, the spokes $\left\{e_{x,j} : x \in [2s-1, 2t-2]\right\}$ can only be made equal to each other.*

Similarly, if $b_{sj} \longleftrightarrow b_{tj}$ for some $1 \le s < t \le q$ and $j \in \{2, 3, \dots, m\}$, then $b_{s(j-1)} \longleftrightarrow b_{t(j-1)}$. Moreover, the spokes $\left\{e_{x,j-1} : x \in [2s, 2t-1]\right\}$ can only be made equal to each other.

Corollary 5.1.16. *If C is a dominant constraint graph in $\mathcal{C}_{\left(\alpha_{Z(m)},2q\right)}$, then the following statements are true:*

1. *If $a_{i1} \longleftrightarrow a_{j1}$ for some $1 \le i < j \le q$, then $a_{ik} \longleftrightarrow a_{jk}$ for all $k \in [m]$.*

2. *Similarly, if $b_{im} \longleftrightarrow a_{jm}$ for some $1 \le i < j \le q$, then $b_{ik} \longleftrightarrow b_{jk}$ for all $k \in [m]$.*

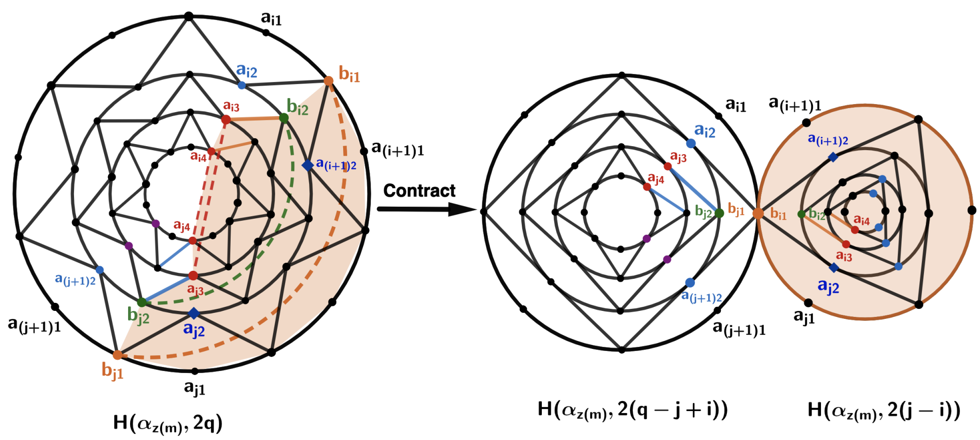


Figure 5.3: Illustration of Corollary 5.1.16: $C \in \mathcal{C}_{\left(\alpha_{Z(m)},2q\right)}$ is dominant. $a_{ik} \longleftrightarrow a_{jk}$ and $b_{i(k-1)} \longleftrightarrow b_{j(k-1)}$. Here $q = 7$, $m = 4$, $k = 3$ and $j - i = 3$.

3. *More generally, if $a_{ik} \longleftrightarrow a_{jk}$ and $b_{i(k-1)} \longleftrightarrow b_{j(k-1)}$ for some $k \in [m+1]$, then $a_{is} \longleftrightarrow a_{js}$ for all $k \leq s \leq m$ and $b_{it} \longleftrightarrow b_{jt}$ for all $1 \leq t \leq k-1$. Note that $k = 1$ corresponds to case 1 above and $k = m+1$ corresponds to case 2 above. See Figure 5.3 for an illustration.*

In all three cases, contracting the constrained vertices splits $H\left(\alpha_{Z(m)}, 2q\right)$ into $H\left(\alpha_{Z(m)}, 2(j-i)\right)$ and $H\left(\alpha_{Z(m)}, 2(q-j+i)\right)$, and the induced constraint graphs $C' \in \mathcal{C}_{\left(\alpha_{Z(m)},2(j-i)\right)}$ and $C'' \in \mathcal{C}_{\left(\alpha_{Z(m)},2(q-j+i)\right)}$ are dominant.

Proof sketch. We focus on the third case as this is the trickiest case. To split $H\left(\alpha_{Z(m)}, 2q\right)$ into $H\left(\alpha_{Z(m)}, 2(j-i)\right)$ and $H\left(\alpha_{Z(m)}, 2(q-j+i)\right)$, imagine doing the following:

1. Cut towards the center of the wheels through the vertices $b_{i1}, b_{i2}, \ldots, b_{i(k-1)}$, then cut along the spoke $\{a_{ik}, b_{i(k-1)}\}$, and then cut towards the center of the wheels through the vertices $a_{ik}, \ldots a_{im}$.

2. Similarly, cut towards the center of the wheels through the vertices $b_{j1}, b_{j2}, \ldots, b_{j(k-1)}$, then cut along the spoke $\{a_{jk}, b_{j(k-1)}\}$, and then cut towards the center of the wheels through the vertices $a_{jk}, \ldots a_{jm}$.

These cuts split $H\left(\alpha_{Z(m)}, 2q\right)$ into two halves. Taking each half and gluing it to itself along the cuts gives us $H\left(\alpha_{Z(m)}, 2(j-i)\right)$ and $H\left(\alpha_{Z(m)}, 2(q-j+i)\right)$.

To see that these the induced constraint graphs C' and C'' are dominant, observe that except for the vertices along the cuts, each vertex appears in one half or the other but not both. Except for the spokes $\{a_{ik}, b_{i(k-1)}\}$ and $\{a_{jk}, b_{j(k-1)}\}$, each edge is incident to a vertex which is not part of the cut, so this implies that all of the edges in $H\left(\alpha_{Z(m)}, 2q\right)$ except for the spokes $\{a_{ik}, b_{i(k-1)}\}$ and $\{a_{jk}, b_{j(k-1)}\}$ (which are made equal to each other) appear in one half or the other but not both.

By Corollary 4.1.16, edges on wheels on one side of the split can only be made equal to the edges on the same side. By Lemma 4.1.29, $b_{ir} \longleftrightarrow b_{jr}$ implies that spokes between wheels W_r and W_{r-1} on one side of the split can only be made equal to the spokes on the same side. By the same lemma, $a_{ir} \longleftrightarrow a_{jr}$ implies that spokes between wheels W_r and W_{r+1} on one side of the split can only be made equal to the spokes on the same side. By the assumptions on the constrained vertices, we conclude that other than the spokes between wheels W_k and W_{k-1}, all other spokes are made equal to the spokes on its own side after splitting.

Thus so far, in both $H\left(\alpha_{Z(m)}, 2(j-i)\right)/C'$ and $H\left(\alpha_{Z(m)}, 2(q-j+i)\right)/C''$, each edge except for the spokes between W_k and W_{k-1} appears an even number of times.

For the spokes between W_k and W_{k-1}, assume there is a spoke e_s on one side of the split such that $e_s \not\longleftrightarrow \{a_{ik}, b_{i(k-1)}\}$ and $e_s \longleftrightarrow e_t$ for some e_t on the other side of the split. Then by Lemma 4.1.27, $e_s \longleftrightarrow e_t \longleftrightarrow e_{2i-1,k-1} \longleftrightarrow e_{2j-1,k-1}$, a contradiction. Thus spokes on one side that are not made equal to $\{a_{ik}, b_{i(k-1)}\}$ or $\{a_{jk}, b_{j(k-1)}\}$ are only made equal to other spokes on the same side. Since the total number of spokes on each side is even, $\{a_{ik}, b_{i(k-1)}\}$ needs to appear even number of times in $H\left(\alpha_{Z(m)}, 2(j-i)\right)/C'$ and $\{a_{jk}, b_{j(k-1)}\}$ needs to appear even number of times in $H\left(\alpha_{Z(m)}, 2(q-j+i)\right)/C''$. Thus in both $H\left(\alpha_{Z(m)}, 2(j-i)\right)/C'$ and $H\left(\alpha_{Z(m)}, 2(q-j+i)\right)/C''$, each spoke between W_k

and W_{k-1} appears an even number of times.

This implies that C' and C'' are nonzero-valued constraint graphs on $H\left(\alpha_{Z(m)}, 2(j-i)\right)$ and $H\left(\alpha_{Z(m)}, 2(q-j+i)\right)$ and are thus dominant, as needed. □

5.1.3 Proof of Theorem 5.1.4

Now we are ready to prove the main result of this section, Theorem 5.1.4. The proof will be similar to the one for the Z-shape case. For the Z-shape, we split $H(\alpha_Z, 2q)$ two times to get the recurrence relation, where first step is based on the constrained vertices on the inner wheel, and the second step is based on the constrained vertices on the outer wheel. For the multi-Z-shape with m layers, we will split $H\left(\alpha_{Z(m)}, 2q\right)$ m times for the recurrence relation, where the r^{th} step will be based on the constrained vertices on the $(m-r+1)^{th}$ wheel, starting from the inner-most wheel.

Definition 5.1.17. For $r \in [m]$, let $\mathcal{D}_{m,q,r}$ denote the set of all dominant constraint graphs in $\mathcal{C}_{\left(\alpha_{Z(m)},2q\right)}$ such that $a_{1m} \longleftrightarrow a_{2m}, a_{1(m-1)} \longleftrightarrow a_{2(m-1)}, \ldots, a_{1(m-r+1)} \longleftrightarrow a_{2(m-r+1)}$ in C. When $r = 0$, define $\mathcal{D}_{m,q,0} =$ the set of all dominant constraint graphs in $\mathcal{C}_{\left(\alpha_{Z(m)},2q\right)}$. Define $D(m,q,r) = \left|\mathcal{D}_{m,q,r}\right|$ for all m, q, r.

We first consider the first step of the splitting.

Lemma 5.1.18. *Let $\mathcal{D}_{m,q,r}$ be as defined in Definition 5.1.17. There is a bijection between $\mathcal{D}_{m,q,0}$ and $\bigcup_{i=0}^{q-1} \mathcal{D}_{m,i,0} \times \mathcal{D}_{m,q-i,1}$. Thus*

$$D(m,q,0) = \sum_{i=0}^{q-1} D(m,i,0) \cdot D(m,q-i,1). \tag{5.4}$$

Proof. Let $C \in \mathcal{D}_{m,q,0}$. i.e., C is a dominant constraint graph on $H\left(\alpha_{Z(m)}, 2q\right)$. Let $i > 0$ be the smallest index such that a_{1m} is constrained to $a_{(i+1)m}$. If a_{1m} is isolated then we take $i = q$.

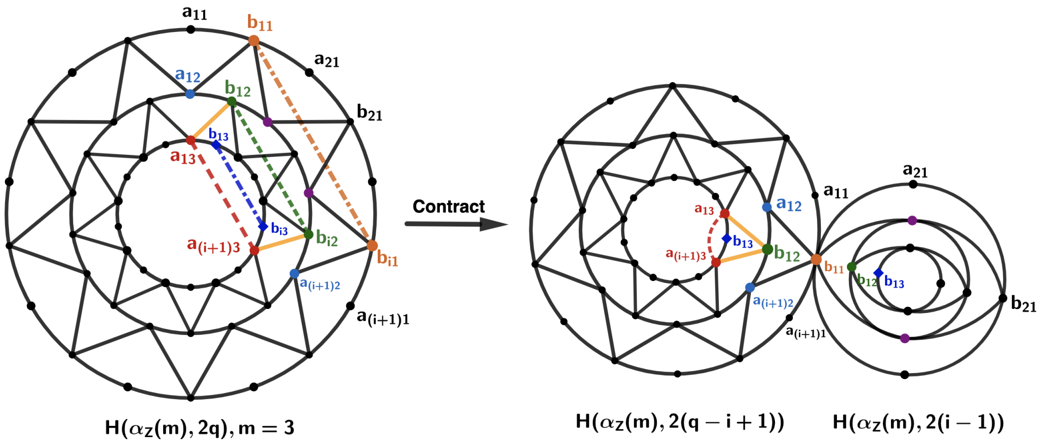


Figure 5.4: Illustration of Lemma 5.1.18: $a_{(i+1)m}$ is the first vertex that a_{1m} is constrained to. Constrained vertices are indicated by dash lines. Here $m = 3$.

Since C is dominant, by Corollary 5.1.14, the induced constraint graph C_m on W_m is dominant. By Lemma 4.1.17 and Lemma 4.1.13 (in the case when a_{1m} is isolated), $b_{1m} \longleftrightarrow b_{im}$. By Corollary 5.1.16, $H\left(\alpha_{Z(m)}, 2q\right)$ splits into two parts, $H\left(\alpha_{Z(m)}, 2(i-1)\right)$ and $H\left(\alpha_{Z(m)}, 2(q-i+1)\right)$ where $a_{1m} \longleftrightarrow a_{(i+1)m}$ (see Figure 5.4 for an illustration). Moreover, the induced constraint graphs $C' \in \mathcal{C}_{\left(\alpha_{Z(m)}, 2(i-1)\right)}$ and $C'' \in \mathcal{C}_{\alpha_{Z(m)}, 2(q-i+1)}$ are dominant.

Since C' is dominant on $H(\alpha_{Z(m)}, 2(i-1))$, we have $C' \in \mathcal{D}_{m,i-1,0}$. Since C'' is dominant on $H\left(\alpha_{Z(m)}, 2(q-i+1)\right)$ and $a_{1m} \longleftrightarrow a_{(i+1)m}$ where a_{1m} and $a_{(i+1)m}$ are adjacent in $H\left(\alpha_{Z(m)}, 2(q-i+1)\right)$ (see Figure 5.4 for an illustration), $C'' \in \mathcal{D}_{m,q-i+1,1}$.

So far we constructed a mapping $C \in \mathcal{D}_{m,q,0} \mapsto (C', C'') \in \mathcal{D}_{m,i-1,0} \times \mathcal{D}_{m,q-i+1,1}$ given that $i \in [q]$ is the smallest index such that $a_{1m} \longleftrightarrow a_{(i+1)m}$ in C. Thus by considering the first vertex that a_{1m} is constrained to in C, we can construct a map from $\mathcal{D}_{m,q,0}$ to $\bigcup_{i=0}^{q-1} \mathcal{D}_{m,i,0} \times \mathcal{D}_{m,q-i,1}$.

The other direction of the map goes in reverse order of the contracting process. It can be verified that this is a bijection. Thus we proved that $D(m,q,0) = \sum_{i=0}^{q-1} D(m,i,0) \cdot D(m,q-i,1)$. $\square$

Now we will consider the later steps of the splitting.

Lemma 5.1.19. *For $r \geq 1$, there exists a bijection between $\mathcal{D}_{m,q,r}$ and $\bigcup_{i=0}^{q-1} \mathcal{D}_{m,i,0} \times \mathcal{D}_{m,q-i,r+1}$. Thus*

$$D(m,q,r) = \sum_{i=0}^{q-1} D(m,i,0) \cdot D(m,q-i,r+1). \tag{5.5}$$

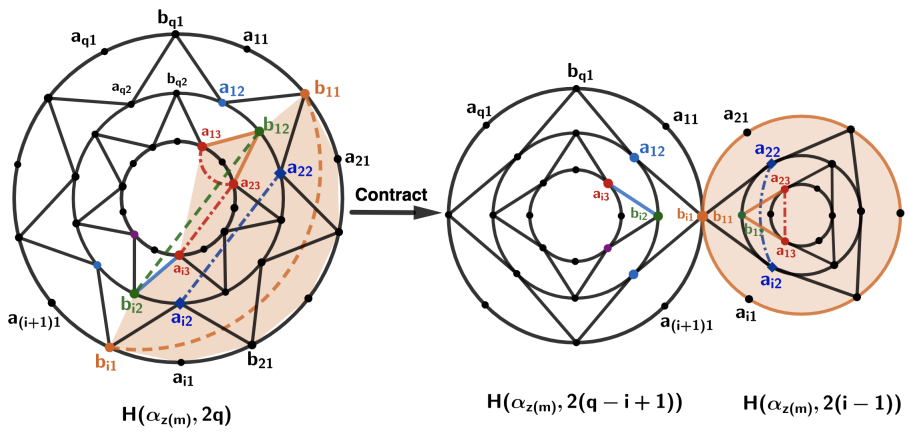


Figure 5.5: Illustration of proof for Lemma 5.1.19: On the left is a $C \in \mathcal{D}_{m,q,r}$, with constrained vertices indicated by dash lines. Here $q = 7$, $m = 3$ and $r = 1$. $b_{i(m-r)}$ is the first vertex that $b_{1(m-r)}$ is constrained to in C. On the right is $H\left(\alpha_{Z(m)}, 2(q-i+1)\right)$ and $H\left(\alpha_{Z(m)}, 2(i-1)\right)$ after splitting.

Proof. Let $C \in \mathcal{D}_{m,q,r}$ for some $r \geq 1$. i.e. C is dominant and $a_{1k} \longleftrightarrow a_{2k}$ for all $m-r+1 \leq k \leq m$. Let $i > 1$ be the first index such that $b_{1(m-r)} \longleftrightarrow b_{i(m-r)}$. If $b_{1(m-r)}$ is isolated then we take $i = q+1$. By Corollary 5.1.14, since C is dominant on $H\left(\alpha_{Z(m)}, 2q\right)$, the induced constraint graph C_{m-r} is dominant, thus by Lemma 4.1.17 and Lemma 4.1.13, $a_{2(m-r)} \longleftrightarrow a_{i(m-r)}$. By Lemma 5.1.15, $a_{2k)} \longleftrightarrow a_{ik}$ for all $m-r \leq k \leq m$. Since $a_{1k} \longleftrightarrow a_{2k}$ for all $m-r+1 \leq k \leq m$ by assumption, $a_{1k} \longleftrightarrow a_{ik}$ for all $m-r+1 \leq k \leq m$. By Lemma 5.1.15, since $b_{1(m-r)} \longleftrightarrow b_{i(m-r)}$, $b_{1k} \longleftrightarrow b_{ik}$ for all $1 \leq k \leq m-r$.

So far we have that

1. $a_{1k} \longleftrightarrow a_{2k} \longleftrightarrow a_{ik}$ for all $m-r+1 \leq k \leq m$.

2. $b_{1k} \longleftrightarrow b_{ik}$ for all $1 \leq k \leq m-r$.

3. $a_{2(m-r)} \longleftrightarrow a_{i(m-r)}$.

Since $b_{1(m-r)} \longleftrightarrow b_{i(m-r)}$ and $a_{1(m-r+1)} \longleftrightarrow a_{i(m-r+1)}$, by Corollary 5.1.16, $H\left(\alpha_{Z(m)}, 2q\right)$ splits into $H\left(\alpha_{Z(m)}, 2(i-1)\right)$ and $H\left(\alpha_{Z(m)}, 2(q-i+1)\right)$. Let $C' \in \mathcal{C}_{\left(\alpha_{Z(m)}, 2(i-1)\right)}$ and $C'' \in \mathcal{C}_{\left(\alpha_{Z(m)}, 2(q-i+1)\right)}$ be the induced constraint graphs. By Corollary 4.1.16, the C' and C'' are dominant. Notice that there is no extra required constrained vertices in C'', so $C'' \in \mathcal{D}_{m,q-i+1,0}$. For C', since $a_{2k} \longleftrightarrow a_{ik}$ for all $m-r+1 \leq k \leq m$ and $a_{2(m-r)} \longleftrightarrow a_{i(m-r)}$, $C' \in \mathcal{D}_{m,i-1,r+1}$ (see Figure 5.5 for an illustration).

So far we have constructed a mapping $C \in \mathcal{D}_{m,q,r} \mapsto (C', C'') \in \mathcal{D}_{m,q-i+1,0} \times \mathcal{D}_{m,i-1,r+1}$ given that $1 < i < q+1$ is the first index such that $b_{1(m-r)} \longleftrightarrow b_{i(m-r)}$ in C. Thus by considering the first vertex that $b_{1(m-r)}$ is constrained to in C, we can construct a mapping from $\mathcal{D}_{m,q,r}$ to $\bigcup_{i=0}^{q-1} \mathcal{D}_{m,i,0} \times \mathcal{D}_{m,q-i,r+1}$.

The other direction of the map simply reverses the the splitting process. It can be verified that this is a bijection. Thus we proved that $D(m,q,r) = \sum_{i=0}^{q-1} D(m,i,0) \cdot D(m,q-i,r+1)$. $\square$

Finally we arrive at the last step, where we identify the last item from the splitting process $\mathcal{D}_{m,q,m}$ with $\mathcal{D}_{m,q-1,0}$.

Lemma 5.1.20. *Let $\mathcal{D}_{m,q,r}$ be defined as in Definition 5.1.17. For any $q \geq 1$ and $m \geq 1$, there is a bijection between $\mathcal{D}_{m,q,m}$ and $\mathcal{D}_{m,q-1,0}$. Thus $D(m,q,m) = D(m,q-1,0)$.*

Proof. Let $C \in \mathcal{D}_{m,q,m}$. Then $a_{1j} \longleftrightarrow a_{2j}$ for all $j \in [m]$. Contracting the constrained vertices we get the induced constraint graph C' on $H(\alpha_{Z(m)}, 2(q-1))$, and C' is dominant. i.e. $C' \in \mathcal{D}_{m,q-1,0}$.

Given $C' \in \mathcal{D}_{m,q-1,0}$, we first relabel the indices by increasing all of them by 1. We then expand a_{2j} to a_{1j} and a_{2j} and make them constrained, for each $j \in [m]$. Adding these constraints $a_{1j} \longleftrightarrow a_{2j}$ for all $j \in [m]$ to C', we get a new constraint graph $C \in \mathcal{D}_{m,q,m}$.

We have constructed maps between $\mathcal{D}_{m,q,m}$ and $\mathcal{D}_{m,q-1,0}$ in both directions. It is easy to see that this is a bijection. Thus we proved that $D(m,q,m) = D(m,q-1,0)$. $\square$

With the above lemmas, we are ready to prove Theorem 5.1.4.

Proof of Theorem 5.1.4. Recall that $D(m,q) = \frac{1}{mq+1}\binom{(m+1)q}{q}$ and we want to prove that $D(m,q,0) =$ the number of dominant constraint graphs in $\mathcal{C}_{\left(\alpha_{Z(m)},2q\right)} = D(m,q)$. For $q=1$, we check that $D(m,1,0) = 1 = D(m,1)$. We will then show that

$$D(m,q,0) = \sum_{\substack{i_1,\ldots,i_{m+1}\geq 0 \\ i_1+\cdots+i_{m+1}=q-1}} D(m,i_1,0)\ldots D(m,i_{m+1},0). \tag{5.6}$$

which is the same recurrence relation for $D(m,q)$ shown in Theorem 5.1.7. Thus by induction on q, we can prove $D(m,q,0) = D(m,q)$ for all $q \geq 1$.

Therefore it suffices to prove Equation (5.6).

Recall that from Lemma 5.1.18, Lemma 5.1.19 and Lemma 5.1.20, we have

1. $D(m,q,0) = \sum_{i=0}^{q-1} D(m,i,0)\cdot D(m,q-i,1)$.
2. $D(m,q,r) = \sum_{i=0}^{q-1} D(m,i,0)\cdot D(m,q-i,r+1)$.
3. $D(m,q,m) = D(m,q-1,0)$.

Thus

$$
\begin{aligned}
D(m,q,0) &= \sum_{i_1=0}^{q-1} D(m,i_1,0)\cdot D(m,q-i_1,1) \\
&= \sum_{i_1=0}^{q-1} D(m,i_1,0)\cdot\left(\sum_{i_2=0}^{q-i_1-1} D(m,i_2,0)\cdot D(q-i_1-i_2,2)\right) \\
&\vdots \\
&= \sum_{\substack{i_1,\dots,i_m\ge 0, i'_{m+1}\ge 1:\\ i_1+\dots+i_m+i'_{m+1}=q}} D(m,i_1,0)\cdot D(m,i_2,0)\dots D(m,i_m,0)\cdot D(m,i'_{m+1},m) \\
&= \sum_{\substack{i_1,\dots,i_m\ge 0, i'_{m+1}\ge 1:\\ i_1+\dots+i_m+i'_{m+1}=q}} D(m,i_1,0)\dots D(m,i_m,0)\cdot D(m,i'_{m+1}-1,0) \\
&= \sum_{i_j\ge 0: i_1+\dots+i_{m+1}=q-1} D(m,i_1,0)\dots D(m,i_m,0)\cdot D(m,i_{m+1},0)
\end{aligned}
$$

This proves Equation (5.6), as needed. □

5.2 The Spectrum of Multi-Z-shaped Graph Matrices

In this section we aim to find the spectrum of the singular values for m-layer Z-shape graph matrices. Let $M_{n,m} = \frac{1}{n^{m/2}} M_{\alpha_{Z(m)}}(G)$ where $\alpha_{Z(m)}$ is the m-layer Z-shape as defined in Definition 2.2.17 and $G \sim G(n,1/2)$. Let $r(n,m) = \frac{n!}{(n-m)!}$ be the dimension of $M_{n,m}$. By Corollary 5.1.6, $\frac{1}{r(n,m)} \cdot \mathbb{E}\left[\operatorname{tr}\left(M_{n,m}M_{n,m}^T\right)^k\right] = D(m,k)$. Thus by Corollary 3.1.9, if we can find a function g_m such that $\int_0^\infty g_m(x)x^{2k}\,\mathrm{dx} = D(m,k)$, then g_m describes the limiting spectrum of singular values for the m-layer Z-shape graph matrix as n goes to ∞.

Recall that $D(m,n) = \frac{1}{mn+1}\binom{(m+1)n}{n}$ as defined in Definition 5.1.2. In this section we will generalize the arguments for the $m = 2$ case in Section 4.2 to the case $m = 3$. The general steps will be:

1. Assume $\int_0^{\infty} f(x)x^{2k}\,\mathrm{dx} = D(3,k)$ and derive a differential equation for $f(x)$.

2. Prove that under this differential equation, the moments of $f(x)$ are indeed $D(3,k)$.

3. Apply Corollary 3.1.9 to conclude that $f(x)$ is the desired spectrum.

Theorem 5.2.1. *Let $a = \lim_{n\to\infty} D(3,n+1)/D(3,n) = \frac{16}{3\sqrt{3}}$. If $f(x)$ is a function such that $\int_0^a f(x)x^{2k}dx = D(3,k)$ for all nonnegative integers k and*

1. *f is three times continuously differentiable on $(0,a)$ and $\lim_{x\to a^-} f(x) = 0$,*

2. *$\lim_{x\to a^-}(x-a)f'(x) = 0$,*

3. *$\lim_{x\to a^-} 2(x-a)f''(x) + f'(x) = 0$,*

4. *$\lim_{x\to 0^+} xf(x) = 0$, $\lim_{x\to 0^+} x^2 f'(x) = 0$, and $\lim_{x\to 0^+} x^3 f''(x) = 0$.*

then f satisfies the following ODE on $(0,a)$:

$$(27x^4 - 256x^2)f'''(x) + (162x^3 - 768x)f''(x) + (177x^2 - 192x)f'(x) + 15xf(x) = 0. \quad (5.7)$$

Lemma 5.2.2. *Let a be some positive constant. If f is continuous on $[0,a]$ and $\int_0^a f(x)x^{2n+1}\,\mathrm{dx} = 0$ for all nonnegative integers n, then $f = 0$ on $(0,a)$.*

Proof. The proof is similar to the proof for Lemma 4.2.14 for the case of all zero even moments. Here we approximate $f(\sqrt{x})\cdot\sqrt{x}$ by a polynomial $p(x)$, so that the odd polynomial $p(x^2)/x$ ($p(x)$ has 0 constant term) approximates $f(x)$. □

Proof of Theorem 5.2.1. Denote the LHS of the ODE by $G(x)$. Let $A(k,n) = \int_0^a f^{(k)}(x)\cdot$

x^n dx and $B(k,n) = \left[f^k(x)\cdot x^n\right]_0^a$. Repeatedly doing integration by parts we get that

$$\begin{aligned}
A(m,n) &= B(m-1,n) - n\cdot A(m-1,n-1)\\
&= B(m-1,n) - nB(m-2,n-1) + n(n-1)A(m-2,n-2)\\
&= B(m-1,n) - nB(m-2,n-1) + n(n-1)B(m-3,n-2)\\
&\quad - n(n-1)(n-2)A(m-3,n-3).
\end{aligned}$$

Also, we have that

$$\frac{D(3,n)}{D(3,n-1)} = \frac{A(0,2n)}{A(0,2n-2)} = \frac{4(4n-3)(4n-2)(4n-1)}{(3n+1)(3n)(3n-1)}. \tag{5.8}$$

The steps for deducing the ODE for $f(x)$ are very similar to the steps used in the proof of Theorem 4.2.9 to deduce the ODE for the Z-shaped graph matrix. We first apply $m = 3$ and $n = 2n+3$ to the first equation and rewrite the term $n(n-1)(n-2)A(m-3,n-3)$ using the second equation (5.8). We then gradually eliminate the non-constant coefficients in front of $A(m,n)$'s using the first equation.

Plugging in $m = 3, n = 2n+3$ into the first equation, we get

$$\begin{aligned}
27A(3,2n+3) &= 27B(2,2n+3) - 27(2n+3)B(1,2n+2) +\\
&\quad 27(2n+3)(2n+2)B(0,2n+1) - 27(2n+3)(2n+2)(2n+1)A(0,2n).
\end{aligned}$$

Rewriting the last term on the RHS above and applying the second equation (5.8), we get

$$
\begin{aligned}
&27(2n+3)(2n+2)(2n+1)A(0,2n) \\
&\qquad = 8(3n+1)(3n)(3n-1)A(0,2n)- \\
&\qquad\qquad 81\cdot 2(2n+2)(2n+1)A(0,2n)+59\cdot 3(2n-1)A(0,2n)-15A(0,2n) \\
&\qquad = 32(4n-3)(4n-2)(4n-1)A(0,2n-2)- \\
&\qquad\qquad 81\cdot 2(2n+2)(2n+1)A(0,2n)+59\cdot 3(2n-1)A(0,2n)-15A(0,2n)
\end{aligned}
$$

We can rewrite the first term on the RHS as

$$
\begin{aligned}
&32(4n-3)(4n-2)(4n-1)A(0,2n-2) \\
&\qquad = 256(2n+1)(2n)(2n-1)A(0,2n-2)- \\
&\qquad\qquad 256\cdot 3(2n)(2n-1)A(0,2n-2)+128\cdot 3(2n-1)A(0,2n-2).
\end{aligned}
$$

Now apply the first equation to all the $(n+1)A(m,n)$, $(n+2)(n+1)A(m,n)$ and $(n+3)(n+2)(n+1)A(m,n)$ above, group together the $A(m,n)$ terms and $B(m,n)$ terms separately, and rewrite the $B(m,n)$ term using the definition of $B(m,n)$. We get that for all $n \geq 1$,

$$
\begin{aligned}
&27A(3,2n+3)-256A(3,2n+1)+2\cdot 81A(2,2n+2)-3\cdot 256A(2,2n) \\
&\qquad +3\cdot 59A(1,2n+1)-3\cdot 128A(1,2n-1)+15A(0,2n) \\
&= \left[\Big((27x^2-256)f''(x)+27xf'(x)\Big)\,x^{2n+1}\right]_0^a \\
&\qquad +\left[\Big((2n-2)(-27x^2+256)\Big)\,f'(x)x^{2n}\right]_0^a+\Big[p(n,x)\cdot f(x)x^{2n-1}\Big]_0^a
\end{aligned}
$$

where $p(n,x)$ is some polynomial in terms of n and x.

Observe that the last term on the RHS is 0 since $\lim_{x\to 0^+} xf(x)=0$ and $\lim_{x\to a^-} f(x)=0$ by assumption. The second last term is 0 since $27x^2-256=27(x+a)(x-a)$, $\lim_{x\to a^-}(x-a)f'(x)=$

0 and $\lim_{x\to 0^+} x^2 f'(x) = 0$. The first term top part is $\lim_{x\to a^-} 27\left((x+a)(x-a)f''(x) + xf'(x)\right)x^{2n+1}$ $= \lim_{x\to a^-} 27\left(2a(x-a)f''(x) + af'(x)\right)a^{2n+1} = 0$ since $\lim_{x\to a^-} 2(x-a)f''(x) + f'(x) = 0$ by assumption. The bottom part is 0 since $\lim_{x\to 0^+} x^3 f''(x) = 0$ by assumption. Thus the RHS is 0.

Expanding out each $A(m,n)$ term by using the definition of $A(m,n)$, we get that $\int_0^a G(x)x^{2n+1} = 0$ for all $n \geq 1$. By Lemma 5.2.2, $G(x) = 0$ on $(0,a)$, which proves that f satisfies the ODE.

□

Theorem 5.2.3. *Let* $a = \lim_{n\to\infty} D(3,n)/D(3,n-1) = \dfrac{16}{3\sqrt{3}}$ *and let* f *be a function satisfying the following ODE*

$$(27x^4 - 256x^2)f'''(x) + (162x^3 - 768x)f''(x) + (177x^2 - 192)f'(x) + 15xf(x) = 0 \quad (5.9)$$

and the conditions listed in Theorem 5.2.1. Moreover, assume $\int_0^a f(x)\,\mathrm{dx} = 1$. *Then for any nonnegative integer* k,

$$A(0,2k) = \int_0^a x^{2k} \cdot f(x)\,\mathrm{dx} = D(3,k). \quad (5.10)$$

The proof is very similar to the proof for Theorem 4.2.16. We integrate the ODE from 0 to a, do integration by parts and use the conditions for f to eliminate the redundant terms and finally arrive at the ratio between $A(0,2(k-1))$ and $A(0,2k)$ which matches the ratio between $D(3,k-1)$ and $D(3,k)$. By induction on k we conclude (5.10).

Corollary 5.2.4. *Let* $M_n = \dfrac{1}{n^{3/2}} \cdot M_{\alpha_{Z(3)}}$ *where* $M_{\alpha_{Z(3)}}$ *is the graph matrix with random input graph* $G \sim G(n, 1/2)$ *and* $\alpha_{Z(3)}$ *is the multi-Z-shape defined in Definition 2.2.17. Let* $r(n) = n(n-1)(n-2)$ *be the dimension of* M_n. *Let* $g(x)$ *be* $f(x)$ *as in Theorem 5.2.3 on* $(0,a)$ *and* 0 *for* $x \geq a$. *Then as* $n \to \infty$, *the distribution of the singular values of* M_n *approaches* $g(x)$ *almost surely.*

Proof. This is true by Corollary 5.1.6, Corollary 3.1.9, and Theorem 5.2.3. □

For this ODE (5.7), WolframAlpha fails to give us an explicit solution. Instead, we solve the ODE numerically by approximating the tail segment of $f(x)$ by $c \cdot (a - x)^r$ for some constants c and r.

Step 1: We analyze the behaviour of the ODE when x is very close to a. Notice that $a = \frac{16}{3\sqrt{3}} \iff 27a^2 - 256 = 0.$

1. $27x^4 - 256x^2 = 27x^2(x - a)(x + a) \sim 52a^3(x - a).$
2. $162x^3 - 768x = 81x^3 + 3x(27x^2 - 256) \sim 81a^3.$
3. $177x^2 - 192 = \frac{3}{4}(209x^2 + 27x^2 - 256) \sim \frac{3 \cdot 209a^2}{4}.$

Thus when x is very close to a, the ODE is

$$52a^3(x - a)f'''(x) + 81a^3 f''(x) + \left(\frac{3 \cdot 209a^2}{4}\right) f'(x) = 0.$$

One can check that $f'(x) = C'\left((a - x)^{-1/2} + \frac{209}{64\sqrt{3}}(a - x)^{1/2}\right)$ is a solution to the above ODE. Thus

$$f(x) \sim g(x) = C\left((a - x)^{1/2} + \frac{209}{64 \cdot 3\sqrt{3}}(a - x)^{3/2}\right) \tag{5.11}$$

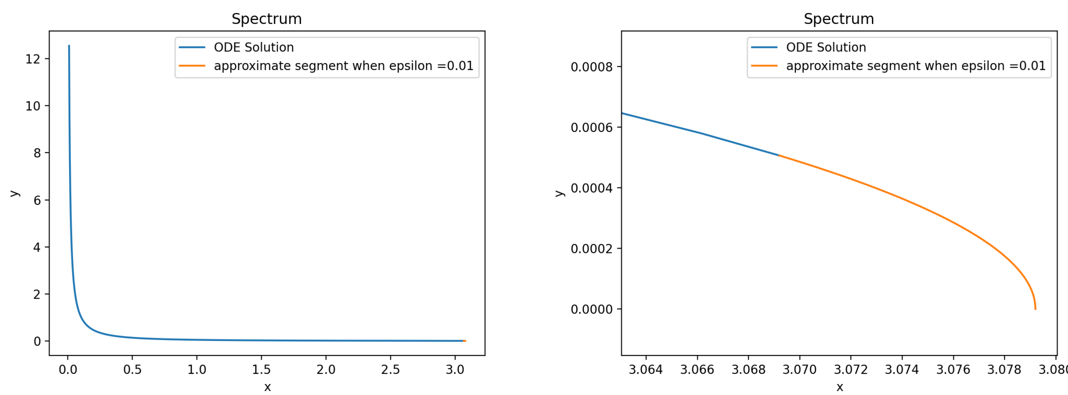
for some constant C when x is very close to a.

Step 2: We approximate the solution of $f(x)$ by approximating the tail segment of $f(x)$ (where $|x - a| < \epsilon$) by $g(x)$ and use this approximation to obtain initial conditions for the ODE for f. In particular, we choose a small $\epsilon > 0$ and set the initial conditions for the ODE as follows:

$$f(a-\epsilon) = g(a-\epsilon), f'(a-\epsilon) = g'(a-\epsilon), f''(a-\epsilon) = g''(a-\epsilon).$$

We calculate the constant C in g by noticing that the integration of f over $(-a, a)$ should be 1.

Setting $\epsilon = 0.01$ and solving the ODE in python, we get the following plot for the solution to the ODE (concatenated with a tail segment where we use g as an approximation):

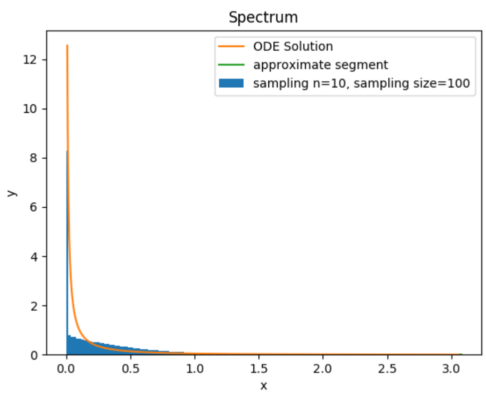


(a) Plot of the ODE solution with the approximated tail segment.

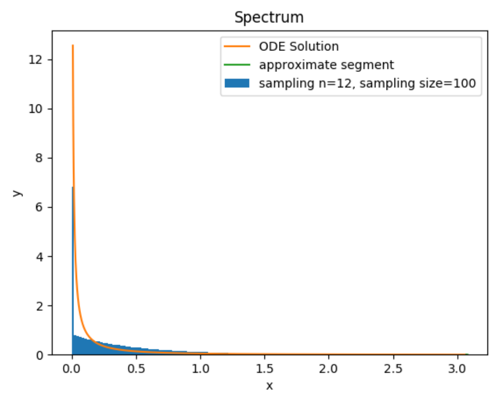
(b) Zoom in at the tail segment.

Figure 5.6: The plot of the spectrum where $x > 0$.

To test this solution experimentally, we can sample from the distribution of singular values of M_n by sampling a random graph G, computing the resulting matrix $M_n(G)$, and computing its singular values. See Figure 5.7 for a plot of the approximated spectrum together with the empirical distribution of the singular values of M_n with $n = 10$ and $n = 12$, respectively (where we sampled 100 random graphs G).



(a) Sampling of the singular values of M_n for $n = 10$.



(b) Sampling of the singular values of M_n for $n = 12$.

Figure 5.7: The approximated Spectrum of singular values of the 3-layer Z-shape graph matrix and some samplings of the singular values of M_n, for $n = 10$ and $n = 12$.

CHAPTER 6

MOMENTS FOR ab WHEN a, b ARE FREELY INDEPENDENT

The remaining chapters are all from Cai and Potechin [2022].

In this chapter, we prove Theorem 2.2.8. For convenience, we restate Theorem 2.2.8 here.

Recall that

$$C\left(\vec{\alpha}, \vec{\beta}\right) = (-1)^{A+B-k-1} \cdot k \cdot \binom{A+B-2}{k-1} \cdot \frac{(A-1)!}{\alpha_1! \ldots \alpha_k!} \cdot \frac{(B-1)!}{\beta_1! \ldots \beta_k!}$$

where $A = \alpha_1 + \cdots + \alpha_k$ and $B = \beta_1 + \cdots + \beta_k$.

Theorem 2.2.8. *If $(\mathcal{A}, \varphi)$ is a non-commutative probability space and $a, b \in \mathcal{A}$ are freely independent then*

$$\varphi\left((ab)^k\right) = \sum_{\vec{\alpha}, \vec{\beta} \in P_k} C\left(\vec{\alpha}, \vec{\beta}\right) \cdot \vec{\varphi_a}^{\vec{\alpha}} \cdot \vec{\varphi_b}^{\vec{\beta}} \tag{2.7}$$

Definition 6.0.1. Let $\pi = \{V_1, \ldots, V_m\} \in NC(k)$. Let $\mathcal{C}_k$ be the cycle graph with vertices $\{1, \ldots, k\}$. Define $\mathcal{C}_k/\pi$ to be the graph obtained by identifying vertices together under π.

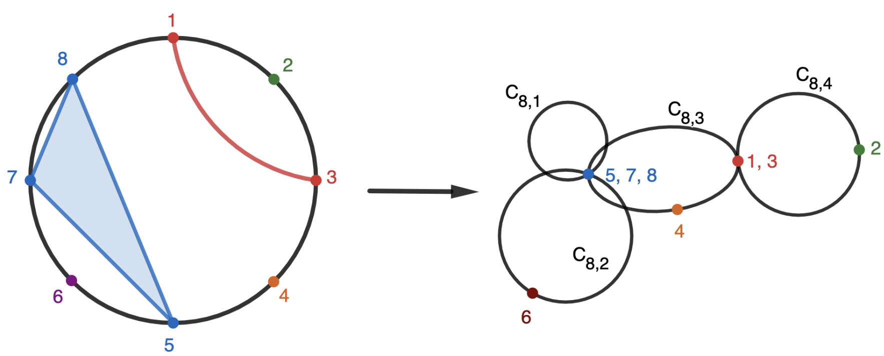


Figure 6.1: Illustration of Proposition 6.0.2 and Definition 6.0.3: $\mathcal{C}_8/\pi = \mathcal{C}_{8,1} \cup \mathcal{C}_{8,2} \cup \mathcal{C}_{8,3} \cup \mathcal{C}_{8,4}$ where $\pi = \{\{2\}, \{4\}, \{6\}, \{1,3\}, \{5,7,8\}\}$. Here $S_\pi = \{1, 2, 3, 2\}$.

Proposition 6.0.2. *Let* $\pi = \{V_1, \ldots, V_m\} \in NC(k)$. *Then* $\mathcal{C}_k/\pi$ *is a union of cycles* $\mathcal{C}_{k,1}, \ldots, \mathcal{C}_{k,p}$ *where*

1. $\left|V\left(\mathcal{C}_{k,i}\right) \cap V\left(\mathcal{C}_{k,i+1}\right)\right| = 1$ *for each* $i \in [p-1]$.

2. $p = k - m + 1$.

3. $\sum_{i=1}^{p} \left|\mathcal{C}_{k,i}\right| = k$.

Definition 6.0.3. Let $\pi \in NC(k)$. Assume $\mathcal{C}_k/\pi = \mathcal{C}_{k,1} \cup \cdots \cup \mathcal{C}_{k,p}$. We define S_π to be the unordered sequence $\left\{i_1, \ldots, i_p\right\}$ where i_j is the size of the cycle $\mathcal{C}_{k,j}$ for each $j \in [p]$.

Proposition 6.0.4. *Let* $k \in \mathbb{N}$ *and* $\pi \in NC(k)$. *Then*

$$[\pi, 1_k] \cong [0_k, K(\pi)] \cong \prod_{V \in K(\pi)} NC\left(|V|\right) = \prod_{x \in S_\pi} NC(x) \tag{6.1}$$

Proposition 6.0.5. *Let* $k \in \mathbb{N}$ *and* $\pi = \{V_1, \ldots, V_m\} \in NC(k)$. *Let* μ *be the Möbius function of* $NC(k)$. *Then*

$$\mu\left(\pi, 1_k\right) = \prod_{x \in S_\pi} (-1)^{x-1} \cdot C_{x-1} = (-1)^{m-1} \cdot \prod_{x \in S_\pi} C_{x-1} \tag{6.2}$$

Proof. By the canonical factorization of $[\pi, 1_k]$ and Proposition 3.4.6,

$$\mu\left(\pi, 1_k\right) = \mu\left(\prod_{x \in S_\pi} NC(x)\right) = \prod_{x \in S_\pi} \mu\left(NC(x)\right) = \prod_{x \in S_\pi} (-1)^{x-1} \cdot C_{x-1}$$

The last equality is because $\sum_{x \in S_\pi} x - 1 = k - p = m - 1$. □

Definition 6.0.6. Given $\pi, \sigma \in NC(k)$, we say that $\pi \oplus \sigma \in NC(k)$ if $\pi \cup \bar{\sigma} \in NC(1, \bar{1}, \ldots, k, \bar{k})$ where $\bar{\sigma} := \{\{\bar{v_1}, \ldots \bar{v_m}\} : W = \{v_1, \ldots, v_m\} \in \sigma\}$.

See Figure 6.2 for an illustration.

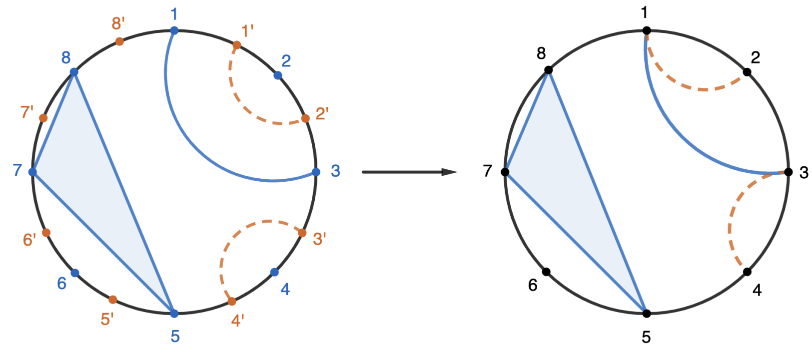


Figure 6.2: Illustration of Definition 6.0.6 and Definition 6.0.7: Blue solid represents π and orange dashed represents σ. Here $\pi \oplus \sigma \in NC(k)$ since $\pi \cup \bar{\sigma}$ is non-crossing in the left figure. Moreover, $S_{\pi\oplus\sigma} = \{1, 1, 1, 1, 2, 2\}$.

Definition 6.0.7. Let $\pi, \sigma \in NC(k)$ such that $\pi \oplus \sigma \in NC(k)$. Let $\mathcal{C}_k/\pi \oplus \sigma$ be the graph obtained by identifying vertices together under $\pi \cup \sigma$. Then $\mathcal{C}_k/\pi \oplus \sigma = \mathcal{C}_{k,1} \cup \cdots \cup \mathcal{C}_{k,p}$ is a union of cycles. We define $S_{\pi\oplus\sigma} = \{i_1, \ldots, i_p\}$ where i_j is the size of $\mathcal{C}_{k,j}$. See Figure 6.2 for an illustration.

Proposition 6.0.8. *Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space. Let $\{a_1, \ldots, a_n\}$, $\{b_1, \ldots, b_n\} \in \mathcal{A}$ be freely independent. Then*

$$\varphi(a_1 b_1 \ldots a_n b_n) = \sum_{\substack{\pi,\sigma \in NC(n):\\ \pi\oplus\sigma\in NC(n)}} (-1)^{|\pi|+|\sigma|-k-1} \cdot \prod_{x \in S_{\pi\oplus\sigma}} C_{x-1} \cdot \prod_{V\in\pi} \varphi(V)[a_1, \ldots, a_n] \cdot \prod_{W\in\sigma} \varphi(W)[b_1, \ldots, b_n] \tag{6.3}$$

In particular, for $a, b \in \mathcal{A}$ freely independent,

$$\varphi\left((ab)^n\right) = \sum_{\substack{\pi,\sigma \in NC(n):\\ \pi\oplus\sigma\in NC(n)}} (-1)^{|\pi|+|\sigma|-k-1} \cdot \prod_{x \in S_{\pi\oplus\sigma}} C_{x-1} \cdot \prod_{V\in\pi} \varphi\left(a^{|V|}\right) \cdot \prod_{W\in\sigma} \varphi\left(b^{|W|}\right) \tag{6.4}$$

Proof. By Theorem 3.4.11,

$$\varphi\left(a_1 b_1 \ldots a_n b_n\right) = \sum_{\pi \in NC(n)} \kappa_\pi[a_1, \ldots, a_n] \cdot \varphi_{K(\pi)}[b_1, \ldots, b_n]$$

$$= \sum_{\pi \in NC(n)} \kappa_{K(\pi)}[a_1, \ldots, a_n] \cdot \varphi_\pi[b_1, \ldots, b_n]$$

$$= \sum_{\pi \in NC(n)} \prod_{V \in K(\pi)} \kappa(V)[a_1, \ldots, a_n] \cdot \prod_{W \in \pi} \varphi(W)[b_1, \ldots, b_n]$$

$$= \sum_{\pi \in NC(n)} \prod_{V \in K(\pi)} \kappa_{|V|}\left(a_{i_1}, \ldots, a_{i_{|V|}}\right) \cdot \prod_{W \in \pi} \varphi(W)[b_1, \ldots, b_n] \text{ where } V = \{i_1, \ldots, i_{|V|}\}$$

$$= \sum_{\pi \in NC(n)} \left(\prod_{V \in K(\pi)} \sum_{\sigma \in NC(|V|)} \varphi_\sigma\left[a_{i_1}, \ldots, a_{i_{|V|}}\right] \cdot \mu\left(\sigma, 1_{|V|}\right) \right) \cdot \prod_{W \in \pi} \varphi(W)[b_1, \ldots, b_n]$$

$$= \sum_{\pi \in NC(n)} \left(\prod_{V_j \in K(\pi)} \sum_{\sigma_j \in NC(\,|V_j|)} \varphi_{\sigma_j}\left[a_{i_{j1}}, \ldots, a_{i_{j\,|V_j|}}\right] \cdot \prod_{x \in S_{\sigma_j}} (-1)^{x-1} C_{x-1} \right) \cdot \prod_{W \in \pi} \varphi(W)[b_1, \ldots, b_n]$$

$$= \sum_{\substack{\pi, \sigma \in NC(n): \\ \pi \oplus \sigma \in NC(n)}} \left(\prod_{x \in S_{\pi \oplus \sigma}} (-1)^{x-1} C_{x-1} \right) \cdot \prod_{V \in \sigma} \varphi(V)[a_1, \ldots, a_n] \cdot \prod_{W \in \pi} \varphi(W)[b_1, \ldots, b_n]$$

$$= \sum_{\substack{\pi, \sigma \in NC(n): \\ \pi \oplus \sigma \in NC(n)}} (-1)^{|\pi|+|\sigma|-k-1} \cdot \prod_{x \in S_{\pi \oplus \sigma}} C_{x-1} \cdot \prod_{V \in \sigma} \varphi(V)[a_1, \ldots, a_n] \cdot \prod_{W \in \pi} \varphi(W)[b_1, \ldots, b_n]$$

□

Definition 6.0.9. For $\vec{\alpha}, \vec{\beta} \in P_k$, we will denote $\mathcal{NP}\left(\vec{\alpha}, \vec{\beta}\right)$ to be

$$\mathcal{NP}\left(\vec{\alpha}, \vec{\beta}\right) = \left\{(\pi, \sigma) : \pi \in \mathcal{NP}(\vec{\alpha}), \sigma \in \mathcal{NP}(\vec{\beta}), \pi \oplus \sigma \in NC(k)\right\} \tag{6.5}$$

Corollary 6.0.10. *Let $(\mathcal{A}, \varphi)$ be a non-commutative probability space. Let $a, b \in \mathcal{A}$ be freely independent. Then*

$$\varphi\left((ab)^k\right) = \sum_{\vec{\alpha}, \vec{\beta} \in P_k} (-1)^{A+B-k-1} \cdot \left(\sum_{(\pi,\sigma) \in \mathcal{NP}\left(\vec{\alpha}, \vec{\beta}\right)} \prod_{x \in S_{\pi \oplus \sigma}} C_{x-1} \right) \cdot \vec{\varphi_a}^{\vec{\alpha}} \cdot \vec{\varphi_b}^{\vec{\beta}} \tag{6.6}$$

where $A = \alpha_1 + \cdots + \alpha_k$ *and* $B = \beta_1 + \cdots + \beta_k$ *for* $\vec{\alpha}, \vec{\beta} \in P_k$.

Proof. This is because for $\pi \in \mathcal{NP}(\vec{\alpha})$ and $\sigma \in \mathcal{NP}(\vec{\beta})$, $|\pi| = A$ and $|\sigma| = B$, and $\prod_{V\in\pi} \varphi\left(a^{|V|}\right) = \prod_{V\in\pi} \varphi(a^i)^{\alpha_i} = \vec{\varphi_a}^{\vec{\alpha}}$, $\prod_{W\in\sigma} \varphi\left(b^{|W|}\right) = \prod_{W\in\sigma} \varphi(b^i)^{\beta_i} = \vec{\varphi_b}^{\vec{\beta}}$. □

Theorem 6.0.11. *Let* $\vec{\alpha}, \vec{\beta} \in P_k$ *Then*

$$\sum_{(\pi,\sigma)\in\mathcal{NP}\left(\vec{\alpha},\vec{\beta}\right)} \prod_{x\in S_{\pi\oplus\sigma}} C_{x-1} = k \cdot \binom{A+B-2}{k-1} \cdot \frac{(A-1)!}{\alpha_1!\dots\alpha_k!} \cdot \frac{(B-1)!}{\beta_1!\dots\beta_k!} \tag{6.7}$$

Note that in the special case when $\vec{\alpha} = \vec{\beta} = (k, 0, \dots, 0) \in P_k$, $C(\vec{\alpha}, \vec{\beta}) = (-1)^{k-1} \cdot \frac{1}{k}\binom{2k-2}{k-1} = (-1)^{k-1} \cdot C_{k-1}$, and $LHS = (-1)^{|0_k|+|0_k|-k-1} \cdot C_{k-1} = (-1)^{k-1} C_{k-1}$.

Theorem 2.2.8 follows directly from Corollary 6.0.10 and Theorem 6.0.11.

Proof.

$$\begin{aligned}
\varphi\left((ab)^k\right) &= \sum_{\vec{\alpha},\vec{\beta}\in P_k} (-1)^{A+B-k-1} \cdot \left(\sum_{(\pi,\sigma)\in\mathcal{NP}\left(\vec{\alpha},\vec{\beta}\right)} \prod_{x\in S_{\pi\oplus\sigma}} C_{x-1} \right) \cdot \vec{\varphi_a}^{\vec{\alpha}} \cdot \vec{\varphi_b}^{\vec{\beta}} \\
&= \sum_{\vec{\alpha},\vec{\beta}\in P_k} (-1)^{A+B-k-1} \cdot k \cdot \binom{A+B-2}{k-1} \cdot \frac{(A-1)!}{\alpha_1!\dots\alpha_k!} \cdot \frac{(B-1)!}{\beta_1!\dots\beta_k!} \cdot \vec{\varphi_a}^{\vec{\alpha}} \cdot \vec{\varphi_b}^{\vec{\beta}} \\
&= \sum_{\vec{\alpha},\vec{\beta}\in P_k} C\left(\vec{\alpha}, \vec{\beta}\right) \cdot \vec{\varphi_a}^{\vec{\alpha}} \cdot \vec{\varphi_b}^{\vec{\beta}}
\end{aligned}$$

□

So to prove Theorem 2.2.8, it suffices to prove Theorem 6.0.11. We will first prove Theorem 6.0.11 when $\vec{\beta} = (k, 0, \dots, 0) \in P_k$ and then prove the theorem in the general case.

6.1 Case when $\vec{\beta} = (k, 0, \dots, 0)$

When $\vec{\beta} = (k, 0, \dots, 0)$, we want to show the following.

Theorem 6.1.1. *Let $\vec{\alpha} = (\alpha_1, \ldots, \alpha_k) \in P_k$ and $a = \alpha_1 + \cdots + \alpha_k$. Then*

$$\sum_{\pi \in \mathcal{NP}(\vec{\alpha})} \prod_{x \in S_\pi} C_{x-1} = \binom{a+k-2}{k-1} \cdot \frac{(a-1)!}{\alpha_1! \ldots \alpha_k!} \tag{6.8}$$

Definition 6.1.2. Given a set of elements $\mathcal{I} = \left\{i_1, \ldots, i_p\right\}$ where i_j's are not necessarily distinct, we denote perm $\left(i_1, \ldots, i_p\right)$, or perm $(\mathcal{I})$, to be the number of permutations of the elements in $\mathcal{I}$.

Example 6.1.3. $\operatorname{perm}(\{1,2,3,4\}) = 4! = 24$. $\operatorname{perm}(\{1,1,2,2\}) = \dfrac{4!}{2!2!} = 6$.

Definition 6.1.4. Let π be a non-crossing partition of $[k]$. Assume $\mathcal{C}_k/\pi = \mathcal{C}_{k,1} \cup \cdots \cup \mathcal{C}_{k,p}$. Let $\mathcal{X} = \{x_1, \ldots, x_p\}$ be such that $x_1 + \cdots + x_p = k$. We say a permutation $\sigma : [p] \to [p]$ is a *label of $\mathcal{C}_k/\pi$ corresponding to $\mathcal{X}$* if $\left|\mathcal{C}_{k,j}\right| = x_{\sigma[j]}$ for all $j \in [p]$.

Definition 6.1.5. Let $\vec{\alpha} \in P_k$. Let $p = k - a + 1$ and $\mathcal{X} = \left\{x_1, \ldots, x_p\right\}$ be such that $x_1 + \cdots + x_p = k$. We denote

1. $\mathcal{NP}(\vec{\alpha}, \mathcal{X})$, or $\mathcal{NP}\left(\vec{\alpha}, x_1, \ldots, x_p\right)$ to be

$$\mathcal{NP}(\vec{\alpha}, \mathcal{X}) = \left\{\pi \in \mathcal{NP}(\vec{\alpha}) : S_\pi = \{x_1, \ldots, x_p\}\right\}, \tag{6.9}$$

   reads "the set of $\mathcal{N}$oncrossing $\mathcal{P}$artitions corresponding to $\vec{\alpha}$ that partitions $\mathcal{C}_k$ into parts of sizes $\mathcal{X}$ ".

2. $\mathcal{NPL}_A(\vec{\alpha}, \mathcal{X})$, or $\mathcal{NPL}_A\left(\vec{\alpha}, x_1, \ldots, x_p\right)$ to be

$$\mathcal{NPL}_A(\vec{\alpha}, \mathcal{X}) = \left\{\begin{aligned} (\pi, \sigma) :& \pi = \{P_1, \ldots, P_a\} \in \mathcal{NP}(\vec{\alpha}, \mathcal{X}), \\ & \sigma : [a] \to [a] \text{ where } |P_i| = \left|P_{\sigma(i)}\right| \text{ for all } i \in [a] \end{aligned}\right\}, \tag{6.10}$$

   reads "the set of $\mathcal{N}$oncrossing $\mathcal{P}$artitions corresponding to $\mathcal{L}$abeled polygons $\vec{\alpha}$, which partitions $\mathcal{C}_k$ into parts of sizes $\mathcal{X}$ ".

3. $\mathcal{NPL}_{\mathcal{X}}(\vec{\alpha},\mathcal{X})$, or $\mathcal{NPL}_{\mathcal{X}}\left(\vec{\alpha},x_1,\ldots,x_p\right)$ to be

$$\mathcal{NPL}_{\mathcal{X}}(\vec{\alpha},\mathcal{X}) = \left\{\begin{aligned}&(\pi,\sigma) : \pi \in \mathcal{NP}(\vec{\alpha},\mathcal{X}),\\ &\qquad\qquad \sigma \text{ is a label of } \mathcal{C}_k/\pi \text{ corresponding to } \mathcal{X}\end{aligned}\right\}, \tag{6.11}$$

reads "the set of $\mathcal{N}$oncrossing $\mathcal{P}$artitions corresponding to $\vec{\alpha}$ that partitions $\mathcal{C}_k$ into parts $\mathcal{L}$abeled with $\mathcal{X}$ ".

4. $\mathcal{NPL}_{A,\mathcal{X}}(\vec{\alpha},\mathcal{X})$, or $\mathcal{NPL}_{A,\mathcal{X}}\left(\vec{\alpha},x_1,\ldots,x_p\right)$ to be

$$\mathcal{NPL}_{A,\mathcal{X}}(\vec{\alpha},\mathcal{X}) = \left\{\begin{aligned}&(\pi,\sigma_1,\sigma_2) : (\pi,\sigma_1) \in \mathcal{NPL}_A(\vec{\alpha},\mathcal{X}),\\ &\qquad\qquad \sigma_2 \text{ is a label of } \mathcal{C}_k/\pi \text{ corresponding to } \mathcal{X}\end{aligned}\right\}, \tag{6.12}$$

reads "the set of $\mathcal{N}$oncrossing $\mathcal{P}$artitions corresponding to $\mathcal{L}$abeled polygons $\vec{\alpha}$, which partitions $\mathcal{C}_k$ into parts $\mathcal{L}$abeled with $\mathcal{X}$ ".

Example 6.1.6.

1. *Let* $\vec{\alpha} \in P_k$ *correspond to* Ω_2^k *and* $\mathcal{X} = \{k\}$. *Then* $|\mathcal{NP}(\vec{\alpha},\mathcal{X})| = 1$, $|\mathcal{NPL}_A(\vec{\alpha},\mathcal{X})| = k!$, $|\mathcal{NPL}_{\mathcal{X}}(\vec{\alpha},\mathcal{X})| = 1$ *and* $\left|\mathcal{NPL}_{A,\mathcal{X}}(\vec{\alpha},\mathcal{X})\right| = k!$.

2. *Let* $\vec{\alpha} \in P_k$ *correspond to* Ω_{2k} *and* $\mathcal{X} = \{1,1,\ldots,1\}$. *Then* $|\mathcal{NP}(\vec{\alpha},\mathcal{X})| = 1$, $|\mathcal{NPL}_A(\vec{\alpha},\mathcal{X})| = 1$, $|\mathcal{NPL}_{\mathcal{X}}(\vec{\alpha},\mathcal{X})| = k!$ *and* $\left|\mathcal{NPL}_{A,\mathcal{X}}(\vec{\alpha},\mathcal{X})\right| = k!$.

3. *Let* $\vec{\alpha} \in P_5$ *correspond to* $\Omega_2\Omega_4^2$ *and* $\mathcal{X} = \{1,1,3\}$. *Then* $|\mathcal{NP}(\vec{\alpha},\mathcal{X})| = 5$, $|\mathcal{NPL}_A(\vec{\alpha},\mathcal{X})| = 10$ *(can exchange the position of the two lines)*, $|\mathcal{NPL}_{\mathcal{X}}(\vec{\alpha},\mathcal{X})| = 10$ *(can exchange the labels* x_1 *and* x_2 *which are both of size* 1*) and* $\left|\mathcal{NPL}_{A,\mathcal{X}}(\vec{\alpha},\mathcal{X})\right| = 20$. *See Figure 6.3, Figure 6.4 and Figure 6.5 for an illustration.*

Observation 6.1.7. Let $\vec{\alpha} = (\alpha_1,\ldots,\alpha_k) \in P_k$, $a = \alpha_1 + \cdots + \alpha_k$ and $p = k - a + 1$. Let

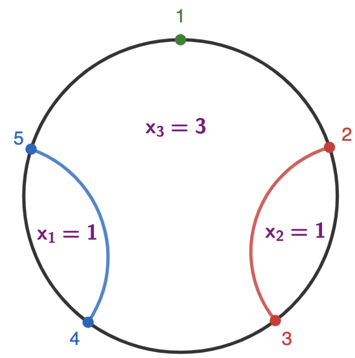


Figure 6.3: Example of $\vec{\Omega}^{\vec{\alpha}} = \Omega_2\Omega_4^2$ and $\mathcal{X} = \{1,1,3\}$ where $\pi = \{\{1\},\{2,3\},\{4,5\}\} \in \mathcal{NP}(\vec{\alpha},\mathcal{X})$. The other four partitions in $\mathcal{NP}(\vec{\alpha},\mathcal{X})$ are $\{\{2\},\{3,4\},\{5,1\}\}$, $\{\{3\},\{4,5\},\{1,2\}\}$, $\{\{4\},\{5,1\},\{2,3\}\}$ and $\{\{5\},\{1,2\},\{3,4\}\}$.

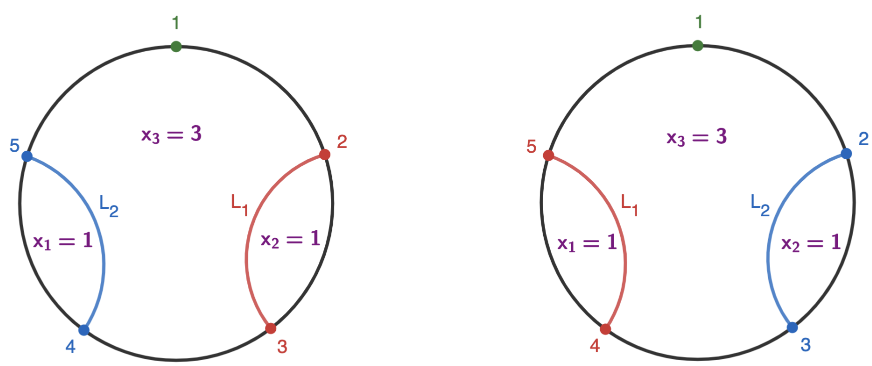


Figure 6.4: For $\mathcal{NPL}_A(\vec{\alpha},\mathcal{X})$, we can swap the position of the lines L_1 and L_2, doubling the counting for $\mathcal{NP}(\vec{\alpha},\mathcal{X})$.

$\mathcal{X} = \{x_1,\ldots,x_p\}$ be such that $x_1 + \cdots + x_p = k$. Then

$$|\mathcal{NP}(\vec{\alpha},\mathcal{X})| = \frac{\text{perm}(\mathcal{X})}{p!} \cdot |\mathcal{NPL}_{\mathcal{X}}(\vec{\alpha},\mathcal{X})| \tag{6.13}$$

and

$$|\mathcal{NP}(\vec{\alpha},\mathcal{X})| = \frac{1}{\alpha_1!\ldots\alpha_k!} \cdot |\mathcal{NPL}_A(\vec{\alpha},\mathcal{X})| \tag{6.14}$$

Theorem 6.1.8. *Let* $\vec{\alpha} = (\alpha_1,\ldots,\alpha_k) \in P_k$ *and* $a = \alpha_1 + \cdots + \alpha_k$, $p = k - a + 1$. *Let*

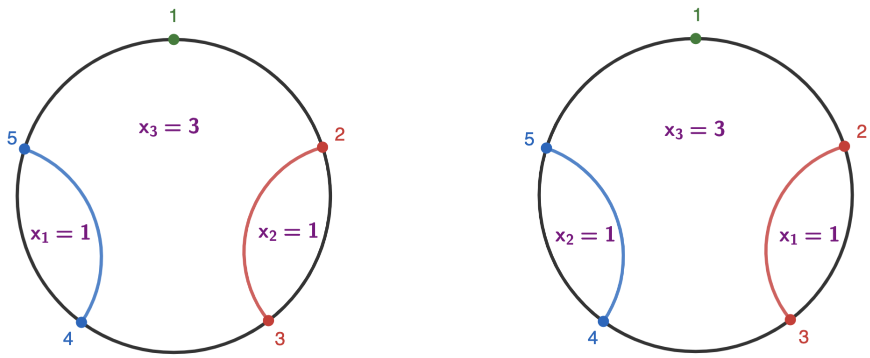


Figure 6.5: For $\mathcal{NPL}_{\mathcal{X}}(\vec{\alpha}, \mathcal{X})$, we can swap the position of the x_1 and x_2, doubling the counting for $\mathcal{NP}(\vec{\alpha}, \mathcal{X})$.

$x_1, \ldots, x_p$ be such that $x_1 + \cdots + x_p = k$. Then

$$\left|\mathcal{NPL}_{A,\mathcal{X}}(\vec{\alpha}, \mathcal{X})\right| = k \cdot (p-1)! \cdot (a-1)!\,. \tag{6.15}$$

A direct corollary is the following.

Corollary 6.1.9. *Let $\vec{\alpha} = (\alpha_1, \ldots, \alpha_k) \in P_k$ and $a = \alpha_1 + \cdots + \alpha_k$, $p = k - a + 1$. Let $x_1, \ldots, x_p$ be such that $x_1 + \cdots + x_p = k$. Then*

$$|\mathcal{NPL}_{\mathcal{X}}(\vec{\alpha}, \mathcal{X})| = k \cdot (p-1)! \cdot \frac{(a-1)!}{\alpha_1! \ldots \alpha_k!}\,. \tag{6.16}$$

To prove Theorem 6.1.8, we will start with the case of a single polygon.

Lemma 6.1.10. *Let $\vec{\alpha}$ corresponds to $\Omega_2^{k-t}\Omega_{2t}$. Let $\mathcal{X} = \{x_1, \ldots, x_t\}$ be such that $x_1 + \cdots + x_t = k$. Then*

$$\left|\mathcal{NPL}_{A,\mathcal{X}}(\vec{\alpha}, \mathcal{X})\right| = k \cdot (t-1)! \cdot (k-t)!\,. \tag{6.17}$$

Proof. Ω_{2t} corresponds to a t-gon on the cycle $\mathcal{C}_k$ and divides $\mathcal{C}_k$ into t parts of size $x_1, \ldots, x_t$. There are k ways to choose where to place the first vertex of the t-gon on $\mathcal{C}_k$, $t!$ ways to arrange the order of the x_i (corresponding to ways to put the remaining $t-1$ vertices on $\mathcal{C}_k$). For each

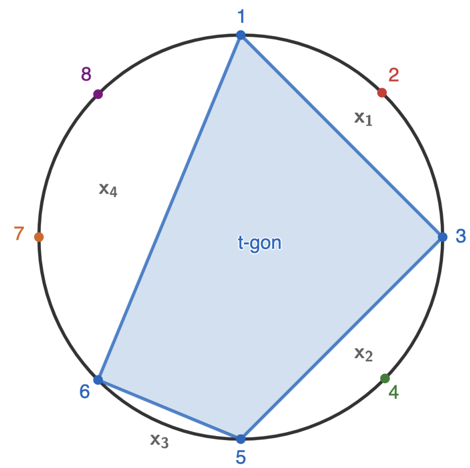


Figure 6.6: Illustration of Lemma 6.1.10. In this case we choose 1 to be the starting point of the t-gon, and x_1, x_2, x_3, x_4 as the order of x_i's. On the other hand, choosing 3 to be the starting point of the t-gon and x_2, x_3, x_4, x_1 as order of the x_i's results in the exact same configuration.

placement of the t-gon on $\mathcal{C}_k$, there are t choices for the first vertex to result in this placement. Since all the dots corresponding to Ω^{k-t} are considered to be distinct, there are $(k-t)!$ ways rearrange the order of these dots. Thus $\left|\mathcal{NPL}_{A,\mathcal{X}}(\vec{\alpha}, \mathcal{X})\right| = k \cdot t!/t \cdot (k-t)! = k \cdot (t-1)! \cdot (k-t)!$ as wanted. □

Lemma 6.1.11. *Let $\vec{\alpha}$ be corresponding to $\Omega_2^{\alpha_1} \Omega_{2t_1} \ldots \Omega_{2t_m}$ and $\vec{\beta}$ be corresponding to $\Omega_2^{\alpha_1+1} \Omega_{2t_1} \ldots \Omega_{2t_{m-2}} \Omega_{2(t_{m-1}+t_m-1)}$. Notice that $a = a = \alpha_1 + m$. Let $p = k - a + 1 = k - \alpha_1 - m + 1$ and $\mathcal{X} = \{x_1, \ldots, x_p\}$ be such that $x_1 + \cdots + x_p = k$. Then*

$$\left|\mathcal{NPL}_{A,\mathcal{X}}(\vec{\alpha}, \mathcal{X})\right| = \left|\mathcal{NPL}_{A,\mathcal{X}}\left(\vec{\beta}, \mathcal{X}\right)\right| . \tag{6.18}$$

Proof. We will show that there is a bijection between $\mathcal{NPL}_{A,\mathcal{X}}(\vec{\alpha}, \mathcal{X})$ and $\mathcal{NPL}_{A,\mathcal{X}}\left(\vec{\beta}, \mathcal{X}\right)$.

1. $\mathcal{NPL}_{A,\mathcal{X}}(\vec{\alpha}, \mathcal{X}) \to \mathcal{NPL}_{A,\mathcal{X}}\left(\vec{\beta}, \mathcal{X}\right)$: Let $\pi = \{P_1, \ldots, P_a\}$ be a non-crossing partition of the cycle $\mathcal{C}_k$ corresponding to distinct $\vec{\alpha}$ and $\mathcal{X}$. Assume P_{a-1} and P_a are the t_{m-1}-gon and t_m-gon, respectively. Assume $v_1, \ldots, v_{t_m}, w_1, \ldots, w_{t_{m-1}}$ are ordered clockwise on $\mathcal{C}_k$ where $P_a = \{v_1, \ldots, v_{t_m}\}$ and $P_{a-1} = \{w_1, \ldots, w_{t_{m-1}}\}$.

We now merge P_{a-1} and P_a to get P_b of size $(t_{m-1} + t_m - 1)$ and shift the other polygons in the following way:

i. Let $P_b = \Big\{v_1, \ldots, v_{t_m}\Big\} \bigcup \Big\{w_i - \Big(w_1 - v_{t_m}\Big) \mod k : i \in [t_{m-1}]\Big\}$. i.e. We shift the P_{a-1} polygon $\Big(w_1 - v_{t_m} \mod k\Big)$ units in the counter-clockwise direction so that the original vertex w_1 touches v_{t_m}.

ii. For any $i \in [a-2]$, let $P_i' = \Big\{p + \delta_1(p) \cdot \Big(w_{t_m} - w_1\Big) - \delta_2(p) \cdot \Big(w_1 - v_{t_m}\Big) \mod k : p \in P_i\Big\}$ where

$$\delta_1(p) = \begin{cases} 1 & \text{if } p \text{ is strictly in between } v_{t_m} \text{ and } w_1 \text{ in the clockwise direction} \\ 0 & \text{otherwise} \end{cases}$$

and

$$\delta_2(p) = \begin{cases} 1 & \text{if } p \text{ is strictly in between } w_1 \text{ and } w_{t_{m-1}} \text{ in the clockwise direction} \\ 0 & \text{otherwise} \end{cases}$$

i.e. For each of the remaining polygons P_i, we shift its vertices that are in between v_{t_m} and w_1 **clockwise** $\Big(w_{t_m} - w_1\Big)$ units and those in between w_1 and $w_{t_{m-1}}$ $\Big(w_1 - v_{t_m} \mod k\Big)$ units in the **counter-clockwise** direction.

iii. Let $s = w_{t_{m-1}}$.

The resulting $\pi' = \Big\{P_1', \ldots, P_{a-2}', \{s\}, P_b\Big\}$ is in $\mathcal{NPL}_{A,\mathcal{X}}\Big(\vec{\beta}, \mathcal{X}\Big)$. See Figure 6.7 for an illustration.

2. $\mathcal{NPL}_{A,\mathcal{X}}\Big(\vec{\beta}, \mathcal{X}\Big) \to \mathcal{NPL}_{A,\mathcal{X}}(\vec{\alpha}, \mathcal{X})$: Let $\pi = \{P_1, \ldots, P_b\}$ be a non-crossing partition of the cycle $\mathcal{C}_k$ corresponding to distinct $\vec{\beta}$ and $\mathcal{X}$. W.O.L.G assume $P_1 = \{s\}$ is a point, call s the special point, and $P_b = \{p_1, \ldots, p_{t_{m-1}+t_m-1}\}$ is the $(t_{m-1}+t_m-1)$-polygon.

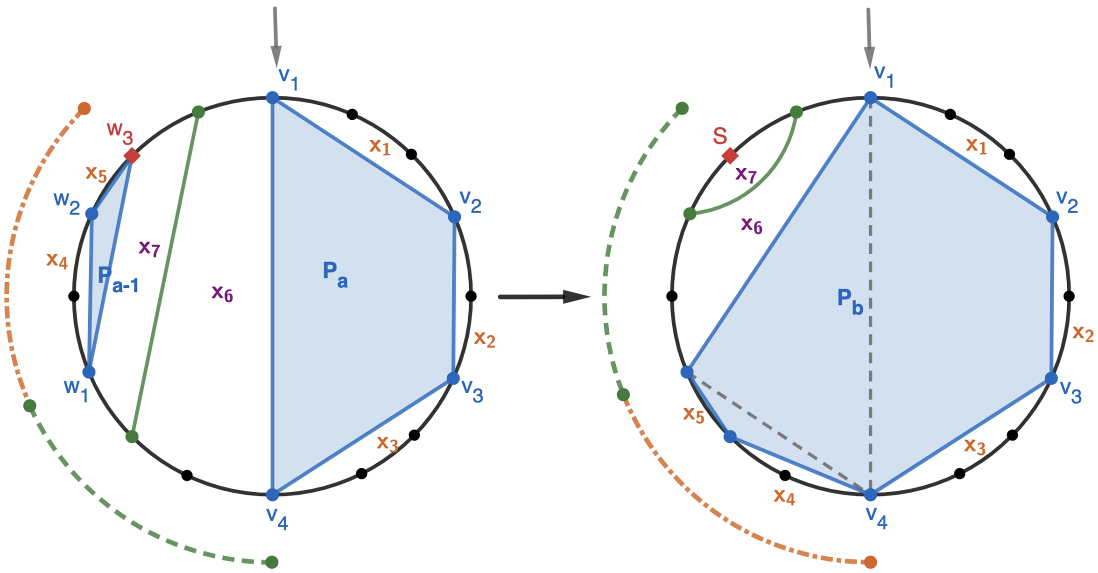


Figure 6.7: Illustration of $\mathcal{NPL}_{A,\mathcal{X}}(\vec{\alpha}, \mathcal{X}) \to \mathcal{NPL}_{A,\mathcal{X}}\left(\vec{\beta}, \mathcal{X}\right)$.

Now we will split P_b into P_{b,t_m} and $P_{b,t_{m-1}}$, a t_m-gon and a t_{m-1}-gon respectively, in the following way:

i. Let $P_{b,t_m} = \{p_1, \ldots, p_{t_m}\} \subseteq P_b$ be the first t_m points in the clockwise direction starting from s.

ii. Let $P_{b,t_{m-1}} = \{q_1, \ldots, q_{t_{m-1}}\}$ where $q_i = p_{t_m+i-1} + \left(s - p_{t_{m-1}+t_m-1}\right) \mod k$. i.e. We shift the t_{m-1}-gon $\{p_{t_m}, \ldots, p_{t_{m-1}+t_m-1}\}$ $\left(s - p_{t_{m-1}+t_m-1} \mod k\right)$ units so that the last vertex of the t_{m-1}-gon replace the original s.

iii. For any $i \in \{2, 3, \ldots, b-1\}$, let $P_i' = \left\{p + \delta_1(p) \cdot \left(s - p_{t_m+t_{m-1}-1}\right) - \delta_2(p) \cdot \left(p_{t_m+t_{m-1}-1} - p_{t_m}\right) \mod k : p \in P_i\right\}$ where

$$\delta_1(p) = \begin{cases} 1 & \text{if } p \text{ is strictly in between } p_{t_m} \text{ and } p_{t_m+t_{m-1}-1} \text{ in the clockwise direction} \\ 0 & \text{otherwise} \end{cases}$$

and

$$\delta_2(p) = \begin{cases} 1 & \text{if } p \text{ is strictly in between } p_{t_m+t_{m-1}-1} \text{ and } s \text{ in the clockwise direction} \\ 0 & \text{otherwise} \end{cases}$$

i.e. For each of the remaining polygons P_i, we shift its vertices that are in between p_{t_m} and $p_{t_m+t_{m-1}-1}$ **clockwise** $\left(s - p_{t_m+t_{m-1}-1}\right)$ units and those in between $p_{t_m+t_{m-1}-1}$ and s $\left(p_{t_m+t_{m-1}-1} - p_{t_m} \mod k\right)$ units in the **counter-clockwise** direction.

The resulting partition $\pi' = \{P'_2, \ldots, P'_{b-1}, P_{b,t_{m-1}}, P_{b,t_m}\}$ is in $\mathcal{NPL}_{A,\mathcal{X}}(\vec{\alpha}, \mathcal{X})$. See Figure 6.8 for an illustration.

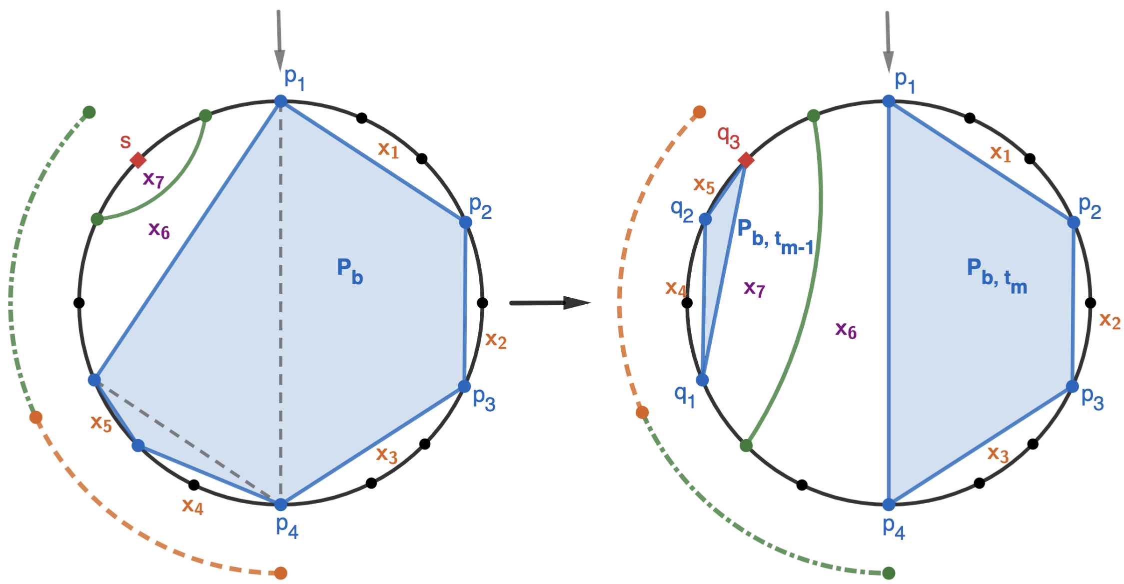


Figure 6.8: Illustration of $\mathcal{NPL}_{A,\mathcal{X}}\left(\vec{\beta}, \mathcal{X}\right) \to \mathcal{NPL}_{A,\mathcal{X}}(\vec{\alpha}, \mathcal{X})$.

3. It is not hard to see that the above two operations are reverses of each other.

□

Corollary 6.1.12. *Let $\vec{\alpha} \in P_k$. Let $a = \alpha_1 + m$ and $p = k - a + 1$. Let $\mathcal{X} = \left\{x_1, \ldots, x_p\right\}$ be such that $x_1 + \cdots + x_p = k$. Then*

$$\left|\mathcal{NPL}_{A,\mathcal{X}}\left(\vec{\alpha}, \mathcal{X}\right)\right| = k \cdot (p-1)! \cdot (a-1)! \,. \tag{6.19}$$

Proof. We can write $\vec{\Omega}^{\vec{\alpha}}$ as $\Omega_2^{\alpha_1}\Omega_{2t_1}\ldots\Omega_{2t_m}$ where t_i's are not necessarily distinct. eg. $\Omega_2^2\Omega_4^3\Omega^6$ can be written as $\Omega_2^2\Omega_4\Omega_4\Omega_4\Omega_6$.

We can prove this by induction on m.

1. Base case $m = 0$: $\mathcal{NPL}_{A,\mathcal{X}}\left(\vec{\alpha}, \mathcal{X}\right) = k!$ and $k \cdot (p-1)! \cdot (a-1)! = k \cdot 0! \cdot (k-1)! = k!$.

2. Base case $m = 1$: $a = k - t + 1$. By Lemma 6.1.10, $\left|\mathcal{NPL}_{A,\mathcal{X}}\left(\vec{\alpha}, \mathcal{X}\right)\right| = k \cdot (t-1)! \cdot (k-t)!$ and $RHS = k \cdot (t-1)! \cdot (a-1)! = k \cdot (t-1)! \cdot (k-t)!$.

3. Inductive case $(m-1) \implies m$: By Lemma 6.1.11 and the inductive hypothesis on the term $\vec{\Omega}^{\vec{\beta}} = \Omega_2^{\alpha_1+1}\Omega_{2t_1}\ldots\Omega_{2t_{m-2}}\Omega_{2(t_{m-1}+t_m-1)}$, which has $b = \alpha_1 + 1 + m - 1 = \alpha_1 + m = a$ and $p' = k - b + 1 = k - a + 1 = p$, we get

$$\begin{aligned}\left|\mathcal{NPL}_{A,\mathcal{X}}\left(\vec{\alpha}, \mathcal{X}\right)\right| &= \left|\mathcal{NPL}_{A,\mathcal{X}}\left(\Omega_2^{\alpha_1+1}\Omega_{2t_1}\ldots\Omega_{2t_{m-2}}\Omega_{2(t_{m-1}+t_m-1)}, \mathcal{X}\right)\right| \\ &= k \cdot (p'-1)! \cdot (b-1)! = k \cdot (p-1)! \cdot (a-1)! \,.\end{aligned}$$

□

To prove Theorem 6.1.1, we still need the following Catalan number identity.

Lemma 6.1.13. *Let $C_n = \dfrac{1}{n+1}\dbinom{2n}{n}$ denote the n^{th} Catalan number. Then*

$$\sum_{i_j \geq 0:\, i_1 + \cdots + i_k = n} C_{i_1}C_{i_2}\ldots C_{i_k} = k \cdot \frac{(2n+k-1)!}{n!(n+k)!} \tag{6.20}$$

Definition 6.1.14. We denote $A_n^{(k)} = \sum_{i_j \geq 0 : i_1 + \cdots + i_k = n} C_{i_1} C_{i_2} \ldots C_{i_k}$.

Proposition 6.1.15. *$A_n^{(1)} = C_n$ and $A_n^{(2)} = C_{n+1}$.*

Proof. $A_n^{(1)} = C_n$ by the definition of $A_n^{(k)}$. $A_n^{(2)} = \sum_{i_1 + i_2 = n} C_{i_1} C_{i_2} = C_{n+1}$. □

Proof of Lemma 6.1.13. We first prove that $A_n^{(k)} = A_{n+1}^{(k-1)} - A_{n+1}^{(k-2)}$, and then prove the result by induction on k.

Claim 6.1.16. $A_n^{(k)} = A_{n+1}^{(k-1)} - A_{n+1}^{(k-2)}$.

Proof.

$$
\begin{aligned}
A_n^{(k)} &= \sum_{i_j \geq 0 : i_1 + \cdots + i_k = n} C_{i_1} C_{i_2} \ldots C_{i_k} = \sum_{\substack{i_j \geq 0, r \geq 0: \\ i_1 + \cdots + i_{k-2} + r = n}} C_{i_1} C_{i_2} \ldots C_{i_{k-2}} \cdot \left(\sum_{i_{k-1} + i_k = r} C_{i_{k-1}} C_{i_k} \right) \\
&= \sum_{\substack{i_j \geq 0, r \geq 0: \\ i_1 + \cdots + i_{k-2} + r = n}} C_{i_1} C_{i_2} \ldots C_{i_{k-2}} \cdot C_{r+1} = \sum_{\substack{i_1, \ldots, i_{k-2} \geq 0, i'_{k-1} \geq 1: \\ i_1 + \cdots + i_{k-2} + i'_{k-1} = n+1}} C_{i_1} C_{i_2} \ldots C_{i_{k-2}} \cdot C_{i'_{k-1}} \\
&= \left(\sum_{\substack{i_j \geq 0: \\ i_1 + \cdots + i_{k-2} + i'_{k-1} = n+1}} C_{i_1} C_{i_2} \ldots C_{i_{k-2}} \cdot C_{i'_{k-1}} \right) - \left(\sum_{\substack{i_j \geq 0: \\ i_1 + \cdots + i_{k-2} = n+1}} C_{i_1} C_{i_2} \ldots C_{i_{k-2}} \cdot C_0 \right) \\
&= A_{n+1}^{(k-1)} - A_{n+1}^{(k-2)}.
\end{aligned}
$$

□

Now we prove the lemma by induction on k.

1. Base case: $A_n^{(0)} = 0 = 0 \cdot \frac{(2n-1)!}{n!n!}$. $A_n^{(1)} = C_n = \frac{1}{n+1} \cdot \frac{(2n)!}{n!n!}$ and $RHS = \frac{(2n)!}{n!(n+1)!}$.

2. Assume the result holds for $k-1$, then

$$A_n^{(k)} = (k-1)\cdot\frac{(2n+k)!}{(n+1)!(n+k)!} - (k-2)\cdot\frac{(2n+k-1)!}{(n+1)!(n+k-1)!}$$

$$= \frac{(2n+k-1)!}{n!(n+k)!}\cdot\left(\frac{(k-1)(2n+k)}{n+1} - \frac{(k-2)(n+k)}{n+1}\right)$$

$$= \frac{(2n+k-1)!}{n!(n+k)!}\cdot\left(\frac{nk+k}{n+1}\right) = k\cdot\frac{(2n+k-1)!}{n!(n+k)!}.$$

□

Corollary 6.1.17. *We can rewrite Lemma 6.1.13 as*

$$\sum_{i_j\geq 0\,:\,i_1+\cdots+i_k=n-k} C_{i_1}\dots C_{i_k} = k\cdot\frac{(2n-k-1)!}{(n-k)!n!} \tag{6.21}$$

or

$$\sum_{i_j\geq 1\,:\,i_1+\cdots+i_k=n} C_{i_1-1}\dots C_{i_k-1} = k\cdot\frac{(2n-k-1)!}{(n-k)!n!}. \tag{6.22}$$

Now we are ready to prove Theorem 6.1.1.

Theorem 6.1.1. *Let* $\vec{\alpha} = (\alpha_1,\dots,\alpha_k) \in P_k$ *and* $a = \alpha_1+\cdots+\alpha_k$. *Then*

$$\sum_{\pi\in\mathcal{NP}(\vec{\alpha})}\prod_{x\in S_\pi} C_{x-1} = \binom{a+k-2}{k-1}\cdot\frac{(a-1)!}{\alpha_1!\dots\alpha_k!} \tag{6.8}$$

Proof. Recall that $p = k - a + 1$. By Theorem 6.1.8 and Corollary 6.1.17,

$$\begin{aligned}
\sum_{\pi \in \mathcal{NP}(\vec{\alpha})} \prod_{i_j \in S_\pi} C_{i_j - 1} &= \sum_{i_j \geq 1 \colon i_1 + \cdots + i_p = k} \frac{\left|\mathcal{NP}\left(\vec{\alpha}, i_1, \ldots, i_p\right)\right|}{\text{perm}\left(i_1, \ldots, i_p\right)} \cdot C_{i_1 - 1} \cdot \cdots \cdot C_{i_p - 1} \\
&= \sum_{i_j \geq 1 \colon i_1 + \cdots + i_p = k} \frac{\left|\mathcal{NPL}_{\mathcal{X}}\left(\vec{\alpha}, i_1, \ldots, i_p\right)\right|}{p\,!} \cdot C_{i_1 - 1} \cdot \cdots \cdot C_{i_p - 1} \\
&= \sum_{i_j \geq 1 \colon i_1 + \cdots + i_p = k} \left(\frac{(a-1)!}{\alpha_1! \ldots \alpha_k!} \cdot k \cdot (p-1)!/p\,!\right) \cdot C_{i_1 - 1} \cdot \cdots \cdot C_{i_p - 1} \\
&= \left(\frac{(a-1)!}{\alpha_1! \ldots \alpha_k!} \cdot k \cdot (p-1)!/p\,!\right) \cdot \left(p \cdot \frac{(2k-p-1)!}{(k-p)!\,k!}\right) \\
&= \frac{(a-1)!}{\alpha_1! \ldots \alpha_k!} \cdot \frac{(k+a-2)!}{(a-1)!(k-1)!} \\
&= \frac{(a-1)!}{\alpha_1! \ldots \alpha_k!} \cdot \binom{k+a-2}{k-1}.
\end{aligned}$$

□

6.2 The General Case

Recall that

$$\mathcal{NP}\left(\vec{\alpha}, \vec{\beta}\right) := \left\{(\pi, \sigma) : \pi \in \mathcal{NP}(\vec{\alpha}), \sigma \in \mathcal{NP}(\vec{\beta}), \pi \oplus \sigma \in NC(k)\right\} \tag{6.23}$$

We want to prove the following.

$$\sum_{(\pi,\sigma) \in \mathcal{NP}\left(\vec{\alpha}, \vec{\beta}\right)} \prod_{x \in S_{\pi \oplus \sigma}} C_{x-1} = k \cdot \binom{A+B-2}{k-1} \cdot \frac{(A-1)!}{\alpha_1! \ldots \alpha_k!} \cdot \frac{(B-1)!}{\beta_1! \ldots \beta_k!} \tag{6.24}$$

Definition 6.2.1. Let $\vec{\alpha}, \vec{\beta} \in P_k$ and $p = k - (a+b) + 1$. Let $\mathcal{X} = \left\{x_1, \ldots, x_p\right\}$ be such that $x_1 + \cdots + x_p = k$. We denote

1. $\mathcal{NP}\left(\vec{\alpha},\vec{\beta},\mathcal{X}\right)$ to be

$$\mathcal{NP}\left(\vec{\alpha},\vec{\beta},\mathcal{X}\right)=\left\{(\pi,\sigma)\in\mathcal{NP}\left(\vec{\alpha},\vec{\beta}\right):S_{\pi\oplus\sigma}=\{x_1,\ldots,x_p\}\right\}, \tag{6.25}$$

reads "the set of $\mathcal{N}$oncrossing $\mathcal{P}$artitions corresponding to $\vec{\alpha}$ and $\vec{\beta}$ that partition $\mathcal{C}_k$ into parts of sizes $\mathcal{X}$ ".

2. $\mathcal{NPL}_A\left(\vec{\alpha},\vec{\beta},\mathcal{X}\right)$ to be

$$\mathcal{NPL}_A\left(\vec{\alpha},\vec{\beta},\mathcal{X}\right)=\left\{\begin{array}{l}(\pi,\sigma,\tau_1,\tau_2):(\pi,\sigma)\in\mathcal{NP}\left(\vec{\alpha},\vec{\beta},\mathcal{X}\right),\\ \hfill\tau_1:[a]\to[a]\text{ where }|P_i|=\left|P_{\tau(i)}\right|\text{ for all }i\in[a]\\ \hfill\tau_2:[b]\to[b]\text{ where }|Q_i|=\left|Q_{\tau(i)}\right|\text{ for all }i\in[b]\end{array}\right\}. \tag{6.26}$$

3. $\mathcal{NPL}_{\mathcal{X}}\left(\vec{\alpha},\vec{\beta},\mathcal{X}\right)$ to be

$$\mathcal{NPL}_A\left(\vec{\alpha},\vec{\beta},\mathcal{X}\right)=\left\{\begin{array}{l}(\pi,\sigma,\rho):(\pi,\sigma)\in\mathcal{NP}\left(\vec{\alpha},\vec{\beta},\mathcal{X}\right),\\ \hfill\rho\text{ is a label of }\mathcal{C}_k/\pi\oplus\sigma\text{ corresponding to }\mathcal{X}\end{array}\right\}, \tag{6.27}$$

reads "the set of $\mathcal{N}$oncrossing $\mathcal{P}$artitions corresponding to $\vec{\alpha}$ and $\vec{\beta}$ that partition $\mathcal{C}_k$ into parts $\mathcal{L}$abeled with $\mathcal{X}$ ".

4. $\mathcal{NPL}_{A,\mathcal{X}}\left(\vec{\alpha},\vec{\beta},\mathcal{X}\right)$ to be

$$\mathcal{NPL}_{A,\mathcal{X}}\left(\vec{\alpha},\vec{\beta},\mathcal{X}\right)=\left\{\begin{array}{l}(\pi,\sigma,\tau_1,\tau_2,\rho):(\pi,\sigma,\tau_1,\tau_2)\in\mathcal{NPL}_A\left(\vec{\alpha},\vec{\beta},\mathcal{X}\right),\\ \hfill\rho\text{ is a label of }\mathcal{C}_k/\pi\oplus\sigma\text{ corresponding to }\mathcal{X}\end{array}\right\}. \tag{6.28}$$

Lemma 6.2.2. *Let $\vec{\alpha}$ and $\vec{\beta}$ be corresponding to $\Omega_2^{k-t_a}\Omega_{2t_a}$ and $\Omega_2^{k-t_b}\Omega_{2t_b}$, respectively. Let $a=k-t_a+1$, $b=k-t_b+1$ and $p=t_a+t_b-1=2k-a-b+1$. Let $\mathcal{X}=\{x_1,\ldots,x_p\}$ be* *such that* $x_1 + \cdots + x_p = k$. *Then*

$$\left|\mathcal{NPL}_{A,\mathcal{X}}\left(\vec{\alpha},\vec{\beta},\mathcal{X}\right)\right| = k^2(p-1)!\cdot(a-1)!\cdot(b-1)!\,. \tag{6.29}$$

Proof. Let $\vec{\gamma}$ corresponds to $\Omega_2^{k-t_a-t_b+1}\Omega_{2(t_a+t_b-1)}$. Note that $p_c = t_a+t_b-1 = p$. We will prove that there is a bijection between $\mathcal{NPL}_{\mathcal{X}}\left(\vec{\alpha},\vec{\beta},\mathcal{X}\right)$ and $\mathcal{NPL}_{\mathcal{X}}\left(\vec{\gamma},\mathcal{X}\right)\times[k]$.

1. $\mathcal{NPL}_{\mathcal{X}}\left(\vec{\alpha},\vec{\beta},\mathcal{X}\right) \to \mathcal{NPL}_{\mathcal{X}}\left(\vec{\gamma},\mathcal{X}\right)\times[k]$: let $\mathcal{P}\cup\mathcal{Q}$ be a non-crossing partition of $[k]$ corresponding to $\vec{\alpha}$ and $\vec{\beta}$ and labels $\mathcal{X}$. In this case $\mathcal{P}$ contains a polygon P_a of size t_a and $\mathcal{Q}$ contains a polygon P_b of size t_b. Assume $P_b = \{p_1,\ldots,p_{t_b}\}$ and $P_a = \{q_1,\ldots,q_{t_a}\}$ such that $p_1,\ldots,p_{t_b},q_1,\ldots,q_{t_a}$ are ordered clockwise on $\mathcal{C}_k$.

   We now merge P_a and P_b to get P_c of size (t_a+t_b-1) in the following way: let $P_c = \left\{p_1,\ldots,p_{t_b}\right\}\cup\left\{q_i-\left(q_1-p_{t_b}\right) \mod k : i\in[t_a]\right\}$. i.e. We shift the P_a polygon $\left(q_1-p_{t_b} \mod k\right)$ units in the counter-clockwise direction so that the original vertex q_1 touches p_{t_b}. We further let $s = q_{t_a}$. The resulting P_c corresponds to a partition in $\mathcal{NPL}_{\mathcal{X}}\left(\vec{\gamma},\mathcal{X}\right)$ and $s\in[k]$. See Figure 6.9 for an illustration.

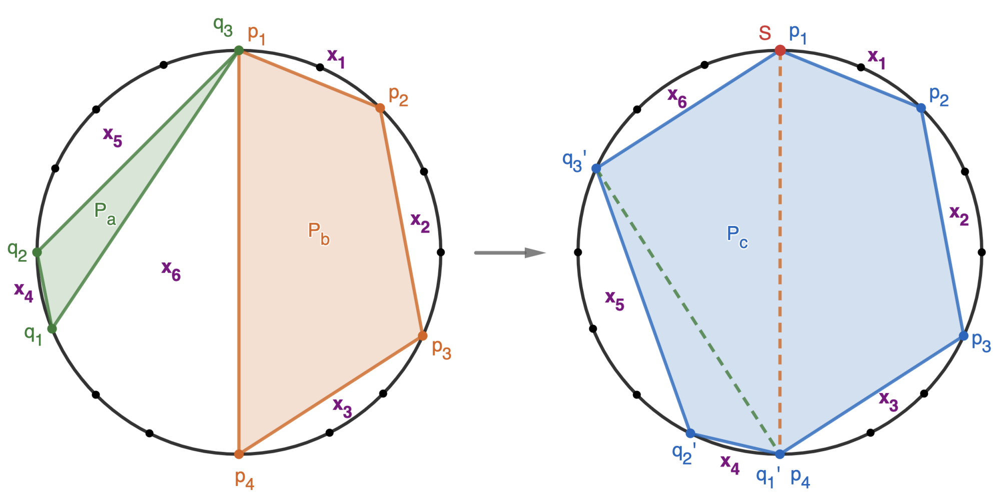


Figure 6.9: Illustration of $\mathcal{NPL}_{\mathcal{X}}\left(\vec{\alpha},\vec{\beta},\mathcal{X}\right)\to\mathcal{NPL}_{\mathcal{X}}\left(\vec{\gamma},\mathcal{X}\right)\times[k]$: here $p_1=q_3=s$, $t_a=3$, $t_b=4$ and $t_c=6$.

2. $\mathcal{NPL}_{\mathcal{X}}(\vec{\gamma}, \mathcal{X}) \times [k] \to \mathcal{NPL}_{\mathcal{X}}\left(\vec{\alpha}, \vec{\beta}, \mathcal{X}\right)$: Let $\mathcal{P}$ be a non-crossing partition of $[k]$ corresponding to $\vec{\gamma}$ and labels $\mathcal{X}$, and s be a point in $[k]$. In this case $\mathcal{P}$ contains a polygon P_c of size $t_c = t_a + t_b - 1$. Assume $P_c = \{p_1, \ldots, p_{t_c}\}$ such that p_i's are ordered in the clockwise direction starting from s. Note that s might be a point in P_c and then in that case $p_1 = s$. Now we will split P_c into P_a and P_b in the following way:

   i. Let $P_b = \{p_1, \ldots, p_{t_b}\}$.

   ii. Let $P_a = \{q_1, \ldots, q_{t_a}\}$ where $q_i = p_{t_b+i-1} + \left(s - p_{t_c}\right) \mod k$ for each $i \in [t_a]$. i.e. We shift the $\{p_{t_b}, \ldots, p_{t_c}\}$ polygon $\left(s - p_{t_c} \mod k\right)$ units in the clockwise direction so that the last vertex of the t_a-gon touches s.

   Note that in the case of $s = p_1$, P_a and P_b touches at s, which is allowed for $\mathcal{NP}\left(\vec{\alpha}, \vec{\beta}\right)$. The resulting P_a and P_b correspond to a partition in $\mathcal{NPL}_{\mathcal{X}}\left(\vec{\alpha}, \vec{\beta}, \mathcal{X}\right)$. See Figure 6.10 for an illustration.

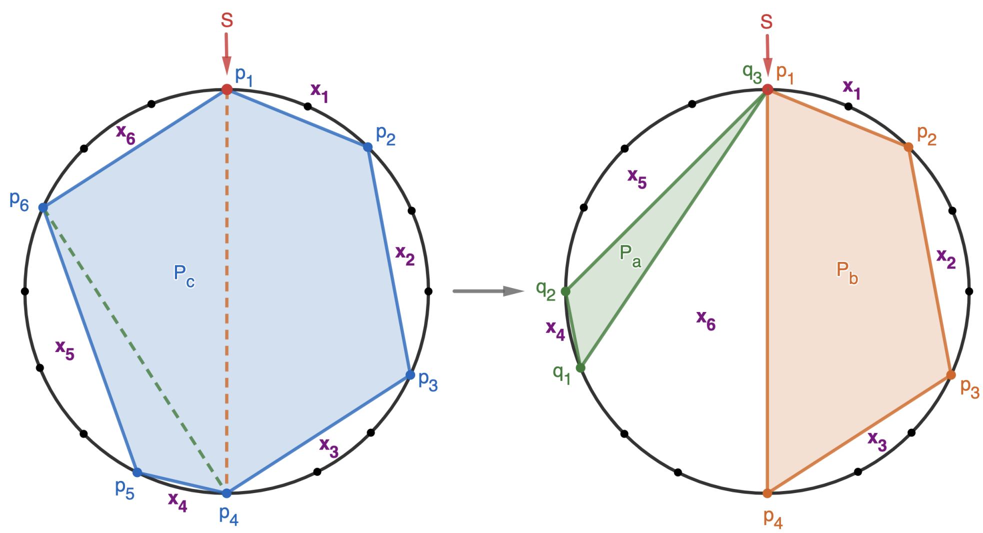


Figure 6.10: Illustration of $\mathcal{NPL}_{\mathcal{X}}(\vec{\gamma}, \mathcal{X}) \times [k] \to \mathcal{NPL}_{\mathcal{X}}\left(\vec{\alpha}, \vec{\beta}, \mathcal{X}\right)$: here $s = p_1$, $t_a = 3$, $t_b = 4$ and $t_c = 6$.

3. It is not hard to see that the above two operations are inverses of each other.

Thus $\left|\mathcal{NPL}_{\mathcal{X}}\left(\vec{\alpha},\vec{\beta},\mathcal{X}\right)\right| = \left|\mathcal{NPL}_{\mathcal{X}}\left(\vec{\gamma},\mathcal{X}\right)\times[k]\right| = \left|\mathcal{NPL}_{\mathcal{X}}\left(\vec{\gamma},\mathcal{X}\right)\right|\cdot k = k\cdot(p-1)!\cdot k = k^2(p-1)!$. To consider $\mathcal{NPL}_{A,\mathcal{X}}\left(\vec{\alpha},\vec{\beta},\mathcal{X}\right)$, we take into account the fact that all the $(a-1)$ points in $\vec{\alpha}$ and $(b-1)$ points in $\vec{\beta}$ can be permuted, thus $\mathcal{NPL}_{A,\mathcal{X}}\left(\vec{\alpha},\vec{\beta},\mathcal{X}\right) = k^2(p-1)!\cdot(a-1)!\cdot(b-1)!$. □

Theorem 6.2.3. *Let* $\vec{\alpha} = (\alpha_1,\ldots,\alpha_k), \vec{\beta} = (\beta_1,\ldots,\beta_k) \in P_k$ *and* $a = \alpha_1+\cdots+\alpha_k$, $b = \beta_1+\ldots,\beta_k$, $p = 2k-a-b+1$. *Let* $\mathcal{X} = \{x_1,\ldots,x_p\}$ *be such that* $x_1+\cdots+x_p = k$. *Then*

$$\left|\mathcal{NPL}_{A,\mathcal{X}}\left(\vec{\alpha},\vec{\beta},\mathcal{X}\right)\right| = k^2(p-1)!\cdot(a-1)!\cdot(b-1)!. \tag{6.30}$$

Proof. We will applying the inductive combining argument for $\mathcal{NPL}_{A,\mathcal{X}}\left(\vec{\alpha},\mathcal{X}\right)$ from the last section to $\mathcal{NPL}_{A,\mathcal{X}}\left(\vec{\alpha},\vec{\beta},\mathcal{X}\right)$, until we reach the case when Ω has only one polygon. Each combining step of a t_1-gon and t_2-gon gives a t_1+t_2-1-gon and an extra point. So in the end we will obtain a polygon of size $2\alpha_2+3\alpha_3+\cdots+k\alpha_k-(\alpha_2+\cdots+\alpha_k-1) = (k-\alpha_1)-(a-\alpha_1-1) = k-a+1$ and $\alpha_1+(\alpha_2+\cdots+\alpha_k-1) = a-1$ points. Let $\vec{\alpha}'$ corresponds to $\Omega_2^{a-1}\Omega_{2(k-a+1)}$. Thus $\mathcal{NPL}_{A,\mathcal{X}}\left(\vec{\alpha},\vec{\beta},\mathcal{X}\right) = \mathcal{NPL}_{A,\mathcal{X}}\left(\overrightarrow{\alpha'},\vec{\beta},\mathcal{X}\right)$.

Similarly we can apply the same argument to $\vec{\beta}$ and result in $\Omega_2^{b-1}\Omega_{2(k-b+1)}$. Let $\vec{\beta}'$ correspond to $\Omega_2^{b-1}\Omega_{2(k-b+1)}$. Then

$$\mathcal{NPL}_{A,\mathcal{X}}\left(\vec{\alpha},\vec{\beta},\mathcal{X}\right) = \mathcal{NPL}_{A,\mathcal{X}}\left(\overrightarrow{\alpha'},\vec{\beta},\mathcal{X}\right) = \mathcal{NPL}_{A,\mathcal{X}}\left(\overrightarrow{\alpha'},\overrightarrow{\beta'},\mathcal{X}\right)$$

Note that $a' = a-1+1 = a$, $b' = b-1+1 = b$ and $p' = (k-a+1)+(k-b+1)-1 = 2k-a-b+1 = p$. By Lemma 6.2.2, $\mathcal{NPL}_{A,\mathcal{X}}\left(\overrightarrow{\alpha'},\overrightarrow{\beta'},\mathcal{X}\right) = k^2(p'-1)!(a'-1)!(b'-1)! = k^2(p-1)!(a-1)!(b-1)!$. □

Corollary 6.2.4. *Let* $\vec{\alpha} = (\alpha_1,\ldots,\alpha_k), \vec{\beta} = (\beta_1,\ldots,\beta_k) \in P_k$ *and* $a = \alpha_1+\cdots+\alpha_k$, $b = \beta_1+\ldots,\beta_k$, $p = 2k-a-b+1$. *Let* $\mathcal{X} = \{x_1,\ldots,x_p\}$ *be such that* $x_1+\cdots+x_p = k$. *Then*

$$\left|\mathcal{NPL}_{\mathcal{X}}\left(\vec{\alpha},\vec{\beta},\mathcal{X}\right)\right| = k^2(p-1)!\cdot\frac{(a-1)!}{\alpha_1!\ldots\alpha_k!}\cdot\frac{(b-1)!}{\beta_1!\ldots\beta_k!}. \tag{6.31}$$

Now we are ready to prove Theorem 6.0.11.

Theorem 6.0.11. *Let* $\vec{\alpha}, \vec{\beta} \in P_k$ *Then*

$$\sum_{(\pi,\sigma)\in\mathcal{NP}\left(\vec{\alpha},\vec{\beta}\right)} \prod_{x\in S_{\pi\oplus\sigma}} C_{x-1} = k \cdot \binom{A+B-2}{k-1} \cdot \frac{(A-1)!}{\alpha_1!\ldots\alpha_k!} \cdot \frac{(B-1)!}{\beta_1!\ldots\beta_k!} \tag{6.7}$$

Proof. By Theorem 6.2.3 and Corollary 6.1.17,

$$\begin{aligned}
&\sum_{\substack{\pi\in\mathcal{NP}(\vec{\alpha}),\sigma\in\mathcal{NP}(\vec{\beta}):\\ \pi\oplus\sigma\in NC(k)}} \prod_{x\in S_{\pi\oplus\sigma}} C_{x-1} = \sum_{(\pi,\sigma)\in\mathcal{NP}\left(\vec{\alpha},\vec{\beta}\right)} \prod_{i_j\in S_{\pi\oplus\sigma}} C_{i_j-1}\\
&= \sum_{i_j\geq 1:\, i_1+\cdots+i_p=k} \frac{\left|\mathcal{NP}\left(\vec{\alpha},\vec{\beta},i_1,\ldots,i_p\right)\right|}{\operatorname{perm}\left(i_1,\ldots,i_p\right)} \cdot C_{i_1-1}\cdot\cdots\cdot C_{i_p-1}\\
&= \sum_{i_j\geq 1:\, i_1+\cdots+i_p=k} \frac{\left|\mathcal{NPL}_{\mathcal{X}}\left(\vec{\alpha},\vec{\beta},i_1,\ldots,i_p\right)\right|}{p\,!} \cdot C_{i_1-1}\cdot\cdots\cdot C_{i_p-1}\\
&= \sum_{i_j\geq 1:\, i_1+\cdots+i_p=k} \left(\frac{(a-1)!}{\alpha_1!\ldots\alpha_k!}\cdot\frac{(b-1)!}{\beta_1!\ldots\beta_k!}\cdot k^2\cdot(p-1)!/p\,!\right)\cdot C_{i_1-1}\cdot\cdots\cdot C_{i_p-1}\\
&= \frac{(a-1)!}{\alpha_1!\ldots\alpha_k!}\cdot\frac{(b-1)!}{\beta_1!\ldots\beta_k!}\cdot k^2\cdot\frac{1}{p}\cdot\left(p\cdot\frac{(2k-p-1)!}{(k-p)!\,k!}\right)\\
&= \frac{(a-1)!}{\alpha_1!\ldots\alpha_k!}\cdot\frac{(b-1)!}{\beta_1!\ldots\beta_k!}\cdot k\cdot\frac{(a+b-2)!}{(a+b-k-1)!\,(k-1)!}\\
&= \frac{(a-1)!}{\alpha_1!\ldots\alpha_k!}\cdot\frac{(b-1)!}{\beta_1!\ldots\beta_k!}\cdot k\cdot\binom{a+b-2}{k-1}.
\end{aligned}$$

□

CHAPTER 7

SPECIAL CASE: $\Omega_{Z(m)} \circ_R \Omega$

In this chapter, we analyze the random matrix $M_{Z(m),\Omega} = D'RD$ where $\Omega' = \Omega_{Z(m)}$ and Ω is an arbitrary distribution. i.e. $\Omega'_{2i} = C(i,m)$ for all i.

Our main result is as follows.

Theorem 7.0.1. *Let $\Omega_{Z(m)}$ be defined as in Definition 2.2.19. Then*

$$\left(\Omega_{Z(m)} \circ_R \Omega\right)_{2k} = \sum_{\vec{\alpha} \in P_k} \binom{mk}{a-1} \cdot \frac{(a-1)!}{\alpha_1! \ldots \alpha_k!} \cdot \vec{\Omega}^{\vec{\alpha}}. \tag{7.1}$$

After proving this result, we prove Theorem 2.2.24 by plugging in $\Omega = \Omega_{Z(m')}$ to the above theorem.

Theorem 2.2.24. *For any $m, m' \in \mathbb{N}$,*

$$\Omega_{Z(m)} \circ_R \Omega_{Z(m')} = \Omega_{Z(m+m')} \tag{2.12}$$

7.1 Proof of the Formula

Definition 7.1.1. Let $\Omega' = \Omega_{Z(m)}$. Given a distribution Ω, we define $A_m(k,0)$ to be

$$A_m(k,0) = \sum_{\vec{\alpha},\vec{\beta} \in P_k} C\left(\vec{\alpha},\vec{\beta}\right) \cdot \vec{\Omega}^{\vec{\alpha}} \cdot \vec{\Omega'}^{\vec{\beta}} = \sum_{\vec{\alpha},\vec{\beta} \in P_k} C\left(\vec{\alpha},\vec{\beta}\right) \cdot \vec{\Omega}^{\vec{\alpha}} \cdot \left(\prod_{i=1}^{k} C(i,m)^{\beta_i}\right). \tag{7.2}$$

where

$$C(\vec{\alpha},\vec{\beta}) = (-1)^{a+b-k-1} \cdot k \cdot \binom{a+b-2}{k-1} \cdot \frac{(a-1)!}{\alpha_1! \ldots \alpha_k!} \cdot \frac{(b-1)!}{\beta_1! \ldots \beta_k!}. \tag{7.3}$$

Note that by Corollary 2.2.16, $A_m(k,0) = \left(\Omega_{Z(m)} \circ_R \Omega\right)_{2k}$. A restatement of Theorem 7.0.1 is the following.

Theorem 7.1.2.

$$A_m(k,0) = \sum_{\vec{\alpha}\in P_k} \binom{mk}{a-1} \cdot \frac{(a-1)!}{\alpha_1! \dots \alpha_k!} \cdot \vec{\Omega}^{\vec{\alpha}}. \tag{7.4}$$

To prove the above theorem, we need the following identities.

Proposition 7.1.3.

$$\sum_{i=0}^{m-k} (-1)^{i+(m-k)} \cdot \binom{m-i}{k} \cdot \binom{n}{i} = \binom{n-k-1}{m-k}. \tag{7.5}$$

Proof. Note that Proposition 7.1.3 is always true when $k > m$ since it becomes $0 = 0$. W.O.L.G we will assume $k \le m$. We prove this by induction on m.

1. $m = 0$: Since $k \le m < n$, $k = 0$ and $n \ge 1$. Then $LHS = \binom{0}{0} \cdot \binom{n}{0} = 1$ and $RHS = \binom{n-1}{0} = 1$.

2. $m \implies (m+1)$: Assume Proposition 7.1.3 is true for all triples (n', m', k') where $m' \le m$ and $k' \le m' < n'$. For any n, k such that $k \le m+1 < n$,

$$\begin{aligned}
&\sum_{i=0}^{m+1-k} (-1)^{i+m+1-k} \cdot \binom{m+1-i}{k} \cdot \binom{n}{i} \\
=&\sum_{i=0}^{m+1-k} (-1)^{i+m+1-k} \cdot \left(\binom{m-i}{k} + \binom{m-i}{k-1}\right) \cdot \binom{n}{i} \\
=&(-1) \cdot \left(\sum_{i=0}^{m-k} (-1)^{i+m-k} \cdot \binom{m-i}{k} \cdot \binom{n}{i}\right) + \binom{k-1}{k} \cdot \binom{n}{m+1-k} \\
&\quad + \sum_{i=0}^{m-(k-1)} (-1)^{i+m-(k-1)} \cdot \binom{m-i}{k-1} \cdot \binom{n}{i} \\
=&-\binom{n-k-1}{m-k} + 0 + \binom{n-(k-1)-1}{m-(k-1)} \\
=&-\binom{n-k-1}{m-k} + \binom{n-k}{m-k+1} = \binom{n-k-1}{m-k+1} = \binom{n-k-1}{(m+1)-k}.
\end{aligned}$$

□

Proposition 7.1.4.

$$\sum_{i=0}^{n} (-1)^{n-i} \cdot \binom{m+i}{k}\binom{n}{i} = \binom{m}{k-n}. \tag{7.6}$$

Proof. We prove this by induction on n.

1. $n = 0$: $LHS = \binom{m+0}{k}\binom{0}{0} = \binom{m}{k} = RHS$.

2. $n \implies (n+1)$: Assume $\sum_{i=0}^{n} (-1)^{n-i} \cdot \binom{m+i}{k}\binom{n}{i} = \binom{m}{k-n}$. We want to prove that $\sum_{i=0}^{n+1} (-1)^{n+1-i} \cdot \binom{m+i}{k}\binom{n+1}{i} = \binom{m}{k-(n+1)}$:

$$\begin{aligned}
&\sum_{i=0}^{n+1} (-1)^{n+1-i} \cdot \binom{m+i}{k}\binom{n+1}{i} \\
&= \sum_{i=0}^{n+1} (-1)^{n+1-i} \cdot \binom{m+i}{k}\left(\binom{n}{i} + \binom{n}{i-1}\right) \\
&= -\sum_{i=0}^{n} (-1)^{n-i} \cdot \binom{m+i}{k}\binom{n}{i} + \sum_{i=1}^{n+1} (-1)^{(n+1)-i} \cdot \binom{(m+1)+(i-1)}{k}\binom{n}{i-1} \\
&= -\binom{m}{k-n} + \binom{m+1}{k-n} = \binom{m}{k-n-1} = \binom{m}{k-(n+1)}.
\end{aligned}$$

□

Lemma 7.1.5. *Assume $k \geq t \geq 0$. Then*

$$\sum_{i_j \geq 1 \,:\, i_1+\cdots+i_t=k} C(i_1, m)\ldots C(i_t, m) = \frac{t}{k} \cdot \binom{(m+1)k}{k-t} \tag{7.7}$$

Proof. By Corollary 3.3.17, $C(k, m)$ is the number of non-crossing partitions of $[(m+1)k]$ with k parts of sizes $(m+1)$. Therefore we can view LHS as the number of ways to put k polygons of size $(m+1)$ and one polygon P_0 of size t on the cycle of length $(m+1)k+t$ non-crossingly, with the following requirements:

1. P_0 must contain point 1, and

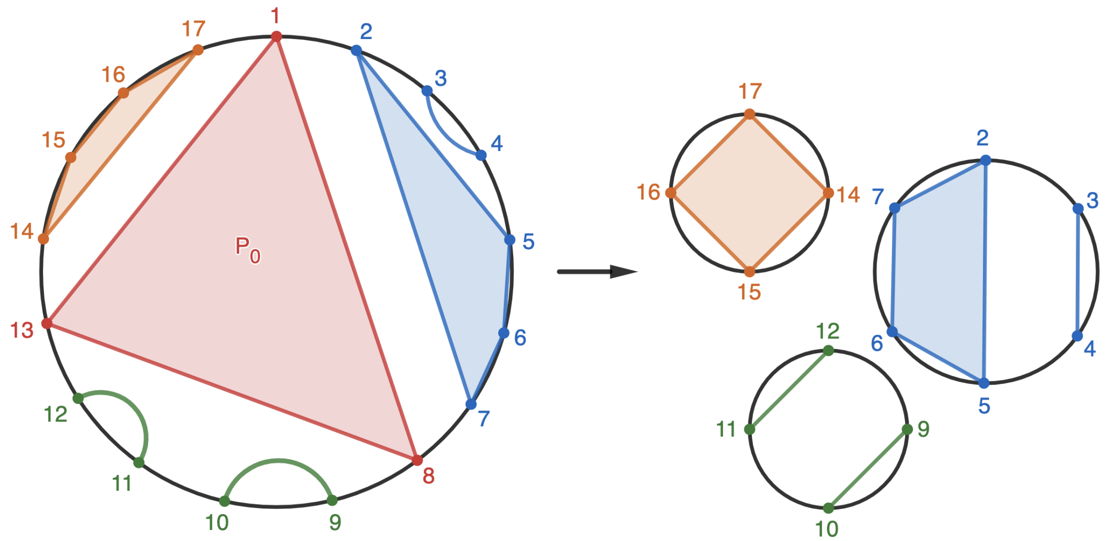


Figure 7.1: Illustration of Lemma 7.1.5: here $t = 3$, $m = 2$, $k = 7$. For this configuration, $i_1 = 3$ (blue), $i_2 = 2$ (green), $i_3 = 2$ (orange).

2. the t parts on the cycle divided by P_0 are nonempty and contain some $(m+1)$-gon. i.e. if $P_0 = \{p_1, \ldots, p_t\}$ where $1 = p_1 < p_2 < \cdots < p_t$, then $p_{i+1} - p_i > 1$ for $i = 1, \ldots, t-1$ and $p_t < k$.

We can count the LHS in an alternative way:

1. Count the number of non-crossing partitions of $[(m+1)k + t]$ using k polygons of size $(m+1)$ and one polygon of size t. By Theorem 3.3.13, this is $\binom{(m+1)k+t}{(k+1)-1} \cdot \frac{k!}{k!1!} = \binom{(m+1)k+t}{k}$.

2. We want to restrict that the t-gon has to contain 1, this gives us a factor of $\frac{t}{(m+1)k+t}$.

3. The above also counts the case when some of the parts from the t-gon are empty, so we will use inclusion-exclusion principle for the counting. If a fixed $(t-i)$ parts out of t parts have to be empty, then it is equivalent to counting such non-crossing partitions of $[(m+1)k+i]$ with k $(m+1)$-gons and one i-gon. This is $\frac{i}{(m+1)k+i} \cdot \binom{(m+1)k+i}{k}$. There are $\binom{t}{t-i} = \binom{t}{i}$ ways for choosing the $(t-i)$ empty parts.

Thus

$$\begin{aligned}
LHS &= \sum_{i=1}^{t} (-1)^{t-i} \cdot \frac{i}{(m+1)k+i} \cdot \binom{(m+1)k+i}{k} \cdot \binom{t}{i} \\
&= \sum_{i=1}^{t} (-1)^{t-i} \cdot \frac{((m+1)k+i-1)!}{k!(mk+i)!} \cdot \frac{t!}{(i-1)!(t-i)!} \\
&= \sum_{i=1}^{t} (-1)^{t-i} \cdot \frac{t}{k} \cdot \binom{(m+1)k+i-1}{k-1}\binom{t-1}{i-1} \\
&= \frac{t}{k} \cdot \left(\sum_{j=0}^{t-1} (-1)^{t-1-j} \cdot \binom{(m+1)k+j}{k-1}\binom{t-1}{j} \right) \qquad \text{(Let } j = i-1\text{)} \\
&= \frac{t}{k} \cdot \binom{(m+1)k}{k-t} \qquad \text{(By } \textit{Proposition } 7.1.4\text{)}
\end{aligned}$$

□

Proof of Theorem 7.1.2.

$$\begin{aligned}
&A_m(k,0) \\
&= \sum_{\vec{\alpha},\vec{\beta} \in P_k} C\left(\vec{\alpha},\vec{\beta}\right) \cdot \vec{\Omega}^{\vec{\alpha}} \cdot \left(\prod_{i=1}^{k} C(i,m)^{\beta_i} \right) \\
&= \sum_{\vec{\alpha},\vec{\beta} \in P_k} (-1)^{a+b-k-1} \cdot k \cdot \binom{a+b-2}{k-1} \cdot \frac{(a-1)!}{\alpha_1! \dots \alpha_k!} \cdot \frac{(b-1)!}{\beta_1! \dots \beta_k!} \cdot \prod_{i=1}^{k} C(i,m)^{\beta_i} \cdot \vec{\Omega}^{\vec{\alpha}} \\
&= \sum_{\vec{\alpha} \in P_k} \left(\sum_{\vec{\beta} \in P_k} (-1)^{a+b-k-1} \cdot k \cdot \binom{a+b-2}{k-1} \cdot \frac{(b-1)!}{\beta_1! \dots \beta_k!} \cdot \prod_{i=1}^{k} C(i,m)^{\beta_i} \right) \cdot \frac{(a-1)!}{\alpha_1! \dots \alpha_k!} \cdot \vec{\Omega}^{\vec{\alpha}}
\end{aligned}$$

Thus it is equivalent to prove that

$$\sum_{\vec{\beta} \in P_k} (-1)^{a+b-k-1} \cdot \frac{(b-1)!}{\beta_1! \dots \beta_k!} \cdot k \cdot \binom{a+b-2}{k-1} \cdot \prod_{i=1}^{k} C(i,m)^{\beta_i} = \binom{mk}{a-1}. \tag{7.8}$$

By Lemma 7.1.5,

$$\begin{aligned}
&\sum_{\vec{\beta}\in P_k} (-1)^{a+b-k-1} \cdot \frac{(b-1)!}{\beta_1!\dots\beta_k!} \cdot k \cdot \binom{a+b-2}{k-1} \cdot \prod_{i=1}^{k} C(i,m)^{\beta_i} \\
=&\sum_{b=1}^{k} \sum_{\vec{\beta}\in P_k : \sum \beta_i = b} (-1)^{a+b-k-1} \cdot k \cdot \binom{a+b-2}{k-1} \cdot \frac{1}{b} \cdot \left(\frac{b!}{\beta_1!\dots\beta_k!} \cdot \prod_{i=1}^{k} C(i,m)^{\beta_i} \right) \\
=&\sum_{b=1}^{k} (-1)^{a+b-k-1} \cdot \binom{a+b-2}{k-1} \cdot \frac{k}{b} \cdot \left(\sum_{i_j \geq 1 \,:\, i_1+\dots+i_b = k} \prod_{j=1}^{b} C\left(i_j, m\right) \right) \\
=&\sum_{b=1}^{k} (-1)^{a+b-k-1} \cdot \binom{a+b-2}{k-1} \cdot \frac{k}{b} \cdot \frac{b}{k} \cdot \binom{(m+1)k}{k-b} \\
=&\sum_{b=k-(a-1)}^{k} (-1)^{a+b-k-1} \cdot \binom{a+b-2}{k-1} \cdot \binom{(m+1)k}{k-b} \\
=&\sum_{i=0}^{a-1} (-1)^{(a-1)-i} \binom{(a-1)+(k-1)-i}{k-1} \cdot \binom{(m+1)k}{i} \qquad \text{(Let } i = k-b\text{)}
\end{aligned}$$

Plugging in $n' = (m+1)k$, $m' = (a-1)+(k-1)$, $k' = (k-1)$ for Proposition 7.1.3,

$$\begin{aligned}
&\sum_{i=0}^{a-1} (-1)^{(a-1)-i} \binom{(a-1)+(k-1)-i}{k-1} \cdot \binom{(m+1)k}{i} \\
=&\sum_{i=0}^{m'-k'} (-1)^{(m'-k')-i} \binom{m'-i}{k'} \cdot \binom{n'}{i} \\
=&\binom{n'-k'-1}{m'-k'} = \binom{(m+1)k-(k-1)-1}{a-1} = \binom{mk}{a-1}
\end{aligned}$$

as needed. □

7.2 Application when $\Omega = \Omega_{Z(m')}$

As a direct application of the results from the previous section, in this section we prove Theorem 2.2.24.

Theorem 2.2.24. *For any $m, m' \in \mathbb{N}$,*

$$\Omega_{Z(m)} \circ_R \Omega_{Z(m')} = \Omega_{Z(m+m')} \tag{2.12}$$

To prove Theorem 2.2.24, we recall the main result from the paper Cai and Potechin [2020].

Theorem 7.2.1. *Let $M_{\alpha_{Z(m)}}$ be the $Z(m)$-shape graph matrix. Let $M_{n,m} = \frac{1}{n^{m/2}} M_{\alpha_{Z(m)}}$ and $r(n,m) = \frac{n!}{(n-m)!}$. Then*

$$\lim_{n \to \infty} \frac{1}{r(n,m)} \cdot \mathbb{E}\left[\operatorname{tr}\left(\left(M_{n,m}^T M_{n,m}\right)^k\right)\right] = C(k,m) \tag{7.9}$$

for all $k \in \mathbb{N}$.

Rewriting the above result in terms of $\Omega_{Z(m)}$, we get the following corollary.

Corollary 7.2.2. *For any $k, m \geq 0$,*

$$\left(\Omega_{Z(m)}\right)_{2k} = C(k,m). \tag{7.10}$$

By the Trace Power Method and Corollary 7.2.2, Theorem 2.2.24 can be proved by proving the following statement.

Theorem 7.2.3.

$$\left(\Omega_{Z(m)} \circ_R \Omega_{Z(m')}\right)_{2k} = C\left(k, m+m'\right) = \left(\Omega_{Z(m+m')}\right)_{2k}. \tag{7.11}$$

By Corollary 2.2.16 and Corollary 7.2.2, it suffices to prove the following.

Theorem 7.2.4. *Let $\Omega_{2k} = C(k,m)$ and $\Omega'_{2k} = C(k,m')$ for all k. Then*

$$\sum_{\vec{\alpha},\vec{\beta}\in P_k} C\left(\vec{\alpha},\vec{\beta}\right)\cdot\vec{\Omega}^{\vec{\alpha}}\cdot\overrightarrow{\Omega'}^{\vec{\beta}} = C\left(k, m+m'\right). \tag{7.12}$$

An easy combinatorial identity is needed for the proof.

Proposition 7.2.5. *For any $m,n,k \in \mathbb{Z}_{\geq 0}$,*

$$\sum_{i=0}^{k}\binom{m}{i}\binom{n}{k-i} = \binom{m+n}{k}. \tag{7.13}$$

Proof. The RHS is the number of ways to choose k elements from a collection of $m+n$ elements. Each summand in the LHS is the number of ways to choose i elements from the first m elements, and $k-i$ elements from the last n elements. Summing up all the possible i's gives the equality. □

Now we are ready to prove the theorem.

Proof of Theorem 7.2.4. By Theorem 7.1.2, when $\Omega'_{2k} = C\left(k,m'\right)$,

$$\sum_{\vec{\alpha},\vec{\beta}\in P_k} C\left(\vec{\alpha},\vec{\beta}\right)\cdot\vec{\Omega}^{\vec{\alpha}}\cdot\overrightarrow{\Omega'}^{\vec{\beta}} = \sum_{\vec{\alpha}\in P_k}\binom{m'k}{a-1}\cdot\frac{(a-1)!}{\alpha_1!\ldots\alpha_k!}\cdot\vec{\Omega}^{\vec{\alpha}}.$$

Plugging in $\Omega_{2k} = C(k,m)$, we get

$$
\begin{aligned}
&\sum_{\vec{\alpha},\vec{\beta}\in P_k} C\left(\vec{\alpha},\vec{\beta}\right)\cdot \vec{\Omega}^{\vec{\alpha}}\cdot \vec{\Omega'}^{\vec{\beta}} \\
&= \sum_{\vec{\alpha}\in P_k} \binom{m'k}{a-1}\cdot \frac{(a-1)!}{\alpha_1!\dots\alpha_k!}\cdot \prod_{i=1}^{k} C(i,m)^{\alpha_i} \\
&= \sum_{a=1}^{k} \binom{m'k}{a-1}\cdot \frac{1}{a}\cdot \left(\sum_{i_j\geq 1: i_1+\dots+i_a=k}\ \prod_{j=1}^{a} C(i_j,m)\right) \\
&= \sum_{a=1}^{k} \binom{m'k}{a-1}\cdot \frac{1}{a}\cdot \frac{a}{k}\cdot \binom{(m+1)k}{k-a} \qquad \text{(By Lemma 7.1.5)} \\
&=\frac{1}{k}\cdot \sum_{a=1}^{k} \binom{m'k}{a-1}\cdot \binom{(m+1)k}{k-a} \\
&=\frac{1}{k}\cdot \sum_{i,j\geq 0: i+j=k-1} \binom{m'k}{i}\cdot \binom{(m+1)k}{j} \\
&=\frac{1}{k}\cdot \binom{(m'+m+1)k}{k-1} \qquad \text{(By Proposition 7.2.5)} \\
&=\frac{\left((m+m'+1)k\right)!}{k!\cdot \left((m+m')k+1\right)!} = \frac{1}{(m+m')k+1}\cdot \binom{(m+m'+1)k}{k} = C\left(k, m+m'\right)\ .
\end{aligned}
$$

□

CHAPTER 8

GENERAL CASE: $\Omega_{Z(m)} \circ_R \Omega^{(1)} \circ_R \ldots \circ_R \Omega^{(s)}$

In this chapter, we generalize the result from the previous chapter.

Definition 8.0.1. Given $\vec{\alpha}_1, \ldots, \vec{\alpha}_s \in P_k$, we will denote $C_m(\vec{\alpha}_1, \ldots, \vec{\alpha}_s)$ to be

$$C_m(\vec{\alpha}_1, \ldots, \vec{\alpha}_s) = \binom{(m-s+1)k}{mk - (a_1 + \cdots + a_s) + 1} \cdot k^{s-1} \cdot \prod_{i=1}^{s} \frac{(a_i - 1)!}{\alpha_{i1}! \ldots \alpha_{ik}!} \tag{8.1}$$

Theorem 8.0.2. *Given* $\Omega^{(1)}, \ldots, \Omega^{(s)}$,

$$\left(\Omega_{Z(m)} \circ_R \Omega^{(1)} \circ_R \ldots \circ_R \Omega^{(s)}\right)_{2k} = \sum_{\vec{\alpha}_i \in P_k} C_m(\vec{\alpha}_1, \ldots, \vec{\alpha}_s) \cdot \overrightarrow{\Omega^{(1)}}^{\vec{\alpha}_1} \ldots \overrightarrow{\Omega^{(s)}}^{\vec{\alpha}_s}. \tag{8.2}$$

We will prove the theorem by induction on s. If we know the formula for $\left(\Omega_{Z(m)} \circ_R \Omega^{(1)} \circ_R \ldots \circ_R \Omega^{(s-1)}\right)_{2k}$, we can compute $\left(\Omega_{Z(m)} \circ_R \Omega^{(1)} \circ_R \ldots \circ_R \Omega^{(s)}\right)_{2k}$ by using the formula from Corollary 2.2.16:

$$\begin{aligned}\left(\Omega_{Z(m)} \circ_R \Omega^{(1)} \circ_R \ldots \circ_R \Omega^{(s)}\right)_{2k} &= \left(\left(\Omega_{Z(m)} \circ_R \Omega^{(1)} \circ_R \ldots \circ_R \Omega^{(s-1)}\right) \circ_R \Omega^{(s)}\right)_{2k} \\ &= \sum_{\vec{\alpha}, \vec{\beta} \in P_k} C\left(\vec{\alpha}, \vec{\beta}\right) \cdot \left(\overrightarrow{\Omega_{Z(m)} \circ_R \Omega^{(1)} \circ_R \ldots \circ_R \Omega^{(s-1)}}\right)^{\vec{\alpha}} \cdot \overrightarrow{\Omega^{(s)}}^{\vec{\beta}}.\end{aligned}$$

8.1 Extended Definition of Non-crossing Partitions

Recall that $\mathcal{NP}(\vec{\alpha})$ is the set of non-crossing partitions of $[k]$ such that there are α_i many parts of size i and $\mathcal{NP}(m\vec{\alpha})$ is the set of non-crossing partitions of $[k]$ such that there are α_i many parts of size mi.

To prove our main results, we need to first extend our definitions and results on $\mathcal{NP}(\vec{\alpha})$ and $\mathcal{NP}(m\vec{\alpha})$.

Definition 8.1.1. Let $V \subseteq [n]$ and $\vec{\alpha} \in P_k$. We say that $\mathcal{P} \in \mathcal{NP}\,(V, m\vec{\alpha})$ if $|V| = mk$ and if $a_1 < \cdots < a_{mk}$ are the elements of V in ascending order, and $\mathcal{P}' = \Big\{\{i_1, \ldots, i_p\} : \{a_{i_1}, \ldots, a_{i_p}\} \in \mathcal{P}\Big\}$, then $\mathcal{P}' \in \mathcal{NP}\,(m\vec{\alpha})$.

Example 8.1.2. *Assume* $V = \{1, 2, 4, 5, 7, 8\} \subseteq [9]$ *and* $m = 2$*,* $k = 3$*,* $\vec{\alpha} = (1, 1, 0)$*. Then* $\mathcal{P} = \{\{1, 2\}, \{4, 5, 7, 8\}\} \in \mathcal{NP}\,(V, m\vec{\alpha})$ *since* $(a_1, \ldots, a_6) = (1, 2, 4, 5, 7, 8)$ *and* $\mathcal{P}' = \{\{1, 2\}, \{3, 4, 5, 6\}\} \in \mathcal{NP}\,(m\vec{\alpha})$*.*

Definition 8.1.3. Given $\vec{\alpha}_1, \ldots, \vec{\alpha}_s \in P_k$ and $m_1, \ldots, m_s \in \mathbb{Z}^+$, let $m = m_1 + \cdots + m_s$. We define $\mathcal{NP}\,(m_1\vec{\alpha}_1, \ldots, m_s\vec{\alpha}_s)$ to be the set of non-crossing partitions of $[mk]$ corresponding to $m_1\vec{\alpha}_1, \ldots, m_s\vec{\alpha}_s$, defined as follows:

- For each $t \in [s]$, let $V_t = \Big\{\,(m_1 + \cdots + m_{t-1}) + m(i-1) + j : i \in [k], j \in [m_t]\Big\}$. Note that $|V_t| = m_t k$.
- $\mathcal{P} \in \mathcal{NP}\,(m_1\vec{\alpha}_1, \ldots, m_s\vec{\alpha}_s) \iff \mathcal{P} = \mathcal{P}_1 \bigsqcup \cdots \bigsqcup \mathcal{P}_s$ where $\mathcal{P}_i \in \mathcal{NP}\,(V_i, m_i\vec{\alpha}_i)$ for each $i \in [s]$, and $\mathcal{P}$ is non-crossing (i.e. $\mathcal{P}_i$'s do not cross).

See Figure 8.1 for an illustration.

Remark 8.1.4. *Notice that when* $s = 1$*, the definition of* $\mathcal{NP}\,(m_1\vec{\alpha}_1, \ldots, m_s\vec{\alpha}_s)$ *coincides with* $\mathcal{NP}\,(m\vec{\alpha})$ *from Chapter 7.*

Definition 8.1.5. Given $\vec{\alpha}_1, \ldots, \vec{\alpha}_s \in P_k$, we denote $\mathcal{NP}_m\,(\vec{\alpha}_1, \ldots, \vec{\alpha}_s)$ to be $\mathcal{NP}\,((m - s + 1)\,\vec{\alpha}_1, \vec{\alpha}_2, \ldots, \vec{\alpha}_s)$.

Definition 8.1.6. We denote

1. $\mathcal{NP}_k = \Big\{\mathcal{P} \in \mathcal{NP}(\vec{\alpha}) : \vec{\alpha} \in P_k\Big\}$ = the set of non-crossing partitions on $[k]$,
2. $\mathcal{NP}_{k,m} = \Big\{\mathcal{P} \in \mathcal{NP}(m\vec{\alpha}) : \vec{\alpha} \in P_k\Big\}$ = the set of non-crossing partitions on $[mk]$ where the size of each partition set is a multiple of $[m]$, and
3. $\mathcal{NP}_{k,m,s} = \Big\{\mathcal{P}_1 \cup \cdots \cup \mathcal{P}_s \in \mathcal{NP}_m(\vec{\alpha}_1, \ldots, \vec{\alpha}_s) : \vec{\alpha}_i \in P_k \text{ for } i \in [s]\Big\}$.

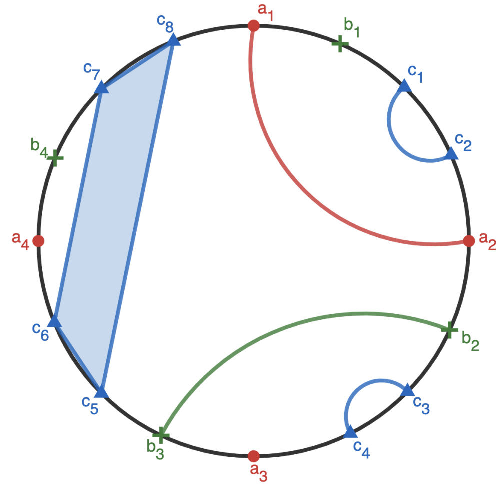


Figure 8.1: Illustration of Definition 8.1.3: here $s = 3, m = 4, k = 4$ and $m_1 = m_2 = 1, m_3 = 2$. $V_1 = \{a_1, a_2, a_3, a_4\}$, $V_2 = \{b_1, b_2, b_3, b_4\}$ and $V_3 = \{c_1, \ldots, c_8\}$.

Remark 8.1.7. *We will add $\mathcal{L}_A$ in the end for all the above notations (i.e. $\mathcal{PL}_A(\vec{\alpha})$, $\mathcal{NPL}_A(\vec{\alpha})$, $\mathcal{NP}_m\mathcal{L}_A(\vec{\alpha}_1, \ldots, \vec{\alpha}_s)$) to indicate that all polygons are distinct, including the ones of the same size. eg. $\left|\mathcal{NPL}_A((k, 0, \ldots, 0))\right| = k!$ while $\left|\mathcal{NP}((k, 0, \ldots, 0))\right| = 1$.*

Theorem 8.1.8. *Let $\vec{\alpha}_1, \ldots, \vec{\alpha}_s \in P_k$.*

$$\left|\mathcal{NP}_m(\vec{\alpha}_1, \ldots, \vec{\alpha}_s)\right| = C_m(\vec{\alpha}_1, \ldots, \vec{\alpha}_s)\ . \tag{8.3}$$

Equivalently,

$$\left|\mathcal{NP}_m\mathcal{L}_A(\vec{\alpha}_1, \ldots, \vec{\alpha}_s)\right| = \binom{(m-s+1)k}{mk - (a_1 + \cdots + a_s) + 1} \cdot k^{s-1} \cdot \prod_{i=1}^{s}(a_i - 1)! \tag{8.4}$$

To prove Theorem 8.1.8, we need the following middle step, which requires an equivalent definition of $\mathcal{NP}_m(\vec{\alpha}_1, \ldots, \vec{\alpha}_s)$, similar to the one we have seen in Section 6.2.

Definition 8.1.9. Let $\mathcal{P}_i = \left\{P_{i1}, \ldots, P_{ik_i}\right\}$ be partitions of $[n]$ for $i = 1, \ldots, s$. We say that $\mathcal{P}_1 \cup \cdots \cup \mathcal{P}_s$ is *non-crossing* if

1. Each $\mathcal{P}_i$ is non-crossing, and

2. For any $P_i \in \mathcal{P}_i$ and $P_j \in \mathcal{P}_j$, they do not cross when placed on the cycle $\mathcal{C}_n$, and they touch at at most one point.

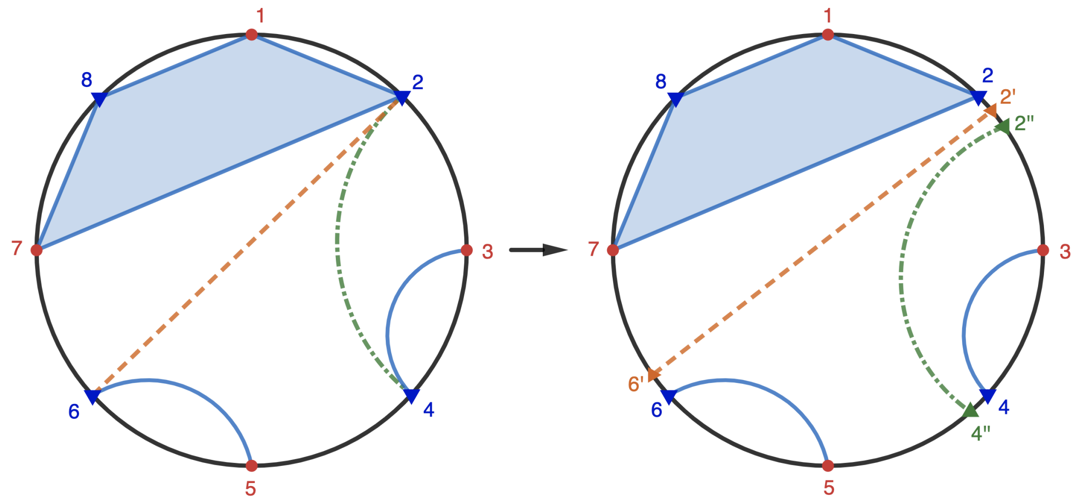


Figure 8.2: Illustration of Definition 8.1.9 and Definition 8.1.10: here $m = 4, s = 3, k = 4$, $V = \{2, 4, 6, 8\}$. $\vec{\alpha}_1 = \vec{\alpha}_2 = \vec{\alpha}_3 = (2, 1, 0, 0)$. $\mathcal{P}_1 = \Big\{\{1, 2, 7, 8\}, \{3, 4\}, \{5, 6\}\Big\} \in \mathcal{NP}\left(2\vec{\alpha}_1\right)$ is marked blue, $\mathcal{P}_2 = \Big\{\{2, 6\}, \{4\}, \{8\}\Big\} \in \mathcal{NP}\left(V, \vec{\alpha}_2\right)$ is marked orange, $\mathcal{P}_3 = \Big\{\{2, 4\}, \{6\}, \{8\}\Big\} \in \mathcal{NP}\left(V, \vec{\alpha}_3\right)$ is marked green.

Definition 8.1.10. Let $\vec{\alpha}_1, \ldots, \vec{\alpha}_s \in P_k$. Let $V = \Big\{(m - s + 1)i : i \in [k]\Big\}$. We say $\mathcal{P} = \mathcal{P}_1 \cup \cdots \cup \mathcal{P}_s \in \mathcal{NP}_m^{(T)}\left(\vec{\alpha}_1, \ldots, \vec{\alpha}_s\right)$ if

1. $\mathcal{P}_1 \in \mathcal{NP}\Big((m - s + 1)\vec{\alpha}_1\Big)$ and $\mathcal{P}_i \in \mathcal{NP}\left(V, \vec{\alpha}_i\right)$ for $i = 2, \ldots, s$,

2. $\mathcal{P}_1 \cup \cdots \cup \mathcal{P}_s$ is non-crossing,

3. For any $P_i \in \mathcal{P}_i$ and $P_j \in \mathcal{P}_j$ where $i < j$ that touch at a point t_0, we can order P_i and Q_j as $P_j = \{t_0, p_1, \ldots, p_x\}$ and $P_i = \{q_1, \ldots, q_y, t_0\}$ such that $t_0, p_1, \ldots, p_x, q_1, \ldots, q_y$ are ordered in the clockwise direction on $\mathcal{C}_k$.

Pictorially, for any point t_0 where some P_j's touch together, if we perturb the "t_0" vertex of each P_j in the clockwise direction $(j-1)\epsilon$ distance for some small enough ϵ, the resulting partition consisting of P_j's is non-crossing.

For an example, see Figure 8.2. $P_1 = \{1, 2, 7, 8\} \in \mathcal{P}_1$ and $P_2 = \{2, 6\} \in \mathcal{P}_2$ touch at 2. If we perturb vertex 2 of P_2 in the clockwise direction to the $2'$ position on the right then P_2' and P_1 do not cross.

Remark 8.1.11. *$\mathcal{NP}_2^{(T)}(\vec{\alpha}_1, \vec{\alpha}_2) = \mathcal{NP}(\vec{\alpha}_1, \vec{\alpha}_2)$ from Section 6.2.*

Proposition 8.1.12.

$$\left|\mathcal{NP}_m(\vec{\alpha}_1, \dots, \vec{\alpha}_s)\right| = \left|\mathcal{NP}_m^{(T)}(\vec{\alpha}_1, \dots, \vec{\alpha}_s)\right|. \tag{8.5}$$

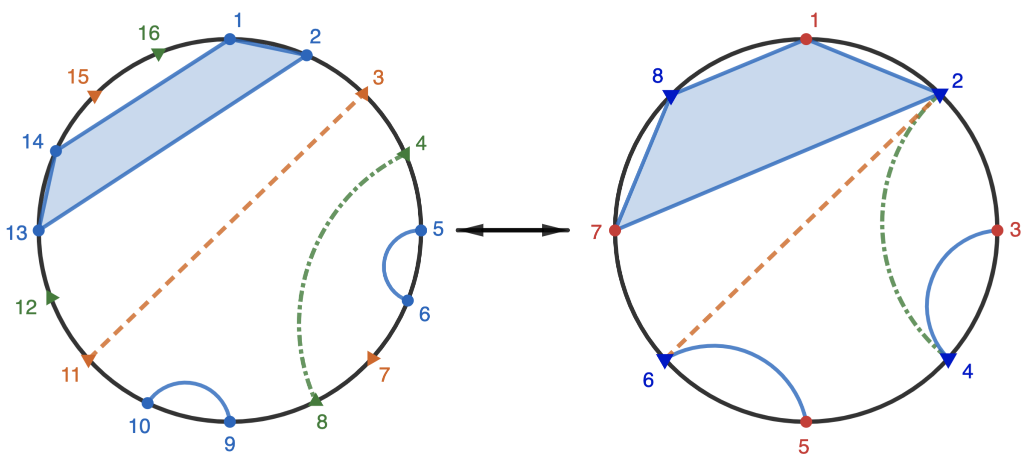


Figure 8.3: Illustration of Proposition 8.1.12: On the left is a partition in $\mathcal{NP}_m(\vec{\alpha}_1, \dots, \vec{\alpha}_s)$ and on the right is a partition in $\mathcal{NP}_m^{(T)}(\vec{\alpha}_1, \dots, \vec{\alpha}_s)$. Here $m = 4$, $s = 3$, $k = 4$.

Proof. We give a bijection between $\mathcal{NP}_m(\vec{\alpha}_1, \dots, \vec{\alpha}_s)$ and $\mathcal{NP}_m^{(T)}(\vec{\alpha}_1, \dots, \vec{\alpha}_s)$. The idea is similar to the pictorial explanation for the third condition in Definition 8.1.10.

1. $\mathcal{NP}_m(\vec{\alpha}_1,\dots,\vec{\alpha}_s) \to \mathcal{NP}_m^{(T)}(\vec{\alpha}_1,\dots,\vec{\alpha}_s)$: Let $\mathcal{P}_1 \bigsqcup \cdots \bigsqcup \mathcal{P}_t \in \mathcal{NP}_m(\vec{\alpha}_1,\dots,\vec{\alpha}_s)$. By definition, they locate on the cycle of length mk. We will shrink the cycle length to $(m-s+1)k$ by grouping together the last s points for every m points. More precisely,

   i. For each $P \in \mathcal{P}_1$, let $P' = \Big\{(m-s+1)i+j : mi+j \in P$ for some $i \in \{0,1,\dots,k-1\}, j \in [m]\Big\}$. Let $\mathcal{P}_1' = \bigcup\limits_{P\in\mathcal{P}_1} P'$.

   ii. For each $P_t \in \mathcal{P}_t$ where $t > 1$, let $P_t' = \Big\{(m-s+1)i : m(i-1)+j \in P$ for some $i \in [k], j \in [m]\Big\}$. For each $t > 1$, let $\mathcal{P}_t' = \bigcup\limits_{P_t\in\mathcal{P}_t} P_t'$.

   iii. We can see that $\mathcal{P}_1' \cup \cdots \cup \mathcal{P}_t' \in \mathcal{NP}_m^{(T)}(\vec{\alpha}_1,\dots,\vec{\alpha}_s)$.

   We can visualize the above procedure as follows: for each $i \in [k]$, take the consecutive s points $m(i-1)+(m-s)+1, m(i-1)+(m-s)+2, \dots, mi$ and push all of them counterclockwise to the first point $m(i-1)+(m-s)+1$. This results in gluing together consecutive vertices of the polygons belonging to $\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_s$ all to one point, which is still allowed under $\mathcal{NP}_m^{(T)}(\vec{\alpha}_1,\dots,\vec{\alpha}_s)$. See See Figure 8.3 for an illustration.

2. $\mathcal{NP}_m^{(T)}(\vec{\alpha}_1,\dots,\vec{\alpha}_s) \to \mathcal{NP}_m(\vec{\alpha}_1,\dots,\vec{\alpha}_s)$: Let $\mathcal{P}_1 \cup \cdots \cup \mathcal{P}_s \in \mathcal{NP}_m^{(T)}(\vec{\alpha}_1,\dots,\vec{\alpha}_s)$. By definition, they locate on the cycle of length $(m-s+1)k$. We can expand the cycle length to mk and separate the touching points in the expanded cycle to get a nontouching version of the non-crossing partition.

   i. For each $P \in \mathcal{P}_1$, let $P' = \Big\{mi+j : (m-s+1)i+j \in P$ for some $i \in \{0,1,\dots,k-1\}, j \in [m-s+1]\Big\}$. Let $\mathcal{P}_1' = \bigcup\limits_{P\in\mathcal{P}_1} P'$.

   ii. For each $P_t \in \mathcal{P}_t$ where $t > 1$, let $P_t' = \Big\{mi-(s-t) : (m-s+1)i \in P$ for some $i \in [k]\Big\}$. For each $t > 1$, let $\mathcal{P}_t' = \bigcup\limits_{P_t\in\mathcal{P}_t} P_t'$.

   iii. We can see that $\mathcal{P}_1' \bigsqcup \cdots \bigsqcup \mathcal{P}_t' \in \mathcal{NP}_m(\vec{\alpha}_1,\dots,\vec{\alpha}_s)$.

We can visualize the above procedure as follows: take each point $(m-s+1)i$ where the vertices of the polygons of different groups $\mathcal{P}_t$ can possibly touch, and expand this point into s points in the clockwise direction, where polygon belonging to $\mathcal{P}_t$ takes the t^{th} expanded point as the new vertex. See Figure 8.3 for an illustration.

3. It is not hard to see that the above procedures are inverses of each other.

□

Definition 8.1.13. Let $\vec{\alpha}_1, \ldots, \vec{\alpha}_s \in P_k$ and $m \geq s$. Let $\mathcal{X} = \left\{x_1, \ldots, x_p\right\}$ be such that $x_1 + \cdots + x_p = (m-s+1)k$. we define

1. $\mathcal{NP}_m^{(T)}(\vec{\alpha}_1, \ldots, \vec{\alpha}_s, \mathcal{X})$, the set of non-crossing partitions corresponding to $(m-s+1)\vec{\alpha}_1, \ldots, \vec{\alpha}_s$ and part sizes $\mathcal{X}$, to be

$$\left\{\mathcal{P} \in \mathcal{NP}_m(\vec{\alpha}_1, \ldots, \vec{\alpha}_s) : S_{\mathcal{P}} = \{x_1, \ldots, x_p\}\right\},$$

2. $\mathcal{NP}_m^{(T)}\mathcal{L}_{\mathcal{X}}(\vec{\alpha}_1, \ldots, \vec{\alpha}_s, \mathcal{X})$, the set of non-crossing partitions corresponding to $\vec{\alpha}_1, \ldots, \vec{\alpha}_s$ and $\mathcal{X}$ as part labels, to be

$$\left\{(\mathcal{P}, L) : \mathcal{P} \in \mathcal{NP}_m(\vec{\alpha}_1, \ldots, \vec{\alpha}_s), S_{\mathcal{P}} = \mathcal{X}, L \text{ is a label of } \mathcal{C}_k/\mathcal{P} \text{ corresponding to } \mathcal{X}\right\}.$$

3. $\mathcal{NP}_m^{(T)}\mathcal{L}_{A,\mathcal{X}}(\vec{\alpha}_1, \ldots, \vec{\alpha}_s, \mathcal{X})$ to be the set of partitions corresponding to $\vec{\alpha}_1, \ldots, \vec{\alpha}_s$ and labels $\mathcal{X}$, but where all the parts are distinct.

Lemma 8.1.14. *Let $\vec{\alpha}_1, \ldots, \vec{\alpha}_s \in P_k$ and $m \geq s$. Let $p = mk - (a_1 + \cdots + a_s) + 1$. Let $\mathcal{X} = \left\{x_1, \ldots, x_p\right\}$ be such that $x_1 + \cdots + x_p = (m-s+1)k$. Then*

$$\mathcal{NP}_m^{(T)}\mathcal{L}_{A,\mathcal{X}}(\vec{\alpha}_1, \ldots, \vec{\alpha}_s, \mathcal{X}) = (m-s+1)\,k^s\,(p-1)!(a_1-1)!\ldots(a_s-1)! \tag{8.6}$$

Proof. We will prove this result using the method from Section 6.2.

By the same argument from Section 6.1, we can reduce the number of polygons from $\vec{\alpha}_2, \ldots, \vec{\alpha}_s$ to one for each $\vec{\alpha}_j$, by combining polygons of size x and y into a single polygon of size $x+y-1$. i.e.

$$\left|\mathcal{NP}_m^{(T)}\mathcal{L}_{A,\mathcal{X}}\left(\vec{\alpha}_1, \ldots, \vec{\alpha}_s, \mathcal{X}\right)\right| = \left|\mathcal{NP}_m^{(T)}\mathcal{L}_{A,\mathcal{X}}\left(\vec{\alpha}_1, {\Omega_2^{(2)}}^{\beta_2}\Omega_{2t_2}^{(2)}, \ldots, {\Omega_2^{(s)}}^{\beta_s}\Omega_{2t_s}^{(s)}\right)\right|$$

where $t_j = \left(\sum_{i=2}^{k} i \cdot \alpha_{ji}\right) - \left(\alpha_{j2} + \cdots + \alpha_{jk} - 1\right) = \left(\sum_{i=1}^{k} i \cdot \alpha_{ji}\right) - \left(\alpha_{j1} + \alpha_{j2} + \cdots + \alpha_{jk}\right) + 1 = k - a_j + 1$ and $\beta_j = k - t_j = a_j - 1$ for each $j = 2, \ldots, s$.

Now we can further combine the $(s-1)$ polygons $\Omega_{2t_j}^{(j)}$ of different types for $j = 2, \ldots, s$ (which can possibly touch each other) into one single polygon, using the method from Lemma 6.2.2. Since we do not combine the type-1 polygons, we keep regarding them as distinct ($\mathcal{L}_{\mathcal{A}}$), whereas we view polygons of types $j = 2, \ldots, s$ as not distinct. We give the special notation $\vec{\alpha}_1\left(\mathcal{L}_{\mathcal{A},\mathcal{Z}}\right)$ for this. i.e.

$$\left|\mathcal{NP}_m^{(T)}\mathcal{L}_{\mathcal{X}}\left(\vec{\alpha}_1(\mathcal{L}_{\mathcal{A},\mathcal{X}}), {\Omega_2^{(2)}}^{\beta_2}\Omega_{2t_2}^{(2)}, \ldots, {\Omega_2^{(s)}}^{\beta_s}\Omega_{2t_s}^{(s)}, \mathcal{X}\right)\right| = k^{s-2} \cdot \left|\mathcal{NP}_m^{(T)}\mathcal{L}_{\mathcal{X}}\left(\vec{\alpha}_1(\mathcal{L}_{\mathcal{A},\mathcal{X}}), {\Omega'_2}^{\gamma}\Omega'_{2t}, \mathcal{X}\right)\right|$$

where $t = t_2 + \cdots + t_s - (s-2)$ and $\gamma = k - t$. Here we obtain a factor of k^{s-2} because each time we combine $\Omega_{2t'_j}^{(j)}$ and $\Omega_{2t_{j-1}}^{(j-1)}$, we get a factor of k and we do the combination step $s-2$ times for $s-1$ polygons. Thus

$$\begin{aligned}&\left|\mathcal{NP}_m^{(T)}\mathcal{L}_{A,\mathcal{X}}\left(\vec{\alpha}_1, {\Omega_2^{(2)}}^{\beta_2}\Omega_{2t_2}^{(2)}, \ldots, {\Omega_2^{(s)}}^{\beta_s}\Omega_{2t_s}^{(s)}, \mathcal{X}\right)\right| \\ =&k^{s-2} \cdot \left|\mathcal{NP}_m^{(T)}\mathcal{L}_{\mathcal{X}}\left(\vec{\alpha}_1(\mathcal{L}_{\mathcal{A},\mathcal{X}}), {\Omega'_2}^{\gamma}\Omega'_{2t}, \mathcal{X}\right)\right| \cdot (a_2-1)! \ldots (a_s-1)!\end{aligned}$$

since we can permute the dots ${\Omega_2^{(j)}}^{\alpha_j - 1}$ for each $j = 2, \ldots, s$.

Now we will count $\mathcal{NP}_m^{(T)}\mathcal{L}_{A,\mathcal{X}}\left(\vec{\alpha}_1, {\Omega'_2}^{\gamma}\Omega'_{2t}, \mathcal{X}\right)$ by counting $\mathcal{NPL}_{\mathcal{X}}\left(\vec{\beta}, {\Omega'_2}^{\gamma'}\Omega'_{2t}, \mathcal{X}\right)$ where if $\vec{\Omega}^{\vec{\beta}} = \prod_{j=1}^{k} \Omega_{2(m-s+1)j}^{\alpha_j}$ and $\gamma' = (m-s+1)k - t$. i.e. We are considering the number of non-crossing partitions corresponding to augmented $\vec{\alpha}_1$ (by a factor of $(m-s+1)$) and a polygon of size t of different type on the cycle of length $(m-s+1)k$. Notice that for $\mathcal{NP}_m^{(T)}\mathcal{L}_{\mathcal{X}}\left(\vec{\alpha}_1, {\Omega'_2}^{\gamma}{\Omega'}_{2t}, \mathcal{X}\right)$, ${\Omega'_2}^{\gamma}{\Omega'}_{2t}$ is only allowed to use vertices from $\{(m-s+1)i : i \in [k]\}$, but in $\mathcal{NPL}_{\mathcal{X}}\left(\vec{\beta}, {\Omega'_2}^{\gamma'}{\Omega'}_{2t}, \mathcal{X}\right)$, ${\Omega'_2}^{\gamma'}{\Omega'}_{2t}$ is allowed to use all vertices from $[(m-s+1)k]$. Thus for each configuration of polygons in $\mathcal{NPL}_{\mathcal{X}}\left(\vec{\alpha}_1, {\Omega'_2}^{\gamma}{\Omega'}_{2t}, \mathcal{X}\right)$, we can rotate it in the clockwise direction $1, 2, \ldots m-s$ units to get an extra $(m-s)$ configurations in $\mathcal{NPL}_{\mathcal{X}}\left(\vec{\beta}, {\Omega'_2}^{\gamma'}{\Omega'}_{2t}, \mathcal{X}\right)$. Thus

$$\left|\mathcal{NP}_m^{(T)}\mathcal{L}_{\mathcal{X}}\left(\vec{\alpha}_1(\mathcal{L}_{\mathcal{A},\mathcal{X}}), {\Omega'_2}^{\gamma}{\Omega'}_{2t}, \mathcal{X}\right)\right| = \frac{1}{m-s+1} \cdot \left|\mathcal{NPL}_{\mathcal{X}}\left(\vec{\beta}(\mathcal{L}_{\mathcal{A},\mathcal{X}}), {\Omega'_2}^{\gamma'}{\Omega'}_{2t}, \mathcal{X}\right)\right|.$$

Further reducing the polygons in $\vec{\alpha}_1$, we get

$$\left|\mathcal{NPL}_{\mathcal{X}}\left(\vec{\beta}(\mathcal{L}_{\mathcal{A},\mathcal{X}}), {\Omega'_2}^{\gamma'}{\Omega'}_{2t}, \mathcal{X}\right)\right| = \left|\mathcal{NPL}_{\mathcal{X}}\left(\Omega_2^{a_1-1}\Omega_{2r}(\mathcal{L}_{\mathcal{A},\mathcal{X}}), {\Omega'_2}^{\gamma'}{\Omega'}_{2t}, \mathcal{X}\right)\right|$$

where $t = (m-s+1)k - a_1 + 1$.

By the counting method in Section 6.2, we know that

$$\left|\mathcal{NPL}_{\mathcal{X}}\left(\Omega_2^{a_1-1}\Omega_{2r}, {\Omega'_2}^{\gamma'}{\Omega'}_{2t}, \mathcal{X}\right)\right| = ((m-s+1)k)^2\,(p-1)!$$

Thus

$$\begin{aligned}
&\left|\mathcal{NPL}_{\mathcal{X}}\left(\vec{\beta}(\mathcal{L}_{\mathcal{A},\mathcal{X}}), {\Omega'_2}^{\gamma'}{\Omega'}_{2t}, \mathcal{X}\right)\right| = ((m-s+1)k)^2\,(p-1)!(a_1-1)! \\
\Longrightarrow &\left|\mathcal{NP}_m^{(T)}\mathcal{L}_{\mathcal{X}}\left(\vec{\alpha}_1(\mathcal{L}_{\mathcal{A},\mathcal{X}}), {\Omega'_2}^{\gamma}{\Omega'}_{2t}, \mathcal{X}\right)\right| = (m-s+1)k^2(p-1)!(a_1-1)!
\end{aligned}$$

and

$$
\begin{aligned}
&\left|\mathcal{NP}_m^{(T)}\mathcal{L}_{A,\mathcal{X}}\left(\vec{\alpha}_1,\ldots,\vec{\alpha}_s,\mathcal{X}\right)\right| \\
=&\left|\mathcal{NP}_m^{(T)}\mathcal{L}_{A,\mathcal{X}}\left(\vec{\alpha}_1,{\Omega_2^{(2)}}^{\beta_2}\Omega_{2t_2}^{(2)},\ldots,{\Omega_2^{(s)}}^{\beta_s}\Omega_{2t_s}^{(s)},\mathcal{X}\right)\right| \\
=&k^{s-2}\cdot\left|\mathcal{NP}_m^{(T)}\mathcal{L}_{\mathcal{X}}\left(\vec{\alpha}_1(\mathcal{L}_{\mathcal{A},\mathcal{X}}),{\Omega'_2}^{\gamma}{\Omega'}_{2t},\mathcal{X}\right)\right|\cdot(a_2-1)!\ldots(a_s-1)! \\
=&k^{s-2}\cdot\Big((m-s+1)k^2(p-1)!(a_1-1)!\Big)\cdot(a_2-1)!\ldots(a_s-1)! \\
=&(m-s+1)k^s(p-1)!(a_1-1)!\ldots(a_s-1)!
\end{aligned}
$$

as needed. □

Proof of Theorem 8.1.8. By Lemma 8.1.14,

$$
\begin{aligned}
\mathcal{NP}_m\mathcal{L}_A\left(\vec{\alpha}_1,\ldots,\vec{\alpha}_s\right) &= \sum_{\substack{x_i\geq 1:\\ x_1+\cdots+x_p=(m-s+1)k}} \frac{\mathcal{NP}_m\mathcal{L}_A\left(\vec{\alpha}_1,\ldots,\vec{\alpha}_s,\mathcal{X}\right)}{\operatorname{perm}\left(x_1,\ldots,x_p\right)} \\
=&\sum_{\substack{x_i\geq 1:\\ x_1+\cdots+x_p=(m-s+1)k}} \frac{\mathcal{NP}_m\mathcal{L}_{A,\mathcal{X}}\left(\vec{\alpha}_1,\ldots,\vec{\alpha}_s,\mathcal{X}\right)}{p!} \\
=&\sum_{\substack{x_i\geq 1:\\ x_1+\cdots+x_p=(m-s+1)k}} (m-s+1)k^s\cdot(p-1)!(a_1-1)!\ldots(a_s-1)!\cdot\frac{1}{p!} \\
=&\binom{(m-s+1)k-1}{p-1}\cdot(m-s+1)k^s\cdot(a_1-1)!\ldots(a_s-1)!\cdot\frac{1}{p} \\
=&\binom{(m-s+1)k-1}{p-1}\cdot\frac{(m-s+1)k}{p}\cdot k^{s-1}\cdot(a_1-1)!\ldots(a_s-1)! \\
=&\binom{(m-s+1)k}{p}\cdot k^{s-1}\cdot(a_1-1)!\ldots(a_s-1)! \\
=&\binom{(m-s+1)k}{mk-(a_1+\cdots+a_s)+1}\cdot k^{s-1}\cdot(a_1-1)!\ldots(a_s-1)!
\end{aligned}
$$

□

We now extend the definition further to $\mathcal{NP}_m\left(\vec{\alpha}_1,\ldots,\vec{\alpha}_s,\Omega_{2c},\mathcal{X}\right)$.

Definition 8.1.15. Let $\vec{\alpha}_1, \ldots, \vec{\alpha}_s \in P_k$. Let $V = \Big\{(m-s+1)i : i \in [k]\Big\}$. We say $\mathcal{P} = \mathcal{P}_1 \cup \cdots \cup \mathcal{P}_s \cup \hat{\mathcal{P}} \in \mathcal{NP}_m^{(T)}(\vec{\alpha}_1, \ldots, \vec{\alpha}_s, \Omega_{2c})$ if

1. $\mathcal{P} \in \mathcal{NP}_{m+1}^{(T)}(\vec{\alpha}_1, \ldots, \vec{\alpha}_s, \vec{\alpha}_{s+1})$ where $\vec{\alpha}_{s+1}$ corresponds to $\Omega_2^{k-c}\Omega_{2c}$.

2. $\hat{\mathcal{P}}$ has to contain $(m-s+1)k$ (i.e. the last element of the cycle).

Definition 8.1.16. We define

$$\mathcal{NP}_{k,m,s}^{(T)}(\Omega_{2c}) = \Big\{\mathcal{P} = \mathcal{P}_1 \cup \cdots \cup \mathcal{P}_s \cup \hat{\mathcal{P}} \in \mathcal{NP}_m^{(T)}(\vec{\alpha}_1, \ldots, \vec{\alpha}_s, \Omega_{2c}) : \vec{\alpha}_i \in \mathcal{P}_k\Big\}$$

.

Theorem 8.1.17.

$$\left|\mathcal{NP}_m^{(T)}(\vec{\alpha}_1, \ldots, \vec{\alpha}_s, \Omega_{2c})\right| = c \cdot \binom{(m-s+1)k}{mk-(a_1+\cdots+a_s)+c} \cdot k^{s-1} \cdot \prod_{i=1}^{s} \frac{(a_i-1)!}{\alpha_{i1}! \ldots \alpha_{ik}!}. \quad (8.7)$$

Proof. We first view the c-gon as adding an extra $\vec{\Omega}^{\vec{\beta}} = \Omega_2^{k-c}\Omega_{2c}$, and then factor out the double counting.

Let $m' = m+1$, $s' = s+1$ and $b = k-c+1$. By Theorem 8.1.8,

$$\begin{aligned}
&\left|\mathcal{NP}_{m'}^{(T)}\left(\vec{\alpha}_1, \ldots, \vec{\alpha}_s, \vec{\beta}\right)\right| \\
=&\binom{(m'-s'+1)k}{m'k-(a_1+\cdots+a_s+b)+1} \cdot k^{s'-1} \cdot \left(\prod_{i=1}^{s} \frac{(a_i-1)!}{\alpha_{i1}! \ldots \alpha_{ik}!}\right) \cdot \frac{(b-1)!}{(k-c)!} \\
=&\binom{(m-s+1)k}{(m+1)k-(a_1+\cdots+a_s)-(k-c+1)+1} \cdot k^{s} \cdot \prod_{i=1}^{s} \frac{(a_i-1)!}{\alpha_{i1}! \ldots \alpha_{ik}!} \\
=&\binom{(m-s+1)k}{mk-(a_1+\cdots+a_s)+c} \cdot k^{s} \cdot \prod_{i=1}^{s} \frac{(a_i-1)!}{\alpha_{i1}! \ldots \alpha_{ik}!}.
\end{aligned}$$

Note that the original c-gon has to contain $(m-s+1)k$, while in the counting with the term $\vec{\Omega}^{\vec{\beta}}$, there is no such requirement on the c-gon. This gives a factor of $\frac{c}{k}$. Thus

$$\begin{aligned}\left|\mathcal{NP}_m^{(T)}\left(\vec{\alpha}_1,\ldots,\vec{\alpha}_s,\Omega_{2c}\right)\right| &= \frac{c}{k}\cdot\left|\mathcal{NP}_{m'}^{(T)}\left(\vec{\alpha}_1,\ldots,\vec{\alpha}_s,\vec{\beta}\right)\right| \\ &= \frac{c}{k}\cdot\binom{(m-s+1)k}{mk-(a_1+\cdots+a_s)+c}\cdot k^s\cdot\prod_{i=1}^{s}\frac{(a_i-1)!}{\alpha_{i1}!\ldots\alpha_{ik}!} \\ &= c\cdot\binom{(m-s+1)k}{mk-(a_1+\cdots+a_s)+c}\cdot k^{s-1}\cdot\prod_{i=1}^{s}\frac{(a_i-1)!}{\alpha_{i1}!\ldots\alpha_{ik}!}\end{aligned}$$

as needed.

□

8.2 Formula for Moments of $\Omega_{Z(m)} \circ_R \Omega^{(1)} \circ_R \ldots, \circ_R \Omega^{(s)}$

First we are going to give another equivalent notation for the expression $\left(\Omega_{Z(m)} \circ_R \Omega^{(1)} \circ_R \ldots \circ_R \Omega^{(s)}\right)_{2k}$ defined inductively based on Corollary 2.2.16.

Definition 8.2.1. We define $A_m^{(s)}\left(k,\underbrace{0,\ldots,0}_{s}\right)$ associated with $\Omega^{(1)},\ldots,\Omega^{(s)}$ inductively as the following:

1. $A_m^{(0)}(k) = C(k,m)$.
2. $A_m^{(1)}(k,0) = \sum_{\vec{\alpha},\vec{\beta}\in P_k} C\left(\vec{\alpha},\vec{\beta}\right)\cdot\left(\prod_{i=1}^{k} C(i,m)^{\alpha_i}\right)\cdot\overrightarrow{\Omega^{(1)}}^{\vec{\beta}}$.
3. Let Ω' be the distribution where $\Omega'_{2i} = A_m^{(s-1)}\left(i,\underbrace{0,\ldots,0}_{s-1}\right)$ for all i. Define

$$\begin{aligned}A_m^{(s)}(k,0,\ldots,0) &= \sum_{\vec{\alpha},\vec{\beta}\in P_k} C\left(\vec{\alpha},\vec{\beta}\right)\cdot\overrightarrow{\Omega'}^{\vec{\alpha}}\cdot\overrightarrow{\Omega^{(s)}}^{\vec{\beta}} \\ &= \sum_{\vec{\alpha},\vec{\beta}\in P_k} C\left(\vec{\alpha},\vec{\beta}\right)\cdot\left(\prod_{i=1}^{k}\left(A_m^{(s-1)}(i,0,\ldots,0)\right)^{\alpha_i}\right)\cdot\overrightarrow{\Omega^{(s)}}^{\vec{\beta}}\end{aligned} \tag{8.8}$$

where

$$C\left(\vec{\alpha},\vec{\beta}\right) = (-1)^{a+b-k-1}\cdot k\cdot\binom{a+b-2}{k-1}\cdot\frac{(a-1)!}{\alpha_1!\dots\alpha_k!}\cdot\frac{(b-1)!}{\beta_1!\dots\beta_k!}$$

Remark 8.2.2. *By Corollary 2.2.16 and Corollary 7.2.2,*
$A_m^{(s)}(k,0,\dots,0) = \left(\Omega_{Z(m)}\circ_R\Omega^{(1)}\circ_R\dots\circ_R\Omega^{(s)}\right)_{2k}$ *for all* k.

Thus to prove Theorem 8.0.2, it is equivalent to prove the following theorem.

Theorem 8.2.3.

$$\begin{aligned} A_m^{(s)}\left(k,\underbrace{0,\dots,0}_{s}\right) &= \sum_{\vec{\alpha}_i\in P_k} C_m(\vec{\alpha}_1,\dots,\vec{\alpha}_s)\cdot\prod_{j=1}^{s}\overrightarrow{\Omega^{(i)}\vec{\alpha}_i} \\ &= \sum_{\vec{\alpha}_i\in P_k}\binom{(m-s+1)k}{mk-(a_1+\dots+a_s)+1}\cdot k^{s-1}\cdot\left(\prod_{j=1}^{s}\frac{(a_i-1)!}{\alpha_{i1}!\dots\alpha_{ik}!}\cdot\overrightarrow{\Omega^{(i)}\vec{\alpha}_i}\right) \end{aligned} \tag{8.9}$$

where $\vec{\alpha}_i = (\alpha_{i1},\dots,\alpha_{ik})$, $a_i = \sum_{j=1}\alpha_{ij}$ *for all* $i = 1,\dots,s$.

Definition 8.2.4. Define $B_m^{(s)}(k)$ to be

$$B_m^{(s)}(k) = \sum_{\vec{\alpha}_i\in P_k} C_m(\vec{\alpha}_1,\dots,\vec{\alpha}_s)\cdot\prod_{j=1}^{s}\overrightarrow{\Omega^{(i)}\vec{\alpha}_i}. \tag{8.10}$$

Definition 8.2.5. Given $\mathcal{P} = \{P_1,\dots,P_a\}\in\mathcal{NP}_k$ and a distribution Ω, we denote $\vec{\Omega}_{\mathcal{P}}$ to be

$$\vec{\Omega}_{\mathcal{P}} = \Omega_{2|P_1|}\cdot\dots\cdot\Omega_{2|P_a|} \tag{8.11}$$

and if the size of every P_i's is a multiple of m, we denote $\vec{\Omega}_{\mathcal{P}/m}$ to be

$$\vec{\Omega}_{\mathcal{P}/m} = \Omega_{2|P_1|/m}\cdot\dots\cdot\Omega_{2|P_a|/m} \tag{8.12}$$

Proposition 8.2.6. *Assume $m \geq s$. Let $\hat{m} = m - s + 1$. Then*

$$B_m^{(s)}(k) = \sum_{\substack{\mathcal{P}_1 \sqcup \cdots \sqcup \mathcal{P}_s \\ \in \mathcal{NP}_{k,m,s}}} \overrightarrow{\Omega^{(1)}}_{\mathcal{P}_1/\hat{m}} \cdot \overrightarrow{\Omega^{(2)}}_{\mathcal{P}_2} \cdot \cdots \cdot \overrightarrow{\Omega^{(s)}}_{\mathcal{P}_s} \tag{8.13}$$

where $\mathcal{P}_i = \left\{P_{i,1}, \ldots, P_{i,a_i}\right\}$ for each $i \in [s]$.

Moreover, we can replace $\mathcal{NP}_{k,m,s}$ in the summand with $\mathcal{NP}^{(T)}_{k,m,s}$.

Proof. By Theorem 3.3.13, $C_m\left(\vec{\alpha}_1, \ldots, \vec{\alpha}_s\right) = \left|\mathcal{NP}_m\left(\vec{\alpha}_1, \ldots, \vec{\alpha}_s\right)\right|$. In particular, the size of any partition set corresponding to $\vec{\alpha}_1$ is a multiple of $\hat{m} = m - s + 1$. Thus

$$\begin{aligned}
B_m^{(s)}(k) &= \sum_{\vec{\alpha}_i \in P_k} C_m(\vec{\alpha}_1, \ldots, \vec{\alpha}_s) \cdot \overrightarrow{\Omega^{(1)}}^{\vec{\alpha}_1} \cdot \cdots \cdot \overrightarrow{\Omega^{(s)}}^{\vec{\alpha}_s} \\
&= \sum_{\vec{\alpha}_i \in P_k} \left|\mathcal{NP}_m\left(\vec{\alpha}_1, \ldots, \vec{\alpha}_s\right)\right| \cdot \overrightarrow{\Omega^{(1)}}^{\vec{\alpha}_1} \cdot \cdots \cdot \overrightarrow{\Omega^{(s)}}^{\vec{\alpha}_s} \\
&= \sum_{\vec{\alpha}_i \in P_k} \sum_{\substack{\mathcal{P}_1 \sqcup \cdots \sqcup \mathcal{P}_s \in \\ \mathcal{NP}_m(\vec{\alpha}_1, \ldots, \vec{\alpha}_s)}} \left({\Omega^{(1)}_2}^{\alpha_{11}} \ldots {\Omega^{(1)}_{2k}}^{\alpha_{1k}}\right) \cdot \cdots \cdot \left({\Omega^{(s)}_2}^{\alpha_{s1}} \ldots {\Omega^{(s)}_{2k}}^{\alpha_{sk}}\right) \\
&= \sum_{\vec{\alpha}_i \in P_k} \sum_{\substack{\mathcal{P}_1 \sqcup \cdots \sqcup \mathcal{P}_s \in \\ \mathcal{NP}_m(\vec{\alpha}_1, \ldots, \vec{\alpha}_s)}} \left(\Omega^{(1)}_{2\,|P_{1,1}|/\hat{m}} \cdot \cdots \cdot \Omega^{(1)}_{2\,|P_{1,a_1}|/\hat{m}}\right) \cdot \prod_{j=2}^{s} \left(\Omega^{(j)}_{2\,|P_{j,1}|} \cdot \cdots \cdot \Omega^{(j)}_{2\,|P_{j,a_j}|}\right) \\
&= \sum_{\substack{\mathcal{P}_1 \sqcup \cdots \sqcup \mathcal{P}_s \\ \in \mathcal{NP}_{k,m,s}}} \left(\Omega^{(1)}_{2\,|P_{1,1}|/\hat{m}} \cdot \cdots \cdot \Omega^{(1)}_{2\,|P_{1,a_1}|/\hat{m}}\right) \cdot \prod_{j=2}^{s} \left(\Omega^{(j)}_{2\,|P_{j,1}|} \cdot \cdots \cdot \Omega^{(j)}_{2\,|P_{j,a_j}|}\right) \\
&= \sum_{\substack{\mathcal{P}_1 \sqcup \cdots \sqcup \mathcal{P}_s \\ \in \mathcal{NP}_{k,m,s}}} \overrightarrow{\Omega^{(1)}}_{\mathcal{P}_1/\hat{m}} \cdot \overrightarrow{\Omega^{(2)}}_{\mathcal{P}_2} \cdot \cdots \cdot \overrightarrow{\Omega^{(s)}}_{\mathcal{P}_s}.
\end{aligned}$$

□

Lemma 8.2.7.

$$\begin{aligned}
&\sum_{\substack{k_i \geq 1: \\ k_1 + \cdots + k_c = k}} B_m^{(s)}(k_1) \ldots B_m^{(s)}(k_c) \\
&= c \cdot \sum_{\alpha_i \in P_k} \binom{(m-s+1)k}{mk - (a_1 + \cdots + a_s) + c} \cdot k^{s-1} \cdot \left(\prod_{i=1}^{s} \frac{(a_i - 1)!}{\alpha_{i1}! \ldots \alpha_{ik}!} \cdot \overrightarrow{\Omega^{(i)}}^{\vec{\alpha}_i}\right)
\end{aligned} \tag{8.14}$$

Proof. By Theorem 8.1.17, $\left|\mathcal{NP}_m^{(T)}(\vec{\alpha}_1,\dots,\vec{\alpha}_s,\Omega_{2c})\right| = c\cdot\binom{(m-s+1)k}{mk-(a_1+\cdots+a_s)+c}\cdot k^{s-1}\cdot\prod_{i=1}^{s}\frac{(a_i-1)!}{\alpha_{i1}!\dots\alpha_{ik}!}$. Thus we can rewrite the RHS as $\sum_{\alpha_i\in P_k}\left|\mathcal{NP}_m^{(T)}(\vec{\alpha}_1,\dots,\vec{\alpha}_s,\Omega_{2c})\right|\cdot\prod_{i=1}^{s}\overrightarrow{\Omega^{(i)}}_{\vec{\alpha}_i}$.

By the same argument as Proposition 8.2.6,

$$\begin{aligned} RHS &= \sum_{\alpha_i\in P_k}\left|\mathcal{NP}_m^{(T)}(\vec{\alpha}_1,\dots,\vec{\alpha}_s,\Omega_{2c})\right|\cdot\prod_{i=1}^{s}\overrightarrow{\Omega^{(i)}}_{\vec{\alpha}_i} \\ &= \sum_{\substack{\mathcal{P}_1\cup\dots\cup\mathcal{P}_s\cup\hat{\mathcal{P}}\\ \in\mathcal{NP}_{k,m,s}^{(T)}(\Omega_{2c})}} \overrightarrow{\Omega^{(1)}}_{\mathcal{P}_1/\hat{m}}\cdot\overrightarrow{\Omega^{(2)}}_{\mathcal{P}_2}\cdot\dots\cdot\overrightarrow{\Omega^{(s)}}_{\mathcal{P}_s}. \end{aligned}$$

where $\hat{m} = m-s+1$.

We are going to prove that there is a bijection between $\bigsqcup_{\substack{k_i\geq 1:\\ k_1+\dots+k_c=k}}\mathcal{NP}_{k_1,m,s}^{(T)}\times\dots\times\mathcal{NP}_{k_c,m,s}^{(T)}$ and $\mathcal{NP}_{k,m,s}^{(T)}(\Omega_{2c})$.

1. $\Phi:\mathcal{NP}_{k,m,s}^{(T)}(\Omega_{2c})\to\bigsqcup_{\substack{k_i\geq 1:\\ k_1+\dots+k_c=k}}\mathcal{NP}_{k_1,m,s}^{(T)}\times\dots\times\mathcal{NP}_{k_c,m,s}^{(T)}$: Let $\mathcal{P}_1\cup\dots\cup\mathcal{P}_s\cup\hat{\mathcal{P}}\in\mathcal{NP}_{k,m,s}^{(T)}(\Omega_{2c})$. $\hat{\mathcal{P}}$ contains a polygon P_c of size c. Moreover, since $\hat{m}k\in P_c$ by definition, we can assume $P_c=\{\hat{m}i_1,\dots,\hat{m}i_c\}$ for some $1\leq i_1<\dots<i_c=k$. Let $i_0=0$. Since $\mathcal{P}_1\cup\dots\cup\mathcal{P}_s\cup\hat{\mathcal{P}}$ is non-crossing, for each $j\in[c]$, everything in between $\hat{m}i_{j-1}$ (exclusive) and $\hat{m}i_j$ (inclusive) can be treated as a collection of non-crossing partitions in $\mathcal{NP}_{k_j,m,s}^{(T)}$ where $k_j=i_j-i_{j-1}$. More precisely, for each $j\in[c]$, let $V_j=\left\{\hat{m}i_{j-1}+1,\hat{m}i_{j-1}+2,\dots\hat{m}i_j\right\}$. Then we can divide $\mathcal{P}_1,\dots,\mathcal{P}_s$ into $\mathcal{P}_{ij}$ for $i\in[s]$, $j\in[c]$ such that

   i. $\mathcal{P}_{i1}\cup\dots\cup\mathcal{P}_{ic}=\mathcal{P}_i$, and

   ii. for each $j\in[c]$, $\mathcal{P}_{1j}\cup\dots\cup\mathcal{P}_{sj}$ only uses vertices in V_j. i.e. $\mathcal{P}_{1j}\cup\dots\cup\mathcal{P}_{sj}\in\mathcal{NP}_{k_j,m,s}^{(T)}$ where $k_j=i_j-i_{j-1}$. See Figure 8.4 for an illustration.

Thus the resulting $\left(\mathcal{P}_{11}\cup\cdots\cup\mathcal{P}_{s1},\ldots,\mathcal{P}_{1c}\cup\cdots\cup\mathcal{P}_{sc}\right)\in\mathcal{NP}^{(T)}_{k_1,m,s}\times\cdots\times\mathcal{NP}^{(T)}_{k_c,m,s}$.

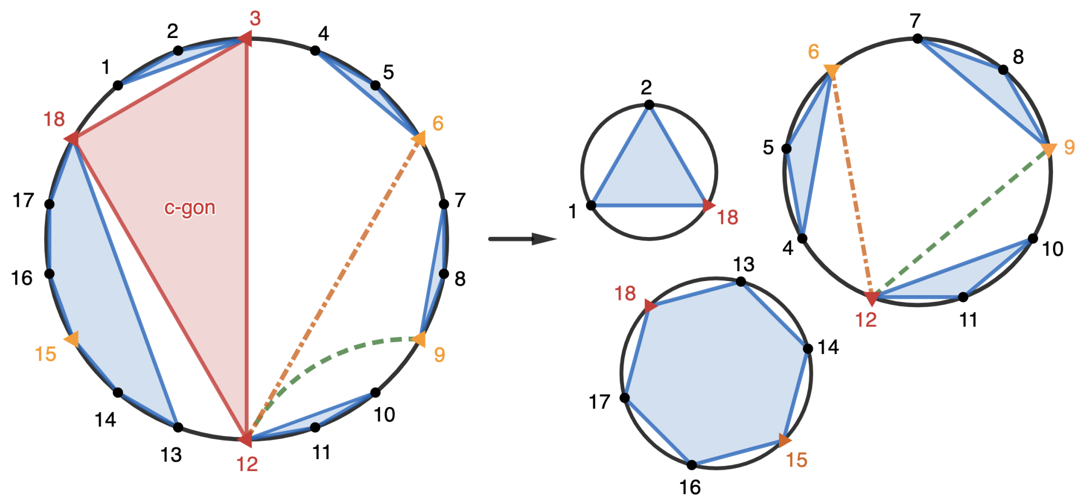


Figure 8.4: Illustration of Lemma 8.2.7: here $m=5, s=3, k=6, c=3$ and $\hat{m}=m-s+1=3$. For this configuration $k_1=1$, $k_2=3$ and $k_3=2$.

2. Ψ : $\displaystyle\bigsqcup_{\substack{k_i\geq 1:\\ k_1+\cdots+k_c=k}} \mathcal{NP}^{(T)}_{k_1,m,s}\times\cdots\times\mathcal{NP}^{(T)}_{k_c,m,s}\to\mathcal{NP}^{(T)}_{k,m,s}(\Omega_{2c})$: Let $\left(\mathcal{P}_{11}\cup\cdots\cup\mathcal{P}_{s1},\ldots,\mathcal{P}_{1c}\cup\cdots\cup\mathcal{P}_{sc}\right)\in\mathcal{NP}^{(T)}_{k_1,m,s}\times\cdots\times\mathcal{NP}^{(T)}_{k_c,m,s}$ for some $k_i\geq 1$ where $k_1+\cdots+k_c=k$. We will "glue" them together in the following way:

   i. For each $i\in[s], j\in[c]$, let $\mathcal{P}'_{ij}=\mathcal{P}_{ij}+\hat{m}(k_1+\cdots+k_{j-1}):=\left\{\{v+\hat{m}(k_1+\cdots+k_{j-1}):v\in P\}:P\in\mathcal{P}_{ij}\right\}$. i.e. we shift the indices of the polygons in each $\mathcal{P}_{ij}$ up by $\hat{m}(k_1+\cdots+k_{j-1})$.

   ii. For each $i\in[s]$, let $\mathcal{P}_i=\mathcal{P}_{i1}\cup\cdots\cup\mathcal{P}_{ic}$.

   iii. Let $P_c=\left\{\hat{m}(k_1+\cdots+k_j):j\in[c]\right\}$ and $\hat{\mathcal{P}}=\left\{\{v\},P_c:v\in[\hat{m}k]\setminus P_c\right\}$.

   We can see that $\mathcal{P}_1\cup\cdots\cup\mathcal{P}_s\cup\hat{\mathcal{P}}\in\mathcal{NP}^{(T)}_{k,m,s}(\Omega_{2c})$.

3. It is easy to see that Φ and Ψ are inverses of each other.

Since the bijection only shifts indices of the polygons, number of indices in the polygons stay the same. i.e. for each $\mathcal{P}_1\cup\cdots\cup\mathcal{P}_s\cup\hat{\mathcal{P}}\in\mathcal{NP}^{(T)}_{k,m,s}(\Omega_{2c})$ where $\mathcal{P}_i=\{P_{i,1},\ldots,P_{i,a_i}\}$,

$\Phi\left(\mathcal{P}_1 \cup \dots \cup \mathcal{P}_s \cup \hat{\mathcal{P}}\right) = \left(\mathcal{P}_{11} \cup \dots \cup \mathcal{P}_{s1}, \dots, \mathcal{P}_{1c} \cup \dots \cup \mathcal{P}_{sc}\right)$ for some $\mathcal{P}_{ij}$ where $\mathcal{P}_{ij} = \{P_{ij,1}, \dots, P_{ij,a_{ij}}\}$. Then

$$\left(\Omega^{(1)}_{2\,|P_{1,1}|/\hat{m}} \cdot \dots \cdot \Omega^{(1)}_{2\,|P_{1,a_1}|/\hat{m}}\right) \cdot \prod_{j=2}^{s} \left(\Omega^{(j)}_{2\,|P_{j,1}|} \cdot \dots \cdot \Omega^{(j)}_{2\,|P_{j,a_j}|}\right)$$
$$= \prod_{t=1}^{c} \left(\left(\Omega^{(1)}_{2\,|P_{1t,1}|/\hat{m}} \cdot \dots \cdot \Omega^{(1)}_{2\,|P_{1t,a_{1t}}|/\hat{m}}\right) \cdot \prod_{j=2}^{s} \left(\Omega^{(j)}_{2\,|P_{jt,1}|} \cdot \dots \cdot \Omega^{(j)}_{2\,|P_{jt,a_{jt}}|}\right)\right)$$

Thus combining everything,

$$\sum_{\substack{k_i \geq 1: \\ k_1 + \dots + k_c = k}} B^{(s)}_m(k_1) \dots B^{(s)}_m(k_c)$$
$$= \sum_{\substack{k_i \geq 1: \\ k_1 + \dots + k_c = k}} \prod_{t=1}^{c} \sum_{\substack{\mathcal{P}_{1t} \sqcup \dots \sqcup \mathcal{P}_{st} \\ \in \mathcal{NP}_{k_t, m, s}}} \overrightarrow{\Omega^{(1)}}_{\mathcal{P}_{1t}/\hat{m}} \cdot \overrightarrow{\Omega^{(2)}}_{\mathcal{P}_{2t}} \cdot \dots \cdot \overrightarrow{\Omega^{(s)}}_{\mathcal{P}_{st}}$$
$$= \sum_{\substack{k_i \geq 1: \\ k_1 + \dots + k_c = k}} \sum_{\substack{(\mathcal{P}_{11} \cup \dots \cup \mathcal{P}_{s1}, \dots, \mathcal{P}_{1c} \cup \dots \cup \mathcal{P}_{sc}) \\ \in \mathcal{NP}^{(T)}_{k_1, m, s} \times \dots \times \mathcal{NP}^{(T)}_{k_c, m, s}}} \prod_{t=1}^{c} \overrightarrow{\Omega^{(1)}}_{\mathcal{P}_{1t}/\hat{m}} \cdot \overrightarrow{\Omega^{(2)}}_{\mathcal{P}_{2t}} \cdot \dots \cdot \overrightarrow{\Omega^{(s)}}_{\mathcal{P}_{st}}$$
$$= \sum_{\substack{k_i \geq 1: \\ k_1 + \dots + k_c = k}} \sum_{\substack{(\mathcal{P}_{11} \cup \dots \cup \mathcal{P}_{s1}, \dots, \mathcal{P}_{1c} \cup \dots \cup \mathcal{P}_{sc}) \\ \in \mathcal{NP}^{(T)}_{k_1, m, s} \times \dots \times \mathcal{NP}^{(T)}_{k_c, m, s}}} \overrightarrow{\Omega^{(1)}}_{\mathcal{P}_1/\hat{m}} \cdot \overrightarrow{\Omega^{(2)}}_{\mathcal{P}_2} \cdot \dots \cdot \overrightarrow{\Omega^{(s)}}_{\mathcal{P}_s}$$

where $\mathcal{P}_1 \cup \dots \cup \mathcal{P}_s \cup \hat{\mathcal{P}} = \Psi\Big((\mathcal{P}_{11} \cup \dots \cup \mathcal{P}_{s1}, \dots, \mathcal{P}_{1c} \cup \dots \cup \mathcal{P}_{sc}) \Big), \mathcal{P}_i = \Big\{P_{i,1}, \dots, P_{i,a_i}\Big\}$

$$= \sum_{\substack{\mathcal{P}_1 \cup \dots \cup \mathcal{P}_s \cup \hat{\mathcal{P}} \\ \in \mathcal{NP}^{(T)}_{k,m,s}(\Omega_{2c})}} \overrightarrow{\Omega^{(1)}}_{\mathcal{P}_1/\hat{m}} \cdot \overrightarrow{\Omega^{(2)}}_{\mathcal{P}_2} \cdot \dots \cdot \overrightarrow{\Omega^{(s)}}_{\mathcal{P}_s}$$
$$= c \cdot \sum_{\alpha_i \in P_k} \binom{(m-s+1)k}{mk - (a_1 + \dots + a_s) + c} \cdot k^{s-1} \cdot \left(\prod_{i=1}^{s} \frac{(a_i - 1)!}{\alpha_{i1}! \dots \alpha_{ik}!} \cdot \overrightarrow{\Omega^{(i)}}^{\vec{\alpha}_i} \right)$$

as needed. □

Now we are ready to prove the main result for this section, Theorem 8.2.3.

Theorem 8.2.3.

$$\begin{aligned} A_m^{(s)}\left(k, \underbrace{0,\ldots,0}_{s}\right) &= \sum_{\vec{\alpha}_i \in P_k} C_m(\vec{\alpha}_1,\ldots,\vec{\alpha}_s) \cdot \prod_{j=1}^{s} \overrightarrow{\Omega^{(i)}\vec{\alpha}_i} \\ &= \sum_{\vec{\alpha}_i \in P_k} \binom{(m-s+1)k}{mk-(a_1+\cdots+a_s)+1} \cdot k^{s-1} \cdot \left(\prod_{j=1}^{s} \frac{(a_i-1)!}{\alpha_{i1}!\ldots\alpha_{ik}!} \cdot \overrightarrow{\Omega^{(i)}\vec{\alpha}_i} \right) \end{aligned} \tag{8.9}$$

where $\vec{\alpha}_i = (\alpha_{i1},\ldots,\alpha_{ik})$, $a_i = \sum_{j=1} \alpha_{ij}$ *for all* $i = 1,\ldots,s$.

Proof of Theorem 8.2.3. We prove this by induction on s.

1. Base case $s = 0$:

$$\begin{aligned} &\binom{(m-s+1)k}{mk-(a_1+\cdots+a_s)+1} \cdot k^{s-1} \cdot \prod_{i=1}^{s} \frac{(a_i-1)!}{\alpha_{i1}!\ldots\alpha_{ik}!} \\ =&\binom{(m+1)k}{mk+1} \cdot k^{-1} = \frac{((m+1)k)!}{(mk+1)!(k-1)!} \cdot \frac{1}{k} = \frac{1}{mk+1} \cdot \frac{((m+1)k)!}{(mk)!k!} \\ =&\frac{1}{mk+1} \cdot \binom{(m+1)k}{k} = C(k,m) \end{aligned}$$

2. Assume $A_m^{(s)}\left(k, \underbrace{0,\ldots,0}_{s}\right) = \sum_{\vec{\alpha}_i \in P_k} C_m\left(\vec{\alpha}_1,\ldots,\vec{\alpha}_s\right) \cdot \prod_{i=1}^{s} \overrightarrow{\Omega^{(i)}\vec{\alpha}_i}$, we will prove

   $A_m^{(s+1)}\left(k, \underbrace{0,\ldots,0}_{s}\right) = \sum_{\vec{\alpha}_i \in P_k} C_m\left(\vec{\alpha}_1,\ldots,\vec{\alpha}_{s+1}\right) \cdot \prod_{i=1}^{s+1} \overrightarrow{\Omega^{(i)}\vec{\alpha}_i}$.

For the LHS, by the definition of $A_m^{(s+1)}(k,0,\dots,0)$,

$$\begin{aligned}
& A_m^{(s+1)}(k,0,\dots,0) \\
= & \sum_{\vec{\gamma},\vec{\beta}\in P_k} C\left(\vec{\alpha},\vec{\beta}\right)\cdot \overrightarrow{A_m^{(s)}}^{\vec{\gamma}} \cdot \overrightarrow{\Omega^{(s+1)}}^{\vec{\beta}} \\
= & \sum_{\vec{\gamma},\vec{\beta}\in P_k} (-1)^{c+b-k-1}\cdot k\cdot \binom{b+c-2}{k-1}\cdot \frac{(c-1)!}{\gamma_1!\dots\gamma_k!}\cdot \overrightarrow{A_m^{(s)}}^{\vec{\gamma}}\cdot \frac{(b-1)!}{\beta_1!\dots\beta_k!}\cdot \overrightarrow{\Omega^{(s+1)}}^{\vec{\beta}} \\
= & \sum_{\vec{\beta}\in P_k} k\cdot \left(\sum_{\vec{\gamma}\in P_k} (-1)^{c+b-k-1}\cdot \binom{b+c-2}{k-1}\cdot \frac{(c-1)!}{\gamma_1!\dots\gamma_k!}\cdot \overrightarrow{A_m^{(s)}}^{\vec{\gamma}}\right)\cdot \frac{(b-1)!}{\beta_1!\dots\beta_k!}\cdot \overrightarrow{\Omega^{(s+1)}}^{\vec{\beta}}
\end{aligned}$$

On the other hand, for the RHS,

$$\begin{aligned}
& \sum_{\vec{\alpha}_i\in P_k} C_m\left(\vec{\alpha}_1,\dots,\vec{\alpha}_{s+1}\right)\cdot \prod_{i=1}^{s+1} \overrightarrow{\Omega^{(i)}}^{\vec{\alpha}_i} \\
= & \sum_{\vec{\alpha}_i\in P_k} \binom{(m-s)k}{mk-(a_1+\dots+a_{s+1})+1}\cdot k^s\cdot \left(\prod_{i=1}^{s+1} \frac{(a_i-1)!}{\alpha_{i1}!\dots\alpha_{ik}!}\cdot \overrightarrow{\Omega^{(i)}}^{\vec{\alpha}_i}\right) \\
= & \sum_{\vec{\beta}\in P_k} k\cdot \left(\sum_{\vec{\alpha}_i\in P_k} \binom{(m-s)k}{mk-(a_1+\dots+a_s+b)+1}\cdot k^{s-1}\cdot \prod_{i=1}^{s} \frac{(a_i-1)!}{\alpha_{i1}!\dots\alpha_{ik}!}\cdot \overrightarrow{\Omega^{(i)}}^{\vec{\alpha}_i}\right) \\
& \cdot \frac{(b-1)!}{\beta_1!\dots\beta_k!}\cdot \overrightarrow{\Omega^{(s+1)}}^{\vec{\beta}}
\end{aligned}$$

Thus it suffices to prove that

$$\begin{aligned}
& \sum_{\vec{\gamma}\in P_k} (-1)^{c+b-k-1}\cdot \binom{b+c-2}{k-1}\cdot \frac{(c-1)!}{\gamma_1!\dots\gamma_k!}\cdot \overrightarrow{A_m^{(s)}}^{\vec{\gamma}} \\
= & \sum_{\vec{\alpha}_i\in P_k} \binom{(m-s)k}{mk-(a_1+\dots+a_s+b)+1}\cdot k^{s-1}\cdot \left(\prod_{i=1}^{s} \frac{(a_i-1)!}{\alpha_{i1}!\dots\alpha_{ik}!}\cdot \overrightarrow{\Omega^{(i)}}^{\vec{\alpha}_i}\right)
\end{aligned}$$

Rewriting the LHS and using Lemma 8.2.7, we get

$$\begin{aligned}
&\sum_{\vec{\gamma}\in P_k} (-1)^{c+b-k-1} \cdot \binom{b+c-2}{k-1} \cdot \frac{(c-1)!}{\gamma_1! \dots \gamma_k!} \cdot \overrightarrow{A_m^{(s)}}^{\vec{\gamma}} \\
&= \sum_{\vec{\gamma}\in P_k} (-1)^{c+b-k-1} \cdot \binom{b+c-2}{k-1} \cdot \frac{1}{c} \cdot \frac{c!}{\gamma_1! \dots \gamma_k!} \cdot \left(\prod_{i=1}^{k} A_m^{(s)}(i,0,\dots,0)^{\gamma_i} \right) \\
&= \sum_{c=1}^{k} (-1)^{c+b-k-1} \cdot \binom{b+c-2}{k-1} \cdot \frac{1}{c} \cdot \left(\sum_{\substack{k_i\geq 1: \\ k_1+\cdots+k_c=k}} A_m^{(s)}(k_1,0,\dots,0) \dots A_m^{(s)}(k_c,0,\dots,0) \right) \\
&= \sum_{c=1}^{k} (-1)^{c+b-k-1} \cdot \binom{b+c-2}{k-1} \\
&\qquad \cdot \left(\sum_{\alpha_i\in P_k} \binom{(m-s+1)k}{a_1+\cdots+a_s-(s-1)k-c} \cdot k^{s-1} \cdot \prod_{i=1}^{s} \frac{(a_i-1)!}{\alpha_{i1}! \dots \alpha_{ik}!} \cdot \overrightarrow{\Omega^{(i)}}^{\vec{\alpha}_i} \right) \\
&= \sum_{\vec{\alpha}_i\in P_k} k^{s-1} \cdot \prod_{i=1}^{s} \frac{(a_i-1)!}{\alpha_{i1}! \dots \alpha_{ik}!} \cdot \overrightarrow{\Omega^{(i)}}^{\vec{\alpha}_i} \\
&\qquad \cdot \left(\sum_{c=1}^{k} (-1)^{c+b-k-1} \cdot \binom{b+c-2}{k-1} \cdot \binom{(m-s+1)k}{a_1+\cdots+a_s-(s-1)k-c} \right)
\end{aligned}$$

Therefore it is equivalent to prove that

$$\begin{aligned}
&\binom{(m-s)k}{mk-(a_1+\cdots+a_s+b)+1} \\
&= \sum_{c=1}^{k} (-1)^{c+b-k-1} \cdot \binom{b+c-2}{k-1} \cdot \binom{(m-s+1)k}{(a_1+\cdots+a_s)-(s-1)k-c}
\end{aligned} \tag{8.15}$$

Recall Proposition 7.1.3

$$\sum_{i=0}^{m-k} (-1)^{i+(m-k)} \cdot \binom{m-i}{k} \cdot \binom{n}{i} = \binom{n-k-1}{m-k}. \tag{8.16}$$

Let $i = (a_1+\cdots+a_s)-(s-1)k-c$ in (8.15). Notice that for the summands to be nonzero, we require

i. $b+c-2 \geq k-1$ which implies that $c \geq k-b+1$

ii. Since $\binom{(m-s+1)k}{(a_1+\cdots+a_s)-(s-1)k-c} = \binom{(m-s+1)k}{mk-(a_1+\cdots+a_s)+c}$, $(m-s+1)k \geq mk-(a_1+\cdots+a_s)+c$ which implies that $c \leq (a_1+\cdots+a_s)-(s-1)k$.

Thus

$$\begin{aligned}
&\sum_{c=1}^{k} (-1)^{c+b-k-1} \cdot \binom{b+c-2}{k-1} \cdot \binom{(m-s+1)k}{(a_1+\cdots+a_s)-(s-1)k-c} \\
=& \sum_{c=k-b+1}^{(a_1+\cdots+a_s)-(s-1)k} (-1)^{c+b-k-1} \cdot \binom{b+c-2}{k-1} \cdot \binom{(m-s+1)k}{(a_1+\cdots+a_s)-(s-1)k-c} \\
=& \sum_{i=0}^{(a_1+\cdots+a_s)+b-sk-1} \binom{b+(a_1+\cdots+a_s)+(s-1)k-2-i}{k-1}\binom{(m-s+1)k}{i} \\
=& \binom{(m-s+1)k-(k-1)-1}{(b+(a_1+\cdots+a_s)+(s-1)k-2)-(k-1)} \\
=& \binom{(m-s)k}{(a_1+\cdots+a_s)+b-sk-1} = \binom{(m-s)k}{mk-(a_1+\cdots+a_s+b)+1}
\end{aligned}$$

as needed.

□

CHAPTER 9

CONNECTION WITH GRAPH MATRICES

Let $M^{(G)}_{Z(m),s}$ be defined as in Definition 2.2.18.

In this chapter, we will prove Theorem 2.2.22.

Theorem 2.2.22. *For all $\Omega^{(1)}, \ldots, \Omega^{(s)} \in \mathbf{\Omega_{0,1}}$ which satisfy Carleman's condition,*

$$\Omega_{Z(m)} \circ_R \Omega^{(1)} \circ_R \ldots \circ_R \Omega^{(s)} = \Omega_{Z(m),\Omega^{(1)},\ldots,\Omega^{(s)}} \tag{2.11}$$

By the Trace Power Method, Theorem 2.2.22 can be proved by proving the following statement.

Theorem 9.0.1. *$\Omega_{Z(m)} \circ_R \Omega^{(1)} \circ_R \ldots \circ_R \Omega^{(s)}$ and the limiting distributions of singular values of $M^{(G)}_{Z(m),s}$ have the same moments. In other words, if we let $r(n,m) = \dfrac{n!}{(n-m)!}$, then*

$$\begin{aligned}\left(\Omega_{Z(m)} \circ_R \Omega^{(1)} \circ_R \ldots \circ_R \Omega^{(s)}\right)_{2k} &= \lim_{n\to\infty} \frac{1}{r(n,m)} \cdot \mathbb{E}\left[\operatorname{tr}\left(\left(M^{(G)}_{Z(m),s} {M^{(G)}_{Z(m),s}}^T\right)^k\right)\right] \\ &= \left(\Omega_{Z(m),\Omega^{(1)},\ldots,\Omega^{(s)}}\right)_{2k}.\end{aligned} \tag{9.1}$$

We are going to prove the above result by proving their moments have the same recurrence relations. First we will prove the recurrence relations for $\left(\Omega_{Z(m)} \circ_R \Omega^{(1)} \circ_R \ldots \circ_R \Omega^{(s)}\right)_{2k}$, and then for the trace power moments of $M^{(G)}_{Z(m),s}$.

9.1 Recurrence Relations for $\left(\Omega_{Z(m)} \circ_R \Omega^{(1)} \circ_R \ldots \circ_R \Omega^{(s)}\right)_{2k}$

We will use the notation $A^{(s)}_m(k,0,\ldots,0)$ from Chapter 8.

$$\begin{aligned}A^{(s)}_m(k,0,\ldots,0) &= \left(\Omega_{Z(m)} \circ_R \Omega^{(1)} \circ_R \ldots \circ_R \Omega^{(s)}\right)_{2k} \\ &= \sum_{\vec{\alpha}_i \in P_k} C_m(\vec{\alpha}_1,\ldots,\vec{\alpha}_s) \cdot \overrightarrow{\Omega^{(1)}}\vec{\alpha}_1 \ldots \overrightarrow{\Omega^{(s)}}\vec{\alpha}_s\end{aligned} \tag{9.2}$$

where

$$C_m\left(\vec{\alpha}_1,\ldots,\vec{\alpha}_s\right)=\binom{(m-s+1)k}{mk-(a_1+\cdots+a_s)+1}\cdot k^{s-1}\cdot\prod_{i=1}^{s}\frac{(a_i-1)!}{\alpha_{i1}!\ldots\alpha_{ik}!}. \tag{9.3}$$

Remark 9.1.1. *Note by definition of $C_m\left(\vec{\alpha}_1,\ldots,\vec{\alpha}_s\right)$, $C_m\left(\vec{\alpha}_1,\ldots,\vec{\alpha}_s\right)\neq 0 \implies m-s+1>0 \implies m\geq s$. Thus we can always assume that $m\geq s$.*

Proposition 9.1.2. *Assume $m\geq s$. Let $\hat{m}=m-s+1$. Then*

$$A_m^{(s)}(k,0,\ldots,0)=\sum_{\vec{\alpha}_i\in P_k}\sum_{\substack{\mathcal{P}_1\sqcup\cdots\sqcup\mathcal{P}_s\in\\ \mathcal{NP}_m(\vec{\alpha}_1,\ldots,\vec{\alpha}_s)}}\left(\Omega^{(1)}_{2\,|P_{1,1}|/\hat{m}}\cdot\cdots\cdot\Omega^{(1)}_{2\,|P_{1,a_1}|/\hat{m}}\right)\cdot\prod_{j=2}^{s}\left(\Omega^{(j)}_{2\,|P_{j,1}|}\cdot\cdots\cdot\Omega^{(j)}_{2\,|P_{j,a_j}|}\right) \tag{9.4}$$

where $\mathcal{P}_i=\left\{P_{i,1},\ldots,P_{i,a_i}\right\}$ for each $i\in[s]$.

Proof. By Theorem 3.3.13, $C_m\left(\vec{\alpha}_1,\ldots,\vec{\alpha}_s\right)=\left|\mathcal{NP}_m\left(\vec{\alpha}_1,\ldots,\vec{\alpha}_s\right)\right|$. In particular, the size of any partition set corresponding to $\vec{\alpha}_1$ is a multiple of $\hat{m}=m-s+1$. Thus

$$\begin{aligned}
A(k,0)&=\sum_{\vec{\alpha}_i\in P_k}C_m(\vec{\alpha}_1,\ldots,\vec{\alpha}_s)\cdot\overrightarrow{\Omega^{(1)}}^{\vec{\alpha}_1}\cdot\cdots\cdot\overrightarrow{\Omega^{(s)}}^{\vec{\alpha}_s}\\
&=\sum_{\vec{\alpha}_i\in P_k}\left|\mathcal{NP}_m\left(\vec{\alpha}_1,\ldots,\vec{\alpha}_s\right)\right|\cdot\overrightarrow{\Omega^{(1)}}^{\vec{\alpha}_1}\cdot\cdots\cdot\overrightarrow{\Omega^{(s)}}^{\vec{\alpha}_s}\\
&=\sum_{\vec{\alpha}_i\in P_k}\sum_{\substack{\mathcal{P}_1\sqcup\cdots\sqcup\mathcal{P}_s\in\\ \mathcal{NP}_m(\vec{\alpha}_1,\ldots,\vec{\alpha}_s)}}\left({\Omega^{(1)}_2}^{\alpha_{11}}\ldots,{\Omega^{(1)}_{2k}}^{\alpha_{1k}}\right)\cdot\cdots\cdot\left({\Omega^{(s)}_2}^{\alpha_{s1}}\ldots,{\Omega^{(s)}_{2k}}^{\alpha_{sk}}\right)\\
&=\sum_{\vec{\alpha}_i\in P_k}\sum_{\substack{\mathcal{P}_1\sqcup\cdots\sqcup\mathcal{P}_s\in\\ \mathcal{NP}_m(\vec{\alpha}_1,\ldots,\vec{\alpha}_s)}}\left(\Omega^{(1)}_{2\,|P_{1,1}|/\hat{m}}\cdot\cdots\cdot\Omega^{(1)}_{2\,|P_{1,a_1}|/\hat{m}}\right)\cdot\prod_{j=2}^{s}\left(\Omega^{(j)}_{2\,|P_{j,1}|}\cdot\cdots\cdot\Omega^{(j)}_{2\,|P_{j,a_j}|}\right).
\end{aligned}$$

□

Next we will define $A_m^{(s)}\left(k,r_1,\ldots,r_s\right)$. For simplicity, we will only give the definition for the case $s=1$ here and defer the full definition to Section 9.1.3 (see Definition 9.1.16).

When $s=1$, we denote $A_m^{(s)}\left(k,r_1,\ldots,r_s\right)$ as $A_m\left(k,r\right)$.

Definition 9.1.3. Let $r \geq 0$. We define $A_m(k, r)$ to be

$$A_m(k,r) = \sum_{\vec{\alpha}\in P_k} \sum_{\substack{\mathcal{P}=\{P_1,\dots,P_a\}\in \\ \mathcal{NP}(m\vec{\alpha}):\,1\in P_1}} \left(\Omega_{2\,(|P_1|/m+r)}\Omega_{2\,|P_2|/m}\cdot\dots\cdot\Omega_{2\,|P_a|/m}\right) \tag{9.5}$$

Remark 9.1.4. *When $m = 1$, we further denote $A_m(k, r)$ as $A(k, r)$. Then*

$$A(k,r) = \sum_{\vec{\alpha}\in P_k} \sum_{\substack{\mathcal{P}=\{P_1,\dots,P_a\}\in \\ \mathcal{NP}(\vec{\alpha}):\,1\in P_1}} \left(\Omega_{2\,(|P_1|+r)}\Omega_{2\,|P_2|}\cdot\dots\cdot\Omega_{2\,|P_a|}\right) \tag{9.6}$$

Remark 9.1.5. *When $r = 0$, $A_m(k, r)$ coincides with $A_m(k, 0)$.*

We will prove the following main result.

Theorem 9.1.6.

$$\begin{aligned} A_m^{(s)}(k, r_1, \dots, r_s) = \sum_{\substack{i_1,\dots,i_{m+1}\geq 0: \\ i_1+\dots+i_{m+1}=k-1}} & A_m^{(s)}(i_1, r_1+1, 0, \dots, 0) \dots A_m^{(s)}(i_s, 0, \dots, 0, r_s+1)\cdot \\ & A_m^{(s)}(i_{s+1}, 0, \dots, 0) \dots A_m^{(s)}(i_{m+1}, 0, \dots, 0) \end{aligned} \tag{9.7}$$

9.1.1 Base Case s=1, m=1

Recall the line shape in Definition 2.1.6.

Remark 9.1.7. *By definition, $\alpha_{Z(1)} = \alpha_0$. We will let Ω_{α_0} denote $\Omega_{Z(1)}$.*

When $m = 1, s = 1$, applying Proposition 9.1.2, we have that

$$\begin{aligned} A(k,0) = \left(\Omega_{Z(1)} \circ_R \Omega\right)_{2k} = \left(\Omega_{\alpha_0} \circ_R \Omega\right)_{2k} &= \sum_{\vec{\alpha}\in P_k} \binom{k}{a-1} \cdot \frac{(a-1)!}{\alpha_1! \dots \alpha_k!} \cdot \vec{\Omega}^{\vec{\alpha}} \\ &= \sum_{\vec{\alpha}\in P_k} \sum_{\mathcal{P}=\{P_1,\dots,P_a\}\in\mathcal{NP}(\vec{\alpha})} \left(\Omega_{2\,|P_1|}\cdot\dots\cdot\Omega_{2\,|P_a|}\right) \end{aligned} \tag{9.8}$$

and

$$A\,(k,r) = \sum_{\vec{\alpha}\in P_k} \sum_{\substack{\mathcal{P}=\{P_1,\dots,P_a\}\in \\ \mathcal{NP}(\vec{\alpha}):\,1\in P_1}} \left(\Omega_{2\,(|P_1|+r)}\Omega_{2\,|P_2|}\cdot\dots\cdot\Omega_{2\,|P_a|}\right) \tag{9.9}$$

We will prove the following recurrence relation.

Theorem 9.1.8.

$$A\,(k,r) = \sum_{\substack{i,j\geq 0: \\ i+j=k-1}} A\,(i,0)\cdot A\,(j,r+1) \tag{9.10}$$

and

$$A(0,r) = \Omega_{2r}\,. \tag{9.11}$$

The following result is not needed for the proof of Theorem 9.1.8, but gives us an alternative way of looking at $A_m(k,r)$ when $s=1$. More precisely, we can deduce an explicit formula for $A_m(k,r)$.

Definition 9.1.9. Given $\vec{\alpha}\in P_k$, we denote $\vec{\Omega}^{\vec{\alpha}}(i,+r)$ to be $\Omega_2^{\alpha_1}\cdot\dots\cdot\left(\Omega_{2i}^{\alpha_i-1}\Omega_{2(i+r)}\right)\cdot\dots\cdot\Omega_{2k}^{\alpha_k}$.

Proposition 9.1.10. *When* $s=1$, $A_m(k,0)=\sum_{\vec{\alpha}\in P_k} C_m\,(\vec{\alpha})\cdot\vec{\Omega}^{\vec{\alpha}}$ *and*

$$A_m(k,r) = \sum_{\vec{\alpha}\in P_k} C_m\,(\vec{\alpha})\cdot\left(\sum_{i}^{k}\frac{i\cdot\alpha_i}{k}\cdot\vec{\Omega}^{\vec{\alpha}}\,(i,+r)\right)\,. \tag{9.12}$$

Example 9.1.11.

1. $A(2,0)=\Omega_2^2+\Omega_4,\ A(2,1)=\Omega_2\Omega_4+\Omega_6,\ A(2,2)=\Omega_2\Omega_6+\Omega_8.$
2. $A(3,0)=\Omega_2^3+3\,\Omega_2\Omega_4+\Omega_6,\ A(3,1)=\Omega_2^2\Omega_4+3\left(\frac{1}{3}\,\Omega_4\Omega_4+\frac{2}{3}\,\Omega_2\Omega_6\right)+\Omega_8=\Omega_2^2\Omega_4+\Omega_4^2+2\,\Omega_2\Omega_6+\Omega_8.$

Proof of Theorem 9.1.8. We obtain this recurrence relation with the following steps:

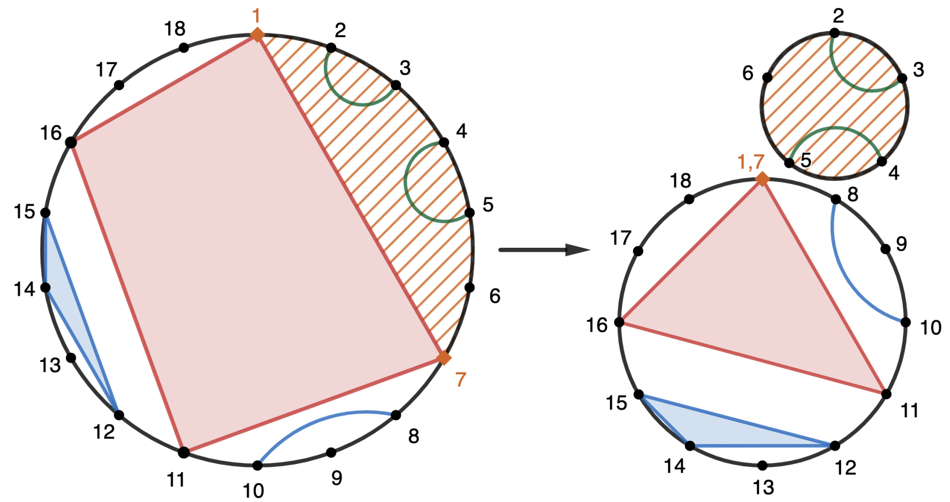


Figure 9.1: Illustration of Theorem 9.1.8: Here $i = 6$. The orange shaded part on the right represents $A(i-1, 0)$, and the other part represents $A(k-i, r+1)$.

i. Given a non-crossing partition $\mathcal{P}$ of $[k]$, we will look at the smallest $i \geq 1$ such that 1 and $i+1$ are in the same partition set (i.e. 1 and $i+1$ are vertices of the same polygon P_1, and $i+1$ is immediately after 1 in the clockwise direction). In the case when 1 is isolated, i is chosen to be k.

ii. Since $\mathcal{P}$ is non-crossing, everything in between 1 and $i+1$ can be viewed a non-crossing partition on the cycle with elements $\{2, 3, \ldots, i\}$ (i.e. cycle of length $i-1$). This gives rise to $A(i-1, 0)$.

iii. Now we identify 1 with $i+1$ and shrink the size of the polygon P_1 by 1. The remaining part from $i+1$ to k is a non-crossing partition on $k-i$ elements, and this gives rise to $A(k-i, r+1)$.

See Figure 9.1 for an illustration.

More precisely,

$$A(k,r) = \sum_{\substack{\mathcal{P}=\{P_1,\ldots,P_a\}\in\mathcal{NP}_k:\\ 1\in P_1}} \Omega_{2\left(|P_1|+r\right)}\Omega_{2|P_2|}\cdot\cdots\cdot\Omega_{2|P_a|}$$

$$=\sum_{i=1}^{k}\sum_{\substack{\mathcal{P}=\{P_1,\ldots,P_a\}\in\mathcal{NP}_k:\\ 1,i+1\in P_1,\, j\notin P_1\forall\, 2\le j\le i}} \Omega_{2\left(|P_1|+r\right)}\Omega_{2|P_2|}\cdot\cdots\cdot\Omega_{2|P_a|}$$

$$=\sum_{i=1}^{k}\sum_{\substack{\mathcal{P}=\{P_1,\ldots,P_a\}\in\mathcal{NP}_k:\\ 1,i+1\in P_1,\, j\notin P_1\forall\, 2\le j\le i}} \left(\Omega_{2\left|P_{i_1}\right|}\cdot\cdots\cdot\Omega_{2\left|P_{i_b}\right|}\right)\cdot\left(\Omega_{2\left(|P_1|+r\right)}\Omega_{2\left|P_{j_1}\right|}\cdot\cdots\cdot\Omega_{2\left|P_{j_{c-1}}\right|}\right)$$

$$\text{where } \{i_1,\ldots,i_b\}\sqcup\{1,j_1,\ldots,j_{c-1}\}=[a],$$

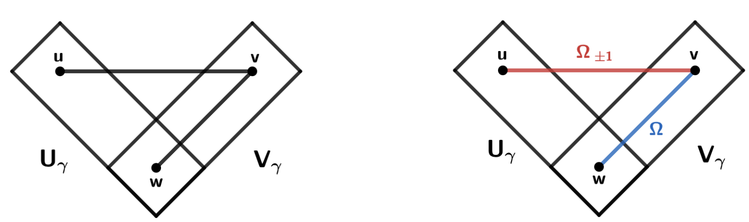
$$P_{i_1}\sqcup\cdots\sqcup P_{i_b}=\{2,\ldots,i\}, P_1\sqcup P_{j_1}\sqcup\cdots\sqcup P_{j_{c-1}}=\{1,i+1,\ldots,k\}$$

$$=\sum_{i=1}^{k}\left(\sum_{\substack{\mathcal{Q}=\{Q_1,\ldots,Q_b\}\\ \in\mathcal{NP}(i-1)}}\Omega_{2|Q_1|}\cdot\cdots\cdot\Omega_{2|Q_b|}\right)\cdot\left(\sum_{\substack{\mathcal{R}=\{R_1,\ldots,R_c\}\in\\ \mathcal{NP}(k-i):\,1\in R_1}}\Omega_{2\left(|R_1|+r+1\right)}\cdot\cdots\cdot\Omega_{2|R_c|}\right)$$

$$=\sum_{i=1}^{k}A(i-1,0)\cdot A(k-i,r+1)=\sum_{i,j\ge 0:\, i+j=k-1}A(i,0)\cdot A(j,r+1)$$

□

Definition 9.1.12. Let γ be the shape with vertices $V(\gamma)=\{u,w,v\}$ and edges $E(\gamma)=\{\{u,v\},\{v,w\}\}$ with distinguished tuples of vertices $U_{\alpha_Z}=(u,w)$ and $V_{\alpha_Z}=(v,w)$. Let $\Omega_{E(\gamma)}$ be the distributions associated with γ where $\Omega_{\{u,v\}}=\Omega_{\pm 1}$ and $\Omega_{\{v,w\}}=\Omega$.



(a) Shape γ.

(b) Associated distributions $\Omega_{E(\gamma)}$ for shape γ.

Figure 9.2: Shape γ and its associated distributions $\Omega_{E(\gamma)}$.

Remark 9.1.13. *When $m = 1$, $\Omega_{Z(1)} = \Omega_{\alpha_0}$, the line shape distribution. $\Omega_{\alpha_0} \circ_R \Omega$ will have the same limiting distribution of singular value moments as $M_{\gamma,\Omega_{E(\gamma)}}$, the graph matrix of the shape γ associated with distributions $\Omega_{E(\gamma)}$. This fact can also be proved by matching their recurrence relations of the moments.*

9.1.2 Base Case s=1

When $s = 1$, applying Proposition 9.1.2, we have that

$$\begin{aligned} A_m(k,0) = \left(\Omega_{Z(m)} \circ_R \Omega\right)_{2k} &= \sum_{\vec{\alpha} \in P_k} \binom{mk}{a-1} \cdot \frac{(a-1)!}{\alpha_1! \dots \alpha_k!} \cdot \vec{\Omega}^{\vec{\alpha}} \\ &= \sum_{\vec{\alpha} \in P_k} \sum_{\substack{\mathcal{P}=\{P_1,\dots,P_a\} \\ \in \mathcal{NP}(m\vec{\alpha})}} \Omega_{2|P_1|/m} \cdot \dots \cdot \Omega_{2|P_a|/m} \end{aligned} \tag{9.13}$$

and

$$A_m(k,r) = \sum_{\vec{\alpha} \in P_k} \sum_{\substack{\mathcal{P}=\{P_1,\dots,P_a\} \\ \in \mathcal{NP}(m\vec{\alpha}):\, 1 \in P_1}} \Omega_{2\left(|P_1|/m+r\right)} \Omega_{2|P_2|/m} \cdot \dots \cdot \Omega_{2|P_a|/m} \,. \tag{9.14}$$

We want to prove the following.

Theorem 9.1.14.

$$A_m(k,r) = \sum_{\substack{i_1,\dots,i_{m+1} \geq 0: \\ i_1+\dots+i_{m+1}=k-1}} A_m(i_1,0) \dots A_m(i_m,0) \cdot A_m(i_{m+1}, r+1) \tag{9.15}$$

and

$$A_m(0,r) = \Omega_{2r} \,. \tag{9.16}$$

Proof. The proof is similar to the proof for Theorem 9.1.8, with the modification that now the size of every polygon in a partition is augmented by a factor of m.

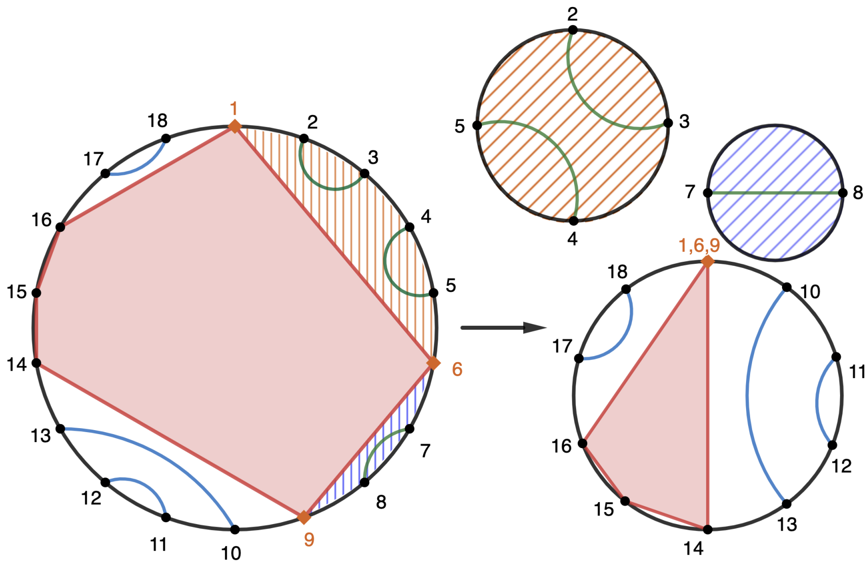


Figure 9.3: Illustration of Theorem 9.1.14: Here $m = 2, k = 9$, $i_1 = 2$, $i_2 = 1$. The orange and blue shaded parts on the right represent $A_2(i_1, 0)$ and $A_2(i_2, 0)$, respectively. The remaining part represents $A_2(k - i_1 - i_2, r + 1)$.

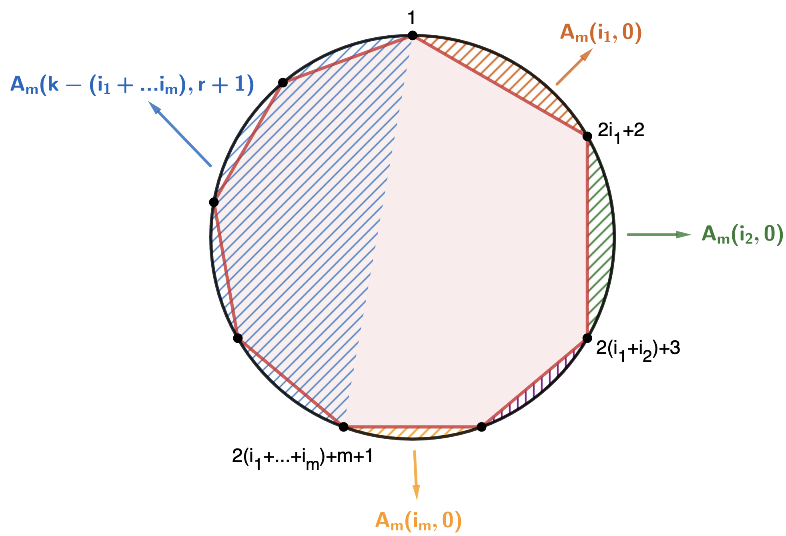


Figure 9.4: Illustration of Theorem 9.1.14:

Consider a $\mathcal{P} = \{P_1, \ldots, P_a\} \in \mathcal{NP}(m\vec{\alpha})$ where $\vec{\alpha} \in P_k$. Then $|P_i|$ is a multiple of m for all i. Since $\mathcal{P}$ is non-crossing, we can deduce that if $x < y$ are adjacent in the same partition set, then $y - x - 1$ must be a multiple of m.

We will obtain this recurrence relation with the following steps:

i. Let $\mathcal{P} \in \mathcal{NP}(m\vec{\alpha})$ for some $\vec{\alpha} \in P_k$. Since size of each partition set in $\mathcal{P}$ is a multiple of m, there are at least m elements in P_1. W.L.O.G., we can assume $1 \in P_1$. Denote $i_0 = 0$. Let $i_1, \ldots, i_m \geq 0$ be such that $V = \{m(i_0 + \cdots + i_s) + s + 1 : s = 0, 1, \ldots, m\}$ are the first $m+1$ elements in the clockwise direction that are in P_1. In the case when there are exactly m elements in P_1, $m(i_1 + \cdots + i_m) + m + 1 = mk + 1 \iff i_m = k - i_1 - \cdots - i_{m-1} - 1$.

ii. Since $\mathcal{P}$ is non-crossing, everything in between $m(i_1 + \cdots + i_{s-1}) + s$ and $m(i_1 + \cdots + i_s) + s + 1$ can be viewed as an augmented non-crossing partition on the cycle with elements $\{m(i_1 + \cdots + i_{s-1}) + s + 1, m(i_1 + \cdots + i_{s-1}) + s + 2, \ldots, m(i_1 + \cdots + i_s) + s\}$ (i.e. cycle of length mi_s). This gives rise to $A(i_s, 0)$.

iii. Now we identify all the vertices in V and shrink the size of the polygon P_1 by m. The remaining part from $m(i_1 + \cdots + i_m) + m + 1$ to mk is an augmented non-crossing partition on $m(k - i_1 - \cdots - i_m - 1)$ elements, and this gives rise to $A(k - i_1 - \cdots - i_m - 1, r + 1)$.

See Figure 9.1 for an illustration.

More precisely, given $i_1, \ldots, i_m \geq 0$, we denote $\vec{i} = (i_1, \ldots, i_m)$ and

- $\left(V_{\vec{i}}\right)_j = \left\{m(i_1 + \cdots + i_{j-1}) + j + 1, \ldots, m(i_0 + \cdots + i_j) + j\right\}$ for $j \in [m]$, and
- $\left(V_{\vec{i}}\right)_0 = \left\{m(i_1 + \cdots + i_j) + j + 1 : j \in [m]\right\} \bigcup \left\{1, m(i_0 + \cdots + i_m) + m + 2, \ldots, mk\right\}$. i.e. $\left(V_{\vec{i}}\right)_0 = [mk] \setminus \bigcup_{j=1}^{m} \left(V_{\vec{i}}\right)_j$.

Note that $\left|\left(V_{\vec{i}}\right)_j\right| = mi_j$ for $j \in [m]$, and $\left|\left(V_{\vec{i}}\right)_0\right| = m(k - i_1 - \cdots - i_m)$.

Then

$$A_m(k,r) = \sum_{\vec{\alpha}\in P_k} \sum_{\substack{\mathcal{P}=\{P_1,\dots,P_a\} \\ \in\mathcal{NP}(m\vec{\alpha}):\, 1\in P_1}} \Omega_{2\,(|P_1|/m+r)}\Omega_{2|P_2|/m}\cdot\dots\cdot\Omega_{2|P_a|/m}$$

$$= \sum_{i_1=0}^{k-1}\cdots\sum_{i_m=0}^{k-1} \sum_{\vec{\alpha}\in P_k} \sum_{\substack{\mathcal{P}=\{P_1,\dots,P_a\} \\ \in\mathcal{NP}(m\vec{\alpha}):1\in P_1}}$$

$$\left(\prod_{j=1}^{m} \Omega_{2\,|P_{i_{j,1}}|/m}\dots\Omega_{2\,|P_{i_{j,a_j}}|/m}\right)\cdot\Omega_{2\,(|P_1|/m+r)}\Omega_{2\,|P_{j_1}|/m}\dots\Omega_{2\,|P_{j_{c-1}}|/m}$$

$$\text{where } \left(\bigsqcup_{j=1}^{m}\left\{i_{j,1},\dots,i_{j,a_j}\right\}\right)\bigsqcup\left\{1,j_1,\dots,j_{c-1}\right\}=[a],\ \vec{i}=(i_1,\dots,i_m),\ \text{and}$$

$$P_{i_{j,1}}\sqcup\dots\sqcup P_{i_{j,a_j}}=\left(V_{\vec{i}}\right)_j \ \text{ for } j\in[m],\ P_1\sqcup P_{j_1}\sqcup\dots\sqcup P_{j_{c-1}}=\left(V_{\vec{i}}\right)_0$$

$$= \sum_{i_1=0}^{k-1}\cdots\sum_{i_m=0}^{k-1}\left(\prod_{j=1}^{m}\sum_{\substack{\mathcal{Q}_j=\{Q_{j,1},\dots,Q_{j,a_j}\} \\ \in\mathcal{NP}(mi_j)}}\Omega_{2\,|Q_{j,1}|/m}\dots\Omega_{2\,|Q_{j,a_j}|/m}\right)\cdot$$

$$\left(\sum_{\substack{\mathcal{R}=\{R_1,\dots,R_b\}\in \\ \mathcal{NP}\left(m(k-i_1-\dots-i_m-1)\right):\, 1\in R_1}}\Omega_{2\,(|R_1|/m+r+1)}\cdot\dots\cdot\Omega_{2|R_c|/m}\right)$$

$$= \sum_{i_1=0}^{k-1}\cdots\sum_{i_m=0}^{k-1} A(i_1,0)\cdot\dots\cdot A(i_m,0)\cdot A(k-i_1-\dots-i_m-1,r+1)$$

$$= \sum_{\substack{i_1,\dots,i_{m+1}\geq 0: \\ i_1+\dots+i_{m+1}=k-1}} A(i_1,0)\cdot\dots\cdot A(i_m,0)\cdot A(i_{m+1},r+1)$$

as needed.

□

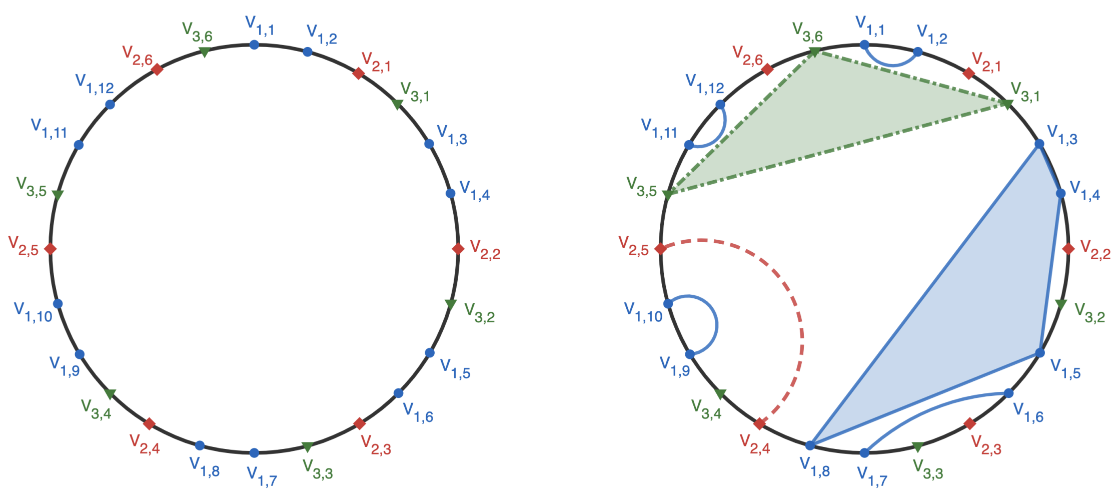
9.1.3 General Case

Recall the definition of $A^{(}_{m}s)(k, 0, \ldots, 0)$ and Proposition 9.1.2.

$$
\begin{aligned}
A_m^{(s)}(k,0,\ldots,0) &= \sum_{\vec{\alpha}_i \in P_k} C_m(\vec{\alpha}_1,\ldots,\vec{\alpha}_s)\cdot \overrightarrow{\Omega^{(1)}}^{\vec{\alpha}_1}\ldots\overrightarrow{\Omega^{(s)}}^{\vec{\alpha}_s} \\
&= \sum_{\vec{\alpha}_i \in P_k} \sum_{\substack{\mathcal{P}_1\sqcup\cdots\sqcup\mathcal{P}_s\in\\ \mathcal{NP}_m(\vec{\alpha}_1,\ldots,\vec{\alpha}_s)}} \left(\Omega^{(1)}_{2\,|P_{1,1}|/\hat{m}}\cdot\cdots\cdot\Omega^{(1)}_{2\,|P_{1,a_1}|/\hat{m}}\right)\cdot\prod_{j=2}^{s}\left(\Omega^{(j)}_{2\,|P_{j,1}|}\cdot\cdots\cdot\Omega^{(j)}_{2\,|P_{j,a_j}|}\right)
\end{aligned}
\tag{9.17}
$$

where $\hat{m} = m - s + 1$ and $\mathcal{P}_i = \left\{P_{i,1},\ldots,P_{i,a_i}\right\}$ for each $i \in [s]$.

Remark 9.1.15. *Let $\hat{m} = m - s + 1$. We will label the vertices in the cycle of length mk by $v_{1,1},\ldots,v_{1,\hat{m}}, v_{2,1},\ldots,v_{s,1},\ v_{1,\hat{m}+1},\ldots,v_{1,2\hat{m}}, v_{2,2},\ldots,v_{s,2},\ldots\ldots v_{1,(k-1)\hat{m}+1},\ldots,v_{1,k\hat{m}}, v_{2,k},\ldots,v_{s,k}$ in the clockwise direction, starting from 1. i.e. The vertices on the cycle are $\left\{v_{1,(t-1)\hat{m}+1},\ldots,v_{1,t\hat{m}},v_{2,t},\ldots,v_{s,t} : t\in[k]\right\}$. More precisely, the set of vertices $\{v_{j,1},\ldots,v_{j,k}\}$ represents the vertices used by the j^{th} type partition. See Figure 9.5 for an illustration.*



(a) Label. (b) An example partition.

Figure 9.5: Labelling of the cycle corresponding to $\mathcal{NP}_m(\vec{\alpha}_1,\ldots,\vec{\alpha}_s)$. Here $s = 3$, $m = 4$, $\hat{m} = m - s + 1 = 2$ and $k = 6$.

Now we will give the full definition of $A_m^{(s)}(k, r_1, \ldots, r_s)$.

Definition 9.1.16. Let $m \geq s$ and $\hat{m} = m - s + 1$. We define $A_m^{(s)}(k, r_1, \ldots, r_s)$ to be

$$\sum_{\vec{\alpha}_i \in P_k} \sum_{\substack{\mathcal{P}_1 \sqcup \cdots \sqcup \mathcal{P}_s \in \\ \mathcal{NP}_m(\vec{\alpha}_1, \ldots, \vec{\alpha}_s)}} \left(\Omega^{(1)}_{2\,(|P_{1,1}|/\hat{m}+r_1)} \Omega^{(1)}_{2\,|P_{1,2}|/\hat{m}} \cdot \cdots \cdot \Omega^{(1)}_{2\,|P_{1,a_1}|/m} \right) \cdot \prod_{j=2}^{s} \left(\Omega^{(j)}_{2\,(|P_{j,1}|+r_j)} \cdot \Omega^{(j)}_{2\,|P_{j,2}|} \cdot \cdots \cdot \Omega^{(j)}_{2\,|P_{j,a_j}|} \right) \tag{9.18}$$

where $\mathcal{P}_i = \left\{ P_{i,1}, \ldots, P_{i,a_i} \right\}$ for each $i \in [s]$, and $P_{i,1}$ are such that

1. $v_{s,1} \in P_{s,1}$.
2. Let v_{s,j_s} be the first vertex in the clockwise direction from $v_{s,1}$ that is in $P_{s,1}$. In the case when $v_{s,1}$ is isolated, let $v_{s,j_s} = v_{s,1}$.
3. Inductively, assume that we have picked $P_{s,1}, P_{s-1,1}, \ldots, P_{i,1}$ and $v_{s,j_s}, v_{s-1,j_{s-1}}, \ldots, v_{i,j_i}$, now we are going to pick $P_{i-1,1}$ and $v_{i-1,j_{i-1}}$. Let $P_{i-1,1}$ be the set containing v_{i-1,j_i} (which is the vertex immediately next to v_{i,j_i} in the counterclockwise direction). Let $v_{i-1,j_{i-1}}$ be the first vertex in the clockwise direction from v_{i-1,j_i}. In the case when v_{i-1,j_i} is isolated, let $v_{i-1,j_{i-1}} = v_{i-1,j_i}$.
4. Do the above process for $i = 3, \ldots, s$, resulting in $P_{2,1}, \ldots, P_{s,1}$ and $v_{2,j_2}, \ldots, v_{s,j_s}$. $v_{1,\hat{m}j_2}$ is the vertex immediately next to v_{2,j_2} in the counterclockwise direction. Now we let $P_{1,1}$ be the set containing $v_{1,\hat{m}j_2}$.

See Figure 9.6 for an illustration.

Remark 9.1.17. *When* $r_1 = \cdots = r_s = 0$, $A_m^{(s)}(k, r_1, \ldots, r_s)$ *coincides with* $A_m^{(s)}(k, 0, \ldots, 0)$.

We are now ready to prove the general case, Theorem 9.1.6.

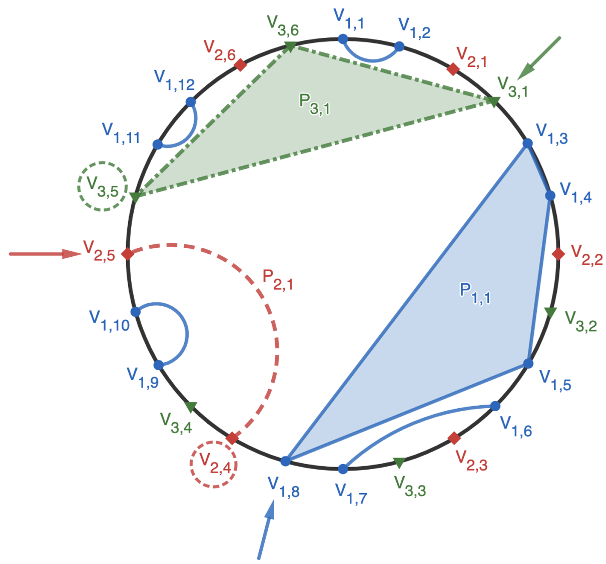


Figure 9.6: Illustration of Definition 9.1.3: Here $s = 3$, $m = 4$, $\hat{m} = m - s + 1 = 2$, and $k = 6$. We start with $v_{s,1} = v_{3,1}$. Then $v_{3,j_3} = v_{3,5}$, $v_{2,j_3} = v_{2,5}$, $v_{2,j_2} = v_{2,4}$ and $v_{1,\hat{m}j_2} = v_{1,8}$. $P_{1,1}, P_{2,1}, P_{3,1}$ are as labeled.

Theorem 9.1.6.

$$A_m^{(s)}(k, r_1, \ldots, r_s) = \sum_{\substack{i_1, \ldots, i_{m+1} \geq 0: \\ i_1 + \cdots + i_{m+1} = k-1}} A_m^{(s)}(i_1, r_1 + 1, 0, \ldots, 0) \ldots A_m^{(s)}(i_s, 0, \ldots, 0, r_s + 1) \cdot$$
$$A_m^{(s)}(i_{s+1}, 0, \ldots, 0) \ldots A_m^{(s)}(i_{m+1}, 0, \ldots, 0) \tag{9.7}$$

Proof. We obtain the recurrence relation through the following steps.

1. Let v_{i,j_i} and $P_{i,1}$ for $i \in [s]$ be as in Definition 9.1.16.

2. For each $i \in \{2, \ldots, s\}$, Identify $v_{i,j_{i+1}}$ with v_{i,j_i}. j_{s+1} is defined to be 1. Since for each i, v_{i,j_i} is the first vertex in the clockwise direction from $v_{i,j_{i+1}}$ in $P_{i,1}$, everything in between v_{i,j_i} and $v_{i,j_{i+1}}$ in the clockwise direction can be viewed as $A_m^{(s)}(k_i, 0, \ldots, r_i + 1, 0 \ldots, 0)$, where $k_i = j_{i+1} - j_i \mod k$.

3. Let $v_{1,t_1\hat{m}+1}, v_{1,t_2\hat{m}+2}, \ldots, v_{1,t_{\hat{m}}\hat{m}+\hat{m}}$ be the first $\hat{m}$ vertices in the clockwise direction that are in the same partition set as $v_{1,j_2\hat{m}}$, for some $1 \leq t_1 \leq \cdots \leq t_{\hat{m}}$. Note that it

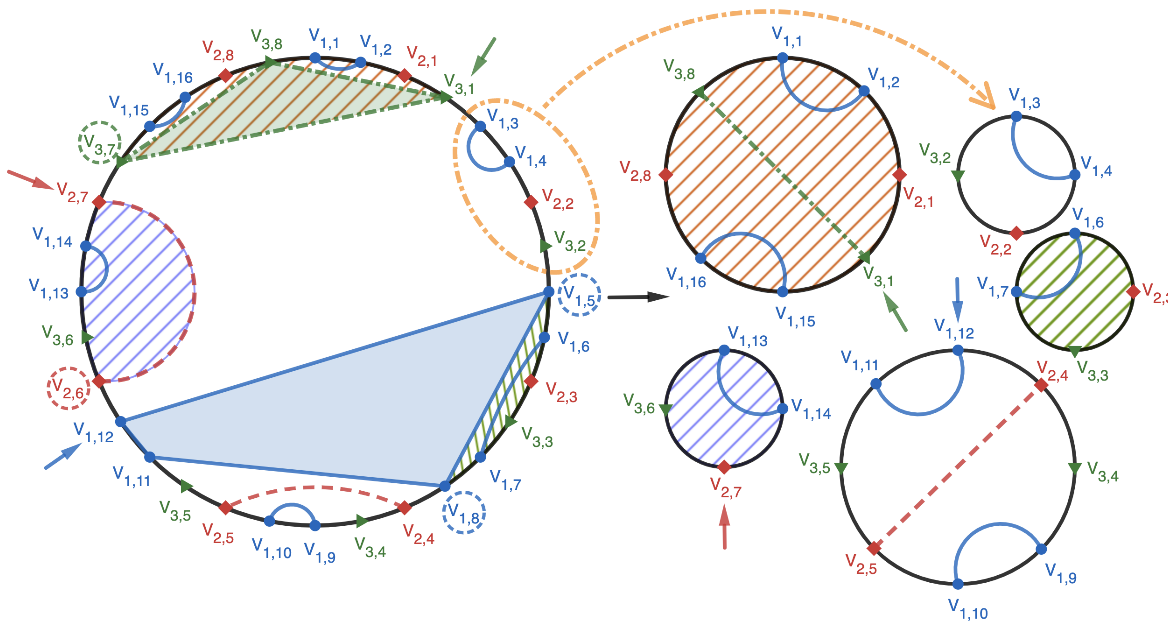


Figure 9.7: Illustration of the proof of Theorem 9.1.6: Here $s = 3$, $m = 4$, $\hat{m} = m - s + 1 = 2$ and $k = 8$. In the clockwise direction, $v_{3,7}$ is the first vertex in the same partition set as $v_{3,1}$, and $v_{2,6}$ is the first vertex in the same partition set as $v_{2,7}$. Thus $j_4 = 1, j_3 = 7$ and $j_2 = 6$. The resulting orange and blue shaded parts correspond to $A_4^{(3)}(k_3, 0, 0, r_3 + 1)$ and $A_4^{(3)}(k_2, 0, r_2 + 1, 0)$, respectively, where $k_3 = 2$, $k_2 = 1$. $v_{1,5}$ and $v_{1,8}$ are the first two vertices in the same partition set as $v_{1,12}$ in the clockwise direction, thus $t_1 = 2, t_2 = 3$. The resulting yellow-circled part and the green shaded part correspond to $A_4^{(3)}(q_1, 0, 0, 0)$ and $A_4^{(3)}(q_2, 0, 0, 0)$, respectively, where $q_1 = q_2 = 1$. The last cycle corresponds to $A_4^{(3)}(j_2 - t_2 - 1, r_1 + 1, 0, 0)$ where $j_2 - t_2 - 1 = 6 - 3 - 1 = 2$.

takes the form $v_{1,t_j\hat{m}+j}$ because the size of any partition set of the type v_1 is a multiple of $\hat{m}$, and we are only considering non-crossing partitions, thus the number of type v_1 vertices strictly in between $v_{1,t_j\hat{m}+j}$ and $v_{1,t_{j+1}\hat{m}+j+1}$ is a multiple of $\hat{m}$.

4. Let $t_0 = j_2$. For each $i \in [\hat{m}]$, identify $v_{1,t_{i-1}\hat{m}+i-1}$ and $v_{1,t_i\hat{m}+i}$. Everything in between $v_{1,t_{i-1}\hat{m}+i-1}$ and $v_{1,t_i\hat{m}+i}$ in the clockwise direction can be viewed as $A_m^{(s)}(q_i, 0, \ldots, 0)$, where $q_i = t_i - t_{i-1}$ for $i > 1$ and $q_1 = t_1 - 1$. The last remaining part can be viewed as $A_m^{(s)}(j_2 - t_{\hat{m}} - 1, r_1 + 1, 0, \ldots, 0)$.

Note that

$$\begin{aligned}
&(k_2 + \cdots + k_s) + (q_1 + \cdots + q_{\hat{m}}) + (j_2 - t_{\hat{m}} - 1) \\
=&\,((k + 1 - j_s) + (j_s - j_{s-1}) + \cdots + (j_3 - j_2)) \\
&\qquad + \Big((t_{\hat{m}} - t_{\hat{m}-1}) + (t_{\hat{m}-1} - t_{\hat{m}-2}) + \cdots + (t_2 - t_1) + (t_1 - 1)\Big) + (j_2 - t_{\hat{m}} - 1) \\
=&\,(k + 1 - j_2) + (t_{\hat{m}} - 1) + (j_2 - t_{\hat{m}} - 1) = k - 1.
\end{aligned}$$

Combining everything, we have

$$\begin{aligned}
&A_m^{(s)}(k, r_1, \ldots, r_s) \\
&= \sum_{0 \leq j_2 \leq \cdots \leq j_s} \sum_{0 \leq t_1 \leq \cdots \leq t_{\hat{m}}} \left(\prod_{i=2}^{s} A_m^{(s)}(k_i, 0, \ldots, r_i + 1, \ldots, 0) \right) \\
&\qquad \cdot \left(\prod_{i=1}^{\hat{m}} A_m^{(s)}(q_i, 0, \ldots, 0) \right) \cdot A_m^{(s)}(j_2 - t_{\hat{m}} - 1, r_1 + 1, 0, \ldots, 0) \\
&\qquad\qquad \text{where } k_i = j_{i+1} - j_i \mod k \text{ and } q_i = t_i - t_{i-1}, q_1 = t_1 - 1 \\
&= \sum_{\substack{i_1, \ldots, i_{m+1} \geq 0: \\ i_1 + \cdots + i_{m+1} = k-1}} \left(\prod_{x=1}^{s} A_m^{(s)}(i_x, 0, \ldots, r_x + 1, \ldots, 0) \right) \cdot \left(\prod_{y=s+1}^{s+\hat{m}} A_m^{(s)}\big(i_y, 0, \ldots, 0\big) \right) \\
&= \sum_{\substack{i_1, \ldots, i_{m+1} \geq 0: \\ i_1 + \cdots + i_{m+1} = k-1}} \left(\prod_{x=1}^{s} A_m^{(s)}(i_x, 0, \ldots, r_x + 1, \ldots, 0) \right) \cdot \left(\prod_{y=s+1}^{m+1} A_m^{(s)}\big(i_y, 0, \ldots, 0\big) \right)
\end{aligned}$$

as needed.

$\square$

9.2 Recurrence Relations for the Trace Power Moments of $M^{(G)}_{Z(m),s}$

Definition 9.2.1. Given distributions $\Omega^{(1)}, \ldots, \Omega^{(s)}$, we define $B^{(s)}_m(k, 0, \ldots, 0)$ to be

$$B^{(s)}_m(k, 0, \ldots, 0) = W\left(\mathcal{D}_{\alpha_{Z(m)}, \Omega_{\alpha_{Z(m)},s}, 2k}\right). \tag{9.19}$$

Definition 9.2.2. We define $B^{(s)}_m(k, r_1, \ldots, r_s)$ to be the weight of dominant constraint graphs of length k associated with $\alpha_{Z(m)}$ and $\Omega_{\alpha_{Z(m)},s}$ where the first spoke on the i^{th} layer has an multi-edge of multiplicity $(2r_i + 1)$ for each $i \in [s]$.

Remark 9.2.3. *When $r_1 = \cdots = r_s = 0$, $B^{(s)}_m(k, r_1, \ldots, r_s)$ coincides with $B^{(s)}_m(k, 0, \ldots, 0)$.*

We will prove the following main result.

Theorem 9.2.4.

$$\begin{aligned} B^{(s)}_m(k, r_1, \ldots, r_s) = \sum_{\substack{i_1,\ldots,i_{m+1}\geq 0:\\ i_1+\cdots+i_{m+1}=k-1}} & B^{(s)}_m(i_1, r_1+1, 0, \ldots, 0) \ldots B^{(s)}_m(i_s, 0, \ldots, 0, r_s+1) \cdot \\ & B^{(s)}_m(i_{s+1}, 0, \ldots, 0) \ldots B^{(s)}_m(i_{m+1}, 0, \ldots, 0) \end{aligned} \tag{9.20}$$

9.2.1 Base Case s=1, m=2

Definition 9.2.5. Given Ω, we will denote $\mathcal{B}_{k,r}$ to be the set of dominant constraint graphs of length k associated with α_Z and $\Omega_{\alpha_Z,1}$ where the first spoke on the first layer has an multi-edge of multiplicity $(2r + 1)$. Note that $\mathcal{B}_{k,0} = \mathcal{D}_{\alpha_Z, \Omega_{\alpha_Z,1}, 2k}$.

When $s = 1$, for simplicity denote $\Omega^{(1)}$ as Ω and $B^{(1)}_m(k, r)$ as $B_m(k, r)$. When $s = 1$ and $m = 2$,

$$B_2(k, r) = W\left(\mathcal{B}_{k,r}\right). \tag{9.21}$$

Theorem 9.2.6.

$$B_2(k,r) = \sum_{\substack{i_1,i_2,i_3\geq 0: \\ i_1+i_2+i_3=k-1}} B_2(i_1,0)\cdot B_2(i_2,0)\cdot B_2(i_3,r+1) \tag{9.22}$$

and

$$B_2(0,r) = \Omega_{2r}\,. \tag{9.23}$$

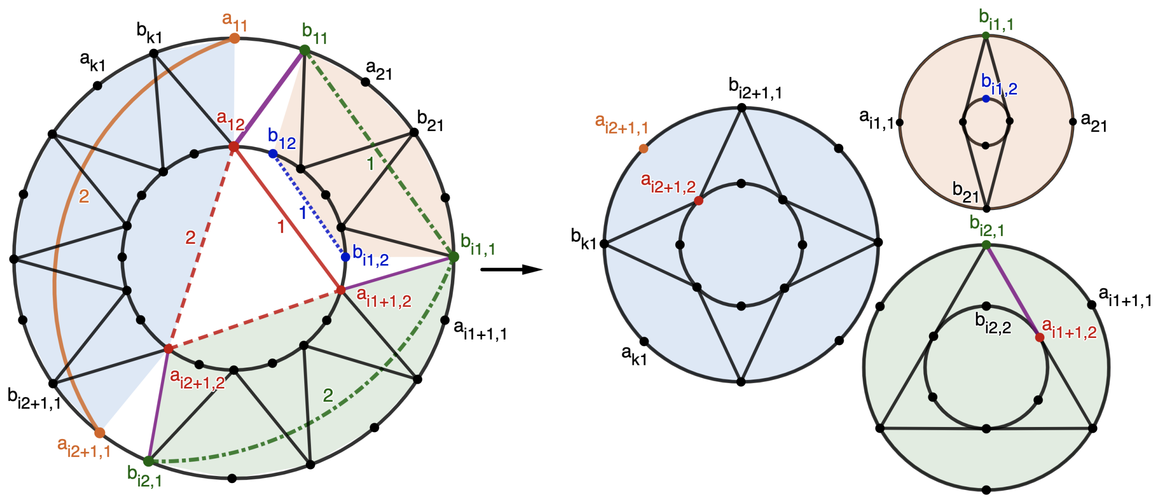


Figure 9.8: Illustration of the proof of Theorem 9.2.6: The spoke $\{a_{12}, b_{11}\}$, colored purple, has multiplicity $2r+1$. For constraint edges, solid constraint edges imply dashed ones with the same numbering. For example, for the constraint edges numbered 1, $a_{1,2} \longleftrightarrow a_{i_1+1,2} \implies b_{1,2} \longleftrightarrow b_{i_1,2} \implies b_{1,1} \longleftrightarrow b_{i_1,1}$.

Proof. We will first prove that there is a bijection between $\mathcal{B}_{k,r}$ and $\bigcup_{\substack{k_i\geq 0: \\ k_1+k_2+k_3=k-1}} \mathcal{B}_{k_1,0} \times \mathcal{B}_{k_2,0} \times \mathcal{B}_{k_3,r+1}$.

1. $\mathcal{B}_{k,r} \to \bigcup_{\substack{k_i\geq 0: \\ k_1+k_2+k_3=k-1}} \mathcal{B}_{k_1,0}\times\mathcal{B}_{k_2,0}\times\mathcal{B}_{k_3,r+1}$: Let $C \in \mathcal{B}_{k,r}$, with the spoke $\{a_{1,2}, b_{1,1}\}$ having multiplicity $2r+1$. We split C into three parts with the following steps.

   i. Let $i_1 \in [k]$ be the first index such that $a_{1,2} \longleftrightarrow a_{i_1+1,2}$. In the case of $a_{1,2}$ being isolated, i_1 is chosen to be k. This implies that $b_{1,2} \longleftrightarrow b_{i_1,2}$, which further implies that $b_{1,1} \longleftrightarrow b_{i_1,1}$.

ii. Let $i_2 \in [k]$ be the first index such that $a_{1,1} \longleftrightarrow a_{i_2+1,1}$. In the case of $a_{1,1}$ being isolated, i_2 is chosen to be k. This implies that $b_{1,1} \longleftrightarrow b_{i_2,1}$ and $a_{1,2} \longleftrightarrow a_{i_2+1,2}$.

iii. Observe that $i_2 \geq i_1$: otherwise if $i_2 < i_1$, $a_{1,1} \longleftrightarrow a_{i_2+1,1}$ and $b_{1,1} \longleftrightarrow b_{i_1,1}$ would imply $a_{1,1} \longleftrightarrow a_{i_2+1,1} \longleftrightarrow b_{1,1} \longleftrightarrow b_{i_1,1}$ since C is dominant and they cross. But $a_{1,1} \longleftrightarrow b_{1,1}$ contradicts that C is parity preserving.

iv. Since $a_{1,2} \longleftrightarrow a_{i_1+1,2}$ and $a_{1,2} \longleftrightarrow a_{i_2+1,2}$, $a_{i_1+1,2} \longleftrightarrow a_{i_2+1,2}$. Since $b_{1,1} \longleftrightarrow b_{i_1,1}$ and $b_{1,1} \longleftrightarrow b_{i_2,1}$, $b_{i_1,1} \longleftrightarrow b_{i_2,1}$.

v. Contracting $b_{1,1}$ with $b_{i_1,1}$ and $b_{1,2}$ with $b_{i_1,2}$ give $H(\alpha_Z, 2(i_1-1))$. The induced constraint graph $C_1 \in \mathcal{C}_{(\alpha_Z,2(i_1-1))}$ is dominant, thus $C_1 \in \mathcal{B}_{i_1-1,0}$.

vi. Contracting $a_{i_1+1,2}$ with $a_{i_2+1,2}$ and $b_{i_1,1}$ with $b_{i_2,1}$ give $H(\alpha_Z, 2(i_2-i_1))$. Moreover, the spokes $\{a_{1,2}, b_{1,1}\}$, $\{a_{i_1+1,2}, b_{i_1,1}\}$ and $\{a_{i_2+1,2}, b_{i_2,1}\}$ are identified together, resulting in multiplicity $(2r+1)+2 = 2(r+1)+1$. thus the induced constraint graph C_2 is in $\mathcal{B}_{k,r+1}$.

vii. Contracting $a_{1,1}$ with $a_{i_2+1,1}$ and $a_{1,2}$ with $a_{i_2+1,2}$ give $H(\alpha_Z, 2(k-i_2))$. The induced constraint graph $C_3 \in \mathcal{C}_{(\alpha_Z,2(k-i_2))}$ is dominant, thus $C_3 \in \mathcal{B}_{k-i_2,0}$.

Since $(i_1-1)+(i_2-i_1)+(k-i_2) = k-1$, we conclude that $(C_1, C_2, C_3) \in \bigcup_{\substack{k_i \geq 0: \\ k_1+k_2+k_3=k-1}} \mathcal{B}_{k_1,0} \times \mathcal{B}_{k_2,0} \times \mathcal{B}_{k_3,r+1}$.

2. $\bigcup_{\substack{k_i \geq 0: \\ k_1+k_2+k_3=k-1}} \mathcal{B}_{k_1,0} \times \mathcal{B}_{k_2,0} \times \mathcal{B}_{k_3,r+1} \to \mathcal{B}_{k,r}$: Conversely, given $C_1 \in \mathcal{B}_{k_1,0}$, $C_2 \in \mathcal{B}_{k_2,0}$ and $C_3 \in \mathcal{B}_{k_3,r+1}$ for some $k_1+k_2+k_3$, we can reverse the above steps and glue them together to get $C \in \mathcal{B}_{k,r}$.

Thus,

$$\begin{aligned}
B(k,r) = W\left(\mathcal{B}_{k,r}\right) &= W\left(\bigcup_{k_i\geq 0\colon k_1+k_2+k_3=k-1} \mathcal{B}_{k_1,0}\times\mathcal{B}_{k_2,0}\times\mathcal{B}_{k_3,r+1}\right)\\
&= \sum_{k_i\geq 0\colon k_1+k_2+k_3=k-1} W\left(\mathcal{B}_{k_1,0}\times\mathcal{B}_{k_2,0}\times\mathcal{B}_{k_3,r+1}\right)\\
&= \sum_{k_i\geq 0\colon k_1+k_2+k_3=k-1} W\left(\mathcal{B}_{k_1,0}\right)\cdot W\left(\mathcal{B}_{k_2,0}\right)\cdot W\left(\mathcal{B}_{k_3,r+1}\right)\\
&= \sum_{k_i\geq 0\colon k_1+k_2+k_3=k-1} B(k_1,0)B(k_2,0)B(k_3,r+1).
\end{aligned}$$

as needed. □

9.2.2 General Case

Definition 9.2.7. Given a distribution $\Omega^{(1)},\dots,\Omega^{(s)}$ and $t\leq s$, we will denote $\mathcal{B}_{m,k,r_1,\dots,r_t}$ to be the set of dominant constraint graphs of length k associated with $\alpha_{Z(m)}$ and $\Omega_{\alpha_{Z(m)},s}$ where the first spoke on the j^{th} layer has an multi-edge of multiplicity $(2r_j+1)$ for each $j\in[t]$. Note that $\mathcal{B}_{m,k,r_1,\dots,r_s}=\mathcal{D}_{\alpha_{Z(m)},\Omega_{\alpha_{Z(m)},s},2k}$ and $\mathcal{B}_{m,k}=\mathcal{D}_{\alpha_{Z(m)},2k}$.

Definition 9.2.8. For $t\in[s]$, we will use $\mathcal{B}_{m,k,r\vec{e}_t}$ to denote $\mathcal{B}_{m,k,r_1,\dots,r_s}$ where $r_t=t$ and $r_j=0$ for all $j\neq t$.

Observation 9.2.9. $B_m^{(s)}(k,r_1,\dots,r_s)=W\left(\mathcal{B}_{m,k,r_1,\dots,r_s}\right)$, $B_m^{(s)}\left(k,0,\dots,r_j,\dots,0\right)$ $=W\left(\mathcal{B}_{m,k,r_j\vec{e}_j}\right)$ and $B_m^{(s)}(k,0,\dots,0)=W\left(\mathcal{B}_{m,k}\right)$.

Now we are ready to prove Theorem 9.2.4.

Theorem 9.2.4.

$$B_m^{(s)}\left(k,r_1,\dots,r_s\right)=\sum_{\substack{i_1,\dots,i_{m+1}\geq 0:\\ i_1+\dots+i_{m+1}=k-1}} B_m^{(s)}\left(i_1,r_1+1,0,\dots,0\right)\dots B_m^{(s)}\left(i_s,0,\dots,0,r_s+1\right)\cdot B_m^{(s)}\left(i_{s+1},0,\dots,0\right)\dots B_m^{(s)}\left(i_{m+1},0,\dots,0\right) \tag{9.20}$$

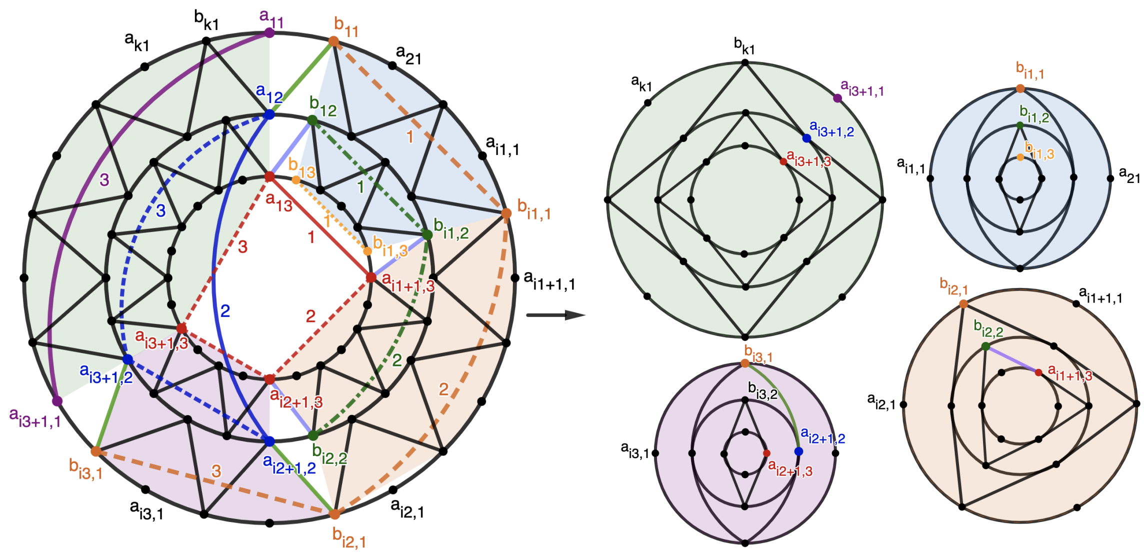


Figure 9.9: Illustration of Theorem 9.2.4. For constraint edges, solid constraint edges imply dashed ones with the same numbering.

Proof. Similar to the proof of Theorem 9.2.6, we now will prove that there is a bijection between $\mathcal{B}_{m,k,r_1,\ldots,r_s}$ and $\bigcup_{\substack{k_i \geq 0: \\ k_1+\cdots+k_{m+1}=k-1}} \mathcal{B}_{m,k_1,(r_1+1)\vec{e}_1} \times \cdots \times \mathcal{B}_{m,k_s,(r_s+1)\vec{e}_s} \times \mathcal{B}_{m,k_{s+1}} \times \ldots \mathcal{B}_{m,k_{m+1}}$.

- $\mathcal{B}_{m,k,r_1,\ldots,r_s} \to \bigcup_{\substack{k_i \geq 0: \\ k_1+\cdots+k_{m+1}=k-1}} \mathcal{B}_{m,k_1,(r_1+1)\vec{e}_1} \times \cdots \times \mathcal{B}_{m,k_s,(r_s+1)\vec{e}_s} \times \mathcal{B}_{m,k_{s+1}} \times \ldots \mathcal{B}_{m,k_{m+1}}$:

  Let $C \in \mathcal{B}_{m,k,r_1,\ldots,r_s}$, with the spokes $\{a_{1,j+1}, b_{1,j}\}$ having multiplicity $2r_j + 1$ for $j \in [s]$. We split C into $m + 1$ parts with the following steps.

  1. Let $i_1 \in [k]$ be the first index such that $a_{1,m} \longleftrightarrow a_{i_1+1,m}$. In the case of $a_{1,m}$ being isolated, i_1 is chosen to be k. This implies that $b_{1,m} \longleftrightarrow b_{i_1,m}$, which further implies that $b_{1,j} \longleftrightarrow b_{i_1,j}$ for all $j \in [m]$.

  2. Let $i_2 \in [k]$ be the first index such that $a_{1,m-1} \longleftrightarrow a_{i_2+1,m-1}$. In the case of $a_{1,m-1}$ being isolated, i_2 is chosen to be k. This implies that

i. $b_{1,m-1} \longleftrightarrow b_{i_2,m-1}$, which further implies that $b_{1,j} \longleftrightarrow b_{i_2+1,j}$ for all $j \in [m-1]$.

ii. $a_{1,m} \longleftrightarrow a_{i_2+1,m}$.

3. Continuing this way, at the t^{th} step where $t \in [m]$, let $i_t \in [k]$ be the first index such that $\alpha_{1,m-t+1} \longleftrightarrow a_{i_t+1,m-t+1}$. In the case of $a_{1,m-t+1}$ being isolated, i_t is chosen to be k. This implies that

   i. $b_{1,m-t+1} \longleftrightarrow b_{i_t,m-t+1}$, which further implies that $b_{1,j} \longleftrightarrow b_{i_t+1,j}$ for all $j \in [m-t+1]$.

   ii. $a_{1,j} \longleftrightarrow a_{i_t+1,j}$ for all $j \in \{m-t+1, \ldots, m\}$.

4. Observe that $i_{j+1} \geq i_j$ for all $j \in [m-1]$: otherwise if $i_{j+1} < i_j$ for some j, $a_{1,m-j} \longleftrightarrow a_{i_{j+1}+1,m-j}$ and $b_{1,m-j} \longleftrightarrow b_{i_j,m-j}$ would imply $a_{1,m-j} \longleftrightarrow a_{i_{j+1}+1,m-j} \longleftrightarrow b_{1,m-j} \longleftrightarrow b_{i_j,m-j}$ since C is dominant and they cross. But $a_{1,m-j} \longleftrightarrow b_{1,m-j}$ contradicts that C is parity preserving.

5. Let $t \in [m]$. Since $a_{1,j} \longleftrightarrow a_{i_t+1,j}$ for all $j \in \{m-t+1, \ldots, m\}$, for each $j \in [m]$, $a_{1,j} \longleftrightarrow a_{i_{m-j+1}+1,j} \longleftrightarrow \ldots \longleftrightarrow a_{i_m+1,j}$. Since $b_{1,j} \longleftrightarrow b_{i_t,j}$ for all $j \in [m-t+1]$, for each $j \in [m]$, $b_{1,j} \longleftrightarrow b_{i_1,j} \longleftrightarrow \ldots \longleftrightarrow b_{i_{m-j+1},j}$.

6. Contracting $b_{1,j}$ with $b_{i_1,j}$ for all $j \in [m]$ gives $H\left(\alpha_{Z(m)}, 2(i_1-1)\right)$. The induced constraint graph $C_0 \in \mathcal{C}_{(\alpha_Z,2(i_1-1))}$ is dominant, thus $C_0 \in \mathcal{B}_{m,i_1-1}$.

7. Let $t \in [m-1]$. Contracting $b_{i_t,j}$ with $b_{i_{t+1},j}$ for all $j \in [m-t]$ and contracting $a_{i_t+1,j}$ with $a_{i_{t+1}+1,j}$ for all $j \in \{m-t+1, \ldots, m\}$ give $H(\alpha_Z, 2(i_{t+1}-i_t))$. Moreover, the spokes $\{a_{1,m-t+1}, b_{1,m-t}\}$, $\{a_{i_t+1,m-t+1}, b_{i_t,m-t}\}$ and $\{a_{i_{t+1}+1,m-t+1}, b_{i_{t+1},m-t}\}$ are identified together, resulting in multiplicity $(2r_{m-t+1}+1)+2 = 2(r_{m-t+1}+1)+1$ if $m-s \leq t \leq m-1$. Thus the induced constraint graph C_t is in $\mathcal{B}_{m,i_{t+1}-i_t}$ if $t \in [m-s-1]$ and in $\mathcal{B}_{m,i_{t+1}-i_t,(r_{m-t}+1)\vec{e}_{m-t}}$ if $m-s \leq t \leq m-1$.

8. Finally, contracting $a_{1,j}$ with $a_{i_m+1,j}$ for all $j \in [m]$ gives $H(\alpha_Z, 2(k-i_m))$. The induced constraint graph $C_m \in \mathcal{C}_{(\alpha_Z,2(k-i_m))}$ is dominant, thus $C_m \in \mathcal{B}_{m,k-i_m}$.

Since $(i_1 - 1) + \sum_{j=1}^{m-1} \left(i_{j+1} - i_j\right) + (k - i_m) = k - 1$, and there are s many C_j's (i.e. $C_{m-s}, \ldots, C_{m-1}$) belonging to $\mathcal{B}_{m,k_1,(r_1+1)\vec{e}_1}, \ldots, \mathcal{B}_{m,k_s,(r_s+1)\vec{e}_s}$ for $k_j = i_{m-j+1} - i_{m-j}$, we conclude that

$$(C_0, C_1, \ldots, C_m) \in \bigcup_{\substack{k_i \geq 0: \\ \sum k_i = k-1}} \mathcal{B}_{m,k_1,(r_1+1)\vec{e}_1} \times \cdots \times \mathcal{B}_{m,k_s,(r_s+1)\vec{e}_s} \times \mathcal{B}_{m,k_{s+1}} \times \cdots \times \mathcal{B}_{m,k_{m+1}}.$$

- $\bigcup_{\substack{k_i \geq 0: \\ \sum k_i = k-1}} \mathcal{B}_{m,k_1,(r_1+1)\vec{e}_1} \times \cdots \times \mathcal{B}_{m,k_s,(r_s+1)\vec{e}_s} \times \mathcal{B}_{m,k_{s+1}} \times \ldots \mathcal{B}_{m,k_{m+1}} \to \mathcal{B}_{m,k,r_1,\ldots,r_s}$:

  Conversely, given $C_i \in \mathcal{B}_{m,k_i,(r_i+1)\vec{e}_j}$ for $i \in [s]$ and $C_j \in \mathcal{B}_{m,k_j}$ for some $j \in \{s+1, \ldots, m+1\}$, we can reverse the above steps and glue them together to get $C \in \mathcal{B}_{m,k,r_1,\ldots,r_s}$.

Thus

$$\begin{aligned}
&B_m(k, r_1, \ldots, r_s) = W\left(\mathcal{B}_{m,k,r_1,\ldots,r_s}\right) \\
=&W\left(\bigcup_{\substack{k_i \geq 0: \\ k_1+\cdots+k_{m+1}=k-1}} \mathcal{B}_{m,k_1,(r_1+1)\vec{e}_1} \times \cdots \times \mathcal{B}_{m,k_s,(r_s+1)\vec{e}_s} \times \mathcal{B}_{m,k_{s+1}} \times \ldots \mathcal{B}_{m,k_{m+1}}\right) \\
=&\sum_{\substack{k_i \geq 0: \\ k_1+\cdots+k_{m+1}=k-1}} W\left(\mathcal{B}_{m,k_1,(r_1+1)\vec{e}_1} \times \cdots \times \mathcal{B}_{m,k_s,(r_s+1)\vec{e}_s} \times \mathcal{B}_{m,k_{s+1}} \times \ldots \mathcal{B}_{m,k_{m+1}}\right) \\
=&\sum_{\substack{k_i \geq 0: \\ k_1+\cdots+k_{m+1}=k-1}} \prod_{i=1}^{s} W\left(\mathcal{B}_{m,k_i,(r_i+1)\vec{e}_i}\right) \cdot \prod_{j=s+1}^{m+1} W\left(\mathcal{B}_{m,k_j}\right) \\
=&\sum_{\substack{k_i \geq 0: \\ k_1+\cdots+k_{m+1}=k-1}} \left(\prod_{i=1}^{s} B_m(m, k_i, 0, \ldots, r_i+1, \ldots, 0)\right) \cdot \left(\prod_{j=s+1}^{m+1} B_m(m, k_j, 0, \ldots, 0)\right)
\end{aligned}$$

as needed.

□

Proof of Theorem 9.0.1. This follows directly from Theorem 9.1.6 and Theorem 9.2.4. □

APPENDIX A

DOMINANT CONSTRAINT GRAPHS ON $H(\alpha_{Z_m}, 2q)$ ARE WELL-BEHAVED

In this section, we prove that dominant constraint graphs on $H(\alpha_{Z_m}, 2q)$ are well-behaved.

A.1 The Set of Graphs $R(H(\alpha_{Z_m}, 2q))$

In order to analyze constraint graphs on $H(\alpha_{Z_m}, 2q)$, we need to analyze a more general class of H. In particular, we need to analyze all H which can be obtained by taking isolated vertices which are not incident to any spokes in $H(\alpha_{Z_m}, 2q)$ and merging their neighbors together.

Definition A.1.1. Define $R(H(\alpha_{Z_m}, 2q))$ to be the set of graphs H which can be obtained by starting from $H(\alpha_{Z_m}, 2q)$ and repeatedly applying the following operation:

1. Choose a vertex $v \in V(H)$ which is in a wheel with at least 4 vertices and is not incident to any spokes. Merge the two neighbors of v, delete v from the graph, and delete any pairs of edges in H which coincide.

Lemma A.1.2. *For any $H \in R(H(\alpha_{Z_m}, 2q))$, we can decompose H as $H = \alpha_1 \circ \ldots \circ \alpha_{2q'}$ where the following statements are true:*

1. *For all odd i, α_i consists of trivial top layers, a multi-Z shape in the middle layers, and trivial bottom layers.*

2. *For all even i, α_i consists of trivial top layers, a multi-Z^T shape in the middle layers, and trivial bottom layers.*

3. *For any two neighboring wheels, the spokes connect with each other and alternate between going up and to the right and down and to the right.*

4. *For any layer, the intervals where this layer is trivial have even length.*

Proof. To prove this, we show that this structure is preserved when we make a new contraction. Without loss of generality, we can assume that the isolated vertex v is the bottom right vertex in some multi-Z shape α_i where i is odd, as cases when the isolated vertex is the top left vertex in a multi-Z shape, the bottom left vertex in a multi-Z^T shape, or the top right vertex in a multi-Z^T shape can be handled in a similar way. Let j be the next index where this wheel is non-trivial. Note that j must be even. Moreover, all layers below v must be trivial in α_i and α_j as otherwise v would not be isolated.

Let u be the vertex preceding v and let w be the vertex after v. We merge u and w together and delete v. This deletes the edges $(u,v),(v,w)$ and may delete spokes incident to u and w. We have the following cases.

1. There is a spoke (u,t) in α_i and a spoke (t,w) in α_j. In this case, merging u and w together also deletes the spokes $(u,t),(t,w)$. We account for this by making v's layer trivial in α_i and α_j, replacing it with the single vertex $u = w$.

2. There is a spoke (u,t) in α_i but no spoke incident to w in α_j. In this case, the edge (v,w) is the only non-trivial part of α_j. Let k be the next index such that this layer is non-trivial in α_k. Observe that α_k is a multi-Z shape where all layers above this layer are trivial. We account for merging u and w together and deleting v as follows:

   (a) Glue α_i and α_k together at the vertex $u = w$.

   (b) Create a copy of $\alpha_i \circ \ldots \circ \alpha_{k-1}$ which only contains the part below the current layer and is trivial at this layer and above. Put these copies to the left of the glued shape.

   (c) Create a copy of $\alpha_{i+1} \circ \ldots \circ \alpha_k$ which only contains the part above the current layer and is trivial at this layer and above. Put these copies to the right of the glued shape.

3. There is no spoke incident to u in α_i but there is a spoke (t, w) in α_j. This can be handled in a similar way to the previous case.

4. There are no spokes incident to u or w. In this case, the edges (u, v) and (v, w) are the only non-trivial parts of α_i and α_j. We again account for this by making v's layer trivial in α_i and α_j, replacing it with the single vertex $u = w$.

□

A.2 Proof that dominant constraint graphs are well-behaved

With this structural result on $R(H(\alpha_{Z_m}, 2q))$, we can now prove that dominant constraint graphs for $H(\alpha_{Z_m}, 2q)$ are well behaved. To do this, we use ideas from Appendix B of Ahn et al. [2020]. First, we modify our constraint graph as follows:

Definition A.2.1. Let $H = \alpha_1 \circ \ldots \circ \alpha_{2q}$ where we set $V_{\alpha_{2q}} = U_{\alpha_1}$. Given a constraint graph C on H, we define the constraint graph C_{aug} as follows:

1. Draw the constraint edges so that all paths in C go from left to right.

2. For each vertex $u \in U_{\alpha_1}$, letting v be the rightmost vertex such that there is a path of constraint edges from u to v, we add an auxiliary constraint edge from v to u. We treat this edge as going from v on the left to u on the right (i.e. we think of u as both on the left side of H and on the right side of H as $u \in U_{\alpha_1} = V_{\alpha_{2q}}$). If u is isolated, this means that we add an auxiliary loop from u to itself.

Definition A.2.2. Given H and C_{aug} as described above, for each α_i we define S_{α_i} to be union of $U_{\alpha_i} \cap V_{\alpha_i}$ and the set of vertices v such that there exists a path P in α_i from U_{α_i} to V_{α_i} such that v is the first vertex on P where either

1. There is a constraint edge from v to the right (i.e. to a vertex in some α_j where $j > i$).

2. $v \in V_{\alpha_i}$.

Proposition A.2.3. *S_{α_i} is a vertex separator of α_i.*

Remark A.2.4. *Alternatively, we could have started from V_{α_i} and taken the first vertex on each path in α_i from V_{α_i} to U_{α_i} which has a constraint edge to the left or is in U_{α_i}.*

Lemma A.2.5. *Let P be a path in α_i from U_{α_i} to V_{α_i} and let v be the first vertex on P where either*

1. *There is a constraint edge from v to the right (i.e. to a vertex in some α_j where $j > i$).*

2. *$v \in V_{\alpha_i}$.*

For any vertex $u \in V(P) \setminus U_{\alpha_i}$ which is equal to v or comes before v, u has an edge to the left.

Proof. Let l be the vertex which comes before u in P. Observe that l does not have any constraint edges to the right. Thus, in order for the edge (l, u) to be duplicated, u must have a constraint edge to the left. $\square$

Corollary A.2.6. *C is a dominant constraint graph for H if and only if the following statements are true for each α_i and S_{α_i}:*

1. *S_{α_i} is a minimum vertex separator of α_i.*

2. *Each vertex in $V(\alpha_i) \setminus (U_{\alpha_i} \cup V_{\alpha_i} \cup S_{\alpha_i})$ is incident with exactly one constraint edge.*

3. *Each vertex in $U_{\alpha_i} \setminus S_{\alpha_i}$ is not incident with any constraint edges to the right.*

4. *Each vertex in $V_{\alpha_i} \setminus S_{\alpha_i}$ is not incident with any constraint edges to the left.*

Theorem A.2.7. *For any $H \in R(H(\alpha_{Z_m}, 2q))$, all dominant constraint graphs C on H are well-behaved (i.e. wheel-respecting and parity preserving).*

Proof sketch. This theorem can be proved by induction using the following lemma.

Lemma A.2.8. *For any $H \in R(H(\alpha_{Z_m}, 2q))$ and any dominant constraint graph C on H, there is a vertex v which is isolated and is not incident to any spokes in H.*

Proof. We prove this lemma with a series of observations.

Proposition A.2.9. *In any dominant constraint graph C, if u precedes v on some wheel then either u does not have a constraint edge to the right or v does not have a constraint edge to the left.*

Proof. Assume that C is dominant, u has an edge to the right, and v has an edge to the left. Consider the separator S for the segment containing u and v. Since u has a constraint edge to the right, $u \in S$. Now either $v \in S$ or $v \notin S$. If $v \in S$ then S is not a minimum vertex separator so C is not dominant. If $v \notin S$ then the constraint edge to the left from v is not accounted for by S so C is not dominant. Thus, in either case C is not dominant, which is a contradiction. □

Corollary A.2.10. *In any dominant constraint graph, if l and r are two vertices on the same wheel such that $l < r$, l has a constraint edge to the right, and r has a constraint edge to the left then there is a vertex m such that $l < m < r$ and m is isolated.*

Proof. Assume that there exist two vertices l and r on the same wheel such that $l < r$, l has a constraint edge to the right, r has a constraint edge to the left, and there is no vertex m such that $l < m < r$ and m is isolated. Choose l and r such that $d(l, r)$ is minimized. Let v be the vertex after l on this wheel.

By Proposition A.2.9, since l precedes v and l has a constraint edge to the right, v does not have a constraint edge to the left. This implies that either v is isolated or v only has a constraint edge to the right. However, we cannot have that v is isolated as otherwise we could take $m = v$ and we would have that $l < m < r$ and m is isolated. Thus, v must only have a constraint edge to the right. But then if we take $l' = v$, l' and r are on the same wheel, l' has a constraint edge to the right, r has a constraint edge to the left, there is no vertex m such that $l < m < r$ and m is isolated, and $d(l', r) < d(l, r)$. This contradicts the fact that we chose l and r to minimize $d(l, r)$. □

With these observations, we can now prove Lemma A.2.8. Consider the highest wheel such that there exist vertices $l < r$ on this wheel satisfying the following properties:

1. l has a constraint edge to the right and r has a constraint edge to the left.

2. For any vertex m between l and r, m is not incident to any spokes in H going from m to the wheel below m.

Observe that the bottom wheel has these properties, so this wheel always exists. By Corollary A.2.10, there is a vertex m such that $l < m < r$ and m is isolated. There are now two cases to consider. Either m is not incident to any spokes in H going from m to the wheel above m, or m is incident to two such spokes. Note that m is not incident to any spokes in H going from m to the wheel below m, so if m is not incident to any spokes in H going from m to the wheel above m, then we have found an isolated vertex which is not incident to any spokes in H. If m is incident to two spokes in H from m to the wheel above m, let $l' < r'$ be the other endpoints of these spokes. Since m is isolated, because of the structure of H, l' must have an edge to the right, r' must have an edge to the left, and there are no vertices m' such that $l' < m' < r'$ and m' is incident to spokes in H going from m' to the wheel below m'. However, this is a contradiction, as this implies that we did not start with the highest wheel such that there exist vertices $l < r$ on this wheel, where l has an edge to the right, r has an edge to this left, and for any vertex m between l and r, m is not incident to any spokes in H going from m to the wheel below m. □

□

APPENDIX B
ANALYZING INNER PRODUCTS WITH RANDOM VECTORS

Definition B.0.1. A *perfect matching $\mathcal{M}$ of* $[n]$ is

$$\mathcal{M} = \left\{ \{a_i, b_i\} : i \in [n/2], a_i, b_i \in [n], \bigsqcup_{i=1}^{n/2} \{a_i\} \sqcup \{b_i\} = [n] \right\}$$

.

We use the following calculation from Jones and Potechin [2022] which is a spherical analogue of Isserlis' Theorem/Wick's Theorem.

Theorem B.0.2. *For any vectors* $\vec{d_1}, \ldots, \vec{d_k} \in \mathbb{R}^n$,

$$\mathbb{E}_{\vec{v} \in S^{n-1}} \left[\prod_{i=1}^{k} \left\langle \vec{v}, \vec{d_i} \right\rangle \right] = \left(\prod_{j=1}^{k/2} n + 2j - 2 \right)^{-1} \cdot \sum_{\substack{\mathcal{M} \text{ perfect} \\ \text{matchings of } [k]}} \prod_{e=(i,j) \in \mathcal{M}} \left\langle \vec{d_i}, \vec{d_j} \right\rangle . \quad \text{(B.1)}$$

Proposition B.0.3. *The number of perfect matchings between* $2k$ *elements is*

$$\left(\binom{2k}{2} \binom{2k-2}{2} \cdots \binom{2}{2} \right) / k! = \frac{(2k)!}{2^k \cdot k!} = (2k-1) \cdot (2k-3) \ldots 3 \cdot 1$$

.

Example B.0.4. *Assume k is even, then*

1. $\mathbb{E}_{\vec{v} \in S^{n-1}} \left\langle \vec{v}, \vec{d_1} \right\rangle \left\langle \vec{v}, \vec{d_2} \right\rangle = \frac{1}{n} \cdot \left\langle \vec{d_1}, \vec{d_2} \right\rangle.$
2. $\mathbb{E}_{\vec{v} \in S^{n-1}} \left\langle \vec{v}, \vec{d_1} \right\rangle^2 \left\langle \vec{v}, \vec{d_2} \right\rangle^2 = \frac{1}{n(n+2)} \left(2 \left\langle \vec{d_1}, \vec{d_2} \right\rangle^2 + \left\langle \vec{d_1}, \vec{d_1} \right\rangle \left\langle \vec{d_2}, \vec{d_2} \right\rangle \right).$
3. $\mathbb{E}_{\vec{v} \in S^{n-1}} \left\langle \vec{v}, \vec{d_1} \right\rangle^k = \frac{(k-1) \cdot (k-3) \ldots 3 \cdot 1}{n(n+2) \ldots (n+k-2)} \cdot \left\langle \vec{d_1}, \vec{d_1} \right\rangle^{k/2}.$

4. *Let* $\vec{d}_1 = \vec{e}_1 = (1, 0, \dots, 0)$ *from 3, then* $\mathbb{E}\left[\, v_1^k \,\right] = \dfrac{(k-1)\cdot(k-3)\dots 3\cdot 1}{n(n+2)\dots(n+k-2)}$ *where* v_1 *is the first entry of a random unit vector* $\vec{v} \in \mathbb{R}^n$*.*

5. *if* x, y *are two entries of different rows and columns from a random orthogonal matrix, then*

$$\mathbb{E}\left[x^2 y^2\right] = \mathop{\mathbb{E}}_{\vec{v}\in S^{n-1}}\left[\langle \vec{v}, \vec{e}_1\rangle^2 \cdot \left(\mathop{\mathbb{E}}_{\vec{w}\in S^{n-2}}\left\langle \vec{w}, \vec{e}_2^{\,\perp}\right\rangle^2\right)\right]$$

where $\vec{e}_2^{\,\perp} = \vec{e}_2 - \langle \vec{e}_2, \vec{v}\rangle\, \vec{v}$*.*

APPENDIX C

ALTERNATIVE PROOF OF $(\Omega \circ_R \Omega')_{2k}$ WITHOUT FREE PROBABILITY

In this section we give an alternative proof for Corollary 2.2.16 without using the results from the free probability theory.

Corollary 2.2.16. *Let Ω and Ω' be two distributions. Then for all $k \in \mathbb{N}$,*

$$\left(\Omega \circ_R \Omega'\right)_{2k} = \sum_{\vec{\alpha},\vec{\beta} \in P_k} C\left(\vec{\alpha},\vec{\beta}\right) \cdot \vec{\Omega}^{\vec{\alpha}} \cdot \vec{\Omega'}^{\vec{\beta}} \tag{2.9}$$

where if we let $a = a_1 + \cdots + a_k$ and $b = b_1 + \cdots + b_k$, then

$$C\left(\vec{\alpha},\vec{\beta}\right) = (-1)^{k+a+b-1} \cdot k \cdot \binom{a+b-2}{k-1} \cdot \frac{(a-1)!}{\alpha_1! \dots \alpha_k!} \cdot \frac{(b-1)!}{\beta_1! \dots \beta_k!}\ . \tag{2.10}$$

Using the Trace Power Method, $\left(\Omega \circ_R \Omega'\right)_{2k} = \lim_{n\to\infty} \frac{1}{n} \cdot \mathbb{E}\left[\operatorname{tr}\left(\left(MM^T\right)^k\right)\right]$ where $M = DRD'$. So proving Corollary 2.2.16 is equivalent to proving the following.

Theorem C.0.1. *Let $M = DRD'$ where R is an $n \times n$ random orthogonal matrix, D and D' are $n \times n$ random diagonal matrices where the diagonal elements are drawn independently from distributions Ω and Ω', respectively. Denote $\vec{\alpha} = (a_1, \dots, \alpha_k)$ and $\vec{\beta} = (\beta_1, \dots, \beta_k)$. Then*

$$\lim_{n\to\infty} \frac{1}{n} \cdot \mathbb{E}\left[\operatorname{tr}\left(\left(MM^T\right)^k\right)\right] = \sum_{\vec{\alpha},\vec{\beta} \in P_k} C\left(\vec{\alpha},\vec{\beta}\right) \cdot \vec{\Omega}^{\vec{\alpha}} \cdot \vec{\Omega'}^{\vec{\beta}}$$

where if we let $\alpha_1 + \cdots + \alpha_k = a$ and $\beta_1 + \cdots + \beta_k = b$, then

$$C\left(\vec{\alpha},\vec{\beta}\right) = (-1)^{k+a+b-1} \cdot k \cdot \binom{a+b-2}{k-1} \cdot \frac{(a-1)!}{\alpha_1! \dots \alpha_k!} \cdot \frac{(b-1)!}{\beta_1! \dots \beta_k!}\ . \tag{C.1}$$

In particular, note that $C\left(\vec{\alpha},\vec{\beta}\right) = 0$ if $a + b \le k$.

More specifically, we are going to prove the equivalent statement of Proposition 6.0.8 in the language of random matrices alternatively, and the rest of the proofs remain the same.

Theorem C.0.2. *Let M be defined as before. Then*

$$\lim_{n\to\infty} \frac{1}{n} \cdot \mathbb{E}\left[\operatorname{tr}\left(\left(MM^T\right)^k\right)\right] = \sum_{\vec{\alpha},\vec{\beta}\in P_k} (-1)^{a+b-k-1} \cdot \left(\sum_{(\pi,\sigma)\in\mathcal{NP}(\vec{\alpha},\vec{\beta})} \prod_{x\in S_{\pi\oplus\sigma}} C_{x-1}\right) \cdot \vec{\Omega}^{\vec{\alpha}} \cdot \vec{\Omega'}^{\vec{\beta}} \quad \text{(C.2)}$$

where $a = \alpha_1 + \cdots + \alpha_k$ and $b = \beta_1 + \cdots + \beta_k$.

Remark C.0.3. *The above theorem is equivalent to the statement*

$$\varphi\left((ab)^k\right) = \sum_{\vec{\alpha},\vec{\beta}\in P_k} (-1)^{A+B-k-1} \cdot \left(\sum_{(\pi,\sigma)\in\mathcal{NP}(\vec{\alpha},\vec{\beta})} \prod_{x\in S_{\pi\oplus\sigma}} C_{x-1}\right) \cdot \vec{\varphi_a}^{\vec{\alpha}} \cdot \vec{\varphi_b}^{\vec{\beta}} \qquad \text{(C.3)}$$

Proof of Theorem C.0.1. With the same steps in Chapter 6, we can prove that

$$\sum_{(\pi,\sigma)\in\mathcal{NP}(\vec{\alpha},\vec{\beta})} \prod_{x\in S_{\pi\oplus\sigma}} C_{x-1} = k \cdot \binom{a+b-2}{k-1} \cdot \frac{(a-1)!}{\alpha_1!\ldots\alpha_k!} \cdot \frac{(b-1)!}{\beta_1!\ldots\beta_k!} \qquad \text{(C.4)}$$

and thus Theorem C.0.2 follows. □

Definition C.0.4. Let $\vec{i} = (i_1, \ldots, i_k) \subseteq [n]^k$ and $\vec{\alpha} = (\alpha_1, \ldots, \alpha_k) \in P_k$. We define $n_j = \left|\left\{s \in [k] : i_s = i_j\right\}\right|$ to be the number of elements in $\vec{i}$ that are equal to i_j, including i_j itself. We say that $P\left(\vec{i}\right) = \vec{\alpha}$ if $\left|\left\{j : n_j = i\right\}\right| = i \cdot \alpha_i$ for all $i \in [k]$.

Example C.0.5. $P(1, 1, 1) = (0, 0, 1)$, $P(1, 2, 1, 4, 4, 3, 1) = (2, 1, 1, 0, 0, 0, 0)$.

Proposition C.0.6. *Let $M = DRD'$ where R is an $n \times n$ random orthogonal matrix, D and D' are $n \times n$ random Ω and Ω'-distribution diagonal matrices, respectively. Then*

$$\mathbb{E}\left[\operatorname{tr}\left(MM^T\right)^k\right] = \sum_{i_m, j_m\in[n]} \mathop{\mathbb{E}}_{d_{i_m}\sim\Omega}\left[d_{i_1}^2 \ldots d_{i_k}^2\right] \cdot \mathbb{E}\left[\prod_{m=1}^{k} R(i_m, j_m)R(i_{m+1}, j_m)\right] \cdot \mathop{\mathbb{E}}_{d'_{j_m}\sim\Omega'}\left[d'^2_{j_1} \ldots d'^2_{j_k}\right]$$

Proposition C.0.7. *Let $M = DRD'$ and $D\left(\vec{\alpha}, \vec{\beta}\right)$ be defined as*

$$D\left(\vec{\alpha}, \vec{\beta}\right) = \lim_{n\to\infty} \frac{1}{n} \cdot \left(\sum_{\substack{\vec{i}, \vec{j} \subseteq [n]^k: \\ P(\vec{i}) = \vec{\alpha}, P(\vec{j}) = \vec{\beta}}} \mathbb{E}\left[\prod_{m=1}^{k} R(i_m, j_m) R(i_{m+1}, j_m) \right] \right) \tag{C.5}$$

Then

$$\lim_{n\to\infty} \frac{1}{n} \cdot \mathbb{E}\left[\operatorname{tr}\left(MM^T\right)^k \right] = \sum_{\vec{\alpha}, \vec{\beta} \in P_k} D\left(\vec{\alpha}, \vec{\beta}\right) \cdot \vec{\Omega}^{\vec{\alpha}} \cdot \vec{\Omega'}^{\vec{\beta}} \tag{C.6}$$

Proof.

$$\begin{aligned}
&\mathbb{E}\left[\operatorname{tr}\left(\left(MM^T\right)^k \right) \right] \\
&= \sum_{i_m \in [n], j_m \in [n]} \mathbb{E}_{d_{i_m} \sim \Omega} \left[d_{i_1}^2 \ldots d_{i_k}^2 \right] \cdot \mathbb{E}\left[\prod_{m=1}^{k} R(i_m, j_m) R(i_{m+1}, j_m) \right] \cdot \mathbb{E}_{d'_{j_m} \sim \Omega'} \left[d'^2_{j_1} \ldots d'^2_{j_k} \right] \\
&= \sum_{\vec{\alpha}, \vec{\beta} \in P_k} \sum_{\substack{P(\vec{i}) = \vec{\alpha}, \\ P(\vec{j}) = \vec{\beta}}} \mathbb{E}_{d_{i_m} \sim \Omega} \left[\prod_{m=1}^{k} d_{i_m}^2 \right] \cdot \mathbb{E}\left[\prod_{m=1}^{k} R(i_m, j_m) R(i_{m+1}, j_m) \right] \cdot \mathbb{E}_{d'_{j_m} \sim \Omega'} \left[\prod_{m=1}^{k} d'^2_{j_m} \right] \\
&= \sum_{\vec{\alpha}, \vec{\beta} \in P_k} \left(\prod_{i=1}^{k} \Omega_{2i}^{\alpha_i} \right) \cdot \left(\sum_{\substack{P(\vec{i}) = \vec{\alpha}, \\ P(\vec{j}) = \vec{\beta}}} \mathbb{E}\left[\prod_{m=1}^{k} R(i_m, j_m) R(i_{m+1}, j_m) \right] \right) \cdot \left(\prod_{j=1}^{k} {\Omega'}_{2j}^{\beta_j} \right)
\end{aligned}$$

Dividing both sides by n and evaluating the limit as $n \to \infty$, we get the conclusion. □

By the above proposition, to prove Theorem C.0.2, it suffices to prove the following.

Theorem C.0.8. *For all $\vec{\alpha}, \vec{\beta} \in P_k$,*

$$D\left(\vec{\alpha}, \vec{\beta}\right) = (-1)^{A+B-k-1} \cdot \left(\sum_{(\pi,\sigma) \in \mathcal{NP}(\vec{\alpha}, \vec{\beta})} \prod_{x \in S_{\pi \oplus \sigma}} C_{x-1} \right) \tag{C.7}$$

Remark C.0.9. *For given $\vec{\alpha}, \vec{\beta}$, we will use notations $D\left(\vec{\alpha}, \vec{\beta}\right)$ and $D\left(\vec{\Omega}^{\vec{\alpha}}, \vec{\Omega'}^{\vec{\beta}}\right)$ interchangeably.*

We will prove Theorem C.0.8 by first proving the base cases first, and then prove it generally.

C.1 Case when $D\left(\vec{\alpha}, \vec{\beta}\right) = D\left(\Omega_2^k, {\Omega'}_2^k\right)$

We want to show that

$$C\left(\Omega_2^k, {\Omega'}_2^k\right) = (-1)^{k-1} \cdot \frac{1}{k} \cdot \binom{2k-2}{k-1} = (-1)^{k-1} \cdot C_{k-1} \tag{C.8}$$

where C_i is the i^{th} Catalan Number.

Definition C.1.1. Given $s_1, \ldots, s_k \in [k]$ such that $s_1 + \cdots + s_m = k$, we define $\operatorname{val}\left(G_{s_1}, \ldots, G_{s_m}\right)$ to be

$$\operatorname{val}\left(G_{s_1}, \ldots, G_{s_m}\right) = \mathbb{E}\left[\prod_{m=1}^{k} R(i_m, j_m) R(i_{m+1}, j_m)\right]$$

where $i_1, \ldots, i_k \in [n]$ and $j_1, \ldots, j_k \in [n]$ are chosen such that $i_{l_1} = i_{l_2} \iff l_r = 1 + \sum_{j=1}^{x_r} s_j$ for some $x_r \in \{0, 1, \ldots, m\}$ for each $r = 1, 2$ and $j_1 \neq j_2 \neq \ldots \neq j_k$. See Figure C.1 for an illustration.

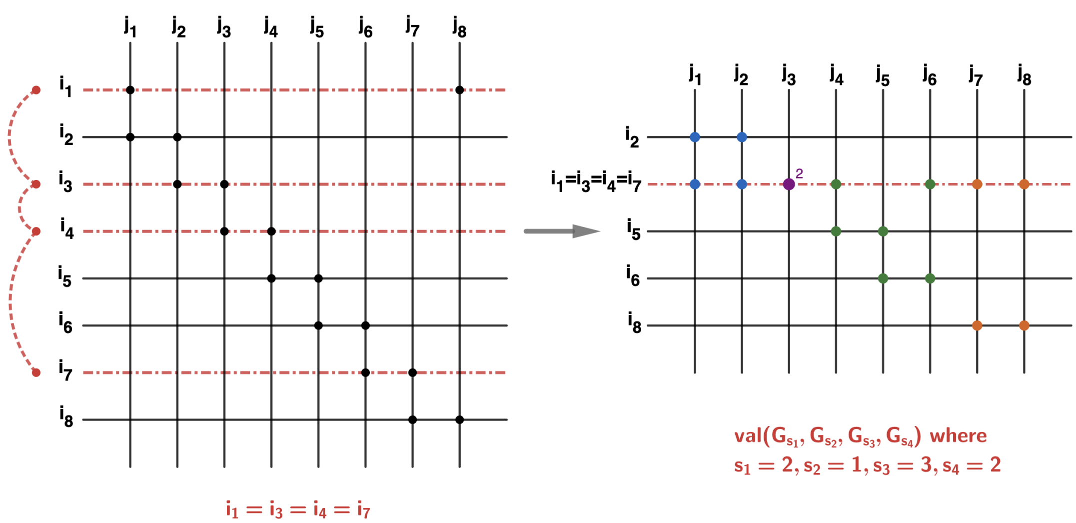


Figure C.1: Example of $\operatorname{val}\left(G_{s_1}, \ldots, G_{s_m}\right)$ when $n = 8$, $m = 4$, $s_1 = 2$, $s_2 = 1$, $s_3 = 3$ and $s_4 = 2$. Note that the purple dot is doubled.

In particular, $\text{val}(G_k) = \mathbb{E}\left[\prod_{m=1}^{k} R(i_m, j_m)R(i_{m+1}, j_m)\right]$ where $i_1 \neq \ldots \neq i_k$ and $j_1 \neq \ldots \neq j_k$.

We can compute $\text{val}\left(G_{s_1}, \ldots, G_{s_m}\right)$ using the following theorem which we show in Section C.4.

Theorem C.1.2.

$$\text{val}\left(G_{s_1}, \ldots, G_{s_m}\right) \sim \prod_{j=1}^{m} \text{val}(G_{s_j}) \qquad (\ \textit{as } n \to \infty)$$

Theorem C.1.3.

$$\text{val}(G_k) \sim (-1)^{k-1} \cdot n^{-2k+1} \cdot C_{k-1} \qquad (\ \textit{as } n \to \infty)$$

where C_{k-1} is the $(k-1)^{th}$ Catalan number.

Proof. We prove this by induction on k.

1. Base case $k = 1$: $\text{val}(G_k) = \mathbb{E}\left[R(i_1, j_1)^2\right] = \mathbb{E}_{\vec{v} \in S^{n-1}}\left[\langle \vec{v}, \vec{e}_1 \rangle^2\right] = \frac{1}{n} \cdot \langle \vec{e}_1, \vec{e}_1 \rangle = \frac{1}{n} = n^{-1} \cdot C_0$.

2. Inductive case: assume $\text{val}(G_i) \sim (-1)^{i-1} \cdot n^{-2i+1} \cdot C_{i-1}$ for all $i < k$, we want to prove that $\text{val}(G_k) \sim (-1)^{k-1} \cdot n^{-2k+1} \cdot C_{k-1}$.

   We will define $\mathcal{E}_1, \ldots, \mathcal{E}_k$ inductively starting from $\mathcal{E}_1$. Note that $\mathcal{E}_j$ is a function of $\vec{v}_1, \ldots, \vec{v}_{j-1}$.

   i. We define $\mathcal{E}_1 = \mathbb{E}_{\vec{v}_1 \in S^{n-k}}\left[\left\langle \vec{v}_1, \vec{e}_k^{\,\perp\times(k-1)} \right\rangle \left\langle \vec{v}_1, \vec{e}_1^{\,\perp\times(k-1)} \right\rangle\right]$ where $\vec{e}^{\,\perp\times i}$ is the orthogonal projection of $\vec{e}$ onto the orthogonal complement of $\text{span}\left(\vec{v}_{k-i+1}, \ldots, \vec{v}_k\right)$, and

   ii. for each $2 \leq i \leq k$, $\mathcal{E}_i = \mathbb{E}_{\vec{v}_i \in S^{n-1-(k-i)}}\left[\left\langle \vec{v}_i, \vec{e}_{i-1}^{\,\perp\times(k-i)} \right\rangle \left\langle \vec{v}_i, \vec{e}_i^{\,\perp\times(k-i)} \right\rangle \cdot \mathcal{E}_{i-1}\right]$.

   Defined this way, $\mathcal{E}_k = \text{val}(G_k)$.

   Since

$$\vec{e}^{\,\perp\times i} = \vec{e}^{\,\perp\times(i-1)} - \left\langle \vec{v}_{k-i+1}, \vec{e}^{\,\perp\times(i-1)} \right\rangle \cdot \vec{v}_{k-i+1}\,,$$

we have that

$$
\begin{aligned}
\mathcal{E}_1 &= \mathop{\mathbb{E}}_{\vec{v}_1 \in S^{n-k}} \left[\left\langle \vec{v}_1, \vec{e}_k^{\,\perp\times(k-1)} \right\rangle \left\langle \vec{v}_1, \vec{e}_1^{\,\perp\times(k-1)} \right\rangle \right] \\
&= \frac{1}{n-k+1} \cdot \left\langle \vec{e}_k^{\,\perp\times(k-1)}, \vec{e}_1^{\,\perp\times(k-1)} \right\rangle \\
&= \frac{1}{n-k+1} \cdot \left(\left\langle \vec{e}_k^{\,\perp\times(k-2)}, \vec{e}_1^{\,\perp\times(k-2)} \right\rangle - \left\langle \vec{v}_2, \vec{e}_k^{\,\perp\times(k-2)} \right\rangle \left\langle \vec{v}_2, \vec{e}_1^{\,\perp\times(k-2)} \right\rangle \right) \\
&= \frac{1}{n-k+1} \cdot \Bigg(\left\langle \vec{e}_k^{\,\perp\times(k-3)}, \vec{e}_1^{\,\perp\times(k-3)} \right\rangle \\
&\qquad - \left\langle \vec{v}_3, \vec{e}_k^{\,\perp\times(k-3)} \right\rangle \left\langle \vec{v}_3, \vec{e}_1^{\,\perp\times(k-3)} \right\rangle - \left\langle \vec{v}_2, \vec{e}_k^{\,\perp\times(k-2)} \right\rangle \left\langle \vec{v}_2, \vec{e}_1^{\,\perp\times(k-2)} \right\rangle \Bigg) \\
&= \ldots\ldots \\
&= -\frac{1}{n-k+1} \cdot \left(\sum_{i=1}^{k-1} \left\langle \vec{v}_{k-i+1}, \vec{e}_k^{\,\perp\times(i-1)} \right\rangle \left\langle \vec{v}_{k-i+1}, \vec{e}_1^{\,\perp\times(i-1)} \right\rangle \right) \\
&= -\frac{1}{n-k+1} \cdot \left(\sum_{i=1}^{k-1} \left\langle \vec{v}_{k-i+1}, \vec{e}_k \right\rangle \left\langle \vec{v}_{k-i+1}, \vec{e}_1 \right\rangle \right)
\end{aligned}
$$

Thus

$$
\mathcal{E}_k = \frac{-1}{n-k+1} \Big(\mathbb{E}\left[\vec{v}_1 \sim \vec{v}_2\right] + \mathbb{E}\left[\vec{v}_1 \sim \vec{v}_3\right] + \cdots + \mathbb{E}\left[\vec{v}_1 \sim \vec{v}_k\right] \Big)
$$

$$
\begin{aligned}
&\sim -n^{-1} \cdot \Big(\mathrm{val}\left(G_1, G_{k-1}\right) + \mathrm{val}\left(G_2, G_{k-2}\right) + \cdots + \mathrm{val}\left(G_{k-1}, G_1\right) \Big) \\
&\sim -n^{-1} \cdot \left(\sum_{i=1}^{k-1} \left((-1)^{i-1} \cdot n^{-2i+1} \cdot C_{i-1} \right) \cdot \left((-1)^{k-i-1} \cdot n^{-2(k-i)+1} \cdot C_{k-i-1} \right) \right) \\
&= (-1)^{k-1} \cdot n^{-2k+1} \cdot \left(\sum_{i=1}^{k-1} C_{i-1} C_{k-i-1} \right) \\
&= (-1)^{k-1} \cdot n^{-2k+1} \cdot C_{k-1} \,.
\end{aligned}
$$

where $\vec{v}_1 \sim \vec{v}_i$ means folding the first row of the matrix to the i^{th} row. See Figure C.2 for an illustration.

□

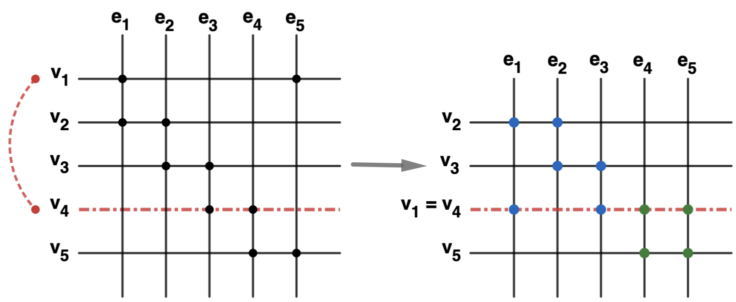


Figure C.2: Illustration of $\mathbb{E}\left[\vec{v}_1 \sim \vec{v}_4\right]$.

Theorem C.1.4. *$D\left(\Omega_2^k, {\Omega'_2}^k\right) = (-1)^{k-1} \cdot C_{k-1}$, where C_k is the $(k-1)^{th}$ Catalan number.*

Proof.

$$\begin{aligned}
D\left(\Omega_2^k, {\Omega'_2}^k\right) &= \lim_{n\to\infty} \frac{1}{n} \cdot \left(\sum_{\substack{i_1 \neq \ldots \neq i_k, \\ j_1 \neq \ldots \neq j_k}} \mathbb{E}\left[\prod_{m=1}^{k} R(i_m, j_m) R(i_{m+1}, j_m) \right] \right) \\
&= \lim_{n\to\infty} \frac{1}{n} \cdot \left(\sum_{\substack{i_1 \neq \ldots \neq i_k, \\ j_1 \neq \ldots \neq j_k}} \mathrm{val}\left(G_k\right) \right) \\
&= \lim_{n\to\infty} \frac{1}{n} \cdot n^{2k} \cdot \mathrm{val}\left(G_k\right) \\
&= \lim_{n\to\infty} n^{2k-1} \cdot n^{-2k+1} \cdot (-1)^{k-1} \cdot C_{k-1} = (-1)^{k-1} \cdot C_{k-1}\,.
\end{aligned}$$

Thus $D\left(\Omega_2^k, {\Omega'_2}^k\right) = (-1)^{k-1} \cdot C_{k-1}$. □

C.2 Case when $D\left(\vec{\alpha}, \vec{\beta}\right) = D\left(\vec{\alpha}, {\Omega'_2}^k\right)$

We want to show that

Theorem C.2.1. *Let $\vec{\alpha} = (\alpha_1, \ldots, \alpha_k) \in P_k$ and $a = \alpha_1 + \cdots + \alpha_k$. Then*

$$D\left(\vec{\Omega}^{\vec{\alpha}}, \, {\Omega_2'}^k\right) = (-1)^{a-1} \cdot \frac{(a-1)!}{\alpha_1! \ldots \alpha_k!} \cdot \binom{a+k-2}{k-1}. \tag{C.9}$$

Definition C.2.2. We say that an assignment $i : [k] \to [n]$ (conventionally denote $i(m)$ as i_m) *respects* a partition $\mathcal{P}$ of $[k]$ if for all $j_1, j_2 \in [k]$, $i_{j_1} = i_{j_2} \iff j_1$ and j_2 are in the same parts under $\mathcal{P}$.

Definition C.2.3. Given $\vec{\alpha} = (\alpha_1, \ldots, \alpha_k), \vec{\beta} = (\beta_1, \ldots, \beta_k) \in P_k$ and $\mathcal{P}_r \in \mathcal{P}(\vec{\alpha})$, $\mathcal{P}_c \in \mathcal{P}\left(\vec{\beta}\right)$, we define

1. $\text{val}(\mathcal{P}_r, \mathcal{P}_c) = \mathbb{E}\left[\prod_{m=1}^{k} R(i_m, j_m) R(i_{m+1}, j_m)\right]$, where $i : [k] \to [n]$ and $j : [k] \to [n]$ are some assignments that respect the partitions $\mathcal{P}_r$ and $\mathcal{P}_c$, respectively.
2. $N(\mathcal{P}_r, \mathcal{P}_c) = \left|\{(i,j) : i : [k] \to [n] \text{ respects } \mathcal{P}_r \text{ and } j : [k] \to [n] \text{ respects } \mathcal{P}_c\}\right|$.

In the special cases when $\mathcal{P}_c$ or $\mathcal{P}_r$ is $\mathcal{P}_0 \in \mathcal{P}(k, 0, \ldots, 0)$, we denote $\text{val}(\mathcal{P}_r)$ to be $\text{val}(\mathcal{P}_r, \mathcal{P}_0)$ and $\text{val}(\mathcal{P}_c)$ to be $\text{val}(\mathcal{P}_0, \mathcal{P}_c)$.

Proposition C.2.4. *Let $\vec{\alpha} = (\alpha_1, \ldots, \alpha_k), \vec{\beta} = (\beta_1, \ldots, \beta_k) \in P_k$ and $\mathcal{P}_r \in \mathcal{P}(\vec{\alpha})$, $\mathcal{P}_c \in \mathcal{P}\left(\vec{\beta}\right)$. Let $a = \alpha_1 + \cdots + \alpha_k$ and $b = \beta_1 + \cdots + \beta_k$. Then*

$$N(\mathcal{P}_r, \mathcal{P}_c) = \frac{n!}{(n-a)!} \cdot \frac{n!}{(n-b)!} \sim n^{a+b}.$$

By definitions of $\text{val}(\mathcal{P}_r, \mathcal{P}_c)$ and $N(\mathcal{P}_r, \mathcal{P}_c)$, we can express $D\left(\vec{\alpha}, \vec{\beta}\right)$ as the following.

Proposition C.2.5. *Let $\vec{\alpha} = (\alpha_1, \ldots, \alpha_k) \in P_k$ and $\vec{\beta} = (\beta_1, \ldots, \beta_k) \in P_k$. Then*

$$D\left(\vec{\alpha}, \vec{\beta}\right) = \lim_{n \to \infty} \frac{1}{n} \cdot \left(\sum_{\mathcal{P}_r \in \mathcal{P}(\vec{\alpha}), \mathcal{P}_c \in \mathcal{P}\left(\vec{\beta}\right)} N(\mathcal{P}_r, \mathcal{P}_c) \cdot \text{val}(\mathcal{P}_r, \mathcal{P}_c)\right). \tag{C.10}$$

As we show in Section C.4, we only need to consider partitions $\mathcal{P}_r$ which are non-crossing.

Theorem C.2.6. *Given $\vec{\alpha} = (\alpha_1, \ldots, \alpha_k) \in P_k$, let $a = \alpha_1 + \cdots + \alpha_k$. Let $\mathcal{P}_r \in \mathcal{P}(\vec{\alpha})$. Then*

$$n^{a+k} \cdot \text{val}(\mathcal{P}_r) = \Theta(n) \iff \mathcal{P}_r \in \mathcal{NP}(\vec{\alpha}). \tag{C.11}$$

Moreover, if $\mathcal{P}_r$ *is not non-crossing, then* $n^{a+k} \cdot \operatorname{val}(\mathcal{P}_r) = O(1)$.

Corollary C.2.7. *Let* $\vec{\alpha} = (\alpha_1, \ldots, \alpha_k) \in P_k$. *Then*

$$D\left(\Omega_2^{\alpha_1} \ldots \Omega_{2k}^{\alpha_k}, \Omega_2'^{k}\right) = \lim_{n\to\infty} \frac{1}{n} \cdot \left(\sum_{\mathcal{P}_r \in \mathcal{NP}(\vec{\alpha})} N(\mathcal{P}_r) \cdot \operatorname{val}(\mathcal{P}_r) \right). \tag{C.12}$$

Definition C.2.8. Let $\mathcal{P} = \{P_1, \ldots, P_m\}$ be a partition of $[k]$. Let $\mathcal{C}_k$ be the cycle graph with vertices $\{1, \ldots, k\}$. Define $\mathcal{C}_k/\mathcal{P}$ to be the graph obtained by identifying vertices together under $\mathcal{P}$.

Proposition C.2.9. *Let* $\vec{\alpha} = (\alpha_1, \ldots, \alpha_k) \in P_k$, $\mathcal{P}_r \in \mathcal{NP}(\vec{\alpha})$ *and* $a = \alpha_1 + \cdots + \alpha_k$. *Let* $S_{\mathcal{P}} = \left\{i_1, \ldots, i_p\right\}$. *Then*

$$\lim_{n\to\infty} n^{a+k-1} \cdot \operatorname{val}(\mathcal{P}_r) = (-1)^{a-1} \cdot \prod_{j=1}^{p} C_{i_j - 1} \tag{C.13}$$

where C_{i_j-1} *is the* $\left(i_j - 1\right)^{th}$ *Catalan number.*

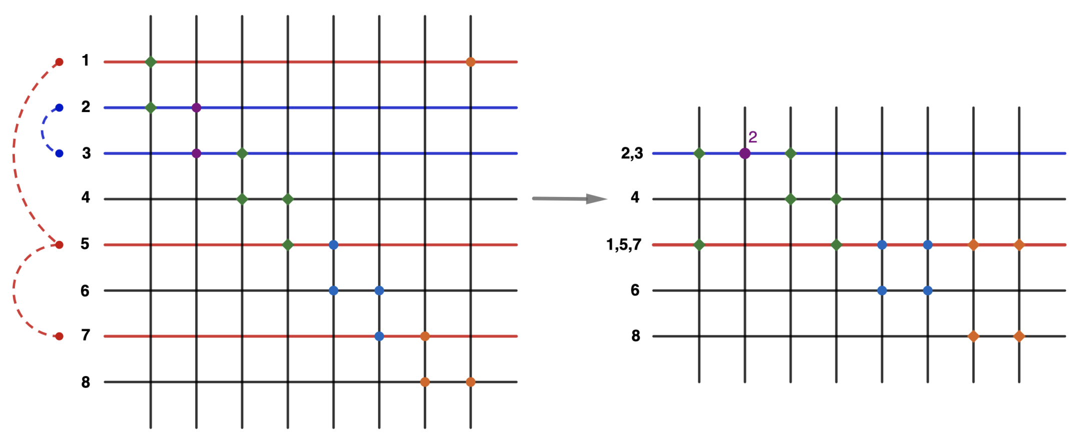


Figure C.3: Illustration of Proposition C.2.9: $\mathcal{P}_r = \{\{4\}, \{6\}, \{8\}, \{2,3\}, \{1,5,7\}\}$, $\operatorname{val}(\mathcal{P}_r) = \operatorname{val}\left(G_{i_1}, \ldots, G_{i_4}\right)$ where $i_1 = 1$, $i_2 = 3$, $i_3 = i_4 = 2$.

Proof. By Proposition 6.0.2, $\mathcal{C}_k/\mathcal{P}_r = \mathcal{C}_{k,1} \cup \cdots \cup \mathcal{C}_{k,p}$ where $p = k - a + 1$ and $i_j = \left|\mathcal{C}_{k,j}\right|$

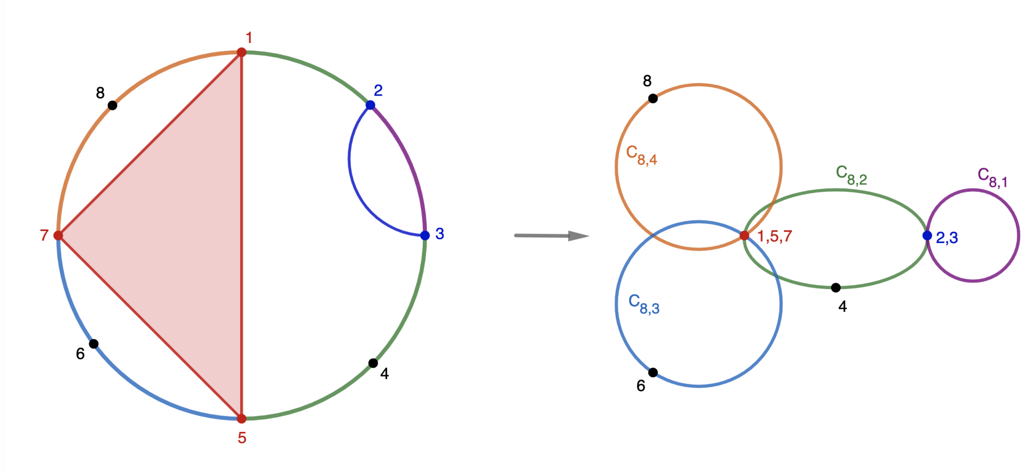


Figure C.4: Illustration of Proposition C.2.9: $\mathcal{P} = \{\{4\},\{6\},\{8\},\{2,3\},\{1,5,7\}\}$, $S_{\mathcal{P}} = \{1,3,2,2\}$

for each $j \in [p]$. Then

$$\begin{aligned}
\lim_{n\to\infty} n^{a+k-1} \cdot \mathrm{val}\left(\mathcal{P}_r\right) &= \lim_{n\to\infty} n^{a+k-1} \cdot \mathrm{val}\left(G_{i_1},\ldots,G_{i_p}\right) \\
&= \lim_{n\to\infty} n^{a+k-1} \cdot \prod_{j=1}^{p} \mathrm{val}\left(G_{i_j}\right) \\
&= \lim_{n\to\infty} n^{a+k-1} \cdot \prod_{j=1}^{p} (-1)^{i_j-1} \cdot n^{-2i_j+1} \cdot C_{i_j-1} \\
&= \lim_{n\to\infty} n^{a+k-1} \cdot (-1)^{k-p} \cdot n^{-2k+p} \cdot \prod_{j=1}^{p} C_{i_j-1} \\
&= (-1)^{a-1} \cdot \lim_{n\to\infty} \left(n^{a-k-1+p} \cdot \prod_{j=1}^{p} C_{i_j-1} \right) = (-1)^{a-1} \cdot \prod_{j=1}^{p} C_{i_j-1}\,.
\end{aligned}$$

□

C.3 The General Case

Now we are ready to show the general case.

Theorem C.3.1. *Let $\vec{\alpha}, \vec{\beta} \in P_k$ and $a = \alpha_1 + \cdots + \alpha_k$, $b = \beta_1 + \cdots + \beta_k$. Then*

$$D\left(\vec{\alpha}, \vec{\beta}\right) = (-1)^{a+b-k-1} \cdot \frac{(a-1)!}{\alpha_1! \ldots \alpha_k!} \cdot \frac{(b-1)!}{\beta_1! \ldots \beta_k!} \cdot k \cdot \binom{a+b-2}{k-1}. \tag{C.14}$$

Definition C.3.2. Let $\mathcal{P} = \{P_1, \ldots, P_k\}$ and $\mathcal{Q} = \{Q_1, \ldots, Q_l\}$ be two partitions of $[n]$. We say that $\mathcal{P} \cup \mathcal{Q}$ is *non-crossing* if

1. $\mathcal{P}$ and $\mathcal{Q}$ are non-crossing, and
2. For any $P \in \mathcal{P}$ and $Q \in \mathcal{Q}$, they do not cross when placing on the cycle $\mathcal{C}_n$, and they touch at at most one point.

See Figure C.5 for an illustration.

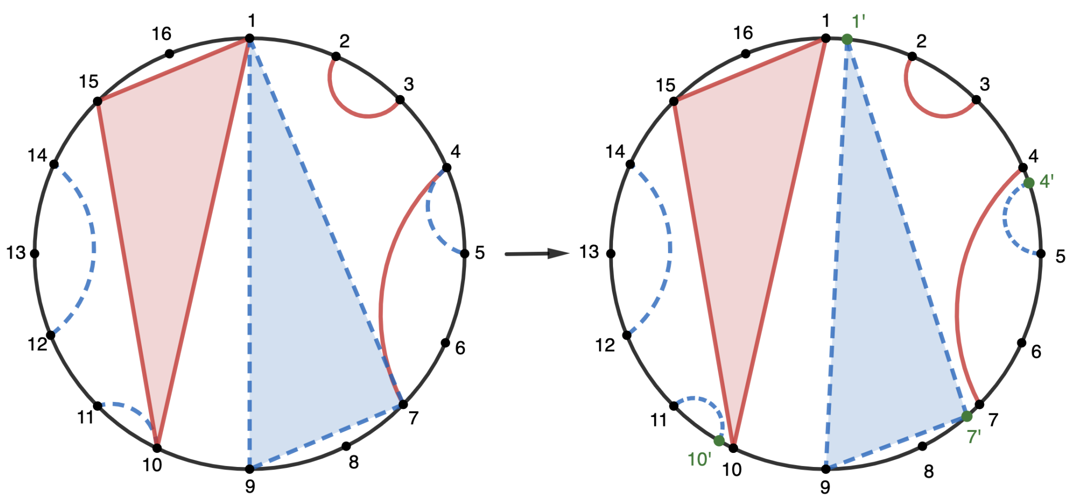


Figure C.5: Illustration of Definition C.3.2 and Definition C.3.3: $\mathcal{P}$ is colored red and labelled with solid lines, $\mathcal{Q}$ is colored blue and labelled with dash lines. $\mathcal{P} \cup \mathcal{Q}$ is non-crossing and moreover, $\in \mathcal{NP}\left(\vec{\alpha}, \vec{\beta}\right)$ where $\vec{\alpha}$ and $\vec{\beta}$ correspond to $\Omega_2^9\Omega_4^2\Omega_6$ and $\Omega_2^7\Omega_4^3\Omega_6$, respectively.

Definition C.3.3. Let $\vec{\alpha}, \vec{\beta} \in P_k$. We say $\mathcal{P} \cup \mathcal{Q} \in \mathcal{NP}\left(\vec{\alpha}, \vec{\beta}\right)$ if

1. $\mathcal{P} \in \mathcal{NP}(\vec{\alpha})$, $\mathcal{Q} \in \mathcal{NP}\left(\vec{\beta}\right)$,
2. $\mathcal{P} \cup \mathcal{Q}$ is non-crossing,
3. For any $P \in \mathcal{P}$ and $Q \in \mathcal{Q}$ that touches at a point t_0, we can order P and Q as $P = \{p_1, \ldots, p_x, t_0\}$ and $Q = \{t_0, q_1, \ldots, q_y, t_0\}$ such that $t_0, q_1, \ldots, q_y, p_1, \ldots, p_x$ are ordered in the clockwise direction on $\mathcal{C}_k$.

Pictorially, for any $P \in \mathcal{P}$ and $Q \in \mathcal{Q}$ that touch at a point t_0, we can perturb the t_0 vertex of Q in the clockwise direction a little so that the perturbed Q does not cross P. See Figure C.5 for an illustration.

As we show in Appendix C.4, we only need to consider non-crossing partitions.

Theorem C.3.4. *Given $\vec{\alpha}, \vec{\beta} \in P_k$, let $a = \alpha_1 + \cdots + \alpha_k$ and $b = \beta_1 + \cdots + \beta_k$. Let $\mathcal{P}_r \in \mathcal{P}(\alpha_1, \ldots, \alpha_k)$ and $\mathcal{P}_c \in \mathcal{P}(\beta_1, \ldots, \beta_k)$. Then*

$$n^{a+b} \cdot \mathrm{val}(P_r, P_c) = \Theta(n) \iff \mathcal{P}_r \cup \mathcal{P}_c \in \mathcal{NP}\left(\vec{\alpha}, \vec{\beta}\right).$$

Moreover, if $\mathcal{P}_r \cup \mathcal{P}_c$ is not non-crossing, then $n^{a+b} \cdot \mathrm{val}(\mathcal{P}_r \cup \mathcal{P}_c) = O(1)$.

Corollary C.3.5. *Let $\vec{\alpha} = (\alpha_1, \ldots, \alpha_k), \vec{\beta} = (\beta_1, \ldots, \beta_k) \in P_k$. Then*

$$D\left(\vec{\alpha}, \vec{\beta}\right) = \lim_{n \to \infty} \frac{1}{n} \cdot \left(\sum_{\mathcal{P}_r \cup \mathcal{P}_c \in \mathcal{NP}\left(\vec{\alpha}, \vec{\beta}\right)} N(\mathcal{P}_r, \mathcal{P}_c) \cdot \mathrm{val}(\mathcal{P}_r, \mathcal{P}_c) \right)$$

Definition C.3.6. Let $\mathcal{P} \cup \mathcal{Q}$ be a non-crossing partition of $[k]$. Assume $\mathcal{C}_k/(\mathcal{P} \cup \mathcal{Q}) = \mathcal{C}_{k,1} \cup \cdots \cup \mathcal{C}_{k,p}$. We define $S_{\mathcal{P},\mathcal{Q}}$ to be the unordered sequence $\left\{i_1, \ldots, i_p\right\}$ where i_j is the size of the cycle $\mathcal{C}_{k,j}$ for each $j \in [p]$.

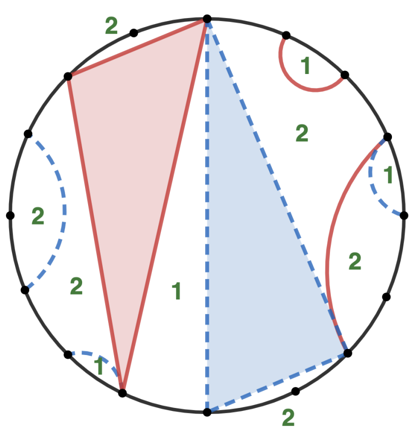


Figure C.6: Illustration of Definition C.3.6.

Example C.3.7. *We can label the sizes for Figure C.5, then $S_{\mathcal{P},\mathcal{Q}} = \{1, 1, 1, 1, 2, 2, 2, 2, 2, 2\}$ as shown in Figure C.6.*

Proposition C.3.8. *Let* $\vec{\alpha} = (\alpha_1, \dots, \alpha_k), \vec{\beta} = (\beta_1, \dots, \beta_k) \in P_k$*,* $\mathcal{P}_r \cup \mathcal{P}_c \in \mathcal{NP}\left(\vec{\alpha}, \vec{\beta}\right)$*, and* $a = \alpha_1 + \cdots + \alpha_k$*,* $b = \beta_1 + \cdots + \beta_k$*. Let* $S_{\mathcal{P},\mathcal{Q}} = \left\{i_1, \dots, i_p\right\}$*. Then*

$$\lim_{n\to\infty} n^{a+b-1} \cdot \mathrm{val}\left(\mathcal{P}_r, \mathcal{P}_c\right) = (-1)^{a+b-k-1} \cdot \prod_{j=1}^{p} C_{i_j-1} \tag{C.15}$$

where C_{i_j-1} *is the* $\left(i_j - 1\right)^{th}$ *Catalan number.*

Proof. By Proposition 6.0.2, $\mathcal{C}_k / \left(\mathcal{P}_r \cup \mathcal{P}_c\right) = \mathcal{C}_{k,1} \cup \cdots \cup \mathcal{C}_{k,p}$ where $p = 2k - a - b + 1$ and $i_j = \left|\mathcal{C}_{k,j}\right|$ for each $j \in [p]$. Then

$$\begin{aligned}
\lim_{n\to\infty} n^{a+b-1} \cdot \mathrm{val}\left(\mathcal{P}_r, \mathcal{P}_c\right) &= \lim_{n\to\infty} n^{a+b-1} \cdot \mathrm{val}\left(G_{i_1}, \dots, G_{i_p}\right) \\
&= \lim_{n\to\infty} n^{a+b-1} \cdot \prod_{j=1}^{p} \mathrm{val}\left(G_{i_j}\right) \\
&= \lim_{n\to\infty} n^{a+b-1} \cdot \prod_{j=1}^{p} (-1)^{i_j-1} \cdot n^{-2i_j+1} \cdot C_{i_j-1} \\
&= \lim_{n\to\infty} n^{a+b-1} \cdot (-1)^{k-p} \cdot n^{-2k+p} \cdot \prod_{j=1}^{p} C_{i_j-1} \\
&= (-1)^{a+b-k-1} \cdot \lim_{n\to\infty} \left(n^{a+b-1-2k+p} \cdot \prod_{j=1}^{p} C_{i_j-1}\right) \\
&= (-1)^{a+b-k-1} \cdot \prod_{j=1}^{p} C_{i_j-1} \,.
\end{aligned}$$

□

C.4 Optimality of Non-Crossing Partitions

We now show Theorems C.1.2, C.2.6, and C.3.4 which say that when we take the limit as $n \to \infty$, we only need to consider non-crossing partitions and we can ignore the interaction between parts of our grid which are not part of the same term.

Our setup is as follows. We start with the set of vertices

$$V_0 = \{(i,i) : i \in [k]\} \cup \{(i+1,i) : i \in [k-1]\} \cup \{(1,k)\}$$

which we think of a staircase of length k (see Definition C.4.4 below). We then perform the following kinds of operations on our current multi-set of vertices V.

Definition C.4.1. In a *row merge* operation, we merge a set of rows $A \subseteq [k]$ together. More precisely, we choose a representative $a \in A$ and then for each $a' \in A \setminus \{a\}$, we replace each vertex $(a', b) \in V$ with (a, b). We define the weight of a row merge operation to be $|A| - 1$. The intuition for this is that the row merge operation reduces the number of distinct indices by $|A| - 1$.

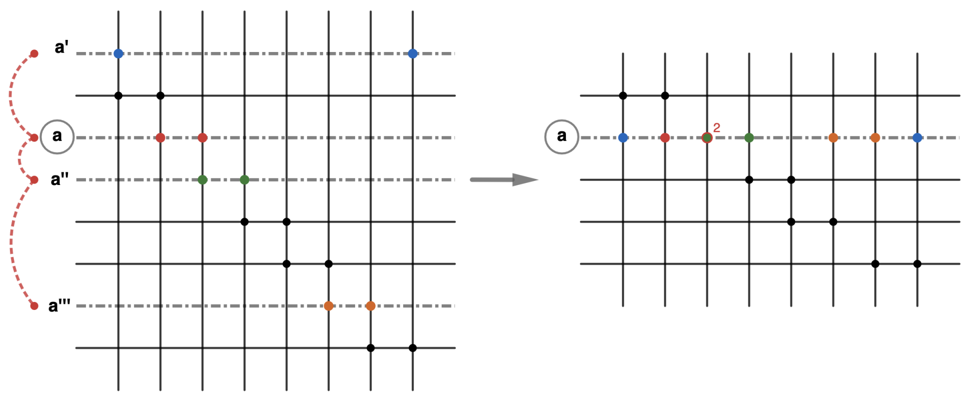


Figure C.7: Illustration of Definition C.4.1: here $A = \{a, a', a'', a'''\}$, and we merge rows A into row a.

We define a *column merge* operation and its weight in the same way.

Definition C.4.2. We define a *shift* operation as follows. We take two vertices (a, b) and (a, b') in the same row a and we shift them to a different row a' by replacing them with the vertices (a', b) and (a', b'). We write this shift operation as $\{(a, b), (a, b')\} \to \{(a', b), (a', b')\}$. We take all shift operations to have weight 1.

In the analysis, we perform these operations in the following way.

1. We first apply a sequence of row and column merge operations. For these operations, we never merge a row or column which was part of a previous merge operation. The order in which we apply these operations does not matter, so we can assume that we first apply all of the column merge operations and then apply the row merge operations.

   These merge operations correspond to creating the parts in the partitions $\mathcal{P}_r$ and $\mathcal{P}_c$.

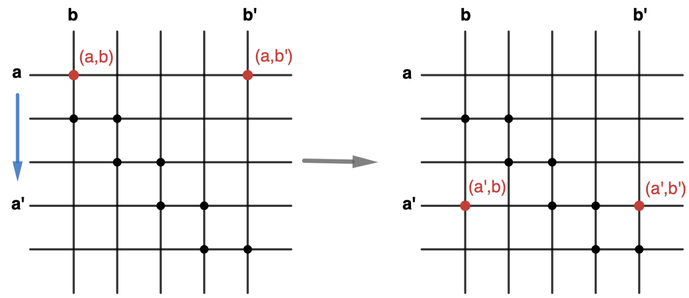


Figure C.8: Illustration of Definition C.4.2: Shift operation $\{(a, b), (a, b')\} \to \{(a', b), (a', b')\}$.

2. We then iteratively choose a row a, pair up the vertices in that row, and shift some or all of the pairs of vertices to different rows which have not yet been chosen.

   A shift operation $\{(a, b), (a, b')\} \to \{(a', b), (a', b')\}$ corresponds to the folding operation $\vec{v}_a \sim \vec{v}_{a'}$ in the proof of Theorem C.1.3.

Since we are taking the limit as $n \to \infty$, we focus on sequences of operations with minimum total weight.

Definition C.4.3. We say that a sequence of row merge, column merge, and shift operations on a starting set of vertices V_0 is *efficient* if it results in a multi-set of vertices where each vertex has multiplicity at least 2 and the total weight of the operations is minimized.

We show that in efficient sequences of operations, each operation is applied to a single staircase and breaks this staircase into smaller staircases.

Definition C.4.4. We define a *staircase of length* j to be a set of vertices of the form

$$\Big\{(a_i, b_i) : i \in [j]\Big\} \cup \Big\{(a_{i+1}, b_i) : i \in [j-1]\Big\} \cup \Big\{(a_1, b_j)\Big\}$$

together with edges $\Big\{\{(a_i, b_i), (a_{i+1}, b_i)\} : i \in [j-1]\Big\} \cup \Big\{\{(a_{i+1}, b_i), (a_{i+1}, b_{i+1})\} : i \in [j-1]\Big\} \cup \Big\{\{(a_j, b_j), (a_1, b_j)\}, \{(a_1, b_j), (a_1, b_1)\}\Big\}$ where $a_1, \ldots, a_j$ are distinct indices in $[k]$ and $b_1, \ldots, b_j$ are distinct indices in $[k]$.

Row merges, column merges, and shifts on a single staircase split this staircase into $w+1$ new staircases where w is the weight of the operation.

1. If we have a staircase on the vertices

$$\{(a_i, b_i) : i \in [j]\} \cup \Big\{(a_{i+1}, b_i) : i \in [j-1]\Big\} \cup \{(a_1, b_j)\}$$

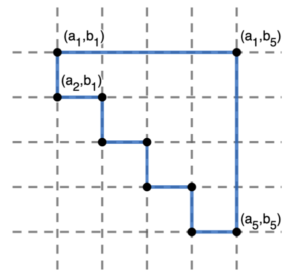


Figure C.9: Illustration of Definition C.4.4.

and apply a row merge operation on the indices $a_{i_1}, \ldots, a_{i_t}$ (where $a_{i_1} < \cdots < a_{i_t}$) then we have the t staircases

$$\left\{\left\{(a_i, b_i) : i \in [i_x, i_{x+1} - 1]\right\} \cup \{(a_{i+1}, b_i) : i \in [i_x, i_{x+1} - 1]\} \cup \left\{(a_{i_x}, b_{i_{x+1}-1}) : x \in [t]\right\}\right\}$$

where we set $a_{i_1} = \cdots = a_{i_t}$, $a_{i+j} = a_i$, and $b_{i+j} = b_i$. Column merges have a similar effect.

2. If we choose a row of the staircase and shift its two vertices to a different row, this has the same effect as merging the two rows.

In order to show Theorems C.1.2, C.2.6, and C.3.4 (i.e. that non-crossing partitions are optimal), we need to show the following. If we start with a staircase of length k, for all efficient sequences of operations,

1. The total weight of the operations is $k - 1$.
2. After all of the operations, we are left with a multi-set of vertices V where each vertex has multiplicity exactly 2.
3. Each operation only affects vertices in a single staircase and breaks this staircase into $w+1$ new staircases where w is the weight of the operation. Thus, at each step we have a set of staircases. Moreover, these staircases are always disjoint, i.e. we never have a vertex which appears multiple times where one copy is in one staircase and another copy is in a different staircase.

We now prove these statements. We start with the second statement.

Lemma C.4.5. *After any efficient sequence of operations, we are left with a multi-set of vertices V where each vertex has multiplicity exactly* 2*.*

Proof. We observe if we have a sequence of operations where we are left with a multi-set of vertices where each vertex has multiplicity at least 2 and some vertex v has multiplicity at least 3 then we can reduce the total weight of the operations as follows.

Let b be the column v is in and modify the sequence of operations so that all of the rows which have a vertex in column b at the end are merged together. Since we have that at the end, the total number of vertices in column b is even, each vertex has multiplicity at least 2, and v has multiplicity at least 3, if we merged j rows together than we must have a vertex of multiplicity at least $2j+2$. Now observe that in order to have a vertex of multiplicity at least $2j+2$, at least $j+1$ columns must have been merged together. By removing this column merge, we can end with vertices with multiplicity 2 and reduce the total weight of the operations by at least j. The additional row merge only increased the total weight of the operations by $j-1$, so the total weight of the operations decreased by at least 1, as needed. ☐

To prove the first and third statements, we add horizontal and vertical edges between the vertices. We show that for any efficient sequence of operations, we can choose these edges so that they give us our disjoint staircases.

Definition C.4.6. Given a multi-set of vertices $V \subseteq \{(a,b) : a,b \in [m]\}$ such that every row and column has an even number of vertices, we define a *row matching* M_r to be a matching between the vertices of V such that every vertex $(a,b) \in V$ is matched with a vertex $(a,b') \in V$ in the same row. Similarly, we define a *column matching* M_c to be a matching between the vertices of V such that every vertex $(a,b) \in V$ is matched with a vertex $(a',b) \in V$ in the same column. If there are multiple copies of the same vertex (a,b) in V then these copies can be matched to each other or to other vertices.

We define (V, M_r, M_c) to be the multi-graph with vertices V and edges $M_r \cup M_c$.

Lemma C.4.7. *Given any sequence of operations where we start with*

$$V_0 = \{(i,i) : i \in [k]\} \cup \{(i+1,i) : i \in [k-1]\} \cup \{(1,k)\}$$

and end with k vertices of multiplicity 2*, we can choose a row matching M_r and a column matching M_c for each step such that*

1. *If we have a row merge operation where we merge rows $a_{i_1}, \ldots, a_{i_t}$ then before the merge, M_r has the two vertices in each row a_{i_x} paired up and after the merge, M_r has a matching between these* $2t$ *vertices. No other edges of the row matching M_r are changed and the column matching M_c is unchanged.*

   The analogous statement holds for column merges.

2. *If we have a shift operation* $\{(a,b),(a,b')\} \to \{(a',b),(a',b')\}$ *then before the shift operation,* $\{(a,b),(a,b')\} \in M_r$*. After the shift operation, we either have the row matching*

$$\left(M_r \setminus \left\{\{(a,b),(a,b')\}\right\}\right) \cup \left\{\{(a',b),(a',b')\}\right\}$$

or the row matching

$$\left(M_r \setminus \left\{\{(a,b),(a,b')\},\{(a',b''),(a',b''')\}\right\}\right) \cup \left\{\{(a',b),(a',b'')\},\{(a',b'),(a',b''')\}\right\}$$

where $\{(a',b''),(a',b''')\}$ *is another edge which was in* M_r*. The column matching* M_c *is unchanged. See Figure C.10 and Figure C.11 for an illustration (we drew these figures from right to left because in the proof of Lemma C.4.7 we will construct the row matchings by starting at the end and working backwards).*

3. *At the beginning,* (V_0, M_r, M_c) *is a staircase of length* k*. At the end,* M_r *and* M_c *consist of loops on the* k *vertices of multiplicity* 2 *(i.e. each vertex is paired with the other copy of that vertex).*

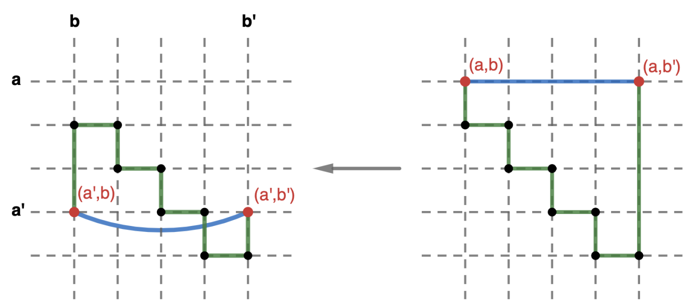


Figure C.10: Illustration of Lemma C.4.7, 2: After we shift row a to row a', we have the row matching $\left(M_r \setminus \left\{\{(a,b),(a,b')\}\right\}\right) \cup \left\{\{(a',b),(a',b')\}\right\}$. Note that while this is allowed in Lemma C.4.7, it will never happen in any efficient sequence of operations because the staircase is still one connected component.

Proof. To prove this lemma, we start from the end and work backwards. More precisely, at the end we take M_r and M_c to be the row and column matchings which consist of loops on the k vertices of multiplicity 2. For each operation, we show that given the row and column matchings after the operation, we can find row and column matchings before the operation which satisfy the needed conditions.

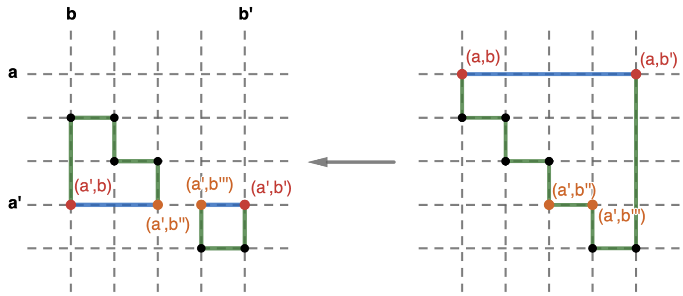


Figure C.11: Illustration of Lemma C.4.7, 2: Shifting row a to row a', we get the row matching $\Big(M_r \setminus \Big\{\{(a,b),(a,b')\},\{(a',b''),(a',b''')\}\Big\}\Big) \cup \Big\{\{(a',b),(a',b'')\},\{(a',b'),(a',b''')\}\Big\}$.

1. If we perform a shift operation $\{(a,b),(a,b')\} \to \{(a',b),(a',b')\}$ and after the shift we have that $\{(a',b),(a',b')\} \in M_r$ then we can take the row matching

$$\Big(M_r \setminus \Big\{\{(a',b),(a',b')\}\Big\}\Big) \cup \Big\{\{(a,b),(a,b')\}\Big\}$$

before the shift. Otherwise, we have that after the shift, $\{(a',b),(a',b'')\},\{(a',b'),(a',b''')\} \in M_r$ for some b'', b''' so we can take the row matching

$$\Big(M_r \setminus \Big\{\{(a',b),(a',b'')\},\{(a',b'),(a',b''')\}\Big\}\Big) \cup \Big\{\{(a,b),(a,b')\},\{(a',b''),(a',b''')\}\Big\}$$

before the shift.

2. If we have a row merge operation where we merge rows $a_{i_1},\dots,a_{i_t}$ then the only option for how to pair up the vertices in these rows in M_r before the merge is to pair up the two vertices in each row a_{i_x}. We can leave the column matching M_c and the edges in M_r not involving the vertices in rows $a_{i_1},\dots,a_{i_t}$ unchanged.

   We handle column merges in a similar way.

Finally, we observe that at the start, each row and column have exactly two vertices so there is only one choice for M_r and M_c and this gives the staircase of length k. □

With these row and column matchings, we can now prove the first and third statements.

Corollary C.4.8. *Efficient sequences of operations on a staircase of length k have total weight $k-1$.*

Proof. We make the following observations about how the multi-graph (V, M_r, M_c) changes at each step.

1. There is one connected component at the start and k connected components at the end.
2. Each operation increases the number of connected components by at most w where w is the weight of the operation.

These observations immediately imply that the sequence of operations must have total weight at least $k - 1$, as needed. □

Corollary C.4.9. *For any efficient sequence of operations on a staircase of length k, each operation only affects vertices in a single staircase and breaks this staircase into $w + 1$ new staircases where w is the weight of the operation. Thus, at each step we have a set of staircases. Moreover, these staircases are always disjoint, i.e. we never have a vertex which appears multiple times where one copy is in one staircase and another copy is in a different staircase.*

Proof. To prove this statement, we consider how the multi-graph (V, M_r, M_c) changes at each step. We show that for each operation of weight w, the only way for this operation to increase the number of connected components by w is if it acts on a single staircase and breaks this staircase into $w + 1$ disjoint staircases. We have the following cases:

1. If we have a row merge operation where we merge rows $a_{i_1}, \ldots, a_{i_t}$ then consider the connected components involving the vertices in these rows. In order for the number of connected components to increase by $t - 1$, there must be exactly one connected component before the merge and exactly t connected components after the merge. Before the merge, we have a collection of disjoint staircases, so all vertices in these rows must be in the same staircase. Since we are acting on a single staircase, it is not hard to show that there is a unique choice for how to pair these vertices up after the merge which results in t connected components and this choice splits the original staircase into t disjoint staircases.

   Similar logic applies to column merges.

2. If we have a shift operation $\{(a, b), (a, b')\} \to \{(a', b), (a', b')\}$, then before the shift we have that $\{(a, b), (a, b')\} \in M_r$ because of the way that we chose M_r and M_c. After the shift, we will either have the row matching

$$\left(M_r \setminus \Big\{\{(a, b), (a, b')\}\Big\}\right) \cup \Big\{\{(a', b), (a', b')\}\Big\}$$

or the row matching

$$\Big(M_r \setminus \Big\{\{(a,b),(a,b')\},\{(a',b''),(a',b''')\}\Big\}\Big) \cup \Big\{\{(a',b),(a',b'')\},\{(a',b'),(a',b''')\}\Big\}$$

for some edge $\{(a',b''),(a',b''')\} \in M_r$. However, the only way that the shift operation can increase the number of connected components is if two edges in the same staircase are removed. This uniquely determines $\{(a',b''),(a',b''')\}$ (assuming that the staircase has row a', otherwise the shift operation cannot increase the number of connected components). There are now two choices for b'' and b''' as these indices can be swapped. One of these choices keeps the staircase as a single connected component while the other splits the staircase into two staircases. Thus, there is a unique choice for M_r after the shift which increases the number of connected components and this choice breaks the staircase containing $\{(a,b),(a,b')\}$ into two staircases.

Finally, we show that we never have two different staircases with the same vertex. To see this, assume that there is an efficient sequence of operations such that at some point, there are two different staircases which each have a copy of some vertex (a,b) and consider the first point where this occurs. Observe that we can swap these copies for the remainder of the sequence, both in terms of which vertices are chosen for shift operations and the edges in M_r and M_c. If we make this swap then looking at the previous step, we will have a different column matching M_c. However, we showed above that for efficient sequences of operations, M_r and M_c are uniquely determined at each step by the operation we perform. This is a contradiction, so we cannot have two different staircases with the same vertex. $\square$

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