Supplementary material - Anomalous Hall effect in single-band chiral superconductors from impurity superlattices

Yu Li,[1,2] Zhiqiang Wang,[3,4] and Wen Huang[1,*]

[1]Shenzhen Institute for Quantum Science and Engineering & Guangdong Provincial Key Laboratory of Quantum Science and Engineering, Southern University of Science and Technology, Shenzhen 518055, Guangdong, China

[2]Kavli Institute for Theoretical Sciences, University of Chinese Academy of Sciences, Beijing 100190, China

[3]Department of Physics and Astronomy, McMaster University, Hamilton, Ontario L8S 4M1, Canada

[4]James Franck Institute, University of Chicago, Chicago, Illinois 60637, USA

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I. IMPURITY-INDUCED BOUND STATES IN CHIRAL SUPERCONDUCTORS

The BdG Hamiltonian for a two-dimensional chiral superconductor with impurities can be written as

$$H_{\mathrm{BdG}} = H_{\mathrm{BdG}}^{(\mathrm{bulk})} + H^{(\mathrm{imp})}, \tag{S1}$$

which is expressed in the Nambu space spanned by the spinor $\phi_{\mathbf{k}} = (c_{\mathbf{k}}, c_{-\mathbf{k}}^{\dagger})^{\intercal}$, where $c_{\mathbf{k}}$ ($c_{\mathbf{k}}^{\dagger}$) is the electron annihilation (creation) operator. In the continuum limit, the bulk Hamiltonian in the momentum space, $H_{\mathbf{k}}^{(\mathrm{bulk})}$, can be expanded in terms of the Pauli matrices τ_i ($i = 1, 2, 3$) as

$$H_{\mathrm{BdG}}^{(\mathrm{bulk})} = \int \frac{d\mathbf{k}}{(2\pi)^2} H_{\mathbf{k}}^{(\mathrm{bulk})} = \int \frac{d\mathbf{k}}{(2\pi)^2} \left[\epsilon_{\mathbf{k}}\tau_3 + \mathrm{Re}\left(\Delta_{\mathbf{k}}\right)\tau_1 - \mathrm{Im}\left(\Delta_{\mathbf{k}}\right)\tau_2\right]. \tag{S2}$$

$\epsilon_{\mathbf{k}} = \mathbf{k}^2/2m - \varepsilon_{\mathrm{F}}$ is the dispersion of electrons relative to the Fermi energy ε_{F}. $\Delta_{\mathbf{k}} = \Delta e^{il\theta_{\mathbf{k}}}$ is the gap function of the chiral pairing, where $\theta_{\mathbf{k}}$ is the azimuthal angle of $\mathbf{k}$, and l and Δ represent, respectively, the Cooper pair angular quantum number and the k independent gap magnitude.

We first solve a single-impurity problem, with a delta-function potential with strength U located at the origin,

$$H^{(\mathrm{imp})}\left(\mathbf{r}\right) = U\tau_3\delta\left(\mathbf{r}\right). \tag{S3}$$

The equation to be solved is $H_{\mathrm{BdG}}\psi\left(\mathbf{r}\right) = E\psi\left(\mathbf{r}\right)$, where E is the eigenvalue. Performing a Fourier transformation $\psi\left(\mathbf{r}\right) = \int \frac{d\mathbf{k}}{(2\pi)^2} e^{i\mathbf{k}\cdot\mathbf{r}}\psi_{\mathbf{k}}$, one obtains

$$\left[E - H_{\mathbf{k}}^{(\mathrm{bulk})}\right]\psi_{\mathbf{k}} = U\tau_3\psi\left(0\right). \tag{S4}$$

Transformed back into the real space, the wavefunction becomes

$$\psi\left(\mathbf{r}\right) = UG\left(E, \mathbf{r}\right)\tau_3\psi\left(0\right), \tag{S5}$$

where $G\left(E, \mathbf{r}\right)$ is the bulk Green's function,

$$\begin{aligned} G\left(E, \mathbf{r}\right) &= \int \frac{d\mathbf{k}}{(2\pi)^2} e^{i\mathbf{k}\cdot\mathbf{r}} \left[E - H_{\mathbf{k}}^{(\mathrm{bulk})}\right]^{-1} = \int \frac{d\mathbf{k}}{(2\pi)^2} e^{i\mathbf{k}\cdot\mathbf{r}} \frac{E\tau_0 + \epsilon_{\mathbf{k}}\tau_3 + \mathrm{Re}\left(\Delta_{\mathbf{k}}\right)\tau_1 - \mathrm{Im}\left(\Delta_{\mathbf{k}}\right)\tau_2}{E^2 - \epsilon_{\mathbf{k}}^2 - \left|\Delta_{\mathbf{k}}\right|^2} \\ &= X_0\tau_0 + X_1\tau_3 + iX_2^{+}\tau_{+} + iX_2^{-}\tau_{-}, \end{aligned} \tag{S6}$$

in which $\tau_{\pm} = \left(\tau_1 \pm i\tau_2\right)/2$, and X_0, X_1, $X_2^{\pm}$ are given by [1, 2],

$$X_0\left(E, \mathbf{r}\right) = -\int \frac{d\mathbf{k}}{(2\pi)^2} \frac{Ee^{i\mathbf{k}\cdot\mathbf{r}}}{\epsilon_{\mathbf{k}}^2 + \left|\Delta_{\mathbf{k}}\right|^2 - E^2} \approx -\frac{2N_{\mathrm{F}}E}{\sqrt{\Delta^2 - E^2}}\,\mathrm{Im}\left\{K_0\left[\left(\kappa - i\right)k_{\mathrm{F}}r\right]\right\}, \tag{S7}$$

$$X_1\left(E, \mathbf{r}\right) = -\int \frac{d\mathbf{k}}{(2\pi)^2} \frac{\epsilon_{\mathbf{k}}e^{i\mathbf{k}\cdot\mathbf{r}}}{\epsilon_{\mathbf{k}}^2 + \left|\Delta_{\mathbf{k}}\right|^2 - E^2} \approx -2N_{\mathrm{F}}\,\mathrm{Re}\left\{K_0\left[\left(\kappa - i\right)k_{\mathrm{F}}r\right]\right\}, \tag{S8}$$

$$X_2^{\pm}(E,\mathbf{r}) = \pm\int \frac{d\mathbf{k}}{(2\pi)^2}\frac{i\Delta e^{\pm il\theta_{\mathbf{k}}}e^{i\mathbf{k}\cdot\mathbf{r}}}{\epsilon_{\mathbf{k}}^2+|\Delta_{\mathbf{k}}|^2-E^2} \approx \pm e^{\pm i(l+1)\pi/2}\frac{2N_{\mathrm{F}}\Delta}{\sqrt{\Delta^2-E^2}}e^{\pm il\theta_{\mathbf{r}}}\,\mathrm{Im}\left\{K_l\left[(\kappa-i)\,k_{\mathrm{F}}r\right]\right\}. \tag{S9}$$

In these expressions, N_{F} is the density of states at ϵ_{F}, $\kappa \equiv \sqrt{\Delta^2-E^2}/(k_{\mathrm{F}}\upsilon_{\mathrm{F}})$, and the function $K_n(x)$ represents the modified Bessel functions of the second kind of order n. Far from the impurity, X_0, X_1, $X_2^{\pm}$ all decay as $e^{-r/\xi}/\sqrt{k_{\mathrm{F}}r}$, in which $\xi = \upsilon_{\mathrm{F}}/\sqrt{\Delta^2-E^2}$ is the effective coherence length. The above equations are valid for $|E| < \Delta$ and $\kappa k_{\mathrm{F}}r = r/\xi \gtrsim 1$. Note that the Bessel functions involved diverge at $r = 0$. Right at $r = 0$, the $\mathbf{k}$ integral can be performed without resorting to Bessel functions, leading to

$$X_0(E,0) = -\frac{\pi N_F E}{\sqrt{\Delta^2-E^2}}, X_1(E,0) = 0, X_2^{\pm}(E,0) = 0. \tag{S10}$$

An ultraviolet energy cut off is needed to regulate the divergence in the $\mathbf{k}$ integrals, in order to obtain the correct behavior of the Green's function at $0 < r/\xi \lesssim 1$. However, we will ignore this short-distance behavior since it is not important for our following discussions.

At $\mathbf{r} = \mathbf{0}$, i.e., right at the impurity site, the eigenvalue equation becomes

$$\left[1-UG(E,0)\,\tau_3\right]\psi(0) = 0. \tag{S11}$$

Using Eqs. (S10) and (S6) we obtain the impurity induced subgap state energies as $E = \pm E_0$ with $E_0 = \Delta/\sqrt{1+\beta^2}$, where $\beta = \pi N_{\mathrm{F}}U$. The two energies are symmetric with respect to $E = 0$, which is not the case in general if the particle-hole asymmetry of the normal state energy dispersion is introduced; also, the expression of E_0 is independent of the sign of U, which needs to be modified if the k-dependence of the gap function is included. However, considering more general cases does not alter the conclusions obtained in the main text. We denote the two eigenvectors corresponding to $E = \pm E_0$ as $\psi_+(\mathbf{r})$ and $\psi_-(\mathbf{r})$, respectively, and consider the $U > 0$ (repulsive)and $U < 0$ (attractive) cases separately in the following.

(1) For $U > 0$ the two eigenvectors at $\mathbf{r} = \mathbf{0}$ are $\psi_+(0) = (1,0)^{\mathsf{T}}$ (particle-like) and $\psi_-(0) = (0,1)^{\mathsf{T}}$ (hole-like). At $\mathbf{r} \neq \mathbf{0}$

$$\begin{aligned}\psi_+(\mathbf{r}) &= \frac{1}{\mathcal{N}}UG(+E_0,\mathbf{r})\,\tau_3\psi_+(0)\\ &= \frac{1}{\sqrt{\left[X_0(E_0,\mathbf{r})+X_1(E_0,\mathbf{r})\right]^2+|X_2^+(E_0,\mathbf{r})|^2}}\begin{pmatrix} X_0(E_0,\mathbf{r})+X_1(E_0,\mathbf{r})\\ iX_2^-(E_0,\mathbf{r})\end{pmatrix} \equiv \begin{pmatrix} u(\mathbf{r})\\ \upsilon(\mathbf{r})\end{pmatrix},\end{aligned} \tag{S12}$$

where $\mathcal{N}$ is a normalization coefficient and, similarly,

$$\psi_-(\mathbf{r}) = \frac{1}{\sqrt{\left[-X_0(-E_0,\mathbf{r})+X_1(-E_0,\mathbf{r})\right]^2+|X_2^+(-E_0,\mathbf{r})|^2}}\begin{pmatrix} -iX_2^+(-E_0,\mathbf{r})\\ -X_0(-E_0,\mathbf{r})+X_1(-E_0,\mathbf{r})\end{pmatrix} = \begin{pmatrix} -\upsilon^*(\mathbf{r})\\ u^*(\mathbf{r})\end{pmatrix}. \tag{S13}$$

Note that $X_0(E,\mathbf{r})$ is odd in E, while $X_1(E,\mathbf{r})$ and $X_2^{\pm}(E,\mathbf{r})$ are both even in E. From Eqns S7-S9 we see that $u(\mathbf{r})$ is real for the given $\psi_+(0)$ and $\psi_-(0)$, and we can write $\psi_+(\mathbf{r}) = (u(\mathbf{r}),\upsilon(\mathbf{r}))^{\mathsf{T}} = \left(u_r, e^{-il\theta_{\mathbf{r}}+\alpha}\upsilon_r\right)^{\mathsf{T}}$, where u_r and υ_r are two real functions of r only, and α is an $\mathbf{r}$-independent phase. For notational simplicity, we will set $\alpha = 0$, which will not qualitatively affect our conclusions.

(2) The eigenvectors for $U < 0$ can be obtained similarly. At $\mathbf{r} = \mathbf{0}$, $\psi_+(0) = (0,1)^{\mathsf{T}}$ (hole-like) and $\psi_-(0) = (1,0)^{\mathsf{T}}$ (particle-like). At $\mathbf{r} \neq \mathbf{0}$,

$$\psi_+(\mathbf{r}) = \frac{1}{\sqrt{\left[-X_0(E_0,\mathbf{r})+X_1(E_0,\mathbf{r})\right]^2+|X_2^+(E_0,\mathbf{r})|^2}}\begin{pmatrix} -iX_2^+(E_0,\mathbf{r})\\ -X_0(E_0,\mathbf{r})+X_1(E_0,\mathbf{r})\end{pmatrix} \equiv \begin{pmatrix} -\upsilon'^*(\mathbf{r})\\ u'^*(\mathbf{r})\end{pmatrix}, \tag{S14}$$

$$\psi_-(\mathbf{r}) = \frac{1}{\sqrt{\left[X_0(-E_0,\mathbf{r})+X_1(-E_0,\mathbf{r})\right]^2+|X_2^+(-E_0,\mathbf{r})|^2}}\begin{pmatrix} X_0(-E_0,\mathbf{r})+X_1(-E_0,\mathbf{r})\\ iX_2^-(-E_0,\mathbf{r})\end{pmatrix} = \begin{pmatrix} u'(\mathbf{r})\\ \upsilon'(\mathbf{r})\end{pmatrix}. \tag{S15}$$

Again, $u'(\mathbf{r})$ is real, and we can write $\psi_-(\mathbf{r}) = (u'(\mathbf{r}),\upsilon'(\mathbf{r}))^{\mathsf{T}} = \left(u'_r, e^{-il\theta_{\mathbf{r}}+i\alpha}\upsilon'_r\right)^{\mathsf{T}}$, where u'_r and υ'_r are real functions of r, and α is again a constant phase we will set to be zero without altering our conclusions.

In the main text and in the following discussions, we only consider the case with repulsive U. The attractive-U scenario produces similar physics.

II. LOW-ENERGY EFFECTIVE MODEL OF THE IMPURITY SUPERLATTICE

In an impurity lattice, the bound states from different impurity sites hybridize through the kinetic hopping and Cooper pairing in the original microscopic BdG Hamiltonian, forming subgap bands. Treating the '$+$' and '$-$' bound states on each impurity site as two independent orbitals, we now construct an effective tight-binding Hamiltonian for the subgap states on an impurity lattice. In the second-quantization formulation, the creation (annihilation) of the orbitals are denoted by the operators $c_{\pm}^{\dagger}$ $(c_{\pm})$. We first consider a two-impurity system with impurities located at $\mathbf{R}_i$ and $\mathbf{R}_j$. In the basis $\hat{\Psi}_i = (c_{i,+}, c_{i,-})^{\intercal}$ where i is the site index, the emergent effective Hamiltonian reads $H = \sum_{i,j} \hat{\Psi}_i^{\dagger} \left[E_0 \delta_{ij} \sigma_3 + \hat{h}_{ij} (1 - \delta_{ij})\right] \hat{\Psi}_j$+H.c., in which

$$h_{ij} = \begin{pmatrix} t_{ij}^{++} & t_{ij}^{+-} \\ t_{ij}^{-+} & t_{ij}^{--} \end{pmatrix}, \tag{S16}$$

where

$$t_{ij}^{\mu\nu} = \int d\mathbf{r} d\mathbf{r}' \psi_{\mu}^{\dagger} (\mathbf{r} - \mathbf{R}_i) H_{\text{BdG}}^{(\text{bulk})} (\mathbf{r}, \mathbf{r}') \psi_{\nu} (\mathbf{r}' - \mathbf{R}_j), \tag{S17}$$

and $\mu, \nu = +, -$. Explicitly,

$$\begin{aligned} t_{ij}^{++} = \int d\mathbf{r} d\mathbf{r}' \bigg\{ & u_{|\mathbf{r}-\mathbf{R}_i|} \left[\delta_{\mathbf{r},\mathbf{r}'} \left(-\frac{\nabla_{\mathbf{r}'}^2}{2m_e} - \mu\right)\right] u_{|\mathbf{r}'-\mathbf{R}_j|} + u_{|\mathbf{r}-\mathbf{R}_i|} \Delta (\mathbf{r} - \mathbf{r}') e^{-il\theta_{\mathbf{r}'-\mathbf{R}_j}} v_{|\mathbf{r}'-\mathbf{R}_j|} \\ & + e^{il\theta_{\mathbf{r}-\mathbf{R}_i}} v_{|\mathbf{r}-\mathbf{R}_i|} \Delta^* (\mathbf{r} - \mathbf{r}') u_{|\mathbf{r}'-\mathbf{R}_j|} + e^{il\theta_{\mathbf{r}-\mathbf{R}_i}} v_{|\mathbf{r}-\mathbf{R}_i|} \left[\delta_{\mathbf{r}',\mathbf{r}} \left(\frac{\nabla_{\mathbf{r}'}^2}{2m_e} + \mu\right)\right] e^{-il\theta_{\mathbf{r}'-\mathbf{R}_j}} v_{|\mathbf{r}'-\mathbf{R}_j|} \bigg\}, \end{aligned} \tag{S18}$$

$$\begin{aligned} t_{ij}^{+-} = \int d\mathbf{r} d\mathbf{r}' \bigg\{ & -u_{|\mathbf{r}-\mathbf{R}_i|} \left[\delta_{\mathbf{r},\mathbf{r}'} \left(-\frac{\nabla_{\mathbf{r}'}^2}{2m_e} - \mu\right)\right] e^{il\theta_{\mathbf{r}'-\mathbf{R}_j}} v_{|\mathbf{r}'-\mathbf{R}_j|} + u_{|\mathbf{r}-\mathbf{R}_i|} \Delta (\mathbf{r} - \mathbf{r}') u_{|\mathbf{r}'-\mathbf{R}_j|} \\ & - e^{il\theta_{\mathbf{r}-\mathbf{R}_i}} v_{|\mathbf{r}-\mathbf{R}_i|} \Delta^* (\mathbf{r} - \mathbf{r}') e^{il\theta_{\mathbf{r}'-\mathbf{R}_j}} v_{|\mathbf{r}'-\mathbf{R}_j|} + e^{il\theta_{\mathbf{r}-\mathbf{R}_i}} v_{|\mathbf{r}-\mathbf{R}_i|} \left[\delta_{\mathbf{r}',\mathbf{r}} \left(\frac{\nabla_{\mathbf{r}'}^2}{2m_e} + \mu\right)\right] u_{|\mathbf{r}'-\mathbf{R}_j|} \bigg\}, \end{aligned} \tag{S19}$$

$$\begin{aligned} t_{ij}^{-+} = \int d\mathbf{r} d\mathbf{r}' \bigg\{ & -e^{-il\theta_{\mathbf{r}-\mathbf{R}_i}} v_{|\mathbf{r}-\mathbf{R}_i|} \left[\delta_{\mathbf{r},\mathbf{r}'} \left(-\frac{\nabla_{\mathbf{r}'}^2}{2m_e} - \mu\right)\right] u_{|\mathbf{r}'-\mathbf{R}_j|} - e^{-il\theta_{\mathbf{r}-\mathbf{R}_i}} v_{|\mathbf{r}-\mathbf{R}_i|} \Delta (\mathbf{r} - \mathbf{r}') e^{-il\theta_{\mathbf{r}'-\mathbf{R}_j}} v_{|\mathbf{r}'-\mathbf{R}_j|} \\ & + u_{|\mathbf{r}-\mathbf{R}_i|} \Delta^* (\mathbf{r} - \mathbf{r}') u_{|\mathbf{r}'-\mathbf{R}_j|} + u_{|\mathbf{r}-\mathbf{R}_i|} \left[\delta_{\mathbf{r}',\mathbf{r}} \left(\frac{\nabla_{\mathbf{r}'}^2}{2m_e} + \mu\right)\right] e^{-il\theta_{\mathbf{r}'-\mathbf{R}_j}} v_{|\mathbf{r}'-\mathbf{R}_j|} \bigg\}, \end{aligned} \tag{S20}$$

$$\begin{aligned} t_{ij}^{--} = \int d\mathbf{r} d\mathbf{r}' \bigg\{ & e^{-il\theta_{\mathbf{r}-\mathbf{R}_i}} v_{|\mathbf{r}-\mathbf{R}_i|} \left[\delta_{\mathbf{r},\mathbf{r}'} \left(-\frac{\nabla_{\mathbf{r}'}^2}{2m_e} - \mu\right)\right] e^{il\theta_{\mathbf{r}'-\mathbf{R}_j}} v_{|\mathbf{r}'-\mathbf{R}_j|} - e^{-il\theta_{\mathbf{r}-\mathbf{R}_i}} v_{|\mathbf{r}-\mathbf{R}_i|} \Delta (\mathbf{r} - \mathbf{r}') u_{|\mathbf{r}'-\mathbf{R}_j|} \\ & - u_{|\mathbf{r}-\mathbf{R}_i|} \Delta^* (\mathbf{r} - \mathbf{r}') e^{il\theta_{\mathbf{r}'-\mathbf{R}_j}} v_{|\mathbf{r}'-\mathbf{R}_j|} + u_{|\mathbf{r}-\mathbf{R}_i|} \left[\delta_{\mathbf{r}',\mathbf{r}} \left(\frac{\nabla_{\mathbf{r}'}^2}{2m_e} + \mu\right)\right] u_{|\mathbf{r}'-\mathbf{R}_j|} \bigg\}. \end{aligned} \tag{S21}$$

From these expressions, one can easily obtain the relations, $t_{ij}^{++} = -\left(t_{ij}^{--}\right)^* \equiv \lambda_{ij}$, $t_{ij}^{+-} = \left(t_{ij}^{-+}\right)^{\dagger} \equiv \eta_{ij}$. The hybridization has three distinct origins: electron-electron hopping, hole-hole hopping and Cooper pairing. Hence we decompose the hopping terms as $\lambda_{ij} = \lambda_{ij}^{e} + \lambda_{ij}^{h} + \lambda_{ij}^{\Delta}$, and $\eta_{ij} = \eta_{ij}^{e} + \eta_{ij}^{h} + \eta_{ij}^{\Delta}$, the details of which we provide below.

Symmetry aspects of the hybridization matrix elements

By changing the variables, one can easily find that λ_{ij} and η_{ij} depend on the relative position of $\mathbf{R}_i$ and $\mathbf{R}_j$, i.e., $\lambda_{ij} \equiv \lambda (\mathbf{R}_j - \mathbf{R}_i)$, $\eta_{ij} \equiv \eta (\mathbf{R}_j - \mathbf{R}_i)$. Define $\mathbf{R}_{\delta} = \mathbf{R}_j - \mathbf{R}_i$, the expressions for $\lambda (\mathbf{R}_{\delta})$ and $\eta (\mathbf{R}_{\delta})$ can be reduced as

$$\begin{aligned}\lambda\left(\mathbf{R}_{\delta}\right)=&\lambda^{e}\left(\mathbf{R}_{\delta}\right)+\lambda^{h}\left(\mathbf{R}_{\delta}\right)+\lambda^{\Delta}\left(\mathbf{R}_{\delta}\right)\\ =&\int d\mathbf{r}d\mathbf{r}'\left\{u_{r}\left[\delta_{\mathbf{r},\mathbf{r}'-\mathbf{R}_{\delta}}\left(-\frac{\nabla_{\mathbf{r}'}^{2}}{2m_{e}}-\mu\right)\right]u_{r'}+e^{il\theta_{\mathbf{r}}}\upsilon_{r}\left[\delta_{\mathbf{r}'-\mathbf{R}_{\delta},\mathbf{r}}\left(\frac{\nabla_{\mathbf{r}'}^{2}}{2m_{e}}+\mu\right)\right]e^{-il\theta_{\mathbf{r}'}}\upsilon_{r'}\right.\\ &\left.+2\,\mathrm{Re}\left[\Delta\left(\mathbf{r}-\mathbf{r}'-\mathbf{R}_{\delta}\right)e^{-il\theta_{\mathbf{r}'}}\right]u_{r}\upsilon_{r'}\right\},\end{aligned}\tag{S22}$$

$$\begin{aligned}\eta\left(\mathbf{R}_{\delta}\right)=&\eta^{e}\left(\mathbf{R}_{\delta}\right)+\eta^{h}\left(\mathbf{R}_{\delta}\right)+\eta^{\Delta}\left(\mathbf{R}_{\delta}\right)\\ =&\int d\mathbf{r}d\mathbf{r}'\left\{\left(1+e^{il\pi}\right)\int d\mathbf{r}d\mathbf{r}'u_{r}\delta_{\mathbf{r},\mathbf{r}'-\mathbf{R}_{\delta}}\left(\frac{\nabla_{\mathbf{r}'}^{2}}{2m_{e}}+\mu\right)e^{il\theta_{\mathbf{r}'}}\upsilon_{r'}+\Delta\left(\mathbf{r}-\mathbf{r}'-\mathbf{R}_{\delta}\right)u_{r}u_{r'}\right.\\ &\left.-e^{il\left(\theta_{\mathbf{r}}+\theta_{\mathbf{r}'}\right)}\Delta^{*}\left(\mathbf{r}-\mathbf{r}'-\mathbf{R}_{\delta}\right)\upsilon_{r}\upsilon_{r'}\right\}.\end{aligned}\tag{S23}$$

To inspect the dependence of $\lambda\left(\mathbf{R}_{\delta}\right)$ and $\eta\left(\mathbf{R}_{\delta}\right)$ on the orientation of $\mathbf{R}_{\delta}$, let us perform a rotation ($\hat{R}$) of arbitrary angle ϕ. Then,

$$\begin{aligned}\lambda\left(\hat{R}_{\phi}\mathbf{R}_{\delta}\right)=&\int d\mathbf{r}d\mathbf{r}'\left\{u_{r}\left[\delta_{\mathbf{r},\mathbf{r}'-\hat{R}_{\phi}\mathbf{R}_{\delta}}\left(-\frac{\nabla_{\mathbf{r}'}^{2}}{2m_{e}}-\mu\right)\right]u_{r'}+e^{il\theta_{\mathbf{r}}}\upsilon_{r}\left[\delta_{\mathbf{r}'-\hat{R}_{\phi}\mathbf{R}_{\delta},\mathbf{r}}\left(\frac{\nabla_{\mathbf{r}'}^{2}}{2m_{e}}+\mu\right)\right]e^{-il\theta_{r'}}\upsilon_{r'}\right.\\ &\left.+2\,\mathrm{Re}\left[\Delta\left(\mathbf{r}-\mathbf{r}'-\hat{R}_{\phi}\mathbf{R}_{\delta}\right)e^{-il\theta_{\mathbf{r}'}}\right]u_{r}\upsilon_{r'}\right\}\\ =&\int d\mathbf{r}d\mathbf{r}'\left\{u_{r}\left[\delta_{\hat{R}_{\phi}\mathbf{r},\hat{R}_{\phi}\left(\mathbf{r}'-\mathbf{R}_{\delta}\right)}\left(-\frac{\nabla_{\hat{R}_{\phi}\mathbf{r}'}^{2}}{2m_{e}}-\mu\right)\right]u_{r'}+e^{il\theta_{\hat{R}_{\phi}\mathbf{r}}}\upsilon_{r}\left[\delta_{\hat{R}_{\phi}\left(\mathbf{r}'-\mathbf{R}_{\delta}\right),\hat{R}_{\phi}\mathbf{r}}\left(\frac{\nabla_{\hat{R}_{\phi}\mathbf{r}'}^{2}}{2m_{e}}+\mu\right)\right]\right.\\ &\left.\times e^{-il\theta_{\hat{R}_{\phi}r'}}\upsilon_{r'}+2\,\mathrm{Re}\left[\Delta\left(\hat{R}_{\phi}\left(\mathbf{r}-\mathbf{r}'-\mathbf{R}_{\delta}\right)\right)e^{-il\theta_{\hat{R}_{\phi}\mathbf{r}'}}\right]u_{r}\upsilon_{r'}\right\}\\ =&\int d\mathbf{r}d\mathbf{r}'\left\{u_{r}\left[\delta_{\mathbf{r},\left(\mathbf{r}'-\mathbf{R}_{\delta}\right)}\left(-\frac{\nabla_{\hat{R}_{\phi}\mathbf{r}'}^{2}}{2m_{e}}-\mu\right)\right]u_{r'}+e^{il\phi}e^{il\theta_{\mathbf{r}}}\upsilon_{r}\left[\delta_{\hat{R}_{\phi}\left(\mathbf{r}'-\mathbf{R}_{\delta}\right),\hat{R}_{\phi}\mathbf{r}}\left(\frac{\nabla_{\hat{R}_{\phi}\mathbf{r}'}^{2}}{2m_{e}}+\mu\right)\right]\right.\\ &\left.\times e^{-il\phi}e^{-il\theta_{r'}}\upsilon_{r'}+2\,\mathrm{Re}\left[e^{il\phi}\Delta\left(\mathbf{r}-\mathbf{r}'-\mathbf{R}_{\delta}\right)e^{-il\phi}e^{-il\theta_{\mathbf{r}'}}\right]u_{r}\upsilon_{r'}\right\}\\ =&\lambda\left(\mathbf{R}_{\delta}\right),\end{aligned}\tag{S24}$$

which is independent of the orientation of $\mathbf{R}_{\delta}$, i.e., $\lambda\left(\mathbf{R}_{\delta}\right)=\lambda\left(\left|\mathbf{R}_{\delta}\right|\right)$. And

$$\begin{aligned}\eta\left(\hat{R}_{\phi}\mathbf{R}_{\delta}\right)=&\int d\mathbf{r}d\mathbf{r}'\left\{\Delta\left(\mathbf{r}-\mathbf{r}'+\hat{R}_{\phi}\mathbf{R}_{\delta}\right)u_{r}u_{r'}-e^{il\left(\theta_{\mathbf{r}}+\theta_{\mathbf{r}'}\right)}\Delta^{*}\left(\mathbf{r}-\mathbf{r}'+\hat{R}_{\phi}\mathbf{R}_{\delta}\right)\upsilon_{r}\upsilon_{r'}\right.\\ &\left.+\left(1+e^{il\pi}\right)\int d\mathbf{r}d\mathbf{r}'u_{r}\left[\delta_{\mathbf{r},\mathbf{r}'+\hat{R}_{\phi}\mathbf{R}_{\delta}}\left(\frac{\nabla_{\mathbf{r}'}^{2}}{2m_{e}}+\mu\right)\right]e^{il\theta_{\mathbf{r}'}}\upsilon_{r'}\right\}\\ =&\int d\mathbf{r}d\mathbf{r}'\left\{\Delta\left(\hat{R}_{\phi}\left(\mathbf{r}-\mathbf{r}'+\mathbf{R}_{\delta}\right)\right)u_{r}u_{r'}-e^{il\left(\theta_{\hat{R}_{\phi}\mathbf{r}}+\theta_{\hat{R}_{\phi}\mathbf{r}'}\right)}\Delta^{*}\left(\hat{R}_{\phi}\left(\mathbf{r}-\mathbf{r}'+\mathbf{R}_{\delta}\right)\right)\upsilon_{r}\upsilon_{r'}\right.\\ &\left.+\left(1+e^{il\pi}\right)\int d\mathbf{r}d\mathbf{r}'u_{r}\left[\delta_{\hat{R}_{\phi}\mathbf{r},\hat{R}_{\phi}\left(\mathbf{r}'-\mathbf{R}_{\delta}\right)}\left(\frac{\nabla_{\hat{R}_{\phi}\mathbf{r}'}^{2}}{2m_{e}}+\mu\right)\right]e^{il\theta_{\mathbf{r}'}}\upsilon_{r'}\right\}\\ =&\int d\mathbf{r}d\mathbf{r}'\left\{e^{il\phi}\Delta\left(\mathbf{r}-\mathbf{r}'+\mathbf{R}_{\delta}\right)u_{r}u_{r'}-e^{i2l\phi}e^{il\left(\theta_{\mathbf{r}}+\theta_{\mathbf{r}'}\right)}e^{-il\phi}\Delta^{*}\left(\mathbf{r}-\mathbf{r}'+\mathbf{R}_{\delta}\right)\upsilon_{r}\upsilon_{r'}\right.\\ &\left.+\left(1+e^{il\pi}\right)\int d\mathbf{r}d\mathbf{r}'u_{r}\left[\delta_{\mathbf{r},\mathbf{r}'+\mathbf{R}_{\delta}}\left(\frac{\nabla_{\mathbf{r}'}^{2}}{2m_{e}}+\mu\right)\right]e^{il\phi}e^{il\theta_{\mathbf{r}'}}\upsilon_{r'}\right\}\\ =&e^{il\phi}\eta\left(\mathbf{R}_{\delta}\right).\end{aligned}\tag{S25}$$

Thus the off-diagonal matrix element t^{+-} inherits the rotaional symmetry property of the chiral pairing in the original bulk BdG Hamiltonian.

Furthermore, in the hybridization between the '$+$' and '$-$' states, the contribution from the electron and hole kinetic processes, $\eta_{ij}^{e}+\eta_{ij}^{h}$, are sensitive to the parity of the Cooper pairing: $\eta_{ij}^{e}+\eta_{ij}^{h}$ vanishes in odd-parity pairing and is finite in even-parity

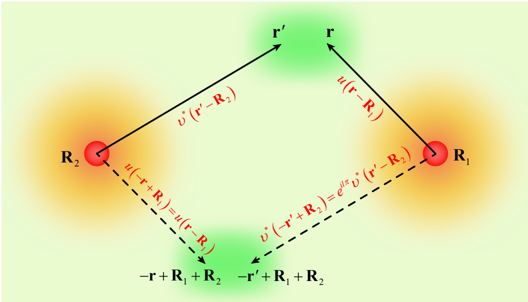

Figure S1: Schematic diagram showing the relation between the integrand of Eq. (4) of the main text at two sets of variables: $(\mathbf{r}, \mathbf{r}')$ indicated by solid arrows and $[\mathbf{R}_i - (\mathbf{r}' - \mathbf{R}_j), \mathbf{R}_j - (\mathbf{r} - \mathbf{R}_i)]$ in dashed arrows. These two sets are related by a 180° rotation about $(\mathbf{R}_1 + \mathbf{R}_2)/2$.

pairing. This is more obvious in the following expression,

$$\begin{aligned}
\eta_{ij}^e + \eta_{ij}^h &= \int d\mathbf{r}d\mathbf{r}' \left\{ -u_{|\mathbf{r}-\mathbf{R}_i|} \left[\delta_{\mathbf{r},\mathbf{r}'} \left(-\frac{\nabla_{\mathbf{r}'}^2}{2m_e} - \mu \right) \right] e^{il\theta_{\mathbf{r}'-\mathbf{R}_j}} \upsilon_{|\mathbf{r}'-\mathbf{R}_j|} + e^{il\theta_{\mathbf{r}-\mathbf{R}_i}} \upsilon_{|\mathbf{r}-\mathbf{R}_i|} \left[\delta_{\mathbf{r}',\mathbf{r}} \left(\frac{\nabla_{\mathbf{r}'}^2}{2m_e} + \mu \right) \right] u_{|\mathbf{r}'-\mathbf{R}_j|} \right\} \\
&= \int d\mathbf{r}d\mathbf{r}' \left\{ u_{|\mathbf{r}-\mathbf{R}_i|} \left[\delta_{\mathbf{r},\mathbf{r}'} \left(\frac{\nabla_{\mathbf{r}'}^2}{2m_e} + \mu \right) \right] e^{il\theta_{\mathbf{r}'-\mathbf{R}_j}} \upsilon_{|\mathbf{r}'-\mathbf{R}_j|} + e^{il\theta_{-\mathbf{r}'+\mathbf{R}_j}} \upsilon_{|-\mathbf{r}'+\mathbf{R}_j|} \left[\delta_{\mathbf{r},\mathbf{r}'} \left(\frac{\nabla_{\mathbf{r}}^2}{2m_e} + \mu \right) \right] u_{|-\mathbf{r}+\mathbf{R}_i|} \right\} \\
&= \left(1 + e^{il\pi}\right) \int d\mathbf{r}d\mathbf{r}' u_r \left[\delta_{\mathbf{r},\mathbf{r}'+\mathbf{R}_j-\mathbf{R}_i} \left(\frac{\nabla_{\mathbf{r}'}^2}{2m_e} + \mu \right) \right] e^{il\theta_{\mathbf{r}'}} \upsilon_{r'},
\end{aligned} \tag{S26}$$

which vanishes for odd l's. To obtain the second equation, we made a substitution of variables, $\mathbf{r} \to \mathbf{R}_i - (\mathbf{r}' - \mathbf{R}_j)$ and $\mathbf{r}' \to \mathbf{R}_j - (\mathbf{r} - \mathbf{R}_i)$. Pictorially, the two terms in the integrand of the second line are depicted in Fig. S1. The final expression was obtained after a partial integration and a substitution of variable.

III. EFFECTIVE TIGHT-BINDING HAMILTONIAN & ANOMALOUS HALL CONDUCTIVITY

We are now in position to formally construct the effective tight-binding Hamiltonian for square, triangular and honeycomb superlattices, and study their anomalous Hall effects.

A). Square impurity superlattice

Let us first consider the case with underlying chiral p-wave pairing. Following Fig. 2 (a) and (b) and by Fourier transformation, in square superlattice, the hybridization matrix in the momentum space can be expressed as $\hat{h}_{\mathbf{k}} = \sum_\delta e^{i\mathbf{k}\cdot\mathbf{R}_\delta} \hat{h}\left(\mathbf{R}_\delta\right)$, in which the matrix elements with only considering the nearest-neighbor terms are expressed as

$$\lambda_{\mathbf{k}} = 2\lambda \left(\cos k_x + \cos k_y\right), \ \eta_{\mathbf{k}} = 2\eta \left(\sin k_x + i \sin k_y\right), \tag{S27}$$

where $\lambda \equiv \lambda(R_0)$, $\eta \equiv \eta(R_0\hat{\mathbf{y}})$ are real constants. The decomposition $\lambda = \lambda^\Delta + \lambda^e + \lambda^h$ and $\eta = \eta^\Delta + \eta^e + \eta^h$ are implicit. Note that, for brevity, we have suppressed R_0 in $\lambda_{\mathbf{k}}$, $\eta_{\mathbf{k}}$ and hereafter. Then, the effective Hamiltonian for the impurity superlattice follows as,

$$H_{\mathbf{k}}^{\text{eff}} = \mathcal{E}_{3\mathbf{k}}\sigma_3 + \mathcal{E}_{1\mathbf{k}}\sigma_1 - \mathcal{E}_{2\mathbf{k}}\sigma_2, \tag{S28}$$

in which

$$\mathcal{E}_{3\mathbf{k}} = E_0 + 2\lambda \left(\cos k_x + \cos k_y\right), \ \ \mathcal{E}_{1\mathbf{k}} = 2\eta \sin k_x, \ \ \mathcal{E}_{2\mathbf{k}} = 2\eta \sin k_y. \tag{S29}$$

This effective Hamiltonian resembles the original chiral p-wave model. As a side remark, the idea to design topological band structure through super-modulations of the order parameter is not new. Besides the present model which also appeared in Ref. [3], a superlattice of magnetic impurities in a conventional superconductor with Rashiba spin-orbit coupling has also been shown to support subgap bands with high Chern numbers [4]. In another context, a pair-density-wave of a chiral p-wave order parameter was shown to generate topologically protected low-energy excitations [5].

Similarly, following the argument given in the main text as well as Fig. 2 (c) and (d), the matrix elements for effective current operators along the x- and y-directions take the following forms:

$$J_{x\mathbf{k}}^{++} = -2\left(\lambda^e - \lambda^h\right)\sin k_x,\; J_{x\mathbf{k}}^{+-} = 2(\eta^e - \eta^h)\cos k_x = 4\eta^e \cos k_x, \tag{S30}$$

$$J_{y\mathbf{k}}^{++} = -2\left(\lambda^e - \lambda^h\right)\sin k_y,\; J_{y\mathbf{k}}^{+-} = 2(\eta^e - \eta^h)\cos k_y = 4\eta^e \cos k_y, \tag{S31}$$

Hence the i-th component of the current operators in terms of the Pauli matrices can be written as:

$$\hat{J}_{i\mathbf{k}}^{\text{eff}} = \mathcal{J}_{3i\mathbf{k}}\sigma_3 + \mathcal{J}_{1i\mathbf{k}}\sigma_1 + \mathcal{J}_{2i\mathbf{k}}\sigma_2, \tag{S32}$$

in which

$$\mathcal{J}_{3x\mathbf{k}} = -2\left(\lambda^e - \lambda^h\right)\sin k_x,\; \mathcal{J}_{1x\mathbf{k}} = 4\eta^e\cos k_x,\; \mathcal{J}_{2x\mathbf{k}} = 0, \tag{S33}$$

for x-direction, and

$$\mathcal{J}_{3y\mathbf{k}} = -2\left(\lambda^e - \lambda^h\right)\sin k_y,\; \mathcal{J}_{1y\mathbf{k}} = 0,\; \mathcal{J}_{2y\mathbf{k}} = -4\eta^e\cos k_y, \tag{S34}$$

for y-direction. Within linear-response theory, the transverse current-current correlation function at one-loop level is given by,

$$\pi_{xy}\left(\mathbf{q}, i\nu_m\right) = T\sum_{\mathbf{k}, i\omega_n} \text{Tr}\left[\hat{J}_{x\mathbf{k}}^{\text{eff}}\hat{G}\left(\mathbf{k}+\mathbf{q}, i\omega_n + i\nu_m\right)\hat{J}_{y\mathbf{k}}^{\text{eff}}\hat{G}\left(\mathbf{k}, i\omega_n\right)\right], \tag{S35}$$

where T is the temperature, $\omega_n = (2n+1)\pi T$ and $\nu_m = 2m\pi T$ are the fermionic and bosonic Matsubara frequencies, respectively. $\hat{G}\left(\mathbf{k}, i\omega_n\right)$ is the single-particle Green's function which can be written as

$$\hat{G}\left(\mathbf{k}, i\omega_n\right) = \left(i\omega_n\sigma_0 - H_{\mathbf{k}}^{\text{eff}}\right)^{-1} = \frac{i\omega_n\sigma_0 + \mathcal{E}_{3\mathbf{k}}\sigma_3 + \mathcal{E}_{1\mathbf{k}}\sigma_1 + \mathcal{E}_{2\mathbf{k}}\sigma_2}{\left(i\omega_n\right)^2 - E_{\mathbf{k}}^2}, \tag{S36}$$

where $E_{\mathbf{k}} = \sqrt{\mathcal{E}_{3\mathbf{k}}^2 + \mathcal{E}_{1\mathbf{k}}^2 + \mathcal{E}_{2\mathbf{k}}^2}$ is the quasiparticle dispersion.

The Hall conductivity is given by the antisymmetric part of the transverse current correlator. After some algebra and an analytical continuation $i\nu_m \to \omega + i\delta$, we arrive at the following,

$$\sigma_{\text{H}}\left(\omega + i\delta\right) = \frac{i}{2\omega}\lim_{\mathbf{q}\to 0}\left[\pi_{xy}\left(\mathbf{q}, \omega + i\delta\right) - \pi_{yx}\left(\mathbf{q}, \omega + i\delta\right)\right] = \sum_{\mathbf{k}}\frac{f\left(\mathbf{k}\right)}{E_{\mathbf{k}}\left[\left(\omega + i\delta\right)^2 - 4E_{\mathbf{k}}^2\right]}, \tag{S37}$$

in which

$$f\left(\mathbf{k}\right) = \sum_{s,m,n}\frac{\epsilon^{smn}}{2}\left[\mathcal{J}_{sx\mathbf{k}}\mathcal{J}_{my\mathbf{k}} - \mathcal{J}_{sy\mathbf{k}}\mathcal{J}_{my\mathbf{k}}\right]\mathcal{E}_{n\mathbf{k}}. \tag{S38}$$

Substituting the expressions, we see that a non-zero anomalous Hall conductivity emerges in the impurity superlattice embedded in a chiral p-wave superconductor.

We now turn to the case of underlying chiral d-wave pairing. We find that, a full description of low-energy model requires a consideration of up to the next-nearest neighboring terms shown in Fig. S2, after which we obtain,

$$\lambda_{\mathbf{k}} = 2\lambda\left(\cos k_x + \cos k_y\right) + 4\tilde{\lambda}\cos k_x\cos k_y,\;\; \eta_{\mathbf{k}} = -2\eta\left(\cos k_x - \cos k_y\right) - i4\tilde{\eta}\sin k_x\sin k_y, \tag{S39}$$

in which $\tilde{\lambda} \equiv \lambda(\sqrt{2}R_0)$ and $\tilde{\eta} \equiv \eta(\sqrt{2}R_0\hat{\mathbf{y}})$ are hopping integrals associated with the next-nearest neighboring contributions. Written in the form of Eq. (S28), the corresponding $\mathcal{E}_{i\mathbf{k}}$ are given by

$$\mathcal{E}_{3\mathbf{k}} = E_0 + 2\lambda\left(\cos k_x + \cos k_y\right) + 4\tilde{\lambda}\cos k_x\cos k_y,\;\; \mathcal{E}_{1\mathbf{k}} = -2\eta(\cos k_x - \cos k_y),\;\; \mathcal{E}_{2\mathbf{k}} = 4\tilde{\eta}\sin k_x\sin k_y. \tag{S40}$$

Turning to the current operators, we have $\hat{J}_{x\mathbf{k}}^{+-} = \hat{J}_{y\mathbf{k}}^{+-} = 0$ on account of the parity constraints ($\eta^e - \eta^h = 0$ for underlying even-parity pairing) discussed in the previous section. Thus $\hat{J}_{i\mathbf{k}}^{\text{eff}} = \mathcal{J}_{3i\mathbf{k}}\sigma_3 = -2\lambda^e\cos k_i\sigma_3$ with $i = x, y$. A straightforward calculation shows that the resultant model generate no anomalous Hall conductivity at the one-loop calculation.

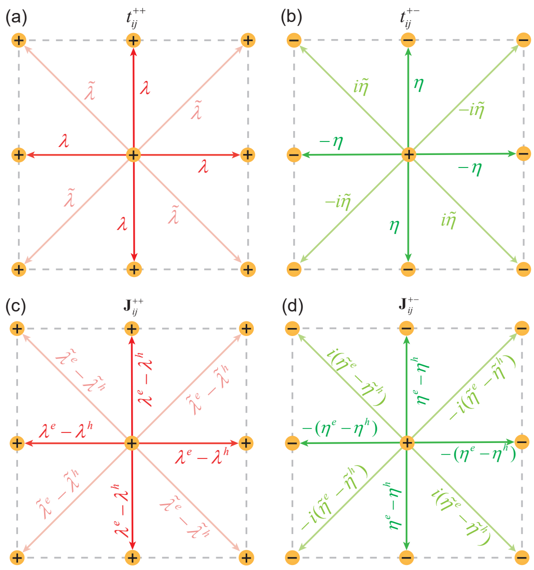

Figure S2: (a) (b) Tight-binding construction of a square impurity superlattice embedded in a chiral d-wave superconductor. Note the relation $\lambda = \lambda^e + \lambda^h + \lambda^\Delta$ and the same for $\tilde{\lambda}$, η and $\tilde{\eta}$. (c) (d) The current operator on the superlattice. The symbols '$+$' and '$-$' symbols designate the impurity bound states, and arrows indicate the reference direction of hopping or current flow.

B). Triangular impurity superlattice

In the case of triangular impurity superlattices, the anomalous Hall conductivity has the same form as in the case of a square superlattice, but with slight modifications. Consider only the nearest-neighbor hoppings, one obtains,

$$\mathcal{E}_{3\mathbf{k}} = E_0 + 2\lambda\left(\cos k_x + 2\cos\frac{k_x}{2}\cos\frac{\sqrt{3}k_y}{2}\right), \tag{S41a}$$

$$\mathcal{E}_{1\mathbf{k}} = \begin{cases} 2\eta\left(\sin k_x + \sin\frac{k_x}{2}\cos\frac{\sqrt{3}k_y}{2}\right), & (l=1) \\ 2\eta\left(\sin k_x - \cos\frac{k_x}{2}\cos\frac{\sqrt{3}k_y}{2}\right), & (l=2) \end{cases} \tag{S41b}$$

$$\mathcal{E}_{2\mathbf{k}} = \begin{cases} 2\sqrt{3}\eta\cos\frac{k_x}{2}\sin\frac{\sqrt{3}k_y}{2}, & (l=1) \\ 2\sqrt{3}\eta\sin\frac{k_x}{2}\sin\frac{\sqrt{3}k_y}{2}, & (l=2) \end{cases}, \tag{S41c}$$

and the associated components of the current operators are,

$$\mathcal{J}_{3x\mathbf{k}} = -2\left(\lambda^e - \lambda^h\right)\left(\sin k_x + \sin\frac{k_x}{2}\cos\frac{\sqrt{3}k_y}{2}\right), \tag{S42a}$$

$$\mathcal{J}_{1x\mathbf{k}} = \begin{cases} 4\eta^e\left(\cos k_x + \frac{1}{2}\cos\frac{k_x}{2}\cos\frac{\sqrt{3}k_y}{2}\right), & (l=1) \\ 0, & (l=2) \end{cases} \tag{S42b}$$

$$\mathcal{J}_{2x\mathbf{k}} = \begin{cases} -6\eta^e\sin\frac{k_x}{2}\cos\frac{\sqrt{3}k_y}{2}, & (l=1) \\ 0, & (l=2) \end{cases} \tag{S42c}$$

and

$$\mathcal{J}_{3y\mathbf{k}} = -2\sqrt{3}\left(\lambda^e - \lambda^h\right)\cos\frac{k_x}{2}\sin\frac{\sqrt{3}k_y}{2}, \tag{S43a}$$

$$\mathcal{J}_{1y\mathbf{k}} = \begin{cases} -6\eta^e\sin\frac{k_x}{2}\sin\frac{\sqrt{3}k_y}{2}, & (l=1) \\ 0, & (l=2) \end{cases} \tag{S43b}$$

$$\mathcal{J}_{2y\mathbf{k}} = \begin{cases} 6\eta^e\cos\frac{k_x}{2}\cos\frac{\sqrt{3}k_y}{2}, & (l=1) \\ 0, & (l=2) \end{cases} \tag{S43c}$$

It can thus be seen that the both the effective tight-binding Hamiltonian and the effective current operators follows the same overall structure as those in the square superlattice models. One thus expects the same outcome as far as the anomalous Hall effect is concerned.

C). Honeycomb impurity superlattice

The honeycomb impurity superlattice is very different from the previous two cases, since the enlargement of the Hilbert space due to an added sublattice degree of freedom. Consider a basis $\Psi_i = \left(c_{i,+}, c'_{i,+}, c_{i,-}, c'_{i,-}\right)^\intercal$, with c and c' representing the two sublattices, a general effective Hamiltonian for impurity superlattice with nearest-neighbor hoppings has the following form

$$\hat{H}^{\text{eff}}_{\mathbf{k}} = \begin{bmatrix} E_0 & \lambda_{\mathbf{k}} & 0 & \eta_{\mathbf{k}} \\ \lambda^*_{\mathbf{k}} & E_0 & (-1)^l \eta_{-\mathbf{k}} & 0 \\ 0 & (-1)^l \eta^*_{-\mathbf{k}} & -E_0 & -\lambda_{\mathbf{k}} \\ \eta^*_{\mathbf{k}} & 0 & -\lambda^*_{\mathbf{k}} & -E_0 \end{bmatrix} \tag{S44}$$

where the matrix elements are given by,

$$\lambda_{\mathbf{k}} = \sum_\delta e^{i\mathbf{k}\cdot\mathbf{R}_\delta} \lambda = \lambda \left(1 + 2e^{-i\frac{3k_x}{2}} \cos\frac{\sqrt{3}k_y}{2}\right), \tag{S45}$$

$$\eta_{\mathbf{k}} = \sum_\delta e^{i\mathbf{k}\cdot\mathbf{R}_\delta} e^{il\theta_{\mathbf{R}_\delta}} \eta \xrightarrow{l=2} \eta \left[1 + 2e^{-i\frac{3k_x}{2}} \cos\left(\frac{\sqrt{3}k_y}{2} - \frac{2\pi}{3}\right)\right]. \tag{S46}$$

in which we have eliminated a prefactor e^{ik_x} by a standard gauge transformation similar to the treatment for monolayer graphene. Note also that in the last expression, we have explicitly taken the example of $l = 2$ for underlying chiral d-wave, and the same below. The current operator follows as,

$$\begin{aligned} \hat{j}^{\text{eff}}_{x\mathbf{k}} &= \begin{pmatrix} 0 & J^{++}_{x\mathbf{k}} & 0 & J^{+-}_{x\mathbf{k}} \\ \left(J^{++}_{x\mathbf{k}}\right)^* & 0 & -J^{+-}_{x,-\mathbf{k}} & 0 \\ 0 & -\left(J^{+-}_{x,-\mathbf{k}}\right)^* & 0 & -J^{++}_{x\mathbf{k}} \\ \left(J^{+-}_{x\mathbf{k}}\right)^* & 0 & -\left(J^{++}_{x\mathbf{k}}\right)^* & 0 \end{pmatrix} \\ &= \mathcal{J}_{1x\mathbf{k}} \varrho_1 \otimes \sigma_3 + \mathcal{J}_{2x\mathbf{k}} \varrho_2 \otimes \sigma_3 + \mathcal{J}_{3x\mathbf{k}} \varrho_1 \otimes \sigma_1 + \mathcal{J}_{4x\mathbf{k}} \varrho_1 \otimes \sigma_2 + \mathcal{J}_{5x\mathbf{k}} \varrho_2 \otimes \sigma_1 + \mathcal{J}_{6x\mathbf{k}} \varrho_2 \otimes \sigma_2, \end{aligned} \tag{S47}$$

or, equivalently,

$$\hat{j}^{\text{eff}}_{y\mathbf{k}} = \mathcal{J}_{1y\mathbf{k}} \varrho_1 \otimes \sigma_3 + \mathcal{J}_{2y\mathbf{k}} \varrho_2 \otimes \sigma_3 + \mathcal{J}_{3y\mathbf{k}} \varrho_1 \otimes \sigma_1 + \mathcal{J}_{4y\mathbf{k}} \varrho_1 \otimes \sigma_2 + \mathcal{J}_{5y\mathbf{k}} \varrho_2 \otimes \sigma_1 + \mathcal{J}_{6y\mathbf{k}} \varrho_2 \otimes \sigma_2, \tag{S48}$$

where

$$J^{++}_{x\mathbf{k}} = \sum_\delta e^{i\mathbf{k}\cdot\mathbf{R}_\delta} J^{++}_x(\mathbf{R}_\delta) = -3\left(\lambda^e - \lambda^h\right)\left(\sin\frac{3k_x}{2}\cos\frac{\sqrt{3}k_y}{2} + i\cos\frac{3k_x}{2}\cos\frac{\sqrt{3}k_y}{2}\right), \tag{S49}$$

$$J^{++}_{y\mathbf{k}} = \sum_\delta e^{i\mathbf{k}\cdot\mathbf{R}_\delta} J^{++}_y(\mathbf{R}_\delta) = -\sqrt{3}\left(\lambda^e - \lambda^h\right)\left(\cos\frac{3k_x}{2}\sin\frac{\sqrt{3}k_y}{2} - i\sin\frac{3k_x}{2}\sin\frac{\sqrt{3}k_y}{2}\right), \tag{S50}$$

$$J^{+-}_{x\mathbf{k}} = \sum_\delta e^{i\mathbf{k}\cdot\mathbf{R}_\delta} J^{+-}_x(\mathbf{R}_\delta) = \begin{cases} -3\left(\eta^e - \eta^h\right)\left(\sin\frac{3k_x}{2} + i\cos\frac{3k_x}{2}\right)\cos\left(\frac{\sqrt{3}k_y}{2} - \frac{2\pi}{3}\right), & (l=1) \\ 0, & (l=2) \end{cases} \tag{S51}$$

$$J^{+-}_{y\mathbf{k}} = \sum_\delta e^{i\mathbf{k}\cdot\mathbf{R}_\delta} J^{++}_y(\mathbf{R}_\delta) = \begin{cases} -\sqrt{3}\left(\eta^e - \eta^h\right)\left(\cos\frac{3k_x}{2} - i\sin\frac{3k_x}{2}\right)\sin\left(\frac{\sqrt{3}k_y}{2} - \frac{2\pi}{3}\right), & (l=1) \\ 0, & (l=2) \end{cases} \tag{S52}$$

and $\mathcal{J}_{1i\mathbf{k}} = \mathrm{Re}(J^{++}_{i\mathbf{k}})$, $\mathcal{J}_{2i\mathbf{k}} = -\mathrm{Im}(J^{++}_{i\mathbf{k}})$, $\mathcal{J}_{3i\mathbf{k}} = \mathrm{Re}(J^{+-}_{i\mathbf{k}} - J^{+-}_{i,-\mathbf{k}})/2$, $\mathcal{J}_{4i\mathbf{k}} = -\mathrm{Im}(J^{+-}_{i\mathbf{k}} - J^{+-}_{i,-\mathbf{k}})/2$, and $\mathcal{J}_{5i\mathbf{k}} = -\mathrm{Im}(J^{+-}_{i\mathbf{k}} + J^{+-}_{i\mathbf{k}})/2$, $\mathcal{J}_{6i\mathbf{k}} = -\mathrm{Re}(J^{+-}_{i\mathbf{k}} + J^{+-}_{i,-\mathbf{k}})/2$, with $i = x, y$.

The Green's function $\hat{G}(\mathbf{k}, i\omega_n) = \left(i\omega_n\sigma_0 - H^{\text{eff}}_{\mathbf{k}}\right)^{-1}$ acquires the following form,

$$\hat{G}(\mathbf{k}, i\omega_n) = \sum_{i,j=0,1,2,3} \frac{g_{ij}\varrho_i \otimes \sigma_j}{\left[(i\omega_n)^2 - E^2_{+,\mathbf{k}}\right]\left[(i\omega_n)^2 - E^2_{-,\mathbf{k}}\right]}, \tag{S53}$$

where $g_{00}=-i\omega_n(\omega_n^2+E_0^2+|\lambda_{\mathbf{k}}|^2+\frac{|\eta_{\mathbf{k}}|^2+|\eta_{-\mathbf{k}}|^2}{2})$, $g_{03}=-E_0(\omega_n^2+E_0^2-|\lambda_{\mathbf{k}}|^2+\frac{|\eta_{\mathbf{k}}|^2+|\eta_{-\mathbf{k}}|^2}{2})$, $g_{33}=-i\omega_n(\frac{|\eta_{\mathbf{k}}|^2-|\eta_{-\mathbf{k}}|^2}{2})$, $g_{30}=-E_0(\frac{|\eta_{\mathbf{k}}|^2-|\eta_{-\mathbf{k}}|^2}{2})$, $g_{11}=-\frac{1}{2}\,\mathrm{Re}[\eta_{-\mathbf{k}}(\omega_n^2+E_0^2+|\eta_{\mathbf{k}}|^2+\lambda_{\mathbf{k}}^2)+\eta_{\mathbf{k}}(\omega_n^2+E_0^2+|\eta_{-\mathbf{k}}|^2+\lambda_{\mathbf{k}}^{*2})]$, $g_{12}=-\frac{1}{2}\,\mathrm{Im}[\eta_{-\mathbf{k}}(\omega_n^2+E_0^2+|\eta_{\mathbf{k}}|^2+\lambda_{\mathbf{k}}^2)+\eta_{\mathbf{k}}(\omega_n^2+E_0^2+|\eta_{-\mathbf{k}}|^2+\lambda_{\mathbf{k}}^{*2})]$, $g_{21}=\frac{1}{2}\,\mathrm{Im}[\eta_{-\mathbf{k}}(\omega_n^2+E_0^2-|\eta_{\mathbf{k}}|^2-\lambda_{\mathbf{k}}^2)+\eta_{\mathbf{k}}(\omega_n^2+E_0^2+|\eta_{-\mathbf{k}}|^2-\lambda_{\mathbf{k}}^{*2})]$, $g_{22}=-\frac{1}{2}\,\mathrm{Re}[\eta_{-\mathbf{k}}(\omega_n^2+E_0^2+|\eta_{\mathbf{k}}|^2-\lambda_{\mathbf{k}}^2)-\eta_{\mathbf{k}}(\omega_n^2+E_0^2+|\eta_{-\mathbf{k}}|^2-\lambda_{\mathbf{k}}^{*2})]$, $g_{13}=-\mathrm{Re}[\lambda_{\mathbf{k}}(\omega_n^2-E_0^2+|\lambda_{\mathbf{k}}|^2-\eta_{\mathbf{k}}^*\eta_{-\mathbf{k}})]$, $g_{23}=-\mathrm{Im}[\lambda_{\mathbf{k}}(\omega_n^2-E_0^2+|\lambda_{\mathbf{k}}|^2-\eta_{\mathbf{k}}^*\eta_{-\mathbf{k}})]$, $g_{10}=2i\omega_nE_0\,\mathrm{Re}\,(\lambda_{\mathbf{k}})$, $g_{20}=2i\omega_nE_0\,\mathrm{Im}\,(\lambda_{\mathbf{k}})$, $g_{31}=-i\omega_n\,\mathrm{Re}\left(\lambda_{\mathbf{k}}\eta_{\mathbf{k}}^*-\lambda_{\mathbf{k}}^*\eta_{-\mathbf{k}}^*\right)$, $g_{32}=i\omega_n\,\mathrm{Im}\left(\lambda_{\mathbf{k}}\eta_{\mathbf{k}}^*-\lambda_{\mathbf{k}}^*\eta_{-\mathbf{k}}^*\right)$, $g_{01}=E_0\,\mathrm{Re}\left(\lambda_{\mathbf{k}}\eta_{\mathbf{k}}^*+\lambda_{\mathbf{k}}^*\eta_{-\mathbf{k}}^*\right)$, $g_{32}=-E_0\,\mathrm{Im}\left(\lambda_{\mathbf{k}}\eta_{\mathbf{k}}^*+\lambda_{\mathbf{k}}^*\eta_{-\mathbf{k}}^*\right)$, and $E_{\pm,\mathbf{k}}=\sqrt{E_0^2+|\lambda_{\mathbf{k}}|^2+\frac{1}{2}\left(|\eta_{\mathbf{k}}|^2+|\eta_{-\mathbf{k}}|^2\right)\pm\sqrt{4\,|\lambda_{\mathbf{k}}|^2\,E_0^2+|\lambda_{\mathbf{k}}^*\eta_{\mathbf{k}}-\lambda_{\mathbf{k}}\eta_{-\mathbf{k}}|^2+\frac{1}{4}\left(|\eta_{\mathbf{k}}|^2-|\eta_{-\mathbf{k}}|^2\right)^2}}$.

We mainly focus on the case with chiral d-wave (even-parity) pairing in which $\mathcal{J}_{3i}=\mathcal{J}_{4i}=\mathcal{J}_{5i}=\mathcal{J}_{6i}=0$ $(i=x,y)$, and study its anomalous Hall conductivity. A lengthy calculation leads to,

$$\begin{aligned}&\pi_{xy}\left(\mathbf{q}=\mathbf{0},i\nu_m\right)-\pi_{yx}\left(\mathbf{q}=\mathbf{0},i\nu_m\right)\\&=\sum_{\mathbf{k}}\frac{\nu_m}{E_+E_-}\left\{\frac{f\left(E_+\right)-f\left(E_-\right)}{\left(E_{+,\mathbf{k}}-E_{-,\mathbf{k}}\right)\left[\left(E_{+,\mathbf{k}}-E_{-,\mathbf{k}}\right)^2+\nu_m^2\right]}+\frac{1-f\left(E_+\right)-f\left(E_-\right)}{\left(E_{+,\mathbf{k}}+E_{-,\mathbf{k}}\right)\left[\left(E_{+,\mathbf{k}}+E_{-,\mathbf{k}}\right)^2+\nu_m^2\right]}\right\}\\&\quad\times E_0\left(|\eta_{\mathbf{k}}|^2-|\eta_{-\mathbf{k}}|^2\right)\left[\mathcal{J}_{1x}\left(\mathbf{k}\right)\mathcal{J}_{2y}\left(\mathbf{k}\right)-\mathcal{J}_{2x}\left(\mathbf{k}\right)\mathcal{J}_{1y}\left(\mathbf{k}\right)\right].\end{aligned}\tag{S54}$$

It returns the following zero-temperature anomalous Hall conductivity at real frequency:

$$\sigma_{\mathrm{H}}\left(\omega+i\delta\right)=\sum_{\mathbf{k}}\frac{E_0\left(|\eta_{\mathbf{k}}|^2-|\eta_{-\mathbf{k}}|^2\right)\left[\mathcal{J}_{1x}\left(\mathbf{k}\right)\mathcal{J}_{2y}\left(\mathbf{k}\right)-\mathcal{J}_{2x}\left(\mathbf{k}\right)\mathcal{J}_{1y}\left(\mathbf{k}\right)\right]}{2E_{+,\mathbf{k}}E_{-,\mathbf{k}}\left(E_{+,\mathbf{k}}+E_{-,\mathbf{k}}\right)\left[\left(E_{+,\mathbf{k}}+E_{-,\mathbf{k}}\right)^2-\left(\omega+i\delta\right)^2\right]}.\tag{S55}$$

This quantity is generically finite. Hence, distinct from cases of square and triangular superlattices, the anomalous Hall conductivity for chiral d-wave and other even-parity chiral states is finite. One can further check that the honeycomb superlattice models with underlying odd-parity chiral pairings also support finite Hall conductance.

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* huangw3@sustech.edu.cn

[1] V. Kaladzhyan, C. Bena, and P. Simon, Characterizing p-wave superconductivity using the spin structure of Shiba states, Phys. Rev. B **93**, 214514 (2016).
[2] V. Kaladzhyan, J. Röntynen, P. Simon, T. Ojanen, Topological state engineering by potential impurities on chiral superconductors, Phys. Rev. B **94**, 060505(R) (2016).
[3] L. Kimme and T. Hyart, Existence of zero-energy impurity states in different classes of topological insulators and superconductors and their relation to topological phase transition, Phys. Rev. B **93**, 035134 (2016).
[4] J. Röntynen and T. Ojanen, Topological superconductivity and high Chern numbers in 2D ferromagnetic Shiba lattices, Phys. Rev. Lett. **114**, 236803 (2015).
[5] L. H. Santos, Y. Wang, E. Fradkin, Pair-density-wave Order and paired fractional quantum Hall fluids, Phys. Rev. X **9**, 021047 (2019).