Supplementary Information: Dynamical transition in controllable quantum neural networks with large depth

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The supplementary Information contains details of derivations.

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From now on, we will omit '^' on operators, as long as it is not confusing.

Supplementary Note 1. DERIVATION OF QNTK DYNAMICS

In this section, we provide details on the derivation of Eq. (8) in the main text. The time difference of QNTK is

$$
\begin{aligned}
\delta K(t) &\equiv K(t+1) - K(t) && (1)\\
&= \sum_\ell \delta\left(\frac{\partial\epsilon}{\partial\theta_\ell}\frac{\partial\epsilon}{\partial\theta_\ell}\right) && (2)\\
&= \sum_\ell \left(\frac{\partial\epsilon(\boldsymbol{\theta})}{\partial\theta_\ell}\frac{\partial\epsilon(\boldsymbol{\theta})}{\partial\theta_\ell}\right)\bigg|_{\boldsymbol{\theta}=\boldsymbol{\theta}(t+1)} - \left(\frac{\partial\epsilon(\boldsymbol{\theta})}{\partial\theta_\ell}\frac{\partial\epsilon(\boldsymbol{\theta})}{\partial\theta_\ell}\right)\bigg|_{\boldsymbol{\theta}=\boldsymbol{\theta}(t)} && (3)\\
&= \sum_\ell \Bigg[\left(\frac{\partial\epsilon(\boldsymbol{\theta})}{\partial\theta_\ell}\frac{\partial\epsilon(\boldsymbol{\theta})}{\partial\theta_\ell}\right)\bigg|_{\boldsymbol{\theta}=\boldsymbol{\theta}(t+1)} - \left(\frac{\partial\epsilon(\boldsymbol{\theta})}{\partial\theta_\ell}\bigg|_{\boldsymbol{\theta}=\boldsymbol{\theta}(t+1)}\frac{\partial\epsilon(\boldsymbol{\theta})}{\partial\theta_\ell}\bigg|_{\boldsymbol{\theta}=\boldsymbol{\theta}(t)}\right) + \left(\frac{\partial\epsilon(\boldsymbol{\theta})}{\partial\theta_\ell}\bigg|_{\boldsymbol{\theta}=\boldsymbol{\theta}(t+1)}\frac{\partial\epsilon(\boldsymbol{\theta})}{\partial\theta_\ell}\bigg|_{\boldsymbol{\theta}=\boldsymbol{\theta}(t)}\right)\\
&\qquad - \left(\frac{\partial\epsilon(\boldsymbol{\theta})}{\partial\theta_\ell}\frac{\partial\epsilon(\boldsymbol{\theta})}{\partial\theta_\ell}\right)\bigg|_{\boldsymbol{\theta}=\boldsymbol{\theta}(t)}\Bigg] && (4)\\
&= \sum_\ell \frac{\partial\epsilon(\boldsymbol{\theta})}{\partial\theta_\ell}\bigg|_{\boldsymbol{\theta}=\boldsymbol{\theta}(t+1)}\delta\left(\frac{\partial\epsilon(\boldsymbol{\theta})}{\partial\theta_\ell}\bigg|_{\boldsymbol{\theta}=\boldsymbol{\theta}(t)}\right) + \delta\left(\frac{\partial\epsilon(\boldsymbol{\theta})}{\partial\theta_\ell}\bigg|_{\boldsymbol{\theta}=\boldsymbol{\theta}(t)}\right)\frac{\partial\epsilon(\boldsymbol{\theta})}{\partial\theta_\ell}\bigg|_{\boldsymbol{\theta}=\boldsymbol{\theta}(t)} && (5)\\
&= \sum_\ell 2\delta\left(\frac{\partial\epsilon(\boldsymbol{\theta})}{\partial\theta_\ell}\bigg|_{\boldsymbol{\theta}=\boldsymbol{\theta}(t)}\right)\frac{\partial\epsilon(\boldsymbol{\theta})}{\partial\theta_\ell}\bigg|_{\boldsymbol{\theta}=\boldsymbol{\theta}(t)} + \frac{\partial\epsilon(\boldsymbol{\theta})}{\partial\theta_\ell}\bigg|_{\boldsymbol{\theta}=\boldsymbol{\theta}(t+1)}\delta\left(\frac{\partial\epsilon(\boldsymbol{\theta})}{\partial\theta_\ell}\bigg|_{\boldsymbol{\theta}=\boldsymbol{\theta}(t)}\right) - \delta\left(\frac{\partial\epsilon(\boldsymbol{\theta})}{\partial\theta_\ell}\bigg|_{\boldsymbol{\theta}=\boldsymbol{\theta}(t)}\right)\frac{\partial\epsilon(\boldsymbol{\theta})}{\partial\theta_\ell}\bigg|_{\boldsymbol{\theta}=\boldsymbol{\theta}(t)} && (6)\\
&= \sum_\ell 2\delta\left(\frac{\partial\epsilon(\boldsymbol{\theta})}{\partial\theta_\ell}\bigg|_{\boldsymbol{\theta}=\boldsymbol{\theta}(t)}\right)\frac{\partial\epsilon(\boldsymbol{\theta})}{\partial\theta_\ell}\bigg|_{\boldsymbol{\theta}=\boldsymbol{\theta}(t)} + \delta\left(\frac{\partial\epsilon(\boldsymbol{\theta})}{\partial\theta_\ell}\bigg|_{\boldsymbol{\theta}=\boldsymbol{\theta}(t)}\right)\delta\left(\frac{\partial\epsilon(\boldsymbol{\theta})}{\partial\theta_\ell}\bigg|_{\boldsymbol{\theta}=\boldsymbol{\theta}(t)}\right). && (7)
\end{aligned}
$$

The second term in the last formula has two δ, so it is in higher orders in η, and we only focus on the first term. We utilize the leading order Taylor expansion on $\delta\partial\epsilon(\boldsymbol{\theta})/\partial\theta_\ell$ as

$$
\delta\left(\frac{\partial\epsilon(\boldsymbol{\theta})}{\partial\theta_\ell}\bigg|_{\boldsymbol{\theta}=\boldsymbol{\theta}(t)}\right) = \sum_{\ell_1}\frac{\partial^2\epsilon(\boldsymbol{\theta})}{\partial\theta_{\ell_1}\partial\theta_\ell}\delta\theta_{\ell_1}\bigg|_{\boldsymbol{\theta}=\boldsymbol{\theta}(t)} + \mathcal{O}(\eta^2) = -\eta\sum_{\ell_1}\frac{\partial^2\epsilon(\boldsymbol{\theta})}{\partial\theta_{\ell_1}\partial\theta_\ell}\frac{\partial\epsilon(\boldsymbol{\theta})}{\partial\theta_{\ell_1}}\epsilon(\boldsymbol{\theta})\bigg|_{\boldsymbol{\theta}=\boldsymbol{\theta}(t)} + \mathcal{O}(\eta^2). \tag{8}
$$

So we have

$$
\sum_\ell \delta\left(\frac{\partial\epsilon(\boldsymbol{\theta})}{\partial\theta_\ell}\bigg|_{\boldsymbol{\theta}=\boldsymbol{\theta}(t)}\right)\frac{\partial\epsilon(\boldsymbol{\theta})}{\partial\theta_\ell}\bigg|_{\boldsymbol{\theta}=\boldsymbol{\theta}(t)} = -\eta\sum_{\ell,\ell_1}\frac{\partial^2\epsilon(\boldsymbol{\theta})}{\partial\theta_\ell\partial\theta_{\ell_1}}\frac{\partial\epsilon(\boldsymbol{\theta})}{\partial\theta_{\ell_1}}\frac{\partial\epsilon(\boldsymbol{\theta})}{\partial\theta_\ell}\epsilon(\boldsymbol{\theta})\bigg|_{\boldsymbol{\theta}=\boldsymbol{\theta}(t)} + \mathcal{O}(\eta^2) = -\eta\epsilon(\boldsymbol{\theta})\mu(\boldsymbol{\theta})|_{\boldsymbol{\theta}=\boldsymbol{\theta}(t)} + \mathcal{O}(\eta^2), \tag{9}
$$

which leads to the gradient descent dynamical equation of $K(t)$ as

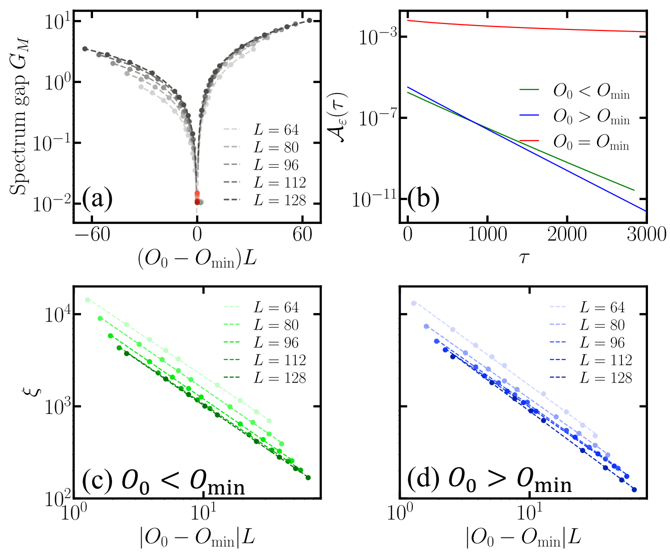
$$
\delta K(t) = -2\eta\epsilon(t)\mu(t) + \mathcal{O}(\eta^2), \tag{10}
$$

where we omit the explicit dependence on $\boldsymbol{\theta}$ and only present the t-dependence. It recovers Eq. (8) of the main text.

Supplementary Note 2. SCHRÖDINGER EQUATION INTERPRETATION

In this section, we provide more details on applying statistical physics tools to study the properties associated with the dynamical transition. If we consider an unnormalized 'differential state' as a superposition of two output states of the QNN,

$$
|\Psi(\boldsymbol{\theta})\rangle = |\psi(\boldsymbol{\theta})\rangle - |\psi(\boldsymbol{\theta}^*)\rangle = N(\boldsymbol{\theta})\left(\boldsymbol{\theta} - \boldsymbol{\theta}^*\right), \tag{11}
$$



Supplementary Figure 1. **Spectrum gap and correlation functions in the Schrödinger equation interpretation.** (a) The spectrum gap versus the scaled target value $(O_0 - O_{\min})L$ with different number of parameters. We choose $O_0 - O_{\min} \in [-0.5, 0.5]$. Dots from light to dark represent gaps with increasing L. The dashed lines with corresponding colors show $G_M \sim (|O_0 - O_{\min}|L)^{\nu_1}$ with $\nu_1 \simeq 1$ from fitting. Red dots represent the critical point $O_0 = O_{\min}$. The nonvanishing gap at $O_0 = O_{\min}$ is due to finite training time. In (b), we plot the decay of autocorrelators $\mathcal{A}_\epsilon(\tau)$ with different $O_0 \lesseqgtr O_{\min}$ (green for '<', red for '=' and blue for '>'). In (c) and (d), we show the scaling of correlation length $\xi \sim (|O_0 - O_{\min}|L)^{-\nu_2}$ with $\nu_2 \simeq 1$ (dashed lines) by fitting for both $O_0 \lessgtr O_{\min}$. Dots from light to dark represent ξ with increasing L variational parameters, and the dashed lines with the corresponding color represent fitting result. Here we choose $O_0 - O_{\min} \in [-0.5, 0.5]$. In (b) the RPA consists of $D = 96$ layers (equivalently $L = 96$ parameters). In all cases, the RPA is applied on a system of $n = 4$ qubits and the parameter in XXZ model is $J = 2$.

where $N(\boldsymbol{\theta}) = \partial|\psi(\boldsymbol{\theta})\rangle/\partial\boldsymbol{\theta}$ at $\boldsymbol{\theta} = \boldsymbol{\theta}^*$ is a $d \times L$ dimensional matrix. Formally, the pseudoinverse of the linear relation is denoted by $N(\boldsymbol{\theta})^{-1}$. From Eq. (18) of the main text, the effective Hamiltonian $H_\infty = NMN^{-1}$ can be obtained from

$$\delta|\Psi(\boldsymbol{\theta})\rangle = N(\boldsymbol{\theta}) \cdot \delta\boldsymbol{\theta} = -\eta N(\boldsymbol{\theta})M(\boldsymbol{\theta})N(\boldsymbol{\theta})^{-1}|\Psi(\boldsymbol{\theta})\rangle. \tag{12}$$

which is the Schrödinger equation with imaginary time $i\eta$. Note that the quantum mechanical Hamiltonian is valid only under the late-time limit $t \to \infty$.

To show that the large depth limit is well-defined, we evaluate the Hessian gap of QNNs at late time for various different circuit depth, or equivalently the number of parameters (i.e. the 'parameter size'). As we see in Supplementary Figure 1(a), the curves of the spectrum gap versus a rescaled target value $(O_0 - O_{\min})L$ collapse well as the parameter size L increases, indicating a well-defined transition at the large-depth limit. We notice a linear-closing gap around the critical point (red triangle in Fig. 4 of the main text), and verify the scaling in Supplementary Figure 1(a) via fitted the gap G_M to

$$G_M \sim (|O_0 - O_{\min}|L)^{\nu_1}, \tag{13}$$

resulting in $\nu_1 = 0.996 \pm 0.004, 1.09 \pm 0.021$ for $O_0 \lessgtr O_{\min}$ (dashed lines). However, we also notice that the minimum gap in the numerical study has no significant dependence on the parameter size L—it is dominated by the finite training time in the numerical simulation which fails to achieve the infinite time limit. As at the critical point $O_0 = O_{\min}$, the QNN training dynamics converges polynomially, which makes accessing the infinite-time limit numerically difficult. However, we do expect that the Hessian gap vanishes exactly at infinite time as both error and QNTK will vanish. Such an exact gap closing within finite size is in contrast to normal phase transitions in statistical physics and therefore we regard the transition not as a conventional phase transition in the statistical physics sense.

Despite not being a genuine phase transition, we can still adopt tools from statistical physics to provide more insight into the gap-closing transition. Regarding the QNN as a Schrödinger system described by a Hamiltonian dependent on both L and t, we then study the correlators. Both $K(t)$ and $\epsilon(t)$ has the same correlator behavior and we focus on ϵ here (see Supplementary Note 3) and define the autocorrelator $\mathcal{A}_\epsilon(\tau) \equiv \mathbb{E}\left[\varepsilon(t)\varepsilon(t+\tau)\right]$, where the average is over ensemble of trajectories and we will consider the $t \gg 1$ region. Here ε is adopted as it captures the residual error for the study of fluctuations. Away from the critical point, from the mean-field dynamics at late time according to Eq. (33) of main text, we expect the autocorrelator

$$\mathcal{A}_\epsilon(\tau) \sim \exp\left[-|\tau|/\xi\right] \tag{14}$$

to decay exponentially with a finite correlation length ξ, which is verified in Supplementary Figure 1(b). In Supplementary Figure 1(c)(d), the correlation length ξ versus the rescaled target value $|O_0 - O_{\min}|L$ also collapses with the increasing number of parameters L, which aligns with the behavior of the spectrum gap in Supplementary Figure 1(a). We further reveal the scaling of correlation length as

$$\xi \sim 1/|C| \sim (|O_0 - O_{\min}|L)^{-\nu_2}, \tag{15}$$

where ν_2 is found to be $0.961 \pm 0.012, 1.01 \pm 0.025$ for $O_0 \lessgtr O_{\min}$, shown in Supplementary Figure 1(c) and (d). The numerical values of ν_1 (Supplementary Figure 1(a)) and ν_2 are indeed identical up to numerical precision, as expected.

At the critical point $O_0 = O_{\min}$, any physical quantity F is expected to exhibit power-law correlation $\mathcal{A}_F(\tau) \sim 1/|\tau|^{2\Delta[F]}$ for $|\tau| \gg t$, defining the scaling dimension $\Delta[F]$. Based on the definitions Eqs. (5), (6) and (9) of the main text, one can establish the following scaling relations

$$\begin{aligned} \Delta[\epsilon] &= 2\Delta[K] - \Delta[\mu] \\ \Delta[\lambda] &= \Delta[\mu] - \Delta[K]. \end{aligned} \tag{16}$$

As shown in Supplementary Figure 1(c), our numerical result suggests $\Delta[\epsilon] = 1/2$. This seems consistent with the solution Eq. (34) of main text, assuming the correlation is factorizable. In Supplementary Note 3, we also find $\Delta[K] = \Delta[\mu] = 1/2$, while $\Delta[\lambda] = 0$ as λ is a constant. We see that the scaling relations in Eqs. (16) are indeed fulfilled.

In summary, we have taken tools from statistical physics to study the dynamical transition, which is shown to mimic certain properties of a continuous gap-closing transition in a quantum mechanical system described by the time-independent effective Hamiltonian H_∞. We also want to mention that reaching the truly infinite-time limit poses challenges both numerically and experimentally. In our estimation of the correlation function in Supplementary Note 3, we rely on taking the subtle limit $\tau \gg t \gg 1$ and the fluctuations being small.

Supplementary Note 3. DETAILS OF AUTOCORRELATORS

In this section, we provide a mean-field approach to provide an insight to the scaling of autocorrelators. For any time-dependent quantity $F(t)$ that has ensemble fluctuations, we define the late-time autocorrelator as

$$\mathcal{A}_F(\tau) \equiv \mathbb{E}\left[\left(F(t) - F(\infty)\right)\left(F(t+\tau) - F(\infty)\right)\right], \tag{17}$$

where the average is over the ensemble of trajectories and we will consider $t \gg 1$ region. Here $F(\infty) = \lim_{T\to\infty} \int_T^{2T} \mathrm{d}t F(t)/T$ is the smoothed late-time value of the function. For $\epsilon(t)$, the definition in Eq. (17) leads to $\mathcal{A}_\epsilon(\tau) \equiv \mathbb{E}\left[\varepsilon(t)\varepsilon(t+\tau)\right]$, which is the one adopted in the main text.

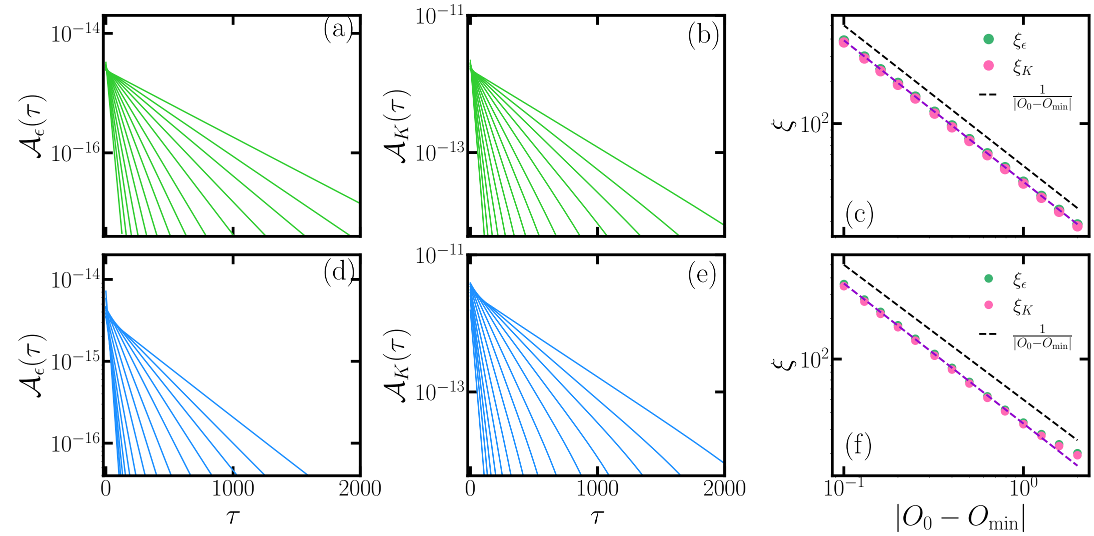
When $O_0 > O_{\min}$ with $C > 0$, utilizing the solution Eq. (33) of the main text, the mean-field approximated autocorrelator in Eq. (17) becomes

$$\mathcal{A}_\epsilon(\tau) = \int d\Gamma(\lambda) d\Gamma(B_1) \frac{C^2/\lambda^2}{\left(B_1 e^{\eta C t} - 2\right)\left(B_1 e^{\eta C(t+\tau)} - 2\right)} \tag{18}$$

$$\simeq \int d\Gamma(\lambda) d\Gamma(B_1) \frac{C^2 \lambda^2}{B_1^2 e^{2\eta C t} e^{\eta C \tau}} \tag{19}$$

$$\sim \frac{C^2/\overline{\lambda}^2}{\overline{B_1}^2 e^{2\eta C t} e^{\eta C \tau}} \sim e^{-\eta C \tau}, \tag{20}$$

where $\Gamma(\lambda), \Gamma(B_1)$ is the distribution of conserved quantity and fitting parameter in different initialization. Similarly,



Supplementary Figure 2. Decay of autocorrelators and corresponding correlation length with O_0 away from critical points as $O_0 < O_{\min}$ (top) and $O_0 > O_{\min}$ (bottom). The first two columns plot the autocorrelators. In (c), (f), the overlapping red and green dots represent correlation length fitted from Eq. (25) and dashed lines with same color show the fitting results. Black dashed lines represent its scaling as $1/|O_0 - O_{\min}|$. The observable is the Hamiltonian of XXZ model with $J = 2$, and circuit ansatz is $n = 2$ qubit RPA with $L = 64$ layers.

for $O_0 < O_{\min}$ with $C < 0$, we have

$$\mathcal{A}_\epsilon(\tau) = \int d\Gamma(\lambda)d\Gamma(B_1)\left(\frac{C/\lambda}{B_1 e^{\eta Ct} - 2} - R\right)\left(\frac{C/\lambda}{B_1 e^{\eta C(t+\tau)} - 2} - R\right) \tag{21}$$

$$= \int d\Gamma(\lambda)d\Gamma(B_1)\left(\frac{-2R}{B_1 e^{\eta Ct} - 2} - R\right)\left(\frac{-2R}{B_1 e^{\eta C(t+\tau)} - 2} - R\right) \tag{22}$$

$$= R^2 \int d\Gamma(\lambda)d\Gamma(B_1)\frac{1}{1 - 2B_1^{-1}e^{-\eta Ct}}\frac{1}{1 - 2B_1^{-1}e^{-\eta C(t+\tau)}} \tag{23}$$

$$\sim \frac{R^2 \overline{B_1}^2 e^{2\eta Ct}e^{\eta C\tau}}{4} \sim e^{\eta C\tau}. \tag{24}$$

We numerically show the decay of autocorrelators with different O_0 in Supplementary Figure 2(a), (d). In both cases, we see the exponential decay of autocorrelators, and the correlation length defined by $\mathcal{A}_F(\tau) \sim \exp(-\tau/\xi)$ is

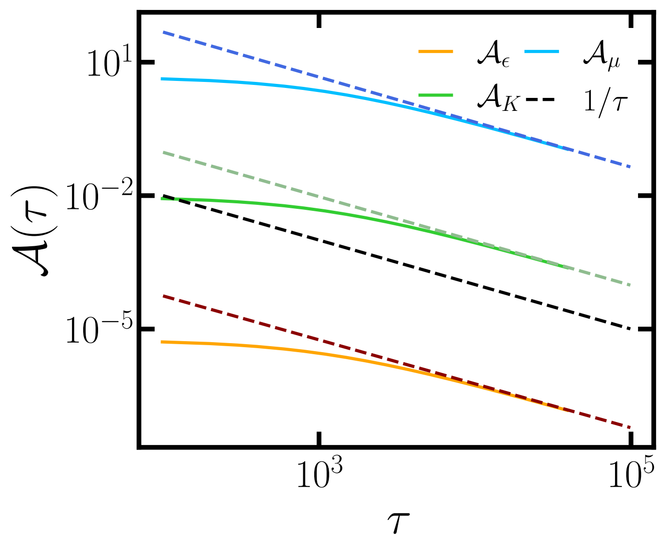
$$\xi \sim 1/(\eta|C|) \sim |O_0 - O_{\min}|^{-\nu_2}. \tag{25}$$

For $O_0 < O_{\min}$, we directly have $|C| \propto |O_0 - O_{\min}|$ which leads to $\nu_2 = 1$. This is verified in Supplementary Figure 2(c), with the fitted exponent $\nu_2 = 1.006$. For $O_0 > O_{\min}$, we have $|C| = |K| = G_M$, with G_M being the spectrum gap of Hessian at late time, which indicates that $\nu_2 = \nu_1$. This is again verified in Supplementary Figure 2(f) with the fitted exponent $\nu_2 = 1.038$.

From the duality between ϵ and K, the autocorrelator of K also decays exponentially, when the system is not at the critical point, shown in Supplementary Figure 2(b), (e). The corresponding correlation length exponent can also be found as $\nu_2 = 1.006, 1.034$ for $O_0 \lessgtr O_{\min}$. In general, we have

$$\nu_2 = 1, \tag{26}$$

within our numerical precision.



Supplementary Figure 3. Decay of autocorrelators for $\epsilon(t), K(t), \mu(t)$ at critical $O_0 = -6$ (solid curves). Dashed lines with corresponding color show the fitting results with $\Delta[\epsilon] = 0.494$, $\Delta[K] = 0.499$ and $\Delta[\mu] = 0.506$. Black dashed line represent the scaling $1/\tau$.

On the other hand, at the critical point $O_0 = O_{\min}$ we have

$$\mathcal{A}_\epsilon(\tau) = \int d\Gamma(\lambda) d\Gamma(B_2) \frac{1/\lambda^2}{\left(B_2^{-1} + 2\eta t\right)\left(B_2^{-1} + 2\eta(t+\tau)\right)} \tag{27}$$

$$\simeq \int d\Gamma(\lambda) d\Gamma(B_2) \frac{1/\lambda^2}{4\eta^2 t(t+\tau)} \tag{28}$$

$$\sim \frac{1/\overline{\lambda}^2}{4\eta^2 t(t+\tau)} \sim \frac{1}{\tau}, \tag{29}$$

which decays polynomially with τ (see Supplementary Figure 3). Note that the scaling of $\mathcal{A}_\epsilon(\tau) \sim 1/\tau$ holds only if $\tau \gg t$, otherwise it is nearly a constant. As $\lambda = \mu/K$ approaches a constant, we have that $\mu \sim K \sim 1/t$, and thus $\mathcal{A}_\mu(\tau) \sim 1/\tau$. From the definition of scaling dimension, $\mathcal{A}_F(\tau) = 1/\tau^{2\Delta[F]}$, one can find that

$$\Delta[\epsilon] = \Delta[K] = \Delta[\mu] = 1/2, \tag{30}$$

which is verified in Supplementary Figure 3, and $\Delta[\mu] = 0$.

Supplementary Note 4. OBSERVABLE TRACE PROPERTIES

In this work, we mainly focus on the traceless observables, where a typical example is the spin Hamiltonian of many-body system. In general, a n-qubit observable can always be written in the form of linear combinations of nontrivial Paulis $O = \sum_{i=1}^N c_i P_i$ with $P_i \in \{\mathbb{I}, \sigma^x, \sigma^y, \sigma^z\}^{\otimes n}/\{\mathbb{I}^{\otimes n}\}$, where $1 \le N \le 4^n - 1$ is the number of unqiue Paulis in the observable. We discuss the scaling of the trace of its powers up to four with respect to Hilbert space dimension d and number of terms N. To begin with,

$$\mathrm{tr}(O) = 0, \tag{31}$$

$$\mathrm{tr}(O^2) = \sum_{i_1, i_2 = 1}^{N} c_{i_1} c_{i_2} \,\mathrm{tr}(P_{i_1} P_{i_2}) = \sum_i c_i^2 \,\mathrm{tr}(P_i^2) + \sum_{i_1 \neq i_2} c_{i_1} c_{i_2} \,\mathrm{tr}(P_{i_1} P_{i_2}) \sim Nd. \tag{32}$$

For higher orders, we focus on some typical cases of observables to provide an insight to its scaling.

Supplementary Note 4.1. One-body observable

For the simplest case, a linear combination of 1-local Paulis $O = \sum_i c_i P_i$, where P_i is nontrivially supported on only one qubit and $c_i \in \mathbb{R}$, the trace of its third and fourth power is

$$\operatorname{tr}\left(O^3_{1-\text{local}}\right) = \sum_{i_1,i_2,i_3} c_{i_1} c_{i_2} c_{i_3} \operatorname{tr}(P_{i_1} P_{i_2} P_{i_3}) = 0, \tag{33}$$

$$\begin{aligned}
\operatorname{tr}\left(O^4_{1-\text{local}}\right) &= \sum_{i_1,i_2,i_3,i_4} c_{i_1} c_{i_2} c_{i_3} c_{i_4} \operatorname{tr}(P_{i_1} P_{i_2} P_{i_3} P_{i_4}) && (34)\\
&= \sum_{i_1,i_2} c^2_{i_1} c^2_{i_2} \operatorname{tr}\left(P^2_{i_1} P^2_{i_2}\right) + 2 \sum_{i_1,i_2 \neq i_3} c^2_{i_1} c_{i_2} c_{i_3} \operatorname{tr}\left(P^2_{i_1} P_{i_2} P_{i_3}\right) + \sum_{\substack{i_1 \neq i_2\\ i_3 \neq i_4}} c_{i_1} c_{i_2} c_{i_3} c_{i_4} \operatorname{tr}(P_{i_1} P_{i_2} P_{i_3} P_{i_4}) && (35)\\
&= \sum_{i_1,i_2} c^2_{i_1} c^2_{i_2} \operatorname{tr}(\boldsymbol{I}) + 0 + 2 \sum_{i_1 \neq i_2} c^2_{i_1} c^2_{i_2} \operatorname{tr}\left(P^2_{i_1} P^2_{i_2}\right) && (36)\\
&\sim N^2 d + 2N(N-1)d \sim 3N^2 d, && (37)
\end{aligned}$$

where the contribution from Paulis nontrivially supported on the same qubit is overestimated. In a special case where O incorporates all possible 1-local Pauli with equal weights, the rigorous result is $\operatorname{tr}\left(O^4\right) = (3N^2 - 6N)d \sim 3N^2 d$, leading to a sub-order correction to the estimation in Eq. (37).

Supplementary Note 4.2. Two-body observable: XXZ model

In the 2-local Pauli case, we consider the Hamiltonian consists of Paulis at most non-trivially supported on two qubit, and specifically, the two qubit are nearest neighbors. Here we take the Heisenberg model as an example,

$$O_{\text{HM}} = -\sum_{i=1}^{n-1} \left(J_x \sigma^x_i \sigma^x_{i+1} + J_y \sigma^y_i \sigma^y_{i+1} + J_z \sigma^z_i \sigma^z_{i+1}\right) - h \sum_{i=1}^{n} \sigma^z_i. \tag{38}$$

Specifically, when $J_x = J_y$ and $J_z = h$ but $J_z \neq J_x$, the general Heisenberg model is reduced to the XXZ model as

$$O_{\text{XXZ}} = -\sum_{i=1}^{n-1} \left(\sigma^x_i \sigma^x_{i+1} + \sigma^y_i y_{i+1} + J \sigma^z_i \sigma^z_{i+1}\right) - J \sum_{i=1}^{n} \sigma^z_i. \tag{39}$$

which is studied in the main text.

The trace of its power from second to fourth can be exactly solved as

$$\begin{aligned}
\operatorname{tr}\left(O^2_{\text{XXZ}}\right) &= \sum_{i,j=1}^{n-1} \operatorname{tr}\left(\sigma^x_i \sigma^x_{i+1} \sigma^x_j \sigma^x_{j+1}\right) + \operatorname{tr}\left(\sigma^y_i \sigma^y_{i+1} \sigma^y_j \sigma^y_{j+1}\right) + J^2 \operatorname{tr}\left(\sigma^z_i \sigma^z_{i+1} \sigma^z_j \sigma^z_{j+1}\right) + \sum_{i,j=1}^{n} J^2 \operatorname{tr}\left(\sigma^z_i \sigma^z_j\right) && (40)\\
&= \sum_{i=1}^{n-1} \left(2 \operatorname{tr}(\boldsymbol{I}) + J^2 \operatorname{tr}(\boldsymbol{I})\right) + \sum_{i=1}^{n} J^2 \operatorname{tr}(\boldsymbol{I}) && (41)\\
&= \left[(J^2+2)(n-1) + J^2 n\right] d && (42)\\
&\simeq 2(J^2+1)nd, && (43)
\end{aligned}$$

where in the second line we only keep the nonzero terms and omit the zero contributions.

With one more step, we can find the trace of its third power as

$$\begin{aligned}
\operatorname{tr}\left(O^3_{\text{XXZ}}\right) &= -6J \sum_{i,j,k=1}^{n-1} \operatorname{tr}\left(\sigma^x_i \sigma^x_{i+1} \sigma^y_j \sigma^y_{j+1} \sigma^z_k \sigma^z_{k+1}\right) - 3J^3 \sum_{i,j,k=1}^{n-1} \operatorname{tr}\left(\sigma^z_i \sigma^z_{i+1} \sigma^z_j \sigma^z_k\right) && (44)\\
&= 6(n-1)Jd - 6J^3(n-1)d && (45)\\
&= 6J(1-J^2)(n-1)d, && (46)\\
&\simeq 6J(1-J^2)nd, && (47)
\end{aligned}$$

where again the first equation is an effective equation for all non-zero contributions. When $J \lessgtr 1$, we have $\mathrm{tr}\left(O_{\mathrm{XXZ}}^3\right) \sim \mp Nd$, and at the critical $J=1$, we have $\mathrm{tr}\left(O_{\mathrm{XXZ}}^3\right) = 0$.

The trace of fourth power is

$$
\begin{aligned}
&\mathrm{tr}\left(O_{\mathrm{XXZ}}^4\right) \\
&= \sum_{\substack{i,j,\\k,l=1}}^{n-1} \left[2\,\mathrm{tr}\left(\sigma_i^x\sigma_{i+1}^x\sigma_j^x\sigma_{j+1}^x\sigma_k^x\sigma_{k+1}^x\sigma_l^x\sigma_{l+1}^x\right) + 4\,\mathrm{tr}\left(\sigma_i^x\sigma_{i+1}^x\sigma_j^x\sigma_{j+1}^x\sigma_k^y\sigma_{k+1}^y\sigma_l^y\sigma_{l+1}^y\right) + 8J^2\,\mathrm{tr}\left(\sigma_i^x\sigma_{i+1}^x\sigma_j^x\sigma_{j+1}^x\sigma_k^z\sigma_{k+1}^z\sigma_l^z\sigma_{l+1}^z\right)\right. \\
&\quad \left. +2\,\mathrm{tr}\left(\sigma_i^x\sigma_{i+1}^x\sigma_j^y\sigma_{j+1}^y\sigma_k^x\sigma_{k+1}^x\sigma_l^y\sigma_{l+1}^y\right) + 4J^2\,\mathrm{tr}\left(\sigma_i^x\sigma_{i+1}^x\sigma_j^z\sigma_{j+1}^z\sigma_k^z\sigma_{k+1}^z\sigma_l^z\sigma_{l+1}^z\right) + J^4\,\mathrm{tr}\left(\sigma_i^z\sigma_{i+1}^z\sigma_j^z\sigma_{j+1}^z\sigma_k^z\sigma_{k+1}^z\sigma_l^z\sigma_{l+1}^z\right)\right] \\
&\quad + \sum_{i,j=1}^{n-1}\sum_{k,l=1}^{n} \left[8J^2\,\mathrm{tr}\left(\sigma_i^x\sigma_{i+1}^x\sigma_j^x\sigma_{j+1}^x\sigma_k^z\sigma_l^z\right) + 8J^2\,\mathrm{tr}\left(\sigma_i^x\sigma_{i+1}^x\sigma_j^y\sigma_{j+1}^y\sigma_k^z\sigma_l^z\right) + 6J^4\,\mathrm{tr}\left(\sigma_i^z\sigma_{i+1}^z\sigma_j^z\sigma_{j+1}^z\sigma_k^z\sigma_l^z\right)\right] \\
&\quad + \sum_{i,k=1}^{n-1}\sum_{j,l=1}^{n} \left[4J^2\,\mathrm{tr}\left(\sigma_i^x\sigma_{i+1}^x\sigma_j^z\sigma_k^x\sigma_{k+1}^x\sigma_l^z\right) + 4J^2\,\mathrm{tr}\left(\sigma_i^x\sigma_{i+1}^x\sigma_j^z\sigma_k^y\sigma_{k+1}^y\sigma_l^z\right)\right] + \sum_{i,j,k,l=1}^{n} J^4\,\mathrm{tr}\left(\sigma_i^z\sigma_j^z\sigma_k^z\sigma_l^z\right) \qquad (48)\\
&= 2(n-1)(3n-5)d + 4(n-1)^2d + 8J^2(n-1)^2d + 2(n-3)^2d + 4J^2(n-3)^2d + J^4(n-1)(3n-5)d \\
&\quad + 8J^2n(n-1)d + 8J^2\left(-2(n-1)d\right) + 6J^4(n^2+3n-8)d + 4J^2(n-1)(n-4)d + 4J^2\left(2(n-1)d\right) + J^4n(3n-2)d \qquad (49)\\
&= \left[12\left(J^2+1\right)^2n^2 + 4\left(2J^4-19J^2-9\right)n - 43J^4 + 68J^2 + 32\right]d \qquad (50)\\
&\simeq 12(J^2+1)^2n^2d \sim N^2d, \qquad (51)
\end{aligned}
$$

where in the first equation we only show the nonzero unique contributions and the coefficient ahead of each term counts its repetitions.

We leave observables with more body interaction for future work though it does not change the major conclusion/scaling of this work.

Supplementary Note 5. METHOD IN ENSEMBLE AVERAGE CALCULATION

To assist the following discussion, we present the expression of first order gradient of residual error by commutators as

$$
\frac{\partial \epsilon}{\partial \theta_\ell} = \partial_{\theta_\ell}\left\langle\psi_0\right|U^\dagger(\boldsymbol{\theta})O\hat{U}(\boldsymbol{\theta})\left|\psi_0\right\rangle = \frac{i}{2}\left\langle\psi_0\right|U_{\ell^-}^\dagger\left[X_\ell, U_{\ell^+}^\dagger O U_{\ell^+}\right]U_{\ell^-}\left|\psi_0\right\rangle = \frac{i}{2}\left\langle\psi_0\right|U_{\ell^-}^\dagger\left[X_\ell, O_{\ell^+}\right]U_{\ell^-}\left|\psi_0\right\rangle, \qquad (52)
$$

where $|\psi_0\rangle$ is the initial pure state of system. Here we define the unitary notations U_{ℓ^-} as

$$
U_{\ell^-} = \prod_{k=1}^{\ell-1} W_kV_k(\theta_k), U_{\ell^+} = \prod_{k=\ell}^{L} W_kV_k(\theta_k), \qquad (53)
$$

and $O_{\ell^+} = U_{\ell^+}^\dagger O U_{\ell^+}$ for simplicity.

The second order gradient assuming $\ell_1 < \ell_2$ and $\ell_1 = \ell_2 = \ell$ can be written in a similar way as

$$
\frac{\partial^2\epsilon}{\partial\theta_{\ell_1}\partial\theta_{\ell_2}} = -\frac{1}{4}\left\langle\psi_0\right|U_{\ell_1^-}^\dagger[X_{\ell_1}, U_{\ell_1\to\ell_2}^\dagger[X_{\ell_2}, U_{\ell_2^+}^\dagger O U_{\ell_2^+}]U_{\ell_1\to\ell_2}]U_{\ell_1^-}\left|\psi_0\right\rangle = -\frac{1}{4}\left\langle\psi_0\right|U_{\ell_1^-}^\dagger[X_{\ell_1}, U_{\ell_1\to\ell_2}^\dagger[X_{\ell_2}, O_{\ell_2^+}]U_{\ell_1\to\ell_2}]U_{\ell_1^-}\left|\psi_0\right\rangle \qquad (54)
$$

$$
\frac{\partial^2\epsilon}{\partial\theta_\ell^2} = -\frac{1}{4}\left\langle\psi_0\right|U_{\ell^-}^\dagger[X_\ell, [X_\ell, O_{\ell^+}]]U_{\ell^-}\left|\psi_0\right\rangle, \qquad (55)
$$

where

$$
U_{\ell_1\to\ell_2} = \prod_{k=\ell_1}^{\ell_2-1} W_kV_k(\theta_k). \qquad (56)
$$

Supplementary Note 6. DEFINITION OF RESTRICTED HAAR RANDOM UNITARY ENSEMBLE

In this section, we provide the definition of restricted Haar (abbreviated as "RH") random ensemble, which will be used in Supplementary Note 7 and Supplementary Note 12. For the optimization problem considered in the main text, we always search for the optimal output state satisfying the constraint from minimum loss function. Equivalently, we can regard it as a state preparation problem with observable $O = |\Phi\rangle\langle\Phi|$. Therefore, the optimal unitary has to be able to map the trivial input state towards the target state while its mapping on the rest states orthogonal to input state can be arbitrary. We can then define the unitary to implement this operation as

$$U_{\mathrm{RH}} = \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & V \end{pmatrix}, \tag{57}$$

where V is a unitary of dimension $d-1$ and we assume $V \sim \mathcal{U}_{\mathrm{Haar}}(d-1)$. Here the columns correspond to orthonormal basis including $|\psi_0\rangle$ while rows correspond to the basis including $|\Phi\rangle$.

Supplementary Note 7. FRAME POTENTIAL WITH RESTRICTED HAAR ENSEMBLE

Supplementary Note 7.1. Frame potential applied to QNN

To quantify the randomness of an ensemble of unitaries, we evaluate the kth frame potential $\mathcal{F}^{(k)}$. For an arbitrary unitary ensemble $\mathcal{E}$,

$$\mathcal{F}_{\mathcal{E}}^{(k)} = \frac{1}{|\mathcal{E}|^2} \sum_{U,U' \in \mathcal{E}} |\operatorname{tr}\left(U^\dagger U'\right)|^{2k} \geq \mathcal{F}_{\mathrm{Haar}}^{(k)} = k!, \tag{58}$$

where the minimum is achieved by Haar ensemble [1].

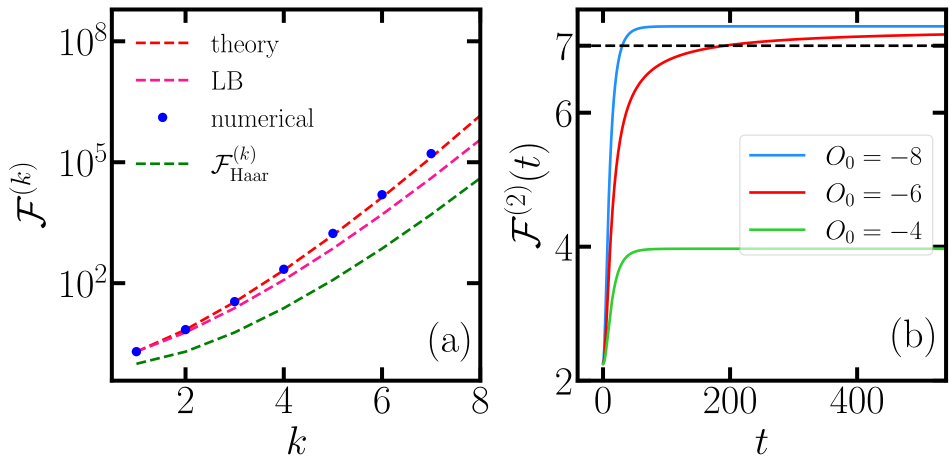
To provide insight into the ensemble in the case of $O_0 \leq O_{\min}$, we evaluate the frame potential of the restricted Haar ensemble

$$\mathcal{F}_{\mathrm{RH}}^{(k)} = \sum_{\substack{k_1,k_2=0 \\ k_1+2k_2 \leq k}}^{k} \frac{k!}{k_1!(k_2!)^2(k-k_1-2k_2)!} \mathcal{F}_{\mathrm{Haar}}^{(k_1+k_2)} \tag{59}$$

$$\geq \mathcal{F}_{\mathrm{Haar}}^{(k+1)} = (k+1)!. \tag{60}$$

We verify the above Eq. (59) and lower bound $(k+1)!$ in Supplementary Figure 4(a).

To verify that the ensemble distribution of unitaries of the QNN satisfy the restricted Haar ensemble in Eq. (17), ideally we want to consider the different unitaries from random initialization that lead to the same converged state, so that the ensemble averaged values can provide insight into a specific training dynamics where a specific converged state is observed. However, this is in general challenging as random initialization in general will lead to convergence to different local optimums, unless in the case of $O_0 \leq O_{\min}$, where the converged state will be the ground state, up to a finite degeneracy. In this case, we can directly evaluate the frame potential over late time unitaries from different initializations. We numerically evaluate the dynamics of 2-order frame potential $\mathcal{F}^{(2)}$ in Supplementary Figure 4(b) where different O_0 is considered. For $k = 2$, the frame potential $\mathcal{F}^{(k)}$ over the restricted Haar unitary ensemble is $\mathcal{F}^{(2)} = 7$ according to Eq. (59) and $\mathcal{F}^{(2)} = 2$ for Haar random ensemble. Indeed, we see that for $O_0 \leq O_{\min}$, the ensemble frame potential approaches to the prediction of restricted Haar ensemble. When $O_0 > O_{\min}$, $\mathcal{F}^{(2)}$ is far away from it, this is because the converged state is not unique for $O_0 > O_{\min}$ and different random initializations fail to provide the ensemble of unitaries with a fixed converged state: When one consider different random initialization, each training trajectory converges to a different state and the entire unitary ensemble under random initialization does not capture the restrictions and in fact approach Haar random. While in each single trajectory, the convergence still places a restriction on the typical unitary that maps the initial state to the final state.



Supplementary Figure 4. (a) Frame potential $F^{(k)}$ for the restricted Haar ensemble with dimension $2^n = 4$. Red dashed line is the exact theory prediction in Eq. (67) and magenta dashed line is its lower bound $(k+1)!$. (b) The evolution of 2-order frame potential $\mathcal{F}^{(2)}$ for ensemble of RPA circuit unitary with different target O_0. The observable is O_{XXZ} with $J = 2$, with $O_{\min} = -6$. Black dashed line is exact value of $\mathcal{F}^{(k)}$ in Eq. 59.

Supplementary Note 7.2. Details of formula

For simplicity, we assume V is a Haar random unitary in Eq. (57). The kth frame potential of the unitary ensemble is thus

$$\mathcal{F}^{(k)}_{\mathrm{RH}} = \frac{1}{|\mathcal{E}|^2}\sum_{U,U'\in\mathcal{E}} |\operatorname{tr}\big(U^\dagger U'\big)|^{2k} \tag{61}$$

$$= \frac{1}{|\mathcal{E}|^2}\sum_{U,U'\in\mathcal{E}} |1+\operatorname{tr}\big(V^\dagger V'\big)|^{2k} \tag{62}$$

$$= \int_{\mathrm{Haar}} dV dV' \left(1+\operatorname{tr}\big(V^\dagger V'\big)+\operatorname{tr}\big(V^\dagger V'\big)^* + |\operatorname{tr}\big(V^\dagger V'\big)|^2\right)^k. \tag{63}$$

where the second line comes from the definition in Eq. (57).

For simplicity, we denote $\operatorname{tr}\big(V^\dagger V'\big) \equiv z$ and then have

$$\mathcal{F}^{(k)}_{\mathrm{RH}} = \int_{\mathrm{Haar}} dV dV' \left(1+z+z^*+|z|^2\right)^k \tag{64}$$

$$= \sum_{\substack{k_1,k_2,k_3=0\\ k_1+k_2+k_3\le k}}^{k} \binom{k}{k_1,k_2,k_3} \int_{\mathrm{Haar}} dV dV' |z|^{2k_1} z^{*k_2} z^{k_3} \tag{65}$$

$$= \sum_{\substack{k_1,k_2=0\\ k_1+2k_2\le k}}^{k} \frac{k!}{k_1!(k_2!)^2(k-k_1-2k_2)!} \int_{\mathrm{Haar}} dV dV' |z|^{2k_1}|z|^{2k_2} \tag{66}$$

$$= \sum_{\substack{k_1,k_2=0\\ k_1+2k_2\le k}}^{k} \frac{k!}{k_1!(k_2!)^2(k-k_1-2k_2)!} \mathcal{F}^{(k_1+k_2)}_{\mathrm{Haar}} \tag{67}$$

$$\ge \mathcal{F}^{(k)}_{\mathrm{Haar}} + k^2 \mathcal{F}^{(k-1)}_{\mathrm{Haar}} = (k+1)\mathcal{F}^{(k)}_{\mathrm{Haar}} = \mathcal{F}^{(k+1)}_{\mathrm{Haar}}. \tag{68}$$

where in Eq. (66) we only keep $k_2 = k_3$ from the above line to keep frame potential to be real.

Supplementary Note 8. DETAILS ON DYNAMICS WITH LINEAR LOSS FUNCTION

In this section, we provide details on derive analytical theories for dynamics with linear loss functions. Following the same formalism, we can still parameterize the quantum neural network via a parameterized quantum circuit, for instance, random Haar ansatz we considered in the main text. The loss function to be considered is

$$\mathcal{L}(\boldsymbol{\theta}) = \langle\psi_0|U^\dagger(\boldsymbol{\theta}) O U(\boldsymbol{\theta})|\psi_0\rangle\,, \tag{69}$$

where ψ_0 is a trivial input state and $U(\boldsymbol{\theta})$ is the parameterized quantum circuit as usual. With sufficiently deep circuit, the loss function will be minimized to the ground state energy $O_{\min}$ of the observable, and thus we describe its converge via the residual error $\varepsilon(t) = \langle O\rangle - O_{\min}$.

Via gradient descent, each parameter is shifted as

$$\delta\theta_\ell(t) = -\eta\frac{\partial\mathcal{L}(\boldsymbol{\theta})}{\partial\theta_\ell} = -\eta\frac{\partial\varepsilon(\boldsymbol{\theta})}{\partial\theta_\ell}. \tag{70}$$

Thus under small learning rate $\eta \ll 1$, the total error is updated as

$$\delta\varepsilon(t) \simeq \sum_\ell \frac{\partial\epsilon(\boldsymbol{\theta})}{\partial\theta_\ell}\delta\theta_\ell + \frac{1}{2}\sum_{\ell_1,\ell_2}\frac{\partial^2\epsilon(\boldsymbol{\theta})}{\partial\theta_{\ell_1}\partial\theta_{\ell_2}}\delta\theta_{\ell_1}\delta\theta_{\ell_2} \tag{71}$$

$$= -\eta K(\boldsymbol{\theta})|_{\boldsymbol{\theta}=\boldsymbol{\theta}(t)} + \frac{1}{2}\eta^2\mu(\boldsymbol{\theta})|_{\boldsymbol{\theta}=\boldsymbol{\theta}(t)}, \tag{72}$$

where the QNTK $K(\boldsymbol{\theta})$ and dQNTK $\mu(\boldsymbol{\theta})$ is defined the same as in the main text. The dynamical equation in the first order of η is

$$\delta\varepsilon(t) = -\eta K(t) + \mathcal{O}(\eta^2). \tag{73}$$

We can also derive the dynamical equation of $K(t)$ as following. Recall from Supplementary Note 1,

$$\delta K(t) = \sum_\ell 2\delta\left(\left.\frac{\partial\varepsilon(\boldsymbol{\theta})}{\partial\theta_\ell}\right|_{\boldsymbol{\theta}=\boldsymbol{\theta}(t)}\right)\left.\frac{\partial\varepsilon(\boldsymbol{\theta})}{\partial\theta_\ell}\right|_{\boldsymbol{\theta}=\boldsymbol{\theta}(t)} + \delta\left(\left.\frac{\partial\varepsilon(\boldsymbol{\theta})}{\partial\theta_\ell}\right|_{\boldsymbol{\theta}=\boldsymbol{\theta}(t)}\right)\delta\left(\left.\frac{\partial\varepsilon(\boldsymbol{\theta})}{\partial\theta_\ell}\right|_{\boldsymbol{\theta}=\boldsymbol{\theta}(t)}\right), \tag{74}$$

where the second term is second order of η, and thus can be omitted in our calculation. The first term becomes

$$\delta\left(\left.\frac{\partial\varepsilon(\boldsymbol{\theta})}{\partial\theta_\ell}\right|_{\boldsymbol{\theta}=\boldsymbol{\theta}(t)}\right) = \sum_{\ell_1}\left.\frac{\partial^2\varepsilon(\boldsymbol{\theta})}{\partial\theta_{\ell_1}\partial\theta_\ell}\delta\theta_{\ell_1}\right|_{\boldsymbol{\theta}=\boldsymbol{\theta}(t)} + \mathcal{O}(\eta^2) = -\eta\sum_{\ell_1}\left.\frac{\partial^2\varepsilon(\boldsymbol{\theta})}{\partial\theta_{\ell_1}\partial\theta_\ell}\frac{\partial\varepsilon(\boldsymbol{\theta})}{\partial\theta_{\ell_1}}\right|_{\boldsymbol{\theta}=\boldsymbol{\theta}(t)} + \mathcal{O}(\eta^2). \tag{75}$$

So we have

$$\sum_\ell\delta\left(\left.\frac{\partial\varepsilon(\boldsymbol{\theta})}{\partial\theta_\ell}\right|_{\boldsymbol{\theta}=\boldsymbol{\theta}(t)}\right)\left.\frac{\partial\varepsilon(\boldsymbol{\theta})}{\partial\theta_\ell}\right|_{\boldsymbol{\theta}=\boldsymbol{\theta}(t)} = -\eta\sum_{\ell,\ell_1}\left.\frac{\partial^2\varepsilon(\boldsymbol{\theta})}{\partial\theta_\ell\partial\theta_{\ell_1}}\frac{\partial\varepsilon(\boldsymbol{\theta})}{\partial\theta_{\ell_1}}\frac{\partial\varepsilon(\boldsymbol{\theta})}{\partial\theta_\ell}\right|_{\boldsymbol{\theta}=\boldsymbol{\theta}(t)} + \mathcal{O}(\eta^2) = -\eta\mu(\boldsymbol{\theta})|_{\boldsymbol{\theta}=\boldsymbol{\theta}(t)} + \mathcal{O}(\eta^2), \tag{76}$$

which leads to the gradient descent dynamical equation of $K(t)$ as

$$\delta K(t) = -2\eta\mu(t) + \mathcal{O}(\eta^2). \tag{77}$$

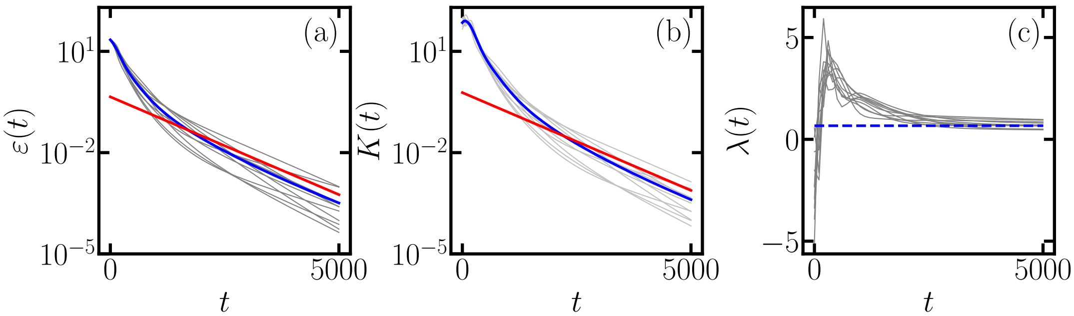
Recall that the relative dQNTK is defined as $\lambda(t) = \mu(t)/K(t)$, and again we assume it converges to a constant λ in late time (see Supplementary Figure 5(c)), the above dynamical equations in Eqs. (73), (77) can be reduced to

$$\begin{cases}\partial_t K(t) &= -2\eta\lambda K(t)\\ \partial_t\varepsilon(t) &= -\eta K(t)\end{cases} \tag{78}$$

Notice that the dynamics of $K(t)$ is self-consistent and $\varepsilon(t)$ is fully controlled by $K(t)$, we can then directly solve the dynamics of $\varepsilon(t)$ and $K(t)$ as

$$2\lambda\varepsilon(t) = K(t) = Ae^{-2\eta\lambda t}, \tag{79}$$

where is exactly the solution we propose in Eq. (25) in the main text. Here A is a free parameter to be fitted. Both residual error and QNTK exponentially decays to zero in the late time at a fixed rate, and the dynamical transition does not persist.



Supplementary Figure 5. Dynamics in QNN in the example of XXZ model with linear loss function. Dynamics of residual error (a) $\varepsilon(t)$ and (b) QNTK $K(t)$ are plotted. Grey lines represent simulation with random initializations and blue lines represent the ensemble average. Red dashed lines show the theory model from Eq. (79). In (c), we show the convergence of dQNTK $\lambda(t)$. Blue dashed line represent the average as $\overline{\lambda} = \overline{\mu}/\overline{K}$. Here the random Pauli ansatz (RPA) consists of $L = 192$ parameters on $n = 6$ qubits, and the observable is XXZ model with $J = 2$.

Supplementary Note 9. MORE DETAILS ON NUMERICAL RESULTS

We first present the training dynamics of another well-established controllable QNN ansatz, hardware efficient ansatz (HEA) [2], and optimize the corresponding quadratic loss function. Suppose the HEA consists of D layers on n qubits, in each layer RY and RZ single-qubit gates are applied on every qubit, where each of them includes a trainable parameters, and single-qubit gates are followed by CNOT gate arranged in brickwall style on nearest neighbors. Therefore, for a D-layer HEA, the total number of trainable parameters is $L = 2nD$. In the optimization problem, we choose the observable to be the Hamiltonian of transverse-field Ising model (TFIM) as

$$O_{\text{TFIM}} = -\sum_i \sigma_i^z \sigma_{i+1}^z - h \sum_i \sigma_i^x, \tag{80}$$

where h describes the strength of transverse field. In Supplementary Figure 6, we show the training dynamics of error and kernel with $O_0 \gtreqless O_{\min}$, where the two branches of dynamics and the critical point can also be identified. We still see alignment between numerical simulation (blue) and our theories (red), indicating the applicable of our theory to characterize the dynamics for general deep controllable QNNs.

Next, we provide numerical evidence to support the assumption in the main text that at late time, relative dQNTK $\lambda(t)$ converges to a constant in a relatively shallow QNN. In Supplementary Figure 7, we show the dynamics of $\lambda(t)$ of random samples in Fig. 6 in the main text, and see that $\lambda(t)$ converges to a constant no matter $O_{\min} \gtreqless O_{\min}(L)$ though the fluctuation among different samples is comparably large in shallow QNNs.

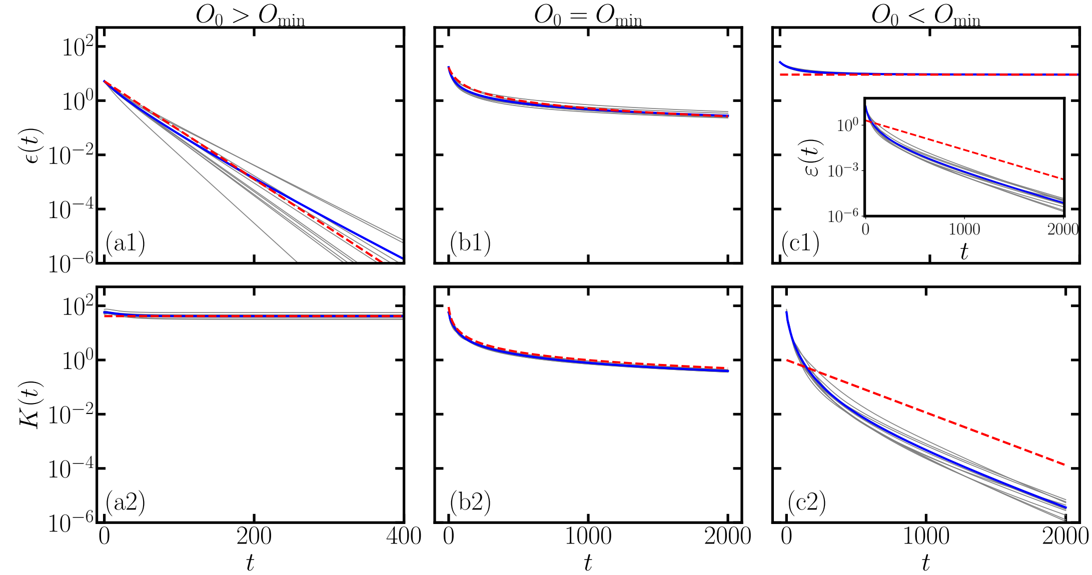
In the following, we show more numerical results associated with the restricted Haar ensemble. Recall the definition of $\lambda_\infty \equiv \mu_\infty/K_\infty$, we can have the average of λ_∞ defined via the following two as $\mathbb{E}[\lambda_\infty] \equiv \overline{\mu_\infty/K_\infty}$, and $\overline{\lambda_\infty} \equiv \overline{\mu_\infty}/\overline{K_\infty}$. In Supplementary Figure 8(a1)(b1), we see that with increasing of L, the discrepancy between $\mathbb{E}[\lambda_\infty]$ and $\overline{\lambda_\infty}$ vanishes for different kind of observables like state projector and XXZ model, which suggests it is free to exchange the definition of average as we have done in definition of relative sample fluctuation of λ_∞ in the main text. Similarly, for $\zeta_\infty \equiv \epsilon_\infty\mu_\infty/K_\infty^2$, we see similar results for $\mathbb{E}[\zeta_\infty] \equiv \overline{\epsilon_\infty\mu_\infty/K_\infty^2}$ and $\overline{\zeta_\infty} \equiv \overline{\epsilon_\infty\mu_\infty}/\overline{K_\infty}^2$. In Supplementary Figure 9, we show the relative sample fluctuation of λ_∞ and ζ_∞ versus parameters L (thus depth D in RPA) with observable to be the XXZ model. The findings discussed in the main text for state projector observable also hold here.

At the end, we provide more details on the training dynamics with linear loss function. In Supplementary Figure 5(a)(b), we see good agreement between our theory (red dashed lines) in Eq. (79) and numerical simulation results (blue lines). Note that the exponential decay of total error is also found in a recent work [3] with similar interest via the Riemann gradient flow formalism.

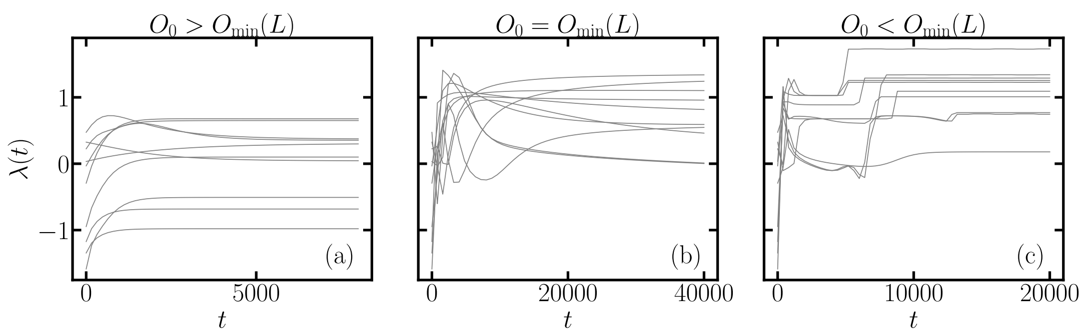
All our numerical simulation is performed with `TensorCircuit` [4] on `Jax` backend.

Supplementary Note 10. DETAILS OF EXPERIMENTS

In this section, we provide our noisy model to characterize the dynamics on noisy devices. Given a noiseless idea prediction $\epsilon_{\text{ideal}}(t) = \text{tr}\big(U\rho_0 U^\dagger O\big) - O_0$, we consider the depolarizing noise model $\mathcal{N}_p(\rho) = (1-p)\rho + p\boldsymbol{I}/d$ where $\boldsymbol{I}$



Supplementary Figure 6. Dynamics in QNN in the example of TFIM model. The top and bottom panel shows the dynamics of total error $\epsilon(t)$ and QNTK $K(t)$ with respect to the three cases $O_0 \gtreqless O_{\rm min}$. Blue solid curves represent numerical ensemble average result. Red dashed curves in panels represents theoretical predictions on the dynamics of total error in Eq. (14), (15), (16) (from left to right). Grey solid lines show the dynamics for each random sample. The inset in (c1) shows the exponentially decay of residual error $\varepsilon(t)$. Here hardware efficient ansatz (HEA) consists of $D = 48$ layers, equivalently $L = 768$ variational parameters, on $n = 8$ qubits, and the parameter in TFIM model is $h = 2$.

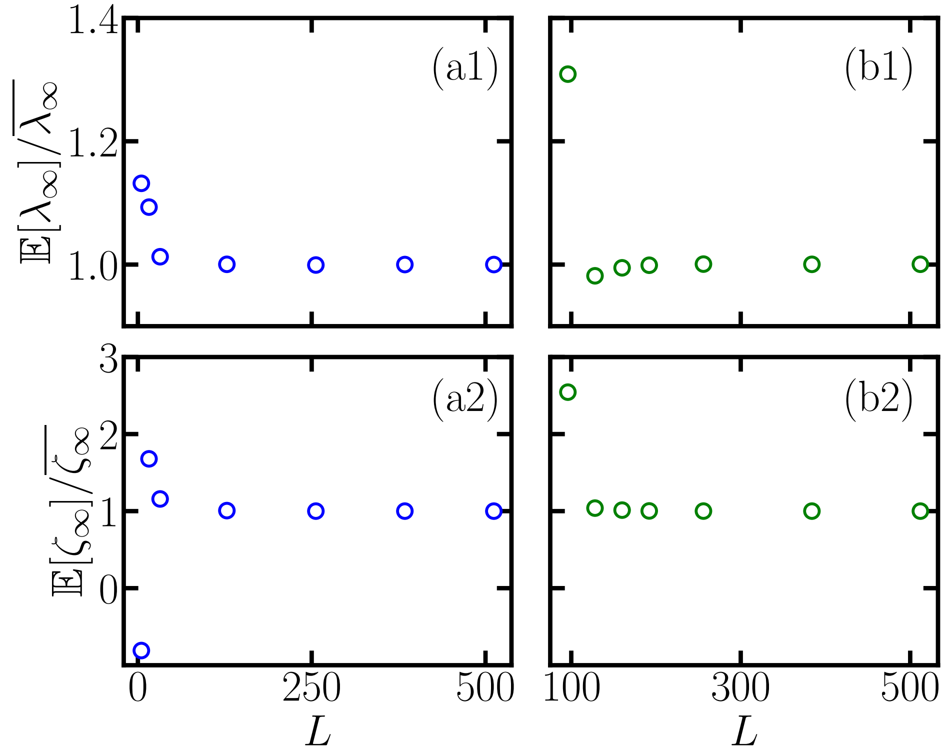


Supplementary Figure 7. Dynamics of relative dQNTK $\lambda(t)$ in shallow QNN in the example of XXZ model with $O_0 \gtreqless O_{\rm min}(L)$ (from left to right). The critical point $O_{\rm min}(L)$ for shallow QNN depends on L. Here random Pulai ansatz (RPA) consists of $L = 6$ variational parameters ($D = L$ for RPA) on $n = 6$ qubits, and the parameter in XXZ model is $J = 2$.

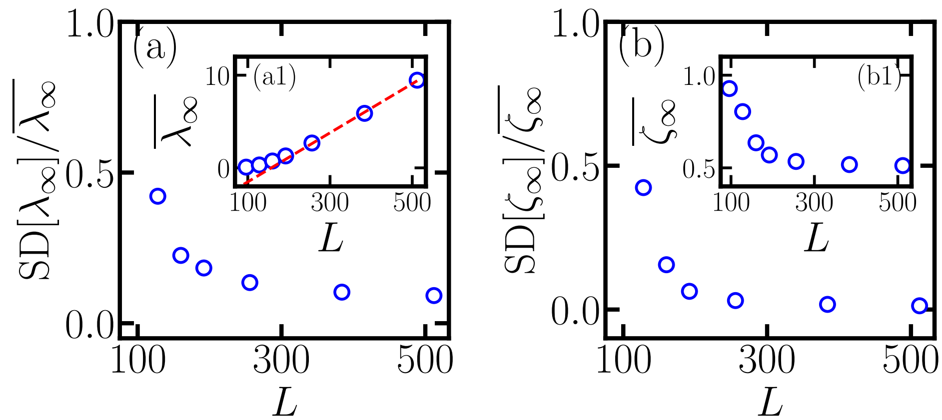
is the identity matrix of dimension d. The depolarizing noisy total error $\epsilon_{\rm dp}(t)$ then becomes

$$\epsilon_{\rm dp}(t) = {\rm tr}\big(\mathcal{N}_p(U\rho_0 U^\dagger)O\big) - O_0 \tag{81}$$
$$= (1-p)\left[{\rm tr}\big(U\rho_0 U^\dagger O\big) - O_0\right] + p\left[{\rm tr}(O)/d - O_0\right] \tag{82}$$
$$= (1-p)\epsilon_{\rm ideal}(t) - pO_0, \tag{83}$$

where in the last line we assume O is a traceless observable. Therefore, compared to ideal case, the total error under depolarizing noise model is simply shifted by a constant depending on target. The residual error can also be studied similarly with $\varepsilon_{\rm dp}(t) = \epsilon_{\rm dp}(t) - R_{\rm dp}$, where $R_{\rm dp} = \lim_{t\to\infty}\epsilon_{\rm dp}(t)$ could be different from ideal case due to noise. For $O_0 \geq O_{\rm min}$, we have $R_{\rm dp} = -pO_0$ while for $O_0 < O_{\rm min}$, we have $R_{\rm dp} = (1-p)O_{\rm min} - O_0$. In Supplementary Figure 10, we provide intuitions via the difference $\epsilon(t) - \epsilon_{\rm ideal}(t)$, and for different O_0, we see that the differences quickly approach to a constant. We can further estimate the depolarizing probability p via least error as follows. For a given O_0, we have multiple data points from ideal simulation $\{\epsilon_{\rm ideal}(t_i)\}_{i=1}^M$, and true experimental data $\{\epsilon(t_i)\}_{i=1}^M$.



Supplementary Figure 8. Comparison between $\mathbb{E}[\lambda_\infty]$ and $\overline{\lambda_\infty}$ (top) as well as ζ_∞ (bottom) versus L. Blue dots left panel represent results for state projector observable in $n = 5$ qubit system with $O_0 = 1$. Green dots in right panel show results for XXZ model with $J = 2$ in $n = 6$ qubit system with $O_0 = O_{\min}$.



Supplementary Figure 9. Late-time scaling of relative dQNTK λ_∞ and dynamical index ζ_∞ on number of parameters D. Relative sample fluctuation of (a) λ_∞ and (b) ζ_∞ versus L at late time. Inset (a1) and (b1) shows the average $\overline{\lambda_\infty} \propto L$ and $\overline{\zeta_\infty} \to 1/2$ separately. Red dashed line in (a1) is the linear fitting of $\overline{\lambda_\infty}$ over L. Here the RPA is applied on $n = 6$ qubits with different L parameters (equivalently D layers). The observable and target is XXZ model with $J = 2$ and the target is $O_0 = O_{\min} = -22$.

From Eq. (83), the SSR (sum squared error) is

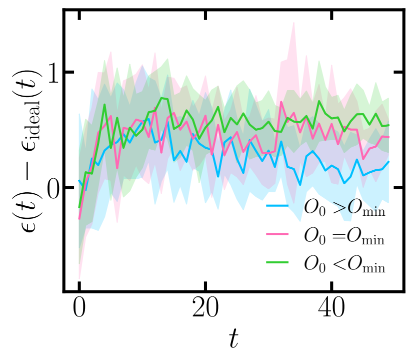
$$\mathrm{SSR} = \sum_{i=1}^{M} \left[\epsilon(t_i) - (1-p)\epsilon_{\mathrm{ideal}}(t_i) + pO_0\right]^2, \tag{84}$$

and the minimum with respect to p is

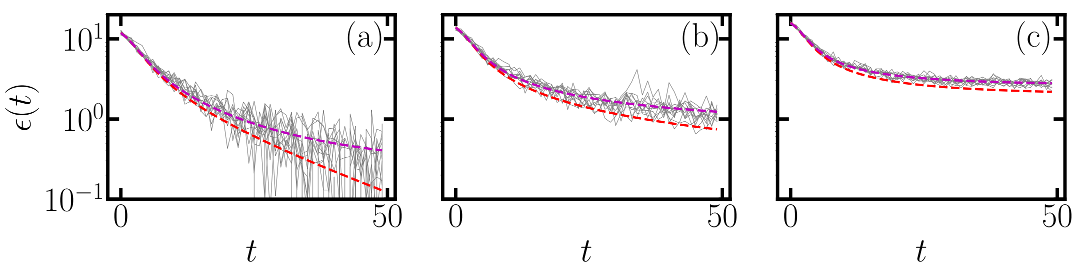
$$0 = \frac{\partial \mathrm{SSR}}{\partial p} = 2\sum_i \left[\epsilon(t_i) - (1-p)\epsilon_{\mathrm{ideal}}(t_i) + pO_0\right]\left[\epsilon_{\mathrm{ideal}}(t_i) + O_0\right] \tag{85}$$

$$p = \frac{\sum_i \left[\epsilon_{\mathrm{ideal}}(t_i) - \epsilon(t_i)\right]\left[\epsilon_{\mathrm{ideal}}(t_i) + O_0\right]}{\sum_i \left[\epsilon_{\mathrm{ideal}}(t_i) + O_0\right]^2}. \tag{86}$$

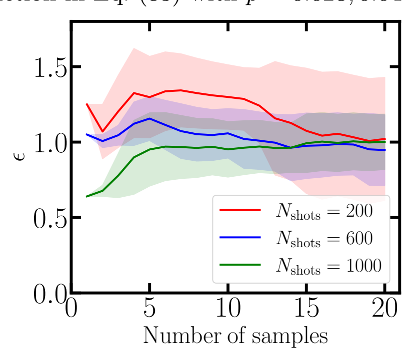
For the three different $O_0 \gtreqless O_{\min}$, we find $p = 0.028 \pm 0.007, 0.044 \pm 0.011, 0.051 \pm 0.005$ which are close and the fluctuation could origin from other types of error other than depolarizing noise and the drift of device. We then verify our noisy model via different experimental trajectories in Supplementary Figure 11 . In contrast to the ideal simulation (red dashed line), the noisy model (magenta) shows a better agreement. We see that each trial also follows the scaling though larger fluctuation exists due to the other circuit and measurement noise.



Supplementary Figure 10. Difference between experimental result $\epsilon(t)$ and ideal noiseless simulation result $\epsilon_{\mathrm{ideal}}(t)$ for different O_0. Shaded areas of different colors represent the fluctuation with corresponding O_0. Here the hardware efficient ansatz (HEA) consists of $D = 4$ layers on $n = 2$ qubits, and the observable is XXZ model with $J = 2$.



Supplementary Figure 11. Dynamical trajectories (grey) of each measurement trial of experiment on `IBM Kolkata` in Fig. 8 of the main text. From (a)-(c) we present $O_0 \gtreqless O_{\min}$. Red dashed lines represent the ideal simulation for reference. Magenta dashed lines represent the noisy model prediction in Eq. (83) with $p = 0.028, 0.044, 0.051$ (from left to right).



Supplementary Figure 12. Experiment result of the error $\epsilon(t)$ for different number of shots in the measurement repetitions for estimating the observable in an individual sample. Multiple samples of training are performed to show the sample fluctuation. As the number of samples increase, the average ϵ becomes almost independent of the number of the shots. The shaded region indicate the sample fluctuation in standard deviation. $O_0 = 10$ and the results are at late time of $t = 40$.

To rule out other noise sources causing the deviation, we also consider changing the number of shots of measurement in the estimation of the observable. As we see in Supplementary Figure 12, the average error all converge to around the same value, regardless of the number of shots in the estimation. For 20 rounds of experiment, we find the error as $\epsilon = 1.02 \pm 0.41, 0.95 \pm 0.23, 1.00 \pm 0.18$ for number of measurement shots $N_{\text{shots}} = 200, 600, 1000$. The difference is much smaller than the sample fluctuation between multiple runs of the experiments.

Supplementary Note 11. RESULTS WITH HAAR RANDOM ENSEMBLE

In this section, we present results evaluated from Haar random unitary ensemble, which provides characterizations of QNN dynamics at early time. The rest of the contents are regarding ensemble averaging over Haar (App. Supplementary Note 11) or restricted Haar ensemble (App. Supplementary Note 12), where we have utilized symbolic tools RTNI from Ref. [5].

Supplementary Note 11.1. Average QNTK under Haar random ensemble

With random initialization, the circuit forms a Haar random ensemble, and the ensemble average of QNTK is $\overline{K_0} = \sum_\ell \mathbb{E}_{\mathcal{U}_{\text{Haar}}}\left[\left(\frac{\partial\epsilon}{\partial\theta_\ell}\right)^2\right]$, where the ensemble average inside the summation is

$$\begin{aligned}
\mathbb{E}_{\mathcal{U}_{\text{Haar}}}\left[\left(\frac{\partial\epsilon}{\partial\theta_\ell}\right)^2\right] &= -\frac{1}{4}\int_{\mathcal{U}_{\text{Haar}}} dU_{\ell-}dU_{\ell+}\,\langle\psi_0|U_{\ell-}^\dagger\,[X_\ell, O_{\ell+}]\,U_{\ell-}|\psi_0\rangle\,\langle\psi_0|U_{\ell-}^\dagger\,[X_\ell, O_{\ell+}]\,U_{\ell-}|\psi_0\rangle && (87)\\
&= -\frac{1}{4}\int_{\mathcal{U}_{\text{Haar}}} dU_{\ell-}dU_{\ell+}\,\mathrm{tr}\left(\rho_0 U_{\ell-}^\dagger\,[X_\ell, O_{\ell+}]\,U_{\ell-}\rho_0 U_{\ell-}^\dagger\,[X_\ell, O_{\ell+}]\,U_{\ell-}\right) && (88)\\
&= -\int_{\mathcal{U}_{\text{Haar}}} dU_{\ell+}\frac{\mathrm{tr}\left([X_\ell, O_{\ell+}]^2\right)}{4(d^2+d)}, && (89)
\end{aligned}$$

where $\rho_0 = |\psi_0\rangle\langle\psi_0|$. Using the trace identity $\mathrm{tr}\big([A,B]^2\big) = 2\,\mathrm{tr}(ABAB) - 2\,\mathrm{tr}(ABBA)$, we then have

$$\begin{aligned}
\mathbb{E}_{\mathcal{U}_{\text{Haar}}}\left[\left(\frac{\partial\epsilon}{\partial\theta_\ell}\right)^2\right] &= -\int_{\mathcal{U}_{\text{Haar}}} dU_{\ell+}\frac{\mathrm{tr}\left([X_\ell, O_{\ell+}]^2\right)}{4(d^2+d)} = -\frac{2}{4(d^2+d)}\left[\int_{\mathcal{U}_{\text{Haar}}} dU_{\ell+}\,\mathrm{tr}\left(X_\ell O_{\ell+} X_\ell O_{\ell+}\right) - \mathrm{tr}\big(O^2\big)\right] && (90)\\
&= \frac{d\,\mathrm{tr}\big(O^2\big) - \mathrm{tr}(O)^2}{2(d-1)(d+1)^2}. && (91)
\end{aligned}$$

Note that here we assume both $U_{\ell-}$ and $U_{\ell+}$ form a Haar random ensemble (2-design) separately. Therefore, the ensemble average of QNTK is

$$\overline{K_0} = \sum_\ell \mathbb{E}_{\mathcal{U}_{\text{Haar}}}\left[\left(\frac{\partial\epsilon}{\partial\theta_\ell}\right)^2\right] = L\frac{d\,\mathrm{tr}\big(O^2\big) - \mathrm{tr}(O)^2}{2(d-1)(d+1)^2} \simeq \frac{L}{2d^3}\left(d\,\mathrm{tr}\big(O^2\big) - \mathrm{tr}(O)^2\right), \tag{92}$$

where we approximate it by $d \gg 1$ in the last equation to simplify expression. Specifically, for the traceless operator we considered in the main text, we have

$$\overline{K_0} \simeq \frac{L}{2d^2}\,\mathrm{tr}\big(O^2\big). \tag{93}$$

Supplementary Note 11.2. Average relative dQNTK under Haar random ensemble

We define the average $\overline{\lambda}$ as

$$\overline{\lambda} = \overline{\mu}/\overline{K}, \tag{94}$$

where dQNTK μ is defined as

$$\mu = \sum_{\ell_1,\ell_2} \frac{\partial^2 \epsilon}{\partial\theta_{\ell_1}\partial\theta_{\ell_2}} \frac{\partial \epsilon}{\partial\theta_{\ell_1}} \frac{\partial \epsilon}{\partial\theta_{\ell_2}}. \tag{95}$$

In the following, we calculate the Haar ensemble average of dQNTK as

$$\overline{\mu_0} = 2\sum_{\ell_1<\ell_2} \mathbb{E}_{\mathcal{U}_{\text{Haar}}}\left[\frac{\partial^2 \epsilon}{\partial\theta_{\ell_1}\partial\theta_{\ell_2}} \frac{\partial \epsilon}{\partial\theta_{\ell_1}} \frac{\partial \epsilon}{\partial\theta_{\ell_2}}\right] + \sum_{\ell} \mathbb{E}_{\mathcal{U}_{\text{Haar}}}\left[\frac{\partial^2 \epsilon}{\partial\theta_{\ell}^2}\left(\frac{\partial \epsilon}{\partial\theta_{\ell}}\right)^2\right] \tag{96}$$

$$= L(L-1)\mathbb{E}_{\mathcal{U}_{\text{Haar}}}\left[\frac{\partial^2 \epsilon}{\partial\theta_{\ell_1}\partial\theta_{\ell_2}} \frac{\partial \epsilon}{\partial\theta_{\ell_1}} \frac{\partial \epsilon}{\partial\theta_{\ell_2}}\right] + L\mathbb{E}_{\mathcal{U}_{\text{Haar}}}\left[\frac{\partial^2 \epsilon}{\partial\theta_{\ell}^2}\left(\frac{\partial \epsilon}{\partial\theta_{\ell}}\right)^2\right]. \tag{97}$$

a. Calculation of $\frac{\partial^2 \epsilon}{\partial\theta_{\ell}^2}\left(\frac{\partial \epsilon}{\partial\theta_{\ell}}\right)^2$ *with Haar random ensemble*

We first evaluate the summation of $\frac{\partial^2 \epsilon}{\partial\theta_{\ell}^2}\left(\frac{\partial \epsilon}{\partial\theta_{\ell}}\right)^2$

$$\mathbb{E}_{\mathcal{U}_{\text{Haar}}}\left[\frac{\partial^2 \epsilon}{\partial\theta_{\ell}^2}\left(\frac{\partial \epsilon}{\partial\theta_{\ell}}\right)^2\right] = \frac{1}{16}\int_{\mathcal{U}_{\text{Haar}}} dU_{\ell^-}dU_{\ell^+}\, \langle\psi_0|U_{\ell^-}^\dagger[X_\ell,[X_\ell,O_{\ell^+}]]U_{\ell^-}|\psi_0\rangle \langle\psi_0|U_{\ell^-}^\dagger\left[X_\ell,O_{\ell^+}\right]U_{\ell^-}|\psi_0\rangle^2 \tag{98}$$

$$= \frac{1}{16}\int_{\mathcal{U}_{\text{Haar}}} dU_{\ell^-}dU_{\ell^+}\, \mathrm{tr}\left(\rho_0 U_{\ell^-}^\dagger[X_\ell,[X_\ell,O_{\ell^+}]]U_{\ell^-}\rho_0 U_{\ell^-}^\dagger\left[X_\ell,O_{\ell^+}\right]U_{\ell^-}\rho_0 U_{\ell^-}^\dagger\left[X_\ell,O_{\ell^+}\right]U_{\ell^-}\right) \tag{99}$$

$$= \int_{\mathcal{U}_{\text{Haar}}} dU_{\ell^+}\frac{\mathrm{tr}\left([X_\ell,[X_\ell,O_{\ell^+}]]\left[X_\ell,O_{\ell^+}\right]^2\right)}{8d(2+3d+d^2)} \tag{100}$$

$$= 0, \tag{101}$$

where the last equation is obtained from the trace cyclic identity.

b. Calculation of $\frac{\partial^2 \epsilon}{\partial\theta_{\ell_1}\partial\theta_{\ell_2}} \frac{\partial \epsilon}{\partial\theta_{\ell_1}} \frac{\partial \epsilon}{\partial\theta_{\ell_2}}$ *with Haar random ensemble*

We next consider the term $\frac{\partial^2 \epsilon}{\partial\theta_{\ell_1}\partial\theta_{\ell_2}} \frac{\partial \epsilon}{\partial\theta_{\ell_1}} \frac{\partial \epsilon}{\partial\theta_{\ell_2}}$ assuming $\ell_1 < \ell_2$ as

$$\mathbb{E}_{\mathcal{U}_{\text{Haar}}}\left[\frac{\partial^2 \epsilon}{\partial\theta_{\ell_1}\partial\theta_{\ell_2}} \frac{\partial \epsilon}{\partial\theta_{\ell_1}} \frac{\partial \epsilon}{\partial\theta_{\ell_2}}\right]$$

$$= \frac{1}{16}\int_{\mathcal{U}_{\text{Haar}}} dU_{\ell_1^-}dU_{\ell_1\to\ell_2}dU_{\ell_2^+}\, \langle\psi_0|U_{\ell_1^-}^\dagger[X_{\ell_1},U_{\ell_1\to\ell_2}^\dagger[X_{\ell_2},O_{\ell_2^+}]U_{\ell_1\to\ell_2}]U_{\ell_1^-}|\psi_0\rangle \langle\psi_0|U_{\ell_1^-}^\dagger[X_{\ell_1},O_{\ell_1^+}]U_{\ell_1^-}|\psi_0\rangle$$
$$\times \langle\psi_0|U_{\ell_1^-}^\dagger U_{\ell_1\to\ell_2}^\dagger[X_{\ell_2},O_{\ell_2^+}]U_{\ell_1\to\ell_2}U_{\ell_1^-}|\psi_0\rangle \tag{102}$$

$$= \frac{1}{16}\int_{\mathcal{U}_{\text{Haar}}} dU_{\ell_1^-}dU_{\ell_1\to\ell_2}dU_{\ell_2^+} \tag{103}$$

$$\mathrm{tr}\left(\rho_0 U_{\ell_1^-}^\dagger[X_{\ell_1},U_{\ell_1\to\ell_2}^\dagger\left[X_{\ell_2},O_{\ell_2^+}\right]U_{\ell_1\to\ell_2}]U_{\ell_1^-}\rho_0 U_{\ell_1^-}^\dagger\left[X_{\ell_1},O_{\ell_1^+}\right]U_{\ell_1^-}\rho_0 U_{\ell_1^-}^\dagger U_{\ell_1\to\ell_2}^\dagger\left[X_{\ell_2},O_{\ell_2^+}\right]U_{\ell_1\to\ell_2}U_{\ell_1^-}\right) \tag{104}$$

$$= \frac{1}{16(d^3+3d^2+2d)}\int_{\mathcal{U}_{\text{Haar}}} dU_{\ell_1\to\ell_2}dU_{\ell_2^+}\left[\mathrm{tr}\left(\left[X_{\ell_1},U_{\ell_1\to\ell_2}^\dagger\left[X_{\ell_2},O_{\ell_2^+}\right]U_{\ell_1\to\ell_2}\right]\left[X_{\ell_1},O_{\ell_1^+}\right]U_{\ell_1\to\ell_2}^\dagger\left[X_{\ell_2},O_{\ell_2^+}\right]U_{\ell_1\to\ell_2}\right)\right.$$
$$\left. + \mathrm{tr}\left(\left[X_{\ell_1},O_{\ell_1^+}\right][X_{\ell_1},U_{\ell_1\to\ell_2}^\dagger\left[X_{\ell_2},O_{\ell_2^+}\right]U_{\ell_1\to\ell_2}]U_{\ell_1\to\ell_2}^\dagger\left[X_{\ell_2},O_{\ell_2^+}\right]U_{\ell_1\to\ell_2}\right)\right], \tag{105}$$

where in the last equation we do the Haar integration (4-design) over $U_{\ell_1^-}$. Next we evaluation the Haar integral over $U_{\ell_1\to\ell_2}, U_{\ell_2^+}$ separately.

The first term $\operatorname{tr}\left(\left[X_{\ell_1}, U^\dagger_{\ell_1\to\ell_2}\left[X_{\ell_2}, O_{\ell_2^+}\right]U_{\ell_1\to\ell_2}\right]\left[X_{\ell_1}, O_{\ell_1^+}\right]U^\dagger_{\ell_1\to\ell_2}\left[X_{\ell_2}, O_{\ell_2^+}\right]U_{\ell_1\to\ell_2}\right)$ is further integrated as

$$\begin{aligned}
&\int_{\mathcal{U}_{\mathrm{Haar}}} dU_{\ell_1\to\ell_2} dU_{\ell_2^+} \operatorname{tr}\left(\left[X_{\ell_1}, U^\dagger_{\ell_1\to\ell_2}\left[X_{\ell_2}, O_{\ell_2^+}\right]U_{\ell_1\to\ell_2}\right]\left[X_{\ell_1}, O_{\ell_1^+}\right]U^\dagger_{\ell_1\to\ell_2}\left[X_{\ell_2}, O_{\ell_2^+}\right]U_{\ell_1\to\ell_2}\right)\\
&= \int_{\mathcal{U}_{\mathrm{Haar}}} dU_{\ell_1\to\ell_2} dU_{\ell_2^+} \left[\operatorname{tr}\left(X_{\ell_1}U^\dagger_{\ell_1\to\ell_2}\left[X_{\ell_2}, O_{\ell_2^+}\right]U_{\ell_1\to\ell_2}X_{\ell_1}U^\dagger_{\ell_1\to\ell_2}O_{\ell_2^+}\left[X_{\ell_2}, O_{\ell_2^+}\right]U_{\ell_1\to\ell_2}\right)\right.\\
&\qquad\qquad + \operatorname{tr}\left(\left[X_{\ell_2}, O_{\ell_2^+}\right]^2 U_{\ell_1\to\ell_2}X_{\ell_1}U^\dagger_{\ell_1\to\ell_2}O_{\ell_2^+}U_{\ell_1\to\ell_2}X_{\ell_1}U^\dagger_{\ell_1\to\ell_2}\right)\\
&\qquad\qquad \left. - \operatorname{tr}\left(X_{\ell_1}U^\dagger_{\ell_1\to\ell_2}\left[X_{\ell_2}, O_{\ell_2^+}\right]O_{\ell_2^+}U_{\ell_1\to\ell_2}X_{\ell_1}U^\dagger_{\ell_1\to\ell_2}\left[X_{\ell_2}, O_{\ell_2^+}\right]U_{\ell_1\to\ell_2}\right) - \operatorname{tr}\left(\left[X_{\ell_2}, O_{\ell_2^+}\right]^2 O_{\ell_2^+}\right)\right] \qquad (106)\\
&= \int_{\mathcal{U}_{\mathrm{Haar}}} dU_{\ell_2^+}\left[\frac{\operatorname{tr}\left(O_{\ell_2^+}\left[X_{\ell_2}, O_{\ell_2^+}\right]^2\right)}{1-d^2} + \frac{d\operatorname{tr}\left(\left[X_{\ell_2}, O_{\ell_2^+}\right]^2\right)\operatorname{tr}(O) - \operatorname{tr}\left(O_{\ell_2^+}\left[X_{\ell_2}, O_{\ell_2^+}\right]^2\right)}{d^2-1} - \frac{\operatorname{tr}\left(O_{\ell_2^+}\left[X_{\ell_2}, O_{\ell_2^+}\right]^2\right)}{1-d^2}\right.\\
&\qquad\qquad \left. - \operatorname{tr}\left(O_{\ell_2^+}\left[X_{\ell_2}, O_{\ell_2^+}\right]^2\right)\right] \qquad (107)\\
&= \int_{\mathcal{U}_{\mathrm{Haar}}} dU_{\ell_2^+}\frac{d\left[\operatorname{tr}\left(\left[X_{\ell_2}, O_{\ell_2^+}\right]^2\right)\operatorname{tr}(O) - d\operatorname{tr}\left(O_{\ell_2^+}\left[X_{\ell_2}, O_{\ell_2^+}\right]^2\right)\right]}{d^2-1} \qquad (108)\\
&= \frac{d}{d^2-1}\left(\operatorname{tr}(O)\frac{2d\left[\operatorname{tr}(O)^2 - d\operatorname{tr}(O^2)\right]}{d^2-1} - d\frac{d\left[\operatorname{tr}(O)\operatorname{tr}(O^2) - d\operatorname{tr}(O^3)\right]}{d^2-1}\right) \qquad (109)\\
&= \frac{d^2\left[d^2\operatorname{tr}(O^3) - 3d\operatorname{tr}(O^2)\operatorname{tr}(O) + 2\operatorname{tr}(O)^3\right]}{(d^2-1)^2}, \qquad (110)
\end{aligned}$$

where we perform Haar unitary integral over $U_{\ell_1\to\ell_2}$ and $U_{\ell_2^+}$ to obtain Eqs. (107), (109) correspondingly.

The second term $\operatorname{tr}\left(\left[X_{\ell_1}, O_{\ell_1^+}\right]\left[X_{\ell_1}, U^\dagger_{\ell_1\to\ell_2}\left[X_{\ell_2}, O_{\ell_2^+}\right]U_{\ell_1\to\ell_2}\right]U^\dagger_{\ell_1\to\ell_2}\left[X_{\ell_2}, O_{\ell_2^+}\right]U_{\ell_1\to\ell_2}\right)$ becomes

$$\begin{aligned}
&\int_{\mathcal{U}_{\mathrm{Haar}}} dU_{\ell_1\to\ell_2} dU_{\ell_2^+} \operatorname{tr}\left(\left[X_{\ell_1}, O_{\ell_1^+}\right]\left[X_{\ell_1}, U^\dagger_{\ell_1\to\ell_2}\left[X_{\ell_2}, O_{\ell_2^+}\right]U_{\ell_1\to\ell_2}\right]U^\dagger_{\ell_1\to\ell_2}\left[X_{\ell_2}, O_{\ell_2^+}\right]U_{\ell_1\to\ell_2}\right)\\
&= \int_{\mathcal{U}_{\mathrm{Haar}}} dU_{\ell_1\to\ell_2} dU_{\ell_2^+} \left[\operatorname{tr}\left(\left[X_{\ell_2}, O_{\ell_2^+}\right]^2 U_{\ell_1\to\ell_2}X_{\ell_1}U^\dagger_{\ell_1\to\ell_2}O_{\ell_2^+}U_{\ell_1\to\ell_2}X_{\ell_1}U^\dagger_{\ell_1\to\ell_2}\right)\right.\\
&\qquad\qquad + \operatorname{tr}\left(X_{\ell_1}U^\dagger_{\ell_1\to\ell_2}\left[X_{\ell_2}, O_{\ell_2^+}\right]U_{\ell_1\to\ell_2}X_{\ell_1}U^\dagger_{\ell_1\to\ell_2}\left[X_{\ell_2}, O_{\ell_2^+}\right]O_{\ell_2^+}U_{\ell_1\to\ell_2}\right)\\
&\qquad\qquad \left. - \operatorname{tr}\left(X_{\ell_1}U^\dagger_{\ell_1\to\ell_2}O_{\ell_2^+}\left[X_{\ell_2}, O_{\ell_2^+}\right]U_{\ell_1\to\ell_2}X_{\ell_1}U^\dagger_{\ell_1\to\ell_2}\left[X_{\ell_2}, O_{\ell_2^+}\right]U_{\ell_1\to\ell_2}\right) - \operatorname{tr}\left(\left[X_{\ell_2}, O_{\ell_2^+}\right]^2 O_{\ell_2^+}\right)\right] \qquad (111)\\
&= \int_{\mathcal{U}_{\mathrm{Haar}}} dU_{\ell_2^+}\left[\frac{d\operatorname{tr}\left(\left[X_{\ell_2}, O_{\ell_2^+}\right]^2\right)\operatorname{tr}(O) - \operatorname{tr}\left(O_{\ell_2^+}\left[X_{\ell_2}, O_{\ell_2^+}\right]^2\right)}{d^2-1} + \frac{\operatorname{tr}\left(O_{\ell_2^+}\left[X_{\ell_2}, O_{\ell_2^+}\right]^2\right)}{1-d^2} - \frac{\operatorname{tr}\left(O_{\ell_2^+}\left[X_{\ell_2}, O_{\ell_2^+}\right]^2\right)}{1-d^2}\right.\\
&\qquad\qquad \left. - \operatorname{tr}\left(O_{\ell_2^+}\left[X_{\ell_2}, O_{\ell_2^+}\right]^2\right)\right] \qquad (112)\\
&= \int_{\mathcal{U}_{\mathrm{Haar}}} dU_{\ell_2^+}\frac{d\left[\operatorname{tr}\left(\left[X_{\ell_2}, O_{\ell_2^+}\right]^2\right)\operatorname{tr}(O) - d\operatorname{tr}\left(O_{\ell_2^+}\left[X_{\ell_2}, O_{\ell_2^+}\right]^2\right)\right]}{d^2-1} \qquad (113)\\
&= \frac{d^2\left[d^2\operatorname{tr}(O^3) - 3d\operatorname{tr}(O^2)\operatorname{tr}(O) + 2\operatorname{tr}(O)^3\right]}{(d^2-1)^2}. \qquad (114)
\end{aligned}$$

Combine Eq. (110) and (114), we then have the average of $\frac{\partial^2\epsilon}{\partial\theta_{\ell_1}\partial\theta_{\ell_2}}\frac{\partial\epsilon}{\partial\theta_{\ell_1}}\frac{\partial\epsilon}{\partial\theta_{\ell_2}}$ as

$$\begin{aligned}\mathbb{E}_{\mathcal{U}_{\text{Haar}}}\left[\frac{\partial^2\epsilon}{\partial\theta_{\ell_1}\partial\theta_{\ell_2}}\frac{\partial\epsilon}{\partial\theta_{\ell_1}}\frac{\partial\epsilon}{\partial\theta_{\ell_2}}\right] &= \frac{1}{16(d^3+3d^2+2d)}\Big[\operatorname{tr}\Big(\Big[X_{\ell_1}, U^\dagger_{\ell_1\to\ell_2}\Big[X_{\ell_2}, O_{\ell_2^+}\Big]U_{\ell_1\to\ell_2}\Big]\Big[X_{\ell_1}, O_{\ell_1^+}\Big]U^\dagger_{\ell_1\to\ell_2}\Big[X_{\ell_2}, O_{\ell_2^+}\Big]U_{\ell_1\to\ell_2}\Big) \\ &\quad + \operatorname{tr}\Big(\Big[X_{\ell_1}, O_{\ell_1^+}\Big]\Big[X_{\ell_1}, U^\dagger_{\ell_1\to\ell_2}\Big[X_{\ell_2}, O_{\ell_2^+}\Big]U_{\ell_1\to\ell_2}\Big]U^\dagger_{\ell_1\to\ell_2}\Big[X_{\ell_2}, O_{\ell_2^+}\Big]U_{\ell_1\to\ell_2}\Big)\Big] \quad (115) \\ &= \frac{d\left[d^2\operatorname{tr}(O^3) - 3d\operatorname{tr}(O^2)\operatorname{tr}(O) + 2\operatorname{tr}(O)^3\right]}{8(d-1)^2(d+1)^3(d+2)}. \quad (116)\end{aligned}$$

c. Summary of average relative dQNTK $\overline{\lambda_0}$ under Haar random ensemble
Summarizing from Eq. (101) and (116), the mean of dQNTK $\overline{\mu_0}$ is

$$\begin{aligned}\overline{\mu_0} &= L(L-1)\mathbb{E}_{\mathcal{U}_{\text{Haar}}}\left[\frac{\partial^2\epsilon}{\partial\theta_{\ell_1}\partial\theta_{\ell_2}}\frac{\partial\epsilon}{\partial\theta_{\ell_1}}\frac{\partial\epsilon}{\partial\theta_{\ell_2}}\right] + L\mathbb{E}_{\mathcal{U}_{\text{Haar}}}\left[\frac{\partial^2\epsilon}{\partial\theta_\ell^2}\left(\frac{\partial\epsilon}{\partial\theta_\ell}\right)^2\right] \quad (117) \\ &= L(L-1)\frac{d\left[d^2\operatorname{tr}(O^3) - 3d\operatorname{tr}(O^2)\operatorname{tr}(O) + 2\operatorname{tr}(O)^3\right]}{8(d-1)^2(d+1)^3(d+2)} \quad (118) \\ &\simeq \frac{L^2}{8d^5}\left[d^2\operatorname{tr}(O^3) - 3d\operatorname{tr}(O^2)\operatorname{tr}(O) + 2\operatorname{tr}(O)^3\right], \quad (119)\end{aligned}$$

where the last line is approximated in the wide QNN limit with $d \gg 1$. Since $\operatorname{tr}(O^3)$ can be nonzero depending on specific choice of O, our result characterizes more of $\overline{\mu_0}$'s scaling compared to the result of $\overline{\mu_0} = 0$ from Ref. [6] where $U_{\ell_1^-}, U_{\ell_1^+}, U_{\ell_2^-}, U_{\ell_2^+}$ is considered to be independent Haar random unitaries, instead, only $U_{\ell_1^-}, U_{\ell_1\to\ell_2}, U_{\ell_2^+}$ should be considered as independent Haar random unitaries.

According to the definition of $\overline{\lambda_0} = \overline{\mu_0}/\overline{K_0}$, we have

$$\begin{aligned}\overline{\lambda_0} &= \overline{\mu_0}/\overline{K_0} \\ &= (L-1)\frac{d\left[d^2\operatorname{tr}(O^3) - 3d\operatorname{tr}(O^2)\operatorname{tr}(O) + 2\operatorname{tr}(O)^3\right]}{4(d-1)(d+1)(d+2)\left[d\operatorname{tr}(O^2) - \operatorname{tr}(O)^2\right]} \quad (120) \\ &\simeq \frac{L}{4d^2}\frac{d^2\operatorname{tr}(O^3) - 3d\operatorname{tr}(O^2)\operatorname{tr}(O) + 2\operatorname{tr}(O)^3}{d\operatorname{tr}(O^2) - \operatorname{tr}(O)^2}. \quad (121)\end{aligned}$$

Specifically for traceless observable, it is further reduced to

$$\overline{\lambda_0} \simeq \frac{L}{4d}\frac{\operatorname{tr}(O^3)}{\operatorname{tr}(O^2)}. \quad (122)$$

Supplementary Note 11.3. Average dynamical index with Haar random ensemble

We define the average $\overline{\zeta}$ as

$$\overline{\zeta} = \overline{\epsilon\mu}/\overline{K}^2. \quad (123)$$

The Haar ensemble average of $\epsilon\mu$ becomes

$$\overline{\epsilon_0\mu_0} = \sum_{\ell_1,\ell_2} \mathbb{E}_{\mathcal{U}_{\rm Haar}} \left[\epsilon \frac{\partial^2\epsilon}{\partial\theta_{\ell_1}\partial\theta_{\ell_2}} \frac{\partial\epsilon}{\partial\theta_{\ell_1}} \frac{\partial\epsilon}{\partial\theta_{\ell_2}}\right] \tag{124}$$

$$= \sum_{\ell_1,\ell_2} \mathbb{E}_{\mathcal{U}_{\rm Haar}} \left[\mathrm{tr}\big(\rho_0 U^\dagger O U\big) \frac{\partial^2\epsilon}{\partial\theta_{\ell_1}\partial\theta_{\ell_2}} \frac{\partial\epsilon}{\partial\theta_{\ell_1}} \frac{\partial\epsilon}{\partial\theta_{\ell_2}}\right] - O_0\overline{\mu_0} \tag{125}$$

$$= 2\sum_{\ell_1<\ell_2} \mathbb{E}_{\mathcal{U}_{\rm Haar}} \left[\mathrm{tr}\big(\rho_0 U^\dagger O U\big) \frac{\partial^2\epsilon}{\partial\theta_{\ell_1}\partial\theta_{\ell_2}} \frac{\partial\epsilon}{\partial\theta_{\ell_1}} \frac{\partial\epsilon}{\partial\theta_{\ell_2}}\right] + \sum_{\ell} \mathbb{E}_{\mathcal{U}_{\rm Haar}} \left[\mathrm{tr}\big(\rho_0 U^\dagger O U\big) \frac{\partial^2\epsilon}{\partial\theta_\ell^2} \left(\frac{\partial\epsilon}{\partial\theta_\ell}\right)^2\right] - O_0\overline{\mu_0} \tag{126}$$

$$= L(L-1)\mathbb{E}_{\mathcal{U}_{\rm Haar}} \left[\mathrm{tr}\big(\rho_0 U^\dagger O U\big) \frac{\partial^2\epsilon}{\partial\theta_{\ell_1}\partial\theta_{\ell_2}} \frac{\partial\epsilon}{\partial\theta_{\ell_1}} \frac{\partial\epsilon}{\partial\theta_{\ell_2}}\right] + L\mathbb{E}_{\mathcal{U}_{\rm Haar}} \left[\mathrm{tr}\big(\rho_0 U^\dagger O U\big) \frac{\partial^2\epsilon}{\partial\theta_\ell^2} \left(\frac{\partial\epsilon}{\partial\theta_\ell}\right)^2\right] - O_0\overline{\mu_0}. \tag{127}$$

where in Eq. (125) we expand $\epsilon = \mathrm{tr}\big(\rho_0 U^\dagger O U\big) - O_0$ following its definition. As $\overline{\mu_0}$ is already solved above, we only need to evaluate the first two parts.

a. Calculation of $\mathrm{tr}\big(\rho_0 U^\dagger O U\big) \frac{\partial^2\epsilon}{\partial\theta_\ell^2}\left(\frac{\partial\epsilon}{\partial\theta_\ell}\right)^2$ *with Haar random ensemble*

The ensemble average of $\mathrm{tr}\big(\rho_0 U^\dagger O U\big) \frac{\partial^2\epsilon}{\partial\theta_\ell^2}\left(\frac{\partial\epsilon}{\partial\theta_\ell}\right)^2$ is

$$\mathbb{E}_{\mathcal{U}_{\rm Haar}} \left[\mathrm{tr}\big(\rho_0 U^\dagger O U\big) \frac{\partial^2\epsilon}{\partial\theta_\ell^2} \left(\frac{\partial\epsilon}{\partial\theta_\ell}\right)^2\right]$$

$$= \frac{1}{16}\int_{\mathcal{U}_{\rm Haar}} dU_{\ell-}dU_{\ell+} \langle\psi_0|U_{\ell-}^\dagger O_{\ell+} U_{\ell-}|\psi_0\rangle \langle\psi_0|U_{\ell-}^\dagger [X_\ell,[X_\ell,O_{\ell+}]]U_{\ell-}|\psi_0\rangle \langle\psi_0|U_{\ell-}^\dagger [X_\ell,O_{\ell+}] U_{\ell-}|\psi_0\rangle^2 \tag{128}$$

$$= \frac{1}{16}\int_{\mathcal{U}_{\rm Haar}} dU_{\ell-}dU_{\ell+}\, \mathrm{tr}\left(\rho_0 U_{\ell-}^\dagger O_{\ell+} U_{\ell-} \rho_0 U_{\ell-}^\dagger [X_\ell,[X_\ell,O_{\ell+}]]U_{\ell-} \rho_0 U_{\ell-}^\dagger [X_\ell,O_{\ell+}]U_{\ell-} \rho_0 U_{\ell-}^\dagger [X_\ell,O_{\ell+}]U_{\ell-}\right) \tag{129}$$

$$\begin{aligned} = \int_{\mathcal{U}_{\rm Haar}} dU_{\ell+} \frac{1}{d(d+1)(d+2)(d+3)} &\left[\mathrm{tr}\big([X_\ell,O_{\ell+}]^2\big)\,\mathrm{tr}([X_\ell,[X_\ell,O_{\ell+}]]O_{\ell+}) + 2\,\mathrm{tr}\big([X_\ell,O_{\ell+}]^2[X_\ell,[X_\ell,O_{\ell+}]]O_{\ell+}\big)\right. \\ &\left. +2\,\mathrm{tr}([X_\ell,O_{\ell+}][X_\ell,[X_\ell,O_{\ell+}]][X_\ell,O_{\ell+}]O_{\ell+}) + 2\,\mathrm{tr}\big([X_\ell,[X_\ell,O_{\ell+}]][X_\ell,O_{\ell+}]^2 O_{\ell+}\big)\right], \end{aligned} \tag{130}$$

where we do the Haar integral (4-design) over $U_{\ell-}$ to obtain the last equation.

The integration over $U_{\ell+}$ of the first term $\mathrm{tr}\big([X_\ell,O_{\ell+}]^2\big)\,\mathrm{tr}([X_\ell,[X_\ell,O_{\ell+}]]O_{\ell+})$ is

$$\begin{aligned} &\int_{\mathcal{U}_{\rm Haar}} dU_{\ell+}\, \mathrm{tr}\big([X_\ell,O_{\ell+}]^2\big)\,\mathrm{tr}([X_\ell,[X_\ell,O_{\ell+}]]O_{\ell+}) \\ &= \int_{\mathcal{U}_{\rm Haar}} dU_{\ell+} \left[8\,\mathrm{tr}(X_\ell O_{\ell+} X_\ell O_{\ell+})\,\mathrm{tr}\big(O^2\big) - 4\,\mathrm{tr}(X_\ell O_{\ell+} X_\ell O_{\ell+})^2 - 4\,\mathrm{tr}\big(O^2\big)^2\right] \\ &= \frac{8\,\mathrm{tr}\big(O^2\big)\left[d\,\mathrm{tr}(O)^2 - \mathrm{tr}\big(O^2\big)\right]}{d^2-1} - 4\,\mathrm{tr}\big(O^2\big)^2 \\ &\quad - \frac{4}{(d^2-1)(d^2-4)(d^2-9)}\left[\left(d^4-d^3-9d^2+4d+20\right)\mathrm{tr}(O)^4 + 2d\left(-3d^2+5d+7\right)\mathrm{tr}\big(O^2\big)\,\mathrm{tr}(O)^2\right. \\ &\qquad \left. + \left(d^5+d^4-12d^3-5d^2+12d+24\right)\mathrm{tr}\big(O^2\big)^2 - 2d\left(d^3+2d^2-14d+2\right)\mathrm{tr}\big(O^4\big)\right] \end{aligned} \tag{131}$$

$$\begin{aligned} = \frac{4}{(d^2-9)(d^2-4)(d^2-1)} &\left[-\left(d^4-d^3-9d^2+4d+20\right)\mathrm{tr}(O)^4 + 2d\left(d^4-10d^2-5d+29\right)\mathrm{tr}\big(O^2\big)\,\mathrm{tr}(O)^2\right. \\ &- 8\left(3d^2-5d-7\right)\mathrm{tr}\big(O^3\big)\,\mathrm{tr}(O) - \left(d^6+d^5-11d^4-12d^3+18d^2+12d+60\right)\mathrm{tr}\big(O^2\big)^2 \\ &\left. +2d\left(d^3+2d^2-14d+2\right)\mathrm{tr}\big(O^4\big)\right]. \end{aligned} \tag{132}$$

where in last line we do simple algebra to reduce all terms to same denominator.

The integration of the second term $\mathrm{tr}\big([X_\ell, O_{\ell^+}]^2[X_\ell,[X_\ell,O_{\ell^+}]]O_{\ell^+}\big)$ becomes

$$\begin{aligned}
&\int_{\mathcal{U}_{\mathrm{Haar}}} dU_{\ell^+}\,\mathrm{tr}\big([X_\ell, O_{\ell^+}]^2[X_\ell,[X_\ell,O_{\ell^+}]]O_{\ell^+}\big)\\
&=\int_{\mathcal{U}_{\mathrm{Haar}}} dU_{\ell^+}\left[8\,\mathrm{tr}\big(X_\ell O_{\ell^+}X_\ell O_{\ell^+}^3\big)-4\,\mathrm{tr}\big(X_\ell O_{\ell^+}^2X_\ell O_{\ell^+}^2\big)-2\,\mathrm{tr}(X_\ell O_{\ell^+}X_\ell O_{\ell^+}X_\ell O_{\ell^+}X_\ell O_{\ell^+})-2\,\mathrm{tr}\big(O^4\big)\right]\\
&=\frac{2d\left[4\left(d^2-7\right)\mathrm{tr}\big(O^3\big)\,\mathrm{tr}(O)+\left(21-2d^2\right)\mathrm{tr}\big(O^2\big)^2-d\left(d^2-7\right)\mathrm{tr}\big(O^4\big)-2d\,\mathrm{tr}\big(O^2\big)\,\mathrm{tr}(O)^2+\mathrm{tr}(O)^4\right]}{(d^2-1)(d^2-9)}.
\end{aligned}\tag{133}$$

The third term $\mathrm{tr}([X_\ell, O_{\ell^+}][X_\ell,[X_\ell,O_{\ell^+}]][X_\ell,O_{\ell^+}]O_{\ell^+})$ becomes

$$\int_{\mathcal{U}_{\mathrm{Haar}}} dU_{\ell^+}\,\mathrm{tr}([X_\ell, O_{\ell^+}][X_\ell,[X_\ell,O_{\ell^+}]][X_\ell,O_{\ell^+}]O_{\ell^+})$$

$$=\int_{\mathcal{U}_{\mathrm{Haar}}} dU_{\ell^+}\left[4\,\mathrm{tr}\big(X_\ell O_{\ell^+}^2X_\ell O_{\ell^+}^2\big)+2\,\mathrm{tr}(X_\ell O_{\ell^+}X_\ell O_{\ell^+}X_\ell O_{\ell^+}X_\ell O_{\ell^+})+2\,\mathrm{tr}\big(O^4\big)-8\,\mathrm{tr}\big(X_\ell O_{\ell^+}X_\ell O_{\ell^+}^3\big)\right]\tag{134}$$

$$=-\frac{2d\left[4\left(d^2-7\right)\mathrm{tr}\big(O^3\big)\,\mathrm{tr}(O)+\left(21-2d^2\right)\mathrm{tr}\big(O^2\big)^2-d\left(d^2-7\right)\mathrm{tr}\big(O^4\big)-2d\,\mathrm{tr}\big(O^2\big)\,\mathrm{tr}(O)^2+\mathrm{tr}(O)^4\right]}{(d^2-1)(d^2-9)}.\tag{135}$$

The last term $\mathrm{tr}\big([X_\ell,[X_\ell,O_{\ell^+}]][X_\ell,O_{\ell^+}]^2O_{\ell^+}\big)$ becomes

$$\int_{\mathcal{U}_{\mathrm{Haar}}} dU_{\ell^+}\,\mathrm{tr}\big([X_\ell,[X_\ell,O_{\ell^+}]][X_\ell,O_{\ell^+}]^2O_{\ell^+}\big)$$

$$=\int_{\mathcal{U}_{\mathrm{Haar}}} dU_{\ell^+}\left[8\,\mathrm{tr}\big(X_\ell O_{\ell^+}X_\ell O_{\ell^+}^3\big)-4\,\mathrm{tr}\big(X_\ell O_{\ell^+}^2X_\ell O_{\ell^+}^2\big)-2\,\mathrm{tr}(X_\ell O_{\ell^+}X_\ell O_{\ell^+}X_\ell O_{\ell^+}X_\ell O_{\ell^+})-2\,\mathrm{tr}\big(O^4\big)\right]\tag{136}$$

$$=\frac{2d\left[4\left(d^2-7\right)\mathrm{tr}\big(O^3\big)\,\mathrm{tr}(O)+\left(21-2d^2\right)\mathrm{tr}\big(O^2\big)^2-d\left(d^2-7\right)\mathrm{tr}\big(O^4\big)-2d\,\mathrm{tr}\big(O^2\big)\,\mathrm{tr}(O)^2+\mathrm{tr}(O)^4\right]}{(d^2-1)(d^2-9)}.\tag{137}$$

Therefore, we have the Haar ensemble average over $\mathrm{tr}\big(\rho_0U^\dagger OU\big)\frac{\partial^2\epsilon}{\partial\theta_\ell^2}\left(\frac{\partial\epsilon}{\partial\theta_\ell}\right)^2$ as

$$\begin{aligned}
&\mathbb{E}_{\mathcal{U}_{\mathrm{Haar}}}\left[\mathrm{tr}\Big(\rho_0U^\dagger OU\Big)\frac{\partial^2\epsilon}{\partial\theta_\ell^2}\left(\frac{\partial\epsilon}{\partial\theta_\ell}\right)^2\right]=-\frac{1}{4d(d-1)(d+1)^2(d-2)(d+2)^2(d-3)(d+3)^2}\left[\left(d^4-2d^3-9d^2+8d+20\right)\mathrm{tr}(O)^4\right.\\
&+2d\left(-d^4+d^3+10d^2+d-29\right)\mathrm{tr}\big(O^2\big)\,\mathrm{tr}(O)^2-4\left(d^5-11d^3-6d^2+38d+14\right)\mathrm{tr}\big(O^3\big)\,\mathrm{tr}(O)\\
&\left.+\left(d^6+3d^5-11d^4-41d^3+18d^2+96d+60\right)\mathrm{tr}\big(O^2\big)^2+d\left(d^5-13d^3-4d^2+56d-4\right)\mathrm{tr}\big(O^4\big)\right].
\end{aligned}\tag{138}$$

b. Calculation $\mathrm{tr}\big(\rho_0U^\dagger OU\big)\frac{\partial^2\epsilon}{\partial\theta_{\ell_1}\partial\theta_{\ell_2}}\frac{\partial\epsilon}{\partial\theta_{\ell_1}}\frac{\partial\epsilon}{\partial\theta_{\ell_2}}$ *with Haar random ensemble*

The Haar ensemble average of $\mathrm{tr}\big(\rho_0U^\dagger OU\big)\frac{\partial^2\epsilon}{\partial\theta_{\ell_1}\partial\theta_{\ell_2}}\frac{\partial\epsilon}{\partial\theta_{\ell_1}}\frac{\partial\epsilon}{\partial\theta_{\ell_2}}$ assuming $\ell_1<\ell_2$ can be written as

$$\begin{aligned}
&\mathbb{E}_{\mathcal{U}_{\mathrm{Haar}}}\left[\mathrm{tr}\big(\rho_0U^\dagger OU\big)\frac{\partial^2\epsilon}{\partial\theta_{\ell_1}\partial\theta_{\ell_2}}\frac{\partial\epsilon}{\partial\theta_{\ell_1}}\frac{\partial\epsilon}{\partial\theta_{\ell_2}}\right]\\
&=\frac{1}{16}\int_{\mathcal{U}_{\mathrm{Haar}}} dU_{\ell_1^-}dU_{\ell_1\to\ell_2}dU_{\ell_2^+}\left[\langle\psi_0|U_{\ell_1^-}^\dagger U_{\ell_1\to\ell_2}^\dagger O_{\ell_2^+}U_{\ell_1\to\ell_2}U_{\ell_1^-}|\psi_0\rangle\,\langle\psi_0|U_{\ell_1^-}^\dagger\left[X_{\ell_1},U_{\ell_1\to\ell_2}^\dagger\left[X_{\ell_2},O_{\ell_2^+}\right]U_{\ell_1\to\ell_2}\right]U_{\ell_1^-}|\psi_0\rangle\right.\\
&\qquad\left.\times\langle\psi_0|U_{\ell_1^-}^\dagger\left[X_{\ell_1},O_{\ell_1^+}\right]U_{\ell_1^-}|\psi_0\rangle\,\langle\psi_0|U_{\ell_1^-}^\dagger U_{\ell_1\to\ell_2}^\dagger\left[X_{\ell_2},O_{\ell_2^+}\right]U_{\ell_1\to\ell_2}U_{\ell_1^-}|\psi_0\rangle\right]
\end{aligned}\tag{139}$$

$$\begin{aligned}
&=\frac{1}{16}\int_{\mathcal{U}_{\mathrm{Haar}}} dU_{\ell_1^-}dU_{\ell_1\to\ell_2}dU_{\ell_2^+}\,\mathrm{tr}\left(\rho_0U_{\ell_1^-}^\dagger U_{\ell_1\to\ell_2}^\dagger O_{\ell_2^+}U_{\ell_1\to\ell_2}U_{\ell_1^-}\rho_0U_{\ell_1^-}^\dagger\left[X_{\ell_1},U_{\ell_1\to\ell_2}^\dagger\left[X_{\ell_2},O_{\ell_2^+}\right]U_{\ell_1\to\ell_2}\right]U_{\ell_1^-}\rho_0\right.\\
&\qquad\left.\cdot U_{\ell_1^-}^\dagger\left[X_{\ell_1},O_{\ell_1^+}\right]U_{\ell_1^-}\rho_0U_{\ell_1^-}^\dagger U_{\ell_1\to\ell_2}^\dagger\left[X_{\ell_2},O_{\ell_2^+}\right]U_{\ell_1\to\ell_2}U_{\ell_1^-}\right).
\end{aligned}\tag{140}$$

The integration of $U_{\ell_1^-}$ over Haar unitary ensemble becomes

$$
\begin{aligned}
&\int_{\mathcal{U}_{\text{Haar}}} dU_{\ell_1^-} dU_{\ell_1 \to \ell_2} dU_{\ell_2^+} \operatorname{tr}\Big(\rho_0 U_{\ell_1^-}^\dagger U_{\ell_1\to\ell_2}^\dagger O_{\ell_2^+} U_{\ell_1\to\ell_2} U_{\ell_1^-} \rho_0 U_{\ell_1^-}^\dagger \Big[X_{\ell_1}, U_{\ell_1\to\ell_2}^\dagger \Big[X_{\ell_2}, O_{\ell_2^+}\Big] U_{\ell_1\to\ell_2}\Big] U_{\ell_1^-} \rho_0 \\
&\qquad\qquad \cdot U_{\ell_1^-}^\dagger \Big[X_{\ell_1}, O_{\ell_1^+}\Big] U_{\ell_1^-} \rho_0 U_{\ell_1^-}^\dagger U_{\ell_1\to\ell_2}^\dagger \Big[X_{\ell_2}, O_{\ell_2^+}\Big] U_{\ell_1\to\ell_2} U_{\ell_1^-}\Big) \\
&= \int_{\mathcal{U}_{\text{Haar}}} dU_{\ell_1\to\ell_2} dU_{\ell_2^+} \frac{1}{d^4+6d^3+11d^2+6d} \Big[\operatorname{tr}(O) \operatorname{tr}\Big(\Big[X_{\ell_1}, O_{\ell_1^+}\Big] U_{\ell_1\to\ell_2}^\dagger \Big[X_{\ell_2}, O_{\ell_2^+}\Big] U_{\ell_1\to\ell_2} \Big[X_{\ell_1}, U_{\ell_1\to\ell_2}^\dagger \Big[X_{\ell_2}, O_{\ell_2^+}\Big] U_{\ell_1\to\ell_2}\Big]\Big) \\
&+ \operatorname{tr}(O) \operatorname{tr}\Big(U_{\ell_1\to\ell_2}^\dagger \Big[X_{\ell_2}, O_{\ell_2^+}\Big] U_{\ell_1\to\ell_2} \Big[X_{\ell_1}, O_{\ell_1^+}\Big] \Big[X_{\ell_1}, U_{\ell_1\to\ell_2}^\dagger \Big[X_{\ell_2}, O_{\ell_2^+}\Big] U_{\ell_1\to\ell_2}\Big]\Big) \\
&+ \operatorname{tr}\Big(U_{\ell_1\to\ell_2}^\dagger \Big[X_{\ell_2}, O_{\ell_2^+}\Big] U_{\ell_1\to\ell_2} \Big[X_{\ell_1}, O_{\ell_1^+}\Big]\Big) \operatorname{tr}\Big(\Big[X_{\ell_1}, U_{\ell_1\to\ell_2}^\dagger \Big[X_{\ell_2}, O_{\ell_2^+}\Big] U_{\ell_1\to\ell_2}\Big] U_{\ell_1\to\ell_2}^\dagger O_{\ell_2^+} U_{\ell_1\to\ell_2}\Big) \\
&+ \operatorname{tr}\Big(\Big[X_{\ell_1}, O_{\ell_1^+}\Big] U_{\ell_1\to\ell_2}^\dagger O_{\ell_2^+} U_{\ell_1\to\ell_2}\Big) \operatorname{tr}\Big(U_{\ell_1\to\ell_2}^\dagger \Big[X_{\ell_2}, O_{\ell_2^+}\Big] U_{\ell_1\to\ell_2} \Big[X_{\ell_1}, U_{\ell_1\to\ell_2}^\dagger \Big[X_{\ell_2}, O_{\ell_2^+}\Big] U_{\ell_1\to\ell_2}\Big]\Big) \\
&+ \operatorname{tr}\Big(\Big[X_{\ell_1}, O_{\ell_1^+}\Big] \Big[X_{\ell_1}, U_{\ell_1\to\ell_2}^\dagger \Big[X_{\ell_2}, O_{\ell_2^+}\Big] U_{\ell_1\to\ell_2}\Big] U_{\ell_1\to\ell_2}^\dagger \Big[X_{\ell_2}, O_{\ell_2^+}\Big] O_{\ell_2^+} U_{\ell_1\to\ell_2}\Big) \\
&+ \operatorname{tr}\Big(\Big[X_{\ell_1}, O_{\ell_1^+}\Big] \Big[X_{\ell_1}, U_{\ell_1\to\ell_2}^\dagger \Big[X_{\ell_2}, O_{\ell_2^+}\Big] U_{\ell_1\to\ell_2}\Big] U_{\ell_1\to\ell_2}^\dagger O_{\ell_2^+} \Big[X_{\ell_2}, O_{\ell_2^+}\Big] U_{\ell_1\to\ell_2}\Big) \\
&+ \operatorname{tr}\Big(\Big[X_{\ell_1}, U_{\ell_1\to\ell_2}^\dagger \Big[X_{\ell_2}, O_{\ell_2^+}\Big] U_{\ell_1\to\ell_2}\Big] \Big[X_{\ell_1}, O_{\ell_1^+}\Big] U_{\ell_1\to\ell_2}^\dagger \Big[X_{\ell_2}, O_{\ell_2^+}\Big] O_{\ell_2^+} U_{\ell_1\to\ell_2}\Big) \\
&+ \operatorname{tr}\Big(\Big[X_{\ell_1}, U_{\ell_1\to\ell_2}^\dagger \Big[X_{\ell_2}, O_{\ell_2^+}\Big] U_{\ell_1\to\ell_2}\Big] \Big[X_{\ell_1}, O_{\ell_1^+}\Big] U_{\ell_1\to\ell_2}^\dagger O_{\ell_2^+} \Big[X_{\ell_2}, O_{\ell_2^+}\Big] U_{\ell_1\to\ell_2}\Big) \\
&+ \operatorname{tr}\Big(\Big[X_{\ell_1}, O_{\ell_1^+}\Big] U_{\ell_1\to\ell_2}^\dagger \Big[X_{\ell_2}, O_{\ell_2^+}\Big] U_{\ell_1\to\ell_2} \Big[X_{\ell_1}, U_{\ell_1\to\ell_2}^\dagger \Big[X_{\ell_2}, O_{\ell_2^+}\Big] U_{\ell_1\to\ell_2}\Big] U_{\ell_1\to\ell_2}^\dagger O_{\ell_2^+} U_{\ell_1\to\ell_2}\Big) \\
&+ \operatorname{tr}\Big(\Big[X_{\ell_1}, U_{\ell_1\to\ell_2}^\dagger \Big[X_{\ell_2}, O_{\ell_2^+}\Big] U_{\ell_1\to\ell_2}\Big] U_{\ell_1\to\ell_2}^\dagger \Big[X_{\ell_2}, O_{\ell_2^+}\Big] U_{\ell_1\to\ell_2} \Big[X_{\ell_1}, O_{\ell_1^+}\Big] U_{\ell_1\to\ell_2}^\dagger O_{\ell_2^+} U_{\ell_1\to\ell_2}\Big) \\
&+ \operatorname{tr}\Big(\Big[X_{\ell_1}, O_{\ell_1^+}\Big] \Big[X_{\ell_1}, U_{\ell_1\to\ell_2}^\dagger \Big[X_{\ell_2}, O_{\ell_2^+}\Big] U_{\ell_1\to\ell_2}\Big]\Big) \operatorname{tr}\Big(\Big[X_{\ell_2}, O_{\ell_2^+}\Big] O_{\ell_2^+}\Big)\Big].
\end{aligned} \tag{141}
$$

We will evaluate the integration over $U_{\ell_1\to\ell_2}, U_{\ell_2^+}$ on each item in the following.

The first two are already solved before (see Eqs. (110) and (114)) up to a cyclic transformation in trace

$$
\begin{aligned}
&\int_{\mathcal{U}_{\text{Haar}}} dU_{\ell_1\to\ell_2} dU_{\ell_2^+} \operatorname{tr}(O) \operatorname{tr}\Big(\Big[X_{\ell_1}, O_{\ell_1^+}\Big] U_{\ell_1\to\ell_2}^\dagger \Big[X_{\ell_2}, O_{\ell_2^+}\Big] U_{\ell_1\to\ell_2} \Big[X_{\ell_1}, U_{\ell_1\to\ell_2}^\dagger \Big[X_{\ell_2}, O_{\ell_2^+}\Big] U_{\ell_1\to\ell_2}\Big]\Big) \\
&= \operatorname{tr}(O) \frac{d^2 \Big[d^2 \operatorname{tr}\big(O^3\big) - 3d \operatorname{tr}\big(O^2\big) \operatorname{tr}(O) + 2 \operatorname{tr}(O)^3\Big]}{\left(d^2-1\right)^2}
\end{aligned} \tag{142}
$$

$$
\begin{aligned}
&\int_{\mathcal{U}_{\text{Haar}}} dU_{\ell_1\to\ell_2} dU_{\ell_2^+} \operatorname{tr}(O) \operatorname{tr}\Big(U_{\ell_1\to\ell_2}^\dagger \Big[X_{\ell_2}, O_{\ell_2^+}\Big] U_{\ell_1\to\ell_2} \Big[X_{\ell_1}, O_{\ell_1^+}\Big] \Big[X_{\ell_1}, U_{\ell_1\to\ell_2}^\dagger \Big[X_{\ell_2}, O_{\ell_2^+}\Big] U_{\ell_1\to\ell_2}\Big]\Big) \\
&= \operatorname{tr}(O) \frac{d^2 \Big[d^2 \operatorname{tr}\big(O^3\big) - 3d \operatorname{tr}\big(O^2\big) \operatorname{tr}(O) + 2 \operatorname{tr}(O)^3\Big]}{\left(d^2-1\right)^2}.
\end{aligned} \tag{143}
$$

The integral of the third one becomes

$$\begin{aligned}
&\int_{\mathcal{U}_{\rm Haar}} dU_{\ell_1\to\ell_2} dU_{\ell_2^+} \operatorname{tr}\left(U^\dagger_{\ell_1\to\ell_2}\left[X_{\ell_2}, O_{\ell_2^+}\right] U_{\ell_1\to\ell_2}\left[X_{\ell_1}, O_{\ell_1^+}\right]\right) \operatorname{tr}\left(\left[X_{\ell_1}, U^\dagger_{\ell_1\to\ell_2}\left[X_{\ell_2}, O_{\ell_2^+}\right] U_{\ell_1\to\ell_2}\right] U^\dagger_{\ell_1\to\ell_2} O_{\ell_2^+} U_{\ell_1\to\ell_2}\right)\\
&=\sum_{i_1,i_2}\int_{\mathcal{U}_{\rm Haar}} dU_{\ell_1\to\ell_2} dU_{\ell_2^+}\Big[2\operatorname{tr}\left(P_{i_2,i_1}X_{\ell_1}U^\dagger_{\ell_1\to\ell_2}O_{\ell_2^+}\left[X_{\ell_2}, O_{\ell_2^+}\right]U_{\ell_1\to\ell_2}P_{i_1,i_2}X_{\ell_1}U^\dagger_{\ell_1\to\ell_2}\left[X_{\ell_2}, O_{\ell_2^+}\right]O_{\ell_2^+}U_{\ell_1\to\ell_2}\right)\\
&\qquad -\operatorname{tr}\left(P_{i_2,i_1}X_{\ell_1}U^\dagger_{\ell_1\to\ell_2}O_{\ell_2^+}\left[X_{\ell_2}, O_{\ell_2^+}\right]U_{\ell_1\to\ell_2}P_{i_1,i_2}X_{\ell_1}U^\dagger_{\ell_1\to\ell_2}O_{\ell_2^+}\left[X_{\ell_2}, O_{\ell_2^+}\right]U_{\ell_1\to\ell_2}\right)\\
&\qquad -\operatorname{tr}\left(P_{i_2,i_1}X_{\ell_1}U^\dagger_{\ell_1\to\ell_2}\left[X_{\ell_2}, O_{\ell_2^+}\right]O_{\ell_2^+}U_{\ell_1\to\ell_2}P_{i_1,i_2}X_{\ell_1}U^\dagger_{\ell_1\to\ell_2}\left[X_{\ell_2}, O_{\ell_2^+}\right]O_{\ell_2^+}U_{\ell_1\to\ell_2}\right)\Big] \qquad (144)\\
&=\int_{\mathcal{U}_{\rm Haar}} dU_{\ell_2^+} 2\left[\frac{d\operatorname{tr}\left(O^2_{\ell_2^+}\left[X_{\ell_2}, O_{\ell_2^+}\right]^2\right)-\operatorname{tr}\left(O_{\ell_2^+}\left[X_{\ell_2}, O_{\ell_2^+}\right]\right)^2}{d^2-1}\right.\\
&\qquad\left.+\frac{\operatorname{tr}\left(O_{\ell_2^+}\left[X_{\ell_2}, O_{\ell_2^+}\right]\right)^2-d\operatorname{tr}\left(O_{\ell_2^+}\left[X_{\ell_2}, O_{\ell_2^+}\right]O_{\ell_2^+}\left[X_{\ell_2}, O_{\ell_2^+}\right]\right)}{d^2-1}\right] \qquad (145)\\
&=\int_{\mathcal{U}_{\rm Haar}} dU_{\ell_2^+}\frac{2d\left[\operatorname{tr}\left(O^2_{\ell_2^+}\left[X_{\ell_2}, O_{\ell_2^+}\right]^2\right)-\operatorname{tr}\left(O_{\ell_2^+}\left[X_{\ell_2}, O_{\ell_2^+}\right]O_{\ell_2^+}\left[X_{\ell_2}, O_{\ell_2^+}\right]\right)\right]}{d^2-1} \qquad (146)\\
&=\frac{2d}{d^2-1}\left[\frac{d\left(d\operatorname{tr}(O^4)+\operatorname{tr}(O^2)^2-2\operatorname{tr}(O)\operatorname{tr}(O^3)\right)}{1-d^2}-\frac{2d\left(\operatorname{tr}(O^2)^2-\operatorname{tr}(O)\operatorname{tr}(O^3)\right)}{d^2-1}\right] \qquad (147)\\
&=\frac{2d^2\left[4\operatorname{tr}(O)\operatorname{tr}(O^3)-d\operatorname{tr}(O^4)-3\operatorname{tr}(O^2)^2\right]}{(d^2-1)^2}, \qquad (148)
\end{aligned}$$

where in Eq. (144) we expand each trace in the first line via an orthonormal basis $|i_1\rangle, |i_2\rangle$ separately, and regroup them together to a single trace operation via projectors $P_{i_2,i_1} = |i_2\rangle\langle i_1|$ and $P_{i_1,i_2} = |i_1\rangle\langle i_2|$ for conveninent calculation with `RTNI` [5].

The fourth one is simply

$$\int dU_{\ell_1\to\ell_2} dU_{\ell_2^+}\operatorname{tr}\left(\left[X_{\ell_1}, O_{\ell_1^+}\right]U^\dagger_{\ell_1\to\ell_2}O_{\ell_2^+}U_{\ell_1\to\ell_2}\right)\operatorname{tr}\left(U^\dagger_{\ell_1\to\ell_2}\left[X_{\ell_2}, O_{\ell_2^+}\right]U_{\ell_1\to\ell_2}\left[X_{\ell_1}, U^\dagger_{\ell_1\to\ell_2}\left[X_{\ell_2}, O_{\ell_2^+}\right]U_{\ell_1\to\ell_2}\right]\right)=0 \qquad (149)$$

by the cyclic property of trace from expansion.

The fifth term through integration is

$$
\begin{aligned}
&\int_{\mathcal{U}_{\mathrm{Haar}}} dU_{\ell_1\to\ell_2} dU_{\ell_2^+}\, \mathrm{tr}\left(\left[X_{\ell_1}, O_{\ell_1^+}\right]\left[X_{\ell_1}, U^\dagger_{\ell_1\to\ell_2}\left[X_{\ell_2}, O_{\ell_2^+}\right] U_{\ell_1\to\ell_2}\right] U^\dagger_{\ell_1\to\ell_2}\left[X_{\ell_2}, O_{\ell_2^+}\right] O_{\ell_2^+} U_{\ell_1\to\ell_2}\right)\\
&= \int_{\mathcal{U}_{\mathrm{Haar}}} dU_{\ell_1\to\ell_2} dU_{\ell_2^+}\Big[\mathrm{tr}\left(X_{\ell_1} U^\dagger_{\ell_1\to\ell_2} O_{\ell_2^+} U_{\ell_1\to\ell_2} X_{\ell_1} U^\dagger_{\ell_1\to\ell_2}\left[X_{\ell_2}, O_{\ell_2^+}\right]^2 O_{\ell_2^+} U_{\ell_1\to\ell_2}\right) - \mathrm{tr}\left(\left[X_{\ell_2}, O_{\ell_2^+}\right]^2 O^2_{\ell_2^+}\right)\\
&\qquad + \mathrm{tr}\left(X_{\ell_1} U^\dagger_{\ell_1\to\ell_2}\left[X_{\ell_2}, O_{\ell_2^+}\right] U_{\ell_1\to\ell_2} X_{\ell_1} U^\dagger_{\ell_1\to\ell_2}\left[X_{\ell_2}, O_{\ell_2^+}\right] O^2_{\ell_2^+} U_{\ell_1\to\ell_2}\right)\\
&\qquad - \mathrm{tr}\left(X_{\ell_1} U^\dagger_{\ell_1\to\ell_2} O_{\ell_2^+}\left[X_{\ell_2}, O_{\ell_2^+}\right] U_{\ell_1\to\ell_2} X_{\ell_1} U^\dagger_{\ell_1\to\ell_2}\left[X_{\ell_2}, O_{\ell_2^+}\right] O_{\ell_2^+} U_{\ell_1\to\ell_2}\right)\Big] \qquad (150)\\
&= \int_{\mathcal{U}_{\mathrm{Haar}}} dU_{\ell_2^+}\Bigg[\frac{d\,\mathrm{tr}(O)\,\mathrm{tr}\left(\left[X_{\ell_2}, O_{\ell_2^+}\right]^2 O_{\ell_2^+}\right) - \mathrm{tr}\left(\left[X_{\ell_2}, O_{\ell_2^+}\right]^2 O^2_{\ell_2^+}\right)}{d^2-1} + \frac{\mathrm{tr}\left(\left[X_{\ell_2}, O_{\ell_2^+}\right]^2 O^2_{\ell_2^+}\right)}{1-d^2} - \frac{\mathrm{tr}\left(\left[X_{\ell_2}, O_{\ell_2^+}\right]^2 O^2_{\ell_2^+}\right)}{1-d^2}\\
&\qquad - \mathrm{tr}\left(\left[X_{\ell_2}, O_{\ell_2^+}\right]^2 O^2_{\ell_2^+}\right)\Bigg] \qquad (151)\\
&= \int_{\mathcal{U}_{\mathrm{Haar}}} dU_{\ell_2^+}\frac{d}{d^2-1}\left[\mathrm{tr}(O)\,\mathrm{tr}\left(\left[X_{\ell_2}, O_{\ell_2^+}\right]^2 O_{\ell_2^+}\right) - d\,\mathrm{tr}\left(\left[X_{\ell_2}, O_{\ell_2^+}\right]^2 O^2_{\ell_2^+}\right)\right] \qquad (152)\\
&= \int_{\mathcal{U}_{\mathrm{Haar}}} dU_{\ell_2^+}\left\{\frac{d}{d^2-1}\mathrm{tr}(O)\left[\mathrm{tr}\left(X_{\ell_2} O_{\ell_2^+} X_{\ell_2} O^2_{\ell_2^+}\right) - \mathrm{tr}\left(O^3\right)\right] - \frac{d^2}{d^2-1}\left[2\,\mathrm{tr}\left(X_{\ell_2} O^3_{\ell_2^+} X_{\ell_2} O_{\ell_2^+}\right) - \mathrm{tr}\left(X_{\ell_2} O^2_{\ell_2^+} X_{\ell_2} O^2_{\ell_2^+}\right) - \mathrm{tr}\left(O^4\right)\right]\right\}\\
&\qquad (153)\\
&= \frac{d}{d^2-1}\mathrm{tr}(O)\frac{d\left[\mathrm{tr}(O)\,\mathrm{tr}\left(O^2\right) - d\,\mathrm{tr}\left(O^3\right)\right]}{d^2-1} + \frac{d^2}{d^2-1}\frac{d}{d^2-1}\left[\mathrm{tr}\left(O^2\right)^2 - 2\,\mathrm{tr}(O)\,\mathrm{tr}\left(O^3\right) + d\,\mathrm{tr}\left(O^4\right)\right] \qquad (154)\\
&= \frac{d^2}{(d^2-1)^2}\left[\mathrm{tr}(O)^2\,\mathrm{tr}\left(O^2\right) - 3d\,\mathrm{tr}(O)\,\mathrm{tr}\left(O^3\right) + d\,\mathrm{tr}\left(O^2\right)^2 + d^2\,\mathrm{tr}\left(O^4\right)\right]. \qquad (155)
\end{aligned}
$$

The sixth item is

$$
\begin{aligned}
&\int_{\mathcal{U}_{\mathrm{Haar}}} dU_{\ell_1\to\ell_2} dU_{\ell_2^+}\, \mathrm{tr}\left(\left[X_{\ell_1}, O_{\ell_1^+}\right]\left[X_{\ell_1}, U^\dagger_{\ell_1\to\ell_2}\left[X_{\ell_2}, O_{\ell_2^+}\right] U_{\ell_1\to\ell_2}\right] U^\dagger_{\ell_1\to\ell_2} O_{\ell_2^+}\left[X_{\ell_2}, O_{\ell_2^+}\right] U_{\ell_1\to\ell_2}\right)\\
&= \int_{\mathcal{U}_{\mathrm{Haar}}} dU_{\ell_1\to\ell_2} dU_{\ell_2^+}\Big[\mathrm{tr}\left(X_{\ell_1} U^\dagger_{\ell_1\to\ell_2} O_{\ell_2^+} U_{\ell_1\to\ell_2} X_{\ell_1} U^\dagger_{\ell_1\to\ell_2}\left[X_{\ell_2}, O_{\ell_2^+}\right] O_{\ell_2^+}\left[X_{\ell_2}, O_{\ell_2^+}\right] U_{\ell_1\to\ell_2}\right) - \mathrm{tr}\left(\left[X_{\ell_2}, O_{\ell_2^+}\right] O_{\ell_2^+}\left[X_{\ell_2}, O_{\ell_2^+}\right] O_{\ell_2^+}\right)\\
&\qquad + \mathrm{tr}\left(X_{\ell_1} U^\dagger_{\ell_1\to\ell_2}\left[X_{\ell_2}, O_{\ell_2^+}\right] U_{\ell_1\to\ell_2} X_{\ell_1} U^\dagger_{\ell_1\to\ell_2} O_{\ell_2^+}\left[X_{\ell_2}, O_{\ell_2^+}\right] O_{\ell_2^+} U_{\ell_1\to\ell_2}\right)\\
&\qquad - \mathrm{tr}\left(X_{\ell_1} U^\dagger_{\ell_1\to\ell_2} O_{\ell_2^+}\left[X_{\ell_2}, O_{\ell_2^+}\right] U_{\ell_1\to\ell_2} X_{\ell_1} U^\dagger_{\ell_1\to\ell_2} O_{\ell_2^+}\left[X_{\ell_2}, O_{\ell_2^+}\right] U_{\ell_1\to\ell_2}\right)\Big] \qquad (156)\\
&= \int_{\mathcal{U}_{\mathrm{Haar}}} dU_{\ell_2^+}\Bigg[\frac{d\,\mathrm{tr}(O)\,\mathrm{tr}\left(\left[X_{\ell_2}, O_{\ell_2^+}\right]^2 O_{\ell_2^+}\right) - \mathrm{tr}\left(\left[X_{\ell_2}, O_{\ell_2^+}\right] O_{\ell_2^+}\left[X_{\ell_2}, O_{\ell_2^+}\right] O_{\ell_2^+}\right)}{d^2-1} + \frac{\mathrm{tr}\left(\left[X_{\ell_2}, O_{\ell_2^+}\right] O_{\ell_2^+}\left[X_{\ell_2}, O_{\ell_2^+}\right] O_{\ell_2^+}\right)}{1-d^2}\\
&\qquad - \frac{\mathrm{tr}\left(\left[X_{\ell_2}, O_{\ell_2^+}\right] O_{\ell_2^+}\left[X_{\ell_2}, O_{\ell_2^+}\right] O_{\ell_2^+}\right)}{1-d^2} - \mathrm{tr}\left(\left[X_{\ell_2}, O_{\ell_2^+}\right] O_{\ell_2^+}\left[X_{\ell_2}, O_{\ell_2^+}\right] O_{\ell_2^+}\right)\Bigg] \qquad (157)\\
&= \int_{\mathcal{U}_{\mathrm{Haar}}} dU_{\ell_2^+}\frac{d}{d^2-1}\left[\mathrm{tr}(O)\,\mathrm{tr}\left(\left[X_{\ell_2}, O_{\ell_2^+}\right]^2 O_{\ell_2^+}\right) - d\,\mathrm{tr}\left(\left[X_{\ell_2}, O_{\ell_2^+}\right] O_{\ell_2^+}\left[X_{\ell_2}, O_{\ell_2^+}\right] O_{\ell_2^+}\right)\right] \qquad (158)\\
&= \int_{\mathcal{U}_{\mathrm{Haar}}} dU_{\ell_2^+}\left\{\frac{d}{d^2-1}\mathrm{tr}(O)\left[\mathrm{tr}\left(X_{\ell_2} O_{\ell_2^+} X_{\ell_2} O^2_{\ell_2^+}\right) - \mathrm{tr}\left(O^3\right)\right] - \frac{2d^2}{d^2-1}\left[\mathrm{tr}\left(X_{\ell_2} O^2_{\ell_2^+} X_{\ell_2} O^2_{\ell_2^+}\right) - \mathrm{tr}\left(X_{\ell_2} O^3_{\ell_2^+} X_{\ell_2} O_{\ell_2^+}\right)\right]\right\} \qquad (159)\\
&= \frac{d}{d^2-1}\mathrm{tr}(O)\frac{d\left[\mathrm{tr}(O)\,\mathrm{tr}\left(O^2\right) - d\,\mathrm{tr}\left(O^3\right)\right]}{d^2-1} - \frac{d^2}{d^2-1}\frac{2d}{d^2-1}\left[\mathrm{tr}\left(O^2\right)^2 - \mathrm{tr}(O)\,\mathrm{tr}\left(O^3\right)\right] \qquad (160)\\
&= \frac{d^2}{(d^2-1)^2}\left[\mathrm{tr}(O)^2\,\mathrm{tr}\left(O^2\right) - 2d\,\mathrm{tr}\left(O^2\right)^2 + d\,\mathrm{tr}(O)\,\mathrm{tr}\left(O^3\right)\right]. \qquad (161)
\end{aligned}
$$

The seventh item is

$$
\begin{aligned}
&\int_{\mathcal{U}_{\mathrm{Haar}}} dU_{\ell_1\to\ell_2} dU_{\ell_2^+}\,\mathrm{tr}\left(\left[X_{\ell_1}, U^\dagger_{\ell_1\to\ell_2}\left[X_{\ell_2}, O_{\ell_2^+}\right]U_{\ell_1\to\ell_2}\right]\left[X_{\ell_1}, O_{\ell_1^+}\right]U^\dagger_{\ell_1\to\ell_2}\left[X_{\ell_2}, O_{\ell_2^+}\right]O_{\ell_2^+}U_{\ell_1\to\ell_2}\right)\\
&=\int_{\mathcal{U}_{\mathrm{Haar}}} dU_{\ell_1\to\ell_2} dU_{\ell_2^+}\left[\mathrm{tr}\left(X_{\ell_1}U^\dagger_{\ell_1\to\ell_2}\left[X_{\ell_2}, O_{\ell_2^+}\right]U_{\ell_1\to\ell_2}X_{\ell_1}U^\dagger_{\ell_1\to\ell_2}O_{\ell_2^+}\left[X_{\ell_2}, O_{\ell_2^+}\right]O_{\ell_2^+}U_{\ell_1\to\ell_2}\right)-\mathrm{tr}\left(\left[X_{\ell_2}, O_{\ell_2^+}\right]O_{\ell_2^+}\left[X_{\ell_2}, O_{\ell_2^+}\right]O_{\ell_2^+}\right)\right.\\
&\qquad\qquad+\mathrm{tr}\left(X_{\ell_1}U^\dagger_{\ell_1\to\ell_2}O_{\ell_2^+}U_{\ell_1\to\ell_2}X_{\ell_1}U^\dagger_{\ell_1\to\ell_2}\left[X_{\ell_2}, O_{\ell_2^+}\right]O_{\ell_2^+}\left[X_{\ell_2}, O_{\ell_2^+}\right]U_{\ell_1\to\ell_2}\right)\\
&\qquad\qquad\left.-\mathrm{tr}\left(X_{\ell_1}U^\dagger_{\ell_1\to\ell_2}\left[X_{\ell_2}, O_{\ell_2^+}\right]O_{\ell_2^+}U_{\ell_1\to\ell_2}X_{\ell_1}U^\dagger_{\ell_1\to\ell_2}\left[X_{\ell_2}, O_{\ell_2^+}\right]O_{\ell_2^+}U_{\ell_1\to\ell_2}\right)\right] \qquad (162)\\
&=\frac{d^2}{(d^2-1)^2}\left[\mathrm{tr}(O)^2\,\mathrm{tr}\left(O^2\right)-2d\,\mathrm{tr}\left(O^2\right)^2+d\,\mathrm{tr}(O)\,\mathrm{tr}\left(O^3\right)\right]. \qquad (163)
\end{aligned}
$$

The eighth item is

$$
\begin{aligned}
&\int_{\mathcal{U}_{\mathrm{Haar}}} dU_{\ell_1\to\ell_2} dU_{\ell_2^+}\,\mathrm{tr}\left(\left[X_{\ell_1}, U^\dagger_{\ell_1\to\ell_2}\left[X_{\ell_2}, O_{\ell_2^+}\right]U_{\ell_1\to\ell_2}\right]\left[X_{\ell_1}, O_{\ell_1^+}\right]U^\dagger_{\ell_1\to\ell_2}O_{\ell_2^+}\left[X_{\ell_2}, O_{\ell_2^+}\right]U_{\ell_1\to\ell_2}\right)\\
&=\int_{\mathcal{U}_{\mathrm{Haar}}} dU_{\ell_1\to\ell_2} dU_{\ell_2^+}\left[\mathrm{tr}\left(X_{\ell_1}U^\dagger_{\ell_1\to\ell_2}\left[X_{\ell_2}, O_{\ell_2^+}\right]U_{\ell_1\to\ell_2}X_{\ell_1}U^\dagger_{\ell_1\to\ell_2}O^2_{\ell_2^+}\left[X_{\ell_2}, O_{\ell_2^+}\right]U_{\ell_1\to\ell_2}\right)-\mathrm{tr}\left(\left[X_{\ell_2}, O_{\ell_2^+}\right]^2O^2_{\ell_2^+}\right)\right.\\
&\qquad\qquad+\mathrm{tr}\left(X_{\ell_1}U^\dagger_{\ell_1\to\ell_2}O_{\ell_2^+}U_{\ell_1\to\ell_2}X_{\ell_1}U^\dagger_{\ell_1\to\ell_2}O_{\ell_2^+}\left[X_{\ell_2}, O_{\ell_2^+}\right]^2U_{\ell_1\to\ell_2}\right)\\
&\qquad\qquad\left.-\mathrm{tr}\left(X_{\ell_1}U^\dagger_{\ell_1\to\ell_2}\left[X_{\ell_2}, O_{\ell_2^+}\right]O_{\ell_2^+}U_{\ell_1\to\ell_2}X_{\ell_1}U^\dagger_{\ell_1\to\ell_2}O_{\ell_2^+}\left[X_{\ell_2}, O_{\ell_2^+}\right]U_{\ell_1\to\ell_2}\right)\right] \qquad (164)\\
&=\int_{\mathcal{U}_{\mathrm{Haar}}} dU_{\ell_2^+}\left[\frac{\mathrm{tr}\left(\left[X_{\ell_2}, O_{\ell_2^+}\right]^2O^2_{\ell_2^+}\right)}{1-d^2}+\frac{d\,\mathrm{tr}(O)\,\mathrm{tr}\left(\left[X_{\ell_2}, O_{\ell_2^+}\right]^2O_{\ell_2^+}\right)-\mathrm{tr}\left(\left[X_{\ell_2}, O_{\ell_2^+}\right]^2O^2_{\ell_2^+}\right)}{d^2-1}-\frac{\mathrm{tr}\left(\left[X_{\ell_2}, O_{\ell_2^+}\right]^2O^2_{\ell_2^+}\right)}{1-d^2}\right.\\
&\qquad\left.-\mathrm{tr}\left(\left[X_{\ell_2}, O_{\ell_2^+}\right]^2O^2_{\ell_2^+}\right)\right] \qquad (165)\\
&=\int_{\mathcal{U}_{\mathrm{Haar}}} dU_{\ell_2^+}\frac{d}{d^2-1}\left[\mathrm{tr}(O)\,\mathrm{tr}\left(\left[X_{\ell_2}, O_{\ell_2^+}\right]^2O_{\ell_2^+}\right)-d\,\mathrm{tr}\left(\left[X_{\ell_2}, O_{\ell_2^+}\right]^2O^2_{\ell_2^+}\right)\right] \qquad (166)\\
&=\frac{d^2}{(d^2-1)^2}\left[\mathrm{tr}(O)^2\,\mathrm{tr}\left(O^2\right)-3d\,\mathrm{tr}(O)\,\mathrm{tr}\left(O^3\right)+d\,\mathrm{tr}\left(O^2\right)^2+d^2\,\mathrm{tr}\left(O^4\right)\right]. \qquad (167)
\end{aligned}
$$

The ninth item is

$$
\begin{aligned}
&\int_{\mathcal{U}_{\mathrm{Haar}}} dU_{\ell_1\to\ell_2} dU_{\ell_2^+} \operatorname{tr}\left(\left[X_{\ell_1}, O_{\ell_1^+}\right] U^\dagger_{\ell_1\to\ell_2}\left[X_{\ell_2}, O_{\ell_2^+}\right] U_{\ell_1\to\ell_2}[X_{\ell_1}, U^\dagger_{\ell_1\to\ell_2}\left[X_{\ell_2}, O_{\ell_2^+}\right] U_{\ell_1\to\ell_2}] U^\dagger_{\ell_1\to\ell_2} O_{\ell_2^+} U_{\ell_1\to\ell_2}\right) \\
&= \int_{\mathcal{U}_{\mathrm{Haar}}} dU_{\ell_1\to\ell_2} dU_{\ell_2^+} \Big[\operatorname{tr}\left(X_{\ell_1} U^\dagger_{\ell_1\to\ell_2} O_{\ell_2^+}\left[X_{\ell_2}, O_{\ell_2^+}\right] U_{\ell_1\to\ell_2} X_{\ell_1} U^\dagger_{\ell_1\to\ell_2}\left[X_{\ell_2}, O_{\ell_2^+}\right] O_{\ell_2^+} U_{\ell_1\to\ell_2}\right) \\
&\qquad + \operatorname{tr}\left(X_{\ell_1} U^\dagger_{\ell_1\to\ell_2}\left[X_{\ell_2}, O_{\ell_2^+}\right]^2 U_{\ell_1\to\ell_2} X_{\ell_1} U^\dagger_{\ell_1\to\ell_2} O^2_{\ell_2^+} U_{\ell_1\to\ell_2}\right) \\
&\qquad - \operatorname{tr}\left(X_{\ell_1} U^\dagger_{\ell_1\to\ell_2} O_{\ell_2^+}\left[X_{\ell_2}, O_{\ell_2^+}\right]^2 U_{\ell_1\to\ell_2} X_{\ell_1} U^\dagger_{\ell_1\to\ell_2} O_{\ell_2^+} U_{\ell_1\to\ell_2}\right) \\
&\qquad - \operatorname{tr}\left(X_{\ell_1} U^\dagger_{\ell_1\to\ell_2}\left[X_{\ell_2}, O_{\ell_2^+}\right] U_{\ell_1\to\ell_2} X_{\ell_1} U^\dagger_{\ell_1\to\ell_2}\left[X_{\ell_2}, O_{\ell_2^+}\right] O^2_{\ell_2^+} U_{\ell_1\to\ell_2}\right)\Big]
\end{aligned} \tag{168}
$$

$$
\begin{aligned}
&= \int_{\mathcal{U}_{\mathrm{Haar}}} dU_{\ell_2^+} \Bigg[\frac{\operatorname{tr}\left(\left[X_{\ell_2}, O_{\ell_2^+}\right]^2 O^2_{\ell_2^+}\right)}{1-d^2} + \frac{d \operatorname{tr}(O^2) \operatorname{tr}\left(\left[X_{\ell_2}, O_{\ell_2^+}\right]^2\right) - \operatorname{tr}\left(\left[X_{\ell_2}, O_{\ell_2^+}\right]^2 O^2_{\ell_2^+}\right)}{d^2-1} \\
&\qquad - \frac{d \operatorname{tr}(O) \operatorname{tr}\left(\left[X_{\ell_2}, O_{\ell_2^+}\right]^2 O_{\ell_2^+}\right) - \operatorname{tr}\left(\left[X_{\ell_2}, O_{\ell_2^+}\right]^2 O^2_{\ell_2^+}\right)}{d^2-1} - \frac{\operatorname{tr}\left(\left[X_{\ell_2}, O_{\ell_2^+}\right]^2 O^2_{\ell_2^+}\right)}{1-d^2}\Bigg]
\end{aligned} \tag{169}
$$

$$
= \int_{\mathcal{U}_{\mathrm{Haar}}} dU_{\ell_2^+} \frac{d}{d^2-1}\left[\operatorname{tr}(O^2) \operatorname{tr}\left(\left[X_{\ell_2}, O_{\ell_2^+}\right]^2\right) - \operatorname{tr}(O) \operatorname{tr}\left(\left[X_{\ell_2}, O_{\ell_2^+}\right]^2 O_{\ell_2^+}\right)\right] \tag{170}
$$

$$
= \frac{d}{d^2-1} \operatorname{tr}(O^2) \frac{2d[\operatorname{tr}(O)^2 - d \operatorname{tr}(O^2)]}{d^2-1} - \frac{d}{d^2-1} \operatorname{tr}(O) \frac{d[\operatorname{tr}(O) \operatorname{tr}(O^2) - d \operatorname{tr}(O^3)]}{d^2-1} \tag{171}
$$

$$
= \frac{d^2}{(d^2-1)^2}\left[\operatorname{tr}(O^2) \operatorname{tr}(O)^2 - 2d \operatorname{tr}(O^2)^2 + d \operatorname{tr}(O) \operatorname{tr}(O^3)\right]. \tag{172}
$$

The tenth item is

$$
\begin{aligned}
&\int_{\mathcal{U}_{\mathrm{Haar}}} dU_{\ell_1\to\ell_2} dU_{\ell_2^+} \operatorname{tr}\left([X_{\ell_1}, U^\dagger_{\ell_1\to\ell_2}\left[X_{\ell_2}, O_{\ell_2^+}\right] U_{\ell_1\to\ell_2}] U^\dagger_{\ell_1\to\ell_2}\left[X_{\ell_2}, O_{\ell_2^+}\right] U_{\ell_1\to\ell_2}\left[X_{\ell_1}, O_{\ell_1^+}\right] U^\dagger_{\ell_1\to\ell_2} O_{\ell_2^+} U_{\ell_1\to\ell_2}\right) \\
&= \int_{\mathcal{U}_{\mathrm{Haar}}} dU_{\ell_1\to\ell_2} dU_{\ell_2^+} \Big[\operatorname{tr}\left(X_{\ell_1} U^\dagger_{\ell_1\to\ell_2}\left[X_{\ell_2}, O_{\ell_2^+}\right]^2 U_{\ell_1\to\ell_2} X_{\ell_1} U^\dagger_{\ell_1\to\ell_2} O^2_{\ell_2^+} U_{\ell_1\to\ell_2}\right) \\
&\qquad + \operatorname{tr}\left(X_{\ell_1} U^\dagger_{\ell_1\to\ell_2}\left[X_{\ell_2}, O_{\ell_2^+}\right] O_{\ell_2^+} U_{\ell_1\to\ell_2} X_{\ell_1} U^\dagger_{\ell_1\to\ell_2} O_{\ell_2^+}\left[X_{\ell_2}, O_{\ell_2^+}\right] U_{\ell_1\to\ell_2}\right) \\
&\qquad - \operatorname{tr}\left(X_{\ell_1} U^\dagger_{\ell_1\to\ell_2}\left[X_{\ell_2}, O_{\ell_2^+}\right]^2 O_{\ell_2^+} U_{\ell_1\to\ell_2} X_{\ell_1} U^\dagger_{\ell_1\to\ell_2} O_{\ell_2^+} U_{\ell_1\to\ell_2}\right) \\
&\qquad - \operatorname{tr}\left(X_{\ell_1} U^\dagger_{\ell_1\to\ell_2}\left[X_{\ell_2}, O_{\ell_2^+}\right] U_{\ell_1\to\ell_2} X_{\ell_1} U^\dagger_{\ell_1\to\ell_2} O^2_{\ell_2^+}\left[X_{\ell_2}, O_{\ell_2^+}\right] U_{\ell_1\to\ell_2}\right)\Big]
\end{aligned} \tag{173}
$$

$$
\begin{aligned}
&= \int_{\mathcal{U}_{\mathrm{Haar}}} dU_{\ell_2^+} \Bigg[\frac{d \operatorname{tr}(O^2) \operatorname{tr}\left(\left[X_{\ell_2}, O_{\ell_2^+}\right]^2\right) - \operatorname{tr}\left(\left[X_{\ell_2}, O_{\ell_2^+}\right]^2 O^2_{\ell_2^+}\right)}{d^2-1} + \frac{\operatorname{tr}\left(\left[X_{\ell_2}, O_{\ell_2^+}\right]^2 O^2_{\ell_2^+}\right)}{1-d^2} \\
&\qquad - \frac{d \operatorname{tr}(O) \operatorname{tr}\left(\left[X_{\ell_2}, O_{\ell_2^+}\right]^2 O_{\ell_2^+}\right) - \operatorname{tr}\left(\left[X_{\ell_2}, O_{\ell_2^+}\right]^2 O^2_{\ell_2^+}\right)}{d^2-1} - \frac{\operatorname{tr}\left(\left[X_{\ell_2}, O_{\ell_2^+}\right]^2 O^2_{\ell_2^+}\right)}{1-d^2}\Bigg]
\end{aligned} \tag{174}
$$

$$
= \frac{d^2}{(d^2-1)^2}\left[\operatorname{tr}(O^2) \operatorname{tr}(O)^2 - 2d \operatorname{tr}(O^2)^2 + d \operatorname{tr}(O) \operatorname{tr}(O^3)\right]. \tag{175}
$$

The eleventh (last) item is

$$
\int_{\mathcal{U}_{\mathrm{Haar}}} dU_{\ell_1\to\ell_2} dU_{\ell_2^+} \operatorname{tr}\left(\left[X_{\ell_1}, O_{\ell_1^+}\right][X_{\ell_1}, U^\dagger_{\ell_1\to\ell_2}\left[X_{\ell_2}, O_{\ell_2^+}\right] U_{\ell_1\to\ell_2}]\right) \operatorname{tr}\left(\left[X_{\ell_2}, O_{\ell_2^+}\right] O_{\ell_2^+}\right) = 0 \tag{176}
$$

due to the cyclic property of trace by expansion of commutators.

Combing the above eleven terms, we have $\operatorname{tr}\left(\rho_0 U^\dagger O U\right) \frac{\partial^2 \epsilon}{\partial \theta_{\ell_1} \partial \theta_{\ell_2}} \frac{\partial \epsilon}{\partial \theta_{\ell_1}} \frac{\partial \epsilon}{\partial \theta_{\ell_2}}$

$$
\begin{aligned}
&\mathbb{E}_{\mathcal{U}_{\text{Haar}}}\left[\operatorname{tr}\left(\rho_0 U^\dagger O U\right) \frac{\partial^2 \epsilon}{\partial \theta_{\ell_1} \partial \theta_{\ell_2}} \frac{\partial \epsilon}{\partial \theta_{\ell_1}} \frac{\partial \epsilon}{\partial \theta_{\ell_2}}\right] \\
&= \frac{d}{8(d-1)^2(d+1)^3(d+2)(d+3)}\left[2 \operatorname{tr}(O)^4 - 3(d-1)\operatorname{tr}(O)^2 \operatorname{tr}\left(O^2\right) - 3(d+1)\operatorname{tr}\left(O^2\right)^2 + (d^2-d+4)\operatorname{tr}(O)\operatorname{tr}\left(O^3\right)\right. \\
&\qquad \left. + (d^2-d)\operatorname{tr}\left(O^4\right)\right]. \qquad (177)
\end{aligned}
$$

c. Summary of average dynamical index $\overline{\zeta_0}$ under Haar random ensemble

Summarizing from Eq. (138) and (177), combined with the result from Eq. (118) we thus have $\overline{\epsilon_0\mu_0}$ as

$$
\overline{\epsilon_0\mu_0} = L(L-1)\mathbb{E}_{\mathcal{U}_{\text{Haar}}}\left[\operatorname{tr}\left(\rho_0 U^\dagger O U\right) \frac{\partial^2 \epsilon}{\partial \theta_{\ell_1} \partial \theta_{\ell_2}} \frac{\partial \epsilon}{\partial \theta_{\ell_1}} \frac{\partial \epsilon}{\partial \theta_{\ell_2}}\right] + L\mathbb{E}_{\mathcal{U}_{\text{Haar}}}\left[\operatorname{tr}\left(\rho_0 U^\dagger O U\right) \frac{\partial^2 \epsilon}{\partial \theta_\ell^2}\left(\frac{\partial \epsilon}{\partial \theta_\ell}\right)^2\right] - O_0\overline{\mu_0} \qquad (178)
$$

$$
\begin{aligned}
&= -\frac{L}{4d(d-1)(d+1)^2(d-2)(d+2)^2(d-3)(d+3)^2}\left[\left(d^4-2d^3-9d^2+8d+20\right)\operatorname{tr}(O)^4\right. \\
&+ 2d\left(-d^4+d^3+10d^2+d-29\right)\operatorname{tr}\left(O^2\right)\operatorname{tr}(O)^2 - 4\left(d^5-11d^3-6d^2+38d+14\right)\operatorname{tr}\left(O^3\right)\operatorname{tr}(O) \\
&+ \left.\left(d^6+3d^5-11d^4-41d^3+18d^2+96d+60\right)\operatorname{tr}\left(O^2\right)^2 + d\left(d^5-13d^3-4d^2+56d-4\right)\operatorname{tr}\left(O^4\right)\right] \\
&+ \frac{L(L-1)d}{8(d-1)^2(d+1)^3(d+2)(d+3)}\left[-2(d+3)O_0\operatorname{tr}(O)^3 + 3d(d+3)O_0\operatorname{tr}(O)\operatorname{tr}\left(O^2\right) + (d^2-d+4)\operatorname{tr}(O)\operatorname{tr}\left(O^3\right)\right. \\
&\left. + d(d-1)\operatorname{tr}\left(O^4\right) - d^2(d+3)O_0\operatorname{tr}\left(O^3\right) - 3(d-1)\operatorname{tr}\left(O^2\right)\operatorname{tr}(O)^2 - 3(d+1)\operatorname{tr}\left(O^2\right)^2 + 2\operatorname{tr}(O)^4\right]. \qquad (179)
\end{aligned}
$$

One can then find

$$
\overline{\zeta_0} = \overline{\epsilon_0\mu_0}/\overline{K_0}^2, \qquad (180)
$$

where $\overline{\epsilon_0\mu_0}$ and $\overline{K_0}$ can be found in Eq. (179) and (92), though too complicated to show it completely here.

In the asymptotic limit of $L \gg 1, d \gg 1$, Eq. (179) can be reduced to

$$
\begin{aligned}
\overline{\epsilon_0\mu_0} \simeq &-\frac{L}{4d^6}\left[\operatorname{tr}(O)^4 - 2d\operatorname{tr}\left(O^2\right)\operatorname{tr}(O)^2 - 4d\operatorname{tr}\left(O^3\right)\operatorname{tr}(O) + d^2\operatorname{tr}\left(O^2\right)^2 + d^2\operatorname{tr}\left(O^4\right)\right] \\
&+ \frac{L^2}{8d^6}\left[-2dO_0\operatorname{tr}(O)^3 + 3d^2O_0\operatorname{tr}(O)\operatorname{tr}\left(O^2\right) + d^2\operatorname{tr}(O)\operatorname{tr}\left(O^3\right) + d^2\operatorname{tr}\left(O^4\right) - d^3O_0\operatorname{tr}\left(O^3\right)\right. \\
&\left. - 3d\operatorname{tr}\left(O^2\right)\operatorname{tr}(O)^2 - 3d\operatorname{tr}\left(O^2\right)^2 + 2\operatorname{tr}(O)^4\right]. \qquad (181)
\end{aligned}
$$

We then have the ratio $\overline{\zeta_0}$ as

$$
\overline{\zeta_0} = \frac{\overline{\epsilon_0\mu_0}}{\overline{K_0}^2} \qquad (182)
$$

$$
\begin{aligned}
&\simeq -\frac{1}{L}\frac{\operatorname{tr}(O)^4 - 2d\operatorname{tr}\left(O^2\right)\operatorname{tr}(O)^2 - 4d\operatorname{tr}\left(O^3\right)\operatorname{tr}(O) + d^2\operatorname{tr}\left(O^2\right)^2 + d^2\operatorname{tr}\left(O^4\right)}{\left(d\operatorname{tr}(O^2) - \operatorname{tr}(O)^2\right)^2} \\
&+ \frac{1}{2\left(d\operatorname{tr}(O^2) - \operatorname{tr}(O)^2\right)^2}\left[-2dO_0\operatorname{tr}(O)^3 + 3d^2O_0\operatorname{tr}(O)\operatorname{tr}\left(O^2\right) + d^2\operatorname{tr}(O)\operatorname{tr}\left(O^3\right) + d^2\operatorname{tr}\left(O^4\right) - d^3O_0\operatorname{tr}\left(O^3\right)\right. \\
&\left. - 3d\operatorname{tr}\left(O^2\right)\operatorname{tr}(O)^2 - 3d\operatorname{tr}\left(O^2\right)^2 + 2\operatorname{tr}(O)^4\right]. \qquad (183)
\end{aligned}
$$

Specifically, for traceless observable, the $\overline{\zeta_0}$ can be further simplified as

$$
\overline{\zeta_0} = \frac{\overline{\epsilon_0\mu_0}}{\overline{K_0}^2} \qquad (184)
$$

$$
\simeq -\frac{1}{L}\frac{\operatorname{tr}\left(O^2\right)^2 + \operatorname{tr}\left(O^4\right)}{\operatorname{tr}(O^2)^2} + \frac{1}{2d\operatorname{tr}(O^2)^2}\left[d\operatorname{tr}\left(O^4\right) - d^2O_0\operatorname{tr}\left(O^3\right) - 3\operatorname{tr}\left(O^2\right)^2\right]. \qquad (185)
$$

Supplementary Note 11.4. Fluctuations of error and QNTK under Haar random ensemble

At the end of discussion about Haar ensemble result, we discuss the standard deviation of total error $\mathrm{SD}[\epsilon_0]$ and QNTK $\mathrm{SD}[K_0]$. To calculate the standard deviation, we first focus on the variance.

Supplementary Note 11.4.1. Relative fluctuation of total error under Haar random ensemble

The variance of total error ϵ over Haar random ensemble is

$$\begin{aligned}
\mathrm{Var}[\epsilon_0] &= \overline{\epsilon_0^2} - \overline{\epsilon_0}^2 && (186)\\
&= \mathbb{E}_{\mathcal{U}_{\mathrm{Haar}}}\left[(\langle O\rangle - O_0)^2\right] - \left(\mathbb{E}_{\mathcal{U}_{\mathrm{Haar}}}[\langle O\rangle] - O_0\right)^2 && (187)\\
&= \mathbb{E}_{\mathcal{U}_{\mathrm{Haar}}}\left[\langle O\rangle^2\right] - \mathbb{E}_{\mathcal{U}_{\mathrm{Haar}}}[\langle O\rangle]^2 && (188)\\
&= \frac{\mathrm{tr}(O)^2 + \mathrm{tr}(O^2)}{d^2+d} - \frac{\mathrm{tr}(O)^2}{d^2} && (189)\\
&= \frac{d\,\mathrm{tr}(O^2) - \mathrm{tr}(O)^2}{d^2(d+1)}. && (190)
\end{aligned}$$

Then the standard deviation of error is

$$\mathrm{SD}[\epsilon_0] = \sqrt{\mathrm{Var}[\epsilon_0]} = \sqrt{\frac{d\,\mathrm{tr}(O^2) - \mathrm{tr}(O)^2}{d^2(d+1)}}, \tag{191}$$

and the relative fluctuation can be obtained directly as

$$\frac{\mathrm{SD}[\epsilon_0]}{\overline{\epsilon_0}} = \frac{1}{\mathrm{tr}(O)/d - O_0}\sqrt{\frac{d\,\mathrm{tr}(O^2) - \mathrm{tr}(O)^2}{d^2(d+1)}}. \tag{192}$$

Specifically, for traceless observable we have $\mathrm{SD}[\epsilon_0] = \sqrt{\mathrm{tr}(O^2)/d(d+1)}$ and $\mathrm{SD}[\epsilon_0]/\overline{\epsilon_0} = -\sqrt{\mathrm{tr}(O^2)/d(d+1)}/O_0$.

Supplementary Note 11.4.2. Relative fluctuation of QNTK under Haar random ensemble

The variance of K is $\mathrm{Var}[K_0] = \overline{K_0^2} - \overline{K_0}^2$, which can be written as

$$\begin{aligned}
\mathrm{Var}[K_0] &= \sum_{\ell_1,\ell_2} \mathbb{E}_{\mathcal{U}_{\mathrm{Haar}}}\left[\left(\frac{\partial\epsilon}{\partial\theta_{\ell_1}}\right)^2\left(\frac{\partial\epsilon}{\partial\theta_{\ell_2}}\right)^2\right] - \overline{K_0}^2 && (193)\\
&= 2\sum_{\ell_1<\ell_2} \mathbb{E}_{\mathcal{U}_{\mathrm{Haar}}}\left[\left(\frac{\partial\epsilon}{\partial\theta_{\ell_1}}\right)^2\left(\frac{\partial\epsilon}{\partial\theta_{\ell_2}}\right)^2\right] + \sum_{\ell} \mathbb{E}_{\mathcal{U}_{\mathrm{Haar}}}\left[\left(\frac{\partial\epsilon}{\partial\theta_{\ell}}\right)^4\right] - \overline{K_0}^2 && (194)\\
&= L(L-1)\mathbb{E}_{\mathcal{U}_{\mathrm{Haar}}}\left[\left(\frac{\partial\epsilon}{\partial\theta_{\ell_1}}\right)^2\left(\frac{\partial\epsilon}{\partial\theta_{\ell_2}}\right)^2\right] + L\mathbb{E}_{\mathcal{U}_{\mathrm{Haar}}}\left[\left(\frac{\partial\epsilon}{\partial\theta_{\ell}}\right)^4\right] - \overline{K_0}^2. && (195)
\end{aligned}$$

Under the random initialization of circuit parameters, we can calculate the variance via Haar integral as following.

a. Calculation of $\mathbb{E}\left[\left(\frac{\partial\epsilon}{\partial\theta_\ell}\right)^4\right]$ *with Haar random ensemble*

We first evaluate $\mathbb{E}_{\mathcal{U}_{\text{Haar}}}\left[\left(\frac{\partial\epsilon}{\partial\theta_\ell}\right)^4\right]$, which can be expanded as

$$\mathbb{E}_{\mathcal{U}_{\text{Haar}}}\left[\left(\frac{\partial\epsilon}{\partial\theta_\ell}\right)^4\right] = \frac{1}{16}\mathbb{E}_{\mathcal{U}_{\text{Haar}}}\left[\left(\langle\psi_0|U_{\ell-}^\dagger\left[X_\ell, O_{\ell+}\right]U_{\ell-}|\psi_0\rangle\right)^4\right] \tag{196}$$

$$= \frac{1}{16}\int_{\mathcal{U}_{\text{Haar}}} dU_{\ell-}dU_{\ell+}\,\text{tr}\left(\rho_0 U_{\ell-}^\dagger\left[X_\ell, O_{\ell+}\right]U_{\ell-}\rho_0 U_{\ell-}^\dagger\left[X_\ell, O_{\ell+}\right]U_{\ell-}\rho_0 U_{\ell-}^\dagger\left[X_\ell, O_{\ell+}\right]U_{\ell-}\rho_0 U_{\ell-}^\dagger\left[X_\ell, O_{\ell+}\right]U_{\ell-}\right) \tag{197}$$

$$= \int_{\mathcal{U}_{\text{Haar}}} dU_{\ell+}\frac{3\,\text{tr}\big([X_\ell, O_{\ell+}]^2\big)^2 + 6\,\text{tr}\left([X_\ell, O_{\ell+}]^4\right)}{16d(d+2)(d^2+4d+3)}, \tag{198}$$

where we evaluate the integral over $U_{\ell-}$ by considering $U_{\ell-}$ forms Haar random (4-design) ensemble.

From the expansion of $\text{tr}\big([X_\ell, O_{\ell+}]^4\big)$, we have

$$\int_{\mathcal{U}_{\text{Haar}}} dU_{\ell+}\,\text{tr}\left([X_\ell, O_{\ell+}]^4\right) =$$
$$\int_{\mathcal{U}_{\text{Haar}}} dU_{\ell+}\left[2\,\text{tr}\left(X_\ell O_{\ell+}X_\ell O_{\ell+}X_\ell O_{\ell+}X_\ell O_{\ell+}\right) - 8\,\text{tr}\left(X_\ell O_{\ell+}^3 X_\ell O_{\ell+}\right) + 4\,\text{tr}\left(X_\ell O_{\ell+}^2 X_\ell U_{\ell+}^\dagger O_{\ell+}^2\right)\right] + 2\,\text{tr}\big(O^4\big). \tag{199}$$

The expansion follows from Supplementary notes of [6]. With the help of `RTNI` [5], we have the integrals as

$$\int_{\mathcal{U}_{\text{Haar}}} dU_{\ell+}\,\text{tr}\left(X_\ell O_{\ell+}X_\ell O_{\ell+}X_\ell O_{\ell+}X_\ell O_{\ell+}\right)$$
$$= \frac{2d^2\,\text{tr}\big(O^2\big)\,\text{tr}(O)^2 + \big(d^2+9\big)\,\text{tr}\big(O^4\big) - d\,\text{tr}(O)^4 - 8d\,\text{tr}\big(O^3\big)\,\text{tr}(O) - 3d\,\text{tr}\big(O^2\big)^2}{d^4 - 10d^2 + 9} \tag{200}$$

$$\int_{\mathcal{U}_{\text{Haar}}} dU_{\ell+}\,\text{tr}\left(X_\ell O_{\ell+}^3 X_\ell O_{\ell+}\right) = \frac{d\,\text{tr}(O)\,\text{tr}\big(O^3\big) - \text{tr}\big(O^4\big)}{d^2-1} \tag{201}$$

$$\int_{\mathcal{U}_{\text{Haar}}} dU_{\ell+}\,\text{tr}\left(X_\ell O_{\ell+}^2 X_\ell O_{\ell+}^2\right) = \frac{d\,\text{tr}\big(O^2\big)^2 - \text{tr}\big(O^4\big)}{d^2-1}, \tag{202}$$

where $U_{\ell+}$ follows Haar random unitary (4-design).

The ensemble average of $\mathbb{E}_{\mathcal{U}_{\text{Haar}}}\left[\text{tr}\big([X_\ell, O_{\ell+}]^4\big)\right]$ is thus

$$\mathbb{E}_{\mathcal{U}_{\text{Haar}}}\left[\text{tr}\big([X_\ell, O_{\ell+}]^4\big)\right]$$
$$= \frac{2d\left[\big(28-4d^2\big)\,\text{tr}\big(O^3\big)\,\text{tr}(O) + \big(2d^2-21\big)\,\text{tr}\big(O^2\big)^2 + \big(d^3-7d\big)\,\text{tr}\big(O^4\big) + 2d\,\text{tr}\big(O^2\big)\,\text{tr}(O)^2 - \text{tr}(O)^4\right]}{d^4-10d^2+9}. \tag{203}$$

On the other hand, $\text{tr}\big([X_\ell, O_{\ell+}]^2\big)^2$ can be expanded as

$$\int_{\mathcal{U}_{\text{Haar}}} dU_{\ell+}\,\text{tr}\left([X_\ell, O_{\ell+}]^2\right)^2 = \sum_{i_1,i_2}\int dU_{\ell+}\,\text{tr}\left(P_{i_2,i_1}[X_\ell, O_{\ell+}]^2 P_{i_1,i_2}[X_\ell, O_{\ell+}]^2\right) \tag{204}$$

$$= \sum_{i_1,i_2}\int_{\mathcal{U}_{\text{Haar}}} dU_{\ell+}\left[4\,\text{tr}\left(P_{i_2,i_1}X_\ell O_{\ell+}X_\ell O_{\ell+}P_{i_1,i_2}X_\ell O_{\ell+}X_\ell O_{\ell+}\right)\right.$$
$$\left. -8\,\text{tr}\left(P_{i_2,i_1}X_\ell O_{\ell+}X_\ell O_{\ell+}P_{i_1,i_2}O_{\ell+}^2\right) + 4\,\text{tr}\left(P_{i_2,i_1}O_{\ell+}^2 P_{i_1,i_2}O_{\ell+}^2\right)\right]. \tag{205}$$

where again in the first equation we introduce $P_{i_1,i_2} = |i_1\rangle\langle i_2|$ to combine the two traces together such that it can be evaluated by `RTNI` [5]. Similar to above calculation, we then can find

$$\sum_{i_1,i_2}\int_{\mathcal{U}_{\text{Haar}}} dU_{\ell^+}\,\mathrm{tr}\left(P_{i_2,i_1}X_\ell O_{\ell^+}X_\ell O_{\ell^+}P_{i_1,i_2}X_\ell O_{\ell^+}X_\ell O_{\ell^+}\right)$$
$$=\frac{\left(d^2-6\right)\mathrm{tr}(O)^4+\left(2d^2-9\right)\mathrm{tr}\left(O^2\right)^2-6d\,\mathrm{tr}\left(O^2\right)\mathrm{tr}(O)^2-6d\,\mathrm{tr}\left(O^4\right)+24\,\mathrm{tr}\left(O^3\right)\mathrm{tr}(O)}{d^4-10d^2+9} \tag{206}$$

$$\sum_{i_1,i_2}\int_{\mathcal{U}_{\text{Haar}}} dU_{\ell^+}\,\mathrm{tr}\left(P_{i_2,i_1}X_\ell O_{\ell^+}X_\ell O_{\ell^+}P_{i_1,i_2}O^2_{\ell^+}\right)=\frac{\mathrm{tr}\left(O^2\right)\left(d\,\mathrm{tr}(O)^2-\mathrm{tr}\left(O^2\right)\right)}{d^2-1} \tag{207}$$

$$\sum_{i_1,i_2}\int_{\mathcal{U}_{\text{Haar}}} dU_{\ell^+}\,\mathrm{tr}\left(P_{i_2,i_1}O^2_{\ell^+}P_{i_1,i_2}O^2_{\ell^+}\right)=\mathrm{tr}\left(O^2\right)^2. \tag{208}$$

Combining the above three equations, the average over $\mathrm{tr}\left([X_\ell, O_{\ell^+}]^2\right)^2$ is

$$\mathbb{E}_{\mathcal{U}_{\text{Haar}}}\left[\mathrm{tr}\left([X_\ell,O_{\ell^+}]^2\right)^2\right]=$$
$$\frac{4\left[\left(d^2-6\right)\mathrm{tr}(O)^4-2\left(d^3-6d\right)\mathrm{tr}\left(O^2\right)\mathrm{tr}(O)^2+\left(d^4-6d^2-18\right)\mathrm{tr}\left(O^2\right)^2-6d\,\mathrm{tr}\left(O^4\right)+24\,\mathrm{tr}\left(O^3\right)\mathrm{tr}(O)\right]}{d^4-10d^2+9}. \tag{209}$$

Therefore, the ensemble average of $\left(\frac{\partial\epsilon}{\partial\theta_l}\right)^4$ is

$$\mathbb{E}_{\mathcal{U}_{\text{Haar}}}\left[\left(\frac{\partial\epsilon}{\partial\theta_l}\right)^4\right]=\int_{\mathcal{U}_{\text{Haar}}} dU_{\ell^+}\frac{3\,\mathrm{tr}\left([X_\ell,O_{\ell^+}]^2\right)^2+6\,\mathrm{tr}\left([X_\ell,O_{\ell^+}]^4\right)}{16d(d+2)(d^2+4d+3)} \tag{210}$$

$$=\frac{3\left[(d^2+3d+3)\,\mathrm{tr}\left(O^2\right)^2+d(d+1)\,\mathrm{tr}\left(O^4\right)+\mathrm{tr}(O)^4-2d\,\mathrm{tr}\left(O^2\right)\mathrm{tr}(O)^2-4(d+1)\,\mathrm{tr}\left(O^3\right)\mathrm{tr}(O)\right]}{4(d-1)d(d+1)^2(d+3)^2} \tag{211}$$

$$\simeq\frac{3\,\mathrm{tr}\left(O^2\right)^2}{4d^4}+\frac{3\,\mathrm{tr}\left(O^4\right)}{4d^4}+\frac{3\,\mathrm{tr}(O)^4}{4d^6}-\frac{3\,\mathrm{tr}\left(O^2\right)\mathrm{tr}(O)^2}{2d^5}-\frac{3\,\mathrm{tr}\left(O^3\right)\mathrm{tr}(O)}{d^5}, \tag{212}$$

where we still approximate by $d\gg 1$.

b. Calculation of $\mathbb{E}_{\mathcal{U}_{\text{Haar}}}\left[\left(\frac{\partial\epsilon}{\partial\theta_{\ell_1}}\right)^2\left(\frac{\partial\epsilon}{\partial\theta_{\ell_2}}\right)^2\right]$ *with Haar random ensemble*

For the evaluation of $\mathbb{E}_{\mathcal{U}_{\text{Haar}}}\left[\left(\frac{\partial\epsilon}{\partial\theta_{\ell_1}}\right)^2\left(\frac{\partial\epsilon}{\partial\theta_{\ell_2}}\right)^2\right]$, there are four different unitaries $U_{\ell_1^-}, U_{\ell_2^-}, U_{\ell_1^+}, U_{\ell_2^+}$ assuming $\ell_1<\ell_2$. Note that $U_{\ell_2^-}=U_{\ell_1\to\ell_2}U_{\ell_1^-}$ and $U_{\ell_1^+}=U_{\ell_2^+}U_{\ell_1\to\ell_2}$ are not fully independent, so the Haar average has to be performed in with respect to $U_{\ell^-}, U_{\ell_1 \to \ell_2}, U_{\ell_2^+}$ individually only.

$$
\begin{aligned}
&\mathbb{E}_{\mathcal{U}_{\text{Haar}}}\left[\left(\frac{\partial\epsilon}{\partial\theta_{\ell_1}}\right)^2\left(\frac{\partial\epsilon}{\partial\theta_{\ell_2}}\right)^2\right]\\
&=\frac{1}{16}\int_{\mathcal{U}_{\text{Haar}}} dU_{\ell_1^-}dU_{\ell_1\to\ell_2}dU_{\ell_2^+}\,\langle\psi_0|U^\dagger_{\ell_1^-}\left[X_{\ell_1},O_{\ell_1^+}\right]U_{\ell_1^-}|\psi_0\rangle^2\,\langle\psi_0|U^\dagger_{\ell_1^-}U^\dagger_{\ell_1\to\ell_2}\left[X_{\ell_2},O_{\ell_2^+}\right]U_{\ell_1\to\ell_2}U_{\ell_1^-}|\psi_0\rangle^2 && (213)\\
&=\frac{1}{16}\int_{\mathcal{U}_{\text{Haar}}} dU_{\ell_1^-}dU_{\ell_1\to\ell_2}dU_{\ell_2^+}\,\mathrm{tr}\left(\rho_0U^\dagger_{\ell_1^-}\left[X_{\ell_1},O_{\ell_1^+}\right]U_{\ell_1^-}\rho_0U^\dagger_{\ell_1^-}\left[X_{\ell_1},O_{\ell_1^+}\right]U_{\ell_1^-}\rho_0U^\dagger_{\ell_1^-}U^\dagger_{\ell_1\to\ell_2}\left[X_{\ell_2},O_{\ell_2^+}\right]U_{\ell_1\to\ell_2}U_{\ell_1^-}\right.\\
&\left.\cdot\rho_0U^\dagger_{\ell_1^-}U^\dagger_{\ell_1\to\ell_2}\left[X_{\ell_2},O_{\ell_2^+}\right]U_{\ell_1\to\ell_2}U_{\ell_1^-}\right) && (214)\\
&=\frac{1}{16d(d+2)\left(d^2+4d+3\right)}\\
&\quad\times\int_{\mathcal{U}_{\text{Haar}}} dU_{\ell_1\to\ell_2}dU_{\ell_2^+}\left[\mathrm{tr}\left(\left[X_{\ell_1},O_{\ell_1^+}\right]^2\right)\mathrm{tr}\left(\left[X_{\ell_1},O_{\ell_2^+}\right]^2\right)+2\,\mathrm{tr}\left(U^\dagger_{\ell_1\to\ell_2}\left[X_{\ell_1},O_{\ell_2^+}\right]U_{\ell_1\to\ell_2}\left[X_{\ell_1},O_{\ell_1^+}\right]\right)^2\right.\\
&\qquad+4\,\mathrm{tr}\left(\left[X_{\ell_1},O_{\ell_1^+}\right]^2U^\dagger_{\ell_1\to\ell_2}\left[X_{\ell_1},O_{\ell_2^+}\right]^2U_{\ell_1\to\ell_2}\right)\\
&\qquad\left.+\,\mathrm{tr}\left(U^\dagger_{\ell_1\to\ell_2}\left[X_{\ell_1},O_{\ell_2^+}\right]U_{\ell_1\to\ell_2}\left[X_{\ell_1},O_{\ell_1^+}\right]U^\dagger_{\ell_1\to\ell_2}\left[X_{\ell_1},O_{\ell_2^+}\right]U_{\ell_1\to\ell_2}\left[X_{\ell_1},O_{\ell_1^+}\right]\right)\right], && (215)
\end{aligned}
$$

where in the first equation we expand the derivative and write $U_{\ell_2^-}$ and $U_{\ell_1^+}$ in terms of $U_{\ell_1\to\ell_2}, U_{\ell_1^-}, U_{\ell_2^+}$. Next we evaluate the average over $U_{\ell_1\to\ell_2}$ and $U_{\ell_2^+}$ of each term separately.

The average of first term, $\mathrm{tr}\left(\left[X_{\ell_1},O_{\ell_1^+}\right]^2\right)\mathrm{tr}\left(\left[X_{\ell_2},O_{\ell_2^+}\right]^2\right)$, becomes

$$
\begin{aligned}
&\int_{\mathcal{U}_{\text{Haar}}} dU_{\ell_1\to\ell_2}dU_{\ell_2^+}\,\mathrm{tr}\left(\left[X_{\ell_1},O_{\ell_1^+}\right]^2\right)\mathrm{tr}\left(\left[X_{\ell_2},O_{\ell_2^+}\right]^2\right) && (216)\\
&=\int_{\mathcal{U}_{\text{Haar}}} dU_{\ell_2^+}\,2\,\mathrm{tr}\left(\left[X_{\ell_2},O_{\ell_2^+}\right]^2\right)\left[\frac{d\,\mathrm{tr}(O)^2-\mathrm{tr}(O^2)}{d^2-1}-\mathrm{tr}(O^2)\right] && (217)\\
&=4\left[\frac{d\,\mathrm{tr}(O)^2-\mathrm{tr}(O^2)}{d^2-1}-\mathrm{tr}(O^2)\right]\left[\frac{d\,\mathrm{tr}(O)^2-\mathrm{tr}(O^2)}{d^2-1}-\mathrm{tr}(O^2)\right] && (218)\\
&=\frac{4d^2\left[\mathrm{tr}(O)^2-d\,\mathrm{tr}(O^2)\right]^2}{(d^2-1)^2}. && (219)
\end{aligned}
$$

The average of second term, $\operatorname{tr}\Big(U^\dagger_{\ell_1\to\ell_2}\left[X_{\ell_2},O_{\ell_2^+}\right]U_{\ell_1\to\ell_2}\left[X_{\ell_1},O_{\ell_1^+}\right]\Big)^2$ is

$$
\begin{aligned}
&\int_{\mathcal{U}_{\rm Haar}} dU_{\ell_1\to\ell_2}dU_{\ell_2^+}\operatorname{tr}\Big(U^\dagger_{\ell_1\to\ell_2}\left[X_{\ell_2},O_{\ell_2^+}\right]U_{\ell_1\to\ell_2}\left[X_{\ell_1},O_{\ell_1^+}\right]\Big)^2\\
&=\sum_{i_1,i_2}\int_{\mathcal{U}_{\rm Haar}} dU_{\ell_1\to\ell_2}dU_{\ell_2^+}\operatorname{tr}\left(P_{i_2,i_1}U^\dagger_{\ell_1\to\ell_2}\left[X_{\ell_2},O_{\ell_2^+}\right]U_{\ell_1\to\ell_2}\left[X_{\ell_1},O_{\ell_1^+}\right]P_{i_1,i_2}U^\dagger_{\ell_1\to\ell_2}\left[X_{\ell_2},O_{\ell_2^+}\right]U_{\ell_1\to\ell_2}\left[X_{\ell_1},O_{\ell_1^+}\right]\right) && (220)\\
&=\sum_{i_1,i_2}\int_{\mathcal{U}_{\rm Haar}} dU_{\ell_1\to\ell_2}dU_{\ell_2^+}\Big[\operatorname{tr}\left(O_{\ell_2^+}\left[X_{\ell_2},O_{\ell_2^+}\right]U_{\ell_1\to\ell_2}X_{\ell_1}U^\dagger_{\ell_1\to\ell_2}\right)^2+\operatorname{tr}\left(\left[X_{\ell_2},O_{\ell_2^+}\right]O_{\ell_2^+}U_{\ell_1\to\ell_2}X_{\ell_1}U^\dagger_{\ell_1\to\ell_2}\right)^2\\
&\qquad -2\operatorname{tr}\left(O_{\ell_2^+}\left[X_{\ell_2},O_{\ell_2^+}\right]U_{\ell_1\to\ell_2}X_{\ell_1}U^\dagger_{\ell_1\to\ell_2}\right)\operatorname{tr}\left(\left[X_{\ell_2},O_{\ell_2^+}\right]O_{\ell_2^+}U_{\ell_1\to\ell_2}X_{\ell_1}U^\dagger_{\ell_1\to\ell_2}\right)\Big] && (221)\\
&=\int_{\mathcal{U}_{\rm Haar}} dU_{\ell_2^+}2\left[\frac{d\operatorname{tr}\Big(O_{\ell_2^+}\left[X_{\ell_2},O_{\ell_2^+}\right]O_{\ell_2^+}\left[X_{\ell_2},O_{\ell_2^+}\right]\Big)-\operatorname{tr}\Big(O_{\ell_2^+}\left[X_{\ell_2},O_{\ell_2^+}\right]\Big)^2}{d^2-1}+\frac{\operatorname{tr}\Big(O_{\ell_2^+}\left[X_{\ell_2},O_{\ell_2^+}\right]\Big)^2-d\operatorname{tr}\Big(\left[X_{\ell_2},O_{\ell_2^+}\right]^2O^2_{\ell_2^+}\Big)}{d^2-1}\right] && (222)\\
&=\int_{\mathcal{U}_{\rm Haar}} dU_{\ell_2^+}2d\frac{\operatorname{tr}\Big(O_{\ell_2^+}\left[X_{\ell_2},O_{\ell_2^+}\right]O_{\ell_2^+}\left[X_{\ell_2},O_{\ell_2^+}\right]\Big)-\operatorname{tr}\Big(\left[X_{\ell_2},O_{\ell_2^+}\right]^2O^2_{\ell_2^+}\Big)}{d^2-1} && (223)\\
&=\frac{2d}{d^2-1}\int_{\mathcal{U}_{\rm Haar}} dU_{\ell_2^+}\left[2\operatorname{tr}\left(O^2_{\ell_2^+}X_{\ell_2}O^2_{\ell_2^+}X_{\ell_2}\right)-2\operatorname{tr}\left(O_{\ell_2^+}X_{\ell_2}O^3_{\ell_2^+}X_{\ell_2}\right)-2\operatorname{tr}\left(X_{\ell_2}O_{\ell_2^+}X_{\ell_2}O^3_{\ell_2^+}\right)+\operatorname{tr}\left(X_{\ell_2}O^2_{\ell_2^+}X_{\ell_2}O^2_{\ell_2^+}\right)+\operatorname{tr}(O^4)\right] && (224)\\
&=\frac{2d}{d^2-1}\int_{\mathcal{U}_{\rm Haar}} dU_{\ell_2^+}\left[3\operatorname{tr}\left(O^2_{\ell_2^+}X_{\ell_2}O^2_{\ell_2^+}X_{\ell_2}\right)-4\operatorname{tr}\left(O_{\ell_2^+}X_{\ell_2}O^3_{\ell_2^+}X_{\ell_2}\right)+\operatorname{tr}(O^4)\right] && (225)\\
&=\frac{2d^2\left[d\operatorname{tr}(O^4)+3\operatorname{tr}(O^2)^2-4\operatorname{tr}(O)\operatorname{tr}(O^3)\right]}{(d^2-1)^2}. && (226)
\end{aligned}
$$

The average of third term, $\operatorname{tr}\left(\left[X_{\ell_1},O_{\ell_1^+}\right]^2U^\dagger_{\ell_1\to\ell_2}\left[X_{\ell_2},O_{\ell_2^+}\right]^2U_{\ell_1\to\ell_2}\right)$ is

$$
\begin{aligned}
&\int_{\mathcal{U}_{\rm Haar}} dU_{\ell_1\to\ell_2}dU_{\ell_2^+}\operatorname{tr}\left(\left[X_{\ell_1},O_{\ell_1^+}\right]^2U^\dagger_{\ell_1\to\ell_2}\left[X_{\ell_2},O_{\ell_2^+}\right]^2U_{\ell_1\to\ell_2}\right)\\
&=\int_{\mathcal{U}_{\rm Haar}} dU_{\ell_1\to\ell_2}dU_{\ell_2^+}\Big[\operatorname{tr}\Big(X_{\ell_1}O_{\ell_1^+}X_{\ell_1}U^\dagger_{\ell_1\to\ell_2}O_{\ell_2^+}\left[X_{\ell_2},O_{\ell_2^+}\right]^2U_{\ell_1\to\ell_2}\Big)+\operatorname{tr}\Big(X_{\ell_1}O_{\ell_1^+}X_{\ell_1}U^\dagger_{\ell_1\to\ell_2}\left[X_{\ell_2},O_{\ell_2^+}\right]^2O_{\ell_2^+}U_{\ell_1\to\ell_2}\Big)\\
&\qquad-\operatorname{tr}\Big(X_{\ell_1}O^2_{\ell_1^+}X_{\ell_1}U^\dagger_{\ell_1\to\ell_2}\left[X_{\ell_2},O_{\ell_2^+}\right]^2U_{\ell_1\to\ell_2}\Big)-\operatorname{tr}\Big(O^2_{\ell_2^+}\left[X_{\ell_2},O_{\ell_2^+}\right]^2\Big)\Big] && (227)\\
&=\int_{\mathcal{U}_{\rm Haar}} dU_{\ell_2^+}\Bigg[2\frac{d\operatorname{tr}(O)\operatorname{tr}\Big(\left[X_{\ell_2},O_{\ell_2^+}\right]^2O_{\ell_2^+}\Big)-\operatorname{tr}\Big(\left[X_{\ell_2},O_{\ell_2^+}\right]^2O^2_{\ell_2^+}\Big)}{d^2-1}-\frac{d\operatorname{tr}\Big(\left[X_{\ell_2},O_{\ell_2^+}\right]^2\Big)\operatorname{tr}(O^2)-\operatorname{tr}\Big(\left[X_{\ell_2},O_{\ell_2^+}\right]^2O^2_{\ell_2^+}\Big)}{d^2-1}\\
&\qquad-\operatorname{tr}\Big(O^2_{\ell_2^+}\left[X_{\ell_2},O_{\ell_2^+}\right]^2\Big)\Bigg] && (228)\\
&=\int_{\mathcal{U}_{\rm Haar}} dU_{\ell_2^+}\frac{d\left[-d\operatorname{tr}\Big(\left[X_{\ell_2},O_{\ell_2^+}\right]^2O^2_{\ell_2^+}\Big)+2\operatorname{tr}(O)\operatorname{tr}\Big(\left[X_{\ell_2},O_{\ell_2^+}\right]^2O_{\ell_2^+}\Big)-\operatorname{tr}\Big(\left[X_{\ell_2},O_{\ell_2^+}\right]^2\Big)\operatorname{tr}(O^2)\right]}{d^2-1} && (229)\\
&=\frac{d}{d^2-1}\left[d\frac{d\left[d\operatorname{tr}(O^4)+\operatorname{tr}(O^2)^2-2\operatorname{tr}(O)\operatorname{tr}(O^3)\right]}{d^2-1}+2\operatorname{tr}(O)\frac{d[\operatorname{tr}(O)\operatorname{tr}(O^2)-d\operatorname{tr}(O^3)]}{d^2-1}-\operatorname{tr}(O^2)\frac{2d\left[\operatorname{tr}(O)^2-d\operatorname{tr}(O^2)\right]}{d^2-1}\right] && (230)\\
&=\frac{d^3\left[d\operatorname{tr}(O^4)+3\operatorname{tr}(O^2)^2-4\operatorname{tr}(O)\operatorname{tr}(O^3)\right]}{(d^2-1)^2} && (231)
\end{aligned}
$$

The average over $U_{\ell_1\to\ell_2}$ in the last term, $\operatorname{tr}\Big(U^\dagger_{\ell_1\to\ell_2}\left[X_{\ell_2},O_{\ell_2^+}\right]U_{\ell_1\to\ell_2}\left[X_{\ell_1},O_{\ell_1^+}\right]U^\dagger_{\ell_1\to\ell_2}\left[X_{\ell_2},O_{\ell_2^+}\right]U_{\ell_1\to\ell_2}\left[X_{\ell_1},O_{\ell_1^+}\right]\Big)$

is

$$\int_{\mathcal{U}_{\rm Haar}} dU_{\ell_1\to\ell_2} dU_{\ell_2^+}\, \mathrm{tr}\Big(U_{\ell_1\to\ell_2}^\dagger \left[X_{\ell_2}, O_{\ell_2^+}\right] U_{\ell_1\to\ell_2}\left[X_{\ell_1}, O_{\ell_1^+}\right] U_{\ell_1\to\ell_2}^\dagger \left[X_{\ell_2}, O_{\ell_2^+}\right] U_{\ell_1\to\ell_2}\left[X_{\ell_1}, O_{\ell_1^+}\right]\Big)$$

$$\begin{aligned}
&= \int_{\mathcal{U}_{\rm Haar}} dU_{\ell_1\to\ell_2} dU_{\ell_2^+} \Big[\mathrm{tr}\left(X_{\ell_1} U_{\ell_1\to\ell_2}^\dagger O_{\ell_2^+}\left[X_{\ell_2}, O_{\ell_2^+}\right] U_{\ell_1\to\ell_2} X_{\ell_1} U_{\ell_1\to\ell_2}^\dagger O_{\ell_2^+}\left[X_{\ell_2}, O_{\ell_2^+}\right] U_{\ell_1\to\ell_2}\right) \\
&\qquad + \mathrm{tr}\left(X_{\ell_1} U_{\ell_1\to\ell_2}^\dagger \left[X_{\ell_2}, O_{\ell_2^+}\right] O_{\ell_2^+} U_{\ell_1\to\ell_2} X_{\ell_1} U_{\ell_1\to\ell_2}^\dagger \left[X_{\ell_2}, O_{\ell_2^+}\right] O_{\ell_2^+} U_{\ell_1\to\ell_2}\right) \\
&\qquad -2\,\mathrm{tr}\left(O_{\ell_2^+}\left[X_{\ell_2}, O_{\ell_2^+}\right] O_{\ell_2^+} U_{\ell_1\to\ell_2} X_{\ell_1} U_{\ell_1\to\ell_2}^\dagger \left[X_{\ell_2}, O_{\ell_2^+}\right] U_{\ell_1\to\ell_2} X_{\ell_1} U_{\ell_1\to\ell_2}^\dagger\right)\Big] &(232) \\
&= \int_{\mathcal{U}_{\rm Haar}} dU_{\ell_2^+}\, 2\left[\frac{d\,\mathrm{tr}\Big(\left[X_{\ell_2}, O_{\ell_2^+}\right] O_{\ell_2^+}\Big)^2 - \mathrm{tr}\Big(\left[X_{\ell_2}, O_{\ell_2^+}\right] O_{\ell_2^+}\left[X_{\ell_2}, O_{\ell_2^+}\right] O_{\ell_2^+}\Big)}{d^2-1} + \frac{\mathrm{tr}\Big(O_{\ell_2^+}\left[X_{\ell_2}, O_{\ell_2^+}\right] O_{\ell_2^+}\left[X_{\ell_2}, O_{\ell_2^+}\right]\Big)}{d^2-1}\right] \\
& &(233) \\
&= \int_{\mathcal{U}_{\rm Haar}} dU_{\ell_2^+} \frac{2d\,\mathrm{tr}\Big(\left[X_{\ell_2}, O_{\ell_2^+}\right] O_{\ell_2^+}\Big)^2}{d^2-1} &(234) \\
&= 0, &(235)
\end{aligned}$$

where the last line can be found by the cyclic property of trace directly.

Therefore, we can conclude on $\mathbb{E}_{\mathcal{U}_{\rm Haar}}\left[\left(\frac{\partial\epsilon}{\partial\theta_{\ell_1}}\right)^2\left(\frac{\partial\epsilon}{\partial\theta_{\ell_2}}\right)^2\right]$ is

$$\begin{aligned}
&\mathbb{E}_{\mathcal{U}_{\rm Haar}}\left[\left(\frac{\partial\epsilon}{\partial\theta_{\ell_1}}\right)^2\left(\frac{\partial\epsilon}{\partial\theta_{\ell_2}}\right)^2\right] \\
&= \frac{1}{16d(d+2)(d^2+4d+3)}\Bigg(\frac{4d^2\left[\mathrm{tr}(O)^2 - d\,\mathrm{tr}(O^2)\right]^2}{(d^2-1)^2} + 2\frac{2d^2\left[d\,\mathrm{tr}(O^4) + 3\,\mathrm{tr}(O^2)^2 - 4\,\mathrm{tr}(O)\,\mathrm{tr}(O^3)\right]}{(d^2-1)^2} \\
&\qquad + 4\frac{d^3\left[d\,\mathrm{tr}(O^4) + 3\,\mathrm{tr}(O^2)^2 - 4\,\mathrm{tr}(O)\,\mathrm{tr}(O^3)\right]}{(d^2-1)^2} + 0\Bigg) &(236) \\
&= \frac{d\left[(d^2+3d+3)\,\mathrm{tr}(O^2)^2 + d(d+1)\,\mathrm{tr}(O^4) + \mathrm{tr}(O)^4 - 2d\,\mathrm{tr}(O^2)\,\mathrm{tr}(O)^2 - 4(d+1)\,\mathrm{tr}(O^3)\,\mathrm{tr}(O)\right]}{4(d-1)^2(d+1)^3(d+2)(d+3)} &(237) \\
&\simeq \frac{\mathrm{tr}(O^2)^2}{4d^4} + \frac{\mathrm{tr}(O^4)}{4d^4} + \frac{\mathrm{tr}(O)^4}{4d^6} - \frac{\mathrm{tr}(O^2)\,\mathrm{tr}(O)^2}{2d^5} - \frac{\mathrm{tr}(O^3)\,\mathrm{tr}(O)}{d^5}. &(238)
\end{aligned}$$

c. Summary of relative fluctuation $\mathrm{SD}[K_0]/\overline{K_0}$ *under random initialization*

To summarize from Eq. (212) and (238), the ensemble average of $\mathrm{Var}[K_0]$ is

$$\begin{aligned}
\mathrm{Var}[K_0] &= L(L-1)\mathbb{E}_{\mathcal{U}_{\rm Haar}}\left[\left(\frac{\partial\epsilon}{\partial\theta_{\ell_1}}\right)^2\left(\frac{\partial\epsilon}{\partial\theta_{\ell_2}}\right)^2\right] + L\mathbb{E}_{\mathcal{U}_{\rm Haar}}\left[\left(\frac{\partial\epsilon}{\partial\theta_{\ell}}\right)^4\right] - \overline{K_0}^2 &(239) \\
&= L(L-1)\frac{d\left[(d^2+3d+3)\,\mathrm{tr}(O^2)^2 + d(d+1)\,\mathrm{tr}(O^4) + \mathrm{tr}(O)^4 - 2d\,\mathrm{tr}(O^2)\,\mathrm{tr}(O)^2 - 4(d+1)\,\mathrm{tr}(O^3)\,\mathrm{tr}(O)\right]}{4(d-1)^2(d+1)^3(d+2)(d+3)} \\
&+ L\frac{3\left[(d^2+3d+3)\,\mathrm{tr}(O^2)^2 + d(d+1)\,\mathrm{tr}(O^4) + \mathrm{tr}(O)^4 - 2d\,\mathrm{tr}(O^2)\,\mathrm{tr}(O)^2 - 4(d+1)\,\mathrm{tr}(O^3)\,\mathrm{tr}(O)\right]}{4(d-1)d(d+1)^2(d+3)^2} \\
&- \left(L\frac{d\,\mathrm{tr}(O^2) - \mathrm{tr}(O)^2}{2(d-1)(d+1)^2}\right)^2. &(240)
\end{aligned}$$

The relative fluctuation is then

$$\mathrm{SD}[K_0]/\overline{K_0} = \sqrt{\mathrm{Var}[K_0]}/\overline{K_0}, \tag{241}$$

where $\mathrm{Var}[K_0]$ and $\overline{K_0}$ can be found in Eq. (240) and (92).

In the asymptotic limit of $L, d \gg 1$, we have

$$\mathrm{Var}[K_0] \simeq \frac{L^2+3L}{4d^6}\left[d^2\,\mathrm{tr}\big(O^2\big)^2 + d^2\,\mathrm{tr}\big(O^4\big) + \mathrm{tr}(O)^4 - 2d\,\mathrm{tr}\big(O^2\big)\,\mathrm{tr}(O)^2 - 4d\,\mathrm{tr}\big(O^3\big)\,\mathrm{tr}(O)\right] - L^2\frac{\left(d\,\mathrm{tr}(O^2) - \mathrm{tr}(O)^2\right)^2}{4d^6}. \tag{242}$$

Thus we have the standard deviation of QNTK as

$$\mathrm{SD}[K_0] \simeq \frac{1}{2d^3}\left((L^2+3L)\left[d^2\,\mathrm{tr}\big(O^2\big)^2 + d^2\,\mathrm{tr}\big(O^4\big) + \mathrm{tr}(O)^4 - 2d\,\mathrm{tr}\big(O^2\big)\,\mathrm{tr}(O)^2 - 4d\,\mathrm{tr}\big(O^3\big)\,\mathrm{tr}(O)\right] - L^2\left(d\,\mathrm{tr}\big(O^2\big) - \mathrm{tr}(O)^2\right)^2\right)^{\frac{1}{2}}. \tag{243}$$

and the relative fluctuation is

$$\frac{\mathrm{SD}[K_0]}{\overline{K_0}} \simeq \left((L^2+3L)\left[d^2\,\mathrm{tr}\big(O^2\big)^2 + d^2\,\mathrm{tr}\big(O^4\big) + \mathrm{tr}(O)^4 - 2d\,\mathrm{tr}\big(O^2\big)\,\mathrm{tr}(O)^2 - 4d\,\mathrm{tr}\big(O^3\big)\,\mathrm{tr}(O)\right] - L^2\left(d\,\mathrm{tr}\big(O^2\big) - \mathrm{tr}(O)^2\right)^2\right)^{1/2} \times \frac{1}{L\left(d\,\mathrm{tr}(O^2) - \mathrm{tr}(O)^2\right)}. \tag{244}$$

For traceless observable O, the standard deviation and relative fluctuation are reduced to

$$\mathrm{SD}[K_0] \simeq \frac{1}{2d^2}\sqrt{L^2\,\mathrm{tr}(O^4) + 3L\,\mathrm{tr}(O^2)^2}, \tag{245}$$

$$\frac{\mathrm{SD}[K_0]}{\overline{K_0}} \simeq \frac{1}{\sqrt{L}}\sqrt{L\frac{\mathrm{tr}(O^4)}{\mathrm{tr}(O^2)^2} + 3}. \tag{246}$$

Remark

A further simplification is considered in [6] by treating the four unitaries $U_{\ell_1^-}, U_{\ell_1,L}, U_{1,\ell_2-1}, U_{\ell_2^+}$ that appears in $\mathbb{E}\left[\left(\frac{\partial\epsilon}{\partial\theta_{\ell_1}}\right)^2\left(\frac{\partial\epsilon}{\partial\theta_{\ell_2}}\right)^2\right]$ are independent sampled from Haar random, which leads to the scaling of $\mathrm{SD}[K_0] \sim \sqrt{L}$ only.

Supplementary Note 12. RESULTS WITH RESTRICTED HAAR ENSEMBLE

We first review the restricted Haar random ensemble here. Recall the unitary in the restricted Haar ensemble $\mathcal{U}_{\mathrm{RH}}$ is defined as (see Supplementary Note 6)

$$U_{\mathrm{RH}} = \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & V \end{pmatrix}, \tag{247}$$

where V is a unitary of dimension $d-1$, and in restricted ensemble, we assume V follows Haar random ensemble. As the overall circuit unitary $U_{\mathrm{RH}} = U_{\ell^-}U_{\ell^+} = U_{\ell_1^-}U_{\ell_1 \to \ell_2}U_{\ell_2^+}$ (see definitions around Eqs. (53), (56)), the form of Eq. (57) will lead to constraint on the unitaries $U_{\ell^-}, U_{\ell^+}, U_{\ell_1^-}, U_{\ell_1\to\ell_2}, U_{\ell_2^+}$ utilized in evaluation. It turns out that the specific distribution of V for the overall unitary ensemble $\{U_{\mathrm{RH}}\}$ does not affect the ensemble averages and thus we do not specify the distribution of V. To implement the constraint, we can assume that the unitaries of a part of circuit including $U_{\ell^-}, U_{\ell_1^-}, U_{\ell_1\to\ell_2} \sim \mathcal{U}_{\mathrm{Haar}}(d)$ follow independent Haar random distribution; while $U_{\ell^+}, U_{\ell_2^+}$ are directly determined by $U_{\ell^+} = U_{\mathrm{RH}}U_{\ell^-}^\dagger$ and $U_{\ell_2^+} = U_{\mathrm{RH}}U_{\ell^-}^\dagger U_{\ell_1\to\ell_2}^\dagger$ due to the constraint.

To prepare a target state $O = |\Phi\rangle\langle\Phi|$, we can keep the loss function in Eq. (1) in the main text with a target value $O_0 \geq 0$, and the total error as

$$\epsilon = |\langle\Phi|U|\psi_0\rangle|^2 - O_0. \tag{248}$$

As fidelity between arbitrary two quantum states is bounded from zero to unity, when $0 < O_0 < 1$, we expect the error can be decreased to zero with sufficiently large L, leading to the *frozen kernel dynamics*; On the other hand if $O_0 > 1$, the error can only be optimized to a negative constant leading to the *frozen error dynamics*; $O_0 = 1$ will become the critical point. To capture late-time dynamics, we have the fidelity

$$F_0 = |\langle\Phi|U|\psi_0\rangle|^2 = O_0 + R, \tag{249}$$

where $R = \lim_{t\to\infty} \epsilon(t) = \min\{1 - O_0, 0\}$ which is consistent with definition in the main text.

Supplementary Note 12.1. Average QNTK under restricted Haar ensemble

We can evaluate average of QNTK $\overline{K_\infty}$ under restricted Haar ensemble. Recall that the QNTK is defined as $\overline{K_\infty} = \sum_\ell \mathbb{E}_{\mathcal{U}_{\rm RH}}\left[\left(\frac{\partial\epsilon}{\partial\theta_\ell}\right)^2\right]$, thus we have

$$\mathbb{E}_{\mathcal{U}_{\rm RH}}\left[\left(\frac{\partial\epsilon}{\partial\theta_\ell}\right)^2\right] = -\frac{1}{4}\int dU_{\ell-}dU_{\ell+}\,\mathrm{tr}\left(\rho_0 U_{\ell-}^\dagger\left[X_\ell, O_{\ell+}\right]U_{\ell-}\rho_0 U_{\ell-}^\dagger\left[X_\ell, O_{\ell+}\right]U_{\ell-}\right) \tag{250}$$

$$\begin{aligned}= -\frac{1}{4}\int_{\mathcal{U}_{\rm RH}} dU_{\rm RH}\int_{\mathcal{U}_{\rm Haar}} dU_{\ell-}\Big[&\mathrm{tr}\left(\rho_0 U_{\ell-}^\dagger X_\ell U_{\ell-} O_{\rm RH}\rho_0 U_{\ell-}^\dagger X_\ell U_{\ell-} O_{\rm RH}\right) + \mathrm{tr}\left(\rho_0 O_{\rm RH} U_{\ell-}^\dagger X_\ell U_{\ell-}\rho_0 O_{\rm RH} U_{\ell-}^\dagger X_\ell U_{\ell-}\right)\\ &- \mathrm{tr}\left(\rho_0 U_{\ell-}^\dagger X_\ell U_{\ell-} O_{\rm RH}\rho_0 O_{\rm RH} U_{\ell-}^\dagger X_\ell U_{\ell-}\right) - \mathrm{tr}\left(\rho_0 O_{\rm RH} U_{\ell-}^\dagger X_\ell U_{\ell-}\rho_0 U_{\ell-}^\dagger X_\ell U_{\ell-} O_{\rm RH}\right)\Big]\end{aligned} \tag{251}$$

$$= -\frac{2}{4}\int_{\mathcal{U}_{\rm RH}} dU_{\rm RH}\left[\frac{d\,\mathrm{tr}(\rho_0 O_{\rm RH})^2 - \mathrm{tr}(\rho_0 O_{\rm RH}\rho_0 O_{\rm RH})}{d^2-1} - \frac{d\,\mathrm{tr}(O_{\rm RH}\rho_0 O_{\rm RH}) - \mathrm{tr}(\rho_0 O_{\rm RH}\rho_0 O_{\rm RH})}{d^2-1}\right] \tag{252}$$

$$= \int_{\mathcal{U}_{\rm RH}} dU_{\rm RH}\frac{2d\left[\mathrm{tr}(\rho_0 O_{\rm RH})^2 - \mathrm{tr}(O_{\rm RH}\rho_0 O_{\rm RH})\right]}{4(d^2-1)} \tag{253}$$

$$= \frac{dF_0(1-F_0)}{2(d^2-1)}, \tag{254}$$

where $O_{\rm RH} = U_{\rm RH}^\dagger O U_{\rm RH}$ is defined for simplicity. In the last line, we utilize the fact that $\mathrm{tr}(\rho_0 O_{\rm RH})^2 = F_0^2$ and $\mathrm{tr}(O_{\rm RH}\rho_0 O_{\rm RH}) = F_0$. Thus the QNTK is

$$\overline{K_\infty} = L\mathbb{E}_{\mathcal{U}_{\rm RH}}\left[\left(\frac{\partial\epsilon}{\partial\theta_\ell}\right)^2\right] = \frac{LdF_0(1-F_0)}{2(d^2-1)} \tag{255}$$

$$= \frac{Ld}{2(d^2-1)}\left(O_0+R\right)\left(1-O_0-R\right) \simeq \frac{L}{2d}\left(O_0+R\right)\left(1-O_0-R\right), \tag{256}$$

where in the last equation we utilize the relation between F_0 and O_0, R in Eq. (249). The approximation in the last line is taken for $d \gg 1$ for direct identification on its scaling. When $O_0 > 1$ with $R = 1 - O_0$, we directly have $\overline{K_\infty} = 0$; when $O_0 = 1$ with $R = 0$, we also have $\overline{K_\infty} = 0$; however when $O_0 < 1$ with $R = 0$, we have a finite nonzero QNTK as $\overline{K_\infty} \propto O_0(1 - O_0)$ and specifically, $\overline{K_\infty} \propto O_0$ in the limit of O_0 close to unity.

Supplementary Note 12.2. Average relative dQNTK under restricted Haar ensemble

In this section, we evaluate the factor $\overline{\lambda_\infty} = \overline{\mu_\infty}/\overline{K_\infty}$ under restricted Haar ensemble. As $\overline{K_\infty}$ is already calculated above, we focus on dQNTK $\overline{\mu_\infty}$ in the following. Recall that $\overline{\mu_\infty} = L(L-1)\mathbb{E}_{\mathcal{U}_{\rm RH}}\left[\frac{\partial^2\epsilon}{\partial\theta_{\ell_1}\partial\theta_{\ell_2}}\frac{\partial\epsilon}{\partial\theta_{\ell_1}}\frac{\partial\epsilon}{\partial\theta_{\ell_2}}\right] + L\mathbb{E}_{\mathcal{U}_{\rm RH}}\left[\frac{\partial^2\epsilon}{\partial\theta_\ell^2}\left(\frac{\partial\epsilon}{\partial\theta_\ell}\right)^2\right]$, we evaluate the two ensemble average separately.

a. $\mathbb{E}_{\mathcal{U}_{\mathrm{RH}}}\left[\frac{\partial^2\epsilon}{\partial\theta_\ell^2}\left(\frac{\partial\epsilon}{\partial\theta_\ell}\right)^2\right]$ *under restricted Haar ensemble*

We first consider $\mathbb{E}_{\mathcal{U}_{\mathrm{RH}}}\left[\frac{\partial^2\epsilon}{\partial\theta_\ell^2}\left(\frac{\partial\epsilon}{\partial\theta_\ell}\right)^2\right]$.

$$\mathbb{E}_{\mathcal{U}_{\mathrm{RH}}}\left[\frac{\partial^2\epsilon}{\partial\theta_\ell^2}\left(\frac{\partial\epsilon}{\partial\theta_\ell}\right)^2\right]=\frac{1}{16}\int dU_{\ell-}dU_{\ell+}\,\mathrm{tr}\left(\rho_0U_{\ell-}^\dagger O_{\ell+}U_{\ell-}\rho_0U_{\ell-}^\dagger[X_\ell,[X_\ell,O_{\ell+}]]U_{\ell-}\rho_0U_{\ell-}^\dagger[X_\ell,O_{\ell+}]U_{\ell-}\rho_0U_{\ell-}^\dagger[X_\ell,O_{\ell+}]U_{\ell-}\right) \tag{257}$$

$$\begin{aligned}
&=\frac{2}{16}\int_{\mathcal{U}_{\mathrm{RH}}}dU_{\mathrm{RH}}\int_{\mathcal{U}_{\mathrm{Haar}}}dU_{\ell-}\Big[\mathrm{tr}\left(O_{\mathrm{RH}}\rho_0O_{\mathrm{RH}}\rho_0U_{\ell+}^\dagger X_\ell U_{\ell+}O_{\mathrm{RH}}\rho_0U_{\ell+}^\dagger X_\ell U_{\ell+}\right)+\mathrm{tr}\left(\rho_0O_{\mathrm{RH}}\rho_0O_{\mathrm{RH}}U_{\ell+}^\dagger X_\ell U_{\ell+}\rho_0O_{\mathrm{RH}}U_{\ell+}^\dagger X_\ell U_{\ell+}\right)\\
&-\mathrm{tr}\left(\rho_0O_{\mathrm{RH}}\rho_0U_{\ell+}^\dagger X_\ell U_{\ell+}O_{\mathrm{RH}}\rho_0O_{\mathrm{RH}}U_{\ell+}^\dagger X_\ell U_{\ell+}\right)-\mathrm{tr}\left(O_{\mathrm{RH}}\rho_0O_{\mathrm{RH}}\rho_0O_{\mathrm{RH}}U_{\ell+}^\dagger X_\ell U_{\ell+}\rho_0U_{\ell+}^\dagger X_\ell U_{\ell+}\right)\\
&+\mathrm{tr}\left(\rho_0U_{\ell+}^\dagger X_\ell U_{\ell+}O_{\mathrm{RH}}U_{\ell+}^\dagger X_\ell U_{\ell+}\rho_0U_{\ell+}^\dagger X_\ell U_{\ell+}O_{\mathrm{RH}}\rho_0O_{\mathrm{RH}}U_{\ell+}^\dagger X_\ell U_{\ell+}\right)\\
&+\mathrm{tr}\left(O_{\mathrm{RH}}\rho_0U_{\ell+}^\dagger X_\ell U_{\ell+}O_{\mathrm{RH}}U_{\ell+}^\dagger X_\ell U_{\ell+}\rho_0O_{\mathrm{RH}}U_{\ell+}^\dagger X_\ell U_{\ell+}\rho_0U_{\ell+}^\dagger X_\ell U_{\ell+}\right)\\
&-\mathrm{tr}\left(O_{\mathrm{RH}}\rho_0U_{\ell+}^\dagger X_\ell U_{\ell+}O_{\mathrm{RH}}U_{\ell+}^\dagger X_\ell U_{\ell+}\rho_0U_{\ell+}^\dagger X_\ell U_{\ell+}O_{\mathrm{RH}}\rho_0U_{\ell+}^\dagger X_\ell U_{\ell+}\right)\\
&-\mathrm{tr}\left(\rho_0U_{\ell+}^\dagger X_\ell U_{\ell+}O_{\mathrm{RH}}U_{\ell+}^\dagger X_\ell U_{\ell+}\rho_0O_{\mathrm{RH}}U_{\ell+}^\dagger X_\ell U_{\ell+}\rho_0O_{\mathrm{RH}}U_{\ell+}^\dagger X_\ell U_{\ell+}\right)\Big].
\end{aligned} \tag{258}$$

One can see that the first two terms equals and they are

$$\begin{aligned}
&\int_{\mathcal{U}_{\mathrm{RH}}}dU_{\mathrm{RH}}\int_{\mathcal{U}_{\mathrm{Haar}}}dU_{\ell-}\Big[\mathrm{tr}\left(O_{\mathrm{RH}}\rho_0O_{\mathrm{RH}}\rho_0U_{\ell+}^\dagger X_\ell U_{\ell+}O_{\mathrm{RH}}\rho_0U_{\ell+}^\dagger X_\ell U_{\ell+}\right)+\mathrm{tr}\left(\rho_0O_{\mathrm{RH}}\rho_0O_{\mathrm{RH}}U_{\ell+}^\dagger X_\ell U_{\ell+}\rho_0O_{\mathrm{RH}}U_{\ell+}^\dagger X_\ell U_{\ell+}\right)\Big]\\
&=\int_{\mathcal{U}_{\mathrm{RH}}}dU_{\mathrm{RH}}\frac{2}{d^2-1}\left[d\,\mathrm{tr}(\rho_0O_{\mathrm{RH}})\,\mathrm{tr}(\rho_0O_{\mathrm{RH}}\rho_0O_{\mathrm{RH}})-\mathrm{tr}(\rho_0O_{\mathrm{RH}}\rho_0O_{\mathrm{RH}}\rho_0O_{\mathrm{RH}})\right]\\
&=\frac{2(d-1)F_0^3}{d^2-1}.
\end{aligned} \tag{259}$$

The third and fourth term also equals and are

$$\begin{aligned}
&\int_{\mathcal{U}_{\mathrm{RH}}}dU_{\mathrm{RH}}\int_{\mathcal{U}_{\mathrm{Haar}}}dU_{\ell-}\Big[\mathrm{tr}\left(\rho_0O_{\mathrm{RH}}\rho_0U_{\ell+}^\dagger X_\ell U_{\ell+}O_{\mathrm{RH}}\rho_0O_{\mathrm{RH}}U_{\ell+}^\dagger X_\ell U_{\ell+}\right)+\mathrm{tr}\left(O_{\mathrm{RH}}\rho_0O_{\mathrm{RH}}\rho_0O_{\mathrm{RH}}U_{\ell+}^\dagger X_\ell U_{\ell+}\rho_0U_{\ell+}^\dagger X_\ell U_{\ell+}\right)\Big]\\
&=\int_{\mathcal{U}_{\mathrm{RH}}}dU_{\mathrm{RH}}\frac{1}{d^2-1}\Big[d\,\mathrm{tr}(\rho_0O_{\mathrm{RH}})^2-\mathrm{tr}(\rho_0O_{\mathrm{RH}}\rho_0O_{\mathrm{RH}}\rho_0O_{\mathrm{RH}})+d\,\mathrm{tr}(\rho_0O_{\mathrm{RH}}\rho_0O_{\mathrm{RH}})-\mathrm{tr}(\rho_0O_{\mathrm{RH}}\rho_0O_{\mathrm{RH}}\rho_0O_{\mathrm{RH}})\Big]
\end{aligned} \tag{260}$$

$$=\frac{2(dF_0^2-F_0^3)}{d^2-1}. \tag{261}$$

The fifth term is

$$\begin{aligned}
&\int_{\mathcal{U}_{\mathrm{RH}}}dU_{\mathrm{RH}}\int_{\mathcal{U}_{\mathrm{Haar}}}dU_{\ell-}\,\mathrm{tr}\left(\rho_0U_{\ell+}^\dagger X_\ell U_{\ell+}O_{\mathrm{RH}}U_{\ell+}^\dagger X_\ell U_{\ell+}\rho_0U_{\ell+}^\dagger X_\ell U_{\ell+}O_{\mathrm{RH}}\rho_0O_{\mathrm{RH}}U_{\ell+}^\dagger X_\ell U_{\ell+}\right)\\
&=\int_{\mathcal{U}_{\mathrm{RH}}}dU_{\mathrm{RH}}\frac{1}{(d^2-9)(d^2-1)}\left[\left(d^2-3\right)\mathrm{tr}(\rho_0O_{\mathrm{RH}}\rho_0O_{\mathrm{RH}}\rho_0O_{\mathrm{RH}})+2(d-3)\,\mathrm{tr}(\rho_0O_{\mathrm{RH}})(-\mathrm{tr}(\rho_0O_{\mathrm{RH}})+d+2)\right.\\
&\qquad\left.-2\,\mathrm{tr}(\rho_0O_{\mathrm{RH}}\rho_0O_{\mathrm{RH}})(d\,\mathrm{tr}(\rho_0O_{\mathrm{RH}})+3d-9)\right]
\end{aligned} \tag{262}$$

$$=\frac{F_0\left(d\left(F_0^2+2\right)+F_0^2-8F_0+4\right)}{d^3+3d^2-d-3}. \tag{263}$$

The sixth term can be found to be equal to the fifth one above. The seventh term is

$$\begin{aligned}
&\int_{\mathcal{U}_{\mathrm{RH}}}dU_{\mathrm{RH}}\int_{\mathcal{U}_{\mathrm{Haar}}}dU_{\ell-}\,\mathrm{tr}\left(O_{\mathrm{RH}}\rho_0U_{\ell+}^\dagger X_\ell U_{\ell+}O_{\mathrm{RH}}U_{\ell+}^\dagger X_\ell U_{\ell+}\rho_0U_{\ell+}^\dagger X_\ell U_{\ell+}O_{\mathrm{RH}}\rho_0U_{\ell+}^\dagger X_\ell U_{\ell+}\right)\\
&=\frac{1}{(d^2-9)(d^2-1)}\Big[\mathrm{tr}(\rho_0O_{\mathrm{RH}})^2(3\,\mathrm{tr}(\rho_0O_{\mathrm{RH}})+d(2d-5)-6)+d\,\mathrm{tr}(\rho_0O_{\mathrm{RH}}\rho_0O_{\mathrm{RH}})(-3\,\mathrm{tr}(\rho_0O_{\mathrm{RH}})+d-4)\\
&\qquad+6(\mathrm{tr}(\rho_0O_{\mathrm{RH}}\rho_0O_{\mathrm{RH}})+\mathrm{tr}(\rho_0O_{\mathrm{RH}}\rho_0O_{\mathrm{RH}}\rho_0O_{\mathrm{RH}}))\Big]
\end{aligned} \tag{264}$$

$$=\frac{3F_0^2(d-F_0)}{d^3+3d^2-d-3}. \tag{265}$$

The eighth term is also equal to the seventh one above.

Conclude from the above calculation, we have

$$
\begin{aligned}
\mathbb{E}_{\mathcal{U}_{\mathrm{RH}}}\left[\frac{\partial^2\epsilon}{\partial\theta_\ell^2}\left(\frac{\partial\epsilon}{\partial\theta_\ell}\right)^2\right] &= \frac{1}{16}\int dU_{\ell-}dU_{\ell+}\,\mathrm{tr}\left(\rho_0 U_{\ell-}^\dagger O_{\ell+}U_{\ell-}\rho_0 U_{\ell-}^\dagger[X_\ell,[X_\ell,O_{\ell+}]]U_{\ell-}\rho_0 U_{\ell-}^\dagger[X_\ell,O_{\ell+}]U_{\ell-}\rho_0 U_{\ell-}^\dagger[X_\ell,O_{\ell+}]U_{\ell-}\right) \\
&= \frac{2}{8}\left[\frac{(d-1)F_0^3}{d^2-1} - \frac{(dF_0^2-F_0^3)}{d^2-1} + \frac{F_0\left[d\left(F_0^2+2\right)+F_0^2-8F_0+4\right]}{d^3+3d^2-d-3} - \frac{3F_0^2(d-F_0)}{d^3+3d^2-d-3}\right] \qquad (266) \\
&= \frac{(d+2)(F_0-1)F_0\left[(d+2)F_0-2\right]}{4(d-1)(d+1)(d+3)}. \qquad (267)
\end{aligned}
$$

b. $\mathbb{E}_{\mathcal{U}_{\mathrm{RH}}}\left[\frac{\partial^2\epsilon}{\partial\theta_{\ell_1}\partial\theta_{\ell_2}}\frac{\partial\epsilon}{\partial\theta_{\ell_1}}\frac{\partial\epsilon}{\partial\theta_{\ell_2}}\right]$ *under restricted Haar ensemble*

The other part $\mathbb{E}_{\mathcal{U}_{\mathrm{RH}}}\left[\frac{\partial^2\epsilon}{\partial\theta_{\ell_1}\partial\theta_{\ell_2}}\frac{\partial\epsilon}{\partial\theta_{\ell_1}}\frac{\partial\epsilon}{\partial\theta_{\ell_2}}\right]$ is

$$
\begin{aligned}
&\mathbb{E}_{\mathcal{U}_{\mathrm{RH}}}\left[\frac{\partial^2\epsilon}{\partial\theta_{\ell_1}\partial\theta_{\ell_2}}\frac{\partial\epsilon}{\partial\theta_{\ell_1}}\frac{\partial\epsilon}{\partial\theta_{\ell_2}}\right] \\
&= \frac{1}{16}\int dU_{\ell_1^-}dU_{\ell_1\to\ell_2}dU_{\ell_2^+} \\
&\qquad \mathrm{tr}\left(\rho_0 U_{\ell_1^-}^\dagger\left[X_{\ell_1},U_{\ell_1\to\ell_2}^\dagger\left[X_{\ell_2},O_{\ell_2^+}\right]U_{\ell_1\to\ell_2}\right]U_{\ell_1^-}\rho_0 U_{\ell_1^-}^\dagger\left[X_{\ell_1},O_{\ell_1^+}\right]U_{\ell_1^-}\rho_0 U_{\ell_1^-}^\dagger U_{\ell_1\to\ell_2}^\dagger\left[X_{\ell_2},O_{\ell_2^+}\right]U_{\ell_1\to\ell_2}U_{\ell_1^-}\right) \\
&= \frac{1}{16}\int_{\mathcal{U}_{\mathrm{RH}}} dU_{\mathrm{RH}}\int_{\mathcal{U}_{\mathrm{Haar}}} dU_{\ell_1^-}dU_{\ell_1\to\ell_2}\Big[\mathrm{tr}\Big(O_{\mathrm{RH}}\rho_0 U_{\ell_1^-}^\dagger X_{\ell_1}U_{\ell_1\to\ell_2}^\dagger X_{\ell_2}U_{\ell_1\to\ell_2}U_{\ell_1^-}O_{\mathrm{RH}}\rho_0 U_{\ell_1^-}^\dagger X_{\ell_1}U_{\ell_1^-}O_{\mathrm{RH}}\rho_0 U_{\ell_1^-}^\dagger U_{\ell_1\to\ell_2}^\dagger X_{\ell_2}U_{\ell_1\to\ell_2}U_{\ell_1^-}\Big) \\
&+ \mathrm{tr}\Big(\rho_0 U_{\ell_1^-}^\dagger X_{\ell_1}U_{\ell_1\to\ell_2}^\dagger X_{\ell_2}U_{\ell_1\to\ell_2}U_{\ell_1^-}O_{\mathrm{RH}}\rho_0 O_{\mathrm{RH}}U_{\ell_1^-}^\dagger X_{\ell_1}U_{\ell_1^-}\rho_0 O_{\mathrm{RH}}U_{\ell_1^-}^\dagger U_{\ell_1\to\ell_2}^\dagger X_{\ell_2}U_{\ell_1\to\ell_2}U_{\ell_1^-}\Big) \\
&- \mathrm{tr}\Big(\rho_0 U_{\ell_1^-}^\dagger X_{\ell_1}U_{\ell_1\to\ell_2}^\dagger X_{\ell_2}U_{\ell_1\to\ell_2}U_{\ell_1^-}O_{\mathrm{RH}}\rho_0 U_{\ell_1^-}^\dagger X_{\ell_1}U_{\ell_1^-}O_{\mathrm{RH}}\rho_0 O_{\mathrm{RH}}U_{\ell_1^-}^\dagger U_{\ell_1\to\ell_2}^\dagger X_{\ell_2}U_{\ell_1\to\ell_2}U_{\ell_1^-}\Big) \\
&- \mathrm{tr}\Big(O_{\mathrm{RH}}\rho_0 U_{\ell_1^-}^\dagger X_{\ell_1}U_{\ell_1\to\ell_2}^\dagger X_{\ell_2}U_{\ell_1\to\ell_2}U_{\ell_1^-}O_{\mathrm{RH}}\rho_0 O_{\mathrm{RH}}U_{\ell_1^-}^\dagger X_{\ell_1}U_{\ell_1^-}\rho_0 U_{\ell_1^-}^\dagger U_{\ell_1\to\ell_2}^\dagger X_{\ell_2}U_{\ell_1\to\ell_2}U_{\ell_1^-}\Big) \\
&+ \mathrm{tr}\Big(O_{\mathrm{RH}}\rho_0 U_{\ell_1^-}^\dagger X_{\ell_1}U_{\ell_1^-}O_{\mathrm{RH}}U_{\ell_1^-}^\dagger U_{\ell_1\to\ell_2}^\dagger X_{\ell_2}U_{\ell_1\to\ell_2}U_{\ell_1^-}\rho_0 O_{\mathrm{RH}}U_{\ell_1^-}^\dagger X_{\ell_1}U_{\ell_1^-}\rho_0 U_{\ell_1^-}^\dagger U_{\ell_1\to\ell_2}^\dagger X_{\ell_2}U_{\ell_1\to\ell_2}U_{\ell_1^-}\Big) \\
&+ \mathrm{tr}\Big(\rho_0 U_{\ell_1^-}^\dagger X_{\ell_1}U_{\ell_1^-}O_{\mathrm{RH}}U_{\ell_1^-}^\dagger U_{\ell_1\to\ell_2}^\dagger X_{\ell_2}U_{\ell_1\to\ell_2}U_{\ell_1^-}\rho_0 U_{\ell_1^-}^\dagger X_{\ell_1}U_{\ell_1^-}O_{\mathrm{RH}}\rho_0 O_{\mathrm{RH}}U_{\ell_1^-}^\dagger U_{\ell_1\to\ell_2}^\dagger X_{\ell_2}U_{\ell_1\to\ell_2}U_{\ell_1^-}\Big) \\
&- \mathrm{tr}\Big(O_{\mathrm{RH}}\rho_0 U_{\ell_1^-}^\dagger X_{\ell_1}U_{\ell_1^-}O_{\mathrm{RH}}U_{\ell_1^-}^\dagger U_{\ell_1\to\ell_2}^\dagger X_{\ell_2}U_{\ell_1\to\ell_2}U_{\ell_1^-}\rho_0 U_{\ell_1^-}^\dagger X_{\ell_1}U_{\ell_1^-}O_{\mathrm{RH}}\rho_0 U_{\ell_1^-}^\dagger U_{\ell_1\to\ell_2}^\dagger X_{\ell_2}U_{\ell_1\to\ell_2}U_{\ell_1^-}\Big) \\
&- \mathrm{tr}\Big(\rho_0 U_{\ell_1^-}^\dagger X_{\ell_1}U_{\ell_1^-}O_{\mathrm{RH}}U_{\ell_1^-}^\dagger U_{\ell_1\to\ell_2}^\dagger X_{\ell_2}U_{\ell_1\to\ell_2}U_{\ell_1^-}\rho_0 O_{\mathrm{RH}}U_{\ell_1^-}^\dagger X_{\ell_1}U_{\ell_1^-}\rho_0 O_{\mathrm{RH}}U_{\ell_1^-}^\dagger U_{\ell_1\to\ell_2}^\dagger X_{\ell_2}U_{\ell_1\to\ell_2}U_{\ell_1^-}\Big) \\
&+ \mathrm{tr}\Big(\rho_0 U_{\ell_1^-}^\dagger U_{\ell_1\to\ell_2}^\dagger X_{\ell_2}U_{\ell_1\to\ell_2}U_{\ell_1^-}O_{\mathrm{RH}}U_{\ell_1^-}^\dagger X_{\ell_1}U_{\ell_1^-}\rho_0 U_{\ell_1^-}^\dagger X_{\ell_1}U_{\ell_1^-}O_{\mathrm{RH}}\rho_0 O_{\mathrm{RH}}U_{\ell_1^-}^\dagger U_{\ell_1\to\ell_2}^\dagger X_{\ell_2}U_{\ell_1\to\ell_2}U_{\ell_1^-}\Big) \\
&+ \mathrm{tr}\Big(O_{\mathrm{RH}}\rho_0 U_{\ell_1^-}^\dagger U_{\ell_1\to\ell_2}^\dagger X_{\ell_2}U_{\ell_1\to\ell_2}U_{\ell_1^-}O_{\mathrm{RH}}U_{\ell_1^-}^\dagger X_{\ell_1}U_{\ell_1^-}\rho_0 O_{\mathrm{RH}}U_{\ell_1^-}^\dagger X_{\ell_1}U_{\ell_1^-}\rho_0 U_{\ell_1^-}^\dagger U_{\ell_1\to\ell_2}^\dagger X_{\ell_2}U_{\ell_1\to\ell_2}U_{\ell_1^-}\Big) \\
&- \mathrm{tr}\Big(O_{\mathrm{RH}}\rho_0 U_{\ell_1^-}^\dagger U_{\ell_1\to\ell_2}^\dagger X_{\ell_2}U_{\ell_1\to\ell_2}U_{\ell_1^-}O_{\mathrm{RH}}U_{\ell_1^-}^\dagger X_{\ell_1}U_{\ell_1^-}\rho_0 U_{\ell_1^-}^\dagger X_{\ell_1}U_{\ell_1^-}O_{\mathrm{RH}}\rho_0 U_{\ell_1^-}^\dagger U_{\ell_1\to\ell_2}^\dagger X_{\ell_2}U_{\ell_1\to\ell_2}U_{\ell_1^-}\Big) \\
&- \mathrm{tr}\Big(\rho_0 U_{\ell_1^-}^\dagger U_{\ell_1\to\ell_2}^\dagger X_{\ell_2}U_{\ell_1\to\ell_2}U_{\ell_1^-}O_{\mathrm{RH}}U_{\ell_1^-}^\dagger X_{\ell_1}U_{\ell_1^-}\rho_0 O_{\mathrm{RH}}U_{\ell_1^-}^\dagger X_{\ell_1}U_{\ell_1^-}\rho_0 O_{\mathrm{RH}}U_{\ell_1^-}^\dagger U_{\ell_1\to\ell_2}^\dagger X_{\ell_2}U_{\ell_1\to\ell_2}U_{\ell_1^-}\Big) \\
&+ \mathrm{tr}\Big(O_{\mathrm{RH}}\rho_0 O_{\mathrm{RH}}U_{\ell_1^-}^\dagger U_{\ell_1\to\ell_2}^\dagger X_{\ell_2}U_{\ell_1\to\ell_2}X_{\ell_1}U_{\ell_1^-}\rho_0 U_{\ell_1^-}^\dagger X_{\ell_1}U_{\ell_1^-}O_{\mathrm{RH}}\rho_0 U_{\ell_1^-}^\dagger U_{\ell_1\to\ell_2}^\dagger X_{\ell_2}U_{\ell_1\to\ell_2}U_{\ell_1^-}\Big) \\
&+ \mathrm{tr}\Big(\rho_0 O_{\mathrm{RH}}U_{\ell_1^-}^\dagger U_{\ell_1\to\ell_2}^\dagger X_{\ell_2}U_{\ell_1\to\ell_2}X_{\ell_1}U_{\ell_1^-}\rho_0 O_{\mathrm{RH}}U_{\ell_1^-}^\dagger X_{\ell_1}U_{\ell_1^-}\rho_0 O_{\mathrm{RH}}U_{\ell_1^-}^\dagger U_{\ell_1\to\ell_2}^\dagger X_{\ell_2}U_{\ell_1\to\ell_2}U_{\ell_1^-}\Big) \\
&- \mathrm{tr}\Big(\rho_0 O_{\mathrm{RH}}U_{\ell_1^-}^\dagger U_{\ell_1\to\ell_2}^\dagger X_{\ell_2}U_{\ell_1\to\ell_2}X_{\ell_1}U_{\ell_1^-}\rho_0 U_{\ell_1^-}^\dagger X_{\ell_1}U_{\ell_1^-}O_{\mathrm{RH}}\rho_0 O_{\mathrm{RH}}U_{\ell_1^-}^\dagger U_{\ell_1\to\ell_2}^\dagger X_{\ell_2}U_{\ell_1\to\ell_2}U_{\ell_1^-}\Big) \\
&+ \mathrm{tr}\Big(O_{\mathrm{RH}}\rho_0 O_{\mathrm{RH}}U_{\ell_1^-}^\dagger U_{\ell_1\to\ell_2}^\dagger X_{\ell_2}U_{\ell_1\to\ell_2}X_{\ell_1}U_{\ell_1^-}\rho_0 O_{\mathrm{RH}}U_{\ell_1^-}^\dagger X_{\ell_1}U_{\ell_1^-}\rho_0 U_{\ell_1^-}^\dagger U_{\ell_1\to\ell_2}^\dagger X_{\ell_2}U_{\ell_1\to\ell_2}U_{\ell_1^-}\Big)\Big]. \qquad (268)
\end{aligned}
$$

The integral over $U_{\ell_1^-}, U_{\ell_1 \to \ell_2}$ of first term is

$$\int_{\mathcal{U}_{\mathrm{RH}}} dU_{\mathrm{RH}} \int_{\mathcal{U}_{\mathrm{Haar}}} dU_{\ell_1^-} dU_{\ell_1 \to \ell_2} \operatorname{tr}\Big(O_{\mathrm{RH}}\rho_0 U_{\ell_1^-}^\dagger X_{\ell_1} U_{\ell_1 \to \ell_2}^\dagger X_{\ell_2} U_{\ell_1 \to \ell_2} U_{\ell_1^-} O_{\mathrm{RH}}\rho_0 U_{\ell_1^-}^\dagger X_{\ell_1} U_{\ell_1^-} O_{\mathrm{RH}}\rho_0 U_{\ell_1^-}^\dagger U_{\ell_1 \to \ell_2}^\dagger X_{\ell_2} U_{\ell_1 \to \ell_2} U_{\ell_1^-}\Big)$$

$$= \int_{\mathcal{U}_{\mathrm{RH}}} dU_{\mathrm{RH}} \frac{1}{(d^2-1)^2} \left[(d^2+1)\operatorname{tr}(\rho_0 O_{\mathrm{RH}}\rho_0 O_{\mathrm{RH}}\rho_0 O_{\mathrm{RH}}) - 2d \operatorname{tr}(\rho_0 O_{\mathrm{RH}}) \operatorname{tr}(\rho_0 O_{\mathrm{RH}}\rho_0 O_{\mathrm{RH}})\right] \tag{269}$$

$$= \frac{F_0^3}{(d+1)^2}. \tag{270}$$

The second term is

$$\int_{\mathcal{U}_{\mathrm{RH}}} dU_{\mathrm{RH}} \int_{\mathcal{U}_{\mathrm{Haar}}} dU_{\ell_1^-} dU_{\ell_1 \to \ell_2} \operatorname{tr}\Big(\rho_0 U_{\ell_1^-}^\dagger X_{\ell_1} U_{\ell_1 \to \ell_2}^\dagger X_{\ell_2} U_{\ell_1 \to \ell_2} U_{\ell_1^-} O_{\mathrm{RH}}\rho_0 O_{\mathrm{RH}} U_{\ell_1^-}^\dagger X_{\ell_1} U_{\ell_1^-} \rho_0 O_{\mathrm{RH}} U_{\ell_1^-}^\dagger U_{\ell_1 \to \ell_2}^\dagger X_{\ell_2} U_{\ell_1 \to \ell_2} U_{\ell_1^-}\Big)$$

$$= \frac{F_0^2(F_0 + d^2 - 2d)}{(d^2-1)^2}. \tag{271}$$

The third term is

$$\int_{\mathcal{U}_{\mathrm{RH}}} dU_{\mathrm{RH}} \int_{\mathcal{U}_{\mathrm{Haar}}} dU_{\ell_1^-} dU_{\ell_1 \to \ell_2} \operatorname{tr}\Big(\rho_0 U_{\ell_1^-}^\dagger X_{\ell_1} U_{\ell_1 \to \ell_2}^\dagger X_{\ell_2} U_{\ell_1 \to \ell_2} U_{\ell_1^-} O_{\mathrm{RH}}\rho_0 U_{\ell_1^-}^\dagger X_{\ell_1} U_{\ell_1^-} O_{\mathrm{RH}}\rho_0 O_{\mathrm{RH}} U_{\ell_1^-}^\dagger U_{\ell_1 \to \ell_2}^\dagger X_{\ell_2} U_{\ell_1 \to \ell_2} U_{\ell_1^-}\Big)$$

$$= \frac{F_0^2(d - F_0)}{(d+1)^2(d-1)}. \tag{272}$$

The fourth one equals the third one above.

The fifth one is

$$\int_{\mathcal{U}_{\mathrm{RH}}} dU_{\mathrm{RH}} \int_{\mathcal{U}_{\mathrm{Haar}}} dU_{\ell_1^-} dU_{\ell_1 \to \ell_2} \operatorname{tr}\Big(O_{\mathrm{RH}}\rho_0 U_{\ell_1^-}^\dagger X_{\ell_1} U_{\ell_1^-} O_{\mathrm{RH}} U_{\ell_1^-}^\dagger U_{\ell_1 \to \ell_2}^\dagger X_{\ell_2} U_{\ell_1 \to \ell_2} U_{\ell_1^-} \rho_0 O_{\mathrm{RH}} U_{\ell_1^-}^\dagger X_{\ell_1} U_{\ell_1^-} \rho_0 U_{\ell_1^-}^\dagger U_{\ell_1 \to \ell_2}^\dagger X_{\ell_2} U_{\ell_1 \to \ell_2} U_{\ell_1^-}\Big)$$

$$= \frac{F_0(F_0 - d)^2}{(d^2-1)^2}. \tag{273}$$

The sixth term equals the first; the seventh and eighth equals the third and fourth; the ninth equals the fifth; the tenth equals the sixth; the eleventh and twelfth equals the seventh and eighth; the thirteenth equals the second; the fourteenth equals the first; the fifteenth and sixteenth equals the third and fourth.

Conclude from the above calculation, we have

$$\begin{aligned}
&\mathbb{E}_{\mathcal{U}_{\mathrm{RH}}}\left[\frac{\partial^2 \epsilon}{\partial\theta_{\ell_1}\partial\theta_{\ell_2}} \frac{\partial \epsilon}{\partial\theta_{\ell_1}} \frac{\partial \epsilon}{\partial\theta_{\ell_2}}\right] \\
&= \frac{1}{16}\int dU_{\ell_1^-} dU_{\ell_1 \to \ell_2} dU_{\ell_2^+} \\
&\qquad \operatorname{tr}\left(\rho_0 U_{\ell_1^-}^\dagger \left[X_{\ell_1}, U_{\ell_1 \to \ell_2}^\dagger \left[X_{\ell_2}, O_{\ell_2^+}\right] U_{\ell_1 \to \ell_2}\right] U_{\ell_1^-} \rho_0 U_{\ell_1^-}^\dagger \left[X_{\ell_1}, O_{\ell_1^+}\right] U_{\ell_1^-} \rho_0 U_{\ell_1^-}^\dagger U_{\ell_1 \to \ell_2}^\dagger \left[X_{\ell_2}, O_{\ell_2^+}\right] U_{\ell_1 \to \ell_2} U_{\ell_1^-}\right) \\
&= \frac{1}{16}\left[\frac{4F_0^3}{(d+1)^2} + \frac{2F_0^2(F_0 + d^2 - 2d)}{(d^2-1)^2} - \frac{8F_0^2(d-F_0)}{(d+1)^2(d-1)} + \frac{2F_0(F_0-d)^2}{(d^2-1)^2}\right] &(274)\\
&= \frac{d^2 F_0(F_0-1)(2F_0-1)}{8(d^2-1)^2}. &(275)
\end{aligned}$$

c. Summary of average relative dQNTK $\overline{\lambda_\infty}$ with restricted Haar ensemble

Combining Eq. (267) and (275), restricted Haar ensemble averaged dQNTK $\overline{\mu_\infty}$ is

$$\begin{aligned}
\overline{\mu_\infty} &= L(L-1)\mathbb{E}_{\mathcal{U}_{\mathrm{RH}}}\left[\frac{\partial^2 \epsilon}{\partial\theta_{\ell_1}\partial\theta_{\ell_2}} \frac{\partial \epsilon}{\partial\theta_{\ell_1}} \frac{\partial \epsilon}{\partial\theta_{\ell_2}}\right] + L\mathbb{E}_{\mathcal{U}_{\mathrm{RH}}}\left[\frac{\partial^2 \epsilon}{\partial\theta_\ell^2}\left(\frac{\partial \epsilon}{\partial\theta_\ell}\right)^2\right] \\
&= L(L-1)\frac{d^2 F_0(F_0-1)(2F_0-1)}{8(d^2-1)^2} + L\frac{(d+2)(F_0-1)F_0\left[(d+2)F_0 - 2\right]}{4(d-1)(d+1)(d+3)}. &(276)
\end{aligned}$$

Combining with the average QNTK calculated above, we have

$$\overline{\lambda_\infty} = \frac{\overline{\mu_\infty}}{\overline{K_\infty}} \tag{277}$$

$$= (L-1)\frac{d(1-2F_0)}{4(d^2-1)} - \frac{(d+2)\left[(d+2)F_0 - 2\right]}{2d(d+3)} \tag{278}$$

$$= \frac{(L-1)d}{4(d^2-1)}(1-2O_0 - 2R) - \frac{(d+2)}{2d(d+3)}\left[(d+2)(O_0+R) - 2\right] \tag{279}$$

$$\simeq \frac{L}{4d}(1-2O_0-2R) - \frac{O_0+R}{2}, \tag{280}$$

where in the last line we still make approximations under $d \gg 1$ for direct understanding on its scaling, and we can see that $\overline{\lambda_\infty}$ is a constant $\propto L/d$ regardless of $O_0 \lesseqgtr 1$.

Supplementary Note 12.3. Average dynamical index under restricted Haar ensemble

We evaluate the restricted Haar ensemble averaged $\overline{\zeta_\infty} = \overline{\epsilon_\infty \mu_\infty}/\overline{K_\infty}^2$ in the following.

$$\begin{aligned}
\overline{\epsilon_\infty\mu_\infty} &= \mathbb{E}_{\mathcal{U}_{\rm RH}}\left[BZ\mathrm{tr}\Big(\rho_0 U_{\rm RH}^\dagger O U_{\rm RH}\Big)\mu\right] - O_0\overline{\mu_\infty}\\
&= L(L-1)\mathbb{E}_{\mathcal{U}_{\rm RH}}\left[\mathrm{tr}\Big(\rho_0 U_{\rm RH}^\dagger O U_{\rm RH}\Big)\frac{\partial^2\epsilon}{\partial\theta_{\ell_1}\partial\theta_{\ell_2}}\frac{\partial\epsilon}{\partial\theta_{\ell_1}}\frac{\partial\epsilon}{\partial\theta_{\ell_2}}\right] + L\mathbb{E}\left[\mathrm{tr}\Big(\rho_0 U_{\rm RH}^\dagger O U_{\rm RH}\Big)\frac{\partial^2\epsilon}{\partial\theta_\ell^2}\left(\frac{\partial\epsilon}{\partial\theta_\ell}\right)^2\right] - O_0\overline{\mu_\infty}.
\end{aligned} \tag{281}$$

a. $\mathbb{E}_{\mathcal{U}_{\rm RH}}\left[\mathrm{tr}\Big(\rho_0 U_{\rm RH}^\dagger O U_{\rm RH}\Big)\frac{\partial^2\epsilon}{\partial\theta_\ell^2}\left(\frac{\partial\epsilon}{\partial\theta_\ell}\right)^2\right]$ *under restricted Haar ensemble*

We first consider $\mathbb{E}_{\mathcal{U}_{\rm RH}}\left[\mathrm{tr}\Big(\rho_0 U_{\rm RH}^\dagger O U_{\rm RH}\Big)\frac{\partial^2\epsilon}{\partial\theta_\ell^2}\left(\frac{\partial\epsilon}{\partial\theta_\ell}\right)^2\right]$.

$$\begin{aligned}
&\mathbb{E}_{\mathcal{U}_{\rm RH}}\left[\mathrm{tr}\Big(\rho_0 U_{\rm RH}^\dagger O U_{\rm RH}\Big)\frac{\partial^2\epsilon}{\partial\theta_\ell^2}\left(\frac{\partial\epsilon}{\partial\theta_\ell}\right)^2\right]\\
&= \frac{1}{16}\int dU_{\ell-}dU_{\ell+}\,\mathrm{tr}\left(\rho_0 U_{\ell-}^\dagger O_{\ell+}U_{\ell-}\rho_0 U_{\ell-}^\dagger[X_\ell,[X_\ell,O_{\ell+}]]U_{\ell-}\rho_0 U_{\ell-}^\dagger[X_\ell,O_{\ell+}]U_{\ell-}\rho_0U_{\ell-}^\dagger[X_\ell,O_{\ell+}]U_{\ell-}\right)\\
&= \frac{2}{16}\int_{\mathcal{U}_{\rm RH}} dU_{\rm RH}\int_{\mathcal{U}_{\rm Haar}} dU_{\ell-}\left[\mathrm{tr}\left(O_{\rm RH}\rho_0 O_{\rm RH}\rho_0 O_{\rm RH}\rho_0 U_{\ell+}^\dagger X_\ell U_{\ell+} O_{\rm RH}\rho_0 U_{\ell+}^\dagger X_\ell U_{\ell+}\right)\right.\\
&+\mathrm{tr}\left(\rho_0 O_{\rm RH}\rho_0 O_{\rm RH}\rho_0 O_{\rm RH} U_{\ell+}^\dagger X_\ell U_{\ell+}\rho_0 O_{\rm RH}U_{\ell+}^\dagger X_\ell U_{\ell+}\right) - \mathrm{tr}\left(\rho_0 O_{\rm RH}\rho_0 O_{\rm RH}\rho_0 U_{\ell+}^\dagger X_\ell U_{\ell+} O_{\rm RH}\rho_0 O_{\rm RH}U_{\ell+}^\dagger X_\ell U_{\ell+}\right)\\
&-\mathrm{tr}\left(O_{\rm RH}\rho_0 O_{\rm RH}\rho_0 O_{\rm RH}\rho_0 O_{\rm RH}U_{\ell+}^\dagger X_\ell U_{\ell+}\rho_0 U_{\ell+}^\dagger X_\ell U_{\ell+}\right)\\
&+\mathrm{tr}\left(\rho_0 O_{\rm RH}\rho_0 U_{\ell+}^\dagger X_\ell U_{\ell+} O_{\rm RH}U_{\ell+}^\dagger X_\ell U_{\ell+}\rho_0 U_{\ell+}^\dagger X_\ell U_{\ell+} O_{\rm RH}\rho_0 O_{\rm RH}U_{\ell+}^\dagger X_\ell U_{\ell+}\right)\\
&+\mathrm{tr}\left(O_{\rm RH}\rho_0 O_{\rm RH}\rho_0 U_{\ell+}^\dagger X_\ell U_{\ell+} O_{\rm RH}U_{\ell+}^\dagger X_\ell U_{\ell+}\rho_0 O_{\rm RH}U_{\ell+}^\dagger X_\ell U_{\ell+}\rho_0 U_{\ell+}^\dagger X_\ell U_{\ell+}\right)\\
&-\mathrm{tr}\left(O_{\rm RH}\rho_0 O_{\rm RH}\rho_0 U_{\ell+}^\dagger X_\ell U_{\ell+} O_{\rm RH}U_{\ell+}^\dagger X_\ell U_{\ell+}\rho_0 U_{\ell+}^\dagger X_\ell U_{\ell+} O_{\rm RH}\rho_0 U_{\ell+}^\dagger X_\ell U_{\ell+}\right)\\
&\left.-\mathrm{tr}\left(\rho_0 O_{\rm RH}\rho_0 U_{\ell+}^\dagger X_\ell U_{\ell+} O_{\rm RH}U_{\ell+}^\dagger X_\ell U_{\ell+}\rho_0 O_{\rm RH}U_{\ell+}^\dagger X_\ell U_{\ell+}\rho_0 O_{\rm RH}U_{\ell+}^\dagger X_\ell U_{\ell+}\right)\right].
\end{aligned} \tag{282}$$

The first and second terms equal and are

$$\begin{aligned}
&\int_{\mathcal{U}_{\mathrm{RH}}} dU_{\mathrm{RH}} \int_{\mathcal{U}_{\mathrm{Haar}}} dU_{\ell-} \left[\operatorname{tr}\left(O_{\mathrm{RH}}\rho_0 O_{\mathrm{RH}}\rho_0 O_{\mathrm{RH}}\rho_0 U_{\ell+}^\dagger X_\ell U_{\ell+} O_{\mathrm{RH}}\rho_0 U_{\ell+}^\dagger X_\ell U_{\ell+}\right)\right.\\
&\qquad\qquad\left. + \operatorname{tr}\left(\rho_0 O_{\mathrm{RH}}\rho_0 O_{\mathrm{RH}}\rho_0 O_{\mathrm{RH}} U_{\ell+}^\dagger X_\ell U_{\ell+}\rho_0 O_{\mathrm{RH}} U_{\ell+}^\dagger X_\ell U_{\ell+}\right)\right]\\
&= \int_{\mathcal{U}_{\mathrm{RH}}} dU_{\mathrm{RH}} \frac{2}{d^2-1}\left[d\operatorname{tr}(\rho_0 O_{\mathrm{RH}})\operatorname{tr}(\rho_0 O_{\mathrm{RH}}\rho_0 O_{\mathrm{RH}}\rho_0 O_{\mathrm{RH}}) - \operatorname{tr}(\rho_0 O_{\mathrm{RH}}\rho_0 O_{\mathrm{RH}}\rho_0 O_{\mathrm{RH}}\rho_0 O_{\mathrm{RH}})\right]\\
&= \frac{2(d-1)F_0^4}{d^2-1}.
\end{aligned} \tag{283}$$

The third and fourth terms equal and are

$$\begin{aligned}
&\int_{\mathcal{U}_{\mathrm{RH}}} dU_{\mathrm{RH}} \int_{\mathcal{U}_{\mathrm{Haar}}} dU_{\ell-} \left[\operatorname{tr}\left(\rho_0 O_{\mathrm{RH}}\rho_0 O_{\mathrm{RH}}\rho_0 U_{\ell+}^\dagger X_\ell U_{\ell+} O_{\mathrm{RH}}\rho_0 O_{\mathrm{RH}} U_{\ell+}^\dagger X_\ell U_{\ell+}\right)\right.\\
&\qquad\left. + \operatorname{tr}\left(O_{\mathrm{RH}}\rho_0 O_{\mathrm{RH}}\rho_0 O_{\mathrm{RH}}\rho_0 O_{\mathrm{RH}} U_{\ell+}^\dagger X_\ell U_{\ell+}\rho_0 U_{\ell+}^\dagger X_\ell U_{\ell+}\right)\right]\\
&= \int_{\mathcal{U}_{\mathrm{RH}}} dU_{\mathrm{RH}} \frac{1}{d^2-1}\left[d\operatorname{tr}(\rho_0 O_{\mathrm{RH}})\operatorname{tr}(\rho_0 O_{\mathrm{RH}}\rho_0 O_{\mathrm{RH}}) - \operatorname{tr}(\rho_0 O_{\mathrm{RH}}\rho_0 O_{\mathrm{RH}}\rho_0 O_{\mathrm{RH}}\rho_0 O_{\mathrm{RH}}) + d\operatorname{tr}(\rho_0 O_{\mathrm{RH}}\rho_0 O_{\mathrm{RH}}\rho_0 O_{\mathrm{RH}})\right.
\end{aligned} \tag{284}$$

$$\left. - \operatorname{tr}(\rho_0 O_{\mathrm{RH}}\rho_0 O_{\mathrm{RH}}\rho_0 O_{\mathrm{RH}}\rho_0 O_{\mathrm{RH}})\right] \tag{285}$$

$$= \frac{2(dF_0^3 - F_0^4)}{d^2-1}. \tag{286}$$

The fifth term is

$$\begin{aligned}
&\int_{\mathcal{U}_{\mathrm{RH}}} dU_{\mathrm{RH}} \int_{\mathcal{U}_{\mathrm{Haar}}} dU_{\ell-} \operatorname{tr}\left(\rho_0 O_{\mathrm{RH}}\rho_0 U_{\ell+}^\dagger X_\ell U_{\ell+} O_{\mathrm{RH}} U_{\ell+}^\dagger X_\ell U_{\ell+}\rho_0 U_{\ell+}^\dagger X_\ell U_{\ell+} O_{\mathrm{RH}}\rho_0 O_{\mathrm{RH}} U_{\ell+}^\dagger X_\ell U_{\ell+}\right)\\
&= \int_{\mathcal{U}_{\mathrm{RH}}} dU_{\mathrm{RH}} \frac{1}{(d^2-9)(d^2-1)}\left\{\left(d^2-3\right)\operatorname{tr}(\rho_0 O_{\mathrm{RH}}\rho_0 O_{\mathrm{RH}}\rho_0 O_{\mathrm{RH}}\rho_0 O_{\mathrm{RH}}) + 2(d-3)(d+2)\operatorname{tr}(\rho_0 O_{\mathrm{RH}})^2 + 3\operatorname{tr}(\rho_0 O_{\mathrm{RH}})^3\right.\\
&- \operatorname{tr}(\rho_0 O_{\mathrm{RH}})\left[(4d-6)\operatorname{tr}(\rho_0 O_{\mathrm{RH}}\rho_0 O_{\mathrm{RH}}) + d\operatorname{tr}(\rho_0 O_{\mathrm{RH}}\rho_0 O_{\mathrm{RH}}\rho_0 O_{\mathrm{RH}})\right] - d\operatorname{tr}(\rho_0 O_{\mathrm{RH}}\rho_0 O_{\mathrm{RH}})^2\\
&\left.+(15-4d)\operatorname{tr}(\rho_0 O_{\mathrm{RH}}\rho_0 O_{\mathrm{RH}}\rho_0 O_{\mathrm{RH}})\right\}
\end{aligned} \tag{287}$$

$$= \frac{F_0^2\left[d\left(F_0^2+2\right) + F_0^2 - 8F_0 + 4\right]}{d^3+3d^2-d-3}. \tag{288}$$

The sixth term can be found to be equal to the fifth one above.

The seventh term is

$$\begin{aligned}
&\int_{\mathcal{U}_{\mathrm{RH}}} dU_{\mathrm{RH}} \int_{\mathcal{U}_{\mathrm{Haar}}} dU_{\ell-} \operatorname{tr}\left(O_{\mathrm{RH}}\rho_0 O_{\mathrm{RH}}\rho_0 U_{\ell+}^\dagger X_\ell U_{\ell+} O_{\mathrm{RH}} U_{\ell+}^\dagger X_\ell U_{\ell+}\rho_0 U_{\ell+}^\dagger X_\ell U_{\ell+} O_{\mathrm{RH}}\rho_0 U_{\ell+}^\dagger X_\ell U_{\ell+}\right)\\
&= \frac{1}{(d^2-9)(d^2-1)}\left[\operatorname{tr}(\rho_0 O_{\mathrm{RH}}\rho_0 O_{\mathrm{RH}})\left(\operatorname{tr}(\rho_0 O_{\mathrm{RH}})\left[3\operatorname{tr}(\rho_0 O_{\mathrm{RH}}) + d(2d-5) - 6\right] - d\operatorname{tr}(\rho_0 O_{\mathrm{RH}}\rho_0 O_{\mathrm{RH}})\right)\right.\\
&\left.+d\operatorname{tr}(\rho_0 O_{\mathrm{RH}}\rho_0 O_{\mathrm{RH}}\rho_0 O_{\mathrm{RH}})(-2\operatorname{tr}(\rho_0 O_{\mathrm{RH}}) + d - 4) + 6(\operatorname{tr}(\rho_0 O_{\mathrm{RH}}\rho_0 O_{\mathrm{RH}}\rho_0 O_{\mathrm{RH}}) + \operatorname{tr}(\rho_0 O_{\mathrm{RH}}\rho_0 O_{\mathrm{RH}}\rho_0 O_{\mathrm{RH}}\rho_0 O_{\mathrm{RH}}))\right]
\end{aligned} \tag{289}$$

$$= \frac{3F_0^3(d-F_0)}{d^3+3d^2-d-3}. \tag{290}$$

The eighth term also equals to the seventh one above.

Summarizing from above, we have

$$\begin{aligned}
&\mathbb{E}_{\mathcal{U}_{\mathrm{RH}}}\left[\operatorname{tr}\left(\rho_0 U_{\mathrm{RH}}^\dagger O U_{\mathrm{RH}}\right)\frac{\partial^2\epsilon}{\partial\theta_\ell^2}\left(\frac{\partial\epsilon}{\partial\theta_\ell}\right)^2\right]\\
&= \frac{2}{8}\left[\frac{(d-1)F_0^4}{d^2-1} - \frac{(dF_0^3 - F_0^4)}{d^2-1} + \frac{F_0^2\left[d\left(F_0^2+2\right) + F_0^2 - 8F_0 + 4\right]}{d^3+3d^2-d-3} - \frac{3F_0^3(d-F_0)}{d^3+3d^2-d-3}\right]
\end{aligned} \tag{291}$$

$$= \frac{(d+2)(F_0-1)F_0^2\left[(d+2)F_0 - 2\right]}{4(d-1)(d+1)(d+3)}. \tag{292}$$

b. $\mathbb{E}_{\mathcal{U}_{\mathrm{RH}}}\left[\mathrm{tr}\Big(\rho_0 U_{\mathrm{RH}}^\dagger O U_{\mathrm{RH}}\Big)\frac{\partial^2\epsilon}{\partial\theta_{\ell_1}\partial\theta_{\ell_2}}\frac{\partial\epsilon}{\partial\theta_{\ell_1}}\frac{\partial\epsilon}{\partial\theta_{\ell_2}}\right]$ *under restricted Haar ensemble*

The other item $\mathbb{E}_{\mathcal{U}_{\mathrm{RH}}}\left[\mathrm{tr}\Big(\rho_0 U_{\mathrm{RH}}^\dagger O U_{\mathrm{RH}}\Big)\frac{\partial^2\epsilon}{\partial\theta_{\ell_1}\partial\theta_{\ell_2}}\frac{\partial\epsilon}{\partial\theta_{\ell_1}}\frac{\partial\epsilon}{\partial\theta_{\ell_2}}\right]$ can be expanded as

$$
\begin{aligned}
&\mathbb{E}_{\mathcal{U}_{\mathrm{RH}}}\left[\mathrm{tr}\Big(\rho_0 U_{\mathrm{RH}}^\dagger O U_{\mathrm{RH}}\Big)\frac{\partial^2\epsilon}{\partial\theta_{\ell_1}\partial\theta_{\ell_2}}\frac{\partial\epsilon}{\partial\theta_{\ell_1}}\frac{\partial\epsilon}{\partial\theta_{\ell_2}}\right]\\
&=\frac{1}{16}\int dU_{\ell_1^-}dU_{\ell_1\to\ell_2}dU_{\ell_2^+}\,\mathrm{tr}\left(\rho_0 U^\dagger_{\ell_1^-}U^\dagger_{\ell_1\to\ell_2}O_{\ell_2^+}U_{\ell_1\to\ell_2}U_{\ell_1^-}\rho_0 U^\dagger_{\ell_1^-}\left[X_{\ell_1},U^\dagger_{\ell_1\to\ell_2}\left[X_{\ell_2},O_{\ell_2^+}\right]U_{\ell_1\to\ell_2}\right]U_{\ell_1^-}\rho_0U^\dagger_{\ell_1^-}\left[X_{\ell_1},O_{\ell_1^+}\right]U_{\ell_1^-}\right.\\
&\left.\cdot\rho_0U^\dagger_{\ell_1^-}U^\dagger_{\ell_1\to\ell_2}\left[X_{\ell_2},O_{\ell_2^+}\right]U_{\ell_1\to\ell_2}U_{\ell_1^-}\right) \qquad (293)\\
&=\frac{1}{16}\int_{\mathcal{U}_{\mathrm{RH}}}dU_{\mathrm{RH}}\int_{\mathcal{U}_{\mathrm{Haar}}}dU_{\ell_1^-}dU_{\ell_1\to\ell_2}\left[\mathrm{tr}\Big(O_{\mathrm{RH}}\rho_0O_{\mathrm{RH}}\rho_0U^\dagger_{\ell_1^-}X_{\ell_1}U^\dagger_{\ell_1\to\ell_2}X_{\ell_2}U_{\ell_1\to\ell_2}U_{\ell_1^-}O_{\mathrm{RH}}\rho_0U^\dagger_{\ell_1^-}X_{\ell_1}U_{\ell_1^-}O_{\mathrm{RH}}\rho_0U^\dagger_{\ell_1^-}U^\dagger_{\ell_1\to\ell_2}X_{\ell_2}U_{\ell_1\to\ell_2}U_{\ell_1^-}\Big)\right.\\
&+\mathrm{tr}\Big(\rho_0O_{\mathrm{RH}}\rho_0U^\dagger_{\ell_1^-}X_{\ell_1}U^\dagger_{\ell_1\to\ell_2}X_{\ell_2}U_{\ell_1\to\ell_2}U_{\ell_1^-}O_{\mathrm{RH}}\rho_0O_{\mathrm{RH}}U^\dagger_{\ell_1^-}X_{\ell_1}U_{\ell_1^-}\rho_0O_{\mathrm{RH}}U^\dagger_{\ell_1^-}U^\dagger_{\ell_1\to\ell_2}X_{\ell_2}U_{\ell_1\to\ell_2}U_{\ell_1^-}\Big)\\
&-\mathrm{tr}\Big(\rho_0O_{\mathrm{RH}}\rho_0U^\dagger_{\ell_1^-}X_{\ell_1}U^\dagger_{\ell_1\to\ell_2}X_{\ell_2}U_{\ell_1\to\ell_2}U_{\ell_1^-}O_{\mathrm{RH}}\rho_0U^\dagger_{\ell_1^-}X_{\ell_1}U_{\ell_1^-}O_{\mathrm{RH}}\rho_0O_{\mathrm{RH}}U^\dagger_{\ell_1^-}U^\dagger_{\ell_1\to\ell_2}X_{\ell_2}U_{\ell_1\to\ell_2}U_{\ell_1^-}\Big)\\
&-\mathrm{tr}\Big(O_{\mathrm{RH}}\rho_0O_{\mathrm{RH}}\rho_0U^\dagger_{\ell_1^-}X_{\ell_1}U^\dagger_{\ell_1\to\ell_2}X_{\ell_2}U_{\ell_1\to\ell_2}U_{\ell_1^-}O_{\mathrm{RH}}\rho_0O_{\mathrm{RH}}U^\dagger_{\ell_1^-}X_{\ell_1}U_{\ell_1^-}\rho_0U^\dagger_{\ell_1^-}U^\dagger_{\ell_1\to\ell_2}X_{\ell_2}U_{\ell_1\to\ell_2}U_{\ell_1^-}\Big)\\
&+\mathrm{tr}\Big(O_{\mathrm{RH}}\rho_0O_{\mathrm{RH}}\rho_0U^\dagger_{\ell_1^-}X_{\ell_1}U_{\ell_1^-}O_{\mathrm{RH}}U^\dagger_{\ell_1^-}U^\dagger_{\ell_1\to\ell_2}X_{\ell_2}U_{\ell_1\to\ell_2}U_{\ell_1^-}\rho_0O_{\mathrm{RH}}U^\dagger_{\ell_1^-}X_{\ell_1}U_{\ell_1^-}\rho_0U^\dagger_{\ell_1^-}U^\dagger_{\ell_1\to\ell_2}X_{\ell_2}U_{\ell_1\to\ell_2}U_{\ell_1^-}\Big)\\
&+\mathrm{tr}\Big(\rho_0O_{\mathrm{RH}}\rho_0U^\dagger_{\ell_1^-}X_{\ell_1}U_{\ell_1^-}O_{\mathrm{RH}}U^\dagger_{\ell_1^-}U^\dagger_{\ell_1\to\ell_2}X_{\ell_2}U_{\ell_1\to\ell_2}U_{\ell_1^-}\rho_0U^\dagger_{\ell_1^-}X_{\ell_1}U_{\ell_1^-}O_{\mathrm{RH}}\rho_0O_{\mathrm{RH}}U^\dagger_{\ell_1^-}U^\dagger_{\ell_1\to\ell_2}X_{\ell_2}U_{\ell_1\to\ell_2}U_{\ell_1^-}\Big)\\
&-\mathrm{tr}\Big(O_{\mathrm{RH}}\rho_0O_{\mathrm{RH}}\rho_0U^\dagger_{\ell_1^-}X_{\ell_1}U_{\ell_1^-}O_{\mathrm{RH}}U^\dagger_{\ell_1^-}U^\dagger_{\ell_1\to\ell_2}X_{\ell_2}U_{\ell_1\to\ell_2}U_{\ell_1^-}\rho_0U^\dagger_{\ell_1^-}X_{\ell_1}U_{\ell_1^-}O_{\mathrm{RH}}\rho_0U^\dagger_{\ell_1^-}U^\dagger_{\ell_1\to\ell_2}X_{\ell_2}U_{\ell_1\to\ell_2}U_{\ell_1^-}\Big)\\
&-\mathrm{tr}\Big(\rho_0O_{\mathrm{RH}}\rho_0U^\dagger_{\ell_1^-}X_{\ell_1}U_{\ell_1^-}O_{\mathrm{RH}}U^\dagger_{\ell_1^-}U^\dagger_{\ell_1\to\ell_2}X_{\ell_2}U_{\ell_1\to\ell_2}U_{\ell_1^-}\rho_0O_{\mathrm{RH}}U^\dagger_{\ell_1^-}X_{\ell_1}U_{\ell_1^-}\rho_0O_{\mathrm{RH}}U^\dagger_{\ell_1^-}U^\dagger_{\ell_1\to\ell_2}X_{\ell_2}U_{\ell_1\to\ell_2}U_{\ell_1^-}\Big)\\
&+\mathrm{tr}\Big(\rho_0O_{\mathrm{RH}}\rho_0U^\dagger_{\ell_1^-}U^\dagger_{\ell_1\to\ell_2}X_{\ell_2}U_{\ell_1\to\ell_2}U_{\ell_1^-}O_{\mathrm{RH}}U^\dagger_{\ell_1^-}X_{\ell_1}U_{\ell_1^-}\rho_0U^\dagger_{\ell_1^-}X_{\ell_1}U_{\ell_1^-}O_{\mathrm{RH}}\rho_0O_{\mathrm{RH}}U^\dagger_{\ell_1^-}U^\dagger_{\ell_1\to\ell_2}X_{\ell_2}U_{\ell_1\to\ell_2}U_{\ell_1^-}\Big)\\
&+\mathrm{tr}\Big(O_{\mathrm{RH}}\rho_0O_{\mathrm{RH}}\rho_0U^\dagger_{\ell_1^-}U^\dagger_{\ell_1\to\ell_2}X_{\ell_2}U_{\ell_1\to\ell_2}U_{\ell_1^-}O_{\mathrm{RH}}U^\dagger_{\ell_1^-}X_{\ell_1}U_{\ell_1^-}\rho_0O_{\mathrm{RH}}U^\dagger_{\ell_1^-}X_{\ell_1}U_{\ell_1^-}\rho_0U^\dagger_{\ell_1^-}U^\dagger_{\ell_1\to\ell_2}X_{\ell_2}U_{\ell_1\to\ell_2}U_{\ell_1^-}\Big)\\
&-\mathrm{tr}\Big(O_{\mathrm{RH}}\rho_0O_{\mathrm{RH}}\rho_0U^\dagger_{\ell_1^-}U^\dagger_{\ell_1\to\ell_2}X_{\ell_2}U_{\ell_1\to\ell_2}U_{\ell_1^-}O_{\mathrm{RH}}U^\dagger_{\ell_1^-}X_{\ell_1}U_{\ell_1^-}\rho_0U^\dagger_{\ell_1^-}X_{\ell_1}U_{\ell_1^-}O_{\mathrm{RH}}\rho_0U^\dagger_{\ell_1^-}U^\dagger_{\ell_1\to\ell_2}X_{\ell_2}U_{\ell_1\to\ell_2}U_{\ell_1^-}\Big)\\
&-\mathrm{tr}\Big(\rho_0O_{\mathrm{RH}}\rho_0U^\dagger_{\ell_1^-}U^\dagger_{\ell_1\to\ell_2}X_{\ell_2}U_{\ell_1\to\ell_2}U_{\ell_1^-}O_{\mathrm{RH}}U^\dagger_{\ell_1^-}X_{\ell_1}U_{\ell_1^-}\rho_0O_{\mathrm{RH}}U^\dagger_{\ell_1^-}X_{\ell_1}U_{\ell_1^-}\rho_0O_{\mathrm{RH}}U^\dagger_{\ell_1^-}U^\dagger_{\ell_1\to\ell_2}X_{\ell_2}U_{\ell_1\to\ell_2}U_{\ell_1^-}\Big)\\
&+\mathrm{tr}\Big(O_{\mathrm{RH}}\rho_0O_{\mathrm{RH}}\rho_0O_{\mathrm{RH}}U^\dagger_{\ell_1^-}U^\dagger_{\ell_1\to\ell_2}X_{\ell_2}U_{\ell_1\to\ell_2}X_{\ell_1}U_{\ell_1^-}\rho_0U^\dagger_{\ell_1^-}X_{\ell_1}U_{\ell_1^-}O_{\mathrm{RH}}\rho_0U^\dagger_{\ell_1^-}U^\dagger_{\ell_1\to\ell_2}X_{\ell_2}U_{\ell_1\to\ell_2}U_{\ell_1^-}\Big)\\
&+\mathrm{tr}\Big(\rho_0O_{\mathrm{RH}}\rho_0O_{\mathrm{RH}}U^\dagger_{\ell_1^-}U^\dagger_{\ell_1\to\ell_2}X_{\ell_2}U_{\ell_1\to\ell_2}X_{\ell_1}U_{\ell_1^-}\rho_0O_{\mathrm{RH}}U^\dagger_{\ell_1^-}X_{\ell_1}U_{\ell_1^-}\rho_0O_{\mathrm{RH}}U^\dagger_{\ell_1^-}U^\dagger_{\ell_1\to\ell_2}X_{\ell_2}U_{\ell_1\to\ell_2}U_{\ell_1^-}\Big)\\
&-\mathrm{tr}\Big(\rho_0O_{\mathrm{RH}}\rho_0O_{\mathrm{RH}}U^\dagger_{\ell_1^-}U^\dagger_{\ell_1\to\ell_2}X_{\ell_2}U_{\ell_1\to\ell_2}X_{\ell_1}U_{\ell_1^-}\rho_0U^\dagger_{\ell_1^-}X_{\ell_1}U_{\ell_1^-}O_{\mathrm{RH}}\rho_0O_{\mathrm{RH}}U^\dagger_{\ell_1^-}U^\dagger_{\ell_1\to\ell_2}X_{\ell_2}U_{\ell_1\to\ell_2}U_{\ell_1^-}\Big)\\
&\left.+\mathrm{tr}\Big(O_{\mathrm{RH}}\rho_0O_{\mathrm{RH}}\rho_0O_{\mathrm{RH}}U^\dagger_{\ell_1^-}U^\dagger_{\ell_1\to\ell_2}X_{\ell_2}U_{\ell_1\to\ell_2}X_{\ell_1}U_{\ell_1^-}\rho_0O_{\mathrm{RH}}U^\dagger_{\ell_1^-}X_{\ell_1}U_{\ell_1^-}\rho_0U^\dagger_{\ell_1^-}U^\dagger_{\ell_1\to\ell_2}X_{\ell_2}U_{\ell_1\to\ell_2}U_{\ell_1^-}\Big)\right]. \qquad (294)
\end{aligned}
$$

The integral over $U_{\ell_1^-},U_{\ell_1\to\ell_2}$ of first term is

$$
\begin{aligned}
&\int_{\mathcal{U}_{\mathrm{RH}}}dU_{\mathrm{RH}}\int_{\mathcal{U}_{\mathrm{Haar}}}dU_{\ell_1^-}dU_{\ell_1\to\ell_2}\rho_0O_{\mathrm{RH}}\rho_0U^\dagger_{\ell_1^-}X_{\ell_1}U^\dagger_{\ell_1\to\ell_2}X_{\ell_2}U_{\ell_1\to\ell_2}U_{\ell_1^-}O_{\mathrm{RH}}\rho_0U^\dagger_{\ell_1^-}X_{\ell_1}U_{\ell_1^-}O_{\mathrm{RH}}\rho_0U^\dagger_{\ell_1^-}U^\dagger_{\ell_1\to\ell_2}X_{\ell_2}U_{\ell_1\to\ell_2}U_{\ell_1^-})\\
&=\int_{\mathcal{U}_{\mathrm{RH}}}dU_{\mathrm{RH}}\frac{1}{(d^2-1)^2}\left[(d^2+1)\,\mathrm{tr}(\rho_0O_{\mathrm{RH}}\rho_0O_{\mathrm{RH}}\rho_0O_{\mathrm{RH}}\rho_0O_{\mathrm{RH}})-d\,\mathrm{tr}(\rho_0O_{\mathrm{RH}})\,\mathrm{tr}(\rho_0O_{\mathrm{RH}}\rho_0O_{\mathrm{RH}}\rho_0O_{\mathrm{RH}})\right.\\
&\left.-d\,\mathrm{tr}(\rho_0O_{\mathrm{RH}}\rho_0O_{\mathrm{RH}})^2\right] \qquad (295)\\
&=\frac{F_0^4}{(d+1)^2}. \qquad (296)
\end{aligned}
$$

The second term is

$$\int_{\mathcal{U}_{\mathrm{RH}}} dU_{\mathrm{RH}} \int_{\mathcal{U}_{\mathrm{Haar}}} dU_{\ell_1^-} dU_{\ell_1 \to \ell_2} \operatorname{tr}\Big(\rho_0 O_{\mathrm{RH}} \rho_0 U^\dagger_{\ell_1^-} X_{\ell_1} U^\dagger_{\ell_1 \to \ell_2} X_{\ell_2} U_{\ell_1 \to \ell_2} U_{\ell_1^-} O_{\mathrm{RH}} \rho_0 O_{\mathrm{RH}} U^\dagger_{\ell_1^-} X_{\ell_1} U_{\ell_1^-} \rho_0 O_{\mathrm{RH}} U^\dagger_{\ell_1^-} U^\dagger_{\ell_1 \to \ell_2} X_{\ell_2} U_{\ell_1 \to \ell_2} U_{\ell_1^-}\Big)$$

$$= \frac{1}{(d^2-1)^2}\left[\operatorname{tr}(\rho_0 O_{\mathrm{RH}} \rho_0 O_{\mathrm{RH}} \rho_0 O_{\mathrm{RH}} \rho_0 O_{\mathrm{RH}}) + d^2 \operatorname{tr}(\rho_0 O_{\mathrm{RH}} \rho_0 O_{\mathrm{RH}} \rho_0 O_{\mathrm{RH}}) - 2d \operatorname{tr}(\rho_0 O_{\mathrm{RH}}) \operatorname{tr}(\rho_0 O_{\mathrm{RH}} \rho_0 O_{\mathrm{RH}})\right] \tag{297}$$

$$= \frac{F_0^3(F_0 + d^2 - 2d)}{(d^2-1)^2}. \tag{298}$$

The third term is

$$\int_{\mathcal{U}_{\mathrm{RH}}} dU_{\mathrm{RH}} \int_{\mathcal{U}_{\mathrm{Haar}}} dU_{\ell_1^-} dU_{\ell_1 \to \ell_2} \operatorname{tr}\Big(\rho_0 O_{\mathrm{RH}} \rho_0 U^\dagger_{\ell_1^-} X_{\ell_1} U^\dagger_{\ell_1 \to \ell_2} X_{\ell_2} U_{\ell_1 \to \ell_2} U_{\ell_1^-} O_{\mathrm{RH}} \rho_0 U^\dagger_{\ell_1^-} X_{\ell_1} U_{\ell_1^-} O_{\mathrm{RH}} \rho_0 O_{\mathrm{RH}} U^\dagger_{\ell_1^-} U^\dagger_{\ell_1 \to \ell_2} X_{\ell_2} U_{\ell_1 \to \ell_2} U_{\ell_1^-}\Big)$$

$$= \frac{1}{(d^2-1)^2}\big[\operatorname{tr}(\rho_0 O_{\mathrm{RH}} \rho_0 O_{\mathrm{RH}} \rho_0 O_{\mathrm{RH}} \rho_0 O_{\mathrm{RH}}) + d^2 \operatorname{tr}(\rho_0 O_{\mathrm{RH}} \rho_0 O_{\mathrm{RH}} \rho_0 O_{\mathrm{RH}})$$
$$- d \operatorname{tr}(\rho_0 O_{\mathrm{RH}}) \left(\operatorname{tr}(\rho_0 O_{\mathrm{RH}} \rho_0 O_{\mathrm{RH}}) + \operatorname{tr}(\rho_0 O_{\mathrm{RH}} \rho_0 O_{\mathrm{RH}} \rho_0 O_{\mathrm{RH}})\right)\big] \tag{299}$$

$$= \frac{F_0^3(d - F_0)}{(d+1)^2(d-1)}. \tag{300}$$

The fourth one equals the third one above.

The fifth one is

$$\int_{\mathcal{U}_{\mathrm{RH}}} dU_{\mathrm{RH}} \int_{\mathcal{U}_{\mathrm{Haar}}} dU_{\ell_1^-} dU_{\ell_1 \to \ell_2} \operatorname{tr}\Big(O_{\mathrm{RH}} \rho_0 O_{\mathrm{RH}} \rho_0 U^\dagger_{\ell_1^-} X_{\ell_1} U_{\ell_1^-} O_{\mathrm{RH}} U^\dagger_{\ell_1^-} U^\dagger_{\ell_1 \to \ell_2} X_{\ell_2} U_{\ell_1 \to \ell_2} U_{\ell_1^-} \rho_0 O_{\mathrm{RH}} U^\dagger_{\ell_1^-} X_{\ell_1} U_{\ell_1^-} \rho_0 U^\dagger_{\ell_1^-} U^\dagger_{\ell_1 \to \ell_2} X_{\ell_2} U_{\ell_1 \to \ell_2} U_{\ell_1^-}\Big)$$

$$= \frac{1}{(d^2-1)^2}\left[\operatorname{tr}(\rho_0 O_{\mathrm{RH}} \rho_0 O_{\mathrm{RH}} \rho_0 O_{\mathrm{RH}} \rho_0 O_{\mathrm{RH}}) + d^2 \operatorname{tr}(\rho_0 O_{\mathrm{RH}} \rho_0 O_{\mathrm{RH}}) - 2d \operatorname{tr}(\rho_0 O_{\mathrm{RH}}) \operatorname{tr}(\rho_0 O_{\mathrm{RH}} \rho_0 O_{\mathrm{RH}})\right] \tag{301}$$

$$= \frac{F_0^2(F_0 - d)^2}{(d^2-1)^2}. \tag{302}$$

The sixth term equals the first; the seventh and eighth equals the third and fourth; the ninth equals the fifth; the tenth equals the sixth; the eleventh and twelfth equals the seventh and eighth; the thirteenth equals the second; the fourteenth equals the first; the fifteenth and sixteenth equals the third and fourth.

Concluding from the above sixteenth terms, we have

$$\mathbb{E}_{\mathcal{U}_{\mathrm{RH}}}\left[\operatorname{tr}\Big(\rho_0 U^\dagger_{\mathrm{RH}} O U_{\mathrm{RH}}\Big) \frac{\partial^2 \epsilon}{\partial\theta_{\ell_1}\partial\theta_{\ell_2}} \frac{\partial \epsilon}{\partial\theta_{\ell_1}} \frac{\partial \epsilon}{\partial\theta_{\ell_2}}\right] = \frac{1}{16}\left[\frac{4F_0^4}{(d+1)^2} + \frac{2F_0^3(F_0 + d^2 - 2d)}{(d^2-1)^2} - \frac{8F_0^3(d - F_0)}{(d+1)^2(d-1)} + \frac{2F_0^2(F_0 - d)^2}{(d^2-1)^2}\right] \tag{303}$$

$$= \frac{d^2 F_0^2 (F_0 - 1)(2F_0 - 1)}{8(d^2-1)^2}. \tag{304}$$

Summarizing from Eq. (292) and (304), $\mathbb{E}\left[\operatorname{tr}\Big(\rho_0 U^\dagger_{\mathrm{RH}} O U_{\mathrm{RH}}\Big)\mu\right]$ becomes

$$\mathbb{E}_{\mathcal{U}_{\mathrm{RH}}}\left[\operatorname{tr}\Big(\rho_0 U^\dagger_{\mathrm{RH}} O U_{\mathrm{RH}}\Big)\mu\right]$$

$$= L(L-1)\mathbb{E}_{\mathcal{U}_{\mathrm{RH}}}\left[\operatorname{tr}\Big(\rho_0 U^\dagger_{\mathrm{RH}} O U_{\mathrm{RH}}\Big) \frac{\partial^2 \epsilon}{\partial\theta_{\ell_1}\partial\theta_{\ell_2}} \frac{\partial \epsilon}{\partial\theta_{\ell_1}} \frac{\partial \epsilon}{\partial\theta_{\ell_2}}\right] + L\mathbb{E}_{\mathcal{U}_{\mathrm{RH}}}\left[\operatorname{tr}\Big(\rho_0 U^\dagger_{\mathrm{RH}} O U_{\mathrm{RH}}\Big) \frac{\partial^2 \epsilon}{\partial\theta_\ell^2}\left(\frac{\partial \epsilon}{\partial\theta_\ell}\right)^2\right] \tag{305}$$

$$= L(L-1)\frac{d^2 F_0^2 (F_0 - 1)(2F_0 - 1)}{8(d^2-1)^2} + L\frac{(d+2)(F_0 - 1)F_0^2\left[(d+2)F_0 - 2\right]}{4(d-1)(d+1)(d+3)}. \tag{306}$$

c. Summary of average dynamical index $\overline{\zeta_\infty}$ with restricted Haar ensemble

By subtracting $O_0\overline{\mu_\infty}$ solved in Eq. (276), the ensemble average of $\epsilon\mu$ under restricted Haar ensemble is

$$\overline{\epsilon_\infty\mu_\infty} = \mathbb{E}\left[\operatorname{tr}\Big(\rho_0 U_{\rm RH}^\dagger O U_{\rm RH}\Big)\mu\right] - O_0\overline{\mu_\infty} \tag{307}$$

$$= L(L-1)\frac{d^2F_0^2(F_0-1)(2F_0-1)}{8(d^2-1)^2} + L\frac{(d+2)(F_0-1)F_0^2\left[(d+2)F_0-2\right]}{4(d-1)(d+1)(d+3)}$$
$$- O_0\left[L(L-1)\frac{d^2F_0(F_0-1)(2F_0-1)}{8(d^2-1)^2} + L\frac{(d+2)(F_0-1)F_0\left[(d+2)F_0-2\right]}{4(d-1)(d+1)(d+3)}\right] \tag{308}$$

$$= L(L-1)\frac{d^2F_0(F_0-1)(2F_0-1)}{8(d^2-1)^2}(F_0-O_0) + L\frac{(d+2)(F_0-1)F_0\left[(d+2)F_0-2\right]}{4(d-1)(d+1)(d+3)}(F_0-O_0) \tag{309}$$

$$\simeq L^2\frac{F_0(F_0-1)(2F_0-1)}{8d^2}(F_0-O_0) + L\frac{(F_0-1)F_0^2}{4d}(F_0-O_0)\,. \tag{310}$$

The ratio $\overline{\zeta_\infty}$ becomes

$$\overline{\zeta_\infty} = \frac{\overline{\epsilon_\infty\mu_\infty}}{\overline{K_\infty}^2} \tag{311}$$

$$= (L-1)\frac{(2F_0-1)}{2LF_0(F_0-1)}(F_0-O_0) + \frac{(d+2)(d^2-1)\left[(d+2)F_0-2\right]}{Ld^2(d+3)F_0(F_0-1)}(F_0-O_0) \tag{312}$$

$$= \frac{L-1}{2L}\frac{(2O_0{+}2R-1)R}{(O_0{+}R)(O_0+R-1)} + \frac{(d+2)(d^2-1)}{Ld^2(d+3)}\frac{\left[(d+2)(O_0{+}R)-2\right]R}{(O_0{+}R)(O_0+R-1)} \tag{313}$$

$$\simeq \frac{R}{O_0+R-1}\left(1-\frac{1}{2(O_0{+}R)}+\frac{d}{L}\right). \tag{314}$$

where in the last line approximate it with $L, d \gg 1$ to simplify the formula. When $O_0 < 1$ with $R = 0$, we directly have $\overline{\zeta_\infty} = 0$. At the critical point of $O_0 = 1$ with $R = 0$, we have $\overline{\zeta_\infty} = 1/2 + d/L$, and in the large limit $L \gg d$, we have $\overline{\zeta_\infty} = 1/2$. However for $O_0 > 1$ with $R = 1 - O_0$, we have $\lim_{R\to(1-O_0)^-}\overline{\zeta_\infty} \to +\infty$ which diverges to infinity.

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