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ORIGINAL ARTICLE

Wittgenstein and concept-extension in mathematics

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Abstract

I begin by attempting to get a perspicuous overview of what Wittgenstein means by saying that a mathematical proof forms concepts. I then distinguish these sorts of cases from those we might call concept-extending proofs, which, rather than introducing new concepts, function to enrich those concepts that have already been given a home in our mathematical practice. At the same time, I also want to argue that the line between these two sorts of proofs is not always clear and will sometimes be blurred. I go on to compare paradigmatic examples of each, concluding with a case in which it is not immediately clear whether we ought to all the proof concept-forming or extending. This last section includes a discussion of Sorin Bangu's recent account of Wittgenstein on mathematical proof.

I. | INTRODUCTION

In nearly every article, chapter or book, dealing with the topic of Wittgenstein and mathematical proof, one will find an in-depth discussion of the associated notion of 'concept-formation'. But concept-formation will not be the main topic of this paper; instead, I would like to discuss concept-*extension*, an idea that is, I think, overshadowed by the significance that is generally attributed to concept-formation.[1] I begin by attempting to get a perspicuous overview of

[1]It was pointed out to me in conversation with Jim Conant that the term 'concept-extension' has the potential to be misleading. He related this discussion in mathematics to what Cavell, in *The Claim of Reason*, calls the *projection* of our concepts into new domains (e.g. 'feed the lion', 'feed the swans', 'feed the meter', 'feed his pride'), and suggested that the notion of 'extension' may carry with it a 'Whiggish' (mis)understanding of the development of our concepts. That is, what occurs when a concept is, in my terms, *extended*, is 'less like what happens when a new room is added onto a house, and more like what happens when the layout of a home is restructured'. As Wittgenstein puts this, '[At the end of the proof] Our way of seeing is remodelled' (*RFM*-IV, §30). Still, some of the mathematical proofs we will consider will be very different from and others very similar to Cavell's notion of concept-projection. In fact, in some cases, I think the Whiggish picture is essentially correct, while in others, such a cumulative picture will be, as Jim correctly pointed out, misleading at best.

what Wittgenstein means by saying that a mathematical proof forms concepts. I then distinguish these sorts of cases from those we might call concept-extending proofs, which, rather than introducing new concepts, function to enrich those concepts that have already been given a home in our mathematical practice. At the same time, I also want to argue that the line between these two sorts of proofs is not always clear and will sometimes be blurred. I go on to compare paradigmatic examples of each, concluding with a case in which it is not immediately clear whether we ought to all the proof concept-forming or extending. This last section includes a discussion of Sorin Bangu's recent account of Wittgenstein on mathematical proof.

II. | CONCEPT-FORMATION

What does it mean to say that the result of a mathematical proof is that we come to accept a particular concept-formation? In the *Remarks on the Foundations of Mathematics*, Wittgenstein offers several characterisations of mathematical proofs, and how we ought to understand their results. Consider the following descriptions given: 'The proof constructs a proposition', The proof *defines* '[such-and-such]', 'In mathematics, we are convinced of *grammatical* propositions' and 'The mathematical proposition is to shew us what it makes *sense* to say'.[2] The idea central to each of these characterisations is that the result of a mathematical proof is the determination of a concept, rather than the description or uncovering of some 'fact of nature'.[3] In a very literal sense, what this sort of proof does is to both introduce and give meaning to a new mathematical concept. Interestingly, Wittgenstein often associates our understanding a mathematical proof, and the concept it forms, with a sense of conviction. 'In producing a concept, the proof convinces me of something: what it convinces me of is expressed in the proposition that it has proved'.[4] In fact, *what* I am convinced of here is a particular use of, or 'track laid down' in, language, and our conviction lies in our going on to use concepts such as 'irrational number' or 'nondenumerable' in our mathematical practice. 'I go through the proof and say: "Yes, this is how it *has* to be; I must fix the use of my language in *this* way."'[5]

Wittgenstein offers many other characterisations of mathematical proof, which may, at first glance, seem disconnected from the characterisation of them as introducing new concepts. It is, however, possible to show how each of these descriptions of mathematical proof can help us understand the sense in which a proof introduces a new concept. Some others are as follows:

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[2]Wittgenstein, *RFM*-III, §§24, 26, 28.

[3]Cf. *RFM*-II, §19.

[4]Ibid.-VII, §72. Later in the remark, he adds: 'The picture (proof-picture) is an instrument producing conviction'.

[5]Ibid.-III, §30. Cf. IV, §8: 'The mathematical proposition determines a path, lays down a path for us'.

'A proof is a picture', 'A proof must of course have the character of a model', 'the proved mathematical proposition…is only the expression of acceptance of a new measure', 'the proposition proved…serves as a rule – and so as a paradigm'.[6] The central idea in each of these is that a mathematical proof plays a special role in our language-games. While it might appear as if the expression of the result of a mathematical proof belonged to the language-game of 'giving a description', Wittgenstein is asking us to see the result, instead, as belonging to the preparation for a description of a fact (and his reason for suggesting this point-of-view is the upshot of an attentiveness to the way that mathematical propositions do, in fact, get used). A mathematical proposition, then, does not constitute a *move* in the language-game of 'giving a description', but would be a part of the description of the language-game itself, that is of what it makes sense to *say* in a particular language-game.[7] To say that the proof functions as a model, paradigm or picture just *is* to distinguish it as having a normative, rather than descriptive, function. 'If you know a mathematical proposition, that's not to say you yet know *anything*. I.e., the mathematical proposition is only supposed to supply a framework for a description'.[8]

Wittgenstein clarifies what he means by saying that 'a proof is a picture' by adding that 'it can be thought of as a cinematographic picture'.[9] Importantly, he elsewhere distinguishes between two different senses of the word 'picture', or rather, two different ways that we might *use* a picture or film. 'Suppose that I film a certain experiment. Then I may use this film as part of an historical sentence: 'Malcom did so-and-so.' But I could also use it in another way. For I could say that I am going to describe all future experiments by saying that they either agree with this experiment or disagree with it by so much. It now serves as a standard'.[10] At *PI* §23, we find a similar point being made: 'Imagine a picture representing a boxer in a particular stance. Now, this picture can be used to tell someone how he should stand, should hold himself; or how he should not hold himself; or how a particular man did stand in such-and-such a place; and so on'.[11] When Wittgenstein says a mathematical proof is a picture, he is using it in the *normative* (rather than

[6]Cf. *RFM*-III, §22–28. 'The effect of the proof is, I believe, that we plunge into the new rule'. (*RFM*-IV, §36) & 'It could be said: a proof helps communication. An experiment presupposes it. Or even: a mathematical proof moulds our language' (*RFM*-III, §71).

[7]Cf. *RFM*-VI, §28: 'Following according to the rule is FUNDAMENTAL to our language-game. It characterizes what we call a description'. & *RFM*-VII, §6 on the way in which the contrast between 'rule of description' and 'descriptive proposition' tends to 'shade off in every direction'.

[8]Wittgenstein, *RFM*-VII, §2. Cf. *TLP* 6.2–6.231, 6.22: 'The logic of the world which the propositions of logic show in tautologies, mathematics shows in equations'.

[9]Wittgenstein, *RFM*-III, §22.

[10]Wittgenstein, *Lectures on the Foundations of Mathematics*, p. 73.

[11]Wittgenstein, *Philosophical Investigations*, §23. Cf. §139 on a picture representing 'an old man walking up a steep path leaning on a stick'.

historical) sense we find available in both of these examples; the proof is not, for example, the picture of some particular calculation, but it *defines* and *pictures* what we mean by 'correctly counting' in the first place. This parallels the point made above that the mathematical proposition functions as a preparation for a move in a language-game but does not constitute one itself. '[The proof] is not an experiment but the picture of an experiment. A picture or film of an ordinary experiment is not the same as an experiment'.[12] So too, giving a description of how one ought to stand in a boxing ring is *not* the same thing as standing in a boxing ring; it is rather a *preparation* for standing in the ring.[13]

Another way of putting the point that mathematical proofs form concepts is to bring out the sense in which they constitute the introduction of primary criteria for the use of our concepts. It will be on this point that we come to see one of the main differences between concept-extension and formation. Wittgenstein gives a quite simple example of this: '["Add 200 + 200."] This process of adding *did* indeed yield 400, but now we take this result as the criterion for the correct addition'.[14] What happens in this case, and in others like it, may have the appearance of being a calculation, but it does not serve the same function as a complicated calculation in which disagreement is conceivable. And this difference might be characterised as one belonging to the *grammar* of each expression. If someone (who had seemingly already mastered our technique of calculating) were suddenly to doubt whether or not '200 + 200' was really '400', we shouldn't even be sure what he meant (similarly, if someone were to express a doubt (on a clear day) as to whether or not the sky is blue, we might wonder whether he had ever learned the English word 'blue'; what we would *not* do, however, is suspect him of having made a mistake). That is, such calculations serve as paradigms—*introducing the very criteria that one might appeal to in performing calculations*. 'If we put 3 things and 2 things, that may yield various counts of things. But we see as a *norm* the procedure that 3 things and 2 things make 5 things. See, *this* is how it looks when they make 5'.[15]

[12]Wittgenstein, *LFM*, p. 72. Cf. *RFM*-III, §23, where Wittgenstein admits that, 'Proof, one might say, must originally be a kind of experiment – but is then taken simply as a picture'. This would distinguish mathematical propositions from ordinary grammatical propositions, which are, in a sense, introduced arbitrarily. Mathematical propositions, on the other hand, become a part of a description of our language-games by way of their proofs (Cf. Schroeder, *Wittgenstein on Mathematics*, pp. 61–62).

[13]Cf. *RFM*-IV, §12: 'We do not judge the pictures, we judge by means of the pictures. We do not investigate them, we use them to investigate something else…The picture of combining is not a combining; the picture of separation is not a separating; the picture of something's fitting not a case of fitting. And yet these pictures are of the greatest significance. *That is what it is like*, if a combination is made; if a separation; and so on'.

[14]*RFM*-III, §24.

[15]Wittgenstein, *RFM*-VI, §9. Cf. *RFM*-III, §§66–67 on 'consensus' belonging 'to the essence of *calculation*'.

In these paradigmatic cases of calculation, the criteria for having calculated correctly *just is* arriving at a certain result, and not an appeal to anything outside the calculation. But to then suggest that the very criteria we use for having 'calculated correctly' might be wrong, is not to make a mistake, but to give up the whole practice of calculating. 'If you suddenly wrote numbers down on the blackboard and then said: 'Now, I'm going to add,' and then said: "2 and 21 is 13," etc. I'd say: "This is no blunder."' Or again: 'For a blunder, that's too big'.[16] Of someone who thought 2 and 21 was 13, we wouldn't say they had calculated *wrongly*, but that they weren't calculating anymore.[17] A confusion such as this would be on par with asking whether or not the Paris meter stick is *really* a meter long; to think that such a question makes sense would not be to make a mistake *in measuring*, but a failure to distinguish what we use to measure from what is measured.[18]

I have tried to give a brief overview of what it means to say that mathematical proofs form new concepts. We quickly saw that this comes out in the particular ways in which we use mathematical propositions and that their forming of concepts is connected to their being normative, rather than descriptive, moves in our language-games. But, one might object, 'how could it be the case that *both* simple calculations and very complicated mathematical proofs introduce new concepts?' Wittgenstein is quite receptive to this worry, which is why he notes that there is something ambiguous about saying that mathematical proofs form concepts: 'When I said that the propositions of mathematics determine concepts, that is *vague*; for "2 + 2 = 4" forms a concept in a different sense from "p ⊃ p," "(x). fx ⊃ fa," or Dedekind's Theorem. The point is, *there is a family of cases'*.[19] Calling each of these proofs an example of 'concept-formation' seems to pass over the various sorts of concepts that are formed by the very different kinds of proofs we employ. In some cases, such as simple calculations, what it means to call them concept-forming is just to emphasise their role as criteria we refer to in more complicated calculations; they define, in a very general way, our *practice* of 'calculating correctly'. In other cases, however, to say that a proof is concept-forming is to recognise that its primary role is to

[16]Wittgenstein, Lectures and Conversations on Aesthetics, Psychology, and Religious Belief, 62.

[17]Cf. *RFM*-IV, §26: 'What is the difference between *not* calculating and calculating *wrong*? – Or: is there a sharp dividing line between *not* measuring time and measuring it *wrong*? Not knowing any measurement of time and knowing a wrong one?'

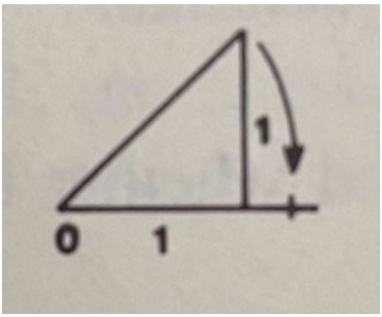
[18]Cf. *RFM*-III, §37: 'What I always do seems to be – to emphasize a distinction between the determination of a sense and the employment of a sense'. & *RFM*-III, §75: 'mathematics as such is always measure, not thing measured'.

[19]Wittgenstein, *RFM*-VII, §42. (Italics added). Cf. *RFM*-V, §15: 'But where is the problem here? Why should I not say that what we call mathematics is family of activities with a family of purposes?', V, §46: 'Mathematics is *not* a sharply delimited concept', & III, §31: 'One would like to say: the proof changes the grammar of our language, changes our concepts. It makes new connexions, and it creates the concept of these connexions (It does not establish that they are there; they do not exist until it makes them)'.

give meaning to an individual concept (Wittgenstein sets up this distinction as follows: 'An equation constructs a conceptual path. But is a conceptual path a concept? And if not, is there a sharp distinction between them?' (*RFM*-V, §42. Cf. VII, §45 & §70 on the word 'concept' being 'too vague by far')). It is precisely the ambiguity in this family of cases that leaves room open for the kinds of concept-*extending* proofs we are concerned with. Before moving on to these, an example of a paradigmatically concept-*forming* proof will serve as a contrast to those we will see in subsequent sections.

III. | LOCATING A POINT ON THE BASELINE AT √2

Consider the following construction that shows that it is possible to find a point on a baseline that is a non-rational distance from zero, and a discussion of it that is extracted from Wittgenstein's 'Philosophy for Mathematicians'.[20] This very simple proof begins by constructing a 1–1 triangle, the hypotenuse of which, we know by way of the Pythagorean theorem, is the √2. We know also that the √2 cannot be expressed as the quotient of two integers by way of the proof of the irrationality of √2. The next step, then, is to pull the hypotenuse of the triangle down to the baseline using a straightedge. By marking the point on the baseline that the end of the hypotenuse touches, one has thus singled-out a point on the baseline that is a non-rational distance from zero.[21].



[20]This section draws on material from Wheeler & Brenner, '"What Line Can't Be Measured With a Ruler?": Riddles and Concept-Formation in Mathematics and Aesthetics' in *Nordic Wittgenstein Review* 13.

[21]Rather than dragging the hypotenuse down to the baseline, one might simply drag the baseline up to the diagonal of the unit square. Still, I take it that this particular construction serves the purpose I want it to; it represents an example of a construction that gives meaning to a concept, and makes sense of the question, 'What point on the baseline is a non-rational distance from zero?' I find nothing wrong with saying that both of these methods do this for us. As Wittgenstein puts this, 'A mathematical proposition stands on four feet, not on three; it is over-determined'. (*RFM*-IV, §7) (Cf. *RFM*-VII, §43: 'The proof of a proposition shews me what I am prepared to stake its truth on. And different proofs can perfectly well cause me to stake the same thing'. & III, §58: 'I can be brought to accept this rule by a variety of paths'.)

What we have now is a unique 'way of measuring a length on the baseline'.[22] The proof that one can find a point on the baseline that is the distance √2 from zero not only extends our concept of what it is to find a specific point on a line, but *gives meaning to* the concept 'a point on a line that is a non-rational distance from zero.' 'Without the construction √2 is not the length'.[23] And this is because the role of the method is to define what it is for something to be that length in the first place. Prior to the introduction of this proof, there was *no* method for producing a point on a baseline whose length could not be expressed as the quotient of two integers. To say that this proof forms a concept is just to say that the question, 'can you find a point on the baseline that is a non-rational distance from zero?' only comes to have a sense once this method is introduced. In other words, *this question cannot be answered by recourse to our ordinary methods of measurement*.

But what of Wittgenstein's earlier insistence that the result of the proof does not constitute a move in the language-game of 'giving a description' but is a preparation for a move in that game? Isn't it the case that this diagram gives us a point on the baseline that is, *in fact*, a non-rational distance from zero? Of this diagram, Wittgenstein goes on to say: 'The idea is that √2 is the result of the construction, i.e., is a certain length, whereas it *is* the construction. It is absurd to say that √2 is the length on the case, for the length is what it measures'.[24] The construction *defines* what it means to say that a point on a line is a non-rational distance from zero, so to say of *that* baseline that it is √2 in length would be like trying to say of the Paris meter stick that *it* is a meter long (Cf. *PI* §50). Additionally, this isn't a case of our having accurately *checked* whether or not the point *is* √2 from zero on our baseline, for the very point of the proof was to define what we mean by 'having checked' in this case.[25] 'Hence accuracy does not come in, for that has to do with measuring rods. Nor is it an approximation'.[26] It is not that such a construction comes as accurately as possible to having found a point on the baseline that is a non-rational distance from zero, but rather, such a construction is a *stipulation* as to what it means to say this at all.

This example was meant to bring to light some of the characteristics of concept-forming proofs discussed earlier. The construction comes to serve as our paradigm for ascertaining whether or not one has found a point on a baseline that is a non-rational distance from zero. It also elucidates the sense in

[22]Wittgenstein, *Wittgenstein's Lectures: Cambridge: 1932-1935,*, p. 215.

[23]Ibid., 221.

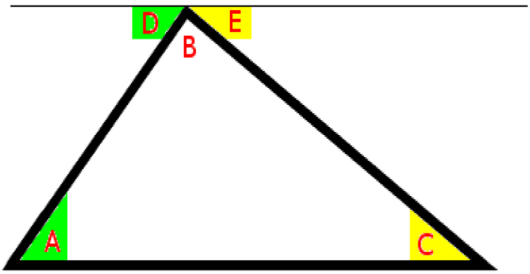
[24]Ibid., 215.

[25]Cf. *On Certainty*, §199: "The reason why the use of the expression 'true or false' has something misleading about it is that it is like saying 'it tallies with the facts or it doesn't', and the very thing that is in question is what 'tallying' is here."

[26]Wittgenstein, *AWL*, 215. Cf. p. 221: 'The length is not an approximation. It has nothing to do with measurement by a foot rule'.

which a mathematical question is given a determinate meaning *by* its proof. That is, in the very act of giving meaning to this concept, the proof also gives meaning to the question of whether or not such a point *can* be found on the baseline. The construction does not present a line that *is* a non-rational distance from zero but belongs to the framework of the *description of a fact*; its function, then, is normative, rather than descriptive.

IV. | CONCEPT-EXTENSION & THE INTERIOR ANGLES OF A TRIANGLE



Consider now, in contrast, the geometrical proof that the interior angles of any triangle will always add up to 180°. This simple proof goes roughly as follows: begin by constructing a line that runs parallel to line AC on a triangle ABC and that cuts through point B. From the fact that a straight line can be formed by putting two 90° angles together, we know that any straight line can be understood as a 180° angle.[27] Further, without knowing *what* the interior angles 'A' or 'C' are, we know that the exterior angle 'D' must be equivalent to 'A' and the exterior angle 'E' must be equivalent to 'C'. If we add angles 'D', 'B' and 'E', they must be equivalent to 180°, since they make up a straight line. As this construction can be applied to *any* triangle, it shows that the interior angles of *every* triangle must add up to 180°.[28].

The suggestion I would like to make now is that such a proof is best characterised as one of concept-*extension* rather than formation and that what gets extended by way of this proof is our concept of a Euclidean triangle. My

[27]Cf. Brenner, 'Natural Law, Motives, and Freedom of the Will', pp. 102–103. Brenner compares the secondary sense of 'motive' that is employed in expressions about 'unconcious motives' to the secondary sense of 'angle' that is employed in out talk about 'straight angles'. He says: 'Compare "unconcious motive" in an explanation of behaviour with "straight angle" in a geometrical proof: in both cases a paradoxical word-formation acquires a use in a special context…[In the case of the proof that the interor angles of a triangle add up to 180°], the interior angles a + b + c, equal the "straight angle," - a' + b + c'. Like "straight angle," "unconcious motiove" is a secondary use of words. "Angle" and "angular" would not have the sense they do were they not at first *contrasted* with "straight" and "line." Nor would "motive" have the sense it does were it not at first *contrasted* with something you might have without being aware of it, such as a cavity or virus'.

[28]Cf. *Philosophical Remarks*, pp. 152 & 218. This image can be located here: "https://complete-concrete-concise.com/mathematics-proving-that-the-angles-in-a-triangle-sum-up-to-180/."

reason for suggesting this is that the proof rests on our already having a certain understanding of what a triangle, or three-sided figure, *is*. In other words, one must already have the concept 'triangle' such that this proof can extend or enrich their concept to include the property, 'having interior angles that add up to 180°'. It is true that 'having interior angles that add up to 180°' does become part of our concept of what it is to be a triangle, and so it is in this sense normative, but this can only come about if one already has the concept of a three-sided figure.[29] We would not say that a person needed to know that the interior angles of any triangle add up to 180° in order to be able to recognise a shape as being a triangle, at least not in the way that we would say a person needed to know that a triangle is a three-sided figure in order to be able to recognise a triangle *as a triangle*.[30] A further point of distinction from the case of concept-formation we discussed above is that this proof does not constitute the introduction of a new method for determining the angles of triangles, but depends on our already having acquired a technique for determining the measure of the angles of a triangle. It was thus possible to ask the question, 'what do the interior angles of this triangle add up to?' prior to our knowing that it *must* be 180°. And this was precisely what we could *not* say in the case of the baseline that is a non-rational distance from zero. In that case, the very function of the proof was to make sense of the question, *not* to treat it as determinable given our current methods of measurement.

My hope is that the differences between concept-forming and concept-extending proofs have been made apparent by our juxtaposing these two cases against one another. To say of a mathematical proof that it functions to *extend* our concepts is to bring out that its task is not primarily a *meaning-giving* one, but one of enriching (or remodelling) the concepts that have already been given a home in our mathematical practice. While the results of such proofs may come to be adopted as definitional criteria for saying, for example, that a

[29]A comparable example can be found in the history of biological investigation; Claude Bernard introduced the idea of *physiologically* carnivorous animals when he discovered that rabbits who had not been fed for long periods of time, and so whose bodies were digesting their own flesh and blood, began to release acidic, as opposed to their typically alkaline, urine (the acidic type being common in flesh-eating animals). But that it is possible to define an animal as being *physiologically* carnivorous when it is deprived of its regularly vegetative appetite, I want to suggest, depends on our having, and being able to use, the standard notion of a carnivorous animal as one that actively consumes the flesh of *other* animals. One of Bernard's achievements, then, was the fruitful extension of the concept 'carnivorous' as it occurs in our specifically physiological investigations. This excellent example was intimated to me, and worked out, during several conversations with William Brenner.

[30]A similar point can be made about many of our ordinary concepts: We would not say that a child must be able to distinguish an 'American short-hair' from a 'domestic short-hair' in order to have the concept 'kitty' or to be able to recognize something as a kitty. Such distinctions only come later and presuppose that one already has the concept 'kitty', and can distinguish cats from things like fur coats.

certain shape is a triangle or not, they do not function as the primary way in which such concepts come to be alive for us.

V. | 'A FAMILY OF CASES'

I want to, in closing, briefly discuss the following account of mathematical proof in Sorin Bangu's recent article, 'Wittgenstein on Proof and Concept-Formation'.[31] This ought to bring out further the relevance of the distinction between concept-formation and extension:

> A way to capture what he [Wittgenstein] has in mind when saying these things [cf. *RFM*-III, §§31, 36, 41, 45, & 71] is, I suggest, to think of a proof as a tool—to modify the domain of mathematical concepts. Thus, the reason why a proof is an 'instrument of language' is that the acceptance of a certain derivation *as a proof* leads—somehow—to *updating* the concepts involved; conversely, updating certain concepts is what allows us to prove certain results. Note the terminology I propose to introduce here: 'Updating' is the more tempered term I shall often use subsequently, instead of Wittgenstein's rather dramatic talk of *changing* the concepts (presumably totally), or of *creating* new concepts (from scratch, as it were)—since…it is somewhat unclear to what extent these radical modifications happen (p. 7).

While Bangu offers an insightful exposition of this topic and *seems* to recognise the distinction between concept-formation and extension, in my view, he fails to capture the subtle differences between them by conflating one with the other. Notice that he refers to the language of '*creating* new concepts' as both overly 'dramatic' and 'radical', but then at times (throughout the rest of the article) *seems* to waffle between rejecting this language and using it himself. Now, in the above passage, there seems to be an indication that Bangu thinks *all* concept-formation is really just concept 'modification' or 'updating'. The motivation for this is likely connected to the worry that, for Wittgenstein, a mathematical question does not have a fully determinate sense until it is given a proof. He then wants to explain Wittgenstein's insistence that the mathematician may work on a 'hunch' when dealing with mathematical 'stimuli' by suggesting that all proofs are modifications of concepts that already have *some* determinate sense. What I mean is, it is much easier to explain how, given the claim that the meaning of a mathematical question is determined by its proof, a mathematical

[31]Bangu, 'Wittgenstein on Proof and Concept-Formation', in *Philosophical Quarterly*. It should be noted that although I will remain quite critical, Bangu's article does offer an excellent account of concept-formation, what he calls the '*creative role*' of mathematical proof, and a powerful criticism of the prevalent notion that mathematical formulae express 'exclusively truth-apt propositions'.

question can *ever* be answered, if the concepts employed are somehow already determinate, and only get extended or modified in the process of the proof (such a view, however, fails to capture how rich the comparison is that Wittgenstein makes between mathematical problems and riddles).[32]

But Bangu seems to be missing something here when he says that ‘updating certain concepts is what allows us to prove certain results’. For wasn't it the proof *itself* that was supposed to bring about the modification of our concepts? On his account, it is only once our concepts have been modified that we can make sense of the result of some proof. He does, of course, also say that ‘the acceptance of a certain derivation *as a proof* leads – somehow – to *updating* the concepts involved’, but my issue is that he seems to turn these into two separate, yet meaningful, acts. On Wittgenstein's account, as I understand it, the result of a proof just *is* that our concepts are changed; such conceptual changes constitute a unity rather than involving two separate acts. This can help us make sense of his insistence that ‘in mathematics, process and result are equivalent’.[33] For our concepts, being changed is not something we get in addition to the result of the proof—it *is* the result of the proof. And it is for this reason that when an imagined interlocutor poses Wittgenstein the question, ‘How can the proposition be separated from its proof?’ His response is that ‘This question betrays a false conception’. (*RFM*-VII, §70; Cf. *PI* §189; it is important that Wittgenstein never responds, ‘the propositon *cannot* be separated from its proof’, as this would be to admit the intelligibility of the interlocutor's question. The response given above is one way of *denying* that; for it is not that separating the proposition from its proof is an overly difficult task we find ourselves unable to accomplish, but rather, there is nothing for one to even try and separate from the proof at all). In other words, the question hinges on a false conception of the relationship between process and result in mathematics.[34] And *what* it is that the question be-

[32]Cf. Säätelä, ‘From Logical Method to “Messing About”: Wittgenstein on “Open Problems” in Mathematics’ & Wheeler & Brenner, ‘*“What Line Can't Be Measured With a Ruler?”*: Riddles and Concept-Formation in Mathematics and Aesthetics’.

[33]*RFM*-I, §82. Cf. *AWL*, pp. 186–191: ‘“Mr. Wisdom is sitting in this chair” and “The Sidgwick Lecturer in Moral Science is sitting in this chair.” These two sentences do not have the same meaning, though they happen to be about the same person. But 4 does not *happen* to be the second place in the development of π; 4 *is* the second place of π. There is no such thing in mathematics as a description of something and its name. That is, there is no such thing as the product of 35 and 45 and the number 1575 which happens to be the number described [as Wisdom *happens* to be the man in the chair *and* the Sidgwick Lecturer in Moral Science]; they are the same number. In this example we have another way of bringing out the fact that a process in mathematics contains its result’. & *LFM*, pp. 49 & 53 on there being *no* distinction between ‘having the construction of a regular pentagon’ and ‘having a regular pentagon’, as there is between the opening of a safe and the jemmy used to open it.

[34]The original German of this response is as follows: ‘Diese Frage zeigt natürlich eine falsche Auffassung’. One might also translate this sentence: ‘This question, of course, *shows* a false conception’. *This* way of translating it, I think, makes clearer the sense in which it is *in the very asking of the question*—in the giving-up and exhibiting something about himself—that the interlocutor reveals his misunderstanding.

trays is its own presupposition ('a myth' [*RFM*-III, §26]) that the concept which is formed or extended by the proof is intelligible independent of the proof.[35]

The example Bangu goes on to give—that of projecting a three-sided plane figure onto a sphere—certainly *is* a case of concept-*extension* (in mathematics, this sort of example is referred to as a 'degenerate' or 'limiting' case, and it is one that may fit closely the model of concept-*projection* mentioned earlier (fn. 1), in which a concept is projected into a new domain).[36] But still, we might wonder, is it really so clear that we ought to call this a case of concept-extension, rather than formation? For isn't the result of the proof an entirely *new* concept, namely 'spherical triangle'? Well, calling it *that* (rather than concept-*formation*) is a way of stressing the logical dependence of 'spherical triangle' on 'Euclidian triangle'.[37] In order to see this point more clearly, consider an explicit case of concept-formation, Cantor's proof regarding the nondenumerability of the real numbers. Prior to Cantor's proof no one knew what it would be for a set to be such that it could not be put into bijection with the natural numbers; the achievement of his proof was that it gave meaning to this expression by stipulating a method for constructing real numbers indefinitely from the set of *all* real numbers. The first example constitutes an extension of our concept 'right-triangle' to include shapes projected onto a sphere, as opposed to a Euclidean plane. The second constitutes the introduction of an entirely *new* concept, namely 'nondenumerability'.[38] What these examples *do* share in common is the fact that neither question could have been answered given our ordinary methods of proof. For had you asked a Euclidean geometer (prior to the discovery of

[35]Cf. *RFM*-III, §25: 'The psychological disadvantage of proofs that construct *propositions* is that they easily make us forget that the *sense* of the result is not to be read off from this by itself, but from the *proof*'.

[36]A different example of degeneracy in mathematics, which is also a case of concept-extension, is the defining of a point as a sphere with a radius of zero. But could there also be cases of concept-*forming* degeneracy? Consider the notion of a *two*-sided polygon, a 'bigon', that can *only* be constructed onto a spherical space. In this case, *unlike* the one of projecting a three-sided figure, there is no concept in Euclidean geometry called a 'bigon' that then gets extended into spherical space. Here, we ought to say, this concept ('bigon') has its *home* in non-Euclidean geometry.

[37]Cf. *PI* §67 on 'family resemblances' and the extension of our concept 'number'.

[38]Cf. Rush Rhees, 'Philosophical Conversations with Wittgenstein', p. 62. For more on Wittgenstein and Cantor, Cf. Wheeler, 'Defending Wittgenstein's Remarks on Cantor from Putnam' & 'A Response to Dehnel's "Defending Wittgenstein,"' Mühlhölzer, 'Wittgenstein on Cantor's Diagonal Method', in Floyd & Mühlhölzer, *Wittgenstein's Annotations To Hardy's Course of Pure Mathematics*, Schroeder, *Wittgenstein on Mathematics* (Ch. 10), & Bold, 'Three Essays on Wittgenstein's Philosophy of Mathematics: Reality, Determination, & Infinity'.

non-Euclidean geometry) to tell you the properties of a spherical triangle, he should have been as lost as the mathematician who was asked (prior to Cantor's proof) to tell you which set of numbers cannot be enumerated. To have asked them *these* questions would have had about as much sense as asking them to point you in the direction of the *East*-Pole (Cf. *LFM*, p. 64).[39]

I mention the example of Cantor because Bangu fails to distinguish between paradigmatic examples of concept-formation such as this and those of concept-extension. In my view, it would be rash to ignore the fact that there *are* distinctive moments of both formation and extension in mathematics, even if there are cases, like the one he does discuss, where the line between these sorts of proofs tends to be blurred.[40] Still, Bangu is right to point out that the language of 'creating new concepts' *can* easily mislead one into (what he calls) a 'made from scratch' picture of the formation of our concepts, in which the 'made from scratch' view treats our concepts like empty shells waiting to be brought to life. My point is that it is possible to maintain both the language of 'concept-formation' and 'extension' while avoiding such a misguided view of *how* our concepts are formed.

In a sense, what I have been arguing for in this paper is just a defence of Wittgenstein's idea that the notion of 'concept-formation' is made of a 'family of cases'. In some of these cases, what needs to be stressed is their dependency on our already having an understanding of the concepts they employ, and the proof's extension of these concepts. In others, we ought to stress their wholly *creative* aspect, in which they produce brand new concepts. My suggestion is that one aspect of recognising this variety comes out in our choosing to distinguish the notions of concept-formation and extension. This is precisely, I take it, what Wittgenstein means by calling mathematics a 'motley of techniques of

[39]Another case that I take to be somewhere *between* the paradigmatic examples of formation and extension we have discussed thus far can also be found in Wittgenstein's 'Philosophy for Mathematicians'. (*AWL*, pp. 217–18) There, Wittgenstein seeks to show that there is an alteration in our concepts 'comparable', 'smaller' and 'larger' when we move from comparing two rationals, such as 4 and 3, or ¾ and ½, to comparing the size of one rational with that of an irrational, such as √2. As he puts this, '"Consider *anything* that is comparable to a rational number." This means nothing, for we have defined "comparable" only for the rationals…To say √2 is smaller or greater than some rational requires a new definition of "comparable"'. And this is because one has not yet stipulated a method for *how* one ought to compare them, as one does have a method for comparing two rationals. My reason for suggesting that such a case constitutes one *between* that of extension and formation is a result of Wittgenstein's concluding remark: 'the old notion of "smaller" comes in, but something different as well'. That is, there is an affinity between the old notion of 'comparable' and our new case, but one need not be mislead (since we call them both 'comparing') into thinking that it is possible to understand the one case (rational and irrational) wholly on the model of the other. (In many of these lectures (esp. pp. 28–30), Wittgenstein does not shy away from connecting the meaning of an expression to its method of verification, but in a way that does *not* amount to a form of verificationism. Mathematics, as I understand him, is a particularly clear case in which this is so.)

[40]This sort of mistake is one that is very often made in discussions about Wittgenstein's notion of family-resemblance. In my view, Wittgenstein's repudiation of the idea that there could be one feature shared by all games, which *make* them all games, does not amount to a rejection of the idea that there *are* paradigmatic examples of games. In fact, that there are characteristic forms of human activity that we would unhesitatingly call 'playing a game' is precisely what allows us to recognize borderline cases as such.

proofs'.[41] In this regard, then, what I have been calling Bangu's 'waffling' between the language of 'creation' and 'updating' should not necessarily be seen as problematic, but rather, as an expression of the great complexity of describing the motley of techniques we call mathematical proofs.

I conclude with a brief discussion of what we might think of as Wittgenstein's own 'waffling' between calling developments in mathematics 'inventions' or 'discoveries'. My reason for doing this is that I have very often found myself dissatisfied with the way in which this topic has either been ignored, or totally misunderstood (and my dissatisfaction is furthered by this latter option not infrequently leading to an unfair attribution of mathematical 'theses' to Wittgenstein). What I would like to suggest is that his willingness to say *both* of these, at different times, and in regard to different mathematical developments, captures just how complex the notions of 'mathematical' and 'mathematics' are ('These things are finer spun than crude hands have any inkling of' (*RFM*-VII, §57)).[42] In other words, some of these developments, he thinks, will share much more in common with what we are ordinarily inclined to call discovery, while others may have more of the face of invention (Cf. *RFM*-IV, §43 on the synthetic character of mathematical propositions being most apparent in the 'unpredictable occurrence of the prime numbers'. These sorts of remarks tend to be overlooked by those I previously characterised as misunderstanding the position that Wittgenstein inhabits). This can help us make sense of the following supposition: 'It might be said: experiment – calculation are poles between which human activities move'. (*RFM*-VII, §30) When Wittgenstein *does* appear to be critical of our using the term 'discovery' in mathematics (e.g. *RFM*-I, §68), this is not because he takes it to be *wrong* to say that discoveries are made in mathematics, but rather, the worry is that this language has the potential to mislead us into failing to distinguish between the distinctive roles of calculation and

[41]Wittgenstein, *RFM*-III, §46. Cf. *RFM*-III, §48 & *The Blue and Brown Books*, pp. 28-29: "Thus we may say of some philosophising mathematicians that they are obviously not aware of the differerence between the many different usages of the word 'proof'; and that they are not clear about the difference between the uses of the word 'kind', when they talk of kinds of numbers, kinds of proofs, as though the word 'kind' here meant the same thing as in the context of 'kinds of apples.' Or, we may say, they are not aware of the different *meanings* of the word 'discovery,' when in one case we talk of the discovery of the construction of the pentagon and in the other case of the discovery of the South Pole."

[42]Cf. *RFM*-IV, §13 & IV, §47: "For the mathematical proposition is the determination of a concept following upon a discovery…You find a new physiognomy. You can now e.g. memorize or copy it. A new form has been found, constructed. But it is used to give a new concept together with the old one."

experiment.[43] Thus, he says: 'Nothing is more fatal to philosophical understanding than the notion of proof and experience as two different but comparable methods of verification'.[44] It is precisely the language of 'discovery' in mathematics that tends to suggest such a misunderstanding.[45]

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[43]To the idea that calculating is a kind of experimentation, by which we get acquainted with the properties of numbers, Wittgenstein responds: 'But do the properties of numbers *exist* outside the calculating?' (*RFM*-III, §58) As we get acquainted with the properties of a rock by performing an experiment on it, don't we *also* perform an experiment when calculating, thereby getting acquainted with the properties of numbers? To that, Wittgenstein would reply that this way of speaking is misleading—misleading because it tends to pass over the different uses of the word 'property'. His question, 'Do the properties of numbers exist outside the calculating?' alludes to one such difference: It is the calculations in which numbers are employed that give them their 'properties', but it is not the descriptions in which predicates are employed that give rocks their properties. (Cf. Wheeler, 'Wittgenstein on Miscalculation and the Foundations of Mathematics', on the distinctive roles of calculation and experiment.)

[44]Wittgenstein, *Philosophical Grammer*, p. 361.

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