THE UNIVERSITY OF CHICAGO

ALGEBRAIC BRAIDS AND GEOMETRIC REPRESENTATION THEORY

A DISSERTATION SUBMITTED TO
THE FACULTY OF THE DIVISION OF THE PHYSICAL SCIENCES
IN CANDIDACY FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

DEPARTMENT OF MATHEMATICS

BY
MINH-TÂM QUANG TRINH

CHICAGO, ILLINOIS
JUNE 2020

To my teachers

One time she found a big white vertebra in the sand. It was too hard to work but could not have been made prettier anyway, so she put it in the magic forest as it was.

—Tove Jansson, *The Summer Book*

Contents

ABSTRACT

This thesis consists of four research papers stapled together. In lieu of an introduction to the whole, we suggest that the reader start with the introduction to the fourth paper, "Algebraic Braids and the Springer Theory of Hitchin Systems." Even though it was written last, it provides (in the author's eyes) the motivation for the other three.

There is significant overlap between the background sections of the four papers. We hope this repetition will be viewed as an aid to the reader, rather than as wasted space.

ACKNOWLEDGMENTS

My work was much inspired by ideas in Vivek Shende's 2013 NSF proposal [94]. In addition, I warmly thank Ron Donagi, Olivier Dudas, Ben Elias, Pavel Etingof, Victor Ginzburg, Eugene Gorsky, Oscar Kivinen, Brian Lawrence, Ian Lê, George Lusztig, Davesh Maulik, Benedict Morrissey, Andrei Neguţ, Bao-Châu Ngô, Emily Norton, Alexei Oblomkov, Jay Taylor, David Treumann, Cheng-Chiang Tsai, and Zhiwei Yun for many helpful discussions. I thank Laura Ball and Bao-Châu Ngô for generously proofreading portions of this thesis.

I am especially grateful to my advisors, Bao-Châu Ngô and Victor Ginzburg, who have shaped my thinking in ways both visible and invisible.

During this work, I've received support from an NSF Graduate Research Fellowship (grant #1746045) and benefited from workshops at MSRI and AIM.

My time at UChicago has been enriched by the companionship of many people here. Written out in full, their names would span pages. I must give special mention to Santiago Chaves Aguilar, Nazerke Bakytzhan, Ishan Banerjee, Owen Barrett, Brian Burns, Daniel Campos, Alan Chang, Jingren Chi, Aygul Galimova, Mathilde Gerbelli-Gauthier, Thomas Hameister, Reid Harris, Quoc Hô, Boming Jia, Lucia Mocz, Olivier Martin, Bryan McGuiggin, Tung Ngûyen, Elia Portnoy, Kostas Psaromiligkos, Antoni Rangachev, Foy Scalf, Aaron Silberstein, Aaron Slipper, Griffin Wang, Yiwen Zhou, and most of all, Rachel McEnroe. Special thanks also go to Benjamin Ihde, Gregory Kufera, Travis Reynolds, Evangelie Zachos, Wentong Zhang, Feng Zhu, and the rest of the McEnroe family—especially my favorite rabbit, Amo(ooo)on.

I thank my parents for their love and their enormous sacrifices on my behalf.

Chapter 1

Annular Homology of Artin Braids I

1.1 Introduction

1.1.1. Let W be a finite Coxeter group and Br_W the corresponding Artin group. We can view Br_W as a generalization of the braid groups studied in geometric topology: If $W = S_n$, the symmetric group on n letters, then $Br_W = Br_n$, the group of topological braids on n strands.

Any element $\beta \in Br_n$ can be represented by a tangle diagram joining n points at the base of a rectangle to n points at the top. Closing up the rectangle into an annulus, we obtain the annular closure of β, a diagram representing a link in a solid torus. Embedding the annulus into a plane, we obtain the planar closure, which represents a link in 3-space. It is usually difficult to decide whether two braids have the same planar closure. By contrast, the conjugacy class of β determines and is determined by its annular closure. In this sense, conjugacy class generalizes the notion of annular closure from braid groups to other Artin groups.

Let $\mathrm{K}_0(W)$ denote the ring of virtual representations of W. We will introduce a class function on Br_W, the **annular character**, that takes values in a graded version of $\mathrm{K}_0(W)$:

$$\textsc{ann} : Br_W \to \mathrm{K}_0(W)[\![\boldsymbol{q}]\!]. \tag{1.1.1}$$

This paper is the first of a trilogy in which we explain how ANN is related to various kinds of representation theory.

(I) In this paper, we show that ANN is a refinement of Y. Gomi's bivariate Markov trace on Br_W. If W is crystallographic, then ANN is itself categorified by a functor on the Hecke category of W that refines triply-graded Khovanov–Rozansky homology, closely related to the "horizontal traces" studied in [51] and [49].

(II) In [106], we show that if W is crystallographic, then ANN can be decomposed into a weighted sum of Green functions, i.e., graded representations of W afforded by the total cohomology of Springer fibers.

(III) In [107], we show that special values of ANN are the graded characters of interesting representations of the rational Cherednik algebra of W.

In a companion paper [108], we will state a conjecture relating ANN with representations of W afforded by the cohomology of affine Springer fibers. Using the previous papers, we will give evidence for the conjecture and relate it with the P = W conjecture of nonabelian Hodge theory.

1.1.2. In the first part of this paper, we give a definition of ANN motivated by knot theory. There is an isotopy invariant of links in 3-space that is valued in bivariate Laurent series. It is named the HOMFLY series after the initials of its discoverers [41]:

$$\textsc{homfly} : \{\text{links in } \mathbf{R}^3\}/\text{isotopy} \to \mathbf{Z}[(\boldsymbol{q}^{\frac{1}{2}} - \boldsymbol{q}^{-\frac{1}{2}})^{\pm 1}, a^{\pm 1}]. \tag{1.1.2}$$

Alexander showed that every link in 3-space is the planar closure of some braid. Ocneanu, extending work of Jones, used this result to construct HOMFLY in terms of annular braid invariants, i.e., class functions on the groups Br_n. These functions factor through linear functions, now known as Markov traces, on corresponding Iwahori–Hecke algebras [56]. Gomi, extending work of Geck–Lambropoulou, gave a uniform generalization of Ocneanu's Markov traces to any Iwahori–Hecke algebra H_W with W a finite Coxeter group [46]:

$$\textsc{tr} : H_W \to \mathbf{Z}[\boldsymbol{q}^{\pm\frac{1}{2}}](a). \tag{1.1.3}$$

Remarkably, Gomi's construction relies on a pairing on the irreducible characters of W derived from Lusztig's exotic Fourier transform (see Section 1.3).

We, too, define ANN purely algebraically in terms of the exotic Fourier transform (see Definition 1.4.3). A priori, it is a function of the form

$$\text{ANN} : Br_W \to \mathrm{K}_0(W) \otimes \mathbf{Q}_W(\boldsymbol{q}^{\frac{1}{2}}), \tag{1.1.4}$$

where $\mathbf{Q}_W$ is a splitting field for W. We show that we can recover TR from ANN by taking certain isotypic components. To make this precise, let $(-,-)_W = \dim \mathrm{Hom}_W(-,-)$, viewed as a bilinear pairing on $\mathrm{K}_0(W) \otimes \mathbf{Q}_W(\boldsymbol{q}^{\frac{1}{2}})$ by linearity, and let V be a (faithful) realization of W of rank r. *In this introduction, we will assume that $V^W = 0$.*

Proposition 1.1.1. *We have*

$$\textit{TR}(\beta) = (-\boldsymbol{q}^{\frac{1}{2}})^{-|\beta|} \left(\frac{1-\boldsymbol{q}}{1-a^2}\right)^r \sum_{0 \leq i \leq r} (-a^2)^i (\Lambda^i(V), \textit{ANN}(\beta))_W, \tag{1.1.5}$$

where $|\beta|$ is the writhe of β (see Section 1.2).

Example 1.1.2. Let $1 \in Br_W$ be the identity. In Section 1.5, we show that $(1-\boldsymbol{q})^r \text{ANN}(1)$ is the regular representation of W, placed in $\boldsymbol{q}$-degree 0. This recovers the fact that $\text{TR}(1) = 1$.

Using the character theory of H_W, we will give a more precise characterization of the image of ANN. In doing so, we will show that ANN retains the well-known $\boldsymbol{q}^{\frac{1}{2}} \to -\boldsymbol{q}^{\frac{1}{2}}$ symmetry of the HOMFLY series.

Theorem 1.1.3. *We have*

$$\textit{ANN}(\beta) \in \mathrm{K}_0(W)[\![\boldsymbol{q}]\!] \cap (\boldsymbol{q}^{\frac{1}{2}})^{|\beta|-r} \mathrm{K}_0(W)(\boldsymbol{q}^{\frac{1}{2}} - \boldsymbol{q}^{-\frac{1}{2}}). \tag{1.1.6}$$

Moreover, $(1, \textit{ANN}(\beta))_W|_{\boldsymbol{q}\to 0} = 1$. Consequently, $\textit{ANN}(\beta)$ is a rational function of $\boldsymbol{q}$ of degree $|\beta| - r$ such that $(\boldsymbol{q}^{\frac{1}{2}})^{r-|\beta|}\textit{ANN}(\beta)$ is invariant under $\boldsymbol{q}^{\frac{1}{2}} \mapsto -\boldsymbol{q}^{-\frac{1}{2}}$.

Our examples in Section 1.5 suggest a positivity conjecture. Let $Br_W^+ \subseteq Br_W$ denote the positive submonoid (see Section 1.2), and let $\mathrm{K}_0^+(W) \subseteq \mathrm{K}_0(W)$ denote the semiring of actual, not virtual, characters of W.

Conjecture 1.1.4. *If $\beta \in Br_W^+$, then* $\textsc{ann}(\beta) \in \mathrm{K}_0^+(W)[\![\boldsymbol{q}]\!]$.

1.1.3. In the second part of this paper, we assume that W is crystallographic and use algebraic geometry to categorify $\textsc{ann}$. This step is again motivated by knot-theoretic ideas.

Dunfield–Gukov–Rasmussen conjectured [32], and Khovanov–Rozansky constructed [62, 61], a refinement of $\textsc{homfly}$ to an isotopy invariant valued in triply-graded vector spaces. Its graded dimension can be expressed as a series in $\boldsymbol{q}^{\frac{1}{2}}, a, t$ that recovers $\textsc{homfly}$ when we set $t = -1$. Michel observed [111, 417] that this invariant can be recast in terms of a categorification of Gomi's Markov trace on H_W.

Recall that H_W is the Grothendieck ring of a much-studied monoidal triangulated category, which we call the Hecke category of W and denote H_W. Gomi's trace can be categorified by an additive functor

$$\mathrm{HHH} : \mathsf{H}_W \to \mathsf{Vect}_{\text{3-gr}}, \tag{1.1.7}$$

where $\mathsf{Vect}_{\text{3-gr}}$ is the category of $\mathbf{Z}^3$-graded vector spaces (over a given field of characteristic zero). This functor is now called Khovanov–Rozansky homology.

Suppose $\mathbf{F}$ is a finite field and W is the Weyl group of a semisimple group G over $\bar{\mathbf{F}}$ with split form G_0 over $\mathbf{F}$. We can define H_W in terms of G_0-equivariant mixed complexes of ℓ-adic sheaves on $\mathcal{B}_0 \times \mathcal{B}_0$, where $\mathcal{B}_0$ is the flag variety of G_0. In [110, 111, 112], Webster–Williamson constructed Khovanov–Rozansky homology via the so-called horocycle correspondence between $\mathcal{B}_0 \times \mathcal{B}_0$ and G_0. Explicitly, they realized HHH as a composition of additive functors between (bounded) homotopy categories:

$$\mathsf{H}_W = \mathsf{K}^b(\mathsf{Pur}(G \backslash (\mathcal{B} \times \mathcal{B}))) \xrightarrow{\mathcal{C}orr} \mathsf{K}^b(\mathsf{Pur}(G \backslash G)) \xrightarrow{\mathrm{Gr}_*^{\mathbf{W}} \, \mathrm{H}^*(G \backslash G, -)} \mathsf{K}^b(\mathsf{Vect}_{\text{2-gr}}), \tag{1.1.8}$$

where $Z = Z_0 \otimes \bar{\mathbf{F}}$ and $\mathsf{Pur}(Z)$ is the full subcategory of $\mathsf{D}^b(Z)$ formed by pullbacks of pure weight-0 complexes on Z_0. Above, $\mathcal{C}orr$ is induced by the horocycle correspondence, and $\mathrm{Gr}_*^{\mathbf{W}} \mathrm{H}^*(G\backslash G, -)$ is induced by the weight filtration on the hypercohomology of objects of $\mathsf{Pur}(G\backslash G)$.

Let T_0 be a split maximal torus of G_0 and form the graded $\bar{\mathbf{Q}}_\ell$-algebra:

$$\mathbf{A} = \bar{\mathbf{Q}}_\ell[W] \ltimes \mathrm{H}^*(\,\cdot\,/T, \bar{\mathbf{Q}}_\ell). \tag{1.1.9}$$

Let $\mathbf{A}\text{-}\mathsf{Mod}_{\mathsf{gr}}$ be the category of graded $\mathbf{A}$-modules. We will introduce a functor

$$\mathcal{A}\mathrm{H} : \mathsf{H}_W \to \mathsf{K}^b(\mathbf{A}\text{-}\mathsf{Mod}_{\mathsf{gr}}) \tag{1.1.10}$$

that we call **annular braid homology** (see Definition 1.7.8). (It is not the annular *link* homology already in the literature.) The construction is close to Webster–Williamson's, but instead of applying $\mathrm{H}^*(G\backslash G, -)$, we apply $\mathrm{Ext}^*_{G\backslash G}(\mathcal{E}_1, -)$, where $\mathcal{E}_1$ is the Grothendieck–Springer sheaf on $G\backslash G$, and keep only the pure part of the weight grading. The $\mathbf{A}$-module structure on $\mathcal{A}\mathrm{H}$ essentially arises from the Yoneda product of Ext-groups.

Rouquier [92] described a map $\beta \mapsto R(\beta)$ from elements of Br_W to objects of H_W, taking braid compositions to monoidal products up to isomorphism. We will show that the functor $\mathcal{A}\mathrm{H}$ refines Khovanov–Rozansky homology, and that the function on Br_W defined by

$$\mathrm{AH}(\beta) = (\boldsymbol{q}^{\frac{1}{2}}t)^{|\beta|}\varepsilon \otimes \bigoplus_{j,k} \boldsymbol{q}^{\frac{j}{2}} t^{-k} \mathrm{H}_k(\mathcal{A}\mathrm{H}^j(R(\beta))) \in \mathrm{K}_0^+(W)[\![\boldsymbol{q}^{\frac{1}{2}}]\!][t] \tag{1.1.11}$$

(where ε is the sign representation of W) is a refinement of ANN, hence also of TR.

Theorem 1.1.5. $\mathcal{A}\mathrm{H}$ *satisfies* $\mathcal{A}\mathrm{H}(K \otimes L) \simeq \mathcal{A}\mathrm{H}(L \otimes K)$ *for all* $K, L \in \mathsf{H}_W$*. In particular,* AH *is a class function on* Br_W*. Up to normalization factors depending on* $|\beta|$*, we have a* *commutative diagram of specializations:*

(1.1.12)
$$\begin{array}{ccc} \mathcal{A}\mathrm{H}(R(\beta)) & \xrightarrow{\mathrm{Hom}_W(\Lambda^*(V), -)} & \mathrm{HHH}(R(\beta)) \\ {\scriptstyle t=-1}\downarrow & & \downarrow{\scriptstyle t=-1} \\ \mathit{ANN}(\beta) & \xrightarrow[(\Lambda^*(V), -)_W]{} & \mathit{TR}(\beta) \end{array}$$

Above, the horizontal arrows amount to taking $\Lambda^*(V)$-isotypic components and the vertical arrows amount to setting $t = -1$.

Example 1.1.6. In Section 1.7, we check that $\mathrm{AH}(1) \simeq \bigoplus_j \boldsymbol{q}^{\frac{j}{2}} t^0 \mathbf{A}^j \simeq \bigoplus_i \boldsymbol{q}^i t^0 \, \mathrm{Sym}^i(V)$.

Remark 1.1.7. In [5, 6], Beliakova *et al.* introduce the notion of the horizontal trace of a bicategory. After one adapts their definition to a monoidal additive category like H_W, it is roughly the universal linear functor under which the images of $K \odot L$ and $L \odot K$ are isomorphic for all objects K, L. In [51], Gorsky–Wedrich show that if $W = S_n$ and $G = \mathrm{SL}_n$, then the horizontal trace of H_W takes values in $\mathsf{K}^b(\mathbf{A}\text{-}\mathsf{Mod}_{\mathsf{gr}})$. (Technically, they work with $G = \mathrm{GL}_n$.)

We will not discuss horizontal traces in this paper; we defer the topic to [51] (and the recent sequel [49]). But it seems natural to speculate that $\mathcal{A}\mathrm{H}$ is the horizontal trace on H_W, up to differences in formalism (e.g., triangulated vs. dg-categorical).

In [106], we conjecture a bivariate generalization of Theorem 1.1.3 with $\mathrm{AH}(\beta)$ in place of $\mathrm{ANN}(\beta)$. It is motivated by the "curious Lefschetz" phenomenon observed by Hausel–Rodriguez-Villegas in the cohomology of character varieties [54].

1.1.4. Lastly, on the submonoid $Br_W^+ \subseteq Br_W$, we will give a more concrete construction of the W-action on AH, using certain varieties studied by Broué–Michel [17] and Deligne [27]. By their work, there is a map $\beta \mapsto O(\beta)_0$ from elements of Br_W^+ to G_0-varieties over $\mathcal{B}_0 \times \mathcal{B}_0$

that takes braid compositions to fiber products up to isomorphism. Set

$$\widetilde{O}(\beta)_0 = O(\beta)_0 \times_{\mathcal{B}_0 \times \mathcal{B}_0} (\mathcal{B}_0 \times G_0), \tag{1.1.13}$$

where $\mathcal{B}_0 \times G_0 \to \mathcal{B}_0 \times \mathcal{B}_0$ is the action map $(x, g) \mapsto (x, gx)$. The variety

$$St_G(\beta)_0 = \widetilde{O}(1)_0 \times_{G_0} \widetilde{O}(\beta)_0 \tag{1.1.14}$$

is an analogue of the Steinberg variety from Springer theory.

Let $Pr_\beta : G_0 \backslash St_G(\beta)_0 \to G_0 \backslash G_0$ be the natural map. In Section 1.8, we will express $\mathcal{A}\mathrm{H}(\beta)$ in terms of the !-pushforward of the constant sheaf along Pr_β, by means of the so-called chromatography functor in Webster–Williamson's work [112]. (This functor was first introduced by Bondarko; see [12, 13].) We will also show how to construct the W-action on $\mathcal{A}\mathrm{H}(\beta)$ from cohomological correspondences on the "Steinberg–like" variety $St_G(\beta)_0$.

Theorem 1.1.8. *Let $\mathcal{C}hr : \mathsf{D}^b_m(\mathsf{Perv}(G \backslash G)) \to \mathsf{K}^b(\mathsf{Pur}(G \backslash G))$ be the chromatography functor in Appendix 1.9. For all $\beta \in Br_W^+$, we have*

$$\mathcal{A}\mathrm{H}^j(\beta) = \mathrm{Gr}^{\mathbf{W}}_j \, \mathrm{H}^j(G \backslash G, \mathcal{C}hr(Pr_{\beta,!} \, \bar{\mathbf{Q}}_{\ell,0})), \tag{1.1.15}$$

where we apply $\mathrm{Gr}^{\mathbf{W}}_j \, \mathrm{H}^j(G \backslash G, -)$ term-by-term to the chromatographic complex. The W-action on $\mathcal{A}\mathrm{H}(\beta)$ is induced by a W-action on $Pr_{\beta,!} \, \bar{\mathbf{Q}}_{\ell,0}$, constructed via cohomological correspondences on $St_G(\beta)_0$.

Remark 1.1.9. We hope that $\mathrm{AH}(\beta)$ admits a much simpler formula in terms of Steinberg-like varieties. Let $\mathcal{U}_0$ be the unipotent locus of G_0, and let

$$St(\beta)_0 = \mathcal{U}_0 \times_{G_0} St_G(\beta)_0. \tag{1.1.16}$$

Let $St(\beta)$ be the pullback of $St(\beta)_0$ to $\bar{\mathbf{F}}$. Again using correspondences, one can construct an W-action on its cohomology. For elements $\beta \in Br_W^+$ such that the image of β in W is *elliptic*, we expect a W-equivariant isomorphism

$$\mathrm{AH}(\beta) \simeq t^{r-|\beta|} \bigoplus_{j,k} (\boldsymbol{q}^{\frac{1}{2}} t)^j t^k \, \mathrm{Gr}_j^{\mathbf{W}} \, \mathrm{H}_c^k(G \backslash St(\beta), \bar{\mathbf{Q}}_\ell). \tag{1.1.17}$$

See [106] for further details, including the expectation for general positive braids.

1.1.5. Acknowledgements I warmly thank Ben Elias, Victor Ginzburg, Eugene Gorsky, Matt Hogancamp, Oscar Kivinen, George Lusztig, Anton Mellit, Andrei Neguţ, Bao-Châu Ngô, Alexei Oblomkov, and Zhiwei Yun for their interest. Also, I'm indebted to Olivier Dudas for explaining Alvis–Curtis–Kawanaka duality to me, and E. Gorsky for pointing me to his paper [51] with Wedrich. I'm grateful to Laura Ball and B.-C. Ngô for proofreading my drafts. During this work, I was supported by an NSF Graduate Research Fellowship (grant #1746045) and benefited from workshops at MSRI and AIM.

Remark 1.1.10. I presented the construction of $\mathcal{A}\mathrm{H}$ in a talk at the FRG conference on "Hilbert Schemes, Categorification and Combinatorics" held at UC-Davis during June 2019. Later at the same conference:

- Matt Hogancamp spoke about his (then ongoing) work with Gorsky and Wedrich on the horizontal trace of H_W, which has since appeared in [49].
- Anton Mellit spoke about a functor very similar to $\mathcal{A}\mathrm{H}$, which will appear in his forthcoming work on the ∇-positivity conjecture of algebraic combinatorics. Mellit's construction stays on the $\mathcal{B} \times \mathcal{B}$ side of the horocycle correspondence.

It would be interesting to study further the relationship between my work, Gorsky–Hogancamp–Wedrich's, and Mellit's.

Remark 1.1.11. I presented results from [106, 107, 108] in talks at MIT in December 2019. After my talks, George Lusztig told me about his joint preprint [77] with Yun, which appeared on arχiv the following day. For $\mathbf{F}$ a finite field and W the Weyl group of a reductive group G over $\bar{\mathbf{F}}$, they introduce a class function $\Psi : W \to \mathrm{K}_0(W)[\boldsymbol{q}]$. It refines the (surjective) map from conjugacy classes of W to unipotent classes in G that Lusztig introduced in [76]. We expect that $\Psi(w)$ and $\textsc{ann}(\beta_w)$ (where $w \mapsto \beta_w$ is defined in Section 1.2) differ by a power of $1 - \boldsymbol{q}$ depending on w.

1.2 The Coxeter System

1.2.1. A useful reference for this section is the book [43] by Geck–Pfeiffer. We especially recommend Chapter 4 for the facts we cite about the Artin braid monoid.

1.2.2. Let (W, S) be a finite Coxeter system. This means W is a finite group and $S \subseteq W$ is a subset that freely generates W modulo relations of the form $(st)^{m_{s,t}} = 1$, where $m_{s,s} = 1$ and $m_{s,t} \geq 1$ for all $s, t \in S$. We say that W is a Coxeter group; it is **irreducible** iff it cannot be written as a product of (nontrivial) Coxeter groups.

If $w \in W$, then a **word** in S for w is a sequence $(s_1, \ldots, s_\ell)$ of elements of S such that $w = s_1 \cdots s_\ell$. The **reduced** words for w are those of minimal length. The **Bruhat order** on W is the partial order where $w' \leq w$ iff some reduced word for w' occurs as a subsequence of a reduced word for w, not necessarily a contiguous one.

The common length of the reduced words for w is called the **Bruhat length** of w and denoted $|w|$. There is a unique element $w_0 \in W$ of maximal Bruhat length, known as the **longest element** of (W, S).

1.2.3. The **Artin braid monoid** of (W, S), which we denote Br_W^+, is the monoid freely generated by formal elements β_s for each $s \in S$, modulo the relations

$$\overbrace{\beta_s\beta_t\beta_s\cdots}^{m_{s,t}\text{ terms}} = \overbrace{\beta_t\beta_s\beta_t\cdots}^{m_{s,t}\text{ terms}} \tag{1.2.1}$$

for *distinct* $s, t \in S$. The **Artin braid group** of (W, S), denoted Br_W, can be defined as the group completion of Br_W^+. We refer to the elements of Br_W as **Artin braids**.

There is a surjective morphism $Br_W \to W$ that sends $\beta_s \mapsto s$. The kernel is generated by the elements of the form β_s^2.

Example 1.2.1. In the Coxeter system of type A_{n-1}, we take W to be the symmetric group S_n and S to consist of the transpositions $(i\ i+1)$ for $i = 1, \ldots, n-1$. Here, we can identify Br_W with the group Br_n of topological braids on n strands.

If $\beta \in Br_W^+$, then a **word** in S for β is a sequence $(s_1, \ldots, s_\ell)$ of elements of S such that $\beta = \beta_{s_1} \cdots \beta_{s_\ell}$. All words for β have the same length, which is called the **writhe** of β and denoted $|\beta|$. Unlike Bruhat length, writhe is additive: It defines a morphism of monoids $|{-}| : Br_W^+ \to \mathbf{Z}_{\geq 0}$, which extends uniquely to a morphism of groups $|{-}| : Br_W \to \mathbf{Z}$.

If $w \in W$ and $(s_1, \ldots, s_\ell)$ is a reduced word for w, then $\beta_w = \beta_{s_1} \cdots \beta_{s_\ell}$ depends only on w, not on the chosen word. The map $w \mapsto \beta_w$ defines a set-theoretic section of the morphism $Br_W^+ \to W$ that satisfies $|\beta_w| = |w|$.

1.2.4. The **Iwahori–Hecke algebra** of (W, S), which we denote H_W, is a variant of the group ring $\mathbf{Z}[W]$ where we modify the relations $s^2 = 1$ to depend on a variable $\boldsymbol{q}^{\frac{1}{2}}$:

$$H_W = \frac{\mathbf{Z}[\boldsymbol{q}^{\pm\frac{1}{2}}][Br_W^+]}{\left\langle (\beta_s - \boldsymbol{q}^{\frac{1}{2}})(\beta_s + \boldsymbol{q}^{-\frac{1}{2}}) : s \in S \right\rangle}. \tag{1.2.2}$$

The set $\{\beta_w\}_{w \in W}$ forms a free $\mathbf{Z}[\boldsymbol{q}^{\pm\frac{1}{2}}]$-basis for H_W. By induction on the Bruhat order, one can show that the multiplication table for this basis is given by

$$\beta_w \beta_s = \begin{cases} \beta_{ws} & ws > w \\ \beta_{ws} + (\boldsymbol{q}^{\frac{1}{2}} - \boldsymbol{q}^{-\frac{1}{2}})\beta_w & ws < w \end{cases} \tag{1.2.3}$$

for all $s \in S$ and $w \in W$. Thus, $\beta_w \mapsto w$ defines an explicit isomorphism $H_W|_{\boldsymbol{q}^{1/2}=1} \simeq \mathbf{Z}[W]$.

1.3 Representation Theory of Coxeter Systems

1.3.1. Characters By "representation," we mean a finite-dimensional linear representation over (a subfield of) $\mathbf{C}$. We write $\mathrm{K}_0(W)$ for the Grothendieck ring of virtual representations of W. Abusing notation, *we treat the elements of* $\mathrm{K}_0(W)$ *interchangeably with their characters.* Let $\mathrm{K}_0^+(W) \subseteq \mathrm{K}_0(W)$ be the semiring of actual, not virtual characters, and let $\mathrm{Irr}(W) \subseteq \mathrm{K}_0^+(W)$ be the set of irreducible characters. Let $\mathbf{Q}_W \subseteq \mathbf{C}$ be the field obtained by adjoining to $\mathbf{Q}$ the character values $\phi(w)$ for all $w \in W$ and $\phi \in \mathrm{Irr}(W)$.

We write 1 and ε for the trivial and sign characters of W, respectively. By definition, $\varepsilon(w) = (-1)^{|w|}$. Let $(-,-)_W$ be the $\mathbf{Z}$-valued pairing on virtual characters defined by

$$(1.3.1) \qquad \begin{aligned} (\phi, \psi)_W &= \dim \mathrm{Hom}(1, \phi \otimes \psi) \\ &= \frac{1}{|W|} \sum_{w \in W} \phi(w)\psi(w^{-1}) \end{aligned}$$

for all $\phi, \psi \in \mathrm{K}_0(W)$. For any ring R, we extend $(-,-)_W$ to a pairing on $\mathrm{K}_0(W) \otimes R$ by linearity.

1.3.2. Realizations If K is a subfield of $\mathbf{C}$, then a **realization** of (W, S) over K is a finite-dimensional K-vector space V together with a (faithful) representation $W \to \mathrm{GL}(V)$ that identifies S with a set of reflections in V. Let $M(\boldsymbol{q} \mid V)$ be the element of $\mathrm{K}_0(W)[\![\boldsymbol{q}]\!] = \mathrm{K}_0(W) \otimes \mathbf{Z}[\![\boldsymbol{q}]\!]$ defined by

$$(1.3.2) \qquad M(\boldsymbol{q} \mid V) = \sum_{i \geq 0} \boldsymbol{q}^i \, \mathrm{Sym}^i(V).$$

For all $\phi \in \mathrm{K}_0(W)$, the power series

$$M_\phi(\boldsymbol{q} \mid V) = (\phi, M(\boldsymbol{q} \mid V))_W = \frac{1}{|W|} \sum_{w \in W} \frac{\phi(w^{-1})}{\det(1 - \boldsymbol{q}w \mid V)}. \tag{1.3.3}$$

is called the **Molien series** of ϕ with respect to V.

Every finite Coxeter system admits a realization over $\mathbf{R}$. We say that (W, S) is **crystallographic** iff it admits a realization V over $\mathbf{Q}$, in which case W is the Weyl group of a root system $\Phi \subseteq V$ and S is the set of reflections corresponding to a system of simple roots in Φ. For crystallographic Coxeter systems, $\mathbf{Q}_W = \mathbf{Q}$ [101, Cor. 4.8].

1.3.3. The Generic Hecke Algebra Consider the following base change of H_W:

$$H_W(\boldsymbol{q}^{\frac{1}{2}}) = H_W \otimes_{\mathbf{Z}[\boldsymbol{q}^{\pm\frac{1}{2}}]} \mathbf{Q}_W(\boldsymbol{q}^{\frac{1}{2}}). \tag{1.3.4}$$

By Thm. 9.3.5 of [43], we have an isomorphism of $\mathbf{Q}_W(\boldsymbol{q}^{\frac{1}{2}})$-algebras $H_W(\boldsymbol{q}^{\frac{1}{2}}) \simeq \mathbf{Q}_W(\boldsymbol{q}^{\frac{1}{2}})[W]$. Thus we obtain an isomorphism of Grothendieck rings:

$$\begin{array}{ccc} \mathrm{K}_0(W) & \xrightarrow{\sim} & \mathrm{K}_0(H_W(\boldsymbol{q}^{\frac{1}{2}})) \\ \phi & \mapsto & \phi_{\boldsymbol{q}} \end{array} \tag{1.3.5}$$

Again, we will identify the representations $\phi_{\boldsymbol{q}}$ with their characters. Thus we write $\phi_{\boldsymbol{q}}(\beta) = \mathrm{tr}(\beta \mid \phi_{\boldsymbol{q}})$ for all $\beta \in H_W(\boldsymbol{q}^{\frac{1}{2}})$ and $\phi \in \mathrm{Irr}(W)$. For example, we have $1_{\boldsymbol{q}}(\beta_w) = (\boldsymbol{q}^{\frac{1}{2}})^{|w|}$ for all $w \in W$.

Let $\tau_{\boldsymbol{q}} : H_W(\boldsymbol{q}^{\frac{1}{2}}) \to \mathbf{Q}_W(\boldsymbol{q}^{\frac{1}{2}})$ be the symmetrizing trace defined by

$$\tau_{\boldsymbol{q}}(\beta_w) = \begin{cases} 1 & w = 1 \\ 0 & w \neq 1 \end{cases} \tag{1.3.6}$$

The corresponding symmetrizer is $\sum_{w \in W} \beta_w \otimes \beta_{w^{-1}} \in H_W \otimes H_W$. By Schur orthogonality, we can write

$$\tau_{\boldsymbol{q}} = \sum_{\phi \in \mathrm{Irr}(W)} \frac{1}{\boldsymbol{s}(\phi_{\boldsymbol{q}})} \phi_{\boldsymbol{q}}, \tag{1.3.7}$$

where the Schur element $\boldsymbol{s}(\phi_{\boldsymbol{q}}) \in \mathbf{Q}_W(\boldsymbol{q}^{\frac{1}{2}})$ is defined by

$$\boldsymbol{s}(\phi_{\boldsymbol{q}}) = \frac{1}{\phi(1)} \sum_{w \in W} \phi_{\boldsymbol{q}}(\beta_w)\phi_{\boldsymbol{q}}(\beta_{w^{-1}}). \tag{1.3.8}$$

In particular, $\boldsymbol{s}(1_{\boldsymbol{q}}) = \sum_{w \in W} \boldsymbol{q}^{|w|}$, the so-called Poincaré polynomial of (W, S).

1.3.4. Fake and Generic Degrees To each $\phi \in \mathrm{Irr}(W)$, we attach two polynomial invariants:

- Fix any realization V. The **fake degree** of ϕ is the ratio

  $$P_\phi(\boldsymbol{q}) = \frac{M_\phi(\boldsymbol{q} \mid V)}{M_1(\boldsymbol{q} \mid V)}. \tag{1.3.9}$$

  Springer showed [101, §2.5-2.6] that $P_\phi(\boldsymbol{q})$ is an element of $\mathbf{Z}[\boldsymbol{q}]$ that does not depend on V.

- The **generic degree** of ϕ is the ratio

  $$D_\phi(\boldsymbol{q}) = \frac{\boldsymbol{s}(1_{\boldsymbol{q}})}{\boldsymbol{s}(\phi_{\boldsymbol{q}})}. \tag{1.3.10}$$

  Benson–Curtis showed [43, Cor. 9.3.6, Rem. 9.3.7] that $D_\phi(\boldsymbol{q}) \in \mathbf{Q}_W[\boldsymbol{q}]$.

Using the fake degrees, Opdam defined an involution $\boldsymbol{o} = \boldsymbol{o}_W$ on the set $\mathrm{Irr}(W)$ [91, 448]. (In Opdam's paper, $\boldsymbol{o}$ is denoted j.) It is uniquely determined by the following properties:

1. If $W = W_1 \times W_2$, then $\boldsymbol{o}_W = \boldsymbol{o}_{W_1} \times \boldsymbol{o}_{W_2}$.

2. $P_{\boldsymbol{o}(\phi)}(\boldsymbol{q}) = \boldsymbol{q}^{2\deg P_\phi} P_\phi(\boldsymbol{q}^{-1})$ for all ϕ.

3. If the coefficients of $P_\phi(\boldsymbol{q})$ are palindromic, then $\boldsymbol{o}$ fixes ϕ.

As it turns out, $\boldsymbol{o}$ is the identity in every irreducible type except E_7, E_8, H_3, and H_4. For these exceptions, the action of $\boldsymbol{o}$ is explicitly stated in [79, 493] (see also [43, 296]). It commutes with sign twist: $\boldsymbol{o}(\varepsilon\phi) = \varepsilon\boldsymbol{o}(\phi)$ for all $\phi \in \mathrm{Irr}(W)$.

1.3.5. The Exotic Fourier Transform Fake and generic degrees are related by a mysterious pairing that we now briefly sketch. In [70, 73, 74], Lusztig assigned to every finite Coxeter group W a finite set $\mathrm{UCh}(W)$, equipped with:

1. An embedding $\mathrm{Irr}(W) \to \mathrm{UCh}(W)$ that we denote $\phi \mapsto \rho_\phi$.

2. A function $\mathrm{UCh}(W) \to \mathbf{Q}_W[\boldsymbol{q}]$ that we denote $\rho \mapsto D_\rho(\boldsymbol{q})$. It satisfies $D_{\rho_\phi}(\boldsymbol{q}) = D_\phi(\boldsymbol{q})$ for all $\phi \in \mathrm{Irr}(W)$.

3. A hermitian unitary pairing

$$\{-,-\} : \mathrm{UCh}(W) \times \mathrm{UCh}(W) \to \mathbf{Q}_W \tag{1.3.11}$$

   known as the **exotic Fourier transform**.

(In type H_4, his work was completed by Malle [78].) For W the Weyl group of a reductive group over a finite field, Lusztig used this data to relate *almost-characters* parametrized by $\mathrm{Irr}(W)$ with *unipotent irreducible characters* parametrized by $\mathrm{UCh}(W)$. We refer to [70] for details. For general W, Lusztig proposed a list of postulates that characterized the data uniquely [73, §2].

Example 1.3.1. In type A, it turns out that $\mathrm{UCh}(W) = \mathrm{Irr}(W)$ and $\{-,-\}$ is the Kronecker delta on $\mathrm{Irr}(W)$.

In this paper, we will only use the restriction of the exotic Fourier transform to a pairing on $\mathrm{Irr}(W)$. Thus we will write $\{\phi, \psi\} = \{\rho_\phi, \rho_\psi\}$ for all $\phi, \psi \in \mathrm{Irr}(W)$:

$$\{-, -\} : \mathrm{Irr}(W) \times \mathrm{Irr}(W) \to \mathbf{Q}_W. \tag{1.3.12}$$

Below, we collect the properties of $\mathrm{UCh}(W)$ and $\{-, -\}$ that we will need.

Property 1.3.2. *If* $W = W_1 \times W_2$*, then* $\mathrm{UCh}(W) = \mathrm{UCh}(W_1) \times \mathrm{UCh}(W_2)$*. The other data attached to* $\mathrm{UCh}(W)$*, including* $\{-, -\}$*, is similarly multiplicative.*

Property 1.3.3. *We have*

$$D_\phi(\boldsymbol{q}) = \sum_{\psi \in \mathrm{Irr}(W)} \{\phi, \psi\} P_{\boldsymbol{o}(\psi)}(\boldsymbol{q}) \tag{1.3.13}$$

for all $\phi \in \mathrm{Irr}(W)$*, where* $\boldsymbol{o}$ *is the involution from the previous subsection.*

Property 1.3.4. *We have* $\{\varepsilon\phi, \varepsilon\psi\} = \{\phi, \psi\}$ *for all* $\phi, \psi \in \mathrm{Irr}(W)$*.*

Remark 1.3.5. We collect references for these statements.

- Property 1.3.2 is Postulate 2.3 in [73].
- Property 3.2.2 is cited as Theorem 4.3 in [79].
- Property 3.2.3 does not seem to be stated in the literature. For Weyl groups, it is a consequence of Alvis–Curtis–Kawanaka duality [19, §8.2]. For general W, we can check it by using Property 1.3.2 to reduce to the irreducible case.

Remark 1.3.6. Property 3.2.2 is stated inaccurately in a few places in the literature. As noted by Marberg in [79], Carter's book [19, §13.6] omits the involution $\boldsymbol{o}$. This omission also occurs in Gomi's paper [46, 580], which we need in the next section.

Recall that in the character decomposition of the symmetrizing trace $\tau_{\boldsymbol{q}}$ on $H_W(\boldsymbol{q}^{\frac{1}{2}})$, the weights take the form $\frac{1}{\boldsymbol{s}(\phi_{\boldsymbol{q}})}$. Using Property 3.2.2, we can re-express these weights in terms of Molien series and the exotic Fourier transform.

Proposition 1.3.7. *Let V be a realization of (W, S) of rank r. Then*

$$\frac{1}{\boldsymbol{s}(\phi_{\boldsymbol{q}})} = (1-\boldsymbol{q})^r \sum_{\psi \in \mathrm{Irr}(W)} \{\phi, \psi\} M_{\boldsymbol{o}(\psi)}(\boldsymbol{q} \mid V) \tag{1.3.14}$$

for all $\phi \in \mathrm{Irr}(W)$.

Proof. By Property 3.2.2, it remains to prove $\boldsymbol{s}(1_{\boldsymbol{q}}) M_1(\boldsymbol{q} \mid V) = (1-\boldsymbol{q})^{-r}$.

Indeed, if $d_1, \dots, d_r$ are the invariant degrees of the W-action on V [43, 149], then

$$\boldsymbol{s}(1_{\boldsymbol{q}}) = \prod_{1 \le i \le r} \frac{1-\boldsymbol{q}^{d_i}}{1-\boldsymbol{q}} \quad \text{and} \quad M_1(\boldsymbol{q} \mid V) = \prod_{1 \le i \le r} \frac{1}{1-\boldsymbol{q}^{d_i}}. \tag{1.3.15}$$

The left-hand side is a formula of Bott–Solomon for the Poincaré polynomial of (W, S) [100]; the right-hand side holds by construction. □

1.4 Numerical Invariants

1.4.1. Markov Traces Recall a link in a 3-manifold is a disjoint union of (finitely many) embedded circles. The **HOMFLY series** of a link $\lambda \subseteq \mathbf{R}^3$ is a bivariate Laurent series that only depends on its isotopy class. We will write it in terms of the variables a and $\boldsymbol{q}^{\frac{1}{2}}$:

$$\textsc{homfly}(\lambda) \in \mathbf{Z}[(\boldsymbol{q}^{\frac{1}{2}} - \boldsymbol{q}^{-\frac{1}{2}})^{\pm 1}, a^{\pm 1}]. \tag{1.4.1}$$

There are many different constructions of this invariant [41]. Ocneanu's approach relies on Alexander's theorem, stating that every link in $\mathbf{R}^3$ is the planar closure of a braid. One constructs for each $n \ge 1$ a certain function on the Iwahori–Hecke algebra of type A_{n-1}, now known as a Markov trace [56]. If λ is the planar closure of $\beta \in Br_n$, then the HOMFLY series of λ is the Markov trace of β, multiplied by a factor depending only on $|\beta|$ and n.

Y. Gomi generalized Ocneanu's Markov traces in a uniform way beyond type A, extending work of Geck–Lambropoulou in type BC [46]. We follow Gomi's conventions here.

Let (W, S) be any finite Coxeter system. If $S' \subseteq S$, then the subgroup $W' \subseteq W$ generated by S' is a parabolic subgroup, meaning (W', S') is again a Coxeter system and $H_{W'} \subseteq H_W$. A **Markov trace** on H_W is a $\mathbf{Z}[\boldsymbol{q}^{\pm\frac{1}{2}}]$-linear function

$$\text{(1.4.2)} \qquad \textsc{tr} : H_W \to \mathbf{Z}[\boldsymbol{q}^{\pm\frac{1}{2}}](a)$$

such that:

1. $\textsc{tr}(1) = 1$.
2. $\textsc{tr}(\beta\gamma) = \textsc{tr}(\gamma\beta)$ for all $\beta, \gamma \in H_W$.
3. We have

$$\text{(1.4.3)} \qquad \textsc{tr}(\beta_s^{\pm 1}\gamma) = -a^{\mp 1}\left(\frac{\boldsymbol{q}^{\frac{1}{2}} - \boldsymbol{q}^{-\frac{1}{2}}}{a - a^{-1}}\right)\textsc{tr}(\gamma)$$

for all $s \in S$ and $\gamma \in H_{W'}$, where W' is the subgroup of W generated by $S \setminus s$.

Axioms (2) and (3) are also known as the first and second Markov moves, respectively.

Theorem 1.4.1 (Ocneanu)**.** *These axioms uniquely define a Markov trace* $\textsc{tr}$ *on the Iwahori–Hecke algebra of type* A_{n-1} *for all* $n \geq 2$*. If we set*

$$\text{(1.4.4)} \qquad \textsc{homfly}(\beta) = (-a)^{|\beta|}\left(\frac{a - a^{-1}}{\boldsymbol{q}^{\frac{1}{2}} - \boldsymbol{q}^{-\frac{1}{2}}}\right)^{n-1}\textsc{tr}(\beta)$$

for $\beta \in Br_n$*, then* $\textsc{homfly}(\beta)$ *only depends on the isotopy class of the planar closure of* β*.*

Suppose a Markov trace on H_W exists. After base changing from $\mathbf{Z}[\boldsymbol{q}^{\pm\frac{1}{2}}]$ to $\mathbf{Q}_W(\boldsymbol{q}^{\frac{1}{2}})$, we can view it as a trace on $H_W(\boldsymbol{q}^{\frac{1}{2}})$. So it can be written

$$\text{(1.4.5)} \qquad \textsc{tr} = \sum_{\phi \in \mathrm{Irr}(W)} \textsc{tr}_\phi \phi_{\boldsymbol{q}}$$

for some weights $\textsc{tr}_\phi \in \mathbf{Q}_W(\boldsymbol{q}^{\frac{1}{2}})(a)$.

The axioms for $\textsc{tr}$ hint that it is a bivariate version of the symmetrizing trace $\tau_{\boldsymbol{q}}$ on $H_W(\boldsymbol{q}^{\frac{1}{2}})$. Proposition 1.3.7 then suggests that its weights $\textsc{tr}_\phi$ should involve bivariate Molien series. These observations motivate the form of Gomi's Markov weights. Below, let

$$\begin{aligned} M_\phi(\boldsymbol{q}, x \mid V) &= \sum_{i,j\geq 0} (-x)^i \boldsymbol{q}^j (\phi, \Lambda^i(V) \otimes \mathrm{Sym}^j(V))_W \\ &= \frac{1}{|W|} \sum_{w \in W} \phi(w^{-1}) \frac{\det(1 - xw \mid V)}{\det(1 - \boldsymbol{q}w \mid V)}. \end{aligned} \tag{1.4.6}$$

Theorem 1.4.2 (Gomi)**.** *Let V be a realization of (W, S) of rank r. Then the weights*

$$\textsc{tr}_\phi = \left(\frac{1-\boldsymbol{q}}{1-a^{-2}}\right)^r \sum_{\psi \in \mathrm{Irr}(W)} \{\phi, \psi\} M_{\boldsymbol{o}(\psi)}(\boldsymbol{q}, a^{-2} \mid V) \tag{1.4.7}$$

define a Markov trace on H_W, independent of the choice of V.

This theorem is a bivariate analogue of Proposition 1.3.7. Above, we have fixed Gomi's statement from [46] to incorporate the involution $\boldsymbol{o}$, *cf.* Remark 1.3.6.

1.4.2. Annular Characters Henceforth, $\textsc{tr}$ *will always denote the Markov trace in Gomi's theorem*, as well as its pullback to a function on Br_W. We fix a realization V of rank r.

We will construct a refinement of $\textsc{tr}$ where the a-grading is replaced by a W-isotypic decomposition. First, let $\mathbb{A} : H_W(\boldsymbol{q}^{\frac{1}{2}}) \to \mathrm{K}_0(W) \otimes \mathbf{Q}_W(\boldsymbol{q}^{\frac{1}{2}})$ be the trace defined by:

$$\mathbb{A}(\beta) = \sum_{\phi, \psi \in \mathrm{Irr}(W)} \{\phi, \psi\} \phi_{\boldsymbol{q}}(\beta) \boldsymbol{o}(\psi). \tag{1.4.8}$$

Definition 1.4.3. Let $\textsc{ann}_V : Br_W \to \mathrm{K}_0(W) \otimes \mathbf{Q}_W(\boldsymbol{q}^{\frac{1}{2}})$ be the class function defined by:

$$\textsc{ann}_V(\beta) = (-\boldsymbol{q}^{\frac{1}{2}})^{|\beta|} \mathbb{A}(\beta) \cdot \varepsilon M(\boldsymbol{q} \mid V). \tag{1.4.9}$$

We say that $\textsc{ann}_V(\beta)$ is the **annular character** of β with respect to V. We also set

$$\textsc{ann} = (1-\boldsymbol{q})^{\dim(V^W)}\textsc{ann}_V. \tag{1.4.10}$$

Equivalently, $\textsc{ann}(\beta)$ is the annular character of β with respect to $V/(V^W)$.

We introduce the trace $\mathbb{A}$ for convenience, even though it is $\textsc{ann}_V$ that appears in our "real-life" applications. Since $M(\boldsymbol{q} \mid V)$ is invertible in the ring $\mathrm{K}_0(W)[\![\boldsymbol{q}]\!]$, we can always recover $\mathbb{A}(\beta)$ from $\textsc{ann}_V(\beta)$.

We now prove Proposition 1.1.1 from the introduction. We must show that:

$$\textsc{tr}(\beta) = (-\boldsymbol{q}^{\frac{1}{2}})^{-|\beta|}\left(\frac{1-\boldsymbol{q}}{1-a^2}\right)^r \sum_{0\leq i\leq r} (-a^2)^i(\Lambda^i(V), \textsc{ann}_V(\beta))_W. \tag{1.4.11}$$

The following lemma lets us write the weights of $\textsc{tr}$ in terms of a^2 rather than a^{-2}:

Lemma 1.4.4. *We have*

$$\frac{1}{(1-x^{-1})^r}\sum_{i\geq 0}(-x^{-1})^i\,\Lambda^i(V) = \frac{\varepsilon}{(1-x)^r}\sum_{i\geq 0}(-x)^i\,\Lambda^i(V) \tag{1.4.12}$$

in $\mathrm{K}_0(W)(x)$.

Proof. On the left-hand side, multiply both the top and the bottom by $(-x)^r$. Then use the identity $\Lambda^{r-i}(V) = \varepsilon\,\Lambda^i(V)$. □

Proof of Proposition 1.1.1. We expand:

$$\begin{aligned}\left(\frac{1-a^2}{1-\boldsymbol{q}}\right)^r \textsc{tr}_\phi &= \sum_{\psi\in\mathrm{Irr}(W)}\{\phi,\psi\}M_{\varepsilon\boldsymbol{o}(\psi)}(\boldsymbol{q}, a^2 \mid V)\\ &= \sum_{\psi\in\mathrm{Irr}(W)}\{\phi,\psi\}\sum_{i,j\geq 0}(-a^2)^i\boldsymbol{q}^j(\Lambda^i(V), \varepsilon\boldsymbol{o}(\psi)\cdot\mathrm{Sym}^j(V))_W.\end{aligned} \tag{1.4.13}$$

The first equality uses the lemma with $x = a^2$; the second uses the self-duality of $\mathrm{Sym}^j(V)$

as a representation of W. By comparing the terms in the last expression with the terms that make up $\textsc{ann}_V(\beta)$, we get the result. $\square$

1.4.3. Proof of Theorem 1.1.3 As motivation for Theorem 1.1.3, recall that the HOMFLY series of a link λ satisfies the following properties:

1. *Symmetry.* $\textsc{homfly}(\lambda)$ is invariant under $\boldsymbol{q}^{\frac{1}{2}} \mapsto -\boldsymbol{q}^{-\frac{1}{2}}$. More strongly, it is an element of $\mathbf{Z}[(\boldsymbol{q}^{\frac{1}{2}} - \boldsymbol{q}^{-\frac{1}{2}})^{\pm 1}, a^{\pm 1}]$.
2. *Parity.* $\textsc{homfly}(\lambda)$ is a $\boldsymbol{q}^{\frac{1}{2}}$-monomial shift of an element of $\mathbf{Z}[\![\boldsymbol{q}]\!][a^{\pm 1}]$, not just $\mathbf{Z}[\![\boldsymbol{q}^{\frac{1}{2}}]\!][a^{\pm 1}]$.
3. *Lowest Degree.* If λ is the planar closure of β, then the term of $\textsc{homfly}(\lambda)$ of lowest degree in both $\boldsymbol{q}^{\frac{1}{2}}$ and a is equal to $(\boldsymbol{q}^{\frac{1}{2}} a^{-1})^{r-|\beta|}$.

Below, Corollaries 1.4.6, 1.4.9, 1.4.11 are respectively the analogues of properties (1), (2), (3) for $\textsc{ann}$. Our proof shows that for Weyl groups, Corollary 1.4.6 is a consequence of Alvis–Curtis–Kawanaka duality via Property 3.2.3.

Lemma 1.4.5. $\mathbb{A}|_{\boldsymbol{q}^{-1/2} \to -\boldsymbol{q}^{1/2}} = \varepsilon \otimes \mathbb{A}$.

Proof. For all $\beta \in H_W(\boldsymbol{q}^{\frac{1}{2}})$ and $\psi \in \mathrm{Irr}(W)$, we have $(\boldsymbol{o}(\psi), \mathbb{A}(\beta))_W = \sum_{\phi,\psi} \{\phi, \psi\} \phi_q(\beta)$. By Property 3.2.3,

$$\begin{aligned} (\boldsymbol{o}(\psi), \varepsilon\mathbb{A}(\beta))_W = (\boldsymbol{o}(\varepsilon\psi), \mathbb{A}(\beta))_W &= \sum_{\phi,\psi} \{\phi, \varepsilon\psi\} \phi_{\boldsymbol{q}}(\beta) \\ &= \sum_{\phi,\psi} \{\varepsilon\phi, \varepsilon\psi\} (\varepsilon \otimes \phi)_{\boldsymbol{q}}(\beta) \\ &= \sum_{\phi,\psi} \{\phi, \psi\} (\varepsilon\phi)_{\boldsymbol{q}}(\beta). \end{aligned} \tag{1.4.14}$$

Since $(\varepsilon\phi)_{\boldsymbol{q}}(\beta)$ and $\phi_{\boldsymbol{q}}(\beta)$ are interchanged under $\boldsymbol{q}^{\frac{1}{2}} \mapsto -\boldsymbol{q}^{-\frac{1}{2}}$, we are done. $\square$

Corollary 1.4.6. $(\boldsymbol{q}^{\frac{1}{2}})^{r-|\beta|} \textsc{ann}_V(\beta) \in \mathrm{K}_0(W) \otimes \mathbf{Q}_W(\boldsymbol{q}^{\frac{1}{2}} - \boldsymbol{q}^{-\frac{1}{2}})$.

Proof. It suffices to show that $(\boldsymbol{q}^{\frac{1}{2}})^{r-|\beta|}\textsc{ann}_V(\beta)$ is invariant under $\boldsymbol{q}^{\frac{1}{2}} \mapsto -\boldsymbol{q}^{-\frac{1}{2}}$. By the previous lemma, we reduce to showing that $\boldsymbol{q}^{\frac{1}{2}} \mapsto -\boldsymbol{q}^{-\frac{1}{2}}$ sends

$$\boldsymbol{q}^{\frac{r}{2}} M(\boldsymbol{q} \mid V) \mapsto \varepsilon \cdot \boldsymbol{q}^{\frac{r}{2}} M(\boldsymbol{q} \mid V). \tag{1.4.15}$$

Observe that the inverse of $M(\boldsymbol{q} \mid V)$ in the multiplicative group $\mathrm{K}_0(W)[\![\boldsymbol{q}]\!]^\times$ is the element $\sum_{i\geq 0} (-\boldsymbol{q})^i \Lambda^i(V)$. So applying Lemma 1.4.4 with $x = \boldsymbol{q}$ concludes the proof. □

Remark 1.4.7. For Weyl groups, we expect that the invariance of ANN under $\boldsymbol{q}^{\frac{1}{2}} \mapsto -\boldsymbol{q}^{-\frac{1}{2}}$ arises from some form of "curious Lefschetz/Poincaré duality" in the sense of Hausel–Rodriguez-Villegas [53, 54], via the geometric interpretation of ANN in Section 1.7. In the papers we have just cited, Alvis–Curtis–Kawanaka duality is used to prove curious Lefschetz for the Hodge numbers of character varieties in type A.

Lemma 1.4.8. *For all $\beta \in Br_W$, we have $\mathbb{A}(\beta) \in \boldsymbol{q}^{-\frac{|\beta|}{2}}\mathrm{K}_0(W)[\boldsymbol{q}]$.*

Proof. Let $\tilde{\beta} = \boldsymbol{q}^{\frac{|\beta|}{2}}\beta$. It suffices to show that the element

$$(\boldsymbol{o}(\psi), \mathbb{A}(\tilde{\beta}))_W = \sum_{\phi,\psi \in \mathrm{Irr}(W)} \{\phi, \psi\}\phi_{\boldsymbol{q}}(\tilde{\beta}) \in \mathbf{Q}_W(\boldsymbol{q}^{\frac{1}{2}}) \tag{1.4.16}$$

actually belongs to $\mathbf{Z}[\boldsymbol{q}]$ for all $\psi \in \mathrm{Irr}(W)$. We make a series of reductions:

By the multiplicativity postulate for $\{-,-\}$, we can assume that (W, S) is irreducible.

By the defining relations of H_W, we can expand $\tilde{\beta}$ as a $\mathbf{Z}[\boldsymbol{q}]$-linear combination of the elements $\tilde{\beta}_w = \boldsymbol{q}^{\frac{|\beta_w|}{2}}\beta_w$ for $w \in W$. So we can assume that $\tilde{\beta} = \tilde{\beta}_w$ for some w.

We can now check the non-crystallographic types case by case. So we can assume that (W, S) is crystallographic.

A formula of Lusztig that we cite in Section 1.7 (Theorem 2.2.5) shows that in the crystallographic case, $(\boldsymbol{o}(\psi), \mathbb{A}(\tilde{\beta}))_W \in \mathbf{Z}[\boldsymbol{q}^{\pm\frac{1}{2}}]$. So it remains to show $(\boldsymbol{o}(\psi), \mathbb{A}(\tilde{\beta}))_W \in \mathbf{Q}[\boldsymbol{q}]$.

Since $\{-,-\}$ takes values in $\mathbf{Q}_W = \mathbf{Q}$, it is enough to show that $\phi_{\boldsymbol{q}}(\tilde{\beta}_w) \in \mathbf{Q}[\boldsymbol{q}]$ for all ϕ. When (W, S) is not of type E_7 or E_8, this is a theorem of Benson–Curtis [7]. Otherwise, it fails:

In types E_7 and E_8, there are two characters ϕ for which we can only ensure $\phi_{\boldsymbol{q}}(\tilde{\beta}_w) \in \mathbf{Q}[\boldsymbol{q}^{\frac{1}{2}}]$. However, it turns out that $(\boldsymbol{o}(\psi), \mathbb{A}(\tilde{\beta}_w))_W \in \mathbf{Q}[\boldsymbol{q}]$ for all ψ and w anyway. ☐

Corollary 1.4.9. *ANN takes values in* $\mathrm{K}_0(W)[\![\boldsymbol{q}]\!]$.

Lemma 1.4.10. *For all* $\beta \in H_W$*, we have* $(\varepsilon, (-\boldsymbol{q}^{\frac{1}{2}})^{|\beta|}\mathbb{A}(\beta))_W|_{\boldsymbol{q}^{1/2}\to 0} = 1$.

Proof. The axioms for $\boldsymbol{o}$ imply $\boldsymbol{o}(\varepsilon) = \varepsilon$, and by [70, Ch. 4] and [79, §2.4], we have $\{\phi, \varepsilon\} = 0$ for all $\phi \neq \varepsilon$. Now the claim follows from the formula $\varepsilon_{\boldsymbol{q}}(\beta) = (-\boldsymbol{q}^{\frac{1}{2}})^{-|\beta|}$. ☐

Corollary 1.4.11. *For all* $\beta \in Br_W$*, we have* $(1, \mathrm{ANN}(\beta))_W|_{\boldsymbol{q}\to 0} = 1$.

Proof. We have $M(0 \mid V) = 1$ from which $(1, \mathrm{ANN}(\beta))_W|_{\boldsymbol{q}\to 0} = (1, \mathrm{ANN}(\beta)|_{\boldsymbol{q}\to 0})_W = 1$. ☐

Corollary 1.4.12. *For all* $\beta \in Br_W$*, the annular character* $\mathrm{ANN}(\beta)$ *is a rational function in* $\boldsymbol{q}$ *of degree* $|\beta| - r$.

Proof. Corollary 1.4.6 shows that $(\boldsymbol{q}^{\frac{1}{2}})^{r-|\beta|}\mathrm{ANN}(\beta)$ is rational in $\boldsymbol{q}^{\frac{1}{2}} - \boldsymbol{q}^{-\frac{1}{2}}$. So $\mathrm{ANN}(\beta)$ is rational in $\boldsymbol{q}^{\frac{1}{2}}$. But Corollary 1.4.9 shows it is also a power series in $\boldsymbol{q}$, so it must be rational in $\boldsymbol{q}$. Finally, Corollary 1.4.11 shows $\mathrm{ANN}(\beta)$ has nonzero constant term, so the invariance of $(\boldsymbol{q}^{\frac{1}{2}})^{r-|\beta|}\mathrm{ANN}(\beta)$ under $\boldsymbol{q}^{\frac{1}{2}} \mapsto -\boldsymbol{q}^{-\frac{1}{2}}$ forces $\deg_{\boldsymbol{q}} \mathrm{ANN}(\beta) = |\beta| - r$. ☐

Together, Corollaries 1.4.6, 1.4.9, 1.4.11 imply Theorem 1.1.3. In the paper [106], we establish even further constraints on the image of ANN for the crystallographic case.

1.5 Examples

1.5.1. The Trivial Braid Let $1 = \beta_1$ denote the identity of Br_W. We will calculate $\mathrm{ANN}_V(1)$ for arbitrary (W, S). For all $\phi \in \mathrm{Irr}(W)$, we observe that:

- $\phi_{\boldsymbol{q}}(1) = \phi(1) = P_\phi(1) = D_\phi(1)$.
- $\boldsymbol{o}(\phi)(1) = \phi(1)$.

Applying Property 3.2.2, we deduce:

$$\mathbb{A}(1) = \sum_{\phi,\psi\in \mathrm{Irr}(W)} \{\phi,\psi\}\phi(1)\boldsymbol{o}(\psi) = \sum_{\psi\in \mathrm{Irr}(W)} \psi(1)\boldsymbol{o}(\psi) = \sum_{\psi\in \mathrm{Irr}(W)} \psi(1)\psi. \tag{1.5.1}$$

That is, $\mathbb{A}(1)$ is the regular representation of W, placed in $\boldsymbol{q}$-degree 0. In particular, it satisfies $\mathbb{A}(1)\cdot\phi = \phi(1)\mathbb{A}(1)$ for all ϕ. Using this property, we find:

$$\textsc{ann}_V(1) = \mathbb{A}(1) \sum_{\phi\in \mathrm{Irr}(W)} \phi(1)M_\phi(\boldsymbol{q} \mid V). \tag{1.5.2}$$

Since $\sum_\phi \phi(1)M_\phi(\boldsymbol{q} \mid V) = \mathrm{tr}(1 \mid M(\boldsymbol{q} \mid V)) = (1-\boldsymbol{q})^{-r}$, we arrive at:

$$\textsc{ann}_V(1) = \frac{1}{(1-\boldsymbol{q})^r} \sum_{\psi\in \mathrm{Irr}(W)} \psi(1)\psi. \tag{1.5.3}$$

In the paper [107], we generalize this example significantly: We calculate $\textsc{ann}_V(\beta)$ for any braid β such that some root of β is also a root of the so-called full twist.

1.5.2. Type A_1 Set $W = S_2$ and $S = \{s\}$. We have $\mathrm{Irr}(W) = \{1, \varepsilon\}$. We compute that:

$$M(\boldsymbol{q} \mid V) = \frac{1+\boldsymbol{q}\varepsilon}{(1-\boldsymbol{q})^r(1+\boldsymbol{q})}. \tag{1.5.4}$$

At the same time, $Br_W = \langle\beta_s\rangle$. By induction, we can obtain a formula for β_s^m in terms of $1 = \beta_1$ and β_s. Since $\{-,-\}$ and $\boldsymbol{o}$ are trivial in type A, we compute:

$$\mathbb{A}(\beta_s^m) = \sum_{\phi\in \mathrm{Irr}(W)} \phi_{\boldsymbol{q}}(\beta_s^m)\phi = (\boldsymbol{q}^{\frac{1}{2}})^m + (-\boldsymbol{q}^{-\frac{1}{2}})^m\varepsilon. \tag{1.5.5}$$

Altogether, we have:

$$\textsc{ann}(\beta_s^m) = \frac{(1-(-\boldsymbol{q})^{m+1}) + (\boldsymbol{q}+(-\boldsymbol{q})^m)\varepsilon}{1-\boldsymbol{q}^2}. \tag{1.5.6}$$

For instance, $\text{ANN}(\beta_s^{2k+1}) = (1 + \boldsymbol{q}^2 + \cdots + \boldsymbol{q}^{2k}) + (\boldsymbol{q} + \boldsymbol{q}^3 + \cdots + \boldsymbol{q}^{2k-1})\varepsilon$.

1.5.3. Type A_2 Set $W = S_3$ and $S = \{s, t\}$. We have $\mathrm{Irr}(W) = \{1, \phi, \varepsilon\}$, where $\phi(1) = 2$. We compute:

$$(1.5.7) \qquad \begin{aligned} M(\boldsymbol{q} \mid V) &= \frac{\sum_{\phi \in \mathrm{Irr}(W)} P_\phi(\boldsymbol{q})\phi}{(1-\boldsymbol{q})^r(1+\boldsymbol{q})(1+\boldsymbol{q}+\boldsymbol{q}^2)} \\ &= \frac{1 + (\boldsymbol{q}+\boldsymbol{q}^2)\phi + \boldsymbol{q}^3\varepsilon}{(1-\boldsymbol{q})^r(1+\boldsymbol{q})(1+\boldsymbol{q}+\boldsymbol{q}^2)}. \end{aligned}$$

The conjugacy classes of W are $\{1\}, \{s, t, w_0\}, \{st, ts\}$. Below, we list $(\psi, \mathbb{A}(\beta_w))_W = \psi_{\boldsymbol{q}}(\beta_w)$ for all $w \in W$ and $\psi \in \mathrm{Irr}(W)$.

(1.5.8)

	$\{1\}$	$\{s, t, w_0\}$	$\{st, ts\}$
1	1	$\boldsymbol{q}^{\frac{1}{2}}$	$\boldsymbol{q}$
ϕ	2	$\boldsymbol{q}^{\frac{1}{2}} - \boldsymbol{q}^{-\frac{1}{2}}$	-1
ε	1	$-\boldsymbol{q}^{-\frac{1}{2}}$	$\boldsymbol{q}^{-1}$

From this matrix, we can compute ANN on the standard basis of H_W.

$$(1.5.9) \qquad \text{ANN}(\beta_w) = \begin{cases} (1-\boldsymbol{q})^{-2}(1+2\phi+\varepsilon) & w = 1 \\ (1-\boldsymbol{q})^{-1}(1+\phi) & w \in \{s, t, w_0\} \\ 1 & w \in \{st, ts\} \end{cases}$$

1.5.4. Type BC_2 Set $W = Dih_4$ and $S = \{s, t\}$. We have $\mathrm{Irr}(W) = \{1, \delta, \phi, \varepsilon\delta, \varepsilon\}$, where $\delta(1) = 1$ and $\phi(1) = 2$. We distinguish δ and $\varepsilon\delta$ by assuming that $\delta(s) = 1$. We compute:

$$(1.5.10) \qquad \begin{aligned} M(\boldsymbol{q} \mid V) &= \frac{\sum_{\phi \in \mathrm{Irr}(W)} P_\phi(\boldsymbol{q})\phi}{(1-\boldsymbol{q})^r(1+\boldsymbol{q})(1+\boldsymbol{q}+\boldsymbol{q}^2+\boldsymbol{q}^3)} \\ &= \frac{1 + \boldsymbol{q}^2\delta + (\boldsymbol{q}+\boldsymbol{q}^3)\phi + \boldsymbol{q}^2\varepsilon\delta + \boldsymbol{q}^4\varepsilon}{(1-\boldsymbol{q})^r(1+\boldsymbol{q})(1+\boldsymbol{q}+\boldsymbol{q}^2+\boldsymbol{q}^3)}. \end{aligned}$$

The conjugacy classes of W are $\{1\}, \{s, tst\}, \{t, sts\}, \{st, ts\}, \{w_0\}$. Below, we list $(\psi, \mathbb{A}(\beta_w))_W$ for all w and ψ. Note that in this setting, $\{-,-\}$ is nontrivial but $\boldsymbol{o}$ remains trivial.

(1.5.11)

	$\{1\}$	$\{s, tst\}$	$\{t, sts\}$	$\{st, ts\}$	$\{w_0\}$
1	1	$\boldsymbol{q}^{\frac{1}{2}}$	$\boldsymbol{q}^{\frac{1}{2}}$	$\boldsymbol{q}$	$\boldsymbol{q}^2$
δ	1	$\boldsymbol{q}^{\frac{1}{2}}$	$-\boldsymbol{q}^{-\frac{1}{2}}$	0	-1
ϕ	2	$\boldsymbol{q}^{\frac{1}{2}} - \boldsymbol{q}^{-\frac{1}{2}}$	$\boldsymbol{q}^{\frac{1}{2}} - \boldsymbol{q}^{-\frac{1}{2}}$	-1	0
$\varepsilon\delta$	1	$-\boldsymbol{q}^{-\frac{1}{2}}$	$\boldsymbol{q}^{\frac{1}{2}}$	0	-1
ε	1	$-\boldsymbol{q}^{-\frac{1}{2}}$	$-\boldsymbol{q}^{-\frac{1}{2}}$	$\boldsymbol{q}^{-1}$	$\boldsymbol{q}^{-2}$

The values of $\textsc{ann}(\beta_w)$ are:

$$\textsc{ann}(\beta_w) = \begin{cases} (1-\boldsymbol{q})^{-2}(1+\delta+2\phi+\varepsilon\delta+\varepsilon) & w = 1 \\ (1-\boldsymbol{q})^{-1}(1+\delta+\phi) & w \in \{s, tst\} \\ (1-\boldsymbol{q})^{-1}(1+\varepsilon\delta+\phi) & w \in \{t, sts\} \\ 1 & w \in \{st, ts\} \\ 1+\boldsymbol{q}^2+\boldsymbol{q}\phi & w = w_0 \end{cases} \tag{1.5.12}$$

1.5.5. An Iterated Torus Braid We only describe one nontrivial example in type A_3. Set $W = S_4$ and $S = \{s, t, u\}$. We have $\mathrm{Irr}(W) = \{1, \phi, \psi, \varepsilon\phi, \varepsilon\}$, where $\phi(1) = 3$ and $\psi(1) = 2$. We distinguish ϕ from $\varepsilon\phi$ by assuming ϕ is the character of a realization of W. Let

$$\beta = (\beta_s\beta_t\beta_u)^6\beta_s \in Br_W = Br_4. \tag{1.5.13}$$

Using Sage, we calculate:

$$\mathbb{A}(\beta) = \boldsymbol{q}^{\frac{19}{2}} - \boldsymbol{q}^{\frac{7}{2}}\phi + (\boldsymbol{q}^{\frac{1}{2}} - \boldsymbol{q}^{-\frac{1}{2}})\psi + \boldsymbol{q}^{-\frac{7}{2}}\varepsilon\phi - \boldsymbol{q}^{-\frac{19}{2}}\varepsilon. \tag{1.5.14}$$

The nonzero coefficients of $\textsc{ann}(\beta)$ are listed below:

(1.5.15)

	$\boldsymbol{q}^0$	$\boldsymbol{q}^1$	$\boldsymbol{q}^2$	$\boldsymbol{q}^3$	$\boldsymbol{q}^4$	$\boldsymbol{q}^5$	$\boldsymbol{q}^6$	$\boldsymbol{q}^7$	$\boldsymbol{q}^8$	$\boldsymbol{q}^9$	$\boldsymbol{q}^{10}$	$\boldsymbol{q}^{11}$	$\boldsymbol{q}^{12}$	$\boldsymbol{q}^{13}$	$\boldsymbol{q}^{14}$	$\boldsymbol{q}^{15}$	$\boldsymbol{q}^{16}$
1	1		1	1	2	1	3	1	3	1	3	1	2	1	1		1
ϕ		1	1	2	2	4	3	5	3	5	3	4	2	2	1	1	
ψ			1		2	1	3	1	4	1	3	1	2		1		
$\varepsilon\phi$				1	1	2	2	3	2	3	2	2	1	1			
ε							1		1		1						

This table offers a vivid illustration of Theorem 1.1.3. It also gives us a measure of belief in Conjecture 1.1.4 (stating that $\beta \in Br_W^+$ implies $\textsc{ann}(\beta) \in \mathrm{K}_0^+(W)[\![\boldsymbol{q}]\!]$), one somewhat more striking than that of the previous examples.

Remark 1.5.1. Let λ be the planar closure of β. As a consequence of Proposition 1.1.1, the $1, \phi, \varepsilon\phi$, and ε rows of the table list the coefficients of $\textsc{homfly}(\lambda)$ up to shifting by $a^{2i}\boldsymbol{q}^{-8}$ for various i.

As it happens, λ is the link of the plane curve singularity given by $y = x^{\frac{3}{2}} + x^{\frac{7}{4}}$ in the complex (x, y)-plane. In particular, it is an example of an iterated torus knot.

1.6 The Hecke Category

1.6.1. Setup Now we assume that (W, S) is *crystallographic*. In this section, we review the categorifications of Br_W and H_W that Webster–Williamson used to construct Khovanov–Rozansky homology, and which we will use to define annular braid homology.

Fix a finite field $\mathbf{F} = \mathbf{F}_q$ and a prime $\ell > 0$ invertible in $\mathbf{F}$. We will make repeated use of the notations established in Appendix 1.9, including D^b, Perv, D^b_m, and Pur (see Definitions 1.9.1-1.9.2). In particular:

- We will use a subscript 0 to indicate stacks over $\mathbf{F}$ and sheaves over them. We omit the subscript to indicate their pullbacks to $\bar{\mathbf{F}}$.

- $\mathcal{C}hr$ is the **chromatography functor** that is the focus of Appendix 1.9.

In addition, we will always assume that the Grothendieck–Verdier six operations on D^b and D^b_m are derived; *we will often drop* R, L*, etc., from our notation.* For example, if f is a map, then we write $f_!, f_*$ for the functors $\mathrm{R}f_!, \mathrm{R}f_*$. Lastly, we fix a square root $q^{\frac{1}{2}} \in \bar{\mathbf{Q}}_\ell$, so that we can define the half-Tate twist $(\frac{1}{2})$, and set $\langle 1\rangle = [1](\frac{1}{2})$.

As W is crystallographic, we can assume that W is the Weyl group of a root datum $(\Phi, \mathbf{X}, \Phi^\vee, \mathbf{X}^\vee)$ and that V is the $\mathbf{Q}$-span of the character lattice $\mathbf{X}$, i.e.,

$$V = \mathbf{X} \otimes_{\mathbf{Z}} \mathbf{Q}. \tag{1.6.1}$$

The root datum defines a split reductive group G_0 over $\mathbf{F}$. Fix a split maximal torus $T_0 \subseteq G_0$ and a Borel subgroup $B_0 \subseteq G_0$ containing T_0. We can assume that $\mathbf{X}$ is the character lattice of T_0 and that S is the set of reflections corresponding to the system of simple roots of Φ defined by B_0.

Let $\mathcal{B}_0 = G_0/B_0$, the flag variety of G_0, and for all $w \in W$, let $O_{w,0}$ be the G_0-orbit of $\mathcal{B}_0 \times \mathcal{B}_0$ indexed by w in the Bruhat decomposition. Recall that the closure order on orbits is precisely the Bruhat order, meaning $\overline{O}_{w,0}$ is the union of all $O_{v,0}$ with $v \leq w$. Let

$$\mathcal{X}_0 = G_0\backslash(\mathcal{B}_0 \times \mathcal{B}_0), \tag{1.6.2}$$

and let $\mathcal{X}_{w,0}$ be the image of $O_{w,0}$ in $\mathcal{X}_0$. We write $j_w : \mathcal{X}_{w,0} \to \mathcal{X}_0$ for the inclusion map. For each $w \in W$, we define the following objects in $\mathsf{D}^b_m(\mathcal{X}_0)$:

$$\Delta_{w,0} = j_{w,!}\bar{\mathbf{Q}}_\ell\langle \dim \mathcal{X}_{w,0}\rangle, \tag{1.6.3}$$

$$\nabla_{w,0} = j_{w,*}\bar{\mathbf{Q}}_\ell\langle \dim \mathcal{X}_{w,0}\rangle, \tag{1.6.4}$$

$$\mathrm{IC}_{w,0} = j_{w,!*}\bar{\mathbf{Q}}_\ell\langle \dim \mathcal{X}_{w,0}\rangle, \tag{1.6.5}$$

where $\dim \mathcal{X}_{w,0} = |w| - \dim B_0$, the *stacky* dimension of $\mathcal{X}_{w,0}$. Since j_w factors through an affine open embedding $\mathcal{X}_0 \to \overline{\mathcal{X}}_0$, all of these complexes are perverse sheaves. Also, $\mathrm{IC}_{w,0}$ is simple and pure of weight 0.

The category $\mathsf{D}^b_m(\mathcal{X}_0)$ is equipped with a monoidal convolution $\odot$. In terms of the three projections

$$G_0\backslash(\mathcal{B}_0 \times \mathcal{B}_0 \times \mathcal{B}_0) \xrightarrow{\mathrm{pr}_{1,2},\mathrm{pr}_{1,3},\mathrm{pr}_{2,3}} G_0\backslash(\mathcal{B}_0 \times \mathcal{B}_0) = \mathcal{X}_0, \tag{1.6.6}$$

it is given by

$$K \odot L = \mathrm{pr}_{1,3,*}(\mathrm{pr}_{1,2}^* K \otimes \mathrm{pr}_{2,3}^* L)\langle \dim B_0 \rangle. \tag{1.6.7}$$

The unit object with respect to $\odot$ is $\Delta_{1,0} = \nabla_{1,0} = \mathrm{IC}_{1,0}$. Deligne's theorem on weights [26] implies that $\odot$ preserves the subcategory of pure complexes of weight 0 over $\mathcal{X}_0$. Thus, it induces a monoidal product on $\mathsf{Pur}(\mathcal{X})$.

1.6.2. Soergel Bimodules Soergel showed that the structure of the monoidal additive category $\mathsf{Pur}(\mathcal{X})$ is largely already encoded in the hypercohomology of its objects. Consider the graded $\bar{\mathbf{Q}}_\ell$-algebra:

$$\mathbf{B} = \mathrm{H}^*(\,\cdot\,/T, \bar{\mathbf{Q}}_\ell). \tag{1.6.8}$$

Explicitly, the cohomology of $\,\cdot\,/T$ is isomorphic to $\mathrm{Sym}^i(V \otimes \bar{\mathbf{Q}}_\ell)$ in degree $2i$ and vanishes in odd degrees. Via the composition of maps

$$\mathcal{X} \xrightarrow{\sim} B\backslash G/B \to T\backslash \cdot /T \xrightarrow{\sim} (\,\cdot\,/T) \times (\,\cdot\,/T), \tag{1.6.9}$$

the hypercohomology of any object of $\mathsf{D}^b(\mathcal{X})$ forms a bimodule over $\mathbf{B}$, equipped with $\mathbf{B}$-module gradings on both sides.

Let $\mathbf{B}\text{-}\mathsf{Mod}_{\mathsf{gr}}$ be the category of $\mathbf{B}$-bimodules equipped with a $\mathbf{B}$-module grading on the left. A **Soergel bimodule** is an object of $\mathbf{B}\text{-}\mathsf{Mod}_{\mathsf{gr}}$ that arises from the hypercohomology of some object of $\mathsf{Pur}(\mathcal{X})$ (by forgetting its grading on the right). Soergel proved [99, Thm. 15-17] that the functor $\mathrm{H}^*(\mathcal{X}, -) : \mathsf{Pur}(\mathcal{X}) \to \mathbf{B}\text{-}\mathsf{Mod}_{\mathsf{gr}}$ satisfies the following properties:

1. It takes the shift [1] to the cohomological shift.

2. It takes the convolution product $(-) \odot (-)$ to the tensor product $(-) \otimes_{\mathbf{B}} (-)$.

3. *(»Erweiterungssatz«)* It is fully faithful (i.e., induces graded isomorphisms of Hom-spaces).

For all $w \in W$, we have a Soergel bimodule

$$\mathbf{B}_w = \mathrm{H}^*(\mathcal{X}, \mathrm{IC}_w). \tag{1.6.10}$$

In particular, $\mathbf{B}_1 = \mathbf{B}$.

1.6.3. The Hecke Category For the purposes of this paper, the **Hecke category** $\mathsf{H} = \mathsf{H}_G$ is the target of the chromatography functor for $\mathcal{X}_0$ (see Appendix 1.9). That is,

$$\mathsf{H} = \mathsf{K}^b(\mathsf{Pur}(\mathcal{X})), \tag{1.6.11}$$

and chromatography is a functor

$$\mathcal{C}hr : \mathsf{D}^b_m(\mathsf{Perv}(\mathcal{X}_0)) \to \mathsf{H}. \tag{1.6.12}$$

The monoidal product $\odot$ on $\mathsf{Pur}(\mathcal{X})$ induces a monoidal product on H, which we again denote by $\odot$. If G_0 is semisimple, then we write H_W in place of H_G, abusing notation.

Let $\mathrm{K}_0(\mathsf{Pur}(\mathcal{X}))$ denote the *split* Grothendieck group of $\mathsf{Pur}(\mathcal{X})$, hence also of its homotopy category H. It forms a $\mathbf{Z}[\boldsymbol{q}^{\pm\frac{1}{2}}]$-algebra via the product $\odot$ and the algebra action

$$\boldsymbol{q}^{\frac{1}{2}} \mapsto [-1]. \tag{1.6.13}$$

The following is [103, Thm. 2.8]:

Theorem 1.6.1. *The monoidal additive category formed by* H *is a categorification of the Iwahori–Hecke algebra* H_W*. Explicitly, there is a* $\mathbf{Z}[\boldsymbol{q}^{\pm\frac{1}{2}}]$*-algebra isomorphism*

$$\mathrm{K}_0(\mathsf{Pur}(\mathcal{X})) \xrightarrow{\sim} H_W \tag{1.6.14}$$

that sends $\mathrm{IC}_s \mapsto \boldsymbol{q}^{-\frac{1}{2}} + \beta_s = \boldsymbol{q}^{\frac{1}{2}} + \beta_s^{-1}$.

Note that the simple objects $\mathrm{IC}_{w,0}\langle n\rangle$ for $w \in W$ and $n \in \mathbf{Z}$ generate the pure complexes of weight 0 over $\mathcal{X}_0$ by taking iterated extensions. Thus, by the semisimplicity theorem, the objects $\mathrm{IC}_w[n]$ generate $\mathsf{Pur}(\mathcal{X})$ merely by taking direct sums.

For general $w \in W$, the image of IC_w in H_W is called the **Kazhdan–Lusztig element** assigned to w. We deduce that the Kazhdan–Lusztig elements span H_W over $\mathbf{Z}[\boldsymbol{q}^{\pm\frac{1}{2}}]$. In fact, they form a basis for H_W as a free $\mathbf{Z}[\boldsymbol{q}^{\pm\frac{1}{2}}]$-module.

1.6.4. Rouquier Complexes For any $s \in S$, we can explicitly describe the images of $\Delta_{s,0}$ and $\nabla_{s,0}$ in the Hecke category. In what follows, *when we write a complex, we enclose the degree-*0 *term within a box.* Let

$$R_s^+ = \mathcal{C}hr(\Delta_{s,0}) \quad \text{and} \quad R_s^- = \mathcal{C}hr(\nabla_{s,0}). \tag{1.6.15}$$

Then R_s^+ and R_s^- can be represented by complexes

$$\begin{aligned} R_s^+ &= 0 \to \boxed{\mathrm{IC}_s} \to \mathrm{IC}_1[1], \\ R_s^- &= \mathrm{IC}_1[-1] \to \boxed{\mathrm{IC}_s} \to 0, \end{aligned} \tag{1.6.16}$$

where the nontrivial morphisms are induced by the canonical weight filtrations of the perverse sheaves $\Delta_{s,0}$ and $\nabla_{s,0}$, respectively. (Above, [1] is the shift internal to $\mathsf{D}^b(\mathcal{X})$.)

Indeed, the weight filtrations of $\Delta_{s,0}$ and $\nabla_{s,0}$ are given by:

$$\begin{array}{c|ccccccc} & \mathbf{W}_{\leq -2} & & \mathbf{W}_{\leq -1} & & \mathbf{W}_{\leq -0} & & \mathbf{W}_{\leq 1} \\ \mathbf{W}_{\leq *}\Delta_{s,0} & 0 & \to & \mathrm{IC}_{1,0} & \to & \Delta_{s,0} & & \\ \mathbf{W}_{\leq *}\nabla_{s,0} & & & 0 & \to & \mathrm{IC}_{s,0} & \to & \nabla_{s,0}. \end{array} \tag{1.6.17}$$

This gives, for instance, $\mathrm{Gr}_0^{\mathbf{W}} \Delta_{s,0} = \mathrm{IC}_{s,0}$ and $\mathrm{Gr}_{-1}^{\mathbf{W}} \Delta_{s,0} = \mathrm{IC}_{1,0}$, from which we can deduce the form of R_s^+. A similar computation works for R_s^-.

Remark 1.6.2. By way of Theorem 1.6.1, the form of these complexes reflects the fact that if γ_s is the Kazhdan–Lusztig element of H_W that corresponds to s, then we simultaneously have $\beta_s = \gamma_s - \boldsymbol{q}^{-\frac{1}{2}}$ and $\beta_s^{-1} = \gamma_s - \boldsymbol{q}^{\frac{1}{2}}$.

Using this explicit computation, Rouquier gave a categorification of the Artin braid group Br_W in terms of H. Observe that any Artin braid can be represented by a word of the form $\beta_{s_1}^{\epsilon_1} \cdots \beta_{s_\ell}^{\epsilon_\ell}$, where $s_1, \ldots, s_\ell$ is a sequence of simple reflections, possibly with repetition, and $\epsilon_i \in \{\pm 1\}$ for all i. Rouquier proved [92] that if $\beta = \beta_{s_1}^{\epsilon_1} \cdots \beta_{s_\ell}^{\epsilon_\ell}$ in Br_W, then

$$R(\beta) = R_{s_1}^{\epsilon_1} \odot \cdots \odot R_{s_\ell}^{\epsilon_\ell} \tag{1.6.18}$$

only depends on β up to isomorphism in H. That is, $\odot$ respects the defining relations of Br_W. In Section 1.8, we will review a similar statement for positive braids due to Deligne.

The hypercohomology functor $\mathrm{H}^*(\mathcal{X}, -) : \mathsf{Pur}(\mathcal{X}) \to \mathbf{B}\text{-}\mathsf{Mod}_{\mathsf{gr}}$ can be lifted to a functor at the level of homotopy categories:

$$\mathrm{H}^*(\mathcal{X}, -) : \mathsf{H} = \mathsf{K}^b(\mathsf{Pur}(\mathcal{X})) \to \mathsf{K}^b(\mathbf{B}\text{-}\mathsf{Mod}_{\mathsf{gr}}). \tag{1.6.19}$$

For all $\beta \in Br_W$, we set:

$$\mathbf{B}(\beta) = \mathrm{H}^*(\mathcal{X}, R(\beta)). \tag{1.6.20}$$

Then $\mathbf{B}(\beta)$ can be viewed as a complex of Soergel bimodules that only depends on β up to homotopy. In the literature on Soergel bimodules, $\mathbf{B}(\beta)$ is generally known as the **Rouquier complex** of β.

1.6.5. The Horocycle Functor The cocenter of an associative algebra A is the largest commutative quotient of A, i.e., the algebra $A/[A, A]$. Roughly, we can categorify the map from the Iwahori–Hecke algebra H_W to its cocenter using the **horocycle correspondence**. It is the diagram

$$\mathcal{X}_0 = G_0\backslash(\mathcal{B}_0 \times \mathcal{B}_0) \xleftarrow{act} G_0\backslash(\mathcal{B}_0 \times G_0) \xrightarrow{pr} G_0\backslash G_0 \tag{1.6.21}$$

where, at the level of orbit representatives,

$$(gB, hgB) \xleftarrow{act} (gB, h) \xmapsto{pr} h. \tag{1.6.22}$$

By Deligne's theorem on weights, we obtain an additive functor:

$$\mathcal{C}orr = pr_* \circ act^* : \mathsf{Pur}(\mathcal{X}) \to \mathsf{Pur}(G\backslash G). \tag{1.6.23}$$

The following well-known result, for which we could not find a concise reference, explains why we view $\mathcal{C}orr$ as a categorified cocenter map.

Proposition 1.6.3. *For all $K, L \in \mathsf{Pur}(\mathcal{X})$, there is an isomorphism*

$$\mathcal{C}orr(K \odot L) \simeq \mathcal{C}orr(L \odot K) \tag{1.6.24}$$

in $\mathsf{Pur}(G\backslash G)$.

Proof. Let $\mathcal{F} : \mathsf{Pur}(\mathcal{X}) \times \mathsf{Pur}(\mathcal{X}) \to \mathsf{Pur}(G\backslash G)$ be the additive functor $(K, L) \mapsto \mathcal{C}orr(K \odot L)$. Then $\mathcal{F}$ is induced by pullback and pushforward along the solid arrows in the diagram below:

$$\begin{array}{ccccc} G_0\backslash(\mathcal{B}_0 \times \mathcal{B}_0 \times G_0) & \dashrightarrow & G_0\backslash(\mathcal{B}_0 \times G_0) & \xrightarrow{pr} & G_0\backslash G_0 \\ \vdots & & \downarrow{\scriptstyle act} & & \\ G_0\backslash(\mathcal{B}_0 \times \mathcal{B}_0 \times \mathcal{B}_0) & \xrightarrow[\mathrm{pr}_{1,3}]{} & \mathcal{X}_0 & & \\ {\scriptstyle \mathrm{pr}_{1,2} \times \mathrm{pr}_{2,3}}\downarrow & & & & \\ \mathcal{X}_0 \times \mathcal{X}_0 & & & & \end{array} \tag{1.6.25}$$

Let the square in the diagram be cartesian. By applying proper base change to that square, we find that $\mathcal{F} = b_* \circ a^*$, where, at the level of orbit representatives:

- $a : G_0\backslash(\mathcal{B}_0 \times \mathcal{B}_0 \times \mathcal{B}_0) \to \mathcal{X}_0 \times \mathcal{X}_0$ sends $(xB, yB, g) \mapsto (xB, yB) \times (yB, gxB)$.

- $b : G_0\backslash(\mathcal{B}_0 \times \mathcal{B}_0 \times \mathcal{B}_0) \to G_0\backslash G_0$ sends $(xB, yB, g) \mapsto g$.

Now consider the diagram

$$\begin{array}{ccccc} \mathcal{X}_0 \times \mathcal{X}_0 & \xleftarrow{a} & G_0\backslash(\mathcal{B}_0 \times \mathcal{B}_0 \times G_0) & & \\ & & & \searrow{\scriptstyle b} & \\ {\scriptstyle i_1}\downarrow & & \uparrow{\scriptstyle i_2} & & G_0\backslash G_0 \\ & & & \nearrow{\scriptstyle b} & \\ \mathcal{X}_0 \times \mathcal{X}_0 & \xleftarrow[a]{} & G_0\backslash(\mathcal{B}_0 \times \mathcal{B}_0 \times G_0) & & \end{array} \tag{1.6.26}$$

where, at the level of orbit representatives:

- $i_1 \in \mathrm{Aut}(\mathcal{X}_0 \times \mathcal{X}_0)$ sends $(xB, yB) \times (x'B, y'B) \mapsto (x'B, y'B) \times (xB, yB)$.

- $i_2 \in \mathrm{Aut}(G_0\backslash(\mathcal{B}_0 \times \mathcal{B}_0 \times G_0))$ sends $(xB, yB, g) \mapsto (yB, gxB, g)$.

One can check directly that the square on the left and the triangle on the right commute. (The former is where we use G_0-equivariance.) Thus the whole diagram commutes, giving us an isomorphism $\mathcal{F}(K, L) = \mathcal{F}(i_1^*(L, K)) \xrightarrow{\sim} \mathcal{F}(L, K)$. □

It will be convenient to write $\mathsf{Ch} = \mathsf{K}^b(\mathsf{Pur}(G\backslash G))$. (The notation Ch is meant to connote "character sheaves," discussed below.) We can lift $\mathcal{C}orr$ to a functor on homotopy categories:

$$\mathcal{C}orr : \mathsf{H} \to \mathsf{Ch}. \tag{1.6.27}$$

By abstract nonsense, we deduce:

Corollary 1.6.4. *The analogue of Proposition 1.6.3 holds with* H *and* Ch *in place of* $\mathsf{Pur}(\mathcal{X})$ *and* $\mathsf{Pur}(G\backslash G)$.

For all $w \in W$, we set:

$$\mathcal{E}_w = \mathcal{C}orr(\mathrm{IC}_w). \tag{1.6.28}$$

A **unipotent character sheaf** of G is any simple perverse sheaf over $G\backslash G$ that, up to shift, forms a summand of $\mathcal{E}_w$ for some w.

Example 1.6.5. $\mathcal{E}_1$ is the !-pushforward of the constant sheaf along $B\backslash B \to G\backslash G$. It is also known as the **Grothendieck–Springer sheaf**. If we write G^{rs} for the locus of regular semisimple elements in G, then $\mathcal{E}_1$ is the IC-extension of a local system over $G\backslash G^{\mathrm{rs}}$ with monodromy group W. Thus, we have a canonical isomorphism $\mathrm{End}_{\bar{\mathbf{Q}}_\ell}(\mathcal{E}_1) \simeq \bar{\mathbf{Q}}_\ell[W]$, by means of which $\mathcal{E}_1$ can be decomposed into W-isotypic components:

$$\mathcal{E}_1 \simeq \bigoplus_{\phi \in \mathrm{Irr}(W)} \phi \otimes \mathcal{L}_\phi, \tag{1.6.29}$$

where $\mathcal{L}_\phi$ is a simple perverse sheaf over $G\backslash G$ for all ϕ.

In a similar way, for all $\beta \in Br_W$, we set:

$$\mathcal{E}(\beta) = \mathcal{C}orr(R(\beta)). \tag{1.6.30}$$

Then $\mathcal{E}(\beta)$ can be viewed as a complex of unipotent character sheaves that only depends on β up to homotopy. By Corollary 1.6.4, the map that sends β to the isomorphism class of $\mathcal{E}(\beta)$ in Ch defines a class function on Br_W. The goal of Section 1.7 will be to study *how various annular invariants of Artin braids factor through this map.*

1.6.6. The Work of Webster–Williamson In [110, 111], Webster–Williamson related the Hochschild homology of Soergel bimodules to the hypercohomology of the objects $\mathcal{E}_w$.

The **Hochschild homology** of any object $M \in \mathbf{B}\text{-}\mathsf{Mod}_{\mathsf{gr}}$ is defined by:

$$\mathrm{HH}_*(M) = \mathrm{Tor}_*^{\mathbf{B}\otimes\mathbf{B}^{\mathrm{op}}}(\mathbf{B}, M). \tag{1.6.31}$$

This vector space is equipped with an *internal grading*, coming from M itself, in addition to its *Hochschild grading.* The hypercohomology of $\mathcal{E}_w$, like that of IC_w, decomposes into a direct sum of pure subspaces under Frobenius. Thus the same holds for the hypercohomology of any direct sum of (shifted) unipotent character sheaves.

Below, we write $\mathrm{Gr}_k^{\mathbf{B}} \mathrm{HH}_* \subseteq \mathrm{HH}_*$ for the summand of internal degree k; and we write $\mathrm{Gr}_k^{\mathbf{W}} \mathrm{H}^* \subseteq \mathrm{H}^*$ for the summand of weight k, i.e., the summand where the eigenvalues of Frobenius have absolute value $q^{\frac{k}{2}}$.

Theorem 1.6.6 (Webster–Williamson)**.** *For all $w \in W$ and $i, j \in \mathbf{Z}$, we have*

$$\mathrm{Gr}_{i+j}^{\mathbf{B}} \mathrm{HH}_i(\mathbf{B}_w) \simeq \mathrm{Gr}_{i+j}^{\mathbf{W}} \mathrm{H}^j(G\backslash G, \mathcal{E}_w), \tag{1.6.32}$$

where $\mathbf{B}_w$ and $\mathcal{E}_w$ are respectively defined by (1.6.10) *and* (1.6.28)*.*

For all $n \geq 0$, we let $\mathsf{Vect}_{n\text{-}\mathsf{gr}}$ denote the category of $\mathbf{Z}^n$-graded $\bar{\mathbf{Q}}_\ell$-vector spaces. Viewing $\mathrm{Gr}_*^{\mathbf{B}} \mathrm{HH}_*$ and $\mathrm{Gr}_*^{\mathbf{W}} \mathrm{H}^*$ as functors to $\mathsf{Vect}_{2\text{-}\mathsf{gr}}$, we lift them to functors:

$$\begin{aligned} \mathrm{Gr}_*^{\mathbf{B}} \mathrm{HH}_* &: & \mathsf{K}^b(\mathbf{B}\text{-}\mathsf{Mod}_{\mathsf{gr}}) &\to \mathsf{K}^b(\mathsf{Vect}_{2\text{-}\mathsf{gr}}), \\ \mathrm{Gr}_*^{\mathbf{W}} \mathrm{H}^* &: & \mathsf{Ch} = \mathsf{K}^b(\mathsf{Pur}(G\backslash G)) &\to \mathsf{K}^b(\mathsf{Vect}_{2\text{-}\mathsf{gr}}). \end{aligned} \tag{1.6.33}$$

By chasing definitions, we arrive at:

Corollary 1.6.7. *For all $\beta \in Br_W$ and i, j, we have*

$$\mathrm{Gr}^{\mathbf{B}}_{i+j} \mathrm{HH}_i(\mathbf{B}(\beta)) \simeq \mathrm{Gr}^{\mathbf{W}}_{i+j} \mathrm{H}^j(G \backslash G, \mathcal{E}(\beta)), \tag{1.6.34}$$

where $\mathbf{B}(\beta)$ and $\mathcal{E}(\beta)$ are respectively defined by (1.6.20) *and* (1.6.30)*.*

1.7 Geometric Invariants

1.7.1. Khovanov–Rozansky Homology In [62], Khovanov–Rozansky gave a categorification of HOMFLY of the form

$$\underline{\text{HOMFLY}} : \{\text{links in } S^3\}/\text{isotopy} \to \mathsf{Vect}_{3\text{-gr}}. \tag{1.7.1}$$

As we will explain below, if the graded dimension of $\underline{\text{HOMFLY}}$ is suitably expressed as a Laurent series in variables $\boldsymbol{q}^{\frac{1}{2}}, a, t$, then HOMFLY is the $t = -1$ limit.

The construction in [62] uses the theory of matrix factorizations. In [61], Khovanov showed how to construct $\underline{\text{HOMFLY}}$ by instead using the theory of Soergel bimodules. To wit, let **Khovanov–Rozansky homology** be the following composition of functors of homotopy categories:

$$\mathrm{HHH} : \mathsf{H} = \mathsf{K}^b(\mathsf{Pur}(\mathcal{X})) \xrightarrow{\mathrm{H}^*(\mathcal{X},-)} \mathsf{K}^b(\mathbf{B}\text{-}\mathsf{Mod}_{\mathsf{gr}}) \xrightarrow{\mathrm{Gr}^{\mathbf{B}}_* \mathrm{HH}_*} \mathsf{K}^b(\mathsf{Vect}_{2\text{-gr}}) \subseteq \mathsf{Vect}_{3\text{-gr}}. \tag{1.7.2}$$

To track the different gradings, we write:

$$\mathrm{HHH}_{i,j,k}(-) = \mathrm{H}_k(\mathrm{Gr}^{\mathbf{B}}_j \mathrm{HH}_i(\mathrm{H}^*(\mathcal{X}, -))). \tag{1.7.3}$$

Restated in words, $\mathrm{HHH}_{i,j,k}(K)$ is the degree-k homology of the component of the complex $\mathrm{H}^*(\mathcal{X}, K) \in \mathsf{K}^b(\mathbf{B}\text{-}\mathsf{Mod}_{\mathsf{gr}})$ where the Hochschild degree is i and the internal degree is j.

It will be very convenient to write the objects of $\mathsf{Vect}_{3\text{-gr}}$ as trivariate Laurent series with coefficients in Vect, so that we can track grading shifts by means of variable substitutions. Given $V = \bigoplus_{i,j,k} V_{i,j,k}$, let us write:

$$[V]_{a,\boldsymbol{q}^{1/2},t} = \bigoplus_{i,j,k} a^i \boldsymbol{q}^{\frac{j}{2}} t^k V_{i,j,k}. \tag{1.7.4}$$

With this notation, we can state:

Theorem 1.7.1 (Khovanov)**.** *Suppose $G_0 = \mathrm{SL}_n$, so that $r = n-1$. If $\underline{\mathit{HOMFLY}}(\beta) \in \mathsf{Vect}_{3\text{-gr}}$ is given by*

$$[\underline{\mathit{HOMFLY}}(\beta)]_{a,\boldsymbol{q}^{1/2},t} = (at)^{|\beta|} a^{-r} \bigoplus_{i,j,k} (a^2 \boldsymbol{q}^{\frac{1}{2}} t)^{r-i} \boldsymbol{q}^{\frac{j}{2}} t^{-k} \mathrm{HHH}_{i,\, i+j,\, k}(R(\beta)) \tag{1.7.5}$$

for all $\beta \in Br_n$, then the isomorphism class of $\underline{\mathit{HOMFLY}}(\beta)$ only depends on the planar closure of β. Indeed, $\underline{\mathit{HOMFLY}}$ is isomorphic to the isotopy invariant of links defined by Khovanov and Rozansky in [62]. It follows that $\dim \underline{\mathit{HOMFLY}}|_{t=-1} = \mathit{HOMFLY}$.

By Corollary 1.6.6 due to Webster–Williamson, we can rewrite $\mathrm{HHH}(R(\beta))$, and hence $\underline{\mathrm{HOMFLY}}(\beta)$, in terms of $\mathcal{E}(\beta)$:

$$\mathrm{HHH}_{i,\, i+j,\, k}(R(\beta)) = \mathrm{H}_k(\mathrm{Gr}^{\mathbf{W}}_{i+j} \mathrm{H}^j(G \backslash G, \mathcal{E}(\beta))). \tag{1.7.6}$$

This observation motivates us to seek a similar refinement of ANN in terms of $\mathcal{E}(\beta)$.

1.7.2. HHH of the Trivial Braid Before moving on, we demonstrate how to use the viewpoint of Webster–Williamson to compute the well-known Khovanov–Rozansky homology of $R(1)$. First, we recall the hypercohomology of the Grothendieck–Springer sheaf.

Lemma 1.7.2. *We have a W-equivariant isomorphism:*

$$\mathrm{Gr}^{\mathbf{W}}_{i+j} \mathrm{H}^j(G \backslash G, \mathcal{E}_1) \simeq \mathrm{H}^i(T) \otimes \mathrm{H}^{j-i}(\,\cdot\,/T). \tag{1.7.7}$$

In particular, $\mathrm{Gr}^{\mathbf{W}}_{i+j}\,\mathrm{H}^j(G\backslash G, \mathcal{L}_\phi)$ *is the* ϕ*-isotypic component of the right-hand side.*

Proof. We know that $\mathrm{H}^*(G\backslash G, \mathcal{E}_1) \simeq \mathrm{H}^*(B\backslash B) \simeq \mathrm{H}^*(T\backslash T)$; these isomorphisms preserve weight gradings. Since the conjugation action of T on itself is trivial,

$$\mathrm{H}^j(T\backslash T) \simeq \bigoplus_i \mathrm{H}^i(T) \otimes \mathrm{H}^{j-i}(T\backslash \cdot\,). \tag{1.7.8}$$

Above, Frobenius acts by $q^{\frac{j-i}{2}}$ on the term $\mathrm{H}^{j-i}(T\backslash \cdot\,)$ and by q^i on the term $\mathrm{H}^i(T)$, so it acts by $q^{\frac{i+j}{2}}$ on the ith summand of the right-hand side. Therefore, this summand must be $\mathrm{Gr}^{\mathbf{W}}_{i+j}\,\mathrm{H}^j(T\backslash T)$.

The W-equivariance of the isomorphism is essentially proven in [111, 416]. From it, the statement about $\mathcal{L}_\phi$ follows. □

Corollary 1.7.3 (Khovanov–Rozansky)**.** *We have*

$$[\underline{\mathit{HOMFLY}}(1)]_{a,\boldsymbol{q}^{1/2},t} = (a^{-1}\boldsymbol{q}^{\frac{1}{2}})^r \bigoplus_{0\leq i\leq r} (a^2t)^{r-i}\mathrm{H}^i(T) \otimes \bigoplus_{j\geq 0} \boldsymbol{q}^{\frac{j}{2}}\mathrm{H}^j(\,\cdot\,/T). \tag{1.7.9}$$

In particular, the graded dimension is

$$\dim\,[\underline{\mathit{HOMFLY}}(1)]_{a,\boldsymbol{q}^{1/2},t} = \left(a^{-1}\boldsymbol{q}^{\frac{1}{2}}\left(\frac{1+a^2t}{1-\boldsymbol{q}}\right)\right)^r \tag{1.7.10}$$

in $\mathbf{Z}(\boldsymbol{q}^{\frac{1}{2}})[a^{\pm 1}, t]$*.*

Proof. Just as $R(1) \in \mathsf{K}^b(\mathsf{Pur}(\mathcal{X}))$ can be represented by the complex

$$\cdots \to 0 \to \boxed{\mathrm{IC}_1} \to 0 \to \cdots \tag{1.7.11}$$

so $\mathcal{E}(1) \in \mathsf{K}^b(\mathsf{Pur}(G\backslash G))$ can be represented by the complex

$$\cdots \to 0 \to \boxed{\mathcal{E}_1} \to 0 \to \cdots \tag{1.7.12}$$

The result now follows from Corollary 1.6.7 and Lemma 1.7.2, after simplifying. □

1.7.3. Annular Braid Homology We introduce the semidirect product algebra:

$$\mathbf{A} = \bar{\mathbf{Q}}_\ell[W] \ltimes \mathbf{B}. \tag{1.7.13}$$

We endow $\mathbf{A}$ with the grading in which $\bar{\mathbf{Q}}_\ell[W]$ is placed in degree zero and $\mathbf{B}$ retains its original grading. Thus, like with $\mathbf{B}$, the grading on $\mathbf{A}$ is concentrated in even degrees. We write $\mathbf{A}\text{-}\mathsf{Mod}_{\mathsf{gr}}$ for the category of graded $\mathbf{A}$-modules.

We will construct a refinement of HHH that takes values in $\mathsf{K}^b(\mathbf{A}\text{-}\mathsf{Mod}_{\mathsf{gr}})$ rather than $\mathsf{Vect}_{3\text{-}\mathsf{gr}}$. Since $\bar{\mathbf{Q}}_\ell[W]$ is a subalgebra of $\mathbf{A}$, the homology of any object of $\mathsf{K}^b(\mathbf{A}\text{-}\mathsf{Mod}_{\mathsf{gr}})$ forms a representation of W equipped with two gradings, one internal and one external. In our construction, these two gradings will correspond to the $\boldsymbol{q}^{\frac{1}{2}}$- and t-gradings in the Khovanov–Rozansky link invariant. Specializing $t = -1$ will give an element of $\mathrm{K}_0(W)[\![\boldsymbol{q}^{\frac{1}{2}}]\!]$ that, up to normalization factors, recovers the annular character of Section 1.4.

To start, we need a way to construct graded $\mathbf{A}$-actions. Observe that

$$\mathrm{H}^*(G\backslash G, -) = \mathrm{Ext}^*_{G\backslash G}(\mathcal{L}_1, -) \quad (= \mathrm{H}^*(\mathrm{RHom}_{\mathsf{D}^b(G\backslash G)}(\mathcal{L}_1, -))), \tag{1.7.14}$$

since $\mathcal{L}_1$ is just the trivial local system. It is natural to replace $\mathcal{L}_1$ with the Grothendieck–Springer sheaf $\mathcal{E}_1$ of which it is a direct summand, forming the functor:

$$\mathrm{Ext}^*_{G\backslash G}(\mathcal{E}_1, -). \tag{1.7.15}$$

Henceforth, we abbreviate by writing $\mathrm{Ext}^* = \mathrm{Ext}^*_{G\backslash G}$.

For all $K \in \mathsf{Pur}(G\backslash G)$, the Yoneda product of Ext-groups defines an associative, bigraded

$\bar{\mathbf{Q}}_\ell$-linear map:

$$\mathrm{Gr}^{\mathbf{W}}_* \operatorname{Ext}^*(\mathcal{E}_1, \mathcal{E}_1) \otimes \mathrm{Gr}^{\mathbf{W}}_* \operatorname{Ext}^*(\mathcal{E}_1, K) \to \mathrm{Gr}^{\mathbf{W}}_* \operatorname{Ext}^*(\mathcal{E}_1, K). \tag{1.7.16}$$

Consequently:

1. $\mathrm{Gr}^{\mathbf{W}}_* \operatorname{Ext}^*(\mathcal{E}_1, \mathcal{E}_1)$ forms a bigraded $\bar{\mathbf{Q}}_\ell$-algebra.
2. $\mathrm{Gr}^{\mathbf{W}}_* \operatorname{Ext}^*(\mathcal{E}_1, K)$ forms a bigraded module over $\mathrm{Gr}^{\mathbf{W}}_* \operatorname{Ext}^*(\mathcal{E}_1, \mathcal{E}_1)$.

However, as it turns out, *we will only need the "pure" parts of these* Ext-*groups for our purposes* (due to Proposition 1.7.12 below).

The following result is a stacky (or G-equivariant) version of [44, Thm. 8.1]. It is the analogue of Lemma 1.7.2 with $\operatorname{Ext}^*(\mathcal{E}_1, -)$ in place of $\mathrm{H}^*(G\backslash G, -)$.

Theorem 1.7.4 (Ginzburg)**.** *We have a W-biequivariant isomorphism of vector spaces:*

$$\mathrm{Gr}^{\mathbf{W}}_{i+j} \operatorname{Ext}^j(\mathcal{E}_1, \mathcal{E}_1) \simeq \bar{\mathbf{Q}}_\ell[W] \otimes \mathrm{H}^i(T) \otimes \mathrm{H}^{j-i}(\,\cdot\,/T). \tag{1.7.17}$$

It induces an isomorphism of bigraded $\bar{\mathbf{Q}}_\ell[W]$-algebras:

$$\mathrm{Gr}^{\mathbf{W}}_* \operatorname{Ext}^*(\mathcal{E}_1, \mathcal{E}_1) \simeq \bar{\mathbf{Q}}_\ell[W] \ltimes (\mathrm{H}^*(T) \otimes \mathrm{H}^*(\,\cdot\,/T)). \tag{1.7.18}$$

(On the right-hand side, $\ltimes$ is the semidirect product with respect to the W-action on $\mathrm{H}^*(T)$ and $\mathrm{H}^*(\,\cdot\,/T)$.)

Remark 1.7.5. To clarify the W-actions on both sides of (1.7.17): On the left, W and W^{op} respectively act on the first and second copies of $\mathcal{E}_1$. On the right, W and W^{op} act via the inclusion of $\bar{\mathbf{Q}}_\ell[W]$ into the tensor product.

Corollary 1.7.6. *We have an isomorphism of graded* $\bar{\mathbf{Q}}_\ell[W]$*-algebras:*

$$\mathbf{A} \simeq \bigoplus_j \mathrm{Gr}_j^{\mathbf{W}} \mathrm{Ext}^j(\mathcal{E}_1, \mathcal{E}_1). \tag{1.7.19}$$

The corollary suggests that we introduce functors

$$\mathcal{E}\mathrm{H}^j(-) = \bigoplus_j \mathrm{Gr}_j^{\mathbf{W}} \mathrm{Ext}^j(\mathcal{E}_1, -) \tag{1.7.20}$$

and $\mathcal{E}\mathrm{H}^* = \bigoplus_j \mathcal{E}\mathrm{H}^j$. Doing so, we find that $\mathbf{A} = \mathcal{E}\mathrm{H}^*(\mathcal{E}_1)$ and that $\mathcal{E}\mathrm{H}^*$ defines a graded additive functor:

$$\mathcal{E}\mathrm{H}^* : \mathsf{Pur}(G\backslash G) \to \mathbf{A}\text{-}\mathsf{Mod}_{\mathsf{gr}}, \tag{1.7.21}$$

where objects of $\mathsf{Pur}(G\backslash G)$ are graded by the shift [1].

Remark 1.7.7. A simple perverse sheaf on $G\backslash G$ is said to be **cuspidal** iff it does not occur in $\mathcal{E}_1$. It follows from the stacky version of Proposition 7.2 of [71] (*cf.* the proof of Proposition 8 in [111]) that if $\mathcal{L}$ is cuspidal, then $\mathrm{Ext}^*(\mathcal{E}_1, \mathcal{L}) = 0$, and thus, $\mathcal{E}\mathrm{H}^*(\mathcal{L}) = 0$. We deduce that $\mathcal{E}\mathrm{H}^*$ factors through the subcategory of $\mathsf{Pur}(G\backslash G)$ spanned by the simple objects of the form $\mathcal{L}_\phi[n]$ for $\phi \in \mathrm{Irr}(W)$ and $n \in \mathbf{Z}$.

Definition 1.7.8. We define **annular braid homology** to be the following composition of functors of homotopy categories:

$$\mathcal{A}\mathrm{H} : \mathsf{H} \xrightarrow{\mathcal{C}orr} \mathsf{Ch} \xrightarrow{\mathcal{E}\mathrm{H}^*} \mathsf{K}^b(\mathbf{A}\text{-}\mathsf{Mod}_{\mathsf{gr}}). \tag{1.7.22}$$

Moreover, for all $\beta \in Br_W$, we set

$$\mathrm{AH}(\beta) = (\boldsymbol{q}^{\frac{1}{2}}t)^{|\beta|}\varepsilon \otimes \bigoplus_{j,k} \boldsymbol{q}^{\frac{j}{2}} t^{-k} \mathrm{H}_k(\mathcal{A}\mathrm{H}^j(R(\beta))) \in \mathrm{K}_0^+(W)[\![\boldsymbol{q}^{\frac{1}{2}}]\!][t], \tag{1.7.23}$$

and say that $\mathrm{AH}(\beta)$ is the annular braid homology of the Artin braid β.

Example 1.7.9. For the trivial braid, we see $\mathcal{A}\mathrm{H}(1) = \mathcal{E}\mathrm{H}^*(\mathcal{E}(1))$ is the complex consisting of $\mathcal{E}\mathrm{H}^*(\mathcal{E}_1) \simeq \mathbf{A}$ in degree zero and no other terms. Thus, $\mathrm{AH}(1) \simeq \bigoplus_j \boldsymbol{q}^{\frac{j}{2}} \mathbf{A}^j$.

It follows immediately from Corollary 1.6.4 that $\mathcal{A}\mathrm{H}(K \odot L) \simeq \mathcal{A}\mathrm{H}(L \odot K)$ for all $K, L \in \mathsf{H}$. In the next two subsections, we prove that $\mathcal{A}\mathrm{H}$ specializes to Khovanov–Rozansky homology and that AH specializes to the annular character ANN, thereby completing the proof of Theorem 1.1.5 from the introduction.

Theorem 1.7.10. *We have an isomorphism of additive functors*

$$\mathrm{HHH}_{i,\, i+j,\, k}(-) \simeq \mathrm{Hom}_W(\Lambda^i(V), \mathrm{H}_k(\mathcal{A}\mathrm{H}^{j-i}(-))) : \mathsf{Ch} \to \mathsf{Vect} \tag{1.7.24}$$

for all i, j.

Theorem 1.7.11. *We have* $\mathrm{AH}(\beta)|_{t=-1} = \textsc{ann}_V(\beta)$ *for all* $\beta \in Br_W$.

1.7.4. Proof of Theorem 1.7.10 By (1.7.6), it is enough to show:

Proposition 1.7.12. *We have* $\mathrm{Hom}_W(\Lambda^i(V), \mathcal{E}\mathrm{H}^{j-i}(-)) \simeq \mathrm{Gr}^{\mathbf{W}}_{i+j} \mathrm{H}^j(G\backslash G, -)$ *as additive functors* $\mathsf{Pur}(G\backslash G) \to \mathsf{Vect}$.

In turn, we will deduce the above result from Theorem 1.7.4. First, taking $(W \times W^{\mathrm{op}})$-isotypic components on both sides of said theorem, we have:

Lemma 1.7.13. *We have* $\mathrm{Gr}^{\mathbf{W}}_{i+j} \mathrm{Ext}^j(\mathcal{L}_\phi, \mathcal{L}_\psi) \simeq \mathrm{Hom}_W(\phi, \psi \otimes \mathrm{H}^i(T) \otimes \mathrm{H}^{j-i}(\,\cdot\,/T))$ *for all* $\phi, \psi \in \mathrm{Irr}(W)$.

The isomorphism in the lemma is functorial in the sense that it identifies the Yoneda product on the left with the composition of Hom's on the right. That is: Given $\phi, \psi, \xi \in \mathrm{Irr}(W)$ and indices j, k, j', k', if we abbreviate

$$\psi^{(j,k)} = \psi \otimes \mathrm{H}^{k-j}(T) \otimes \mathrm{H}^{2j-k}(\,\cdot\,/T), \tag{1.7.25}$$

then the following diagram commutes:

$$
\begin{array}{ccc}
\mathrm{Gr}^{\mathbf{W}}_{k} \operatorname{Ext}^{j}(\mathcal{L}_{\phi}, \mathcal{L}_{\psi}) \otimes \mathrm{Gr}^{\mathbf{W}}_{k'} \operatorname{Ext}^{j'}(\mathcal{L}_{\psi}, \mathcal{L}_{\xi}) & \longrightarrow & \mathrm{Gr}^{\mathbf{W}}_{k+k'} \operatorname{Ext}^{j+j'}(\mathcal{L}_{\phi}, \mathcal{L}_{\xi}) \\
\downarrow & & \\
\mathrm{Hom}_{W}(\phi, \psi^{(j,k)}) \otimes \mathrm{Hom}_{W}(\psi, \xi^{(j',k')}) & & \downarrow \simeq \\
\downarrow {\scriptstyle \mathrm{id} \otimes (-)^{(j',k')}} & & \\
\mathrm{Hom}_{W}(\phi, \psi^{(j,k)}) \otimes \mathrm{Hom}_{W}(\psi^{(j,k)}, \xi^{(j+j',k+k')}) & \longrightarrow & \mathrm{Hom}_{W}(\phi, \xi^{(j+j',k+k')})
\end{array} \tag{1.7.26}
$$

Proof of Proposition 1.7.12. We need to check a (functorial) isomorphism for every object of $\mathsf{Pur}(G\backslash G)$. By Remark 1.7.7, it suffices to consider objects of the form $\mathcal{L}_{\phi}[n]$ for some ϕ and n. Since $\mathcal{L}_{\phi}[n]$ is the base-change to $\bar{\mathbf{F}}$ of the shift-twist $\mathcal{L}_{\phi,0}\langle n\rangle$, we have $\mathcal{E}\mathrm{H}^{j}(\mathcal{L}_{\phi}[n]) = \mathcal{E}\mathrm{H}^{j+n}(\mathcal{L}_{\phi})$. So it suffices to assume $n = 0$. Using Lemma 1.7.13, we now compute:

$$
\begin{aligned}
\mathrm{Hom}_{W}(\Lambda^{i}(V), \mathcal{E}\mathrm{H}^{j-i}(\mathcal{E}_{1}, \mathcal{L}_{\phi})) &\simeq \mathrm{Hom}_{W}(\Lambda^{i}(V), \phi \otimes \mathrm{H}^{j-i}(\,\cdot\,/T)) \\
&\simeq \mathrm{Hom}_{W}(1, \phi \otimes \mathrm{H}^{i}(T) \otimes \mathrm{H}^{j-i}(\,\cdot\,/T)) \\
&\simeq \mathrm{Gr}^{\mathbf{W}}_{i+j} \mathrm{H}^{j}(G\backslash G, \mathcal{L}_{\phi}),
\end{aligned} \tag{1.7.27}
$$

as needed. Finally, the functoriality of this isomorphism follows from (1.7.26). □

1.7.5. Proof of Theorem 1.7.11 Explicitly, we must check that

$$
\sum_{j,k} (-1)^{k} \boldsymbol{q}^{\frac{j}{2}} \mathrm{H}_{k}(\mathcal{E}\mathrm{H}^{j}(\mathcal{E}(\beta))) = \mathbb{A}(\beta) \cdot M(\boldsymbol{q} \mid V) \tag{1.7.28}
$$

in $\mathrm{K}_{0}(W)(\boldsymbol{q}^{\frac{1}{2}})$. Throughout, we rely on the fact that $\mathcal{E}\mathrm{H}^{*} : \mathsf{Pur}(G\backslash G) \to \mathbf{A}\text{-}\mathsf{Mod}_{\mathsf{gr}}$, being a graded additive functor, descends to a $\mathbf{Z}[\boldsymbol{q}^{\frac{1}{2}}]$-linear morphism

$$
\sum_{j} \boldsymbol{q}^{\frac{j}{2}}\, \mathcal{E}\mathrm{H}^{j}(-) : \mathrm{K}_{0}(\mathsf{Pur}(G\backslash G)) \to \mathrm{K}_{0}(W)(\boldsymbol{q}^{\frac{1}{2}}), \tag{1.7.29}
$$

where the $\mathbf{Z}[\boldsymbol{q}^{\frac{1}{2}}]$-module structure of $\mathrm{K}_{0}(\mathsf{Pur}(G\backslash G))$ is given by $\boldsymbol{q}^{\frac{1}{2}} \mapsto [-1]$ (*cf.* (1.6.13)).

Let $\{\gamma_w\}_{w\in W}$ denote the Kazhdan–Lusztig basis of H_W. The following lemma lets us reduce (1.7.28) to the case where $\beta = \gamma_w$ for some w.

Lemma 1.7.14. *Suppose $\beta = \sum_{w\in W} m_w(\beta)\gamma_w$, where $m_w(\beta) \in \mathbf{Z}[\boldsymbol{q}^{\pm\frac{1}{2}}]$ for all w. Then*

$$\sum_{j,k} (-1)^k \boldsymbol{q}^{\frac{j}{2}} \mathrm{H}_k(\mathcal{E}\mathrm{H}^j(\mathcal{E}(\beta))) = \sum_j \sum_{w\in W} \boldsymbol{q}^{\frac{j}{2}} m_w(\beta)\, \mathcal{E}\mathrm{H}^j(\mathcal{E}_w) \tag{1.7.30}$$

in $\mathrm{K}_0(W)[\boldsymbol{q}^{\pm\frac{1}{2}}]$.

Proof. Let $\mathcal{E}(\beta)_k$ denote the kth term of a complex that represents $\mathcal{E}(\beta)$. By the Hopf trace formula,

$$\sum_{j,k} (-1)^k \boldsymbol{q}^{\frac{j}{2}} \mathrm{H}_k(\mathcal{E}\mathrm{H}^j(\mathcal{E}(\beta))) = \sum_{j,k} (-1)^k \boldsymbol{q}^{\frac{j}{2}}\, \mathcal{E}\mathrm{H}^j(\mathcal{E}(\beta)_k) \tag{1.7.31}$$

in $\mathrm{K}_0(W)[\boldsymbol{q}^{\pm\frac{1}{2}}]$. Now we simplify the right-hand side:

Viewing $\mathrm{K}_0(\mathsf{H})$ as a module over $\mathbf{Z}[\boldsymbol{q}^{\pm\frac{1}{2}}]$ via $\boldsymbol{q}^{\frac{1}{2}} \mapsto [-1]$, we know by Theorem 1.6.1 that:

$$\sum_k (-1)^k [R(\beta)_k] = \sum_w m_w(\beta)[\mathrm{IC}_w] \tag{1.7.32}$$

in $\mathrm{K}_0(\mathsf{H})$. Applying $\mathcal{C}orr$, we get:

$$\sum_k (-1)^k [\mathcal{E}(\beta)_k] = \sum_w m_w(\beta)[\mathcal{E}_w] \tag{1.7.33}$$

in $\mathrm{K}_0(\mathsf{Ch})$. Finally, applying $\sum_j \boldsymbol{q}^{\frac{j}{2}}\, \mathcal{E}\mathrm{H}^j$, we get the result. ☐

To handle (1.7.28) in the case of Kazhdan–Lusztig elements, we use the following formula quoted in [111, 413]:

Theorem 1.7.15 (Lusztig)**.** *We have*

$$[\mathcal{E}_w] = \sum_{\phi,\psi\in\mathrm{Irr}(W)} \{\phi,\psi\}\phi_{\boldsymbol{q}}(\gamma_w)[\mathcal{L}_\psi] + \textit{cuspidal terms} \tag{1.7.34}$$

in the $\mathbf{Z}[\boldsymbol{q}^{\pm\frac{1}{2}}]$*-module formed by* $\mathsf{K}_0(\mathsf{Ch})$*. (Above, "cuspidal terms" refers to the contribution of simple objects not of the form* $\mathcal{L}_\psi$*.)*

Corollary 1.7.16. *We have*

$$\sum_j \boldsymbol{q}^{\frac{j}{2}}\, \mathcal{E}\mathrm{H}^j(\mathcal{E}_w) = \mathbb{A}(\gamma_w)\cdot M(\boldsymbol{q}\mid V) \tag{1.7.35}$$

in $\mathrm{K}_0(W)[\boldsymbol{q}^{\pm\frac{1}{2}}]$.

Proof. Applying $\sum_j \boldsymbol{q}^{\frac{j}{2}}\, \mathcal{E}\mathrm{H}^j$ to both sides of Theorem 2.2.5, we obtain:

$$\begin{aligned}\sum_j \boldsymbol{q}^{\frac{j}{2}}\, \mathcal{E}\mathrm{H}^j(\mathcal{E}_w) &= \sum_{\phi,\psi\in\mathrm{Irr}(W)} \{\phi,\psi\}\phi_{\boldsymbol{q}}(\gamma_w)\sum_j \boldsymbol{q}^{\frac{j}{2}}\, \mathcal{E}\mathrm{H}^j(\mathcal{L}_\psi)\\ &= \sum_{\phi,\psi\in\mathrm{Irr}(W)} \{\phi,\psi\}\phi_{\boldsymbol{q}}(\gamma_w)\sum_j \boldsymbol{q}^{\frac{j}{2}}\psi\otimes\mathrm{H}^j(\,\cdot\,/T)\\ &= \sum_{\phi,\psi\in\mathrm{Irr}(W)} \{\phi,\psi\}\phi_{\boldsymbol{q}}(\gamma_w)\psi\otimes M(\boldsymbol{q}\mid V).\end{aligned} \tag{1.7.36}$$

The first equality follows from Remark 1.7.7; the second follows from Lemma 1.7.13 and the self-duality of $\mathrm{H}^j(\,\cdot\,/T)$ as a representation of W. To finish the proof, we claim that

$$(\boldsymbol{o}(\psi), \mathbb{A}(\gamma_w))_W = (\psi, \mathbb{A}(\gamma_w))_W \tag{1.7.37}$$

for all w and ψ. Indeed, by Property 1.3.2 of the exotic Fourier transform, we can reduce to checking this identity by hand for the irreducible types E_7 and E_8. □

1.8 Positive Braids

1.8.1. Broué–Michel Varieties The positive monoid Br_W^+ affords a "representation" valued in G_0-varieties over $\mathcal{B}_0 \times \mathcal{B}_0$. Roughly, each positive braid β defines a variety $O(\beta)_0$ in such a way that the composition of braids corresponds to the fiber product of varieties. As we will explain below, the complex $R(\beta)$ has a simpler geometric interpretation in terms of $O(\beta)$ (or rather, the quotient stack $G_0\backslash O(\beta)_0$).

Remark 1.8.1. The varieties $O(\beta)_0$ are called "open Bott–Samelson spaces" in [95]. We suggest that they be named after Broué–Michel, because they first appear in their paper [17]. Deligne explicitly attributes them to Broué–Michel in [27].

Given varieties O_0 and O_0' over $\mathcal{B}_0 \times \mathcal{B}_0$, we write $O_0 \times_{\mathcal{B}_0} O_0'$ for the fiber product in which $O_0 \to \mathcal{B}_0$ is the right projection and $O_0' \to \mathcal{B}_0$ is the left projection. This endows the category of $(\mathcal{B}_0 \times \mathcal{B}_0)$-varieties with a monoidal product with unit $O(1)_0 \simeq \mathcal{B}_0$.

Given $(x, y) \in \mathcal{B}_0 \times \mathcal{B}_0$, we write $x \xrightarrow{w} y$ to indicate that $(x, y) \in O(w)_0$. Deligne proved [27, 1.11] that if $\beta = \beta_{s_1} \cdots \beta_{s_\ell}$ in Br_W^+, then the variety

$$\begin{aligned} O(\beta)_0 &= O(\beta_{s_1})_0 \times_{\mathcal{B}_0} O(\beta_{s_2})_0 \times_{\mathcal{B}_0} \cdots \times_{\mathcal{B}_0} O(\beta_{s_\ell})_0 \\ &= \{(x_1, \ldots, x_\ell, y) \in \mathcal{B}_0^{\ell+1} : x_1 \xrightarrow{s_1} x_2 \xrightarrow{s_2} \cdots \to x_\ell \xrightarrow{s_\ell} y\}, \end{aligned} \tag{1.8.1}$$

equipped with the map $O(\beta)_0 \to \mathcal{B}_0 \times \mathcal{B}_0$ that sends

$$(x_1, \ldots, x_\ell, y) \mapsto (x_1, y), \tag{1.8.2}$$

only depends on β up to isomorphism over $(\mathcal{B}_0 \times \mathcal{B}_0)$. That is, $\times_{\mathcal{B}_0}$ respects the defining relations of Br_W^+.

Remark 1.8.2. A more precise statement of Deligne's result goes as follows. There is a map $\beta \mapsto O(\beta)_0$ from positive Artin braids to varieties over $\mathcal{B}_0 \times \mathcal{B}_0$ such that:

1. For all $w \in W$, we have $O(\beta_w) = O_w$ and $j_{\beta_w} = j_w$, *cf.* Section 1.6.

2. If $\beta = \beta'\beta''$ in Br_W^+, then there is a *fixed* isomorphism

$$O(\beta)_0 \xrightarrow{\sim} O(\beta'')_0 \times_{\mathcal{B}_0} O(\beta'') \tag{1.8.3}$$

of varieties over $\mathcal{B}_0 \times \mathcal{B}_0$.

3. If $\beta = \beta'\beta''\beta'''$ in Br_W^+, then there is a commutative diagram

$$\begin{array}{ccc} O(\beta)_0 & \longrightarrow & O(\beta'\beta'')_0 \times_{\mathcal{B}_0} O(\beta''')_0 \\ \downarrow & & \downarrow \\ O(\beta')_0 \times_{\mathcal{B}_0} O(\beta''\beta''')_0 & \longrightarrow & O(\beta')_0 \times_{\mathcal{B}_0} O(\beta'')_0 \times_{\mathcal{B}_0} O(\beta''')_0 \end{array} \tag{1.8.4}$$

where the arrows are the isomorphisms of (2).

There is a diagonal G_0-action on $O(\beta)_0$ given by $g \cdot (x_1, \ldots, x_\ell, y) = (gx_1, \ldots, gx_\ell, gy)$. With respect to this action, the map (1.8.2) and all of the isomorphisms in the preceding remark are G_0-equivariant. Thus, everything above descends to the level of quotient stacks. We let $\mathcal{X}(\beta)_0 = G_0 \backslash O(\beta)_0$, and let

$$j_\beta : \mathcal{X}(\beta)_0 \to \mathcal{X}_0 \tag{1.8.5}$$

be the map of stacks induced by (1.8.2). The following result is asserted on page 1106 of [95]:

Proposition 1.8.3. *The chromatography functor* $\mathcal{C}hr : \mathsf{D}_m^b(\mathsf{Perv}(\mathcal{X}_0)) \to \mathsf{H}$ *preserves* $\odot$.

Corollary 1.8.4. *For all* $\beta \in Br_W^+$*, we have* $R(\beta) \simeq \mathcal{C}hr(j_{\beta,!}\bar{\mathbf{Q}}_{\ell,0})$.

Proof. We can check that $\beta = \beta_{s_1} \cdots \beta_{s_\ell}$ implies $j_{\beta,!}\bar{\mathbf{Q}}_{\ell,0} \simeq \Delta(s_1)_0 \odot \cdots \odot \Delta(s_\ell)_0$ by chasing diagrams. □

Just as we applied the horocycle functor $\mathcal{C}orr : \mathsf{H} \to \mathsf{Ch}$ to the objects $R(\beta) \in \mathsf{H}$, we can pull/push the spaces $O(\beta)_0$ and $\mathcal{X}(\beta)_0$ through the horocycle correspondence, *cf.* (1.6.21).

To this end, we form the commutative diagram

$$
\begin{array}{ccccc}
O(\beta)_0 & \longleftarrow & \widetilde{O}(\beta)_0 & & \\
\downarrow & & \downarrow & \searrow & \\
\mathcal{B}_0 \times \mathcal{B}_0 & \xleftarrow[(x,gx) \leftarrow\!\shortmid (x,g)]{} & \mathcal{B}_0 \times G_0 & \xrightarrow[(x,g) \mapsto g]{} & G_0
\end{array}
\tag{1.8.6}
$$

where $\widetilde{O}(\beta)_0$ is defined by making the square cartesian. Explicitly,

$$
\widetilde{O}(\beta)_0 = \{(x_1, \ldots x_\ell, g) \in \mathcal{B}_0^\ell \times G_0 : x_1 \xrightarrow{s_1} x_2 \xrightarrow{s_2} \cdots \to x_\ell \xrightarrow{s_\ell} gx_1\}. \tag{1.8.7}
$$

There is a natural G_0-action on $\widetilde{O}(\beta)_0$ making the whole diagram equivariant. We let $\widetilde{\mathcal{X}}(\beta)_0 = G_0 \backslash \widetilde{O}(\beta)_0$ and write

$$
pr_\beta : \widetilde{\mathcal{X}}(\beta)_0 \to G_0 \backslash G_0 \tag{1.8.8}
$$

for the map induced by the dashed arrow. By smooth base change, we deduce:

Corollary 1.8.5. *For all $\beta \in Br_W^+$, we have $\mathcal{E}(\beta) \simeq \mathcal{C}hr(pr_{\beta,!}\bar{\mathbf{Q}}_{\ell,0})$.*

Example 1.8.6. On the level of points, we can check:

$$
\widetilde{\mathcal{X}}(1)_0 = G_0 \backslash \{(xB, g) \in \mathcal{B}_0 \times G_0 : x^{-1}gx \in B\} \simeq B_0 \backslash B_0. \tag{1.8.9}
$$

Thus, $pr_1 : \widetilde{\mathcal{X}}(1)_0 \to G_0 \backslash G_0$ is precisely the (stacky) Grothendieck–Springer map. In particular, pr_1 is proper and $\mathcal{C}hr(pr_{1,*}\bar{\mathbf{Q}}_{\ell,0})$ can be represented by a complex consisting of $\mathcal{E}_1 \simeq pr_{1,*}\bar{\mathbf{Q}}_\ell$ in degree zero and no other terms, as we saw earlier.

1.8.2. Steinberg-Like Spaces From Corollary 1.8.5, we deduce that the Khovanov–Rozansky homology of a positive braid β is the homology of the chromatographic complex associated with the map $pr_\beta : \widetilde{\mathcal{X}}(\beta)_0 \to G_0 \backslash G_0$. We can describe the annular braid homology of β in an analogous way, after replacing $\widetilde{\mathcal{X}}(\beta)_0$ with a certain fiber product reminiscent of the Steinberg variety.

Definition 1.8.7. For all $\beta \in Br_W^+$, we define the **Steinberg-like variety** of β to be

$$St_G(\beta)_0 = \tilde{O}(1)_0 \times_{G_0} \tilde{O}(\beta). \tag{1.8.10}$$

We define the corresponding **Steinberg-like stack** to be

$$G_0 \backslash St_G(\beta)_0 \simeq \widetilde{\mathcal{X}}(1)_0 \times_{G_0 \backslash G_0} \widetilde{\mathcal{X}}(\beta)_0, \tag{1.8.11}$$

and write

$$Pr_\beta : G_0 \backslash St_G(\beta)_0 \to G_0 \backslash G_0 \tag{1.8.12}$$

for the natural map.

Theorem 1.8.8. *For all $\beta \in Br_W^+$ and $j \geq 0$, we have*

$$\mathcal{A}\mathrm{H}^j(\beta) \simeq \mathrm{Gr}_j^{\mathbf{W}} \mathrm{H}^j(G \backslash G, \mathcal{C}hr(Pr_{\beta,!}\, \bar{\mathbf{Q}}_{\ell,0})) \tag{1.8.13}$$

in $\mathsf{K}^b(\mathsf{Vect})$, where the hypercohomology functor is applied to $\mathcal{C}hr(Pr_{\beta,!}\, \bar{\mathbf{Q}}_{\ell,0})$ term by term.

To start the proof: Observe that by Corollary 1.8.5,

$$\mathcal{A}\mathrm{H}^j(\beta) = \mathcal{E}\mathrm{H}^j(\mathcal{C}hr(pr_{\beta,!}\bar{\mathbf{Q}}_{\ell,0})). \tag{1.8.14}$$

The following lemma lets us relate the two chromatographic complexes at hand:

Lemma 1.8.9. *We have $\mathcal{C}hr(Pr_{\beta,!}\, \bar{\mathbf{Q}}_{\ell,0}) \simeq \mathcal{E}_1 \otimes^{\mathrm{L}} \mathcal{C}hr(pr_{\beta,!}\bar{\mathbf{Q}}_{\ell,0})$, where the functor $\mathcal{E}_1 \otimes^{\mathrm{L}} (-)$ is applied to $\mathcal{C}hr(pr_{\beta,!}\bar{\mathbf{Q}}_{\ell,0})$ term by term.*

Proof. Let $\mathcal{E}_{1,0} = pr_* act^*(\mathrm{IC}_{1,0})$, the mixed perverse sheaf underlying $\mathcal{E}_1$. Using the relative Künneth formula, we expand:

$$\begin{aligned}\mathcal{E}_{1,0} \otimes^{\mathrm{L}} pr_{\beta,!}\bar{\mathbf{Q}}_{\ell,0} &= pr_{1,*}\bar{\mathbf{Q}}_{\ell,0} \otimes^{\mathrm{L}} pr_{\beta,!}\bar{\mathbf{Q}}_{\ell,0} \\ &\simeq Pr_{\beta,!}(\bar{\mathbf{Q}}_{\ell,0} \boxtimes^{\mathrm{L}}_{\mathcal{S}(\beta)} \bar{\mathbf{Q}}_{\ell,0}) \\ &= Pr_{\beta,!}\,\bar{\mathbf{Q}}_{\ell,0}.\end{aligned} \tag{1.8.15}$$

It remains to show that $\mathcal{C}hr(\mathcal{E}_{1,0} \otimes^{\mathrm{L}} K_0) \simeq \mathcal{E}_1 \otimes^{\mathrm{L}} \mathcal{C}hr(K_0)$, where $K_0 = pr_{\beta,!}\bar{\mathbf{Q}}_{\ell,0}$.

In the notation of Appendix 1.9, it is enough to show that if an exact triangle is an α-distillation of K_0, then it remains an α-distillation after tensoring with $\mathcal{E}_{1,0}$. This holds because $\mathcal{E}_{1,0}$ is pure of weight 0 and Verdier self-dual. □

Proof of Theorem 1.8.8. By [44, 86], there is an isomorphism of derived functors:

$$pr_{1,*}pr_1^!(-) \simeq \mathcal{E}_1 \otimes^{\mathrm{L}} (-) : \mathsf{D}^b(G\backslash G) \to \mathsf{D}^b(G\backslash G). \tag{1.8.16}$$

By adjunction, we obtain an isomorphism of functors:

$$\mathrm{Ext}^*_{G\backslash G}(\mathcal{E}_1, -) \simeq \mathrm{H}^*(G\backslash G, \mathcal{E}_1 \otimes^{\mathrm{L}} (-)) \tag{1.8.17}$$

One can check that everything above admits an analogue over $G_0\backslash G_0$ in the *mixed* setting, where the isomorphisms preserve weights. Then, taking the jth pure part on both sides,

$$\mathcal{E}\mathrm{H}^j(-) \simeq \mathrm{Gr}^{\mathbf{W}}_j\,\mathrm{H}^j(G\backslash G, \mathcal{E}_1 \otimes^{\mathrm{L}} (-)). \tag{1.8.18}$$

Combining the last isomorphism with the lemma finishes the proof. □

Example 1.8.10. For the trivial braid, Theorem 1.8.8 specializes to:

$$\mathrm{AH}(1) \simeq \bigoplus_j \boldsymbol{q}^{\frac{j}{2}}\,\mathrm{Gr}^{\mathbf{W}}_j\,\mathrm{H}^j(G\backslash G, Pr_{\beta,!}\,\bar{\mathbf{Q}}_\ell). \tag{1.8.19}$$

The right-hand side is very close to the pure part of the compactly-supported cohomology of $G\backslash \mathit{St}_G(1)$, except that above, we have H^j instead of H^j_c. Recall from Example 1.7.9 that $\mathrm{AH}(1) \simeq \bigoplus_j \boldsymbol{q}^{\frac{j}{2}} \mathbf{A}^j$.

It is interesting to compare these results with the main theorems of [31] and [64]. These papers essentially show that $\mathbf{A}$, under its *opposite* grading and up to a shift, is the compactly-supported cohomology of $G\backslash \mathit{St}(1)$, where $\mathit{St}(1)$ is the pullback of $\mathit{St}_G(1)$ to the unipotent locus of G. (Note that they only discuss the variety $\mathit{St}(1)$, not the stack $G\backslash \mathit{St}(1)$.)

1.8.3. Cohomological Correspondences Theorem 1.8.8 lets us describe the W-action on $\mathcal{A}\mathrm{H}(\beta)$ via cohomological correspondences on $G_0\backslash \mathit{St}_G(\beta)_0$. For a beautifully concise exposition of the formalism of correspondences, see [**?**, Appendix A].

First, we recall the construction of the W-action on the Grothendieck–Springer sheaf by means of correspondences, *cf.* [21, §3]. For the later application to $\mathit{St}_G(\beta)_0$, we write everything in the setting of G_0 rather than G. Let

$$\widetilde{O}(1)_0^{\mathrm{rs}} = \{(xB, g) \in \widetilde{O}(1)_0 : g \in G_0^{\mathrm{rs}}\} \subseteq \widetilde{O}(1)_0. \tag{1.8.20}$$

Springer observed that $\widetilde{O}(1)_0^{\mathrm{rs}} \to G_0^{\mathrm{rs}}$ is the pullback of the W-cover $T_0^{\mathrm{rs}} \to T_0^{\mathrm{rs}} /\!\!/ W$, whence W acts on $\widetilde{O}(1)_0^{\mathrm{rs}}$. We can decompose

$$\mathit{St}_G(1)_0^{\mathrm{rs}} = \widetilde{O}(1)_0^{\mathrm{rs}} \times_{G_0} \widetilde{O}(1)_0^{\mathrm{rs}} \subseteq \mathit{St}_G(1)_0 \tag{1.8.21}$$

into the disjoint union of the action graphs $\Gamma_{w,0}$ as w runs over W. Let $\mathit{St}_G(1)_{w,0}$ be the closure of $\Gamma_{w,0}$ in $\mathit{St}_G(1)_0$. In the terminology of [**?**],

$$\begin{array}{ccccc} & & \mathit{St}_G(1)_0 & & \\ & \swarrow^{\mathrm{pr}_1} & & \searrow^{\mathrm{pr}_2} & \\ \widetilde{O}(1)_0 & \longrightarrow & G_0 & \longleftarrow & \widetilde{O}(1)_0 \end{array} \tag{1.8.22}$$

is a graph-like self-correspondence of $\widetilde{O}(1)_0$ over G. Everything is G_0-equivariant, so these correspondences descend to self-correspondences of $\widetilde{\mathcal{X}}(1)_0$ over $G_0\backslash G_0$. The resulting action on $\mathcal{E}_{1,0} = \mathrm{pr}_{1,*}\, \bar{\mathbf{Q}}_{\ell,0}$ is the Springer action.

We bootstrap the Springer action to a W-action on $Pr_{\beta,!}\, \bar{\mathbf{Q}}_{\ell,0}$. Below, let

$$St_G(1,\beta)_0 = \widetilde{O}(1)_0 \times_{G_0} \widetilde{O}(1)_0 \times_{G_0} \widetilde{O}(\beta)_0. \tag{1.8.23}$$

By pullback,

$$\begin{array}{ccccc} & & St_G(1,\beta)_0 & & \\ & \swarrow^{\mathrm{pr}_{1,3}} & & \searrow^{\mathrm{pr}_{2,3}} & \\ St(\beta)_0 & \longrightarrow & G_0 & \longleftarrow & St(\beta)_0 \end{array} \tag{1.8.24}$$

is a graph-like self-correspondence of $St_G(\beta)_0$ over G_0. For each w, we let $St_G(1,\beta)_{w,0}$ be the pullback of $St(1)_{w,0}$ to $St_G(1,\beta)_0$. These correspondences descend to self-correspondences on $G_0\backslash St_G(\beta)_0$, so we obtain a W-action on $Pr_{\beta,!}\, \bar{\mathbf{Q}}_{\ell,0}$.

Since everything is weight-preserving, we get an action on $\mathcal{C}hr(Pr_{\beta,!}\, \bar{\mathbf{Q}}_{\ell,0})$. Then by Theorem 1.8.8, we get an action on $\mathcal{A}\mathrm{H}(\beta)$. We conclude by affirming:

Proposition 1.8.11. *For all $\beta \in Br_W^+$, the following coincide:*

1. *The W-part of the $\mathbf{A}$-action on $\mathcal{A}\mathrm{H}(\beta)$ defined by means of the Yoneda product.*

2. *The W-action on $\mathcal{A}\mathrm{H}(\beta)$ constructed above by means of correspondences.*

Proof. One can check that the isomorphism of functors (1.8.17) identifies:

1. The W-part of the $\mathrm{Ext}^*_{G\backslash G}(\mathcal{E}_1, \mathcal{E}_1)$-action on $\mathrm{Ext}^*_{G\backslash G}(\mathcal{E}_1, -)$ defined by the Yoneda product.

2. The W-action on $\mathrm{H}^*(G\backslash G, \mathcal{E}_1 \otimes^{\mathrm{L}} (-))$ induced from the W-action on $\mathcal{E}_1$ by the functoriality of $\mathcal{E}_1 \otimes^{\mathrm{L}} (-)$.

At the same time, diagram chasing shows that under the isomorphism in Lemma 1.8.9, the W-action on $\mathcal{E}_1 \otimes^{\mathrm{L}} \mathcal{C}hr(pr_{\beta,!}\bar{\mathbf{Q}}_\ell)$ corresponds to the W-action on $\mathcal{C}hr(Pr_{\beta,!}\,\bar{\mathbf{Q}}_{\ell,0})$ that we constructed via correspondences. □

1.9 Appendix: Chromatography

1.9.1. Let $\boldsymbol{k}$ be a field. Fix a prime ℓ invertible in $\boldsymbol{k}$.

If X is a $\boldsymbol{k}$-scheme of finite type (later, an Artin stack), then we write $\mathsf{D}^b(X)$ for the derived category of bounded constructible complexes of étale $\bar{\mathbf{Q}}_\ell$-sheaves over X. We write $\mathsf{Perv}(X)$ for the full subcategory of $\mathsf{D}^b(X)$ of perverse sheaves (with respect to the middle perversity), and $\mathsf{D}^b(\mathsf{Perv}(X))$ for the bounded derived category of $\mathsf{Perv}(X)$.

1.9.2. Fix a finite field $\mathbf{F} = \mathbf{F}_q$. Henceforth, *we assume that* $\boldsymbol{k} = \bar{\mathbf{F}}$ *and that* X *is equipped with an* $\mathbf{F}$*-structure, i.e., an* $\mathbf{F}$*-scheme* X_0 *such that* $X = X_0 \otimes \boldsymbol{k}$. Following [4], we also use the subscript 0 to distinguish sheaves on X_0 from their pullbacks to X.

Let $\mathsf{D}^b_m(X_0)$ be the full subcategory of $\mathsf{D}^b(X_0)$ of mixed complexes. For all $\alpha \in \mathbf{R}$, let $\mathsf{D}^b_{\leq\alpha}(X_0)$, resp. $\mathsf{D}^b_{\geq\alpha}(X_0)$, be the full subcategory of $\mathsf{D}^b_m(X_0)$ of complexes of weight $\leq \alpha$, resp. $\geq \alpha$. The notations $\mathsf{D}^b_{<\alpha}$, $\mathsf{D}^b_{>\alpha}$ are defined similarly.

We refer to Chapitre 5 of Beilinson–Bernstein–Deligne–Gabber's book [4] for the suite of fundamental theorems describing how perverse sheaves over X_0 interact with weights, mixedness, and purity. We introduce two notations that are not standard:

Definition 1.9.1. Let $\mathsf{D}^b_m(\mathsf{Perv}(X_0))$ be the full subcategory of $\mathsf{D}^b(\mathsf{Perv}(X_0))$ of mixed objects, i.e., complexes whose cohomology sheaves are mixed perverse sheaves.

Definition 1.9.2. Let $\mathsf{Pur}(X)$ be the full subcategory of $\mathsf{D}^b(X)$ formed by the pullbacks of pure complexes over X_0 of weight 0.

1.9.3. The chromatography functor, introduced by Bondarko in [12, 13] under the name of "weight complex functor" and by Webster–Williamson in [112], is a certain additive functor

$$\mathcal{C}hr : \mathsf{D}^b_m(\mathsf{Perv}(X_0)) \to \mathsf{K}^b(\mathsf{Pur}(X)) \tag{1.9.1}$$

of triangulated categories, where $\mathsf{K}^b(\mathsf{Pur}(X))$ is the bounded homotopy category of complexes of objects of $\mathsf{Pur}(X)$. We will sketch its construction, following Bondarko.

Remark 1.9.3. A further exposition of $\mathcal{C}hr$ can be found in Section 6 of the paper [95]. As the authors explain on page 1103, the name comes from chemistry. It connotes the separation of a mixture into pure substances.

The existence of the chromatography functor is motivated by the result below from [4, Th. 5.3.5]:

Theorem 1.9.4 (Beilinson–Bernstein–Deligne–Gabber)**.** *Let E_0 be a mixed perverse sheaf over X_0. Then there is an increasing filtration $\{\mathbf{W}_{\leq i}E_0\}_i$ of E_0 by perverse sheaves, such that for all i, the quotient*

$$\mathrm{Gr}_i^{\mathbf{W}} E_0 = \mathbf{W}_{\leq i+1}E_0/\mathbf{W}_{\leq i}E_0 \tag{1.9.2}$$

is either zero or a pure perverse sheaf of weight i. Such a filtration is strictly compatible with all morphisms of mixed perverse sheaves. If we require the $\mathrm{Gr}_i^{\mathbf{W}} E_0$ to be nonzero, then it is unique up to isomorphism.

Above, it turns out that the shifted quotients $\mathrm{Gr}_i^{\mathbf{W}} E_0[-i]$ can be assembled into a single complex, thereby giving an object of $\mathsf{K}^b(\mathsf{Pur}(X))$. We will explain this procedure in the course of the general construction of the chromatographic complex.

Let $K_0 \in \mathsf{D}^b_m(X_0)$. We define an α**-distillation** of K_0 to be an exact triangle of the form

$$K_{\leq\alpha,0} \to K_0 \to K_{\geq\alpha+1,0} \to K_0'[1] \tag{1.9.3}$$

in which $K_{\leq \alpha,0} \in \mathsf{D}^b_{\leq \alpha}(X_0)$ and $K_{\geq \alpha+1,0} \in \mathsf{D}^b_{\geq \alpha+1}(X_0)$. The lemma below follows from (3)-(4) of [13, Prop. 3.5.3]:

Lemma 1.9.5 (Bondarko). *Let $K_0, L_0 \in \mathsf{D}^b_m(X_0)$, and let $i, j \in \mathbf{Z}$, where $i \leq j$. Fix an i-distillation of K_0 and a j-distillation of L_0. Let $\phi : K_0 \to L_0$ be a morphism of mixed complexes.*

1. *If $i < j$, then there is a morphism of exact triangles of the form*

$$\begin{array}{ccccccc} K_{\leq i,0} & \longrightarrow & K_0 & \longrightarrow & K_{\geq i+1,0} & \longrightarrow & K_{\leq i,0}[1] \\ \downarrow & & \downarrow \phi & & \downarrow & & \downarrow \\ L_{\leq j,0} & \longrightarrow & L_0 & \longrightarrow & L_{\geq j+1,0} & \longrightarrow & L_{\leq j,0}[1] \end{array} \tag{1.9.4}$$

   Moreover, the induced maps $K_{\leq i} \to L_{\leq j}$ and $K_{\geq i+1} \to L_{\geq j+1}$ are uniquely determined by the original data.

2. *If $i = j$, then there is a commutative diagram of the form*

$$\begin{array}{ccccc} K_{\leq i} & \longrightarrow & K & \longrightarrow & K_{\geq i+1} \\ \downarrow & & \downarrow \phi & & \downarrow \\ L_{\leq j} & \longrightarrow & L & \longrightarrow & L_{\geq j+1} \end{array} \tag{1.9.5}$$

We define a **distillation** of K_0 to be a choice of i-distillation of K_0 for all i, along with a system of compatible morphisms of exact triangles between them, such that $K_{\leq i,0} = 0$ for $i \ll 0$ and $K_{\geq i,0} = K_0$ for $i \gg 0$. We can visualize this data as a pair of sequences of morphisms:

$$\cdots \to K_{\leq i-1,0} \to K_{\leq i,0} \to \cdots \quad \text{and} \quad \cdots \to K_{\geq i,0} \to K_{\geq i+1,0} \to \cdots \tag{1.9.6}$$

Abusing notation, we write

$$\operatorname{Gr}_i K_0 = \operatorname{cone}(K_{0,\leq i-1} \to K_{0,\leq i}). \tag{1.9.7}$$

Since $\mathsf{D}^b_{\leq i}(X_0)$ and $\mathsf{D}^b_{\geq i}(X_0)$ are stable under extensions, $\operatorname{Gr}_i K_0$ is pure of weight i. Also, it vanishes for $i \ll 0$ and $i \gg 0$.

We use the $\operatorname{Gr}_i K_0$ to assemble an object of $\mathsf{K}^b(\mathsf{Pur}(X))$. To start, consider the exact triangles

$$\operatorname{Gr}_i K_0 \to \operatorname{cone}(K_{0,\,\leq i-1} \to K_{0,\,\leq i+1}) \to \operatorname{Gr}_{i+1} K_0 \xrightarrow{\partial_i} \operatorname{Gr}_i K_0[1]. \tag{1.9.8}$$

Let $\operatorname{Gr}_i K$ denote the pullback of $\operatorname{Gr}_i K_0$ to X. The **chromatographic sequence** associated with the distilling filtration is the sequence of morphisms

$$\cdots \to \operatorname{Gr}_{i+1} K[-(i+1)] \xrightarrow{\partial_i} \operatorname{Gr}_i K[-i] \xrightarrow{\partial_{i-1}} \operatorname{Gr}_{i-1} K[-(i-1)] \to \cdots \tag{1.9.9}$$

where $\operatorname{Gr}_i K[-i]$ occurs in degree i. Using the octahedral axiom, one can check that this is actually a complex [112, Lem. 3.5]. Also, every term belongs to $\mathsf{Pur}(X)$ by construction.

Lemma 1.9.6. *Every mixed complex admits a distillation.*

Proof. By Theorem 1.9.4, the statement is true for mixed perverse sheaves. So by induction on the filtration of a general mixed complex by perverse truncations, it suffices to show the following: If $L_0 \to K_0 \to M_0 \to L_0[1]$ is an exact triangle and there exist distillations of L_0 and M_0, then there exists a distillation of K_0. By Lemma 1.9.5(1), we further reduce to the following: If there exist 0-distillations

$$L_{\leq 0,0} \to L_0 \to L_{\geq 1,0} \to L_{\leq 0,0}[1], \tag{1.9.10}$$

$$M_{\leq 0,0} \to M_0 \to M_{\geq 1,0} \to M_{\leq 0,0}[1], \tag{1.9.11}$$

then there exists a 0-distillation of K_0. This follows from an argument using the octahedral axiom and the stability of $\mathsf{D}^b_{\leq 0}$ and $\mathsf{D}^b_{\geq 1}$ under extensions [112, 2571-2572]. □

The following theorem is essentially proved in [12], using Lemma 1.9.5 to check the functoriality of the chromatographic complex. However, instead of $\mathsf{D}^b_m(\mathsf{Perv}(X_0))$, that paper considers a triangulated category satisfying more restrictive conditions. Bondarko generalized the results to $\mathsf{D}^b_m(\mathsf{Perv}(X_0))$ in [13].

Theorem 1.9.7 (Bondarko)**.** *If $K_0 \in \mathsf{D}^b_m(\mathsf{Perv}(X_0))$, then any two distillations of K_0 yield homotopic chromatographic complexes. The map that sends K_0 to its chromatographic complex up to homotopy extends to a functor $\mathsf{D}^b_m(\mathsf{Perv}(X_0)) \to \mathsf{K}^b(\mathsf{Pur}(X))$.*

Formally, the **chromatography functor** $\mathcal{C}hr$ is the functor established by this theorem. For any $K_0 \in \mathsf{D}^b_m(X_0)$, the object $\mathcal{C}hr(K_0)$ is called the **chromatographic complex** of K_0.

Remark 1.9.8*.* In [13], Bondarko refers to 0-distillations as *weight decompositions* and to distillations as *bounded Postnikov towers*.

1.9.4. Bondarko's paper [13] only deals with the setting where X_0 is a scheme of finite type over $\mathbf{F}$. In his work, the domain of $\mathcal{C}hr$ is actually $\mathsf{D}^b_m(X_0)$ instead of $\mathsf{D}^b_m(\mathsf{Perv}(X_0))$. Below, we explain why the construction of $\mathcal{C}hr$ still works when X_0 is an Artin stack of finite type, *as long as we take the domain to be* $\mathsf{D}^b_m(\mathsf{Perv}(X_0))$.

The Grothendieck–Verdier yoga of six operations on Artin stacks locally of finite type over a finite-dimensional affine excellent base was worked out by Laszlo–Olsson in [65], along with a yoga of t-structures and perverse sheaves in [66]. Notably, Theorem 9.2 in the latter paper is the stacky generalization of Theorem 1.9.4. So Lemma 1.9.6 still holds for stacks.

In both [13] and [112], the proof that any two distillations of K_0 give rise to homotopic chromatographic complexes relies on being able to express K_0 as a complex of perverse sheaves in $\mathsf{D}^b_m(X_0)$. When X_0 is a scheme of finite type, $\mathsf{D}^b(\mathsf{Perv}(X_0)) \simeq \mathsf{D}^b(X_0)$ by a theorem of Beilinson. But this theorem fails when X_0 is an Artin stack (e.g., when $X_0 = (\,\cdot\,/\mathbf{G}_m)$).

Altogether, it is essential that the domain is $\mathsf{D}^b(\mathsf{Perv}(X_0))$, not $\mathsf{D}^b(X_0)$, in Theorem 1.9.7. Besides this change, the proofs in *ibid.* go through.

Chapter 2

Annular Homology of Artin Braids II: Springer Fibers

2.1 Introduction

2.1.1. Let W be a finite, crystallographic Coxeter group. Let $\mathrm{K}_0(W)$ be its ring of virtual representations, and let Br_W be its Artin group. In the paper [105], we introduced a class function on Br_W called **annular braid homology**, taking the form

$$\mathrm{AH} : Br_W \to \mathrm{K}_0(W)[\![\boldsymbol{q}^{\frac{1}{2}}]\!][t]. \tag{2.1.1}$$

We motivated the existence of AH using ideas from geometric topology. When $W = S_n$, the elements of Br_W can be viewed as topological braids on n strands, which can be closed up end-to-end to form links in 3-space. In this way, AH can be used to reconstruct an isotopy invariant of links valued in $\mathbf{Z}(\!(\boldsymbol{q}^{\frac{1}{2}})\!)[a^{\pm 1}, t^{\pm 1}]$, due to Khovanov–Rozansky [62, 61]. In this paper and the sequels [107, 108], we turn to relating annular braid homology with other phenomena in the geometric representation theory of W.

Let $\mathbf{F} = \mathbf{F}_q$ be a finite field and fix a prime $\ell > 0$ invertible in $\mathbf{F}$. Suppose that W is the Weyl group of a split semisimple group G_0 over $\mathbf{F}$. Throughout, *we will use the subscript* 0 *to denote spaces over* $\mathbf{F}$*, and omit the subscript to denote their base change to* $\bar{\mathbf{F}}$. In particular, $G = G_0 \otimes \bar{\mathbf{F}}$. The notation F will denote geometric Frobenius maps.

Let $\mathcal{B}_0$ be the flag variety of G_0 and $\mathcal{U}_0 \subseteq G_0$ the unipotent locus. Springer observed in [102] that for any $u \in \mathcal{U}^F$, there is a W-action on the ℓ-adic cohomology of the proper variety

$$\mathcal{B}_u = \{B \in \mathcal{B} : u \in B\}. \tag{2.1.2}$$

The varieties $\mathcal{B}_u$ are now called Springer fibers; the graded characters

$$Q_u = \sum_{i \geq 0} \boldsymbol{q}^i \mathrm{H}^{2i}(\mathcal{B}_u, \bar{\mathbf{Q}}_\ell) \in \mathrm{K}_0(W)[\boldsymbol{q}] \tag{2.1.3}$$

are generally called total Springer representations. The functions of the form $u \mapsto Q_u(w) = \mathrm{tr}(w \mid Q_u)$, for $w \in W$, play a major role in the character theory of G^F. In this paper, we will show that for any $\beta \in Br_W$, we can decompose

$$\textsc{ann}(\beta) = \mathrm{AH}(\beta)|_{t \to -1} \tag{2.1.4}$$

as a $\mathbf{Z}[\![\boldsymbol{q}]\!]$-linear combination of the characters Q_u. We note that for $W = S_n$, a certain sum of isotypic components of $\textsc{ann}$ can be used to reconstruct the HOMFLY series: the $t \to -1$ limit of the Khovanov–Rozansky link invariant.

We also fix a Coxeter presentation of W, which fixes a positive submonoid $Br_W^+ \subseteq Br_W$. When $\beta \in Br_W^+$ and we specialize $\boldsymbol{q} \to q$, we can interpret the coefficients of the Springer decomposition of $\textsc{ann}$ as *point counts* on certain stacks over $\mathbf{F}$ attached to β. Among other applications, this relationship between $\textsc{ann}$ and point-counting gives evidence toward a conjectural relationship between AH and cohomological weight filtrations.

2.1.2. To motivate our results, we briefly sketch how AH is related to algebraic geometry. The Artin group Br_W is categorified by a certain monoidal category of sheaf-theoretic objects over $G_0 \backslash (\mathcal{B}_0 \times \mathcal{B}_0)$, discussed in [105, §6]. The horocycle correspondence

$$\begin{array}{ccccc} G_0\backslash(\mathcal{B}_0 \times \mathcal{B}_0) & \xleftarrow{act} & G_0\backslash(\mathcal{B}_0 \times G_0) & \xrightarrow{pr} & G_0\backslash G_0 \\ (B, {}^gB) & \longleftarrow\!\shortmid & (B, g) & \longmapsto & g \end{array} \tag{2.1.5}$$

(where G_0 acts on itself by conjugation) is roughly an algebro-geometric incarnation of the passage from elements of Br_W to their conjugacy classes. Accordingly, AH is defined in terms of the pullback-pushforward of sheaves through this diagram.

In the rest of this introduction, we focus attention on the submonoid $Br_W^+ \subseteq Br_W$ to make things more concrete. The work of Broué–Michel [17] and Deligne [27] gives a map

$$\beta \mapsto O(\beta)_0 \tag{2.1.6}$$

from elements of Br_W^+ to G_0-varieties over $\mathcal{B}_0 \times \mathcal{B}_0$ up to isomorphism, which takes the composition in Br_W^+ to the fiber product of varieties over $\mathcal{B}_0$. We let $\tilde{O}(\beta)_0$ be the pullback of $O(\beta)_0$ along *act*. Then there is a commutative diagram of G-equivariant maps:

$$\begin{array}{ccccc} O(\beta)_0 & \longleftarrow & \tilde{O}(\beta)_0 & \dashrightarrow & \\ \downarrow & & \downarrow & & \\ \mathcal{B}_0 \times \mathcal{B}_0 & \longleftarrow & \mathcal{B}_0 \times G_0 & \longrightarrow & G_0 \end{array} \tag{2.1.7}$$

We note that (the underlying reduced schemes of) the fibers of $\tilde{O}(1) \to G$ over $\mathcal{U}^F$ are precisely the Springer fibers $\mathcal{B}_u$. Indeed, Springer's W-action on the cohomology of $\mathcal{B}_u$ arises from an action on the pushforward of the constant sheaf along this map.

In [105], we give a formula for $\mathrm{AH}(\beta)$ in terms of the **Steinberg-like variety**

$$St_G(\beta)_0 = \tilde{O}(\beta)_0 \times_{G_0} \tilde{O}(1)_0. \tag{2.1.8}$$

However, the formula is somewhat technical: Most notably, it involves the *chromatography* functor of Bondarko [12, 13] and Webster–Williamson [112], applied to the pushforward of the constant sheaf along $G_0 \backslash St_G(\beta)_0 \to G_0 \backslash G_0$. Loosely, this functor turns a mixed complex of sheaves into a complex where the terms themselves are pure complexes of weight 0.

We seek a cleaner formula for $\mathrm{AH}(\beta)$, relating it directly with the cohomology of some stack. To this end, for any Artin stack X_0 over $\mathbf{F}$, let

$$\mathrm{E}(\boldsymbol{q}^{\frac{1}{2}}, t \mid X) = \bigoplus_{j,k} \boldsymbol{q}^{\frac{j}{2}} t^k \, \mathrm{Gr}_j^{\mathbf{W}} \, \mathrm{H}_c^k(X, \bar{\mathbf{Q}}_\ell), \tag{2.1.9}$$

where $\mathrm{H}_c^*(X, \bar{\mathbf{Q}}_\ell)$ is the compactly-supported ℓ-adic cohomology of X and $\mathbf{W}_{\leq *}$ is its weight filtration. Let

$$\mathit{St}(\beta)_0 = \mathcal{U}_0 \times_{G_0} \mathit{St}_G(\beta)_0. \tag{2.1.10}$$

Via the factor $G\backslash\widetilde{O}(1)$, there is a weight-preserving W-action on $\mathrm{H}_c^*(G\backslash \mathit{St}(\beta), \bar{\mathbf{Q}}_\ell)$. Thus, we can view $\mathrm{E}(\boldsymbol{q}^{\frac{1}{2}}, t \mid G\backslash \mathit{St}(\beta))$ as an element of $\mathrm{K}_0(W)[\![\boldsymbol{q}^{\frac{1}{2}}]\!][t]$. We conjecture that it differs from $\mathrm{AH}(\beta)$ only by a change of variables and a normalization factor.

Conjecture 2.1.1. *For all $\beta \in Br_W^+$, we have*

$$\mathrm{AH}(\beta)|_{\boldsymbol{q}^{1/2} \to \boldsymbol{q}^{-1/2} t^{-1}} = (\boldsymbol{q} t^2)^{r - |\beta|} \, \mathrm{E}(\boldsymbol{q}^{\frac{1}{2}}, t \mid G\backslash \mathit{St}(\beta)) \tag{2.1.11}$$

in $\mathrm{K}_0(W)[\![\boldsymbol{q}^{\frac{1}{2}}]\!][t]$, where r is the rank of G and $|\beta|$ is the writhe of β (see [105, §2]).

Example 2.1.2. Let $1 \in Br_W^+$ be the identity. Its writhe is 0. In [105, §7], we computed that

$$\mathrm{AH}(1) \simeq \frac{\bar{\mathbf{Q}}_\ell[W]}{(1 - \boldsymbol{q})^r}. \tag{2.1.12}$$

By a stacky version of the main result of [31, 64], we have

$$\mathrm{E}(\boldsymbol{q}^{\frac{1}{2}}, t \mid G\backslash \mathit{St}(1)) \simeq \frac{1}{(\boldsymbol{q} t^2)^r} \cdot \frac{\bar{\mathbf{Q}}_\ell[W]}{(1 - \boldsymbol{q}^{-1} t^{-2})^r}. \tag{2.1.13}$$

This verifies the conjecture for $\beta = 1$.

2.1.3. We will prove that Conjecture 2.1.1 holds in the $t \to -1$ limit. For any X_0 over $\mathbf{F}$, let

$$\mathrm{E}(\boldsymbol{q}^{\frac{1}{2}} \mid X) = \mathrm{E}(\boldsymbol{q}^{\frac{1}{2}}, t \mid X)|_{t \to -1}, \tag{2.1.14}$$

an element of $\mathrm{K}_0(W)[\![\boldsymbol{q}^{\frac{1}{2}}]\!]$. In the literature, the graded dimension of $\mathrm{E}(\boldsymbol{q}^{\frac{1}{2}} \mid X)$ is known as the virtual Poincaré series of X_0. By the results of [54, Appendix], we can show that $\mathrm{E}(\boldsymbol{q}^{\frac{1}{2}} \mid G \backslash \mathit{St}(\beta))$ specializes to the $\mathbf{F}$-point count of $G \backslash \mathit{St}(\beta)$ when we take $\boldsymbol{q} \to q$. In this way, we are led to study the fibers of $\mathit{St}(\beta)$ over $\mathcal{U}^F$, which take the form $\tilde{O}(\beta)_u \times \mathcal{B}_u$ for $u \in \mathcal{U}^F$. Ultimately, we will prove:

Theorem 2.1.3. *For all $\beta \in Br_W^+$ and $q \gg 0$, we have*

$$\textsc{ann}(\beta)|_{\boldsymbol{q} \to q} = \frac{(-1)^{r-|\beta|}}{|G^F|} \sum_{u \in \mathcal{U}^F} |\tilde{O}(\beta)_u^F| \cdot Q_u|_{\boldsymbol{q} \to q} \tag{2.1.15}$$

in $\mathrm{K}_0(W)[q][\frac{1}{|G^F|}]$, *where* $\textsc{ann} = \mathrm{AH}|_{t \to -1}$.

This identity is the relationship between ANN, total Springer representations, and point-counting described at the beginning of the paper. To prove it, we apply Deligne–Lusztig theory to the purely algebraic definition of ANN given in [105], which involves Lusztig's exotic Fourier transform. As a first application, we deduce:

Theorem 2.1.4. *For all $\beta \in Br_W^+$, we have*

$$\textsc{ann}(\beta) = (-1)^{r-|\beta|}\, \mathrm{E}(\boldsymbol{q}^{\frac{1}{2}} \mid G \backslash \mathit{St}(\beta)) \tag{2.1.16}$$

in $\mathrm{K}_0(W)[\![\boldsymbol{q}]\!]$.

In [105, §4], we proved that the shifted annular character $(\boldsymbol{q}^{\frac{1}{2}})^{r-|\beta|}\textsc{ann}(\beta)$ is invariant under the substitution $\boldsymbol{q}^{\frac{1}{2}} \mapsto -\boldsymbol{q}^{-\frac{1}{2}}$, or equivalently,

$$\textsc{ann}(\beta)|_{\boldsymbol{q}^{1/2} \to -\boldsymbol{q}^{-1/2}} = (-\boldsymbol{q})^{r-|\beta|}\textsc{ann}(\beta). \tag{2.1.17}$$

This shows Theorem 2.1.4 is the $t \to -1$ limit of Conjecture 2.1.1.

As a second application of Theorem 2.1.3, we show that $\textsc{ann}$ is compatible with parabolic induction. Observe that if W' is a parabolic subgroup of W with respect to the chosen Coxeter presentation, then the inclusion $W' \to W$ induces an inclusion of Artin groups $Br_{W'} \to Br_W$. Let $\textsc{ann}_W$ and $\textsc{ann}_{W'}$ denote the appropriate versions of $\textsc{ann}$.

Theorem 2.1.5. *For any parabolic subgroup $W' \to W$, we have a commutative diagram*

(2.1.18)
$$\begin{array}{ccc} Br_{W'} & \xrightarrow{\textsc{ann}_{W'}} & \mathrm{K}_0(W')[\![\boldsymbol{q}]\!] \\ \downarrow & & \downarrow {\scriptstyle (1-\boldsymbol{q})^{r'-r}\,\mathrm{Ind}_{W'}^{W}} \\ Br_W & \xrightarrow{\textsc{ann}_W} & \mathrm{K}_0(W)[\![\boldsymbol{q}]\!] \end{array}$$

where r and r' are the ranks of W and W', respectively.

In [105, §4], we proved that $\textsc{ann}(\beta)$ is a rational function of $\boldsymbol{q}$. As a third application of Theorem 2.1.3, we combine it with Lusztig's techniques in [76, §5] to show that the only pole of $\textsc{ann}(\beta)$ occurs at $\boldsymbol{q} = 1$. In what follows, let $\mathbf{X}$ be the character lattice of G and let $V = \mathbf{X} \otimes \mathbf{Q}$, a representation of W. Let $r = \dim V$, and for all $w \in W$, let

(2.1.19)
$$r(w) = \dim V^w.$$

Note that $V^W = 0$ because we assume G_0 is semisimple.

Theorem 2.1.6. *If $\beta \in Br_W^+$ maps to $w \in W$ under the surjection $Br_W \twoheadrightarrow W$, then*

(2.1.20)
$$\textsc{ann}(\beta) \in \frac{1}{(1-\boldsymbol{q})^{r(w)}} \mathrm{K}_0(W)[\boldsymbol{q}].$$

In particular, the pole of $\textsc{ann}(\beta)$ at $\boldsymbol{q} = 1$ has order at most $r(1) = r$.

Remark 2.1.7. While this work was in preparation, Lusztig–Yun uploaded the preprint [77] to the arχiv. It introduces a class function

$$\Psi : W \to \mathrm{K}_0(W)[\boldsymbol{q}] \tag{2.1.21}$$

that refines Lusztig's map in [76] from conjugacy classes of W to unipotent orbits of G. For all $w \in W$, we expect that $\Psi(w) = (1-\boldsymbol{q})^{r(w)}\mathrm{ANN}(\beta_w)$, where $\beta_w \in Br_W^+$ is the positive lift of w of minimal writhe.

Using the function $w \mapsto r(w)$, we can formulate a new conjecture that simultaneously refines Conjecture 2.1.1 and Theorem 2.1.6.

Conjecture 2.1.8. *If $\beta \in Br_W^+$ maps to $w \in W$, then*

$$(1-\boldsymbol{q})^{r(w)}\mathrm{AH}(\beta) = (\boldsymbol{q}t^2 - 1)^{r(w)}\mathrm{E}(\boldsymbol{q}^{\frac{1}{2}}, t \mid G\backslash St(\beta)). \tag{2.1.22}$$

Moreover, both sides belong to $\mathrm{K}_0(W)[\boldsymbol{q}^{\frac{1}{2}}][t]$, *not just* $\mathrm{K}_0(W)[\![\boldsymbol{q}^{\frac{1}{2}}]\!][t]$.

At the very least, this new conjecture is compatible with Example 2.1.2. Note that it is compatible with Conjecture 2.1.1 only if the integer

$$|\beta| - r + r(w) \tag{2.1.23}$$

is even. As we point out in [108, §9], this can be checked by induction on the writhe of β.

2.1.4. To conclude this introduction, we suggest two further directions of study.

The Full Twist Let $\pi \in Br_W^+$ be the central element known as the full twist; let $\beta \in Br_W$ be arbitrary. When $W = S_n$, the elements β and $\beta\pi$ are topological braids that, when closed up, form distinct links in 3-space. In this setting, Kálmán discovered a remarkable coincidence of coefficients between the HOMFLY series of the links attached to β and $\beta\pi$ [58]. His theorem can be generalized to arbitrary finite Coxeter groups by replacing HOMFLY series with Markov traces. When $\beta \in Br_W^+$, we can use Theorem 2.1.3, together with the relation between ANN and Y. Gomi's Markov trace proved in [105, §4], to reinterpret the result as a coincidence of point counts:

Theorem 2.1.9. *If $\beta \in Br_W^+$, then $|\tilde{O}(\beta\pi)_1^F| = |(\mathcal{U} \times_G \tilde{O}(\beta))^F|$ for $q \gg 0$.*

Together, Theorems 2.1.4 and 2.1.9 imply a result that Shende–Treumann–Zaslow proved using Legendrian knot theory [95, Thm. 6.34].[1]

Corollary 2.1.10 (Shende–Treumann–Zaslow)**.** *If $G = \mathrm{PGL}_{r+1}$ and $\beta \in Br_W^+$, then*

$$[(\boldsymbol{q}^{\frac{1}{2}} a^{-1})^{|\beta|-r} \textit{HOMFLY}(\beta)]|_{(\boldsymbol{q}^{1/2},a)\to(q^{1/2},0)} = \pm \frac{|\tilde{O}(\beta\pi)_1^F|}{|G^F|} \tag{2.1.24}$$

for $q \gg 0$, where $\textit{HOMFLY}(\beta) \in \mathbf{Z}(\boldsymbol{q}^{\frac{1}{2}})[a]$ *is the HOMFLY series of the link attached to β.*

It seems natural to ask whether Theorem 2.1.9 can be categorified to an isomorphism of weight gradings in cohomology. In [48], Gorsky–Hogancamp–Mellit–Nakagane categorify Kálmán's theorem in the language of Soergel bimodules. Essentially, they show that the complex of Soergel bimodules attached to π is a dualizing object for the Soergel category, which implies that Kálmán's work can be lifted from the level of HOMFLY series to the level of Khovanov–Rozansky homology. In light of Conjecture 2.1.1, together with the relation between annular braid homology and Khovanov–Rozansky homology proved in [105, §7], we expect the geometric counterpart of [48] to be:

Conjecture 2.1.11. *If $\beta \in Br_W^+$, then $\mathrm{E}(\boldsymbol{q}^{\frac{1}{2}}, t \mid G\backslash\tilde{O}(\beta\pi)_1) = \mathrm{E}(\boldsymbol{q}^{\frac{1}{2}}, t \mid G\backslash(\mathcal{U} \times_G \tilde{O}(\beta)))$.*

Curious Lefschetz Symmetry Mellit has recently established the so-called *curious Lefschetz property* for generic character varieties with structure group $G = \mathrm{GL}_n$ [82] (see also

1. In particular, we answer a question of Shende on MathOverflow: https://mathoverflow.net/questions/160810/the-markov-trace-via-bott-samelson-fibers

[54]). For an Artin stack X_0 over $\mathbf{F}$, this property amounts to the existence of isomorphisms

$$\mathrm{Gr}^{\mathbf{W}}_{\dim X - j}\, \mathrm{H}^k_c(X, \bar{\mathbf{Q}}_\ell) \xrightarrow{\sim} \mathrm{Gr}^{\mathbf{W}}_{\dim X - j}\, \mathrm{H}^{k+2j}_c(X, \bar{\mathbf{Q}}_\ell) \tag{2.1.25}$$

for all j, k. This means the series $(\boldsymbol{q}^{\frac{1}{2}})^{-\dim X}\mathrm{E}(\boldsymbol{q}^{\frac{1}{2}}, t \mid X)$ is invariant under the substitution $\boldsymbol{q}^{\frac{1}{2}} \mapsto \boldsymbol{q}^{-\frac{1}{2}}t^{-1}$. Mellit's argument relies on decomposing a vector bundle over the given character variety into vector bundles over the varieties $\widetilde{O}(\beta)$ for various $\beta \in Br_W^+$. Roughly, he shows curious Lefschetz for the former follows from curious Lefschetz for the latter.

We likewise hope that the compactly-supported cohomology of $G\backslash St(\beta)$ satisfies curious Lefschetz after we "ignore the stacky part." More precisely, we expect that

$$(\boldsymbol{q}^{\frac{1}{2}})^{r-|\beta|-r(w)}(\boldsymbol{q}t^2 - 1)^{r(w)}\mathrm{E}(\boldsymbol{q}^{\frac{1}{2}}, t \mid G\backslash St(\beta)) \tag{2.1.26}$$

is invariant under $\boldsymbol{q}^{\frac{1}{2}} \mapsto \boldsymbol{q}^{-\frac{1}{2}}t^{-1}$. In light of Conjecture 2.1.8, we can recast this statement in terms of AH, where it can be generalized from elements of Br_W^+ to elements of Br_W:

Conjecture 2.1.12. *If $\beta \in Br_W$ maps to $w \in W$, then*

$$(\boldsymbol{q}^{\frac{1}{2}})^{r-|\beta|-r(w)}(1 - \boldsymbol{q})^{r(w)}\mathrm{AH}(\beta) \tag{2.1.27}$$

is invariant under $\boldsymbol{q}^{\frac{1}{2}} \mapsto \boldsymbol{q}^{-\frac{1}{2}}t^{-1}$.

In [105, §4], we used a form of Alvis–Curtis–Kawanaka duality to show that this conjecture holds in the $t \to -1$ limit.

2.1.5. Acknowledgements I warmly thank Victor Ginzburg, George Lusztig, Andrei Neguţ, Bao-Châu Ngô, Jay Taylor, David Treumann, and Zhiwei Yun for their input. In particular, J. Taylor pointed me to the paper [76], and G. Lusztig told me about his preprint [77], which drew my attention to the order of the pole of ANN at $\boldsymbol{q} = 1$. Sections 2.4-2.5 closely resemble Lusztig–Yun's arguments in [77]. I'm grateful to Laura Ball for proofreading and O. Kivinen for catching faulty conjectures in an earlier draft.

During this work, I was supported by an NSF Graduate Research Fellowship (grant #1746045) and benefited from workshops at MSRI and AIM.

2.1.6. Notation Throughout this paper, we will retain all of the notation from Sections 6-8 of the prequel [105], except that *we will assume G_0 is semisimple, not just reductive.*

The Group W Recall that we fix a Coxeter presentation of W, with respect to which:

- If $w \in W$, then $|w|$ is the Bruhat length of w.
- If $\beta \in Br_W$, then $|\beta|$ is the writhe of β.
- If $w \in W$, then $\beta_w \in Br_W^+$ is the unique lift of w such that $|\beta_w| = |w|$.
- H_W is the Iwahori–Hecke algebra of W over $\mathbf{Z}[\boldsymbol{q}^{\pm\frac{1}{2}}]$. We write $\{\beta_w\}_{w \in W}$ for its standard basis.

We treat the elements of $\mathrm{K}_0(W)$ interchangeably with their characters.

- $\mathrm{Irr}(W) \subseteq \mathrm{K}_0(W)$ is the set of irreducible characters of W.
- 1 and ε are the trivial and sign characters of W, respectively.
- $(-,-)_W$ is the multiplicity pairing on $\mathrm{K}_0(W)$.
- If $\phi \in \mathrm{Irr}(W)$, then $\phi_{\boldsymbol{q}} : H_W \to \mathbf{Q}(\boldsymbol{q}^{\frac{1}{2}})$ is the Tits deformation of ϕ.

The Group G If X_0 is a stack over $\mathbf{F}$, then we let X denote its base change to $\bar{\mathbf{F}}$. We write $F : X \to X$ for the geometric Frobenius. We also fix a square root $q^{\frac{1}{2}} \in \bar{\mathbf{Q}}_\ell$.

- G_0 is a split semisimple group over $\mathbf{F}$ with Weyl group W.
- $V = \mathbf{X} \otimes_{\mathbf{Z}} \mathbf{Q}$, where $\mathbf{X}$ is the character lattice in the root datum of G.
- $\mathcal{B}_0$ is the flag variety of G_0.

- If $w \in W$, then $O_{w,0}$ is the G_0-orbit of $\mathcal{B}_0 \times \mathcal{B}_0$, i.e., Bruhat cell, indexed by w. We write $B \xrightarrow{w} B'$ to indicate that $(B, B') \in O_w$.

- If $\beta \in Br_W^+$, then $O(\beta)_0$ is the Broué–Michel variety attached to β.

- $\mathcal{U}_0$ is the unipotent locus of G_0.

2.2 Preliminaries

2.2.1. Deligne–Lusztig Theory For $w \in W$, the **Deligne–Lusztig variety** $X_w \subseteq \mathcal{B} \times \mathcal{B}$ is the (transverse) intersection of O_w with the graph of $F : \mathcal{B} \to \mathcal{B}$. Since X_w is stable under the action of G^F on $\mathcal{B} \times \mathcal{B}$, it gives rise to a virtual representation of G^F, namely,

$$R_w = \sum_i (-1)^i \mathrm{H}_c^i(X_w, \bar{\mathbf{Q}}_\ell), \tag{2.2.1}$$

where H_c^* denotes étale cohomology with compact support. A virtual representation of G^F is called **unipotent** iff it occurs with nonzero multiplicity in R_w for some w.

Example 2.2.1. When $w = 1$, we see that $X_1 = \mathcal{B}^F$, a finite set of points, and that R_1 is the $\bar{\mathbf{Q}}_\ell$-vector space of functions on $\mathcal{B}^F$. If B is an F-stable Borel of G, then $\mathcal{B}^F \simeq G^F/B^F$. It follows that we can construct R_1 by inducing the trivial character from B^F to G^F.

For all $B \in \mathcal{B}^F$, let $\mathbf{1}_B \in R_1$ be the indicator function on B. The **Hecke operator** T_w is the G^F-equivariant endomorphism of R_1 defined by

$$T_w(\mathbf{1}_B) = \sum_{B \xrightarrow{w} B'} \mathbf{1}_{B'}. \tag{2.2.2}$$

The following well-known result implies the Hecke operators span $\mathrm{End}_{G^F}(R_1)$.

Theorem 2.2.2 (Iwahori)**.** *There is an isomorphism of $\bar{\mathbf{Q}}_\ell$-algebras*

$$H_W \otimes_{\mathbf{Z}[\boldsymbol{q}^{\pm\frac{1}{2}}]} \bar{\mathbf{Q}}_\ell \xrightarrow{\sim} \mathrm{End}_{G^F}(R_1) \tag{2.2.3}$$

that sends $\beta_w \mapsto q^{-\frac{|w|}{2}} T_w$, where above, the tensor product is defined by $\boldsymbol{q}^{\frac{1}{2}} \mapsto q^{\frac{1}{2}}$.

We recall two ways that irreducible representations of W give rise to unipotent virtual representations of G^F:

1. Deligne and Lusztig showed that $(R_w, R_{w'})_{G^F} \neq 0$ if and only if w and w' are conjugate in W. Therefore, for any $\phi \in \mathrm{Irr}(W)$, the $\mathbf{Q}$-linear combination

   $$R_\phi = \frac{1}{|W|} \sum_{w \in W} \phi(w) R_w \tag{2.2.4}$$

   is a unipotent virtual representation of G^F. Its character is called the **almost-character** attached to ϕ.

2. It turns out that for all $\phi \in \mathrm{Irr}(W)$, the Iwahori–Hecke character $\phi_{\boldsymbol{q}}$ maps H_W into $\mathbf{Z}[\boldsymbol{q}^{\pm\frac{1}{2}}]$ [43, Ch. 9-10]. Thus, we can set

   $$\phi_q = \phi_{\boldsymbol{q}}|_{\boldsymbol{q}^{1/2} \to q^{1/2}} : H_W \to \mathbf{Z}[q^{\frac{1}{2}}]. \tag{2.2.5}$$

   By Iwahori's theorem and the double centralizer theorem, we have an isomorphism of (H_W, G^F)-bimodules

   $$R_1 \simeq \bigoplus_{\phi \in \mathrm{Irr}(W)} \phi_q \otimes \rho_\phi, \tag{2.2.6}$$

   where ρ_ϕ is a unipotent irreducible representation of G^F called the **unipotent principal series** attached to ϕ.

Remark 2.2.3. Let $P_\phi(\boldsymbol{q})$, resp. $D_\phi(\boldsymbol{q})$, be the fake, resp. generic, degree of ϕ, as exposited in

[105, §3]. Then $R_\phi(1) = P_\phi(q)$ and $\rho_\phi(1) = D_\phi(q)$.

There is another way in which ϕ_q and ρ_ϕ are related. Let $(-)|_{H_W}$ be the additive functor from representations of G^F to representations of $\mathrm{End}_{G^F}(R_1) \simeq H_W \otimes \bar{\mathbf{Q}}_\ell$ defined by

$$\rho|_{H_W} = \mathrm{Hom}_{G^F}(R_1, \rho). \tag{2.2.7}$$

Applying $(-)|_{H_W}$ to both sides of (2.2.6), we get an isomorphism of H_W-bimodules:

$$H_W \otimes \bar{\mathbf{Q}}_\ell \simeq \bigoplus_{\phi \in \mathrm{Irr}(W)} \phi_q \otimes (\rho_\phi)|_{H_W}. \tag{2.2.8}$$

By Artin–Wedderburn for $H_W \otimes \bar{\mathbf{Q}}_\ell$, we deduce:

Proposition 2.2.4. *For all $\phi \in \mathrm{Irr}(W)$, we have $\rho_\phi|_{H_W} = \phi_q$ as representations of H_W.*

2.2.2. The Exotic Fourier Transform As Lusztig showed [70], the unipotent irreducible characters of G^F can be indexed by a set that only depends on W, which we will denote by $\mathrm{UCh}(W)$. The map $\phi \mapsto \rho_\phi$ defines an embedding $\mathrm{Irr}(W) \to \mathrm{UCh}(W)$. It is a bijection in type A, but not in general.

In his work on the classification of the unipotent irreducible representations of G^F, Lusztig introduced a hermitian unitary pairing

$$\{-,-\} : \mathrm{UCh}(W) \times \mathrm{UCh}(W) \to \mathbf{Q} \tag{2.2.9}$$

now known as the **exotic Fourier transform**. Very roughly, the matrix formed by $\{-,-\}$ takes the vector of almost-characters of G^F to the vector of unipotent irreducible characters of G^F and vice versa. The actual statement, below, involves an involution

$$\boldsymbol{o} : \mathrm{Irr}(W) \to \mathrm{Irr}(W), \tag{2.2.10}$$

which was implicit in Lusztig's work and formally defined in Opdam's paper [91] (where it is denoted j).

Theorem 2.2.5 (Lusztig)**.** *For any unipotent irreducible character ρ of G^F, we have*

$$\rho = \sum_{\psi \in \mathrm{Irr}(W)} \{\rho, \rho_\psi\} R_{\boldsymbol{o}(\psi)} \quad \text{and} \quad R_{\boldsymbol{o}(\phi)} = \sum_{\rho \in \mathrm{UCh}(W)} \{\rho_\phi, \rho\} \rho \tag{2.2.11}$$

in $\mathrm{K}_0(G^F) \otimes \mathbf{Q}$.

A brief summary of some properties of $\{-,-\}$ and $\boldsymbol{o}$ was given in [105, §3], where we used them to state an algebraic definition of ANN. We recall this definition in Section 2.3.

2.2.3. Broué–Michel Varieties For $w \in W$, we define $\widetilde{O}_{w,0}$ by requiring the square in the commutative diagram

$$\begin{array}{ccccc} O_{w,0} & \longleftarrow & \widetilde{O}_{w,0} & & \\ \downarrow & & \downarrow & \searrow & \\ \mathcal{B}_0 \times \mathcal{B}_0 & \longleftarrow & \mathcal{B}_0 \times G_0 & \longrightarrow & G_0 \end{array} \tag{2.2.12}$$

to be cartesian. At the level of points,

$$\widetilde{O}_{w,0} = \{(B, g) \in \mathcal{B}_0 \times G_0 : B \xrightarrow{w} {}^gB\}. \tag{2.2.13}$$

For all $g \in G$, we let $(\widetilde{O}_w)_g$ be the fiber of the map $\widetilde{O}_w \to G$ above g. Using the definition of the Hecke operators, we can check:

Proposition 2.2.6. *We have* $\mathrm{tr}(T_w \times g \mid R_1) = |(\widetilde{O}_w)_g^F|$ *for all* $w \in W$ *and* $g \in G^F$.

Recall from [105, §8] that for $\beta \in Br_W^+$, the **Broué–Michel variety** $O(\beta)_0$ can be defined up to isomorphism in terms of the varieties $O_{w,0}$ by the following properties:

1. $O(\beta_w)_0 = O_{w,0}$ for all $w \in W$.

2. $O(\beta_1\beta_2)_0 \simeq O(\beta_1)_0 \times_{\mathcal{B}_0} O(\beta_2)_0$ for all $\beta_1, \beta_2 \in Br_W^+$.

In addition, recall that we define $\tilde{O}(\beta)_0$ in terms of $O(\beta)_0$ in analogy to the way we define $\tilde{O}_{w,0}$ in terms of $O_{w,0}$: namely, by requiring the square in (4.1.20) to be cartesian. Then by Iwahori's theorem, we deduce:

Corollary 2.2.7. *We have* $q^{\frac{|\beta|}{2}} \operatorname{tr}(\beta \times g \mid R_1) = |\tilde{O}(\beta)_g^F|$ *for all* $\beta \in Br_W^+$ *and* $g \in G^F$.

2.2.4. Springer Fibers The proper (in fact, small) map $pr_1 : \tilde{O}_{1,0} \to G_0$ is known as the **Grothendieck–Springer resolution**. We let $\mathcal{B}_g$ be the underlying reduced scheme of $(\tilde{O}_1)_g$ for all $g \in G$. The varieties $\mathcal{B}_g$ are called **Springer fibers**.

As explained in [105, §7], there is a W-action on the pushforward of the constant sheaf along $pr_1 : \tilde{O}_1 \to G$. This action induces a W-action on the cohomology of $\mathcal{B}_g$, which is weight-preserving when $g \in G^F$. The latter was first constructed by Springer [102] and later by many other authors, including Lusztig [69]. Throughout this paper, *we follow Lusztig's conventions, which differ from Springer's by a sign twist.* We set

$$Q_g = \sum_j (\boldsymbol{q}^{\frac{1}{2}})^j \mathrm{H}^j(\mathcal{B}_g, \bar{\mathbf{Q}}_\ell) \in \mathrm{K}_0(W)[\boldsymbol{q}^{\frac{1}{2}}]. \tag{2.2.14}$$

Springer showed that the characters Q_u, as we run over elements u of the unipotent locus $\mathcal{U}^F$, satisfy a certain orthogonality formula (anticipated by Deligne–Lusztig in [28, Thm. 6.9]). We will use the version of this formula quoted by Shoji on page 510 of [97]:

Theorem 2.2.8 (Orthogonality)**.** *For all* $\phi \in \operatorname{Irr}(W)$*, we have*

$$\boldsymbol{q}^{\dim \mathcal{B}} \phi \cdot \varepsilon Q_1 = \sum_{u \in \mathcal{U}^F} (Q_u, \phi)_W Q_u \tag{2.2.15}$$

in $\mathrm{K}_0(W)[\boldsymbol{q}]$.

Remark 2.2.9. We see that $\mathcal{B}_1 = \mathcal{B}$. It is a theorem of Borel that the cohomology ring of $\mathcal{B}$ is isomorphic to the algebra of W-coinvariants of V: That is,

$$\mathrm{H}^*(\mathcal{B}, \bar{\mathbf{Q}}_\ell) \simeq \operatorname{Sym}^*(V)/\operatorname{Sym}^*(V)_+^W \otimes \bar{\mathbf{Q}}_\ell \tag{2.2.16}$$

as graded $\bar{\mathbf{Q}}_\ell$-algebras, where V is placed in degree 2. Using this fact, one can check that

$$\boldsymbol{q}^{\dim \mathcal{B}} Q_1 = (-1)^r \cdot |G^F| \cdot \sum_{i \geq 0} \boldsymbol{q}^i \operatorname{Sym}^i(V) \tag{2.2.17}$$

in $\mathrm{K}_0(W)[\![\boldsymbol{q}]\!]$.

Springer conjectured [102], and Kazhdan proved [59], the following relationship between the characters Q_g and the unipotent representations of G^F.

Theorem 2.2.10 (Kazhdan)**.** *If $q \gg 0$, then we have*

$$Q_g(w)|_{\boldsymbol{q}^{1/2} \to q^{1/2}} = R_w(g) \tag{2.2.18}$$

for all $w \in W$ and $g \in G^F$.

Lastly, we will also need the following properties of Springer fibers proved in [104] and [24], respectively:

Theorem 2.2.11 (Springer)**.** *If $g \in G^F$, then $\mathrm{H}^j(\mathcal{B}_g, \bar{\mathbf{Q}}_\ell)$ is pure of weight j under Frobenius.*

Theorem 2.2.12 (de Concini–Lusztig–Procesi)**.** *If j is odd, then $\mathrm{H}^j(\mathcal{B}_g, \bar{\mathbf{Q}}_\ell) = 0$.*

Corollary 2.2.13. *If $g \in G^F$, then we have $\mathrm{E}(\boldsymbol{q}^{\frac{1}{2}}, t \mid \mathcal{B}_g) = \bigoplus_{i \geq 0} (\boldsymbol{q}t^2)^i \mathrm{H}^{2i}(\mathcal{B}_g, \bar{\mathbf{Q}}_\ell)$.*

2.3 Point Counting and Weights

2.3.1. Proof of Theorem 2.1.3 In the prequel [105], the function ANN : $Br_W \to \mathrm{K}_0(W)[\![\boldsymbol{q}]\!]$ was called the **annular character** and initially defined in terms of $\{-,-\}$

and $\boldsymbol{o}$. First, we introduced a function $\mathbb{A} : H_W \to \mathrm{K}_0(W) \otimes \mathbf{Q}(\boldsymbol{q}^{\frac{1}{2}})$ defined by:

$$\mathbb{A}(\beta) = \sum_{\phi,\psi \in \mathrm{Irr}(W)} \{\rho_\phi, \rho_\psi\} \phi_{\boldsymbol{q}}(\beta) \boldsymbol{o}(\psi). \tag{2.3.1}$$

Then ANN was defined by:

$$\textsc{ann}(\beta) = (-\boldsymbol{q}^{\frac{1}{2}})^{|\beta|} \mathbb{A}(\beta) \cdot \varepsilon \cdot \sum_{i \geq 0} \boldsymbol{q}^i \,\mathrm{Sym}^i(V). \tag{2.3.2}$$

To prove Theorem 2.1.3, we start by giving an analogue of Kazhdan's identity for $\mathbb{A}(\beta)$.

Lemma 2.3.1. *We have*

$$\mathrm{tr}(w \mid \mathbb{A}(\beta))|_{\boldsymbol{q}^{1/2} \to q^{1/2}} = R_w|_{H_W}(\beta) \tag{2.3.3}$$

for all $\beta \in H_W$ and $w \in W$.

Proof. By Theorem 2.2.5 and Proposition 2.2.4, the left-hand side can be written:

$$\sum_{\phi,\psi \in \mathrm{Irr}(W)} (\rho_\phi, R_\psi)_{G^F} \rho_\phi|_{H_W}(\beta)\psi(w). \tag{2.3.4}$$

If ρ does not take the form ρ_ϕ for some ϕ, then $\rho|_{H_W} = 0$. Therefore, the sum over $\phi \in \mathrm{Irr}(W)$ above can be extended to a sum over $\rho \in \mathrm{UCh}(W)$. Altogether,

$$\begin{aligned} \mathrm{tr}(w \mid \mathbb{A}(\beta))|_{\boldsymbol{q}^{1/2} \to q^{1/2}} &= \sum_{\psi \in \mathrm{Irr}(W)} \sum_{\rho \in \mathrm{UCh}(W)} (\rho, R_\psi)_{G^F} \rho|_{H_W}(\beta)\psi(w) \\ &= \sum_{\psi \in \mathrm{Irr}(W)} R_\psi|_{H_W}(\beta)\psi(w). \end{aligned} \tag{2.3.5}$$

By Schur orthogonality for W, the last expression equals $R_w|_{H_W}(\beta)$. □

Via Proposition 2.2.6, the next lemma is essentially a point-counting interpretation of the expressions $R_w|_{H_W}(\beta)$ from the previous lemma.

Lemma 2.3.2. *We have*

$$\frac{1}{|W|}\sum_{w\in W} R_w|_{H_W}(\beta)R_w(g) = \mathrm{tr}(\beta\times g \mid R_1) \tag{2.3.6}$$

for all $\beta \in H_W$ *and* $g \in G^F$.

Proof. By Schur orthogonality for W, the left-hand side can be written:

$$\sum_{\psi\in \mathrm{Irr}(W)} R_\psi|_{H_W}(\beta)R_\psi(g). \tag{2.3.7}$$

After we expand $R_\psi = \sum_{\rho\in \mathrm{UCh}(W)}\{\rho,\rho_\psi\}\rho$ using Theorem 2.2.5, this becomes:

$$\sum_{\rho,\rho'\in \mathrm{UCh}(G^F)} \rho|_{H_W}(\beta)\rho'(g)\sum_{\psi\in\mathrm{Irr}(W)}\{\rho,\rho_\psi\}\{\rho',\rho_\psi\}. \tag{2.3.8}$$

Since $\{-,-\}$ is a hermitian unitary pairing on $\mathrm{UCh}(W)$, the last expression simplifies to:

$$\sum_{\rho\in \mathrm{UCh}(G^F)} \rho|_{H_W}(\beta)\rho(g). \tag{2.3.9}$$

Again, $\rho|_{H_W} \neq 0$ only if $\rho = \rho_\phi$ for some ϕ. So the sum over $\rho \in \mathrm{UCh}(W)$ above can be restricted to a sum over $\phi \in \mathrm{Irr}(W)$. By (2.2.6), the outcome is $\mathrm{tr}(\beta\times g \mid R_1)$. □

We now finish the proof of Theorem 2.1.3. Let $\beta \in Br_W$ be arbitrary, and let $\tilde{\beta} = \boldsymbol{q}^{\frac{|\beta|}{2}}\beta$. By Theorem 2.2.8 and Remark 2.2.9,

$$\textsc{ann}(\beta) = \frac{(-1)^{r-|\beta|}}{|G^F|}\sum_{u\in\mathcal{U}^F}(Q_u, \mathbb{A}(\tilde{\beta}))_W Q_u. \tag{2.3.10}$$

By Schur orthogonality, Theorem 2.2.10, and Lemmata 2.3.1-2.3.2,

$$
\begin{aligned}
(Q_u, \mathbb{A}(\tilde{\beta}))_W|_{\boldsymbol{q}^{1/2} \to q^{1/2}} &= \frac{1}{|W|} \sum_{w \in W} \operatorname{tr}(w \mid \mathbb{A}(\tilde{\beta})) Q_u(w) \\
&= \frac{1}{|W|} \sum_{w \in W} R_w|_{H_W}(\tilde{\beta}) R_w(u) \\
&= \operatorname{tr}(\tilde{\beta} \times u \mid R_1)
\end{aligned}
\tag{2.3.11}
$$

for $q \gg 0$. By Proposition 2.2.6, we can conclude:

Theorem 2.3.3. *If $\beta \in Br_W$ satisfies $\tilde{\beta} = \sum_{w \in W} m_w(\boldsymbol{q}) \tilde{\beta}_w$ in H_W for some $m_w(\boldsymbol{q}) \in \mathbf{Z}[\boldsymbol{q}]$, then for $q \gg 0$, we have*

$$
\text{ANN}(\beta)|_{\boldsymbol{q} \to q} = \frac{(-1)^{r - |\beta|}}{|G^F|} \sum_{u \in \mathcal{U}^F} \sum_{w \in W} m_w(q) |(\tilde{O}_w)_u^F| Q_u|_{\boldsymbol{q} \to q}
\tag{2.3.12}
$$

in $\mathrm{K}_0(W)[\![q]\!]$.

When β belongs to the positive submonoid $Br_W^+ \subseteq Br_W$, we can sharpen this statement by relying on Corollary 2.2.7 rather than Proposition 2.2.6. Theorem 2.1.3 follows.

2.3.2. Virtual Poincaré Series We prepare the ground for the proof of Theorem 2.1.4. For any Artin stack X_0 over $\mathbf{F}$, we set

$$
\begin{aligned}
\mathrm{E}(\boldsymbol{q}^{\frac{1}{2}} \mid X) &= \mathrm{E}(\boldsymbol{q}^{\frac{1}{2}}, t \mid X)|_{t=-1} \\
&= \sum_{j,k} (-1)^k \boldsymbol{q}^{\frac{j}{2}} \operatorname{Gr}_j^{\mathbf{W}} \mathrm{H}_c^k(X, \bar{\mathbf{Q}}_\ell).
\end{aligned}
\tag{2.3.13}
$$

A priori, this series defines an element of $\mathrm{K}_0(\mathsf{Vect})[\![\boldsymbol{q}]\!]$, where Vect is the category of finite-dimensional $\bar{\mathbf{Q}}_\ell$-vector spaces. If X_0 admits a (left) G_0-action, then

$$
\mathrm{E}(\boldsymbol{q}^{\frac{1}{2}} \mid G \backslash X) = \frac{\mathrm{E}(\boldsymbol{q}^{\frac{1}{2}} \mid X)}{\mathrm{E}(\boldsymbol{q}^{\frac{1}{2}} \mid G)}
\tag{2.3.14}
$$

holds in $\mathrm{K}_0(\mathsf{Vect})[\![\boldsymbol{q}]\!]$, *cf.* [57, §4].

If there is a weight-preserving W-action on $\mathrm{H}_c^*(X, \bar{\mathbf{Q}}_\ell)$, then we can promote $\mathrm{E}(\boldsymbol{q}^{\frac{1}{2}} \mid X)$ to an element of $\mathrm{K}_0(W)[\![\boldsymbol{q}]\!]$. In our context, the W-action will arise via a map $f : X_0 \to Y_0$ over $\mathbf{F}$ and a W-action on the pushforward $\mathrm{R}f_! \bar{\mathbf{Q}}_{\ell,0}$, where $\bar{\mathbf{Q}}_{\ell,0}$ is the constant sheaf over X_0. If f is equivariant with respect to G_0-actions on X_0 and Y_0, then (2.3.14) lifts to $\mathrm{K}_0(W)[\![\boldsymbol{q}]\!]$ once we take the $\mathrm{K}_0(W)$-structure of $\mathrm{E}(\boldsymbol{q}^{\frac{1}{2}} \mid G)$ to be trivial.

Using (2.3.14), we will reduce our analysis of $\mathrm{E}(\boldsymbol{q}^{\frac{1}{2}} \mid X)$ to the setting where X is a scheme of finite type, not a stack. Here, the cohomology of X is bounded, so we can relate $\mathrm{E}(\boldsymbol{q}^{\frac{1}{2}} \mid X)$ to point-counting via the Lefschetz trace formula, following Katz in [54, Appendix]. First, if X_0 admits a model over a finitely-generated $\mathbf{Z}$-algebra R such that, for all finite fields $\mathbf{F}_q$ and morphisms $R \to \mathbf{F}_q$, the point count $X^F = |X_0(\mathbf{F}_q)|$ is a polynomial in q, then

$$\dim \mathrm{E}(q^{\frac{1}{2}} \mid X) = |X^F|. \tag{2.3.15}$$

To handle W-actions, we introduce a refinement. For all $w \in W$, we define the twisted Lefschetz number:

$$\begin{aligned} \Lambda(w \mid X) &= \sum_{i \geq 0} (-1)^i \operatorname{tr}(Fw \mid \mathrm{H}^i(Y, \mathrm{R}f_! \bar{\mathbf{Q}}_\ell)) \\ &= \sum_{y \in Y^F} \sum_{i \geq 0} (-1)^i \operatorname{tr}(Fw \mid \mathrm{H}_c^i(X_y, \bar{\mathbf{Q}}_\ell)). \end{aligned} \tag{2.3.16}$$

If $f : X_0 \to Y_0$ admits a model over R such that for all $\mathbf{F}_q$ and morphisms $R \to \mathbf{F}_q$, the Lefschetz number $\Lambda(w \mid X)$ is a polynomial in q, then

$$\operatorname{tr}(w \mid \mathrm{E}(q^{\frac{1}{2}} \mid X)) = \Lambda(w \mid X). \tag{2.3.17}$$

2.3.3. Proof of Theorem 2.1.4 By Theorem 2.1.3, we want to show that for all $\beta \in Br_W^+$ and $q \gg 0$, we have

$$\mathrm{E}(q^{\frac{1}{2}} \mid G \backslash St(\beta)) = \frac{1}{|G^F|} \sum_{u \in \mathcal{U}^F} |\tilde{O}(\beta)_u^F| \cdot Q_u|_{\boldsymbol{q} \to q} \tag{2.3.18}$$

in $\mathrm{K}_0(W)[q][\frac{1}{|G^F|}]$. Since the point count of G is a polynomial in q for all $q \gg 0$, we have

$$\dim \mathrm{E}(q^{\frac{1}{2}} \mid G) = |G^F|. \tag{2.3.19}$$

Thus we can use (2.3.14) to reduce to showing

$$\mathrm{E}(q^{\frac{1}{2}} \mid St(\beta)) = \sum_{u \in \mathcal{U}^F} |\tilde{O}(\beta)_u^F| \cdot Q_u|_{\boldsymbol{q} \to q} \tag{2.3.20}$$

in $\mathrm{K}_0(W)[q]$.

For convenience, let $\mathcal{V}(\beta)_0 = \mathcal{U}_0 \times_{G_0} \tilde{O}(\beta)_0$ and let the map

$$p_\beta : \mathcal{V}(\beta)_0 \to \mathcal{U}_0 \tag{2.3.21}$$

be induced by the map $\tilde{O}(\beta)_0 \to G_0$. Let $\mathcal{F}_0 = p_{1,!}\bar{\mathbf{Q}}_{\ell,0}$, the Springer sheaf over $\mathcal{U}_0$. Applying base change across the cartesian square

$$\begin{array}{ccc} St(\beta)_0 & \longrightarrow & \mathcal{V}(1)_0 \\ \downarrow & & \downarrow{\scriptstyle p_1} \\ \mathcal{V}(\beta)_0 & \xrightarrow{p_\beta} & \mathcal{U}_0 \end{array} \tag{2.3.22}$$

we see that $\mathrm{R}p_\beta^! \mathcal{F}_0$ coincides with the pushforward of the constant sheaf along $St(\beta)_0 \to \mathcal{V}(\beta)_0$. So the W-action on the cohomology of $St(\beta)$ arises from the W-action that $\mathrm{R}p_\beta^! \mathcal{F}_0$ inherits from $\mathcal{F}_0$.

Since everything in sight has polynomial point-count for $q \gg 0$, we can use (2.3.17) to get:

$$\begin{aligned} \operatorname{tr}(w \mid \mathrm{E}(q^{\frac{1}{2}} \mid \mathit{St}(\beta))) &= \Lambda(w \mid \mathit{St}(\beta)) \\ &= \sum_{v \in \mathcal{V}(\beta)^F} \Lambda(w \mid \mathit{St}(\beta)_v). \end{aligned} \tag{2.3.23}$$

If $p_\beta : \mathcal{V}(\beta) \to \mathcal{U}$ sends $v \mapsto u$, then $\mathit{St}(\beta)_v = \mathcal{V}(1)_u$, and by the discussion in the previous paragraph, $\Lambda(w \mid \mathit{St}(\beta)_v) = \Lambda(w \mid \mathcal{V}(1)_u)$. Thus we arrive at:

$$\operatorname{tr}(w \mid \mathrm{E}(q^{\frac{1}{2}} \mid \mathit{St}(\beta))) = \sum_{u \in \mathcal{U}^F} |\mathcal{V}(\beta)_u^F| \cdot \Lambda(w \mid \mathcal{V}(1)_u). \tag{2.3.24}$$

Since $\mathcal{V}(\beta)_u = \widetilde{O}(\beta)_u$ and $\mathcal{V}(1)_u^{\mathrm{red}} = \mathcal{B}_u$ for all $u \in \mathcal{U}$, it remains to check that

$$Q_u(w)|_{\boldsymbol{q} \to q} = \Lambda(w \mid \mathcal{B}_u). \tag{2.3.25}$$

Indeed, the left-hand side equals $\mathrm{E}(q^{\frac{1}{2}} \mid \mathcal{B}_u)$ by Corollary 2.2.13 and the right-hand side equals $\mathrm{E}(q^{\frac{1}{2}} \mid \mathcal{B}_u)$ by (2.3.17).

2.3.4. Proof of Theorem 2.1.9 Recall from [105, §4] that the Ocneanu–Gomi Markov trace on H_W is a function of the form:

$$\textsc{tr} : H_W \to \mathbf{Z}[\boldsymbol{q}^{\pm\frac{1}{2}}](a). \tag{2.3.26}$$

As in that paper, we normalize it so that $\textsc{tr}(1) = 1$.

When W runs over the symmetric groups, the corresponding Markov traces can be used to construct the HOMFLY series invariant of topological links in 3-space. Namely, Ocneanu showed that if $W = S_n$ and λ denotes the planar closure of an n-strand topological braid

$\beta \in Br_W$, then

$$\text{HOMFLY}(\beta) = (-a)^{|\beta|} \left(\frac{a - a^{-1}}{\boldsymbol{q}^{\frac{1}{2}} - \boldsymbol{q}^{-\frac{1}{2}}} \right)^{n-1} \text{TR}(\beta) \tag{2.3.27}$$

only depends on the isotopy class of the link λ.

For general W, not even necessarily crystallographic, the Markov trace TR can still be defined by a formula involving the exotic Fourier transform. In [105, §4], we related TR to our function ANN by the identity

$$\sum_i (-a^2)^i (\Lambda^i(V), \text{ANN}(\beta))_W = (-\boldsymbol{q}^{-\frac{1}{2}})^{|\beta|} \left(\frac{1 - a^2}{1 - \boldsymbol{q}} \right)^r \text{TR}(\beta), \tag{2.3.28}$$

where V is any faithful realization of W for which $V^W = 0$. By comparing (2.3.27) and (2.3.28), we see that for any topological braid β, each term of $\text{HOMFLY}(\beta)$ appears as a term of $\text{ANN}(\beta)$ up to an overall monomial shift.

For general W, we let $\pi \in Br_W^+$ denote the **full twist**. In terms of the map $W \to Br_W^+$ denoted $w \mapsto \beta_w$, it is defined by

$$\pi = \beta_{w_0}^2, \tag{2.3.29}$$

where w_0 is the longest element of W. The theorem below is the main result of [58]:

Theorem 2.3.4 (Kálmán)**.** *For all topological braids β, the following series in $\boldsymbol{q}^{\frac{1}{2}}$ are equal:*

1. *The coefficient of $a^{|\beta|+r}$ in HOMFLY$(\beta\pi)$.*
2. *The coefficient of $a^{|\beta|-r}$ in HOMFLY(β).*

Kálmán's article gives two proofs. The second proof (suggested by the referee of [58]) can be generalized in a straightforward way to prove:

Theorem 2.3.5. *Let W be an arbitrary finite Coxeter group of rank r, and let*

$$\overline{\text{TR}} = \left(\frac{1-a^2}{1-\boldsymbol{q}}\right)^r \text{TR}. \tag{2.3.30}$$

Then for all $\beta \in Br_W$, the following series in $\boldsymbol{q}^{\frac{1}{2}}$ are equal:

1. *The coefficient of a^{2r} in $\overline{\text{TR}}(\beta\pi)$.*

2. *The coefficient of a^0 in $\overline{\text{TR}}(\beta)$.*

Recall that 1 and ε respectively denote the trivial and sign characters of W. Applying (2.3.28), we deduce:

Corollary 2.3.6. *For all $\beta \in Br_W$, we have*

$$(\varepsilon, \text{ANN}(\beta\pi))_W = \boldsymbol{q}^{|w_0|}(1, \text{ANN}(\beta))_W \tag{2.3.31}$$

in $\mathbf{Z}[\![\boldsymbol{q}]\!]$.

We can finally conclude the proof of Theorem 2.1.9. Recall that $|w_0| = \dim \mathcal{B}$, and recall the following facts about total Springer representations:

- $(1, Q_u)_W = 1$ for all $u \in \mathcal{U}^F$.

- $(\varepsilon, Q_1)_W = \boldsymbol{q}^{\dim \mathcal{B}}$.

- $(\varepsilon, Q_u)_W = 0$ for all $u \neq 1$.

Applying these identities to Theorem 2.1.3 gives

$$(\varepsilon, \text{ANN}(\beta\pi))_W = \frac{(-1)^{r-|\beta|}}{|G^F|} \cdot |\tilde{O}(\beta\pi)_1^F| \cdot \boldsymbol{q}^{\dim \mathcal{B}}, \tag{2.3.32}$$

$$(1, \text{ANN}(\beta))_W = \frac{(-1)^{r-|\beta|}}{|G^F|} \cdot |\mathcal{U}^F \times_{G^F} \tilde{O}(\beta)^F|. \tag{2.3.33}$$

Now, Theorem 2.1.9 follows from Corollary 2.3.6.

To prove Corollary 2.1.10, observe that the left-hand side of (2.1.24) equals the $\boldsymbol{q}^{\frac{1}{2}}$-series coefficient of $a^{|\beta|-r}$ in HOMFLY(β).

2.4 Parabolic Induction

2.4.1. Parabolic Data Let S be the set of simple reflections in the Coxeter presentation of W. A subgroup of W is S-**parabolic** iff it is generated by a subset of S.

Suppose that $W' \subseteq W$ is an S-parabolic subgroup. Recall that S determines the positive submonoid $Br_W^+ \subseteq Br_W$ and the set-theoretic map $W \to Br_W^+$ denoted $w \mapsto \beta_w$. Via this lifting map, the inclusion $W' \to W$ induces an inclusion of positive monoids $Br_{W'}^+ \to Br_W^+$ that preserves writhe. Since $Br_{W'}$ and Br_W are the group completions of $Br_{W'}^+$ and Br_W^+, we obtain an inclusion of Artin groups $Br_{W'} \to Br_W$ that again preserves writhe.

In terms of the W-action on $V^\vee = \mathbf{X}^\vee \otimes \mathbf{Q}$, we can realize S as the set of reflections corresponding to a system of simple roots in $\mathbf{X}$. After we fix a split torus $T_0 \subseteq G_0$, this defines a Borel $B_0 \supseteq T_0$. If P_0 is a parabolic subgroup of G_0 containing B_0, then the Weyl group of the Levi quotient of P_0 is an S-parabolic subgroup of W. Conversely, every S-parabolic subgroup of W arises in this way.

Henceforth, we fix a parabolic subgroup $P_0 \subseteq G_0$ with unipotent radical U_0 and Levi quotient L_0. We distinguish the data attached to L_0 with a subscript; thus,

- W_L is the Weyl group of L_0.
- $\mathcal{B}_{L,0}$ is the flag variety of L_0.
- $\mathcal{U}_{L,0}$ is the unipotent locus of L_0.
- If $w \in W_L$, then:
  - $R_{L,w}$ is the virtual Deligne–Lusztig representation attached to w as an element of W_L, not W.

– $(\widetilde{O}_{L,w})_0$ is the variety attached to w over L_0, not over G_0.

- If $u \in \mathcal{U}_L$, then $Q_{L,u}$ is the total Springer representation attached to u as an element of $\mathcal{U}_L$, not $\mathcal{U}$.

By definition, parabolic induction with respect to P^F is the functor from representations of L^F to representations of G^F that sends

$$\rho_L \mapsto \mathrm{Ind}_{P^F}^{G^F} \mathrm{Res}_{L^F}^{P^F} \rho_L. \tag{2.4.1}$$

Its left adjoint is the functor that sends

$$\rho \mapsto (\mathrm{Res}_{P^F}^{G^F} \rho)^{U^F}, \tag{2.4.2}$$

where $(-)^{U^F}$ means we take U^F-invariants.

The following Frobenius-type formula is Prop. 7.4.4 in [19]. It gives a precise sense in which the Deligne–Lusztig characters of L^F and G^F are compatible with induction.

Theorem 2.4.1 (Deligne–Lusztig)**.** *For all $w \in W$, we have*

$$(R_w)^{U^F} = \frac{1}{|W_L|} \sum_{\substack{x \in W \\ xwx^{-1} \in W_L}} R_{L,xwx^{-1}} \tag{2.4.3}$$

as $\mathbf{Q}$-linear virtual characters of L^F.

2.4.2. Proof of Theorem 2.1.5 Note that $r - r' = \dim V^{W_L}$. We will abbreviate by writing $d_L = \dim V^{W_L}$ and $\textsc{ann}_{W_L}^W = \mathrm{Ind}_{W_L}^W \textsc{ann}_{W_L}$. Then we must show that

$$\textsc{ann}_W(\beta) = \frac{\textsc{ann}_{W_L}^W(\beta)}{(1-\boldsymbol{q})^{d_L}} \tag{2.4.4}$$

for all $\beta \in Br_{W_L}$. It suffices to show equality in the $\boldsymbol{q} \to q$ limit for all $q \gg 0$.

By Theorem 2.3.3, we reduce to the case where $\beta = \beta_w$ for some $w \in W_L$. Here,

$$\mathrm{ANN}_W(\beta)|_{\boldsymbol{q}\to q} = \frac{(-1)^{r-|w|}}{|G^F|} \sum_{u \in \mathcal{U}^F} |(\tilde{O}_w)^F_u| Q_u|_{\boldsymbol{q}\to q}. \tag{2.4.5}$$

The exact analogue of this formula for L is not true, because L may not be semisimple. However, we can fix it by introducing a factor of $(1-\boldsymbol{q})^{\dim V^{W_L}}$, *cf.* [105, §4]:

$$\frac{\mathrm{ANN}_{W_L}(\beta)|_{\boldsymbol{q}\to q}}{(1-q)^{d_L}} = \frac{(-1)^{r-|w|}}{|L^F|} \sum_{u' \in \mathcal{U}_L^F} |(\tilde{O}_{L,w})^F_{u'}| Q_{L,u}|_{\boldsymbol{q}\to q}. \tag{2.4.6}$$

Now, by Theorem 2.2.10,

$$\mathrm{tr}(x \mid \mathrm{ANN}_W(\beta))|_{\boldsymbol{q}\to q} = \frac{(-1)^{r-|w|}}{|G^F|} \sum_{u \in \mathcal{U}^F} |(\tilde{O}_w)^F_u| R_x(u), \tag{2.4.7}$$

$$\frac{\mathrm{tr}(x \mid \mathrm{ANN}^W_{W_L}(\beta))|_{\boldsymbol{q}\to q}}{(1-q)^{d_L}} = \frac{(-1)^{r-|w|}}{|W_L|} \sum_{\substack{y \in W \\ yxy^{-1} \in W_L}} \frac{1}{|L^F|} \sum_{u' \in \mathcal{U}_L^F} |(\tilde{O}_{L,w})^F_{u'}| R_{L,yxy^{-1}}(u') \tag{2.4.8}$$

for all $x \in W$. By Theorem 2.4.1, it remains to show:

$$\frac{1}{|G^F|} \sum_{u \in \mathcal{U}^F} |(\tilde{O}_w)^F_u| R_x(u) = \frac{1}{|L^F|} \sum_{u' \in \mathcal{U}_L^F} |(\tilde{O}_{L,w})^F_{u'}| (\mathrm{Res}^{G^F}_{P^F} R_x)^{U^F}(u'). \tag{2.4.9}$$

In what follows, let $\{\mathcal{U}_i^F\}_i$ be the set of G^F-orbits of $\mathcal{U}^F$. Instead of working with $\tilde{O}_w$, it will be more convenient to work with

$$\tilde{G}_w = \{g \in G : B \xrightarrow{w} {}^g B\}, \tag{2.4.10}$$

where B_0 is the Borel of G_0 contained in P_0.

Lemma 2.4.2. *We have*

$$\frac{|\mathcal{U}_i^F \times_{\mathcal{U}^F} \widetilde{O}_w^F|}{|G^F|} = \frac{|\mathcal{U}_i^F \cap \widetilde{G}_w^F|}{|B^F|} \tag{2.4.11}$$

for all $w \in W$ and $u \in \mathcal{U}_i^F$.

Proof. Since $\mathcal{B}^F = G^F/B^F$, we compute:

$$\begin{aligned}\frac{1}{|G^F|}\,|\mathcal{U}_i^F \times_{\mathcal{U}^F} \widetilde{O}_w^F| &= \frac{1}{|G^F|}\,|\{(g,u) \in G^F \times \mathcal{U}_i^F : {}^gB \xrightarrow{w} {}^{ug}B\}| \\ &= \frac{1}{|G^F|}\,|\{(g,u) \in G^F \times \mathcal{U}_i^F : B \xrightarrow{w} {}^{g^{-1}ug}B\}| \\ &= |\{u \in \mathcal{U}_i^F : B \xrightarrow{w} {}^uB\}|.\end{aligned} \tag{2.4.12}$$

The last expression is $|\mathcal{U}_i^F \cap \widetilde{G}_w^F|$. □

Let $B_L = B \cap L$, so that B_L is a Borel subgroup of L. For all $w \in W_L$, let $\widetilde{L}_w$ be the analogue of $\widetilde{G}_w$ with L, B_L in place of G, B.

Lemma 2.4.3. *For all $w \in W_L$, we have*

$$\widetilde{G}_w = \widetilde{L}_w \cdot U \simeq \widetilde{L}_w \times U. \tag{2.4.13}$$

In particular, $\mathcal{U} \cap \widetilde{G}_w = (\mathcal{U}_L \cap \widetilde{L}_w) \cdot U$.

Proof. To show the first statement, lift w to a point $\dot{w} \in \widetilde{L}$. Then

$$\widetilde{G}_w = B\dot{w}B = UB_L\dot{w}B_LU = (B_L\dot{w}B_L)U = \widetilde{L}_w \cdot U, \tag{2.4.14}$$

as claimed. For the second statement, observe that $U \subseteq \mathcal{U}$ already. □

As above, let $\{\mathcal{U}_i^F\}_i$ be the set of G^F-orbits of $\mathcal{U}^F$. For each i such that $\mathcal{U}_i^F \cap \widetilde{G}_w^F \neq \emptyset$, pick a point $u_i \in \mathcal{U}_i^F$. Using Lemma 2.4.2, we can expand:

$$\begin{aligned}\frac{1}{|G^F|}\sum_{u\in\mathcal{U}^F}|(\tilde{O}_w)_u^F|R_x(u) &= \frac{1}{|B^F|}\sum_i|\mathcal{U}_i^F\cap\tilde{G}_w^F|R_x(u_i)\\ &= \frac{1}{|B^F|}\sum_{u\in\mathcal{U}^F\cap\tilde{G}_w^F}R_x(u).\end{aligned}\tag{2.4.15}$$

Let $\{\mathcal{U}_{L,j}^F\}_j$ be the set of L^F-orbits of $\mathcal{U}_L^F$. For all j, we pick a point $u'_j \in \mathcal{U}_{L,j}^F \cap \tilde{L}_w^F$. Then we can also expand:

$$\begin{aligned}\frac{1}{|L^F|}\sum_{u'\in\mathcal{U}_L^F}|(\tilde{O}_{L,w})_{u'}^F|(R_x)^{U^F}(u') &= \frac{1}{|B_L^F|}\sum_j|\mathcal{U}_{L,j}^F\cap\tilde{L}_w^F|(R_x)^{U^F}(u'_j)\\ &= \frac{1}{|B_L^F|}\sum_{u'\in\mathcal{U}_L^F\cap\tilde{L}_w^F}(R_x)^{U^F}(u')\\ &= \frac{1}{|B_L^F||U^F|}\sum_{\substack{u'\in\mathcal{U}_L^F\cap\tilde{L}_w^F\\ u''\in U^F}}R_x(u'u'').\end{aligned}\tag{2.4.16}$$

Since $|B^F| = |B_L^F||U^F|$, we are done by Lemma 2.4.3.

Remark 2.4.4. The group $Br_{W'}$ is usually much smaller than the preimage of W' along $Br_W \to W$, which we will denote $W' \times_W Br_W$. For example, if $W = S_2$ and $W' = \{1\}$, then $Br_{W'} = \{1\}$ whereas $W' \times_W Br_W$ is the index-2 subgroup of $Br_W \simeq \mathbf{Z}$. It would be interesting to find a generalization of Theorem 2.1.5 from $Br_{W'}$ to $W' \times_W Br_W$. Such a generalization could yield an alternative to the proof of Theorem 2.1.6 below.

2.5 The Pole at 1

2.5.1. Elliptic Elements Recall that V is a faithful realization of W for which $V^W = 0$. An element $w \in W$ is **elliptic** iff $V^w = 0$, or equivalently,

$$\det(1 - w \mid V) \neq 0.\tag{2.5.1}$$

The hypothesis on V ensures that this notion only depends on w. The following fact is Theorem 3.2.12 in [43]:

Theorem 2.5.1 (Geck–Pfeiffer)**.** *For all $w \in W$, we can find a unique subset $S' \subseteq S$ up to W-conjugacy such that if $W' \subseteq W$ is the parabolic subgroup generated by S', then w is W-conjugate to an elliptic element of W'.*

Keeping the notation of the theorem, pick $w' \in W$ such that w' is elliptic in W' and conjugate to w in W. Let $V' = V/(V^{W'})$, and let

$$n(w) = \det(1 - w' \mid V'). \tag{2.5.2}$$

From the factorization of characteristic polynomials

$$\det(1 - \boldsymbol{q}w \mid V) = (1 - \boldsymbol{q})^{r(w)} \det(1 - \boldsymbol{q}w' \mid V'), \tag{2.5.3}$$

we see that the number $n(w)$ only depends on w, not on w'.

2.5.2. Proof of Theorem 2.1.6 Our key tool is the following result. Its proof is similar to, and generalizes, the proof of Theorem 5.2 of [76].

Theorem 2.5.2. *If $\beta \in Br_W^+$ maps to $w \in W$, then the isotropy groups of the G-action on $\widetilde{O}(\beta)$ are diagonalizable groups of order dividing $(1-q)^{r(w)}n(w)$.*

We keep the notation above. By combining Theorems 2.1.3 and 2.5.2, we deduce that for all $q \gg 0$, the specialization $\textsc{ann}(\beta)|_{\boldsymbol{q} \to q}$ is an element of

$$\frac{1}{(1-q)^{r(w)}n(w)} \mathrm{K}_0(W)[q]. \tag{2.5.4}$$

The number $n(w)$ does not depend on q. Therefore, $\textsc{ann}(\beta)$ is an element of

$$\frac{1}{(1-\boldsymbol{q})^{r(w)}n(w)}\mathrm{K}_0(W)[\boldsymbol{q}]. \tag{2.5.5}$$

But we also know $\textsc{ann}(\beta) \in \mathrm{K}_0(W)[\![\boldsymbol{q}]\!]$. Therefore we can dispense with the term $n(w)$ in the denominator. Theorem 2.1.6 follows.

2.6 HOMFLY Revisited

In this section, we demonstrate through examples how the HOMFLY series of a link can be recovered from a linear combination of total Springer representations by way of Theorem 2.1.3. Our examples build on ones introduced in [105, §5].

2.6.1. Type A_1 Set $G = \mathrm{SL}_2$, so that $W = S_2$ and $S = \{s\}$. We have $\mathrm{Irr}(W) = \{1, \varepsilon\}$. In [105], we computed that

$$\textsc{ann}(\beta_s^m) = \frac{1}{1-\boldsymbol{q}^2}((1-(-\boldsymbol{q})^{m+1}) + (\boldsymbol{q} + (-\boldsymbol{q})^m)\varepsilon), \tag{2.6.1}$$

corresponding to the formula

$$\textsc{homfly}(T_{2,m}) = \frac{(\boldsymbol{q}^{-\frac{1}{2}})^{m-1}}{1-\boldsymbol{q}^2}(a^{m-1}(1-(-\boldsymbol{q})^{m+1}) - a^{m+1}(\boldsymbol{q} + (-\boldsymbol{q})^m)\varepsilon). \tag{2.6.2}$$

for the $(2, m)$-torus link $T_{2,m}$. Note that in this setting, $\textsc{homfly}$ carries as much information as $\textsc{ann}$ because there are so few irreducible characters of W.

For $m \geq 0$, it is illustrative to simplify the formula for $\textsc{ann}(\beta_s^m)$ by breaking into even and odd cases:

$$\textsc{ann}(\beta_s^{2k}) = \left(1 + \boldsymbol{q}^2 + \cdots + \boldsymbol{q}^{2k-2} + \tfrac{\boldsymbol{q}^{2k}}{1-\boldsymbol{q}}\right) + \left(\boldsymbol{q} + \cdots + \boldsymbol{q}^{2k-3} + \tfrac{\boldsymbol{q}^{2k-1}}{1-\boldsymbol{q}}\right)\varepsilon, \tag{2.6.3}$$

$$\textsc{ann}(\beta_s^{2k+1}) = (1 + \boldsymbol{q}^2 + \cdots + \boldsymbol{q}^{2k}) + (\boldsymbol{q} + \cdots + \boldsymbol{q}^{2k-1})\varepsilon \tag{2.6.4}$$

The unipotent locus $\mathcal{U}^F$ admits two G^F-orbits, which can be represented by $u = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $1 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. The respective Springer fibers are isomorphic to a point and a projective line. The respective total Springer representations are:

$$Q_u = 1, \tag{2.6.5}$$

$$Q_1 = 1 + \boldsymbol{q}\varepsilon. \tag{2.6.6}$$

Altogether, we find that

$$\text{ANN}(\beta_s^{2k}) = -\boldsymbol{q}^{2k-1}Q_u + \left(1 + \boldsymbol{q}^2 + \cdots + \boldsymbol{q}^{2k-4} + \tfrac{\boldsymbol{q}^{2k-2}}{1-\boldsymbol{q}}\right) Q_1, \tag{2.6.7}$$

$$\text{ANN}(\beta_s^{2k+1}) = \boldsymbol{q}^{2k}Q_u + (1 + \boldsymbol{q}^2 + \cdots + \boldsymbol{q}^{2k-2})Q_1, \tag{2.6.8}$$

exhibiting $\text{ANN}(\beta_s^m)$ as a linear combination of Q_1 and Q_u. As expected, the pole at $\boldsymbol{q} = 1$ only depends on the parity of m.

To go further, we can interpret the $\boldsymbol{q}$-coefficients as point counts. The orbit of u is a $\mathbf{G}_m^F$-torsor over $\mathcal{B}^F \simeq (\mathbf{P}^1)^F$ and its stabilizer is isomorphic to $\mathbf{G}_a^F$. The orbit of 1 consists of itself and its stabilizer is G^F. So Theorem 2.1.3 asserts that:

$$\frac{|\widetilde{O}(\beta_s^{2k})_u^F|}{|\mathbf{G}_a^F|} = q^{2k-1}, \tag{2.6.9}$$

$$\frac{|\widetilde{O}(\beta_s^{2k})_1^F|}{|G^F|} = \frac{q^{2k-2}}{q-1} - q^{2k-4} - \cdots - q^2 - 1, \tag{2.6.10}$$

$$\frac{|\widetilde{O}(\beta_s^{2k+1})_u^F|}{|\mathbf{G}_a^F|} = q^{2k}, \tag{2.6.11}$$

$$\frac{|\widetilde{O}(\beta_s^{2k+1})_1^F|}{|G^F|} = q^{2k-2} + \cdots + q^2 + 1. \tag{2.6.12}$$

Since $|\mathbf{G}_a^F| = q$ and $|G^F| = q(q-1)(q+1)$, these identities are equivalent to:

$$|\tilde{O}(\beta_s^m)_u^F| = q^m, \tag{2.6.13}$$

$$|\tilde{O}(\beta_s^m)_1^F| = q^m + (-1)^m q. \tag{2.6.14}$$

For general m, it is easiest to compute the left-hand sides by expanding β_s^m in the standard basis of the Iwahori–Hecke algebra and applying Proposition 2.2.6.

2.6.2. An Iterated Torus Braid Set $G = \mathrm{SL}_4$, so that $W = S_4$ and $S = \{s, t, u\}$. We have $\mathrm{Irr}(W) = \{1, \phi, \psi, \varepsilon\phi, \varepsilon\}$, where $\phi(1) = 3$ and $\psi(1) = 2$. In [105], we computed the isotypic decomposition of $\textsc{ann}(\beta)$ for the braid

$$\beta = (\beta_s\beta_t\beta_u)^6\beta_s \in Br_W. \tag{2.6.15}$$

The G^F-orbits of $\mathcal{U}^F$ are indexed by the partitions of the integer 4. If the identity orbit corresponds to the partition $4 = 1+1+1+1$, then the total Springer representations are:

$$\begin{array}{r|l} Q_4 & 1 \\ Q_{3,1} & 1 + \boldsymbol{q}\phi \\ Q_{2,2} & 1 + \boldsymbol{q}\phi + \boldsymbol{q}^2\psi \\ Q_{2,1,1} & 1 + (\boldsymbol{q} + \boldsymbol{q}^2)\phi + \boldsymbol{q}^2\psi + \boldsymbol{q}^3\varepsilon\phi \\ Q_{1,1,1,1} & 1 + (\boldsymbol{q} + \boldsymbol{q}^2 + \boldsymbol{q}^3)\phi + (\boldsymbol{q}^2 + \boldsymbol{q}^4)\psi + (\boldsymbol{q}^3 + \boldsymbol{q}^4 + \boldsymbol{q}^5)\varepsilon\phi + \boldsymbol{q}^6\varepsilon \end{array} \tag{2.6.16}$$

Comparing against the data in [105], we calculate directly that:

$$\begin{aligned}\textsc{ann}(\beta) = \boldsymbol{q}^{16} Q_1 &+ (\boldsymbol{q}^{10} + \boldsymbol{q}^{11} + \boldsymbol{q}^{12} + \boldsymbol{q}^{13} + \boldsymbol{q}^{14}) Q_{3,1} \\ &+ (\boldsymbol{q}^{6} - \boldsymbol{q}^{7} + \boldsymbol{q}^{8} + \boldsymbol{q}^{10} + \boldsymbol{q}^{12}) Q_{2,2} \\ &+ (\boldsymbol{q}^{3} + \boldsymbol{q}^{4} + \boldsymbol{q}^{5} + 2\boldsymbol{q}^{6} + 2\boldsymbol{q}^{7} + 2\boldsymbol{q}^{8} + \boldsymbol{q}^{9} + \boldsymbol{q}^{10}) Q_{2,1,1} \\ &+ (1 + \boldsymbol{q}^{2} + \boldsymbol{q}^{4}) Q_{1,1,1,1}\end{aligned} \tag{2.6.17}$$

By way of (2.3.27) and (2.3.28), the 1-, ϕ-, $\varepsilon\phi$-, and ε-isotypic components of $\textsc{ann}(\beta)$ recover the coefficients of the HOMFLY polynomial of the planar closure of β. However, the ψ-isotypic component is novel.

Chapter 3

Annular Homology of Artin Braids III: Rational Cherednik Algebras

3.1 Introduction

3.1.1. Let W be a finite Coxeter group, not necessarily crystallographic, and let Br_W be the corresponding group of Artin braids. There is a distinguished central element of Br_W called the full twist, which we denote by π. If $W = S_n$, then $Br_W = Br_n$, the group of topological braids on n strands; here, π is the braid given by twisting the row of strands through a full revolution. For general W, we will say that an Artin braid $\beta \in Br_W$ is a **fractional twist** of slope $\frac{m}{n} \in \mathbf{Q}$ iff there exists $\gamma \in Br_W$ such that $\beta = \gamma^m$ and $\pi = \gamma^n$.

In this paper, we will describe a precise relationship between fractional twists in Br_W and the representation theory of a certain ring $\mathbf{A}^{\mathrm{rat}}$ known as the rational Cherednik algebra or rational double affine Hecke algebra (DAHA) of W. The relationship relies on the **annular character** that we introduced in [105]: a class function on Br_W of the form

$$\text{(3.1.1)} \qquad \textsc{ann} : Br_W \to \mathrm{K}_0(W)[\![\boldsymbol{q}]\!],$$

where $\mathrm{K}_0(W)$ is the ring of virtual representations of W. This function was inspired by, and refines, Markov traces used to construct the HOMFLY series invariant of links in 3-space [41, 56]. We will show that if $\beta \in Br_W$ is a fractional twist, then $\textsc{ann}(\beta)$ is the graded W-character of a virtual $\mathbf{A}^{\mathrm{rat}}$-module, and in many cases, it is representable by an actual module with interesting properties. Our results specialize to an identity observed by Gorsky–Oblomkov–Rasmussen–Shende [50], relating HOMFLY polynomials of torus knots with simple modules of the rational Cherednik algebras of symmetric groups.

The authors of *ibid.* conjecture that their identity can upgraded to a statement about Khovanov–Rozansky homology, a categorification of the HOMFLY polynomial [62, 61]. In a similar way, we expect that for crystallographic W, our work can be refined in terms of a conjectural $\mathbf{A}^{\mathrm{rat}}$-action on the cohomology of the "Steinberg-like" stack attached to β [105], which is expected to categorify $\textsc{ann}(\beta)$ [106].

3.1.2. To explain our results, we review rational Cherednik algebras in more detail.

Let $\mathbf{Q}_W \subseteq \mathbf{C}$ be any splitting field of W (see [105, §3]). Let V be a faithful realization of W over $\mathbf{Q}_W$ such that $V^W = 0$. For all $\nu \in \mathbf{C}$, we let $\mathbf{A}_\nu^{\mathrm{rat}} = \mathbf{A}_\nu^{\mathrm{rat}}(V)$ be the rational Cherednik algebra of (constant) central charge ν defined by V. It is a deformation of $\mathbf{A}_0^{\mathrm{rat}} = \mathbf{C}[W] \ltimes \mathcal{D}(V_{\mathbf{C}})$, where $\mathcal{D}(V_{\mathbf{C}})$ denotes the Weyl algebra of differential operators on $V_{\mathbf{C}}$. Notably, there is a canonical element $\boldsymbol{h} \in \mathbf{A}_\nu^{\mathrm{rat}}$ that can be interpreted as an Euler vector field; it commutes with the subalgebra $\mathbf{C}[W] \subseteq \mathbf{A}_\nu^{\mathrm{rat}}$.

It was discovered in [8] and [34] that the representation theory of $\mathbf{A}_\nu^{\mathrm{rat}}$ is remarkably similar to that of a finite-dimensional semisimple Lie algebra. In particular, $\mathbf{A}_\nu^{\mathrm{rat}}$ admits a "category O" of well-behaved modules. In the same way that a semisimple Lie algebra contains a Cartan subalgebra, and modules for the former decompose semisimply under the action of the latter, $\mathbf{A}_\nu^{\mathrm{rat}}$ contains $\mathbf{C}[W]$ as a subalgebra, and any $\mathbf{A}_\nu^{\mathrm{rat}}$-module decomposes semisimply under the action of W. In contrast to the story for semisimple Lie algebras, however, an $\mathbf{A}_\nu^{\mathrm{rat}}$-module admits a further decomposition into W-stable eigenspaces under the action of $\boldsymbol{h}$. We will only consider modules for which the $\boldsymbol{h}$-eigenvalues are half-integers. Given such a module M, we obtain a formal Laurent series

$$[M]_{\boldsymbol{q}} = \sum_j \boldsymbol{q}^{\frac{j}{2}} M_{j/2} \in \mathrm{K}_0(W)(\!(\boldsymbol{q}^{\frac{1}{2}})\!), \tag{3.1.2}$$

where $M_\alpha \subseteq M$ is the $\boldsymbol{h}$-eigenspace with eigenvalue α.

The simple objects of category O are parametrized by the set $\mathrm{Irr}(W)$ of irreducible characters of W. For all $\phi \in \mathrm{Irr}(W)$, we let $L_\nu(\phi)$ denote the corresponding simple $\mathbf{A}_\nu^{\mathrm{rat}}$-module and $\Delta_\nu(\phi)$ the corresponding Verma module. A major problem in the representation theory of $\mathbf{A}_\nu^{\mathrm{rat}}$ is to understand how the graded W-characters $[\Delta_\nu(\phi)]_{\boldsymbol{q}}$, which are explicit, can be used to express the graded W-characters $[L_\nu(\phi)]_{\boldsymbol{q}}$, which are a priori mysterious. The problem is highly sensitive to the value of ν: When $\nu \notin \mathbf{Q}$, category O is semisimple and $[L_\nu(\phi)]_{\boldsymbol{q}} = [\Delta_\nu(\phi)]_{\boldsymbol{q}}$; when $\nu \in \mathbf{Q}$, category O grows more complex as the denominator of ν grows smaller.

We will relate the annular characters of fractional twists in Br_W with the characters $[\Delta_\nu(\phi)]_{\boldsymbol{q}}$ by way of polynomials known as generic degrees, which we also discussed in [105, §3]. For all ϕ, let $D_\phi(\boldsymbol{q}) \in \mathbf{Q}_W[\boldsymbol{q}]$ be the generic degree of ϕ. It specializes to $\phi(1)$ in the $\boldsymbol{q} \to 1$ limit.

Theorem 3.1.1. *If $\nu \in \mathbf{Q}$ and $\beta \in Br_W$ is a fractional twist of slope ν, then*

$$\boldsymbol{q}^{\frac{r}{2} - N\nu} \textsc{ann}(\beta) = \sum_{\phi \in \mathrm{Irr}(W)} D_\phi(e^{2\pi i \nu}) [\Delta_\nu(\phi)]_{\boldsymbol{q}} \tag{3.1.3}$$

in $\mathrm{K}_0(W)(\!(\boldsymbol{q}^{\frac{1}{2}})\!)$, *where r is the rank of V and N is the number of reflections in W.*

Remark 3.1.2. On the left-hand side, the factor $\boldsymbol{q}^{\frac{r}{2} - N\nu}$ ensures the expression is invariant under the substitution $\boldsymbol{q}^{\frac{1}{2}} \mapsto -\boldsymbol{q}^{-\frac{1}{2}}$. On the right-hand side, the numbers $D_\phi(e^{2\pi i\nu})$ are all rational integers, *cf.* [105, §4].

Corollary 3.1.3. *If $\beta \in Br_W$ is a fractional twist of slope ν, then $[L_\nu(1)]_{\boldsymbol{q}}$ occurs in* $\textsc{ann}(\beta)$ *with multiplicity one.*

Recall that we define $\textsc{ann}(\beta)$ in terms of a pairing on $\mathrm{Irr}(W)$, derived from the so-called exotic Fourier transform of Lusztig, and the values $\phi_{\boldsymbol{q}}(\beta)$ for $\phi \in \mathrm{Irr}(W)$, where $\phi_{\boldsymbol{q}}$ is the character of the Iwahori–Hecke algebra of W that corresponds to ϕ (see Section 3.4). To prove Theorem 3.1.1, we use an idea of Jones in [56, §9] to express $\phi_{\boldsymbol{q}}(\beta)$ in terms of $\phi(w)$, when w is the image of β under $Br_W \to W$.

Following [90], we say that $\nu \in \mathbf{Q}$ is a **regular elliptic slope** iff it is the eigenvalue of an elliptic element of W acting on a regular eigenvector in V. When W is crystallographic, we can combine Corollary 3.1.3 with a theorem from [106] to obtain a new proof that:

Theorem 3.1.4 (Varagnolo–Vasserot, Etingof)**.** *If ν is a regular elliptic slope, then $L_\nu(1)$ is finite-dimensional.*

Remark 3.1.5. The converse also holds. Varagnolo–Vasserot proved the biconditional statement in the crystallographic case [109]. Etingof generalized it to arbitrary finite Coxeter groups (and rational Cherednik algebras of nonconstant central charge) [39].

As the denominator n shrinks, the extent to which simple modules other than $L_\nu(1)$ occur in $\textsc{ann}(\beta)$ grows. The cleanest situation arises when the denominator of ν in lowest terms is *cuspidal* in the sense of Bezrukavnikov–Etingof [11, §3.9], meaning it divides only one of the invariant degrees of the W-action on V. In this case, we say that ν is a **cuspidal slope**. It is a strictly stronger condition than ν being regular elliptic. We will prove:

Theorem 3.1.6. *If $\beta \in Br_W$ is a fractional twist of slope $\nu \in \mathbf{Q}$, and ν is cuspidal, then*

$$\boldsymbol{q}^{\frac{r}{2}-N\nu}\textsc{ann}(\beta) = \sum_{\psi \in \mathrm{Irr}(W)_1} [L_\nu(\psi)]_{\boldsymbol{q}} \tag{3.1.4}$$

in $\mathrm{K}_0(W)(\!(\boldsymbol{q}^{\frac{1}{2}})\!)$*, where* $\mathrm{Irr}(W)_1 \subseteq \mathrm{Irr}(W)$ *is the set of elements of greatest $\boldsymbol{a}$-value in $e^{2\pi i\nu}$-blocks of defect* 1 *of the Iwahori–Hecke algebra of W (see Section 3.4).*

Corollary 3.1.7. *We keep the hypotheses of Theorem 3.1.6. Suppose that W is irreducible and at least one of the following is true:*

- *W is not of type E_8 or H_4.*
- *$\nu \notin \frac{1}{15}\mathbf{Z}$.*

Then $\boldsymbol{q}^{\frac{r}{2}-N\nu}\textsc{ann}(\beta) = [L_\nu(1)]_{\boldsymbol{q}}$.

The key tool in the proof of Theorem 3.1.6 is the Knizhnik–Zamolodchikov functor from [45]. If H_W denotes the Iwahori–Hecke algebra of W over $\mathbf{Z}[\boldsymbol{q}^{\pm\frac{1}{2}}]$, then this functor relates representations of $\mathbf{A}_\nu^{\mathrm{rat}}$ with representations of $H_W(e^{\pi i\nu}) = H_W|_{\boldsymbol{q}^{1/2}\to e^{\pi i\nu}}$. We will also need results from the block theory of $H_W(e^{\pi i\nu})$, worked out by Geck [42].

Let $(-,-)_W$ be the multiplicity pairing on (graded, virtual) representations of W. When $W = S_n$, Corollary 3.1.7 can be specialized further to the following identity, which was proved in [50] by combinatorial methods:

Theorem 3.1.8 (Gorsky–Oblomkov–Rasmussen–Shende)**.** *Let $m, n > 0$ be coprime integers, and let $W = S_n$. Then, as an element of $\mathbf{Z}[\boldsymbol{q}, a^2]$, the (reduced) HOMFLY polynomial of the (m,n)-torus knot equals*

$$a^{(m-1)(n-1)} \sum_i (-a^2)^i (\Lambda^i, [L_{\frac{m}{n}}(1)]_{\boldsymbol{q}})_{S_n}, \tag{3.1.5}$$

where Λ^i is the Specht module of S_n associated with the ith hook partition of n.

3.1.3. In the rest of the paper [50], the authors keep the hypotheses of Theorem 4.8.8 and construct three different filtrations on $L_{\frac{m}{n}}(1)$ that are compatible with the $\boldsymbol{h}$-grading. They conjecture that all three are the same filtration $\mathcal{F}$, and that Theorem 4.8.8 can be promoted to an isomorphism between the Khovanov–Rozansky homology of the (m,n)-torus knot and the associated graded module of the filtration $\mathcal{F}$ on $L_{\frac{m}{n}}(1)$.

Now assume that W is merely crystallographic. In this setting, in [105], we constructed a common refinement of both ANN and Khovanov–Rozansky homology, which we called **annular braid homology** (distinct from the annular link homology in knot-theory literature). We expect it to shed new light on the conjectures of Gorsky, Oblomkov, Rasmussen, and Shende, as well as on the results of this paper.

Suppose that $\mathbf{F}$ is a finite field and that W is the Weyl group of a reductive group G over $\bar{\mathbf{F}}$. Let $\mathcal{B}$ be the flag variety of G. The Hecke category of W is a certain categorification of the Iwahori–Hecke algebra H_W, defined in terms of complexes of sheaves over $G\backslash(\mathcal{B} \times \mathcal{B})$ equipped with a convolution product. A priori, annular braid homology is a trace on the

Hecke category taking values in $\mathsf{K}^b(\mathbf{A}\text{-}\mathsf{Mod}_{\mathsf{gr}})$, where

$$\mathbf{A} = \mathbf{C}[W] \ltimes \mathrm{Sym}^*(V_{\mathbf{C}}) \tag{3.1.6}$$

and $\mathbf{A}\text{-}\mathsf{Mod}_{\mathsf{gr}}$ is the category of graded $\mathbf{A}$-modules. (Here $\mathrm{Sym}^i(V)$ is placed in degree $2i$.) We note that regardless of ν, there is an embedding of *filtered* algebras $\mathbf{A} \hookrightarrow \mathbf{A}_\nu^{\mathrm{rat}}$. After partially decategorifying, we can view annular braid homology as a class function

$$\mathrm{AH} : \mathit{Br}_W \to \mathrm{K}_0(\mathbf{A}\text{-}\mathsf{Mod}_{\mathsf{gr}})[t], \tag{3.1.7}$$

where $\mathrm{K}_0(\mathbf{A}\text{-}\mathsf{Mod}_{\mathsf{gr}})$ denotes the split Grothendieck group of $\mathbf{A}\text{-}\mathsf{Mod}_{\mathsf{gr}}$, equipped with the grading it inherits from $\mathbf{A}$.

It would be natural to conjecture that for any fractional twist β of slope ν, the graded $\mathbf{A}$-action on $\mathrm{AH}(\beta)$ extends to a filtered $\mathbf{A}_\nu^{\mathrm{rat}}$-action compatible with Theorem 3.1.1 and Corollary 3.1.7 under decategorification. Unfortunately, this cannot work:

Example 3.1.9. If $W = S_2$, then there is a unique fractional twist $\beta \in \mathit{Br}_W = \mathit{Br}_2$ of slope $\frac{3}{2}$. As a topological braid, its planar closure is the trefoil knot. For any generator $\alpha \in V_{\mathbf{C}}$, we calculate that:

1. α acts by zero on $\mathrm{AH}(\beta)$.

2. α acts by a 3-step nilpotent operator on $L_{\frac{3}{2}}(1)$.

So there cannot be an isomorphism $\mathrm{AH}(\beta) \simeq L_{\frac{3}{2}}(1)$ that categorifies Corollary 3.1.7.

We speculate that the mismatch can be fixed when β belongs to the so-called positive submonoid $\mathit{Br}_W^+ \subseteq \mathit{Br}_W$ (see [105, §2]). The fix involves a certain **Steinberg-like variety** $\mathit{St}(\beta)$ over the unipotent locus of G, introduced in [106].

The compactly-supported ℓ-adic cohomology of the stack $G \backslash \mathit{St}(\beta)$ admits a Springer-type action of W. At the same time, it admits a W-stable weight filtration $\mathbf{W}_{\leq *}$, via the

$\mathbf{F}$-structure on $St(\beta)$ induced by the split form of G over $\mathbf{F}$. In [106], we conjectured that up to certain renormalizations, the bigraded W-representation formed by

$$\mathrm{E}(\boldsymbol{q}^{\frac{1}{2}}, t \mid \beta) = \bigoplus_{j,k} \boldsymbol{q}^{\frac{j}{2}} t^k \, \mathrm{Gr}_j^{\mathbf{W}} \, \mathrm{H}_c^k(G \backslash St(\beta), \bar{\mathbf{Q}}_\ell) \tag{3.1.8}$$

matches that formed by $\mathrm{AH}(\beta)$. We proved the conjecture in the $t \to -1$ limit.

Conjecture 3.1.10. *If $\beta \in Br_W^+$ is a positive fractional twist of slope $\nu \in \mathbf{Q}$, then the bigraded W-action on $\mathrm{E}(\boldsymbol{q}^{\frac{1}{2}}, t \mid \beta)$ extends to a bifiltered $\mathbf{A}_\nu^{\mathrm{rat}}$-action. The resulting $\mathbf{A}_\nu^{\mathrm{rat}}$-module:*

1. *Is semisimple and contains $L_\nu(1)$ with multiplicity one.*

2. *Decategorifies in the $t \to -1$ limit to a virtual $\mathbf{A}_\nu^{\mathrm{rat}}$-module possessing the graded W-character in Theorem 3.1.1, possibly up to sign.*

3. *Is isomorphic to $L_\nu(1)$ in the situation of Corollary 3.1.7.*

When $W = S_n$, the variable t corresponds to a splitting of the filtration $\mathcal{F}$ in [50].

3.1.4. Acknowledgements I warmly thank Pavel Etingof, Victor Ginzburg, Eugene Gorsky, Oscar Kivinen, Andrei Neguţ, Bao-Châu Ngô, Emily Norton, and Alexei Oblomkov, for helpful discussions; and Laura Ball, for her generous proofreading. During this work, I have received support from an NSF Graduate Research Fellowship (grant #1746045) and benefited from workshops at MSRI and AIM.

3.1.5. Notation Throughout this paper, we will retain the notation used in [105] and [106]. Thus, we fix a Coxeter presentation of W, with respect to which:

- If $w \in W$, then $|w|$ is the Bruhat length of w.
- $Br_W^+ \subseteq Br_W$ is the positive submonoid of Br_W.

- If $\beta \in Br_W$, then $|\beta|$ is the writhe of β.
- If $w \in W$, then $\beta_w \in Br_W^+$ is the unique lift of w such that $|\beta_w| = |w|$.
- H_W is the Iwahori–Hecke algebra of W over $\mathbf{Z}[\boldsymbol{q}^{\pm\frac{1}{2}}]$. We write $\{\beta_w\}_{w \in W}$ for its standard basis.

We treat the elements of $\mathrm{K}_0(W)$ interchangeably with their characters.

- $\mathrm{Irr}(W) \subseteq \mathrm{K}_0(W)$ is the set of irreducible characters of W. We write $\mathbf{Q}_W \subseteq \mathbf{C}$ for the field obtained by adjoining to $\mathbf{Q}$ the character values $\phi(w)$ for all $w \in W$ and $\phi \in \mathrm{Irr}(W)$.
- 1 and ε are the trivial and sign characters of W, respectively.
- $(-,-)_W$ is the multiplicity pairing on $\mathrm{K}_0(W)$. We extend it to a pairing on $\mathrm{K}_0(W)[\![\boldsymbol{q}]\!]$ by linearity.
- V is a realization of W over $\mathbf{Q}_W$ (see [105, §3]) such that $V^W = 0$.
- If $\phi \in \mathrm{Irr}(W)$, then:
  - $\phi_{\boldsymbol{q}} : H_W \to \mathbf{Q}(\boldsymbol{q}^{\frac{1}{2}})$ is the Tits deformation of ϕ.
  - $\boldsymbol{s}(\phi_{\boldsymbol{q}}) \in \mathbf{Q}_W(\boldsymbol{q}^{\frac{1}{2}})$ is the Schur element attached to $\phi_{\boldsymbol{q}}$, defined by

$$\boldsymbol{s}(\phi_{\boldsymbol{q}}) = \frac{1}{\phi(1)} \sum_{w \in W} \phi_{\boldsymbol{q}}(\beta_w)\phi_{\boldsymbol{q}}(\beta_{w^{-1}}). \tag{3.1.9}$$

  - $P_\phi(\boldsymbol{q}) \in \mathbf{Z}[\boldsymbol{q}]$ is the fake degree of ϕ, defined by

$$P_\phi(\boldsymbol{q}) = \frac{(\phi, M(\boldsymbol{q} \mid V))_W}{(1, M(\boldsymbol{q} \mid V))_W}. \tag{3.1.10}$$

  - $D_\phi(\boldsymbol{q}) \in \mathbf{Q}_W[\boldsymbol{q}]$ is the generic degree of ϕ, defined by

$$D_\phi(\boldsymbol{q}) = \frac{\boldsymbol{s}(1_{\boldsymbol{q}})}{\boldsymbol{s}(\phi_{\boldsymbol{q}})}. \tag{3.1.11}$$

- $\{-,-\}$ is Lusztig's exotic Fourier transform (see [105, §3]). Like in [105], we will only use its restriction to a pairing on $\mathrm{Irr}(W)$:

$$\{-,-\} : \mathrm{Irr}(W) \times \mathrm{Irr}(W) \to \mathbf{Q}_W. \tag{3.1.12}$$

- $\boldsymbol{o} : \mathrm{Irr}(W) \to \mathrm{Irr}(W)$ is Opdam's involution on $\mathrm{Irr}(W)$ (see [105, §3]).

We do *not* assume that W is crystallographic.

3.2 Families of Characters

3.2.1. Using induction on parabolic subgroups of W, Lusztig described a partition of $\mathrm{Irr}(W)$ into subsets called **families**. In this section, we collect properties of this partition that we need in Sections 3.3 and 3.6.

First, families appear in the definitions of the exotic Fourier transform $\{-,-\}$ and the involution $\boldsymbol{o}$. As a result, if $\{-,-\}$ and $\boldsymbol{o}$ are expressed as $\mathrm{Irr}(W) \times \mathrm{Irr}(W)$ matrices, then these matrices are block diagonal with respect to Lusztig's partition.

Remark 3.2.1. If W is an irreducible Coxeter group of type E_7, E_8, H_3, or H_4, then $\boldsymbol{o}$ permutes a single two-element family in $\mathrm{Irr}(W)$ and preserves all other elements. If W is irreducible but not of these types, then the involution $\boldsymbol{o}$ is trivial.

The following properties of $\{-,-\}$ and $\boldsymbol{o}$ appeared in [105, §3]. Note that the presence of $\boldsymbol{o}$ is mistakenly omitted in several places in the literature, *cf.* the remark in *loc. cit.*

Property 3.2.2. *We have*

$$D_\phi(\boldsymbol{q}) = \sum_{\psi \in \mathrm{Irr}(W)} \{\phi, \psi\} P_{\boldsymbol{o}(\psi)}(\boldsymbol{q}) \tag{3.2.1}$$

for all $\phi \in \mathrm{Irr}(W)$.

Property 3.2.3. *We have* $\{\varepsilon\phi, \varepsilon\psi\} = \{\phi, \psi\}$ *for all* $\phi, \psi \in \mathrm{Irr}(W)$.

Property 3.2.4. *We have* $\boldsymbol{o}(\varepsilon\phi) = \varepsilon\boldsymbol{o}(\phi)$ *for all* $\phi \in \mathrm{Irr}(W)$.

3.2.2. In [70], Lusztig introduced functions $\boldsymbol{a}, \boldsymbol{A} : \mathrm{Irr}(W) \to \mathbf{Z}_{\geq 0}$ defined by

$$\boldsymbol{a}(\phi) = \mathrm{val}_{\boldsymbol{q}}\, D_\phi(\boldsymbol{q}), \tag{3.2.2}$$

$$\boldsymbol{A}(\phi) = \deg_{\boldsymbol{q}} D_\phi(\boldsymbol{q}). \tag{3.2.3}$$

Lemma 3.2.5 (Lusztig)**.** *The functions* $\boldsymbol{a}$ *and* $\boldsymbol{A}$ *are constant in families.*

Later we will use $\boldsymbol{a}$ directly. We only will use $\boldsymbol{A}$ by way of the auxiliary function that follows. Let $\mathrm{Ref}(W)$ be the set of elements of W that act on V by reflections. (It is independent of V and its elements are usually just called reflections.) Motivated by the theory of partitions, we introduce:

Definition 3.2.6. For all $\phi \in \mathrm{Irr}(W)$, the **content** of ϕ is the number

$$\boldsymbol{c}(\phi) = \frac{1}{\phi(1)} \sum_{t \in \mathrm{Ref}(W)} \phi(t). \tag{3.2.4}$$

We set $N = \boldsymbol{c}(1) = |\mathrm{Ref}(W)|$.

If $W = S_n$, then there is a bijective correspondence between irreducible representations of W and partitions of n. Under this bijection, $\boldsymbol{c}(\phi)$ is precisely the content of the partition attached to ϕ. For general W, the following result about $\boldsymbol{c}$ is essentially [17, Cor. 4.2], *cf.* [45, §6.2]:

Lemma 3.2.7 (Broué–Michel)**.** *We have* $\boldsymbol{c}(\phi) = N - \boldsymbol{a}(\phi) - \boldsymbol{A}(\phi)$ *for all* $\phi \in \mathrm{Irr}(W)$.

Corollary 3.2.8. *The function* $\boldsymbol{c}$ *is constant in families.*

3.3 Fractional Twists

3.3.1. Let w_0 be the longest element of W with respect to the Bruhat length. The **full twist** of Br_W can be defined as the positive braid

$$\pi = \beta_{w_0}^2 \in Br_W^+. \tag{3.3.1}$$

It is central in Br_W.

Definition 3.3.1. We say that an Artin braid $\beta \in Br_W$ is:

- **Periodic** iff $\beta^n = \pi^m$ for some $\frac{m}{n} \in \mathbf{Q}$.
- A **fractional twist** iff there exist $\gamma \in Br_W$ and $\frac{m}{n} \in \mathbf{Q}$ such that $\beta = \gamma^m$ and $\pi = \gamma^n$. Note that we can always assume $\gamma \in Br_W^+$.

In either case, we say that $\frac{m}{n}$ is the **slope** of β.

Conjecture 3.3.2. *An Artin braid is periodic of slope $\frac{m}{n}$ only if it is a fractional twist of slope $\frac{m}{n}$. (The converse is clear.)*

Remark 3.3.3. Conjecture 3.3.2 holds in type A (that is to say, for topological braids) as a consequence of a theorem of Kerékjártó–Eilenberg [35] (see also [47]).

Remark 3.3.4. Since Br_n/π forms the mapping class group of a sphere with n punctures, there is a Nielsen–Thurston classification of its elements according to their action on Teichmüller space. An element of Br_n is periodic in the sense of Definition 3.3.1 if and only if its image in Br_n/π is periodic in the Nielsen–Thurston sense, i.e., has a Teichmüller fixed point. It would be interesting to know whether a Nielsen–Thurston classification exists for elements of arbitrary Artin groups.

3.3.2. The following result, an incarnation of Schur's lemma, is Theorem 9.4.3 in [43].

Theorem 3.3.5 (Springer)**.** *If $\phi \in \mathrm{Irr}(W)$, then the full twist acts on the underlying H_W-module of $\phi_{\boldsymbol{q}}$ by the scalar $\boldsymbol{q}^{\boldsymbol{c}(\phi)}$. In particular, $\phi_{\boldsymbol{q}}(\pi) = \boldsymbol{q}^{\boldsymbol{c}(\phi)}\phi(1)$.*

The following corollary, and its proof, are inspired by Lemma 9.4 of [56], which Jones uses to calculate the HOMFLY polynomials of torus knots.

Corollary 3.3.6. *If $\phi \in \mathrm{Irr}(W)$ and $\beta \in Br_W$ is periodic of slope $\nu \in \mathbf{Q}$, then*

$$\phi_{\boldsymbol{q}}(\beta) = \boldsymbol{q}^{\nu\boldsymbol{c}(\phi)}\phi(w), \tag{3.3.2}$$

where $w \in W$ is the image of β under the surjection $Br_W \twoheadrightarrow W$.

Proof. It suffices to prove the result when $\boldsymbol{q}^{\frac{1}{2}}$ has been specialized to a generic complex number $q^{\frac{1}{2}}$. We set $H_W[q^{\frac{1}{2}}] = H_W \otimes_{\mathbf{Z}[\boldsymbol{q}^{\pm 1/2}]} \mathbf{C}$, where $\mathbf{Z}[\boldsymbol{q}^{\pm\frac{1}{2}}] \to \mathbf{C}$ sends $\boldsymbol{q}^{\frac{1}{2}} \mapsto q^{\frac{1}{2}}$, and

$$\phi_q = \phi_{\boldsymbol{q}}|_{\boldsymbol{q}^{1/2} \to q^{1/2}}. \tag{3.3.3}$$

Write $\nu = \frac{m}{n}$ in lowest terms. By the theorem, $q^{-m\boldsymbol{c}(\phi)}\pi^m$ acts on the underlying $H_W[q^{\frac{1}{2}}]$-module of ϕ_q by the identity operator I. Since $\beta^n = \pi^m$, we deduce that $q^{-\nu\boldsymbol{c}(\phi)}\beta$ acts by an nth root of I.

Recall that $H_W|_{q\to 1} = \mathbf{Z}[W]$. Choose a generic path in the complex plane from q to 1. Since the (diagonalizable) roots of I do not deform, the deformation of q along this path shows that the following operators must have the same eigenvalue spectrum:

1. The action of $q^{-\nu\boldsymbol{c}(\phi)}\beta$ on the underlying $H[q^{\frac{1}{2}}]$-module of ϕ_q.

2. The action of w on the underlying $\mathbf{C}[W]$-module of ϕ.

The result follows. □

3.3.3. Fractional twists in Br_W are closely related to the so-called regular elements of W, which we now review.

A vector of V is **regular** with respect to W iff its stabilizer in W is trivial, or equivalently, it avoids the reflecting hyperplanes that generate the W-action. Suppose that an element $w \in W$ admits a regular eigenvector in $V \otimes \mathbf{C}$ with eigenvalue $\zeta \in \mathbf{C}^\times$. In this case, ζ must be a root of unity. We say that ζ is **regular** with respect to W and that w is a **ζ-regular element** of W. Springer proved [101]:

Theorem 3.3.7 (Springer)**.** *For any ζ, the set of ζ-regular elements of W is either empty or forms a single conjugacy class.*

Theorem 3.3.8 (Springer)**.** *If $w \in W$ is a ζ-regular element, then*

$$\phi(w) = P_\phi(\zeta^{-1}) = P_\phi(\zeta) \tag{3.3.4}$$

for all $\phi \in \mathrm{Irr}(W)$.

Remark 3.3.9. When ζ is regular, Springer refers to the order of ζ as a **regular number** of W. In the last sentence on page 170 of [101], he claims that if n is a regular number, then the regular elements of W of order n form a single conjugacy class. This claim, which is stronger than Theorem 3.3.7, does *not* hold when W is not crystallographic. For example, let W be the Coxeter group of type $I_2(5)$ and let w be a Coxeter element. Then w and w^2 are both regular elements of order 5, but they are not conjugate.

The following results are Proposition 3.11 and Théorème 3.12 of [17], respectively.

Proposition 3.3.10 (Broué–Michel)**.** *If $w \in W$ is regular of order n, then $\beta_w^n = \pi$.*

Theorem 3.3.11 (Broué–Michel)**.** *If $\gamma \in Br_W^+$ satisfies $\gamma^n = \pi$ for some n, then its image in W is a $e^{\frac{2\pi i}{n}}$-regular element.*

We emphasize that Theorem 3.3.11 is deeper than Proposition 3.3.10: Its proof relies on results used by Charney to prove the biautomaticity of Artin groups [17, 3.19]. It would be very desirable to extend the following corollary of Theorem 3.3.11 from fractional twists to all periodic braids.

Corollary 3.3.12. *If $\beta \in Br_W$ is a fractional twist of slope $\frac{m}{n}$, then the image of β in W is a $e^{2\pi i \frac{m}{n}}$-regular element.*

3.3.4. Recall that in [105], the annular character ANN is defined in terms of an auxiliary function $\mathbb{A} : H_W \to \mathrm{K}_0(W) \otimes \mathbf{Q}_W(\boldsymbol{q}^{\frac{1}{2}})$, namely,

$$\mathbb{A}(\beta) = \sum_{\phi,\psi \in \mathrm{Irr}(W)} \{\phi, \psi\} \phi_{\boldsymbol{q}}(\beta) \mathbf{o}(\psi). \tag{3.3.5}$$

By assembling all of the discussion above, we can show:

Proposition 3.3.13. *If β is a fractional twist of slope $\nu \in \mathbf{Q}$, then*

$$\mathbb{A}(\beta) = \sum_{\psi \in \mathrm{Irr}(W)} \boldsymbol{q}^{\nu \boldsymbol{c}(\psi)} D_\psi(e^{2\pi i \nu}) \psi \tag{3.3.6}$$

in $\mathrm{K}_0(W)[\boldsymbol{q}^{\pm\frac{1}{2}}]$.

Proof. By Corollary 3.3.6 and Theorem 3.3.8,

$$\phi_{\boldsymbol{q}}(\beta) = \boldsymbol{q}^{\nu \boldsymbol{c}(\phi)} P_\phi(e^{2\pi i \nu}). \tag{3.3.7}$$

So, by Property 3.2.2 and Corollary 3.2.8,

$$\begin{aligned} \sum_{\phi \in \mathrm{Irr}(W)} \{\phi, \psi\} \phi_{\boldsymbol{q}}(\beta) &= \boldsymbol{q}^{\nu \boldsymbol{c}(\psi)} \sum_{\phi \in \mathrm{Irr}(W)} \{\phi, \psi\} P_\phi(e^{2\pi i \nu}) \\ &= \boldsymbol{q}^{\nu \boldsymbol{c}(\psi)} D_{\boldsymbol{o}(\psi)}(e^{2\pi i \nu}). \end{aligned} \tag{3.3.8}$$

Finally, again by Corollary 3.2.8, $\boldsymbol{c}(\psi) = \boldsymbol{c}(\boldsymbol{o}(\psi))$. ☐

In Section 3.6, we revisit this result to finish the proof of Theorem 3.1.1.

3.4 Blocks of Symmetric Algebras

3.4.1. In this section, we review the block theory of Iwahori–Hecke algebras at roots of unity. For clarity, we first explain what happens in a general setting. Throughout, we write $\mathrm{K}_0(-)$ to denote Grothendieck groups of modules.

Let E be a field of characteristic zero, and let $A \subseteq E$ be a subring. Let H be a free A-algebra of finite rank such that the E-algebra

$$H_E = H \otimes_A E \tag{3.4.1}$$

is split semisimple. Let k be another field and $\theta : A \to k$ a ring morphism. We set

$$H_\theta = H \otimes_A k. \tag{3.4.2}$$

We will explain how Brauer's theory of decomposition maps gives a direct relation between the representation theories of H_E and H_θ, or more precisely, between $\mathrm{K}_0(H_E)$ and $\mathrm{K}_0(H_\theta)$, as long as A is integrally closed in E.

First, let $\mathrm{K}_0(H_E)_{\geq 0} \subseteq \mathrm{K}_0(H_E)$ be the subset of elements that can be represented by actual, not just virtual, modules. This is a multiplicative submonoid of $\mathrm{K}_0(H_E)$. Letting t be an indeterminate over K, we have a map:

$$\begin{array}{rcl} \mathrm{K}_0(H_E)_{\geq 0} & \xrightarrow{\boldsymbol{p}} & \prod_H (1 + tK[t]) \\ M & \mapsto & \{\det_E(1 - t\alpha \mid M)\}_{\alpha \in H} \end{array} \tag{3.4.3}$$

The Brauer–Nesbitt theorem [43, Lem. 7.3.2] implies that $\boldsymbol{p}$ is injective. The following result is Prop. 7.3.8 of [43]:

Lemma 3.4.1. *If A is integrally closed in E, then $\boldsymbol{p}$ factors through $(1 + tA[t])^H$.*

We define $\mathrm{K}_0(H_\theta)_{\geq 0}$ and $\boldsymbol{p}_\theta : \mathrm{K}_0(H_\theta)_{\geq 0} \to (1 + k[t])^H$ analogously to $\mathrm{K}_0(H_E)_{\geq 0}$ and $\boldsymbol{p}$.

We can now state Theorem 7.4.3 of [43]:

Theorem 3.4.2 (Brauer)**.** *Suppose A is integrally closed in E. Then there is a unique additive map $\boldsymbol{d}_\theta : \mathrm{K}_0(H_E)_{\geq 0} \to \mathrm{K}_0(H_\theta)_{\geq 0}$ such that the following diagram commutes:*

(3.4.4)
$$\begin{array}{ccc} \mathrm{K}_0(H_E)_{\geq 0} & \xrightarrow{\boldsymbol{p}} & (1 + tA[t])^H \\ \boldsymbol{d}_\theta \downarrow & & \downarrow \theta \\ \mathrm{K}_0(H_\theta)_{\geq 0} & \xrightarrow{\boldsymbol{p}_\theta} & (1 + tk[t])^H \end{array}$$

In particular, $\boldsymbol{d}_\theta$ is compatible with the formation of characters from modules.

Above, $\boldsymbol{d}_\theta$ is called the **decomposition map** of θ. Note that if $\phi \in \mathrm{K}_0(H_E)_{\geq 0}$ is irreducible, then $\boldsymbol{d}_\theta(\phi) \in \mathrm{K}_0(H_\theta)_{\geq 0}$ is usually no longer irreducible.

Let $\mathrm{Irr}(H_E) \subseteq \mathrm{K}_0(H_E)_{\geq 0}$ be the set of irreducible characters. The **Brauer graph** of H_θ is the (undirected) graph in which:

1. The vertex set is $\mathrm{Irr}(H_E)$.
2. There is an edge between M and M' iff there is a simple module shared in common by the decompositions of $\boldsymbol{d}_\theta(M)$ and $\boldsymbol{d}_\theta(M')$ in $\mathrm{K}_0(H_\theta)_{\geq 0}$.

The connected components of this graph are called the θ**-blocks** of H_E. The partition of $\mathrm{Irr}(H_E)$ into blocks corresponds to a direct-sum decomposition of H_θ into idempotent ideals called **block ideals**, such that the simple modules over any block ideal are in bijection with the characters in the corresponding block.

3.4.2. Recall that $\mathbf{Q}_W \subseteq \mathbf{C}$ is a splitting field of W. As in [105], let

(3.4.5)
$$H_W(\boldsymbol{q}^{\frac{1}{2}}) = H_W \otimes_{\mathbf{Z}[\boldsymbol{q}^{\pm 1/2}]} \mathbf{Q}_W(\boldsymbol{q}^{\frac{1}{2}}).$$

Fix a number $\nu \in \mathbf{Q}$. Let $\zeta^{\frac{1}{2}} = e^{\pi i \nu}$, and let $H_W(\zeta^{\frac{1}{2}})$ be the $\mathbf{Q}_W(\zeta^{\frac{1}{2}})$-algebra

$$H_W(\zeta^{\frac{1}{2}}) = H_W \otimes_{\mathbf{Z}[\boldsymbol{q}^{\pm\frac{1}{2}}]} \mathbf{Q}_W(\zeta^{\frac{1}{2}}), \tag{3.4.6}$$

where the map $\mathbf{Z}[\boldsymbol{q}^{\pm\frac{1}{2}}] \to \mathbf{Q}_W(\zeta^{\frac{1}{2}})$ sends $\boldsymbol{q}^{\frac{1}{2}} \mapsto \zeta^{\frac{1}{2}}$.

Let $\mathbf{Z}_W$ be the ring of integers of $\mathbf{Q}_W$. Then we can apply the formalism in the previous subsection to the setup where:

- $A = \mathbf{Z}_W[\boldsymbol{q}^{\pm\frac{1}{2}}]$.
- $E = \mathbf{Q}_W(\boldsymbol{q}^{\frac{1}{2}})$.
- $H = H_W \otimes_{\mathbf{Z}} \mathbf{Z}_W$.
- $H_E = H_W(\boldsymbol{q}^{\frac{1}{2}})$.
- $k = \mathbf{Q}_W(\zeta^{\frac{1}{2}})$.
- $\theta : A \to k$ is the specialization map $\boldsymbol{q}^{\frac{1}{2}} \mapsto \zeta^{\frac{1}{2}}$.
- $H_\theta = H_W(\zeta^{\frac{1}{2}})$.

So, Brauer's theorem yields a decomposition map:

$$\boldsymbol{d}_\theta : \mathrm{K}_0(H_W(\boldsymbol{q}^{\frac{1}{2}}))_{\geq 0} \to \mathrm{K}_0(H_W(\zeta^{\frac{1}{2}}))_{\geq 0}. \tag{3.4.7}$$

We will refer to the θ-blocks of $H_W(\boldsymbol{q}^{\frac{1}{2}})$ as **ζ-blocks**. The **principal block** is the block that contains the trivial character 1.

3.4.3. By Proposition 7.3.9 of [43], we have

$$\boldsymbol{s}(\phi_{\boldsymbol{q}}) \in \mathbf{Z}_W[\boldsymbol{q}^{\pm\frac{1}{2}}] \tag{3.4.8}$$

for all $\phi \in \mathrm{Irr}(W)$. The **ζ-defect** of ϕ is the multiplicity with which $\boldsymbol{q}^{\frac{1}{2}} = \zeta^{\frac{1}{2}}$ occurs as a root of the Laurent polynomial $\boldsymbol{s}(\phi_{\boldsymbol{q}})$ (after we extend scalars). This number measures the complexity of the ζ-block containing ϕ in the following sense:

Theorem 3.4.3 (Geck)**.** *The characters within a given ζ-block have the same ζ-defect, so we can speak of the defect of the block itself. Moreover:*

1. *If the defect is* 0*, then the block is a singleton.*

2. *If the defect is* 1*, then the block is a line of characters ordered by $\boldsymbol{a}$-value. Its block ideal is isomorphic to a Brauer tree algebra of type A (see below).*

Proof. This restates Prop. 7.4, Prop. 8.2, and Thm. 9.6 of [42]. □

Remark 3.4.4. By contrast, we do not know of a structure theorem for blocks of defect ≥ 2.

Example 3.4.5. Suppose that we are in type A_{n-1}, meaning $W \simeq S_n$, and that $\nu = m/n$ with m coprime to n.

Here, the principal ζ-block is a line. If V is the standard representation of S_n, then the principal block consists of the exterior powers $\Lambda^i(V)$, ordered along the line from $i = 0$ to $i = n-1$. We refer to the associated block ideal as the **Brauer tree algebra of type** A_{n-1}. All other blocks are singletons.

3.5 Representations of Rational Cherednik Algebras

3.5.1. Let $\langle -, - \rangle : V \times V^\vee \to \mathbf{Q}_W$ denote the evaluation pairing. For every reflection $t \in \mathrm{Ref}(W)$, we fix nonzero vectors $\alpha_t \in V$ and $\alpha_t^\vee \in V^\vee$ such that

1. $\langle \alpha_t, - \rangle = 0$ is the hyperplane of $V^\vee$ fixed by t.

2. $\langle -, \alpha_t^\vee \rangle = 0$ is the hyperplane of V fixed by t.

3. $\langle \alpha_t, \alpha_t^\vee \rangle = 2$.

In the literature, the vectors α_t and $\alpha_t^\vee$ are called roots and coroots even when W is not crystallographic. Condition (3) will ensure that in what follows, our constructions do not depend on the exact choices of $\alpha_t, \alpha_t^\vee$.

Below, we write $A \otimes B$ to denote a *non*commutative tensor product of **C**-algebras and $A \cdot B$ to denote a commutative one. The generic **rational Cherednik algebra** or **rational DAHA** of (W, V) is the graded **C**-algebra

$$\mathbf{A}^{\mathrm{rat}} = \frac{\mathbf{C}[u, \omega] \cdot (\mathbf{C}[W] \ltimes (\mathrm{Sym}^*(V_{\mathbf{C}}) \otimes \mathrm{Sym}^*(V_{\mathbf{C}}^\vee)))}{\left\langle yx - xy - \omega\langle x, y\rangle - u \sum_{t \in \mathrm{Ref}(W)} \langle x, \alpha^\vee\rangle\langle \alpha, y\rangle t : x \in V_{\mathbf{C}},\, y \in V_{\mathbf{C}}^\vee \right\rangle}, \tag{3.5.1}$$

where the grading is given by:

- $\deg u = \deg \omega = 2$.
- $\deg x = 2$ for all $x \in V$.
- $\deg y = 0$ for all $y \in V^\vee$.
- $\deg w = 0$ for all $w \in W$.

If $\nu \in \mathbf{C}$, then the **rational Cherednik algebra** of (W, V) of **central charge** ν is

$$\mathbf{A}_\nu^{\mathrm{rat}} = \mathbf{A}^{\mathrm{rat}} / \langle u + \nu,\, \omega - 1\rangle. \tag{3.5.2}$$

Although this algebra is no longer graded, the grading on $\mathbf{A}^{\mathrm{rat}}$ descends to an increasing filtration on $\mathbf{A}_\nu^{\mathrm{rat}}$.

Remark 3.5.1. The embedding of filtered algebras $\mathbf{A} \hookrightarrow \mathbf{A}_\nu^{\mathrm{rat}}$ mentioned in Subsection 3.1.3 of the introduction is the natural composition

$$\mathbf{A} \hookrightarrow \mathbf{A}^{\mathrm{rat}} \twoheadrightarrow \mathbf{A}_\nu^{\mathrm{rat}}. \tag{3.5.3}$$

The first map is grading-preserving.

Remark 3.5.2. For $\nu = 0$, the defining relations of $\mathbf{A}_\nu^{\mathrm{rat}}$ simplify to $[x, y] = (x, y)$ for all x and y. This shows why $\mathbf{A}_0^{\mathrm{rat}}$ is isomorphic to $\mathbf{C}[W] \ltimes \mathcal{D}(V_{\mathbf{C}})$, the semidirect product of $\mathbf{C}[W]$ with the Weyl algebra of $V_{\mathbf{C}}$.

Like the universal enveloping algebra of a semisimple Lie algebra, $\mathbf{A}_\nu^{\mathrm{rat}}$ admits a triangular decomposition in the sense of the Poincaré–Birkhoff–Witt theorem. It consequently admits an analogue of the Bernstein–Gelfand–Gelfand category O, which we will denote by O_ν. The Verma modules in O_ν are indexed by $\mathrm{Irr}(W)$. For all $\phi \in \mathrm{Irr}(W)$, we write $\Delta_\nu(\phi)$ for the corresponding Verma module of $\mathbf{A}_\nu^{\mathrm{rat}}$, and we write $L_\nu(\phi)$ for its simple quotient. Explicitly, $\Delta_\nu(\phi)$ can be constructed by inflating ϕ from $\mathbf{C}[W]$ up to $\mathbf{A} = \mathbf{C}[W] \ltimes \mathrm{Sym}^*(V_{\mathbf{C}})$, then tensoring up to $\mathbf{A}_\nu^{\mathrm{rat}}$.

Remark 3.5.3. In the literature, $\Delta_\nu(1)$ is known as the **polynomial module** of $\mathbf{A}_\nu^{\mathrm{rat}}$ and $L_\nu(1)$ is known as the **simple spherical module**.

Like the Weyl algebra of $V_{\mathbf{C}}$, the rational Cherednik algebra contains a canonical "Euler element" $\boldsymbol{h}$. (For the precise definition, see [38, Ch. 11].) It commutes with the subalgebra $\mathbf{C}[W] \subseteq \mathbf{A}_\nu^{\mathrm{rat}}$, and its action on any object of O_ν is locally finite. Thus, each object $M \in \mathsf{O}$ is endowed with a decomposition into finite-dimensional, W-stable $\boldsymbol{h}$-eigenspaces

$$M_\alpha = \{m \in M : \boldsymbol{h}m = \alpha m\}. \tag{3.5.4}$$

In all situations that we will encounter, the eigenvalues will be half-integers. We define the **graded W-character** of M to be:

$$[M]_{\boldsymbol{q}} = \sum_j \boldsymbol{q}^{\frac{j}{2}} M_{j/2} \in \mathrm{K}_0(W)(\!(\boldsymbol{q}^{\frac{1}{2}})\!). \tag{3.5.5}$$

By construction [38, Ch. 11], we have

$$[\Delta_\nu(\phi)]_{\boldsymbol{q}} = \boldsymbol{q}^{\frac{r}{2} - \nu \boldsymbol{c}(\phi)} \phi \cdot \sum_{i \geq 0} \boldsymbol{q}^i \, \mathrm{Sym}^i(V) \tag{3.5.6}$$

for all $\phi \in \mathrm{Irr}(W)$.

3.5.2. Any $\mathbf{A}_\nu^{\mathrm{rat}}$-module $M \in \mathsf{O}_\nu$ can be viewed as a W-equivariant quasicoherent sheaf on $V^\vee = \mathrm{Spec}\,\mathrm{Sym}^*(V)$. The **support** of M is the W-stable subvariety of $V^\vee$ that forms its support as a sheaf.

Let $\mathsf{O}_\nu^{\mathrm{tor}} \subseteq \mathsf{O}_\nu$ be the Serre subcategory of modules that are torsion over $\mathrm{Sym}^*(V_{\mathbf{C}}^\vee) \subseteq \mathbf{A}_\nu^{\mathrm{rat}}$. Then, by [45, Lem. 5.1], $\mathsf{O}_\nu^{\mathrm{tor}}$ is also the full subcategory of O_ν of modules supported on the discriminant locus of $V^\vee$, i.e., the common complement of the reflection hyperplanes of the W-action.

Remark 3.5.4. One can show that an object of O_ν is finite-dimensional if and only if its support is $\{0\}$, but we will not use this fact.

Let $\zeta^{\frac{1}{2}} = e^{\pi i \nu}$, and let $\mathsf{Mod}(H_W(\zeta^{\frac{1}{2}}))$ be the category of $H_W(\zeta^{\frac{1}{2}})$-modules. By the discussion in Section 3.4, each character $\phi \in \mathrm{Irr}(W)$ gives rise to an object

$$\phi_\zeta = \boldsymbol{d}_\theta(\phi_{\boldsymbol{q}}) \in \mathsf{Mod}(H_W(\zeta^{\frac{1}{2}})) \tag{3.5.7}$$

up to isomorphism. In [45], Ginzburg–Guay–Opdam–Rouquier introduced the **Knizhnik–Zamolodchikov (KZ) functor**, an exact monoidal functor

$$\mathcal{KZ} : \mathsf{O}_\nu \to \mathsf{Mod}(H_W(\zeta^{\frac{1}{2}})). \tag{3.5.8}$$

We summarize its properties [45, Thm. 5.14, Cor. 5.18, Thm. 6.8]:

Theorem 3.5.5 (Ginzburg–Guay–Opdam–Rouquier)**.** *For all $\nu \in \mathbf{C}$,*

1. *$\mathcal{KZ}$ induces an equivalence $\mathsf{O}_\nu/\mathsf{O}_\nu^{\mathrm{tor}} \simeq \mathsf{Mod}(H_W(\zeta^{\frac{1}{2}}))$.*
2. *$\mathcal{KZ}$ induces an isomorphism between the (vertical) center of O_ν, i.e., the algebra of endomorphisms of its unit object, and the center of $H_W(\zeta^{\frac{1}{2}})$.*
3. *$\mathcal{KZ}(\Delta_\nu(\phi)) = \phi_\zeta$.*

3.5.3. As preparation for the final section, we introduce several subsets of $\mathbf{Q}$ of increasing specificity and describe what we will prove when ν belongs to each subset. Below, let $\{d_i\}_{i \in I(W)}$ be the multiset of invariant degrees of the W-action on V [43, 149], and let

$$I_\nu(W) = \{i \in I(W) : \nu \in \tfrac{1}{d_i}\mathbf{Z}\}. \tag{3.5.9}$$

Concretely, if n is the denominator of ν in lowest terms, then $I_\nu(W)$ is the set of indices i such that n divides the invariant degree d_i.

Definition 3.5.6. If $\nu \in \mathbf{Q}$ and $\zeta = e^{2\pi i \nu}$, then we say that ν is:

1. A **singular slope** iff $|I_\nu(W)| \geq 1$.
2. A **regular slope** iff W contains a ζ-regular element.
3. A **regular elliptic slope** iff W contains a ζ-regular element w such that $V^w = 0$. (Recall that we assume $V^W = 0$.)
4. A **cuspidal slope** iff $|I_\nu(W)| = 1$.

We say that an integer $n > 0$ is a **singular number**, etc., iff it is the denominator in lowest terms of a singular slope, etc.

Lemma 3.5.7. *There is a chain of implications:*

$$\textit{cuspidal} \implies \textit{regular elliptic} \implies \textit{regular} \implies \textit{singular}. \tag{3.5.10}$$

Proof. It is tautological that regular elliptic implies regular. That regular implies singular follows from [101, Thm. 4.2(iii)]. That cuspidal implies regular elliptic is [11, Rem. 3.28]. □

Example 3.5.8. In type A_r, the invariant degrees are $2, 3, 4, \ldots, r+1$. An integer is regular if and only if it is a divisor of either r or $r+1$ [101, §5.1]. The only regular elliptic number is the Coxeter number $r+1$ [109, Ex. 1.1.2].

Example 3.5.9. In type BC_r, the invariant degrees are $2, 4, 6, \dots, 2r$. An integer is regular, resp. regular elliptic, if and only if it is a divisor of $2r$ [101, §5.2], resp. an even divisor of $2r$ [109, Ex. 1.1.2]. To demonstrate: In type BC_5, we find that:

- The singular numbers are $1, 2, 3, 4, 5, 6, 8, 10$.
- The regular numbers are $1, 2, 5, 10$.
- The regular elliptic numbers are $2, 10$.
- The only cuspidal number is 10, the Coxeter number.

This is a minimal example where each implication in Lemma 3.5.7 is strict.

Theorem 3.5.10. *Let $\nu \in \mathbf{Q}$, and let*

$$\Omega_\nu = \sum_{\phi \in \mathrm{Irr}(W)} D_\phi(e^{2\pi i\nu})\Delta_\nu(\phi), \tag{3.5.11}$$

a virtual $\mathbf{A}_\nu^{\mathrm{rat}}$-module. Then:

1. *$L_\nu(1)$ occurs in Ω_ν with multiplicity one.*
2. *If ν is a singular slope, then $\Omega_\nu \in \mathrm{K}_0(\mathsf{O}_\nu^{\mathrm{tor}})$.*
3. *If ν is a regular slope, then*

$$[\Omega_\nu]_{\boldsymbol{q}} = \boldsymbol{q}^{\frac{r}{2} - N\nu} ANN(\beta) \tag{3.5.12}$$

for any fractional twist $\beta \in Br_W$ of slope ν, where $r = \dim V$.

4. *If ν is a regular elliptic slope, then Ω_ν is a difference of finite-dimensional modules.*
5. *If ν is a cuspidal slope, then:*

$$[\Omega_\nu]_{\boldsymbol{q}} = \sum_{\psi \in \mathrm{Irr}(W)_1} [L_\nu(\psi)]_{\boldsymbol{q}} \tag{3.5.13}$$

where $\mathrm{Irr}(W)_1 \subseteq \mathrm{Irr}(W)$ *is the set of elements of greatest* $\boldsymbol{a}$*-value in* ζ*-blocks of* $H_W(\boldsymbol{q}^{\frac{1}{2}})$ *that have defect* 1*.*

Compared to the statements in the introduction, part (3) is Theorem 3.1.1, part (1) implies Corollary 3.1.3, parts (1) and (4) imply Theorem 3.1.4, and part (5) implies Theorem 3.1.6. We will discuss Corollary 3.1.7 and Theorem 4.8.8 in the subsection after the proof of part (5).

Remark 3.5.11. By combining parts (1) and (2) of Theorem 3.5.10, we recover the result of Dunkl–de Jeu–Opdam [33] that ν is a singular slope if and only if $L_\nu(1) \in \mathsf{O}_\nu^{\mathrm{tor}}$.

3.6 Rational Slopes

3.6.1. Proof of Theorem 3.5.10(1) We have $D_1(\boldsymbol{q}) = 1$, and for any ν, the only Verma module that contains $L_\nu(1)$ as a constituent is $\Delta_\nu(1)$.

3.6.2. Singular Slopes We prove Theorem 3.5.10(2). First we note that, in the proof of the lemma below, we will freely use the background discussed in Section 2 of [106].

Lemma 3.6.1. *We have*

$$\sum_{\phi \in \mathrm{Irr}(W)} D_\phi(\boldsymbol{q})\phi_{\boldsymbol{q}}(\beta_w) = \begin{cases} \boldsymbol{s}(1_{\boldsymbol{q}}) & w = 1 \\ 0 & w \neq 1 \end{cases} \tag{3.6.1}$$

in $\mathbf{Q}_W(\boldsymbol{q}^{\frac{1}{2}})$*.*

Proof. If $W \simeq W_1 \times W_2$, then $D_{\phi_1 \times \phi_2}(\boldsymbol{q}) = D_{\phi_1}(\boldsymbol{q}) \cdot D_{\phi_2}(\boldsymbol{q})$ for all $\phi_i \in \mathrm{Irr}(W_i)$. So we can assume that W is irreducible. After checking the non-crystallographic types by hand, we can also assume that W is crystallographic.

Suppose that $\mathbf{F} = \mathbf{F}_q$ and that W is the Weyl group of a reductive group G over $\bar{\mathbf{F}}$. Fix a prime ℓ invertible in $\mathbf{F}$ and a square root $q^{\frac{1}{2}} \in \bar{\mathbf{Q}}_\ell$. Let $F : G \to G$ be the Frobenius map with respect to the split form of G over $\mathbf{F}$, and let R_1 be the unipotent ℓ-adic Deligne–Lusztig representation of G^F attached to $1 \in W$. Explicitly, $R_1 = \mathrm{H}^0(\mathcal{B}^F, \bar{\mathbf{Q}}_\ell)$, where $\mathcal{B}$ is the flag variety of G. Via the Iwahori isomorphism

$$H_W(q^{\frac{1}{2}}) \simeq \mathrm{End}_{G^F}(R_1), \tag{3.6.2}$$

we have an isomorphism of H_W-modules (see [106, §2]):

$$R_1 \simeq \sum_{\phi \in \mathrm{Irr}(W)} D_\phi(q)\phi_q. \tag{3.6.3}$$

Since the Iwahori isomorphism sends $\beta_w \mapsto q^{-\frac{|w|}{2}} T_w$, where T_w is the Hecke operator on R_1 attached to w, we conclude that:

1. β_1 acts on R_1 by the identity operator, which has trace $|\mathcal{B}^F|$.

2. If $w \neq 1$, then β_w acts on R_1 by an operator with zeros along the diagonal.

Since $|\mathcal{B}^F| = \boldsymbol{s}(1_{\boldsymbol{q}})|_{\boldsymbol{q}\to q}$, we are done. □

To prove Theorem 3.5.10(1), it suffices to show that the $H_W(\zeta^{\frac{1}{2}})$-module $\mathcal{KZ}(\Omega_\nu)$ is the zero module. Indeed, by Theorem 3.5.5(1), the functor $\mathcal{KZ}$ restricts to a bijection between the simple objects of $\mathsf{O}_\nu/\mathsf{O}_\nu^{\mathrm{tor}}$ and the simple modules of $H_W(\zeta^{\frac{1}{2}})$. By Theorem 3.5.5(3), we have

$$\mathcal{KZ}(\Omega_\nu) = \sum_{\phi \in \mathrm{Irr}(W)} D_\phi(\zeta)\phi_\zeta. \tag{3.6.4}$$

This is the image, under the decomposition map from Section 3.4, of the virtual $H_W(\boldsymbol{q}^{\frac{1}{2}})$-module $\sum_\phi D_\phi(\boldsymbol{q})\phi_{\boldsymbol{q}}$. So by Lemma 3.6.1, it suffices to show that

$$\boldsymbol{s}(1_{\boldsymbol{q}})|_{\boldsymbol{q}^{\frac{1}{2}} \to \zeta^{\frac{1}{2}}} = 0. \tag{3.6.5}$$

This follows from the Bott–Solomon formula [100], which says

$$\boldsymbol{s}(1_{\boldsymbol{q}}) = \prod_{i \in I(W)} \frac{1 - \boldsymbol{q}^{d_i}}{1 - \boldsymbol{q}}, \tag{3.6.6}$$

and the definition of singular slope.

3.6.3. Regular Slopes We prove Theorem 3.5.10(3).

At the end of Section 3.3, we restated the definition of the function $\mathbb{A} : H_W \to \mathbf{Q}_W(\boldsymbol{q}^{\frac{1}{2}})$ from [105]. The annular character ANN is defined in terms of $\mathbb{A}$ by the formula:

$$\text{ANN}(\beta) = (-\boldsymbol{q}^{\frac{1}{2}})^{|\beta|}\mathbb{A}(\beta) \cdot \varepsilon \sum_{i \geq 0} \boldsymbol{q}^i \operatorname{Sym}^i(V), \tag{3.6.7}$$

where $|\beta|$ denotes the writhe of β. Note that $|\pi| = 2N$, for example, by Theorem 3.3.8, so if $\beta \in Br_W$ is a fractional twist of slope $\nu \in \mathbf{Q}$, then $|\beta| = 2N\nu$. We want to show:

$$\sum_{\phi \in \operatorname{Irr}(W)} D_\phi(e^{2\pi i \nu})[\Delta_\nu(\phi)]_{\boldsymbol{q}} = \boldsymbol{q}^{\frac{r}{2} - N\nu} \text{ANN}(\beta). \tag{3.6.8}$$

The proof will involve a subtle point about signs, which we address with the lemma below.

Lemma 3.6.2. *If ν is a regular slope, then $D_{\varepsilon\psi}(\zeta) = (-1)^{2N\nu} D_\psi(\zeta)$ for all $\psi \in \operatorname{Irr}(W)$.*

Proof. Let w be a ζ-regular element of W. By Properties 3.2.3-3.2.4 in Section 3.2 and the discussion in Section 3.3,

$$\begin{aligned} D_{\varepsilon\psi}(\zeta) &= \sum_{\phi \in \operatorname{Irr}(W)} \{\phi, \varepsilon\psi\} \boldsymbol{o}(\phi)(w) \\ &= \sum_{\phi \in \operatorname{Irr}(W)} \{\phi, \psi\} \boldsymbol{o}(\varepsilon\phi)(w) \\ &= \varepsilon(w) \cdot D_\psi(\zeta). \end{aligned} \tag{3.6.9}$$

Finally, $\varepsilon(w) = (-1)^{|w|} = (-1)^{2N\nu}$ because $|w| = |\beta_w| = 2N(\nu - \lfloor \nu \rfloor)$. □

To lighten notation below, we set $M(\boldsymbol{q} \mid V) = \sum_{i \geq 0} \boldsymbol{q}^i \operatorname{Sym}^i(V) \in \mathrm{K}_0(W)[\![\boldsymbol{q}]\!]$. Using the lemma, we compute:

$$\begin{aligned} \sum_{\phi \in \mathrm{Irr}(W)} D_\phi(\zeta)[\Delta_\nu(\phi)] &= \boldsymbol{q}^{\frac{r}{2}} \sum_{\phi \in \mathrm{Irr}(W)} \boldsymbol{q}^{-\nu \boldsymbol{c}(\phi)} D_\phi(\zeta) \phi \cdot M(\boldsymbol{q} \mid V) \\ &= \boldsymbol{q}^{\frac{r}{2}} \sum_{\phi \in \mathrm{Irr}(W)} \boldsymbol{q}^{-\nu \boldsymbol{c}(\varepsilon\phi)} D_{\varepsilon\phi}(\zeta) \varepsilon\phi \cdot M(\boldsymbol{q} \mid V). \end{aligned} \tag{3.6.10}$$

Using the formula for $\mathbb{A}(\beta)$ in Proposition 3.3.13, along with the fact that $\boldsymbol{c}(\varepsilon\phi) = -\boldsymbol{c}(\phi)$ for all ϕ, we can simplify the last expression to

$$(-1)^{2N\nu} \boldsymbol{q}^{\frac{r}{2}} \mathbb{A}(\beta) \cdot \varepsilon M(\boldsymbol{q} \mid V), \tag{3.6.11}$$

which completes the proof.

3.6.4. Regular Elliptic Slopes We prove Theorem 3.5.10(4). It suffices to show that the graded W-character of Ω_ν is finite-dimensional, i.e., that

$$[\Omega_\nu]_{\boldsymbol{q}} \in \mathrm{K}_0(W)[\boldsymbol{q}]. \tag{3.6.12}$$

Write $\nu = \frac{m}{n}$. By hypothesis, we can pick a $e^{\frac{2\pi i}{n}}$-regular element $w \in W$ such that $V^w = 0$. By Proposition 3.3.10, $\beta_w^m \in Br_W^+$ is a fractional twist of slope $\frac{m}{n}$, so by Theorem 3.5.10(3), proved in the previous subsection,

$$[\Omega_\nu]_{\boldsymbol{q}} = \boldsymbol{q}^{\frac{r}{2} - N\frac{m}{n}} \mathrm{ANN}(\beta_w^m). \tag{3.6.13}$$

We are done by the following result proved in [106, §5]:

Theorem 3.6.3. *If $\beta \in Br_W^+$ maps to $w \in W$, then $(1 - \boldsymbol{q})^{\dim V^w} \mathrm{ANN}(\beta) \in \mathrm{K}_0(W)[\boldsymbol{q}]$.*

3.6.5. Cuspidal Slopes We prove Theorem 3.5.10(5). We will freely use the terminology of Section 3.4.

Lemma 3.6.4. *If $\zeta^{\frac{1}{2}} = e^{\pi i \nu}$, then the ζ-defect of the principal ζ-block of $H_W(\boldsymbol{q}^{\frac{1}{2}})$ is equal to $|I_\nu(W)|$.*

Proof. Observe that the Bott–Solomon formula (3.6.6) can be rewritten

$$\boldsymbol{s}(1_{\boldsymbol{q}}) = \prod_{n \geq 1} \Phi_n(\boldsymbol{q})^{|\{i \in I(W) \,:\, n | d_i\}|}, \tag{3.6.14}$$

where Φ_n is the minimal polynomial of $e^{\frac{2\pi i}{n}}$ over $\mathbf{Q}_W$. □

Proposition 3.6.5. *The ζ-defect of any ζ-block of $H_W(\boldsymbol{q}^{\frac{1}{2}})$ is bounded above by $I_\nu(W)$. Equality holds if and only if $D_\phi(\zeta) \neq 0$. Moreover, if $|I_\nu(W)| \leq 1$, then there are two kinds of blocks:*

1. *Any block of defect 0 is a singleton. If ϕ belongs to such a block, then $D_\phi(\zeta) = 0$.*
2. *Any block of defect 1 is a line of characters ordered by $\boldsymbol{a}$-value. If*

$$\{\phi_0 > \phi_1 > \cdots > \phi_{n-1}\} \tag{3.6.15}$$

is such a block, then $D_{\phi_j}(\zeta) = (-1)^j$ for all j.

Proof. The only claim that does not follow from Theorem 3.4.3 and Lemma 3.6.4 is the claim about generic degrees in case (2). Recall that, by definition,

$$D_{\phi_j}(\zeta) = \left.\frac{\boldsymbol{s}(1_{\boldsymbol{q}})}{\boldsymbol{s}(\phi_{j,\boldsymbol{q}})}\right|_{\boldsymbol{q}^{1/2} \to \zeta^{1/2}}. \tag{3.6.16}$$

Work of Broué [16, §3.7] shows that such ratios of Schur elements at roots of unity are Morita invariants of the block ideal. (Strictly speaking, Broué worked with group algebras, not Iwahori–Hecke algebras, but see the footnote on page 286 of [20].) Thus, by Theorem 3.4.3(2), these ratios match the corresponding ratios for the Brauer tree algebra of type A_{n-1}. In this way, we reduce to the setting where $W = S_n$ and $\nu = \frac{1}{n}$. Here, we have $\phi_j = \Lambda^j(V)$, so it remains to show that

$$D_{\Lambda^j(V)}(e^{\frac{2\pi i}{n}}) = (-1)^j. \tag{3.6.17}$$

Let w be an $e^{\frac{2\pi i}{n}}$-regular element of W. By Remark 3.7 in [56], the right-hand side above equals $\mathrm{tr}(w \mid \Lambda^j(V))$. By Property 3.2.2 and Theorem 3.3.8, the left-hand side also equals $\mathrm{tr}(w \mid \Lambda^j(V))$. □

We return to the proof of Theorem 3.5.10(5). For cuspidal ν, we can combine part (3) of Theorem 3.5.10 with Proposition 3.6.5 to arrive at:

$$[\Omega_\nu]_{\boldsymbol{q}} = \sum_{\Gamma = \{\phi_j^\Gamma\}_j} \sum_j (-1)^j [\Delta_\nu(\phi_j^\Gamma)]_{\boldsymbol{q}}, \tag{3.6.18}$$

where the outer sum runs over ζ-blocks of $H_W(\boldsymbol{q}^{\frac{1}{2}})$ of defect 1.

To conclude, we must show that the inner sum on the right-hand side simplifies to $[L_\nu(\phi_0^\Gamma)]_{\boldsymbol{q}}$, where ϕ_0^Γ is the element of greatest $\boldsymbol{a}$-value in the block Γ. This follows from the result below, which is itself a corollary of Theorem 3.5.5(2).

Theorem 3.6.6. *Suppose that $\{\phi_0 > \phi_1 > \cdots > \phi_{n-1}\}$ is a ζ-block of $H_W(\boldsymbol{q}^{\frac{1}{2}})$, ordered by $\boldsymbol{a}$-value. Then, in O_ν, there is a "Bernstein–Gelfand–Gelfand" resolution of $L_\nu(\phi_0)$ of the form*

$$\Delta_\nu(\phi_{n-1}) \to \cdots \to \Delta_\nu(\phi_1) \to \Delta_\nu(\phi_0) \to L_\nu(\phi_0) \to 0. \tag{3.6.19}$$

In particular, we have

$$[L_\nu(\phi_0)]_{\boldsymbol{q}} = \sum_{0 \leq j \leq n-1} (-1)^j [\Delta_\nu(\phi_j)]_{\boldsymbol{q}} \tag{3.6.20}$$

in $\mathrm{K}_0(W)[\![\boldsymbol{q}]\!]$.

Remark 3.6.7. We believe that the first appearance of this theorem in the literature is [93, Thm. 5.15]. The type A case was used much earlier, in [9].

3.6.6. Proofs of Corollary 3.1.7 and Theorem 4.8.8 There are only two cases where W is an irreducible Coxeter group, ν is a cuspidal slope, and there is a ζ-block of $H_W(\boldsymbol{q}^{\frac{1}{2}})$ of defect 1 besides the principal block. Namely, W must be of type E_8 or type H_4 and the denominator of ν in lowest terms must be 15.

In these cases, there is one other ζ-block of defect 1 and its element of greatest $\boldsymbol{a}$-value is the character of V, assuming as before that $V^W = 0$. We refer to these as the two **exceptional** cases. Theorems 3.1.4-3.1.6 now imply the following, more complete version of Corollary 3.1.7:

Corollary 3.6.8. *If W is irreducible and ν is cuspidal, then:*

$$[\Omega_\nu]_{\boldsymbol{q}} = \begin{cases} [L_\nu(1)]_{\boldsymbol{q}} + [L_\nu(V)]_{\boldsymbol{q}} & (W,\nu) \text{ is exceptional} \\ [L_\nu(1)]_{\boldsymbol{q}} & \text{else} \end{cases} \tag{3.6.21}$$

in $\mathrm{K}_0(W)[\boldsymbol{q}]$. *In particular, the right-hand side is finite-dimensional.*

Remark 3.6.9. This statement also recovers the observation of Bezrukavnikov–Etingof [11, Rem. 3.31] that for irreducible W and cuspidal ν, the only finite-dimensional simple $\mathbf{A}_\nu^{\mathrm{rat}}$-modules are those appearing on the right-hand side of (3.6.21).

Remark 3.6.10. If $W \simeq W' \times W''$ and $V' \simeq V' \oplus V''$, such that V' and V'' are realizations of W' and W'', respectively, then:

- $|I_\nu(W)| = |I_\nu(W')| + |I_\nu(W'')|$, where the right-hand side is defined in terms of V' and V'' rather than V.

- Category O for the rational Cherednik algebra of W is the tensor product of the corresponding categories for W' and W''.

Using these facts, Corollary 3.1.7 can be generalized to any pair (W, ν) such that ν is cuspidal with respect to each irreudcible factor of W.

To conclude, we explain how to get the Gorsky–Oblomkov–Rasmussen–Shende identity (Theorem 4.8.8) from Corollary 3.1.7. For $W = S_n$, we see that

$$r = n - 1, \tag{3.6.22}$$

$$N = n(n-1)/2. \tag{3.6.23}$$

The (m, n)-torus knot is the planar closure of a fractional twist $\beta \in Br_W^+$ of slope $\nu = \frac{m}{n}$. Since n is the Coxeter number of S_n, it is cuspidal. We obtain:

$$|\beta| = m(n-1), \tag{3.6.24}$$

$$\begin{aligned} [L_\nu(1)]_{\boldsymbol{q}} &= [\Omega_\nu]_{\boldsymbol{q}} \\ &= (\boldsymbol{q}^{-\frac{1}{2}})^{(m-1)(n-1)} \mathrm{ANN}(\beta). \end{aligned} \tag{3.6.25}$$

Finally, we have the following comparison from [105] between the annular character and the HOMFLY polynomial in variables $\boldsymbol{q}^{\frac{1}{2}}$ and a:

Theorem 3.6.11. *For all $\beta \in Br_n$, we have*

$$\mathit{HOMFLY}(\beta) = (\boldsymbol{q}^{-\frac{1}{2}} a)^{|\beta| - r} \sum_{0 \leq i \leq r} (-a^2)^i (\Lambda^i, \mathrm{ANN}(\beta))_{S_n}, \tag{3.6.26}$$

where $\Lambda^i = \Lambda^i(V)$ is the Specht module associated with the ith hook partition of n.

Chapter 4

Algebraic Braids and the Springer Theory of Hitchin Systems

4.1 Introduction

The origin of this paper is a striking relationship between algebraic geometry and the theory of knots and links. We will situate both sides within the gestures of representation theory. In doing so, we will introduce some conjectures that relate:

1. Symmetries of affine Springer fibers and Hitchin fibers [115].
2. Categorified trace functions on Artin braid groups, introduced in our paper [105] and inspired by HOMFLY-type link invariants [41, 62].

Crucial to the statements is the notion of an **algebraic braid**, which we introduce to pin down the Artin braids that arise from algebraic geometry. Our results from [106, 107] will allow us to obtain some evidence for our conjectures.

4.1.1. At the risk of anachronism, we start with the representation-theoretic motivation for our work.

Let G be a complex semisimple group with Lie algebra $\mathfrak{g}$ and Weyl group W. Fix a Borel $B \subseteq G$, so that $\mathcal{B} = G/B$ is the flag variety of G. Each element $\gamma \in \mathfrak{g}(\mathbf{C})$ induces a vector field on $\mathcal{B}$, whose fixed-point locus is the projective variety

$$\mathcal{B}_\gamma = \{gB \in \mathcal{B} : \gamma \in \mathrm{Lie}(gBg^{-1})\}. \tag{4.1.1}$$

Springer discovered [102] that W acts on the (Betti) cohomology $\mathrm{H}^*(\mathcal{B}_\gamma, \mathbf{C})$, and moreover, that these graded representations of W bear application to the unipotent representation theory of G. The varieties $\mathcal{B}_\gamma$ are now called Springer fibers.

We are interested in the loop analogue of Springer's construction. Below, we let $D = \operatorname{Spec} \mathbf{C}[\![\varpi]\!]$, the infinitesimal disk. Let $\eta = D \setminus 0 = \operatorname{Spec} \mathbf{C}(\!(\varpi)\!)$ and $\bar{\eta} = \operatorname{Spec} \mathbf{C}(\!(\varpi^{\frac{1}{\infty}})\!)$. At the level of points, the loop group of G is defined by $\mathcal{L}G(\mathbf{C}) = G(\eta)$. The Borel subgroup $B \subseteq G$ lifts to an Iwahori subgroup $\mathbf{I} \subseteq \mathcal{L}G$, in terms of which the affine flag variety of G is the infinite-dimensional quotient

$$\mathcal{F}l_B = \mathcal{L}G/\mathbf{I}. \tag{4.1.2}$$

To get an analogue of Springer's construction, let $\mathfrak{g}^{\mathrm{rs}} \subseteq \mathfrak{g}$ be the locus of regular semisimple elements. Kazhdan–Lusztig showed [60] that if $\gamma \in \mathfrak{g}(D) \cap \mathfrak{g}^{\mathrm{rs}}(\bar{\eta})$, then

$$\mathcal{F}l_{B,\gamma} = \{g\mathbf{I} \in \mathcal{F}l_B : \gamma \in \mathrm{Lie}(g\mathbf{I}g^{-1})\} \tag{4.1.3}$$

is a nonempty finite-dimensional scheme, now called an (Iwahori) affine Springer fiber. It need not be of finite type, and in general, its geometry is richer than that of the non-loop case. Lusztig showed [75] that the affine Weyl group of G acts on $\mathrm{H}^*(\mathcal{F}l_{B,\gamma}, \mathbf{C})$.

It will be convenient to set $\mathfrak{c} = \mathfrak{g} /\!\!/ G$ and $\mathfrak{c}^\circ = \mathfrak{g}^{\mathrm{rs}} /\!\!/ G$. The natural map $\mathfrak{g} \to \mathfrak{c}$ sends γ to an element $f \in \mathfrak{c}(D) \cap \mathfrak{c}^\circ(\eta)$. Up to isomorphism, $\mathcal{F}l_{B,\gamma}$ only depends on f.

At the same time, f defines a conjugacy class in the étale fundamental group of $\mathfrak{c}^\circ$: namely, the class of the image of 1 under $\hat{\mathbf{Z}} \simeq \pi_1^{\text{ét}}(\eta) \to \pi_1^{\text{ét}}(\mathfrak{c}^\circ)$, which we will denote

$$[\beta_f^{\text{ét}}] \subseteq \pi_1^{\text{ét}}(\mathfrak{c}^\circ). \tag{4.1.4}$$

We emphasize that $[\beta_f^{\text{ét}}]$ *is far coarser than a* as an invariant of γ. We therefore ask:

Question 4.1.1. To what extent is the affine Springer fiber $\mathcal{F}l_{B,\gamma}$, or at least its topology, encoded in the conjugacy class $[\beta_f^{\text{ét}}] \subseteq \pi_1^{\text{ét}}(\mathfrak{c}^\circ)$?

Suppose that $f : D \to \mathfrak{c}$ can be extended to a map from a curve into $\mathfrak{c}$. Then we can extend $f(\eta)$ to an unbased *topological* loop in the analytification of $\mathfrak{c}^\circ$. As a consequence, $[\beta_f^{\text{ét}}]$ is the image of a specific conjugacy class in the *topological* fundamental group of $(\mathfrak{c}^\circ)^{\mathrm{an}}$.

To denote it, we simply omit the superscript:

$$[\beta_f] \subseteq \pi_1((\mathfrak{c}^\circ)^{\mathrm{an}}). \tag{4.1.5}$$

The group $\pi_1((\mathfrak{c}^\circ)^{\mathrm{an}})$ is very well-studied. By theorems of Brieskorn and Chevalley, it is isomorphic to the Artin group of W: an infinite group

$$Br_W \twoheadrightarrow W, \tag{4.1.6}$$

whose defining relations generalize those of the braid groups studied in geometric topology. Indeed, if $W = S_n$, then $Br_W = Br_n$, the classical braid group on n strands.

We will refer to the elements of Br_W as Artin braids. Those that have the form β_a for some map $f : (D, \eta) \to (\mathfrak{c}, \mathfrak{c}^\circ)$ will be called **algebraic**. As it turns out, this notion only depends on W, not G. A subsidiary question that we defer to the future is:

Question 4.1.2. What are the algebraic Artin braids for a given W?

4.1.2. As an initial answer to Question 4.1.1, we will use a formula of Bezrukavnikov [10] to show that the dimension of $\mathcal{F}_{B,\gamma}$ only depends on $[\beta_f]$: See Theorem 4.6.4.

As a more comprehensive answer, we present a suite of conjectures that relate the cohomology of affine Springer fibers to homological invariants of algebraic braids.

To describe the latter, let $\mathrm{K}_0(W)$ be the ring of virtual representations of W. In [105], we constructed a class function on Br_W called **annular braid homology**, taking the form

$$\mathrm{AH} : Br_W \to \mathrm{K}_0(W)[\![\boldsymbol{q}^{\frac{1}{2}}]\!][t]. \tag{4.1.7}$$

Its existence was motivated by knot-theoretic ideas. Recall that a link is a closed 1-manifold (tamely) embedded in a 3-manifold; it is a knot iff it is connected, i.e., a circle. We showed that when the elements of Br_W are topological braids, AH can be used to reconstruct the isotopy invariant of links in 3-space known as Khovanov–Rozansky homology [62, 61].

In this paper, we need a slight variant of AH. First, let $\mathbf{X}$ be the character lattice of the root datum of G, so that $\mathbf{X}_{\mathbf{C}} = \mathbf{X} \otimes \mathbf{C}$ is a representation of W. If the map $Br_W \to W$ sends $\beta \mapsto w$, then we set $r(w) = \dim\,(\mathbf{X}_{\mathbf{C}})^w$ and

$$\mathfrak{A}\mathrm{H}(\beta) = \left(\frac{1-\boldsymbol{q}}{1+\boldsymbol{q}t}\right)^{r(w)} \mathrm{AH}(\beta) \in \mathrm{K}_0(W)[\![\boldsymbol{q}^{\frac{1}{2}}, t]\!]. \tag{4.1.8}$$

In [105, §4], we showed that the specialization

$$\mathfrak{A}\mathrm{H}(\beta)|_{t\to -1} = \mathrm{AH}(\beta)|_{t\to -1}, \tag{4.1.9}$$

which we called the **annular character** of β, is a rational function of $\boldsymbol{q}$. In [106, §5], we showed that $r(w)$ is the order of its pole at $\boldsymbol{q} = 1$.

We would like to compare $\mathrm{H}^*(\mathcal{F}l_{B,\gamma}, \mathbf{C})$ and $\mathfrak{A}\mathrm{H}(\beta_f)$ as representations of W. Roughly, we expect that *the t-grading on $\mathfrak{A}\mathrm{H}(\beta_f)$ corresponds to the weight grading on* $\mathrm{H}^*(\mathcal{F}l_{B,\gamma}, \mathbf{C})$. However, to obtain a correspondent of the $\boldsymbol{q}$-grading on $\mathfrak{A}\mathrm{H}(\beta_a)$, it turns out that we must pass from the environment of the local field $\mathbf{C}(\!(\varpi)\!)$ to that of the global field of functions on a proper curve.

In 1987, in his work on Yang–Mills theory, Hitchin introduced [55] an algebro-geometric integrable system depending on a semisimple group G, a smooth projective curve Σ, and a line bundle $L \to \Sigma$ (originally, the cotangent bundle $T_\Sigma^\vee \to \Sigma$):

$$\boldsymbol{h} : \mathcal{M}(G, \Sigma, L) \to \mathcal{A}(G, \Sigma, L). \tag{4.1.10}$$

The base $\mathcal{A}$ is a weighted vector space, and the total space $\mathcal{M}$ is a moduli space of decorated G-bundles over Σ, known as Higgs bundles. The generic fiber of the map $\boldsymbol{h}$ is roughly a (stacky) abelian variety; other fibers can be highly singular. Maps of this form have found broad application across complex algebraic geometry and representation theory, including the solution of the Langlands–Shelstad Fundamental Lemma by B.-C. Ngô in 2008 [87]. One of the innovations of Ngô's proof is a homeomorphism between a Hitchin fiber and a product of (spherical, rather than Iwahori) affine Springer fibers, up to the action of certain commutative group stack on both sides.

Fix a basepoint $0 \in \Sigma(\mathbf{C})$. We will work with a parabolic version of $\mathcal{M}$, denoted

$$\tilde{\mathcal{M}} = \tilde{\mathcal{M}}(G, \Sigma, L), \tag{4.1.11}$$

that classifies Higgs bundles that are further decorated with a B-flag at 0. As explained in Section 4.5, each point $a \in \mathcal{A}(\mathbf{C})$ defines a section of a certain $\mathfrak{c}$-bundle $\mathfrak{c}_L \to \Sigma$, and in a neighborhood of 0, the section restricts to a map

$$a_0 : D \xrightarrow{\sim} \operatorname{Spec} \hat{\mathcal{O}}_{\Sigma,0} \xrightarrow{a} \mathfrak{c}. \tag{4.1.12}$$

Assuming a_0 sends η into $\mathfrak{c}^\circ$, a mild generalization of Ngô's product formula [113, §2.4] relates the (parabolic) Hitchin fiber $\tilde{\mathcal{M}}_a$ to the affine Springer fiber $\mathcal{F}l_{B,\gamma}$ whenever $a_0 \in \mathfrak{c}(D)$ lifts to $\gamma \in \mathfrak{g}(D)$. Moreover, there is an affine Weyl action on $\mathrm{H}^*(\tilde{\mathcal{M}}_a, \mathbf{C})$ compatible with the action on $\mathrm{H}^*(\mathcal{F}l_{B,\gamma}, \mathbf{C})$, as shown by Z. Yun [113]. In a slogan, the Hitchin fiber is the analogue of the affine Springer fiber when the curve Σ replaces the disk D.

The advantage of $\tilde{\mathcal{M}}_a$ is that its cohomology admits a *perverse filtration* $\mathbf{P}_{\leq *}$, defined in terms of $\boldsymbol{h}$ via the machinery of perverse sheaves [4]. We expect that *the $\boldsymbol{q}$-grading on* $\mathfrak{A}\mathrm{H}$ *corresponds to the* $\mathbf{P}$*-grading on the associated graded vector space* $\mathrm{Gr}_*^{\mathbf{P}} \mathrm{H}^*(\tilde{\mathcal{M}}_a)$. As we explain later, a similar perverse filtration on the moduli space $\mathcal{M}$ is known to play an important role in the "nonabelian" Hodge theory of Σ.

In Section 4.7, we gather our conjectures relating the cohomology of Hitchin fibers and annular braid homology. The main conjecture is:

Conjecture 4.1.3. *If a is a $\mathbf{C}$-point of the* anisotropic *locus of $\mathcal{A}$ (see Section 4.5), then*

$$\sum_{i,j,k} (-1)^{i+k} \boldsymbol{q}^j t^k \operatorname{Gr}_j^{\mathbf{P}} \operatorname{Gr}_k^{\mathbf{W}} \mathrm{H}^i(\tilde{\mathcal{M}}_a)_{st} = \mathfrak{A}\mathrm{H}(\beta_{a,0}) \cdot \prod_{v \in \Sigma^{\mathrm{an}} \backslash 0} (1, \mathfrak{A}\mathrm{H}(\beta_{a,v}))_W^{d\chi(v)}, \tag{4.1.13}$$

where both sides are viewed as elements of $\mathrm{K}_0(W)[\![\boldsymbol{q}^{\frac{1}{2}}, t]\!]$. In detail,

- $\mathrm{H}^*(\tilde{\mathcal{M}}_a)_{st}$ *is the stable summand of $\mathrm{H}^*(\tilde{\mathcal{M}}_a)$ in the sense of Ngô's work [87, §6].*
- $\mathbf{P}_{\leq *}$ *and $\mathbf{W}_{\leq *}$ are the (compatible) perverse and weight filtrations on $\mathrm{H}^*(\tilde{\mathcal{M}}_a)_{st}$.*
- *g is the genus of the smooth projective curve Σ.*
- *$\beta_{a,v} \in Br_W$ is the algebraic braid, up to conjugacy, attached to the map*

  $$a_v : D \xrightarrow{\sim} \operatorname{Spec} \hat{\mathcal{O}}_{\Sigma,v} \xrightarrow{a} \mathfrak{c}. \tag{4.1.14}$$

  Here we use the fact that if a is anisotropic, then a_v sends η into $\mathfrak{c}^\circ$.
- *$(1, \mathfrak{A}\mathrm{H}(\beta_{a,v}))_W \in \mathbf{Z}[\![\boldsymbol{q}^{\frac{1}{2}}, t]\!]$ is the bigraded dimension of the space of W-invariants of $\mathfrak{A}\mathrm{H}(\beta_{a,v})$.*
- *The notation $\prod_{v \in \Sigma^{\mathrm{an}} \backslash 0} (-)^{d\chi(v)}$ denotes a* product integral *over the underlying complex-analytic space of $\Sigma \setminus 0$, i.e.,*

  $$\prod_{v \in \Sigma^{\mathrm{an}} \backslash 0} (-)^{d\chi(v)} = \exp \int_{\Sigma^{\mathrm{an}} \backslash 0} \log{(-)}\, d\chi, \tag{4.1.15}$$

  where $d\chi$ is the Lebesgue measure defined by Euler characteristic (see Section 4.7).

4.1.3. At this point, we discuss the history that inspired our conjectures, and situate the results of the paper in this context.

Links of Plane Curves At the turn of the twentieth century, Wirtinger observed [37] that the study of topological links is closely intertwined with the study of complex algebraic functions. If C is a locally-planar algebraic curve, possibly singular, and $p \in C(\mathbf{C})$, then in a local chart, the intersection of C with a small 3-sphere around p is a link

$$\lambda_{C,p} = S^3 \cap C^{\mathrm{an}} \subseteq S^3. \tag{4.1.16}$$

For instance, if p is the cusp of $y^2 = x^3$, then $\lambda_{C,p}$ is a trefoil knot.

The Hilbert scheme of C is roughly a space $\mathrm{Hilb}(C)$ whose points parametrize the finitely-supported subschemes of C. At the beginning of the past decade, Oblomkov and Shende conjectured [89] a remarkable numerical identity relating the topology of $\mathrm{Hilb}(C)$ with the links $\lambda_{C,p}$ as the point p runs over the singularities of C. We give the precise statement in Section 4.8. It involves a link invariant called the HOMFLY series, discovered independently by several mathematicians building on work of Jones [41, 56]. Some years later, in joint work with Rasmussen [88], Oblomkov–Shende proposed a refinement involving Khovanov–Rozansky homology, which categorifies the HOMFLY series.

The original Oblomkov–Shende conjecture, an identity in variables $\boldsymbol{q}^{\frac{1}{2}}$ and $\boldsymbol{a}$, was proved by Maulik [80], building on work of Diaconescu–Hua–Soibelman [29] that recasts $\mathrm{Hilb}(C)$ as a moduli space of framed sheaves on a Calabi–Yau variety. Maulik's proof uses Donaldson–Thomas theory and skein theory to induct on the sequence of blowups that resolve each singularity of C. Ultimately, he reduces to the case where $C = \{y^n = x^m\}$ for some m, n, which can be checked by hand combinatorially. This proof does not seem to generalize to the Oblomkov–Rasmussen–Shende (ORS) conjecture, an identity in variables $\boldsymbol{q}^{\frac{1}{2}}$, $\boldsymbol{a}$, t.

In an unpublished research statement [94], Shende proposed a radically different approach to the ORS conjecture, based on interpreting $\mathrm{Hilb}(C)$ in terms of Hitchin systems. First, observe that if C is singular, then its Jacobian is no longer an abelian variety; but as long as C is integral and projective, its Jacobian admits an integral, projective compactification

$\overline{\mathrm{Jac}}(C)$ [1], equipped with a generalized Abel–Jacobi map

$$\mathrm{Hilb}(C) \to \overline{\mathrm{Jac}}(C). \tag{4.1.17}$$

It is essentially an observation of Hitchin [55] (see also [2]) that when $G = \mathrm{SL}_n$ and $\Sigma = \mathbf{P}^1$, well-behaved fibers of the Hitchin map for (G, Σ, L) are stacky versions of these compactified Jacobians. Namely, each anisotropic point $a \in \mathcal{A}(\mathbf{C})$ defines a branched n-fold cover

$$\Sigma_a \to \Sigma \tag{4.1.18}$$

embedded in L, known as the associated spectral cover, such that $\mathcal{M}_a \simeq \overline{\mathrm{Jac}}(\Sigma_a)$ up to stackiness. In this situation, the Abel–Jacobi map lets us relate the cohomology of $\mathrm{Hilb}(\Sigma_a)$ to the cohomology of $\mathcal{M}_a$, as explained in [81, 84].

Altogether, the Oblomkov–Rasmussen–Shende conjecture becomes an identity relating the cohomology of Hitchin fibers for SL_n and the Khovanov–Rozansky homology of certain links. It is explained in [50, §9] that to incorporate the variable $\boldsymbol{a}$ correctly, we must replace $\mathrm{H}^*(\mathcal{M}_a)$ with a sum of certain S_n-isotypic summands of $\mathrm{H}^*(\tilde{\mathcal{M}}_a)$.

In Section 4.8, we show that the $(G, \Sigma) = (\mathrm{SL}_n, \mathbf{P}^1)$ case of Conjecture 4.1.3 *specializes* to the ORS conjecture once we restrict to such isotypics. The proof uses the comparison between annular braid homology and Khovanov–Rozansky homology established in [105, §7].

Nonabelian Hodge Theory For general G, a significant feature of the moduli space $\mathcal{M}(G, \Sigma, T_\Sigma^\vee)$ is the *nonabelian Hodge correspondence*, established in the 1980s principally by Corlette, Donaldson, and Simpson [98]. This theorem asserts that there is a transcendental homeomorphism between the so-called semistable locus of the Hitchin moduli space and the moduli space of G-local systems over Σ, also known as the character variety or Betti moduli space of (G, Σ). It can be viewed as a multiplicative analogue of the comparison between Dolbeault and Betti cohomology in the classical Hodge theory of Σ.

The effect of the homeomorphism on cohomological structure is especially mysterious. The P = W conjecture of de Cataldo–Hausel–Migliorini [22] says it should identify:

$$\text{perverse filtration on Hitchin side} \quad \leftrightarrow \quad \textit{halved} \text{ weight filtration on Betti side.}$$

In [94], Shende speculated that the ORS conjecture would follow from a "wild" variant of the P = W conjecture for SL_n (or an isogenous group). Since then, he and his collaborators have proved theorems that relate the virtual weight polynomials of wild character varieties to the HOMFLY series of certain links [95, 96].

We only discuss a portion of their work. There is a map

$$\beta \mapsto O(\beta) \tag{4.1.19}$$

from *positive* Artin braids β (see Section 4.3) to G-varieties over $\mathcal{B} \times \mathcal{B}$, originally due to Broué–Michel [17] and Deligne [27]. It takes braid compositions to fiber products over $\mathcal{B}$ up to fixed isomorphisms. Let $act : \mathcal{B} \times G \to \mathcal{B} \times \mathcal{B}$ be the action map, and let $\widetilde{O}(\beta)$ be defined by the cartesian square:

$$\begin{array}{ccc} \widetilde{O}(\beta) & \longrightarrow & O(\beta) \\ \downarrow & & \downarrow \\ \mathcal{B} \times G & \xrightarrow{act} & \mathcal{B} \times \mathcal{B} \end{array} \tag{4.1.20}$$

As explained in [96], the quotient stack $G\backslash\widetilde{O}(\beta)$ is a moduli space of decorated G-local systems. For example, if $1 \in Br_W$ is the identity, then the composition

$$\widetilde{O}(1) \to \mathcal{B} \times G \to G \tag{4.1.21}$$

is the Grothendieck–Springer resolution of G, and the quotient $G\backslash\widetilde{O}(1)$ classifies G-local systems over a circle with a B-flag at a basepoint.

For $W = S_n$, we know that $Br_W = Br_n$ and β can be viewed as a topological braid. By closing up β, we get a link in 3-space. In this setting, Shende–Treumann–Zaslow showed [95] that the virtual weight polynomial of the fiber of $G\backslash\widetilde{O}(\beta) \to G\backslash G$ over 1 forms part of the HOMFLY series of the link.

For general W, consider the **Steinberg-like variety**

$$St(\beta) = \mathcal{U} \times_G \widetilde{O}(1) \times_G \widetilde{O}(\beta), \tag{4.1.22}$$

where $\mathcal{U}$ is the unipotent locus of G. In [106, §3], we proved that up to a sign, the annular character of β defined by (4.1.9) matches the virtual weight series of the stack $G\backslash St(\beta)$, equipped with the W-action it inherits from the Grothendieck–Springer map. Via Springer theory, this result refines Shende–Treumann–Zaslow's.

In Section 4.9, we use our theorem to show that Conjecture 4.1.3 would imply numerical $\mathrm{P} = \mathrm{W}$ identities of a new kind. Suppose that $[\beta_{a,0}]$ admits a positive representative and some other restrictive conditions, discussed in §4.7.3, apply to $|\beta_{a,v}|$ for all $v \neq 0$. Then the $t \to -1$ limit of the conjecture would imply

$$\sum_{i,j} (-1)^i \boldsymbol{q}^j \operatorname{Gr}_j^{\mathbf{P}} \mathrm{H}^i(\tilde{\mathcal{M}}_a)_{st} = \pm(1 - \boldsymbol{q})^{N_a} \sum_{i,j} (-1)^i \boldsymbol{q}^{\frac{j}{2}} \operatorname{Gr}_j^{\mathbf{W}} \mathrm{H}_c^i(G\backslash St(\beta_{a,0})) \tag{4.1.23}$$

for some integer N_a. The normalization factor $(1 - \boldsymbol{q})^{N_a}$ essentially encodes the contribution from the abelian part of Ngô's Picard stack (see Section 4.5). The identity above is equivariant with respect to Springer-type actions of W on both sides.

Keeping the hypotheses above, let $\gamma \in \mathfrak{g}(D)$ be a lift of $a_0 : D \to \mathfrak{c}$. If γ is elliptic (see Section 4.4), then $\mathcal{F}l_{B,a}$ is a proper variety. In this setting, the Ngô–Yun product formula lets us replace the left-hand side of (4.1.23), up to the normalization factor, with a series

$$\sum_{i,j} (-1)^i \boldsymbol{q}^j \operatorname{Gr}_j^{\mathbf{P}} \mathrm{H}^i(\mathcal{F}l_{B,\gamma})_{st} \tag{4.1.24}$$

for some summand $\mathrm{H}^*(\mathcal{F}l_{B,a})_{st} \subseteq \mathrm{H}^*(\mathcal{F}l_{B,a})$ and filtration $\mathbf{P}_{\leq *}$ on $\mathrm{H}^*(\mathcal{F}l_{B,\gamma})_{st}$. We expect, but do not know how to prove, that this filtration is intrinsic to $\mathcal{F}l_{B,\gamma}$.

Suppose further that γ is topologically nilpotent. In the Grothendieck ring of varieties, we can decompose $\mathcal{F}l_{B,\gamma}$ into a linear combination of Springer fibers $\mathcal{B}_e$, as e runs over representatives of the nilpotent orbits of $\mathfrak{g}$. Based on the conjectural purity of affine Springer fibers, we speculate that we can thereby decompose (4.1.24) into a $\mathbf{Z}[\boldsymbol{q}]$-linear combination of total Springer representations

$$\sum_i \boldsymbol{q}^i \mathrm{H}^{2i}(\mathcal{B}_e). \tag{4.1.25}$$

Significantly, we showed in [106] that the right-hand side of (4.1.23) admits an analogous decomposition, albeit stated in terms of the unipotent orbits of G. Recall that there is a G-equivariant isomorphism between the nilpotent locus of $\mathfrak{g}$ and the unipotent locus of G. This observation suggests that our putative $\mathrm{P} = \mathrm{W}$ identities *are compatible with, i.e., factor through, the simultaneous decomposition of $\mathcal{F}l_{B,\gamma}$ and $G\backslash St(\beta_{a,0})$ into Springer fibers.* We refer to Section 4.9 for further details.

Rational Cherednik Algebras For any $\nu \in \mathbf{C}$, the rational Cherednik algebra or rational double affine Hecke algebra (DAHA) of W of central charge ν is an associative algebra $\mathbf{A}_\nu^{\mathrm{rat}}$, given by a certain quotient map

$$\mathbf{C}[W \ltimes (\mathbf{X}_\mathbf{C} \otimes \mathbf{X}_\mathbf{C}^\vee)] \twoheadrightarrow \mathbf{A}_\nu^{\mathrm{rat}}. \tag{4.1.26}$$

Its representation theory closely parallels that of a semisimple Lie algebra. In particular, it admits a "category O" of well-behaved modules [45]. Independently of ν, the simple objects of category O are parametrized by the irreducible characters of W. We will denote the simple, resp. Verma, object that corresponds to the trivial character by L_ν, resp. M_ν.

Each object $M \in \mathsf{O}$ defines a graded W-character. For our purposes, it can be denoted

$$[M]_{\boldsymbol{q}} \in \mathrm{K}_0(W)(\!(\boldsymbol{q}^{\frac{1}{2}})\!). \tag{4.1.27}$$

The construction of M_ν essentially shows that

$$[M_\nu]_{\boldsymbol{q}} = \boldsymbol{q}^{\frac{r}{2} - N\nu} \sum_{i \geq 0} \boldsymbol{q}^i \operatorname{Sym}^i(\mathbf{X}_{\mathbf{C}}), \tag{4.1.28}$$

where $r = \dim V$ and $N = \dim \mathcal{B}$. On the other hand, one of the basic problems in the representation theory of $\mathbf{A}_\nu^{\mathrm{rat}}$ is to determine $[L_\nu]_{\boldsymbol{q}}$.

If $\nu \notin \mathbf{Q}$, then the simple objects and the Verma objects of category O coincide. If $\nu \in \mathbf{Q}$, then the relation between simples and Vermas depends strongly on the denominator of ν in lowest terms. Notably, L_ν is finite-dimensional if and only if the denominator is a *regular elliptic number* in the sense of [109] (see also Section 4.11).

In [90], Oblomkov and Yun construct $\mathbf{A}_\nu^{\mathrm{rat}}$-actions on the cohomology of parabolic Hitchin fibers, or more accurately, on vector spaces of the form

$$\mathrm{Gr}_*^{\mathbf{P}} \mathrm{H}_{\omega=1}^*(\tilde{\mathcal{M}}_a). \tag{4.1.29}$$

Above, $\tilde{\mathcal{M}}_a$ is a special kind of Hitchin fiber that they call homogeneous of slope ν. This condition is highly restrictive: It essentially forces the underlying curve Σ to be a weighted projective line, while ensuring the existence of a certain $\mathbf{G}_m$-action on $\tilde{\mathcal{M}}_a$. The subscript $\omega = 1$ indicates that we take the $\mathbf{G}_m$-equivariant cohomology of $\tilde{\mathcal{M}}_a$, then specialize the equivariant parameter ω to 1. Oblomkov–Yun show that in the regular elliptic case, L_ν appears as a summand of their construction.

Remark 4.1.4. As an $\mathbf{A}_\nu^{\mathrm{rat}}$-module, $\mathrm{Gr}_*^{\mathbf{P}} \mathrm{H}_{\omega=1}^*(\tilde{\mathcal{M}}_a)$ has an underlying graded W-character. This grading turns out to coincide with its $\mathbf{P}$-grading. Thus, our use of $\boldsymbol{q}$ in (4.1.27) is consistent with our use of $\boldsymbol{q}$ in Conjecture 4.1.3.

In Section 4.11, we combine Oblomkov–Yun's work with our results from [107] to exhibit infinitely many examples where the $t \to -1$ limit of Conjecture 4.1.3 holds. The key is to observe that, in their setup, the algebraic braid $\beta_{a,0}$ is conjugate to a **fractional twist** in the sense of [107]. We proved in *ibid.* that if W is irreducible and the denominator of ν is the Coxeter number of W, then the annular character of $\beta_{a,0}$ matches $[L_\nu]_{\boldsymbol{q}}$.

In type A, the resulting examples recover the cases of the original Oblomkov–Shende conjecture where $C = \{y^n = x^m\}$ and m, n are coprime integers.

4.1.4. Itinerary Here is a summary of what we do in each section.

Section 4.2 For a $\mathbf{C}$-scheme U of finite type, we define algebraic loops in both the étale and the topological setting. We explain how this notion relates to the classical notion of algebraic links when $U = \mathrm{Conf}^n(\mathbf{A}^1)$. In **Proposition 4.2.12**, we show the existence of a special class of algebraic loops, which we call fractional loops, essentially when U admits a free $\mathbf{G}_m$-action.

Section 4.3 We introduce the Coxeter group W. We review the Artin group Br_W and its topological interpretation as $\pi_1((\mathfrak{c}^\circ)^{\mathrm{an}})$ via Brieskorn's theorem. Then we define an algebraic braid for W as a $\mathfrak{c}$-algebraic loop in $\mathfrak{c}^\circ$. Fractional twists, e.g., roots of the full twist, are fractional loops, hence algebraic. Simple twists are algebraic braids arising from disks transverse to the discriminant locus. In **Theorem 4.3.12**, we essentially show that the writhe of an algebraic braid equals the intersection number between its associated disk and the discriminant. The argument prefigures much of the global geometry we introduce later in Sections 4.5-4.6.

Section 4.4 We introduce the reductive group G and its loop version. We review finite and affine Springer theory, including the interpretation of affine Springer fibers as moduli of decorated G-torsors.

Section 4.5 We introduce the base curve Σ. Following Ngô and Z. Yun, we review the

Hitchin fibration, its parabolic variants, and its relation to finite and affine Springer theory. In the last section, we introduce the bigraded virtual character of W on the left-hand side of Conjecture 4.1.3.

Section 4.6 We prepare the ground to state our conjectures. First, we explain how a point of the Hitchin base gives rise to a collection of algebraic Artin braids. Next, by combining Theorem 4.3.12 with formulæ of Bezrukavnikov and Ngô, we show that the local and global δ-invariants of Hitchin fibers can be expressed entirely in braid-theoretic terms: See **Theorem 4.6.4** and **Corollaries 4.6.7** and **4.6.9**. Finally, we discuss the technical adjustments needed to extend our ideas to orbifold curves.

Section 4.7 To begin, we summarize the basics of annular braid homology from [105]. Then we state several versions of our main conjecture, which relates the bigraded W-character from Section 4.5 and the annular braid homology of the elements of the corresponding collection of Artin braids. **Conjectures A** is the most general version. **B** is the version from this introduction, while **Conjectures C and D** are useful special cases. Notably, Conjecture D is a statement for affine Springer fibers rather than Hitchin fibers.

Section 4.8 We review the Oblomkov–Rasmussen–Shende conjecture in detail. We also explain the reduction of the conjecture to a conjecture about parabolic Hitchin fibers for $G = \mathrm{SL}_n$, carried out in [50, §9]. In **Theorem 4.8.9**, we show that our Conjecture C implies the ORS conjecture. The proof relies on a well-known identity relating the Milnor number and δ-invariant of a plane curve singularity.

Section 4.9 We review the Broué–Michel varieties and Steinberg-like varieties attached to positive Artin braids, following [105, 106]. We also review the formula we proved in [106], expressing the annular character of $\beta \in Br_W$ in terms of the weight series of a Steinberg-like stack, and a conjectural formula for the annular braid homology of β itself. Via these formulæ, our main conjectures imply identities that match (1) perverse gradings on Hitchin fibers and affine Springer fibers with (2) weight gradings on Steinberg-like stacks. These are summarized in **Propositions 4.9.4 and 4.9.5**. In the nil-elliptic case of Proposition 4.9.5, we deduce a surprising consequence for *substrata* of affine Springer fibers, stated in **Corollary 4.9.7**.

Section 4.10 We introduce the degenerate affine Hecke algebra $\mathbf{H}$ and its rational degeneration $\mathbf{A}$. Following Yun, we review how the bigraded W-character from Section 4.5 can be upgraded to a bigraded $\mathbf{A}$-module. We speculate about a refinement of our conjectures at the level of $\mathbf{A}$-modules.

Section 4.11 We introduce the rational Cherednik algebra $\mathbf{A}_\nu^{\mathrm{rat}} \supseteq \mathbf{A}$. The work of Oblomkov–Yun realizes certain $\mathbf{A}_\nu^{\mathrm{rat}}$-modules in the cohomology of Hitchin fibers and affine Springer fibers. By combining their work with our results from [107] and Sections 4.2-4.3, we deduce **Corollary 4.11.8**: The $t \to -1$ limit of our conjecture holds for almost-simple, simply-connected G and "Coxeter" fractional twists.

4.1.5. Acknowledgements This work was much inspired by ideas in Vivek Shende's 2013 NSF proposal [94]. In addition, I warmly thank Ron Donagi, Victor Ginzburg, Eugene Gorsky, Oscar Kivinen, Davesh Maulik, Benedict Morrissey, Andrei Neguţ, Bao-Châu Ngô, Alexei Oblomkov, Cheng-Chiang Tsai, and Zhiwei Yun for helpful discussions.

During this work, I've received support from an NSF Graduate Research Fellowship (grant #1746045) and benefited from workshops at MSRI and AIM.

4.1.6. Notation We work over $\mathbf{C}$. We will write $- /\!\!/ -$ to denote GIT quotients and $-/-$ to denote stack quotients in the fpqc topology. A curve is a 1-dimensional, reduced, separated $\mathbf{C}$-scheme of finite type. Later, we introduce orbifold curves, which have stacky structure.

A priori, all torsors are fpqc. If G is any algebraic group stack, not necessarily affine, and E, X are stacks that admit actions of G, then we set

$$E \wedge^G X = (E \times X)/G. \tag{4.1.30}$$

If $E \to S$ is a G-bundle over some base S, then we can view $E \wedge^G X$ as the associated bundle over S with fiber X.

For any Artin stack X over $\mathbf{C}$, we let $\mathsf{D}^b(X)$ be the bounded derived category of constructible complexes of sheaves over X. All constructible sheaves will be valued in $\mathbf{C}$-vector spaces. *We always assume that the Grothendieck–Verdier six operations are derived.* Thus we write f_* instead of $\mathrm{R}f_*$, etc.

Let $\mathbf{W}_{\leq *}$ denote the weight filtration on the (compactly-supported) cohomology of X. The **virtual weight series** of X is defined by

$$\mathrm{E}(x \mid X) = \sum_{i,k} (-1)^i x^k \operatorname{Gr}_k^{\mathbf{W}} \mathrm{H}_c^i(X, \mathbf{C}). \tag{4.1.31}$$

Thus, $\mathrm{E}(1 \mid X) = \chi(X)$, the Euler characteristic of X. In the course of the paper, we will also introduce modifications of $\mathrm{E}(x \mid X)$.

4.2 Loops

4.2.1. Étale Loops We write $\mathbf{C}(\!(\varpi^{\frac{1}{\infty}})\!) = \bigcup_{n\geq 1} \mathbf{C}(\!(\varpi^{\frac{1}{n}})\!)$. Let

$$\eta = \operatorname{Spec} \mathbf{C}(\!(\varpi)\!), \tag{4.2.1}$$

$$\bar{\eta} = \operatorname{Spec} \mathbf{C}(\!(\varpi^{\frac{1}{\infty}})\!). \tag{4.2.2}$$

In the étale topology, we can view η as a point in the shape of a circle and $\bar{\eta}$ as its universal cover. The étale fundamental group of η with basepoint $\bar{\eta}$ is

$$\pi_1^{\text{ét}}(\eta, \bar{\eta}) \simeq \varprojlim_n \mu_n, \tag{4.2.3}$$

where $\mu_n \subseteq \mathbf{C}$ is the group of nth roots of unity and the inverse system is given by the transition maps $\zeta \mapsto \zeta^d : \mu_{nd} \to \mu_n$ for $n, d \geq 1$. Since we are working over $\mathbf{C}$, there is a canonical generator of $\varprojlim_n \mu_n$, namely, $(e^{\frac{2\pi i}{n}})_n$. It allows us to identify $\pi_1^{\text{ét}}(\eta, \bar{\eta})$ with $\hat{\mathbf{Z}}$, the profinite completion of the group of integers.

Let U be a connected scheme of finite type over $\mathbf{C}$, and fix once and for all a basepoint $x \in U(\mathbf{C})$. A map $f : \eta \to U$ induces a morphism of profinite groups:

$$f_* : \hat{\mathbf{Z}} \simeq \pi_1^{\text{ét}}(\eta, \bar{\eta}) \to \pi_1^{\text{ét}}(U, f(\bar{\eta})). \tag{4.2.4}$$

Any étale path (i.e., isomorphism of fiber functors) from $f(\bar{\eta})$ to x defines an isomorphism from $\pi_1^{\text{ét}}(U, f(\bar{\eta}))$ onto $\pi_1^{\text{ét}}(U, x)$. Any two such isomorphisms differ at most by conjugation. So the element $f_*(1) \in \pi_1^{\text{ét}}(U, f(\bar{\eta}))$ gives rise to a well-defined conjugacy class

$$[\beta_f^{\text{ét}}] \subseteq \pi_1^{\text{ét}}(U, x). \tag{4.2.5}$$

If f is regarded as an unbased infinitesimal loop in U, then $[\beta_f^{\text{ét}}]$ may be regarded as the isotopy class of this loop.

Definition 4.2.1. In general, we refer to conjugacy classes of $\pi_1^{\text{ét}}(U)$ as **étale loops** of U. An étale loop is **algebraic** iff it takes the form $[\beta_f^{\text{ét}}]$ for some map $f : \eta \to U$.

Let $D = \operatorname{Spec} \mathbf{C}[\![\varpi]\!]$. We regard D as an infinitesimal disk with generic point η. Let X be another $\mathbf{C}$-scheme of finite type, equipped with an open embedding $U \hookrightarrow X$.

Definition 4.2.2. In the setting of Definition 4.2.1, we say that $[\beta_f^{\text{ét}}]$ is X**-algebraic** iff we can extend $f : \eta \to U$ to a map $D \to X$.

Example 4.2.3. Let $X = \mathbf{A}^1$ and $U = \mathbf{A}^1 \setminus 0$. Then a map $f : \eta \to U$ is equivalent to an element $f^\# \in \mathbf{C}(\!(\varpi)\!)^\times$. We can extend f to a map $D \to X$ if and only if $f^\# \in \mathbf{C}[\![\varpi]\!] \setminus 0$.

We know that $\pi_1^{\text{ét}}(U) \simeq \hat{\mathbf{Z}}$. Under this isomorphism, β_f corresponds to the integer $\operatorname{val}_\varpi(f^\#)$. Thus, the algebraic loops of U are the classes of the elements that come from $\pi_1(\mathbf{C}^\times) \simeq \mathbf{Z}$. The X-algebraic loops are those that come from $\mathbf{Z}_{\geq 0}$.

Example 4.2.4. If X is proper, then by the valuation criterion of properness, "algebraic" and "X-algebraic" are equivalent conditions on the étale loops of U. That is, X-algebraicity is only interesting when X is *not* proper. As an extreme case: If U is itself proper, then the only algebraic loop of U is the class of the identity.

4.2.2. Topological Loops Let $(-)^{\mathrm{an}}$ denote the analytification functor from $\mathbf{C}$-schemes of finite type to complex-analytic spaces. We have a morphism of groups:

$$\pi_1(U^{\mathrm{an}}, x) \to \pi_1^{\text{ét}}(U, x). \tag{4.2.6}$$

It induces an isomorphism $\hat{\pi}_1(U^{\mathrm{an}}) \simeq \pi_1^{\text{ét}}(U)$, where $\hat{\pi}_1$ is the profinite completion of π_1.

Definition 4.2.5. We refer to conjugacy classes of $\pi_1(U^{\mathrm{an}})$ as **loops** of U^{an}. We say that a loop is **algebraic**, resp. X**-algebraic**, iff its image in $\pi_1^{\text{ét}}(U)$ is contained in an algebraic, resp. X-algebraic étale loop.

In future work, we will prove that every algebraic loop of U is the image of some loop of U^{an}. For the purposes of this paper, it is enough to consider the following situation:

Definition 4.2.6. Given a map $f : (D, \eta) \to (X, U)$, a **model of finite type** for f consists of the following data:

1. A complex integral curve Σ, not necessarily proper.

2. A nonsingular basepoint $v \in \Sigma(\mathbf{C})$. We let $D_v = \operatorname{Spec} \hat{\mathcal{O}}_{\Sigma,v}$, the completed germ of Σ at v.

3. A factorization of f as a composition

$$D \xrightarrow{\sim} D_v \to \Sigma \to X, \tag{4.2.7}$$

where the first map is an isomorphism and the second map is the natural inclusion.

We denote the model by (Σ, v). It will be convenient to let η_v and $\bar{\eta}_v$ be the images of η and $\bar{\eta}$, respectively, under the isomorphism $D \xrightarrow{\sim} D_v$.

Lemma 4.2.7. *If $f : (D, \eta) \to (X, U)$ admits a model of finite type, then $[\beta_f^{\text{ét}}]$ is the image of a distinguished loop $[\beta_f]$ that depends on f but not on the model.*

Proof. We first show that to any model of finite type (Σ, v), we can attach a loop $[\beta_{\Sigma,v}] \subseteq \pi_1(U^{\text{an}})$ whose image in $\pi_1^{\text{ét}}(U)$ is contained in $[\beta_f]$. We then show that $[\beta_{\Sigma,v}]$ does not depend on (Σ, x).

Fix a small analytic disk $D^{\text{an}} \subseteq \Sigma^{\text{an}}$ containing v, and fix a basepoint $o \in D^{\text{an}} \setminus v$. Let τ be the positively-oriented generator of $\pi_1(D^{\text{an}} \setminus v, o) \simeq \mathbf{Z}$. Then, up to a choice of étale path from $\bar{\eta}_v$ to o in $\Sigma \setminus v$, the following are conjugate:

1. The image of τ under the composition

$$\pi_1(D^{\text{an}} \setminus v, o) \to \pi_1(\Sigma^{\text{an}} \setminus v, o) \to \hat{\pi}_1(\Sigma^{\text{an}} \setminus v, o) \xrightarrow{\sim} \pi_1^{\text{ét}}(\Sigma \setminus v, o). \tag{4.2.8}$$

2. The image of $1 \in \hat{\mathbf{Z}} \simeq \pi_1^{\text{ét}}(\eta, \bar{\eta})$ under the composition

$$\pi_1^{\text{ét}}(\eta, \bar{\eta}) \xrightarrow{\sim} \pi_1^{\text{ét}}(\eta_v, \bar{\eta}_v) \to \pi_1^{\text{ét}}(\Sigma \setminus v, \bar{\eta}_v) \xrightarrow{\sim} \pi_1^{\text{ét}}(\Sigma \setminus v, o). \tag{4.2.9}$$

Let y be the image of o in U. Then, up to a choice of topological path from y to x, (1) and (2) map into the same conjugacy class of $\pi_1^{\text{ét}}(U, x)$. But the composition

$$\pi_1(D^{\text{an}} \setminus v, o) \to \pi_1^{\text{ét}}(\Sigma \setminus v, o) \to \pi_1^{\text{ét}}(U, y) \simeq \pi_1^{\text{ét}}(U, x) \tag{4.2.10}$$

factors through $\pi_1(U^{\text{an}}, y) \simeq \pi_1(U^{\text{an}}, x)$, so the image of τ in $\pi_1(U^{\text{an}}, x)$ defines a conjugacy class $[\beta_{\Sigma,v}] \subseteq \pi_1(U^{\text{an}}, x)$ whose image in $\pi_1^{\text{ét}}(U, x)$ is contained in $[\beta_f^{\text{ét}}]$.

Suppose that (Σ', v') is another model of finite type for f. We want to show that $[\beta_{\Sigma,v}] = [\beta_{\Sigma',v'}]$. After shrinking Σ, resp. Σ', to a smaller neighborhood containing x, resp.

x', we can assume that both schemes embed into X. Then $v = v'$. We find that $(\Sigma \cap \Sigma', v)$ is again a model of finite type for f and that $[\beta_{\Sigma,v}] = [\beta_{\Sigma\cap\Sigma',v}] = [\beta_{\Sigma',v}]$. $\square$

Remark 4.2.8. Following [40, 178-179], a discrete group Π is said to be:

1. **Residually finite** iff the profinite completion map $\Pi \to \hat{\Pi}$ is injective.

2. **Conjugacy separable** iff the map from conjugacy classes of Π to those of $\hat{\Pi}$ is injective.

Property (2) is strictly stronger than property (1).

A priori, we do not know that $\pi_1(U^{\mathrm{an}})$ satisfies either property. In other words, there could be multiple loops of U^{an} that map to a given étale loop of U. The purpose of Lemma 4.2.7 is to allow us to pick out topological loops without ambiguity when we have a model of finite type.

Example 4.2.9. Let $X = \mathrm{Sym}^n(\mathbf{A}^1)$ and $U = \mathrm{Conf}^n(\mathbf{A}^1)$. The $\mathbf{C}$-points of X parametrize the length-n divisors of $\mathbf{A}^1$, and the $\mathbf{C}$-points of U parametrize the divisors where no point occurs with multiplicity. The **universal spectral cover** is the hypersurface

$$\mathcal{C} \subseteq X \times \mathbf{A}^1 \tag{4.2.11}$$

such that for all closed $x \in X$, the fiber $\mathcal{C}_x \subseteq \mathbf{A}^1$ is the divisor corresponding to x.

We can describe $\mathcal{C}$ more rigorously as follows. Fix coordinates $\mathbf{A}^n \simeq \operatorname{Spec} \mathbf{C}[x_1, \ldots, x_n]$ and $\mathbf{A}^1 \simeq \operatorname{Spec} \mathbf{C}[z]$. We have $X \simeq \mathbf{A}^n /\!\!/ S_n$, so by Chevalley's restriction theorem (stated more generally in Section 4.3), we have $X = \operatorname{Spec} \mathbf{C}[\sigma_1, \ldots, \sigma_n]$, where σ_i is the ith symmetric polynomial in $x_1, \ldots, x_n$. Now, $\mathcal{C}$ is given by

$$z^n + \sigma_1 z^{n-1} + \cdots + \sigma_{n-1} z + \sigma_n = 0 \tag{4.2.12}$$

in these coordinates on $X \times \mathbf{A}^1$.

Observe that $\mathcal{C} \to X$ is an n-fold cover, unramified over U. Given a map $f : (D, \eta) \to (X, U)$, we can pull back $\mathcal{C}$ along f to obtain an n-fold cover $D_f \to D$ embedded in $D \times \mathbf{A}^1$ and unramified over η:

$$\begin{array}{ccc} \mathcal{C} & \longrightarrow & X \times \mathbf{A}^1 \\ & \searrow & \downarrow \\ & & X \end{array} \quad \Longrightarrow \quad \begin{array}{ccc} D_f & \longrightarrow & D \times \mathbf{A}^1 \\ & \searrow & \downarrow \\ & & D \end{array} \tag{4.2.13}$$

Conversely, any n-fold cover of this form is the pullback of $\mathcal{C} \to X$ along a unique map $(D, \eta) \to (X, U)$. Explicitly, if D_f is given by

$$z^n + a_1(\varpi) z^{n-1} + \cdots + a_{n-1}(\varpi) z + a_n = 0, \tag{4.2.14}$$

then $f : D \to X$ is given by $f(\sigma_i) = a_i(\varpi)$.

Suppose that f admits a model of finite type, say (Σ, v). The cover $D_f \to D$ extends to a cover $\Sigma_f \to \Sigma$ embedded in $\Sigma \times \mathbf{A}^1$, possibly after we shrink the model. Let $D^{\mathrm{an}} \subseteq \Sigma^{\mathrm{an}}$ be a small analytic disk at v, and let γ be a loop that represents the positive generator of $\pi_1(D^{\mathrm{an}} \setminus v)$. Then the restriction of the cover $\Sigma_f^{\mathrm{an}} \to \Sigma^{\mathrm{an}}$ to γ is an unramified n-fold cover $\gamma_f \to \gamma$, where γ_f is contained in the annulus

$$\bigcirc_v = (D^{\mathrm{an}} \setminus v) \times \mathbf{C}. \tag{4.2.15}$$

We may regard γ_f as the *annular closure* of a topological braid on n strands.

Let Br_n be the n-strand braid group. The annular closure of a braid determines and is determined by its conjugacy class in Br_n, i.e., they are equivalent data. At the same time, Br_n is isomorphic to $\pi_1(U^{\mathrm{an}}) = \pi_1(\mathrm{Conf}^n(\mathbf{C}))$, essentially by definition. Under this equivalence, the annular link $\gamma_f \subseteq \bigcirc_v$ corresponds to the conjugacy class $[\beta_f] \subseteq Br_n$.

4.2.3. Algebraic Links If $C \subseteq \mathbf{A}^2$ is a plane algebraic curve and $p \in C(\mathbf{C}) \subseteq \mathbf{C}^2$, then the **link** of C in any 3-sphere centered at p is defined by:

$$\lambda_{C,p} = S^3 \cap C^{\mathrm{an}} \subseteq S^3. \tag{4.2.16}$$

If the radius is small enough, then the isotopy class of $\lambda_{C,p}$ does not depend it. The links in 3-space that arise this way are called **algebraic links** [36, 67]. Using Example 4.2.9, we will explain how this notion inspires the notion of an algebraic loop.

In the example, the curve Σ_f is embedded in $\Sigma \times \mathbf{A}^1$, so it is locally planar. Suppose that the fiber of Σ_f above v consists of a single point, say v itself, and let $\lambda_f = \lambda_{\Sigma_f,v}$ with respect to a fixed 3-sphere at v. Then λ_f is contained in the annulus $\bigcirc_v$, possibly after we shrink the 3-sphere further. By pushing λ_f around horizontally, we can isotope it within $\bigcirc_v$ to coincide with γ_f. Altogether, λ_f is the *planar closure* of the annular link γ_f, a fortiori of the braid β_f.

We refer to [36] for the definitions of the knot-theoretic terms that follow. If the plane curve germ $D_f = \operatorname{Spec} \hat{\mathcal{O}}_{\Sigma_f,v}$ is unibranch, i.e., irreducible, then λ_f is an iterated torus knot and its structure can be read off from the Newton–Puiseux expansion of D_f by the procedure described in Appendix A of [36]. To wit, suppose that

$$\hat{\mathcal{O}}_{\Sigma_f,v} \simeq \mathbf{C}[\![\varpi, \varpi^{\frac{e_1}{d_1}} + \varpi^{\frac{e_2}{d_1 d_2}} + \cdots + \varpi^{\frac{e_r}{d_1 d_2 \cdots d_r}}]\!], \tag{4.2.17}$$

where each fraction $e_i/(d_1 d_2 \cdots d_i)$ is in lowest terms and

$$d_1 < d_1 d_2 < \cdots < d_1 d_2 \cdots d_r = n. \tag{4.2.18}$$

Then the n-strand braid β_f is the (d_r, e_r)-cable of… the (d_2, e_2)-cable of the (d_1, e_1)-cable of the identity braid. It follows that λ_f is the (d_r, e_r')-cable of… the (d_2, e_2')-cable of the (d_1, e_1')-torus knot in its blackboard framing, where $e_1' = e_1$ and $e_{i+1}' = e_{i+1} + d_i d_{i+1} e_i$ for all $i > 1$. We refer to *loc. cit.* for details.

We conclude that: *The existence of Puiseux expansions for plane curve germs entails strong constraints on the structure of algebraic loops in* $\mathrm{Conf}^n(\mathbf{A}^1)$.

4.2.4. Homogeneity In this subsection, we assume that:

- X and U are affine varieties.
- X is equipped with a $\mathbf{G}_m$-action that restricts to a free action on U.

We write $\mathbf{G}_m \cdot x$ for the subscheme of U formed by the orbit of $x \in U(\mathbf{C})$, and $\mathbf{C}^\times \cdot x$ for the analytification of $\mathbf{G}_m \cdot x$.

Let $\underline{x}$ be the image of x in the quotient stack $U/\mathbf{G}_m$. From the long exact sequence of homotopy groups, we get the following exact sequence:

$$1 \to \pi_1(\mathbf{C}^\times \cdot x, x) \to \pi_1(U^{\mathrm{an}}, x) \to \pi_1(U^{\mathrm{an}}/\mathbf{C}^\times, \underline{x}) \tag{4.2.19}$$

Let $\tau \in \pi_1(U^{\mathrm{an}})$, resp. $\tau^{\text{ét}} \in \pi_1^{\text{ét}}(U)$, denote the image of $1 \in \mathbf{Z} \simeq \pi_1(\mathbf{C}^\times \cdot x)$ in $\pi_1(U^{\mathrm{an}})$, resp. $\pi_1^{\text{ét}}(U)$.

Definition 4.2.10. We say that a loop of U^{an} is **fractional of slope** $\frac{m}{n} \in \mathbf{Q}$ iff it can be represented by an element $\beta \in \pi_1(U^{\mathrm{an}})$ such that $\beta = \alpha^m$ and $\tau = \alpha^n$ for some $\alpha \in \pi_1(U^{\mathrm{an}})$.

We will show that fractional loops are algebraic, and furthermore, we will identify an explicit construction for them. To this end, we introduce some notation and terminology inspired by Oblomkov–Yun [90, §3].

Observe that η embeds into $\mathbf{G}_m$ as the punctured disk at the origin. Thus, the action of $\mathbf{G}_m$ on itself by multiplication can be pulled back to a $\mathbf{G}_m$-action on η. We consequently get a $(\mathbf{G}_m \times \mathbf{G}_m)$-action on the space of maps $\eta \to U$, given by

$$(s, t) \cdot f(\varpi) = t \cdot f(s\varpi) \tag{4.2.20}$$

for all $(s,t) \in \mathbf{G}_m \times \mathbf{G}_m$ and $f \in U(\eta)$.[1]

For all $\frac{m}{n} \in \mathbf{Q}$, written in lowest terms, let $\mathbf{G}_m(\nu)$ be the subtorus of $\mathbf{G}_m \times \mathbf{G}_m$ defined by the exact sequence

$$0 \to \mathbf{Z} \xrightarrow{(m,n)} \mathbf{Z} \oplus \mathbf{Z} \simeq \mathbf{X}(\mathbf{G}_m \times \mathbf{G}_m) \to \mathbf{X}(\mathbf{G}_m(\nu)) \to 0. \tag{4.2.21}$$

(The assumption that $\frac{m}{n}$ is in lowest terms ensures that $\mathbf{G}_m(\nu)$ is reduced.)

Definition 4.2.11. We say that $f \in U(\eta)$ is **homogeneous of slope** ν iff it is a fixed point of the $\mathbf{G}_m(\nu)$-action on $U(\eta)$.

Proposition 4.2.12. *If $f : (D, \eta) \to (X, U)$ is homogeneous of slope $\nu \in \mathbf{Q}$, then f admits a model of finite type and $[\beta_f] \subseteq \pi_1(U^{\mathrm{an}})$ (defined by Lemma 4.2.7) is fractional of slope ν.*

Proof. First, we establish the case where $\nu = 1$. Let $v = f(0) \in X(\mathbf{C})$, and let $\Sigma \subseteq X$ be the subvariety of points that contract onto v under the $\mathbf{G}_m$-action. Then f factors through Σ. We claim that (Σ, v) is a model of finite type for f:

1. Since Σ intersects U and the $\mathbf{G}_m$-action on U is free, we must have $(\Sigma, v) \simeq (\mathbf{A}^1, 0)$. In particular, Σ is a curve.

2. Write $D_v = \operatorname{Spec} \hat{\mathcal{O}}_{U,v}$. Then the map $(D, \eta) \to (U, U \setminus v)$ factors through

$$(D, \eta) \to (D_v, \eta_v). \tag{4.2.22}$$

   That f is homogeneous of slope 1 means that (4.2.22) is equivariant with respect to the $\mathbf{G}_m$-actions on D and D_v. This forces the degree of the map $\eta \to \eta_v$ to be one, which in turn forces (4.2.22) to be an isomorphism.

To finish the proof for $\nu = 1$, it remains to show $[\beta_f] = [\tau]$. Fix a basepoint $o \in \Sigma(\mathbf{C}) \setminus v$, so that $\Sigma \setminus v$ is the $\mathbf{G}_m$-orbit of o. Up to a choice of topological path from o to x, the loop $[\beta_f]$

1. More rigorously, the $(\mathbf{G}_m \times \mathbf{G}_m)$-action is defined on the loop space $\mathcal{L}U$, where $\mathcal{L}U(R) = U(R(\!(\varpi)\!))$ for all $\mathbf{C}$-algebras R, *cf.* §4.4.3. Here, however, we only use the $\mathbf{C}$-points of $\mathcal{L}U$.

defined by Lemma 4.2.7 can be represented by the image of $1 \in \mathbf{Z}$ under

$$\mathbf{Z} \simeq \pi_1(\mathbf{C}^\times \cdot o, o) = \pi_1(\Sigma^{\mathrm{an}} \setminus v, o) \to \pi_1(U^{\mathrm{an}}, o) \simeq \pi_1(U^{\mathrm{an}}, x). \tag{4.2.23}$$

Since the $\mathbf{G}_m$-action on U is free, this element is conjugate to the image of $1 \in \mathbf{Z}$ under

$$\mathbf{Z} \simeq \pi_1(\mathbf{C}^\times \cdot x, x) \to \pi_1(U^{\mathrm{an}}, x), \tag{4.2.24}$$

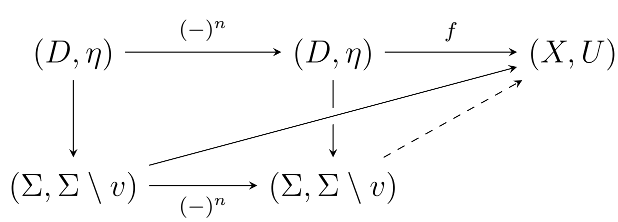
giving $[\beta_f] = [\tau]$ as desired.

Next, we establish the case where $\nu = \frac{1}{n}$. Consider the composition of maps

$$g : (D, \eta) \xrightarrow{(-)^n} (D, \eta) \xrightarrow{f} (X, U), \tag{4.2.25}$$

where the first arrow is the branched n-fold cover of D by itself. Then g is homogeneous of slope $\frac{n}{n} = 1$. So by our preceding work, g admits a model of finite type $(\Sigma, v) \simeq (\mathbf{A}^1, 0)$, and moreover, $[\beta_g] = [\tau]$. The n-fold cover $(-)^n : D \to D$ extends canonically to an n-fold cover $(-)^n : \Sigma \to \Sigma$, given in coordinates by the same nth-power map. It follows that we can fill in the dashed arrow in the commutative diagram below:

(4.2.26)



Thus, f also admits a model of finite type of the form $(\Sigma, v) \simeq (\mathbf{A}^1, 0)$, and by construction, $[\beta_f^n] = [\beta_g]$.

Finally, we consider the general case $\nu = \frac{m}{n}$. We claim that there is a map $g : \eta \to U$ such that g admits a model of finite type and $[\beta_f] = [\beta_g^m]$ and $[\tau] = [\beta_g^n]$. Without loss of generality, assume m and n are coprime. We have

$$f(t^n \varpi) = t^m \cdot f(\varpi) \tag{4.2.27}$$

for all $t \in \mathbf{C}^\times$. After choosing coordinates on the affine variety U, we can express this condition in terms of a collection of power series that define f. Then, by coprimality of m and n, we see that it can hold only if $f(\varpi)$ is not just given by power series in ϖ, but by power series in ϖ^m. That is, we can factor f as a composition

$$f : (D, \eta) \xrightarrow{(-)^m} (D, \eta) \xrightarrow{g} (X, U) \tag{4.2.28}$$

for some $g : \eta \to U$. By construction, g is homogeneous of slope $\frac{1}{n}$, so by the previous case, it admits a model of finite type $(\Sigma, v) \simeq (\mathbf{A}^1, 0)$ and furthermore satisfies $[\beta_g^n] = [\tau]$. By pulling back (Σ, v) along the m-fold cover $(-)^m : D \to D$, we get the model of finite type for f and the identity $[\beta_g^m] = [\beta_f]$, finishing the proof. □

Example 4.2.13. Fix coprime integers $m, n \geq 1$. The plane curve $C \subseteq \mathbf{A}^2$ defined by

$$y^n = x^m \tag{4.2.29}$$

is smooth at the origin when at least one of m or n equals 1, and has a cusp at the origin otherwise. Projection on the x-coordinate defines an n-fold cover $C \to \mathbf{A}^1$ ramified over 0, so in the notation of Example 4.2.9, there is a map $f : D \to \mathrm{Sym}^n(\mathbf{A}^1)$ for which the cover $D_f \to D$ is isomorphic to the cover $\mathrm{Spec}\, \hat{\mathcal{O}}_{C,0} \to \mathrm{Spec}\, \hat{\mathcal{O}}_{\mathbf{A}^1,0}$. In the coordinates on $\mathrm{Sym}^n(\mathbf{A}^1)$ discussed there, we have $f(\sigma_n) = \varpi^m$ and $f(\sigma_i) = 0$ for $i < n$.

We observe that C contracts to the origin under the $\mathbf{G}_m$-action defined by

$$t \cdot (x, y) = (t^n x, t^m y). \tag{4.2.30}$$

The existence of this action corresponds to the fact that f is homogeneous of slope $\frac{m}{n}$.

4.3 Artin Braids

4.3.1. The Group W Let (W, S) be a finite Coxeter system. That is, W is a finite group generated by S modulo relations of the form $(st)^{m(s,t)} = 1$, where $m(s,s) = 1$ and $m(s,t) \geq 1$ for all $s, t \in S$. We write $|-| : W \to \mathbf{Z}_{\geq 0}$ for the Bruhat length with respect to S.

Let $\{\beta_s\}_{s \in S}$ be a copy of the set S. The **Artin group** of (W, S), denoted Br_W, is the group freely generated by the elements β_s modulo the relations

$$\overbrace{\beta_s \beta_t \beta_s \cdots}^{m(s,t) \text{ terms}} = \overbrace{\beta_t \beta_s \beta_t \cdots}^{m(s,t) \text{ terms}} \tag{4.3.1}$$

for *distinct* s and t. There is a surjective morphism $Br_W \to W$ that sends $\beta_s \mapsto s$. Its kernel, denoted PBr_W, is generated by the elements β_s^2 for $s \in S$. We refer to elements of Br_W, resp. PBr_W, as **Artin braids**, resp. **pure Artin braids**. The elements β_s are sometimes called **simple twists**.

Example 4.3.1. If $W = S_n$ and S is the set of transpositions $(i\ i+1)$ for $i = 1, \ldots, n-1$, then the pair (W, S) is a Coxeter system. The Artin group Br_W is isomorphic to Br_n, and the map $Br_W \to W$ sends a topological braid on n strands to its underlying permutation. The kernel is the subgroup of pure topological braids, usually denoted PBr_n.

The **positive Artin monoid** is the submonoid $Br_W^+ \subseteq Br_W$ generated by the elements β_s but not their inverses. If $\beta \in Br_W^+$, then every sequence $(s_1, \ldots, s_\ell)$ of elements of S such that $\beta = \beta_{s_1} \cdots \beta_{s_\ell}$ has the same length $|\beta|$, known as the **writhe** of β. The map $s \mapsto \beta_s$ extends to a set-theoretic section $w \mapsto \beta_w$ of the morphism $Br_W^+ \to W$, uniquely characterized by the following properties:

1. $|\beta_w| = |w|$ for all $w \in W$.

2. If $|w_1 w_2| = |w_1| + |w_2|$, then $\beta_{w_1 w_2} = \beta_{w_1} \beta_{w_2}$.

Brieskorn–Saito [15] and Deligne [25] showed that Br_W is isomorphic to the group completion of Br_W^+, so the writhe function extends uniquely to a map $|-| : Br_W \to \mathbf{Z}$.

Let $w_0 \in W$ be the longest element with respect to Bruhat length. The **full twist** of Br_W is the positive braid $\pi = \beta_{w_0}^2 \in Br_W^+$. Since $w_0^2 = 1$, we know that $\pi \in PBr_W$.

4.3.2. Brieskorn's Theorem Let $\mathfrak{t}$ be a **realization** of W, meaning a faithful and finite-dimensional complex representation on which S acts by reflections. Let

$$\mathfrak{c} = \mathfrak{t} /\!\!/ W. \tag{4.3.2}$$

The Chevalley–Shephard–Todd theorem states that $\mathfrak{c}$ forms a vector space and the map $\mathfrak{t} \to \mathfrak{c}$ takes the dilation action of $\mathbf{G}_m$ on $\mathfrak{t}$ to a weighted action of $\mathbf{G}_m$ on $\mathfrak{c}$.

Let $\mathfrak{c}^\circ \subseteq \mathfrak{c}$ be the unbranched locus of the cover $\mathfrak{t} \to \mathfrak{c}$, and let $\mathfrak{t}^\circ \subseteq \mathfrak{t}$ be the preimage of $\mathfrak{c}^\circ$. The following theorem is essentially proved in [14]:

Theorem 4.3.2 (Brieskorn)**.** *We have an isomorphism of exact sequences:*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & PBr_W & \longrightarrow & Br_W & \longrightarrow & W & \longrightarrow & 1 \\ & & \wr\big| & & \wr\big| & & \big\| & & \\ 1 & \longrightarrow & \pi_1((\mathfrak{t}^\circ)^{\mathrm{an}}) & \longrightarrow & \pi_1((\mathfrak{c}^\circ)^{\mathrm{an}}) & \longrightarrow & W & \longrightarrow & 1 \end{array} \tag{4.3.3}$$

In particular, the groups $\pi_1((\mathfrak{t}^\circ)^{\mathrm{an}})$ and $\pi_1((\mathfrak{c}^\circ)^{\mathrm{an}})$ only depend on W, not on $\mathfrak{t}$.

Remark 4.3.3. For $W = S_n$, Brieskorn's theorem recovers the isomorphisms $Br_W \simeq Br_n$ and $PBr_W \simeq PBr_n$ in the following sense. Taking $\mathfrak{t} = \mathbf{A}^n$, we find that:

1. $\mathfrak{c}^{\mathrm{an}} = \mathrm{Sym}^n(\mathbf{C})$, the space of unordered n-tuples of points in $\mathbf{C}$.

2. $(\mathfrak{c}^\circ)^{\mathrm{an}} = \mathrm{Conf}^n(\mathbf{C})$, the space of unordered n-tuples of *distinct* points in $\mathbf{C}$.

3. $(\mathfrak{t}^\circ)^{\mathrm{an}} = \mathrm{PConf}^n(\mathbf{C})$, the space of *ordered* n-tuples of distinct points in $\mathbf{C}$.

Now recall that $Br_n = \pi_1(\mathrm{Conf}^n(\mathbf{C}))$ and $PBr_n = \pi_1(\mathrm{PConf}^n(\mathbf{C}))$.

Brieskorn's theorem lets us characterize the full twist π topologically. Observe that the $\mathbf{G}_m$-action on $\mathfrak{t}$ restricts to a free action on $\mathfrak{t}^\circ$, so §4.2.4 applies to $(X, U) = (\mathfrak{t}, \mathfrak{t}^\circ)$. The discussion in [18, §2] shows that:

Lemma 4.3.4. *The isomorphism $PBr_W \xrightarrow{\sim} \pi_1((\mathfrak{t}^\circ)^{\mathrm{an}})$ in Theorem 4.3.2 sends π to the image of $1 \in \mathbf{Z}$ under the embedding $\mathbf{Z} \simeq \pi_1(\mathbf{C}^\times) \to \pi_1((\mathfrak{t}^\circ)^{\mathrm{an}})$ of §4.2.4.*

Remark 4.3.5. For $W = S_n$, this lemma says that the full twist corresponds to the topological braid formed by twisting a row of n strands through a full revolution.

4.3.3. Algebraic Braids

Henceforth, we only study algebraic loops of the following form.

Definition 4.3.6. An **algebraic braid** for W is an element of Br_W whose conjugacy class is a $\mathfrak{c}$-algebraic loop of $(\mathfrak{c}^\circ)^{\mathrm{an}}$, or, by abuse of terminology, the loop itself.

Example 4.3.7. For $W = S_n$, the algebraic braids are precisely the elements of Br_n that can appear in Example 4.2.9. That is, they are the topological braids that arise from branched n-fold covers over a disk.

Remark 4.3.8. We may ask: Does Br_W have the properties listed in Remark 4.2.8? The Artin group Br_W is known to be linear [30]. Hence, by a theorem of Malčev [40, 179], it is residually finite. By contrast, it is an open problem whether Br_W is conjugacy separable, even in the case where $W = S_n$ and $n \geq 4$.[2]

Fractional Twists In [107], we defined a **fractional twist of slope $\frac{m}{n} \in \mathbf{Q}$** to be an element $\beta \in Br_W$ such that $\alpha^m = \beta$ and $\alpha^n = \pi$ for some $\alpha \in Br_W$. From Lemma 4.3.4, we infer that Brieskorn's isomorphism $Br_W \xrightarrow{\sim} \pi_1((\mathfrak{c}^\circ)^{\mathrm{an}})$ sends any fractional twist into a fractional loop of $(\mathfrak{c}^\circ)^{\mathrm{an}}$ with the same slope. By Proposition 4.2.12, we conclude:

2. See `https://mathoverflow.net/questions/237029/are-braid-groups-conjugacy-separable`.

Proposition 4.3.9. *If $f : (D, \eta) \to (\mathfrak{c}, \mathfrak{c}^\circ)$ is homogeneous of slope $\nu \in \mathbf{Q}$, then the loop $[\beta_f] \subseteq \pi_1((\mathfrak{c}^\circ)^{\mathrm{an}}) \simeq Br_W$ (defined by Proposition 4.2.12) can be represented by a fractional twist of slope ν.*

Example 4.3.10. A **Coxeter element** of W is an element conjugate to a product of the form $s_1 \cdots s_r$, where $s_1, \dots, s_r$ is an enumeration of the generating set $S \subseteq W$. The Coxeter elements of W form a single conjugacy class [43, Thm. 3.1.4]. Their common order is called the **Coxeter number** of W and denoted h. For instance, if $W = S_n$, then the Coxeter elements are the n-cycles and $h = n$.

Prop. 4.3.4 of [43] shows that if W is irreducible, i.e., not a product of smaller Coxeter groups, and $w \in W$ is a Coxeter element, then β_w is a fractional twist of slope $\frac{1}{h}$. It is claimed in *loc. cit.* that the result extends to the reducible case, but this is not correct: If $W = S_2 \times S_3$, then $h = 6$ but the full twist is a braid of writhe 8.

Simple Twists and the Discriminant We will show that simple twists are algebraic as well. To this end, define the **discriminant divisor** $\mathfrak{D} \subseteq \mathfrak{c}$ to be the branch locus of $\mathfrak{t} \to \mathfrak{c}$, i.e., the complement of $\mathfrak{c}^\circ$. For any map $f : (D, \eta) \to (\mathfrak{c}, \mathfrak{c}^\circ)$, we set

$$\mathfrak{D}_f = f^{-1}(\mathfrak{D}). \tag{4.3.4}$$

By definition, f is transverse to $\mathfrak{D}$ if and only if $\mathrm{val}_\varpi(\mathfrak{D}_f) = 1$.

Lemma 4.3.11. *If $f : (D, \eta) \to (\mathfrak{c}, \mathfrak{c}^\circ)$ is transverse to $\mathfrak{D}$, then there exists $s \in S$ such that $\beta_s \in Br_W \simeq \pi_1((\mathfrak{c}^\circ)^{\mathrm{an}})$ maps into $[\beta_f^{\text{ét}}] \subseteq \pi_1^{\text{ét}}(\mathfrak{c}^\circ)$. If, in addition, f admits a model of finite type, then $[\beta_f] = [\beta_s]$.*

As preparation for the proof, we describe the irreducible components of $\mathfrak{D}$, following the exposition in [18, §2].

Let $\mathrm{Ref}(W)$ be the set of elements of W that act on $\mathfrak{t}$ by reflections, or equivalently, the set of all conjugates of elements of S. For each $t \in \mathrm{Ref}(W)$, we can pick a root $\alpha_t \in \mathfrak{t}^\vee$, i.e., a linear functional on $\mathfrak{t}$ that cuts out the corresponding hyperplane. The conjugacy class $[t] \subseteq \mathrm{Ref}(W)$ gives rise to a W-invariant closed subscheme, cut out by:

$$\prod_{t' \in [t]} \alpha_{t'}^2 = 0. \tag{4.3.5}$$

This subscheme descends to a divisor $\mathfrak{D}_{[t]} \subseteq \mathfrak{c}$. It turns out that these divisors are precisely the irreducible components of $\mathfrak{D}$.

Proof of Lemma 4.3.11. In what follows, we will only write out the proof of the statement about $[\beta_f^{\text{ét}}]$, as the statement about $[\beta_f]$ does not use anything further.

For f to be transverse to $\mathfrak{D}$, it must intersect only one of the irreducible components of $\mathfrak{D}$, say $\mathfrak{D}_{[t]}$, and the intersection must occur at a nonsingular point $f(0) \in \mathfrak{D}_{[t]}$. At this point, the pair $(\mathfrak{c}, \mathfrak{D}_{[t]})$ can be locally modeled by $(\mathbf{A}^r, 0 \times \mathbf{A}^{r-1})$, where $r = \dim \mathfrak{c}$, in the sense of an isomorphism of pairs of germs:

$$(\mathrm{Spec}\, \hat{\mathcal{O}}_{\mathfrak{c}, f(0)}, \mathrm{Spec}\, \hat{\mathcal{O}}_{\mathfrak{D}_{[t]}, f(0)}) \simeq (\mathrm{Spec}\, \hat{\mathcal{O}}_{\mathbf{A}^r, 0}, 0 \times \mathrm{Spec}\, \hat{\mathcal{O}}_{\mathbf{A}^{r-1}, 0}), \tag{4.3.6}$$

In this way, we can reduce to the situation in Example 4.2.3. We deduce that up to conjugation, β_f is a multiple of a generator of the monodromy around $\mathfrak{D}_{[t]}$ at $f(0)$. Transversality of f means that it is conjugate to the positive generator. By the discussion preceding the proof, the latter is conjugate to β_s for any $s \in S$ in the class $[t]$ (see also [18, 142]). □

Theorem 4.3.12. *If $f : (D, \eta) \to (\mathfrak{c}, \mathfrak{c}^\circ)$ admits a model of finite type, then $|\beta_f| = \mathrm{val}_\varpi \mathfrak{D}_f$.*

Proof. We use a deformation argument that presages the geometry introduced in Section 4.5. A number of objects in the proof will be studied more formally in that section.

Let $(\Sigma_0, 0)$ be a model of finite type for f, and let Σ be a smooth compactification of Σ_0, i.e., a smooth proper curve into which Σ_0 embeds. After choosing a line bundle $L \to \Sigma$ of high-enough degree, we can extend the map $\Sigma_0 \to \mathfrak{c}$ to a section of the vector bundle

$$\mathfrak{c}_L = \mathfrak{c} \otimes L. \tag{4.3.7}$$

We will denote the section by $a : \Sigma \to \mathfrak{c}_L$. It defines a $\mathbf{C}$-point of the vector space of global sections $\mathcal{A} = \Gamma(\Sigma, \mathfrak{c}_L)$.

Since $\mathfrak{c}^\circ$ and $\mathfrak{D}$ are stable under the $\mathbf{G}_m$-action on $\mathfrak{c}$, they give rise to sub-bundles $\mathfrak{c}_L^\circ \subseteq \mathfrak{c}_L$ and $\mathfrak{D}_L \subseteq \mathfrak{c}_L$ with fibers $\mathfrak{c}^\circ$ and $\mathfrak{D}$, respectively. Let $\mathcal{A}^\heartsuit \subseteq \mathcal{A}$ be the dense open locus of sections $b : \Sigma \to \mathfrak{c}_L$ that send the generic point of Σ into $\mathfrak{c}_L^\circ$, or equivalently, for which

$$\mathfrak{D}_b = b^{-1}(\mathfrak{D}_L) \tag{4.3.8}$$

is a divisor on Σ. We note that a belongs to $\mathcal{A}^\heartsuit(\mathbf{C})$ because it sends η_0 into $\mathfrak{c}_L^\circ$. For all $b \in \mathcal{A}^\heartsuit(\mathbf{C})$ and $v \in \Sigma(\mathbf{C})$, we can form a map

$$b_v : D \xrightarrow{\sim} D_v \to \mathfrak{c}, \tag{4.3.9}$$

up to choosing a uniformization $D \simeq D_v$ and a trivialization of $\mathfrak{c}_L$ in a Zariski neighborhood of v. We will show in Lemma 4.6.1 that the algebraic braid $[\beta_{b_v}] \subseteq \pi_1((\mathfrak{c}^\circ)^{\mathrm{an}})$ is well-defined for all (b, v).

Let $D^{\mathrm{an}} \subseteq \Sigma^{\mathrm{an}}$ be an analytic disk at 0, small enough that 0 is the only point of $\mathfrak{D}_a \cap D^{\mathrm{an}}$. Using Rouché's principle, *a.k.a.* conservation of number, we check that the following remain constant as we perturb b in a small-enough neighborhood of a in $(\mathcal{A}^\heartsuit)^{\mathrm{an}}$:

1. $\sum_{v \in D^{\mathrm{an}}} |\beta_{b_v}|$.

2. $\sum_{v \in D^{\mathrm{an}}} \mathrm{val}_{\varpi_v} \mathfrak{D}_b$, where ϖ_v is a uniformizer of D_v.

When $b = a$, we see that (1) equals $|\beta_f|$ and (2) equals $\mathrm{val}_\varpi \mathfrak{D}_f$. When b represents a section of $\mathfrak{c}_L$ everywhere transverse to $\mathfrak{D}_L$, we see that (1) and (2) both equal $|\mathfrak{D}_b \cap D^{\mathrm{an}}|$. Since the latter case is generic when $\deg L \gg 0$, we can always find a small perturbation b of a that realizes it. We then have $|\beta_f| = |\mathfrak{D}_b \cap D^{\mathrm{an}}| = \mathrm{val}_\varpi \mathfrak{D}_f$. □

4.4 Local Springer Theory

4.4.1. The Group G Fix a root datum $(\mathbf{X}, \mathbf{X}^\vee, \Phi, \Phi^\vee)$ of rank r. Let G be the (connected) reductive group it defines over $\mathbf{C}$, and let $\mathfrak{g}$ be the Lie algebra of G. Later, we will assume that G is semisimple for certain applications.

Let T be the Cartan torus defined by the character lattice $\mathbf{X}$. Henceforth, *we assume that W is the Weyl group of the root system Φ and that $\mathfrak{t}$ is the Lie algebra of T*. We have identifications $W = N_G(T)/T$ and $\mathfrak{t} = \mathbf{X}^\vee \otimes \mathbf{C}$, under which the W-action on $\mathfrak{t}$ corresponds to the $N_G(T)$-action on $\mathbf{X}$ by conjugation. By definition, $r = \dim T = \dim \mathfrak{t}$.

The Chevalley restriction theorem states that restriction of functions from $\mathfrak{g}$ to $\mathfrak{t}$ induces an isomorphism of rings $\mathbf{C}[\mathfrak{g}]^G \xrightarrow{\sim} \mathbf{C}[\mathfrak{t}]^W$, hence an isomorphism of GIT quotients

$$\mathfrak{c} = \mathfrak{t} /\!\!/ W \xrightarrow{\sim} \mathfrak{g} /\!\!/ G. \tag{4.4.1}$$

The **Chevalley map** is the induced map $\chi : \mathfrak{g} \to \mathfrak{c}$. It is $\mathbf{G}_m$-equivariant with respect to the dilation action on $\mathfrak{g}$ and the weighted action on $\mathfrak{c}$.

The **universal centralizer** of $\mathfrak{g}$ is the group scheme $I = I_G$ over $\mathfrak{g}$ that, at the level of $\mathbf{C}$-points, classifies pairs $(\xi, g) \in \mathfrak{g} \times G$ and $\mathrm{Ad}(g)\xi = \xi$. In terms of I,

- The regular locus $\mathfrak{g}^{\mathrm{reg}} \subseteq \mathfrak{g}$ consists of points ξ such that I_ξ is of minimal dimension, or equivalently, commutative.
- The regular semisimple locus $\mathfrak{g}^{\mathrm{rs}} \subseteq \mathfrak{g}^{\mathrm{reg}}$ consists of points ξ such that I_ξ is a torus. As a more rigorous definition, we may set $\mathfrak{g}^{\mathrm{rs}} = \chi^{-1}(\mathfrak{c}^\circ)$ [115, §1.2.2].

4.4.2. Springer Fibers The main reference for this subsection is [115, Lect. I].

Let $\mathcal{B}$ be the flag variety of G. In terms of a fixed Borel $B \subseteq G$, we have $\mathcal{B} = G/B$. We write $\tilde{\mathfrak{g}} \subseteq \mathcal{B} \times \mathfrak{g}$ for the subvariety of pairs (xB, γ) such that $\gamma \in \mathrm{Lie}(xBx^{-1})$. The **Grothendieck–Springer resolution** is the projection map

$$\boldsymbol{s} : \tilde{\mathfrak{g}} \to \mathfrak{g}. \tag{4.4.2}$$

As suggested by the name, $\boldsymbol{s}$ is a proper map and restricts to an isomorphism over an open locus of $\mathfrak{g}$, namely, $\mathfrak{g}^{\mathrm{rs}}$. In fact, setting $\tilde{\mathfrak{g}}^{\mathrm{rs}} = \mathfrak{g}^{\mathrm{rs}} \times_{\mathfrak{g}} \tilde{\mathfrak{g}}$ and $\boldsymbol{s}^{\mathrm{rs}} = \boldsymbol{s}|_{\tilde{\mathfrak{g}}^{\mathrm{rs}}}$, one can show that $\boldsymbol{s}^{\mathrm{rs}}$ fits into a cartesian diagram:

$$\begin{CD} \tilde{\mathfrak{g}}^{\mathrm{rs}} @>>> \mathfrak{t}^{\circ} \\ @V{\boldsymbol{s}^{\mathrm{rs}}}VV @VVV \\ \mathfrak{g}^{\mathrm{rs}} @>{\chi|_{\mathfrak{g}^{\mathrm{rs}}}}>> \mathfrak{c}^{\circ} \end{CD} \tag{4.4.3}$$

In [102], Springer showed that W acts on the derived pushforward $\boldsymbol{s}_*\mathbf{C} \in \mathsf{D}^b(\mathfrak{g})$. In [69], Lusztig explained how to see this action in terms of the diagram above. First, the vertical arrow on the right is an unramified W-cover, so by pullback, there is a W-action on $\boldsymbol{s}^{\mathrm{rs}}_*\mathbf{C}$. Next, in the language of perverse sheaves [4], $\boldsymbol{s}_*\mathbf{C}$ turns out to be the intermediate extension of $\boldsymbol{s}^{\mathrm{rs}}_*\mathbf{C}$ along the inclusion $\mathfrak{g}^{\mathrm{rs}} \hookrightarrow \mathfrak{g}$, so it too receives a W-action. We point out that Lusztig's action differs from Springer's by a sign twist. Throughout the rest of the paper, *we will adopt Lusztig's convention*.

For all $\gamma \in \mathfrak{g}$, the **Springer fiber** above γ, denoted $\mathcal{B}_\gamma$, is the underlying reduced scheme of the fiber of $\boldsymbol{s}$ above γ. At the level of $\mathbf{C}$-points, we have:

$$\mathcal{B}_\gamma(\mathbf{C}) = \{gB \in \mathcal{B}(\mathbf{C}) : \mathrm{Ad}(g^{-1})\gamma \in \mathrm{Lie}(B)(\mathbf{C})\}. \tag{4.4.4}$$

By proper base change, the W-action on $\boldsymbol{s}_*\mathbf{C}$ restricts to an action on $\mathrm{H}^*(\mathcal{B}_\gamma, \mathbf{C})$.

We will also work with the stacky version of the Grothendieck–Springer map, where we quotient by the adjoint action of G on $\mathfrak{g}$:

$$\boldsymbol{s} : \tilde{\mathfrak{g}}/G \to \mathfrak{g}/G. \tag{4.4.5}$$

Let $\mathfrak{b}$ be the Lie algebra of B. Since the map $\tilde{\mathfrak{g}} \to \mathcal{B}$ is a vector bundle with fiber $\mathfrak{b}$ above B, we find that $\tilde{\mathfrak{g}}/G \simeq \mathfrak{b}/B$ as quotient stacks.

4.4.3. The Loop Group Let $\mathcal{L}G$ be the loop group of G. Recall that it is the ind-group over $\mathbf{C}$ defined by the functor

$$\begin{aligned} \mathcal{L}G(R) &= G(\eta \mathbin{\hat{\otimes}} R) \\ &= G(R(\!(\varpi)\!)) \end{aligned} \tag{4.4.6}$$

on $\mathbf{C}$-algebras R, where $R(\!(\varpi)\!) = R[\![\varpi]\!][\varpi^{-1}]$.

Each parabolic subgroup $P \subseteq G$ defines a parahoric subgroup $\mathbf{P} \subseteq \mathcal{L}G$. At the level of $\mathbf{C}$-points, $\mathcal{L}G(\mathbf{C}) = G(\eta)$ and $\mathbf{P}(\mathbf{C})$ is the preimage of $P(\mathbf{C})$ along the map $G(D) \to G(\mathbf{C})$. The corresponding **affine flag variety** is the fpqc sheaf quotient

$$\mathcal{F}l_P = \mathcal{L}G/\mathbf{P}. \tag{4.4.7}$$

We will focus on the cases $P = B$ and $P = G$. The parahoric attached to B will be called an **Iwahori subgroup** of $\mathcal{L}G$ and denoted $\mathbf{I}$. The ind-scheme $\mathcal{F}l_G$ is also known as the **loop** or **affine Grassmannian**.

Remark 4.4.1. If G is simply-connected, then $\mathcal{F}l_B = \mathcal{L}G/\mathbf{I}$ can be identified with the moduli space of Iwahori subgroups of $\mathcal{L}G$. In general, however, $\mathbf{I}$ is not self-normalizing in $\mathcal{L}G$ and this identification fails [115, §2.1.6].

Following [116, Lect. I], we can interpret the affine flag variety $\mathcal{F}l_P$ as a moduli space of decorated G-bundles over the disk D. Recall that $\mathbf{P}$ defines an integral model of $G \otimes \eta$ over D with special fiber P. This fact can be used to show that, for any $\mathbf{C}$-algebra R, the set

$\mathcal{F}l_P(R)$ classifies triples (E, ι, E_0^P) such that:

1. $E \to D \mathbin{\hat{\otimes}} R$ is a G-torsor.
2. ι is a trivialization $G \times (\eta \mathbin{\hat{\otimes}} R) \xrightarrow{\sim} E|_{\eta \hat{\otimes} R}$.
3. E_0^P is a reduction-of-structure-group from G to P on $E|_{\operatorname{Spec} R}$.

On $\mathbf{C}$-points, the sheafification map $\mathcal{L}G/\mathbf{P} \to \mathcal{F}l_P$ sends a coset $g\mathbf{P}$ to the tuple (E, ι, E_0^P) in which E is trivial and $\iota : G \times \eta \to G \times \eta = E|_\eta$ is just multiplication by g. Note that in this language, the map $\mathcal{F}l_P \to \mathcal{F}l_G$ amounts to forgetting the datum E_0^P.

4.4.4. Affine Springer Fibers The main references for this subsection are [90, §5] and [115, Lect. II].

Let $\gamma \in \mathfrak{g}^{\mathrm{rs}}(\eta)$, or equivalently, $\gamma \in \mathfrak{g}(\eta) \cap \mathfrak{g}^{\mathrm{rs}}(\bar{\eta})$. Let $\mathcal{F}l_{P,\gamma}^{\mathrm{unr}}$ be the closed sub-ind-scheme of $\mathcal{F}l_P$ defined by

$$\mathcal{F}l_{P,\gamma}^{\mathrm{unr}}(R) = \{g\mathbf{P} \in \mathcal{F}l_P(R) : \operatorname{Ad}(g^{-1})\gamma \in \operatorname{Lie}(\mathbf{P})(R)\}. \tag{4.4.8}$$

The **(P-type) affine Springer fiber** of γ is the underlying reduced ind-scheme

$$\mathcal{F}l_{P,\gamma} \subseteq \mathcal{F}l_{P,\gamma}^{\mathrm{unr}}. \tag{4.4.9}$$

In [60], Kazhdan–Lusztig showed that $\mathcal{F}l_{P,\gamma}$ forms a (possibly infinite) union of projective varieties of bounded overall dimension.

Let $\mathcal{L}G_\gamma \subseteq \mathcal{L}G$ be the centralizer of γ. It acts on $\mathcal{F}l_{P,\gamma}^{\mathrm{unr}}$, hence also on $\mathcal{F}l_{P,\gamma}$, by left multiplication. Note that we can identify $\mathcal{L}G_\gamma(\mathbf{C})$ with the set of η-points of a torus

$$G_\gamma \subseteq G \times \eta. \tag{4.4.10}$$

The **split rank** of G_γ is the rank of any maximal $\mathbf{C}(\!(\varpi)\!)$-split subtorus of G_γ. We say that

γ is **elliptic** iff the split rank of G_γ equals zero. In this case, Kazhdan–Lusztig showed that $\mathcal{F}l_{P,\gamma}$ is of finite type and projective.

We can interpret the affine Springer fiber $\mathcal{F}l_{P,\gamma}$ in terms of decorated G-bundles, like we did for the affine flag variety. For any $\mathbf{C}$-algebra R and G-torsor $E \to D \hat{\otimes} R$, let

$$\mathfrak{g}_E = E \wedge^G \mathfrak{g}, \tag{4.4.11}$$

the associated bundle over $D \hat{\otimes} R$ with fiber $\mathfrak{g}$. Then the set $\mathcal{F}l_{B,\gamma}(R)$ classifies triples $(E, \iota, E_0^P) \in \mathcal{F}l_P(R)$ such that, in addition to conditions (1)-(3) listed in §4.4.3,

4. ι induces an isomorphism $\iota_{\mathfrak{g}} : \mathfrak{g} \times (\eta \hat{\otimes} R) \xrightarrow{\sim} \mathfrak{g}_E|_{\eta \hat{\otimes} R}$ with the following effect on γ:
   (a) $\iota_{\mathfrak{g}}(\gamma)$ extends to a section of $\mathfrak{g}_E$ over D.
   (b) $\iota_{\mathfrak{g}}(\gamma)$ is compatible with the P-structure on $E|_{\operatorname{Spec} R}$ in the sense that it sends $\operatorname{Spec} R$ into $\mathfrak{p}/P$, where $\mathfrak{p} = \operatorname{Lie}(P)$.

In Section 4.5, we will see that conditions (1)-(4) correspond to similar conditions in the definition of a parabolic Hitchin fiber.

Note that condition (4b) makes it apparent that there is an evaluation map ev_P : $\mathcal{F}l_{P,\gamma} \to \mathfrak{p}/P$ defined by $ev_P(E, \iota, E_0^P) = \iota_{\mathfrak{g}}(\gamma)(0)$ on $\mathbf{C}$-points. In the coset interpretation, it corresponds to $ev_P(g\mathbf{P}) = \operatorname{Ad}(g^{-1})\gamma$.

Let $f = \chi(\gamma) \in \mathfrak{c}^\circ(\eta)$. We claim that the isomorphism class of $\mathcal{F}l_{P,\gamma}$ only depends on f. First, if $\gamma' = \operatorname{Ad}(g)\gamma$ for some $g \in G(\eta)$, then in the coset interpretation, left multiplication by g defines an isomorphism $\mathcal{F}l_{P,\gamma} \xrightarrow{\sim} \mathcal{F}l_{P,\gamma'}$. This shows the isomorphism class only depends on the $G(\eta)$-orbit of γ in $\mathfrak{g}(\eta)$. Using the Galois cohomology of G_γ over $\mathbf{C}(\!(\varpi)\!)$ and the fact that γ is regular semisimple, we can check that the Chevalley fiber above f consists of a single $G(\eta)$-orbit, as claimed.

The Chevalley map admits a section valued in the regular locus, called the **Kostant**

section and denoted $\kappa : \mathfrak{c} \to \mathfrak{g}^{\text{reg}}$. It will be convenient to set

$$\mathcal{F}l_{P,f} = \mathcal{F}l_{P,\kappa(f)}. \tag{4.4.12}$$

As explained in [115, §2.2.8], $\mathcal{F}l_{P,f}$ is nonempty if and only if $f \in \mathfrak{c}(D)$.

The Local Picard Action Ngô showed [87, §2] that there is a commutative, smooth, affine group scheme J over $\mathfrak{c}$, equipped with an $\mathrm{Ad}(G)$-equivariant morphism

$$\chi^* J \to I \tag{4.4.13}$$

of group schemes over $\mathfrak{g}$ that restricts to an isomorphism over $\mathfrak{g}^{\text{reg}}$. Since J is commutative, its classifying stack $\cdot/J$ forms a commutative group stack over $\mathfrak{c}$. Via (4.4.13), $\cdot/J$ acts on $\mathfrak{g}/G$ fiberwise over $\mathfrak{c}$ in such a way that the map $\mathfrak{g}^{\text{reg}}/G \to \mathfrak{c}$ forms a J-gerbe.

Suppose that $f \in \mathfrak{c}(D) \cap \mathfrak{c}^{\text{rs}}(\eta)$. We let J_f be the group scheme over D formed by the pullback of J along f. Then J_f is an integral model of G_γ over D. We let $\mathcal{L}^+ J_f$ be the arc group of J_f, i.e., the scheme

$$\begin{aligned} \mathcal{L}^+ J_f(R) &= J_f(D \mathbin{\hat{\otimes}} R) \\ &= J_a(R[\![\varpi]\!]). \end{aligned} \tag{4.4.14}$$

The **local Picard stack** of f is the fpqc sheaf quotient

$$P_f = \mathcal{L}G_\gamma / \mathcal{L}^+ J_f, \tag{4.4.15}$$

where $\gamma = \kappa(f)$. For any $\mathbf{C}$-algebra R, the groupoid $P_f(R)$ classifies pairs (Q, ι), where $Q \to D \mathbin{\hat{\otimes}} R$ is a J_f-torsor and ι is a trivialization $G_\gamma \times (\eta \mathbin{\hat{\otimes}} R) \xrightarrow{\sim} Q|_{\eta \hat{\otimes} R}$.

Ngô proved [87, §3.3] that the action of $\mathcal{L}G_\gamma$ on $\mathcal{F}l^{\text{unr}}_{P,\gamma}$ factors through P_f. Hence the same is true for $\mathcal{F}l_{P,\gamma}$. In particular, the P_f-action on $\mathcal{F}l_{G,f}$ admits a dense open orbit [87, Lem. 3.3.1, Cor. 3.10.2].

The Local W-Action We now focus on the case where P is a Borel B.

Observe that the map $\mathcal{F}l_{B,f} \to \mathcal{F}l_{G,f}$, which we will denote by $\boldsymbol{s}_f$, is the pullback of the stacky Grothendieck–Springer map. That is, we have a cartesian square:

(4.4.16)
$$\begin{array}{ccc} \mathcal{F}l_{B,f} & \xrightarrow{ev_B} & \mathfrak{b}/B \\ \downarrow{\scriptstyle \boldsymbol{s}_f} & & \downarrow{\scriptstyle \boldsymbol{s}} \\ \mathcal{F}l_{G,f} & \xrightarrow{ev_G} & \mathfrak{g}/G \end{array}$$

By pullback, the W-action on $\boldsymbol{s}_*\mathbf{C}$ induces an action on $\boldsymbol{s}_{f,*}\mathbf{C}$. Therefore, at least in the case where $\mathcal{F}l_{G,f}$ is of finite type, we obtain a W-action on the Betti cohomology $\mathrm{H}^*(\mathcal{F}l_{B,f}, \mathbf{C})$. It commutes with the action of the local Picard stack. (In general, $\mathcal{F}l_{G,f}$ need not be of finite type and the definition of $\mathrm{H}^*(\mathcal{F}l_{B,f}, \mathbf{C})$ requires more care [90, §7], but the action still exists.)

In [75], Lusztig extended this action of W to an action of $\widetilde{W} = W \ltimes \mathbf{X}^\vee$, the extended affine Weyl group of G. However, we will not use the $\mathbf{X}^\vee$-part of $\widetilde{W}$ in this paper.

4.5 Global Springer Theory

This section is largely based on Yun's exposition in [113]. We keep the notation of the previous section.

4.5.1. The Curve Σ Fix a smooth projective curve Σ over $\mathbf{C}$ of genus g. Later, we will allow Σ to develop orbifold structure, in which case g will denote the *coarse* genus of Σ.

Fix a line bundle $L \to \Sigma$. The frame bundle of L, which we denote $L^\times \to \Sigma$, is the $\mathbf{G}_m$-torsor formed by the complement of the zero section in L. Any $\mathbf{C}$-stack X equipped with a $\mathbf{G}_m$-action gives rise to an associated bundle over Σ with fiber X, namely,

$$X_L = X \wedge^{\mathbf{G}_m} L^\times. \tag{4.5.1}$$

Given a $\mathbf{G}_m$-equivariant map $f : Y \to X$, we write f_L to denote the induced map $Y_L \to X_L$. We let $\Gamma(\Sigma, X_L)$ denote the stack that, at the level of points, parametrizes sections of the projection $\Sigma_L \to \Sigma$.

4.5.2. The Hitchin Fibration An **L-twisted Higgs bundle** over Σ is a pair (E, θ) consisting of a G-torsor $E \to \Sigma$ and a section θ of the vector bundle $\mathfrak{g}_E \otimes L$, where

$$\mathfrak{g}_E = E \wedge^G \mathfrak{g}. \tag{4.5.2}$$

The section θ is also known as a **Higgs field**.

The **L-twisted Hitchin moduli space** of Σ is the stack $\mathcal{M} = \mathcal{M}(G, \Sigma, L)$ over $\mathbf{C}$ that, at the level of points, classifies L-twisted G-Higgs pairs over Σ. Equivalently,

$$\mathcal{M} = \Gamma(\Sigma, [\mathfrak{g}/G]_L). \tag{4.5.3}$$

The **Hitchin base** $\mathcal{A} = \mathcal{A}(G, \Sigma, L)$ is

$$\mathcal{A} = \Gamma(\Sigma, \mathfrak{c}_L), \tag{4.5.4}$$

the (weighted) vector space of global sections of $\mathfrak{c}_L \to \Sigma$. Since the Chevalley map $\chi : \mathfrak{g} \to \mathfrak{c}$ transforms the scaling action of $\mathbf{G}_m$ on $\mathfrak{g}$ into the weighted action of $\mathbf{G}_m$ on $\mathfrak{c}$, we get a map $[\mathfrak{g}/G]_L \to \mathfrak{c}_L$. This, in turn, induces a map

$$\boldsymbol{h} : \mathcal{M} \to \mathcal{A} \tag{4.5.5}$$

that we call the **Hitchin fibration**.

Let $ev : \mathcal{M} \times \Sigma \to [\mathfrak{g}/G]_L$ be the evaluation map $ev(E, \theta, v) = \theta(v)$. For any parabolic subgroup $P \subseteq G$ with Lie algebra $\mathfrak{p}$, the associated **parabolic Hitchin moduli space** is the stack $\mathcal{M}_P = \mathcal{M}_P(G, \Sigma, L)$ that we obtain by pulling back $[\mathfrak{p}/P]_L$ along ev. That is, we have a cartesian diagram:

$$\begin{array}{ccc} \mathcal{M}_P & \xrightarrow{ev_P} & [\mathfrak{p}/P]_L \\ \downarrow & & \downarrow \\ \mathcal{M} \times \Sigma & \xrightarrow{ev} & [\mathfrak{g}/G]_L \end{array} \tag{4.5.6}$$

Explicitly, $\mathcal{M}_P$ classifies tuples (E, θ, v, E_v^P), where $(E, \theta, v) \in \mathcal{M} \times \Sigma$ and E_v^P is a reduction-of-structure-group from G to P on the fiber E_v, compatible with the Higgs field θ in the sense that $\theta(v) \in \mathfrak{p}$ in any local trivialization of E around v. The associated **parabolic Hitchin fibration** is the composition

$$\boldsymbol{h}_P : \mathcal{M}_P \to \mathcal{M} \to \mathcal{A}. \tag{4.5.7}$$

Note that, with our notation, $\mathcal{M}_G = \mathcal{M} \times \Sigma$ and $\boldsymbol{h}_G = \boldsymbol{h}$.

The Global Picard Action Recall the group scheme J over $\mathfrak{c}$ from §4.4.4. The $\mathbf{G}_m$-action on $\mathfrak{c}$ induces an action on J, letting us form $[\,\cdot\,/J]_L$.

The **global Picard stack** $\mathcal{P} = \mathcal{P}(G, C, L)$ is

$$\mathcal{P} = \Gamma(C, [\,\cdot\,/J]_L), \tag{4.5.8}$$

the commutative group stack of global sections of $[\,\cdot\,/J]_L \to \Sigma$. Observe that the map $[\,\cdot\,/J]_L \to \mathfrak{c}_L$ induces a map $\mathcal{P} \to \mathcal{A}$. To describe the fibers more explicitly, let $a \in \mathcal{A}$, and let J_a be the group scheme over Σ obtained by pulling back J_L along $a : \Sigma \to \mathfrak{c}_L$. Then $\mathcal{P}_a$ is the stack that classifies J_a-torsors over Σ. For sufficiently generic a, its connected components can be pictured as abelian varieties with stacky structure. Roughly, this happens because the generic fiber of $J \to \mathfrak{c}$ is a torus.

Let $I_P = I_G|_\mathfrak{p}$. Essentially by [113, Lem 2.3.1], the composition $(\chi^* J)|_\mathfrak{p} \to \chi^* I \to I_G$ factors through I_P. Therefore, the $[\,\cdot\,/J]$-action on $\mathfrak{g}/G$ can be lifted to an action on $\mathfrak{p}/P$. We deduce that $[\,\cdot\,/J]_L$ acts on $[\mathfrak{p}/P]_L$ fiberwise over $\mathfrak{c}_L$, and hence, $\mathcal{P} \times \Sigma$ acts on $\mathcal{M}_P$ fiberwise over $\mathcal{A} \times \Sigma$.

Observe that there is a dense open substack $\mathcal{M}^{\mathrm{reg}} \subseteq \mathcal{M}_a$ that classifies Higgs bundles $(E, \theta) : \Sigma \to [\mathfrak{g}/G]_L$ that factor through $[\mathfrak{g}^{\mathrm{reg}}/G]_L$. Since $\mathfrak{g}^{\mathrm{reg}}/G \to \mathfrak{c}$ is a J-gerbe, we deduce that $\mathcal{M}^{\mathrm{reg}} \to \mathcal{A}$ is a $\mathcal{P}$-torsor. As a result, the $\mathcal{P}_a$-action on $\mathcal{M}_a$ admits a dense open orbit for all a. We explain below that for sufficiently generic a, we can view $\mathcal{M}_a$ as a compactification of this dense orbit. (In general, the analogous statement for the $\mathcal{P}_a$-action on $\mathcal{M}_{P,a,v}$ is false [113, Rem. 2.3.5].)

Loci in the Hitchin Base Throughout the rest of this section, *we assume that G is semisimple.*

Recall that the discriminant divisor $\mathfrak{D} \subseteq \mathfrak{c}$ is the branch locus of $\mathfrak{t} \to \mathfrak{c}$, hence stable under the action of $\mathbf{G}_m$. For all $a \in \mathcal{A}$, we set

$$\mathfrak{D}_a = a^{-1}(\mathfrak{D}_L), \tag{4.5.9}$$

analogously to the definition of $\mathfrak{D}_f$ in §4.3.3. The **hyperbolic locus** $\mathcal{A}^\heartsuit \subseteq \mathcal{A}$ is the locus of points a such that $\mathfrak{D}_a$ is a divisor of Σ, i.e., a finite set.

By [87, Prop. 5.2], the group stack $\mathcal{P}_a$ is smooth for all $a \in \mathcal{A}^\heartsuit$. Note that $\pi_0(\mathcal{P}_a)$ is a finitely-generated commutative group. The **anisotropic locus** $\mathcal{A}^{\mathrm{ani}} \subseteq \mathcal{A}^\heartsuit$ is the locus of points a such that $\pi_0(\mathcal{P}_a)$ is finite.

The inclusions $\mathcal{A}^{\mathrm{ani}} \subseteq \mathcal{A}^\heartsuit \subseteq \mathcal{A}$ are open embeddings, essentially by the upper semi-continuity of dimension. Moreover, $\mathcal{A}^\heartsuit$ is dense in $\mathcal{A}$, and if G is semisimple, then $\mathcal{A}^{\mathrm{ani}}$ is nonempty and thus dense in $\mathcal{A}^\heartsuit$ [113, 280]. We set

$$\mathcal{M}_P^\heartsuit = \mathcal{M}_P|_{\mathcal{A}^\heartsuit \times \Sigma}, \tag{4.5.10}$$

$$\boldsymbol{h}_P^\heartsuit = \boldsymbol{h}_P|_{\mathcal{M}_P^\heartsuit}, \tag{4.5.11}$$

$$\mathcal{P}^\heartsuit = \mathcal{P}|_{\mathcal{A}^\heartsuit} \tag{4.5.12}$$

We define $\mathcal{M}_P^{\mathrm{ani}}, \mathcal{P}^{\mathrm{ani}}, \boldsymbol{h}_P^{\mathrm{ani}}$ similarly.

Essentially by [113, Prop. 2.5.1], the stack $\mathcal{M}_P^{\mathrm{ani}}$ is smooth and Deligne–Mumford. By Cor. 2.5.2 of *loc. cit.*, the map $\boldsymbol{h}_P^{\mathrm{ani}} : \mathcal{M}_P^{\mathrm{ani}} \to \mathcal{A}^{\mathrm{ani}} \times \Sigma$ is proper. Taking $P = G$, this explains why we can regard $\mathcal{M}_a$ as a compactification of the $\mathcal{P}_a$-torsor $\mathcal{M}_a^{\mathrm{reg}}$ for $a \in \mathcal{A}^{\mathrm{ani}}$.

Remark 4.5.1. If G is not semisimple, then $\mathcal{A}^{\mathrm{ani}}$ is empty. To fix this, one might introduce the **elliptic locus** $\mathcal{A}^{\mathrm{ell}} \subseteq \mathcal{A}^\heartsuit$, defined as the locus of points a such that $\mathrm{rk}_{\mathbf{Z}}\, \pi_0(\mathcal{P}_a) = \dim Z_G$. We defer its exploration to a future paper.

The Product Formula We review the product formula from [113, §2.4], relating parabolic Hitchin fibers to parahoric affine Springer fibers.

Fix $a \in \mathcal{A}^\heartsuit(\mathbf{C})$. For all $v \in \Sigma(\mathbf{C})$, we write $D_v = \operatorname{Spec} \hat{\mathcal{O}}_{\Sigma,v}$, in accordance with the notation of Section 4.2. After we fix a uniformization $D_v \simeq D$ and a trivialization of the line bundle L in a Zariski neighborhood of D_v, the section $a : \Sigma \to \mathfrak{c}_L$ defines a map:

$$a_v : D \xrightarrow{\sim} D_v \to \mathfrak{c}. \tag{4.5.13}$$

The condition $a \in \mathcal{A}^\heartsuit(\mathbf{C})$ implies that a_v sends η into $\mathfrak{c}^\circ$. Thus, we can define the affine Springer fiber $\mathcal{F}l_{P,a_v}$ and the local Picard stack P_{a_v}. Up to isomorphism, these do not depend on either the uniformization of D_v or the trivialization of L around D_v.

Fix a basepoint $0 \in \Sigma(\mathbf{C})$. The Ngô–Yun product formula [113, 2.4.1] states that there is a homeomorphism of stacks:

$$\left(\mathcal{F}l_{P,a_0} \times \prod_{v\in\mathfrak{D}_a\setminus 0} \mathcal{F}l_{G,a_v}\right) \wedge^{\prod_{v\in\mathfrak{D}_a\cup 0} P^{\mathrm{red}}_{a_0}} \mathcal{P}_a \xrightarrow{\approx} \mathcal{M}_{P,a}. \tag{4.5.14}$$

Above, the operation $\wedge$ is defined via an embedding $\prod_{v\in\mathfrak{D}_a\cup 0} P_{a_v} \to \mathcal{P}_a$. Roughly, this embedding sends a collection of J_{a_v}-torsors $Q_v \to D_v$ to the J_a-torsor over Σ that we get by gluing each of the Q_v to the trivial J_a-torsor over $\Sigma \setminus (\mathfrak{D}_a \cup 0)$, using Beauville–Laszlo descent.

The Global W-Action We now focus on the case where $P = B$.

The map $\mathcal{M}_B^\heartsuit \to \mathcal{M}^\heartsuit \times \Sigma$, which we will denote $\boldsymbol{s}^\heartsuit$, is the pullback of the L-twisted Grothendieck–Springer map $\boldsymbol{s}_L : [\mathfrak{b}/B]_L \to [\mathfrak{g}/G]_L$. Thus we have a commutative diagram:

$$\begin{array}{ccc}
\mathcal{M}_B^\heartsuit & \xrightarrow{ev_B} & [\mathfrak{b}/B]_L \\
\downarrow{\scriptstyle \boldsymbol{s}^\heartsuit} & & \downarrow{\scriptstyle \boldsymbol{s}_L} \\
\mathcal{M}^\heartsuit \times \Sigma & \xrightarrow{ev} & [\mathfrak{g}/G]_L \\
\downarrow{\scriptstyle \boldsymbol{h}^\heartsuit} & & \downarrow{\scriptstyle \chi} \\
\mathcal{A}^\heartsuit \times \Sigma & \longrightarrow & \mathfrak{c}_L
\end{array} \qquad (\boldsymbol{h}_B^\heartsuit : \mathcal{M}_B^\heartsuit \to \mathcal{A}^\heartsuit \times \Sigma) \tag{4.5.15}$$

Since the W-action on $\boldsymbol{s}_*\mathbf{C}$ is equivariant under $\mathbf{G}_m$, it induces an action on the twisted version $\boldsymbol{s}_{L,*}\mathbf{C}$. The top square in the diagram is cartesian, so by pullback, we get a W-action on $\boldsymbol{s}_*^\heartsuit\mathbf{C}$. By pushforward, we arrive at an action on $\boldsymbol{h}_{B,*}^\heartsuit\mathbf{C}$. For the most part, we will only use its restriction to $\boldsymbol{h}_{B,*}^{\mathrm{ani}}\mathbf{C}$.

By pullback we obtain a W-action on $\mathrm{H}^*(\mathcal{M}_{B,a,v}, \mathbf{C})$ for all $(a, v) \in \mathcal{A}^{\mathrm{ani}} \times \Sigma$. As we will see in §4.7.3, this global W-action is compatible with the local W-action on $\mathrm{H}^*(\mathcal{F}l_{B,a_v}, \mathbf{C})$ described in §4.4.4.

To quote Yun [113, 269], parabolic Hitchin fibers "carr[y] a richer symmetry" than finite or affine Springer fibers. In *ibid.*, Yun extended the action of W on $\boldsymbol{h}_{B,*}^{\mathrm{ani}}\mathbf{C}$ to an action of a trigonometric Cherednik algebra. We will only make use of a subalgebra known as the graded or degenerate affine Hecke algebra, introduced by Lusztig in [72]. We defer the discussion of its action to Section 4.10.

4.5.3. A Bigraded W-Character Let $\mathrm{K}_0(W)$ be the Grothendieck ring of virtual characters of W. Abusing notation, *we treat the elements of* $\mathrm{K}_0(W)$ *interchangeably with their underlying complex representations.*

The aim of this subsection is to define, for all $(a, v) \in \mathcal{A}^{\mathrm{ani}}(\mathbf{C})$, the series

$$\mathrm{E}^{\mathbf{P}}(-t, \boldsymbol{q} \mid \mathcal{M}_{B,a,0})_{st} \in \mathrm{K}_0(W)[\boldsymbol{q}, t] \tag{4.5.16}$$

that will be our subject later in the paper.

The Perverse Filtration Let ${}^p\tau_{\leq j}$ and ${}^p\tau_{\geq j}$ be the jth perverse truncation functors, and let ${}^p\mathcal{H}^j = {}^p\tau_{\leq j}{}^p\tau_{\geq j}$, the jth perverse cohomology sheaf.

By the decomposition theorem of [4], $\boldsymbol{h}^{\mathrm{ani}}_{B,*}\mathbf{C}$ is isomorphic to a direct sum of shifts of simple perverse sheaves over $\mathcal{A}^{\mathrm{ani}} \times \Sigma$. Thus the morphism

$${}^p\tau_{\leq j}\boldsymbol{h}^{\mathrm{ani}}_{B,*}\mathbf{C} \to \boldsymbol{h}^{\mathrm{ani}}_{B,*}\mathbf{C} \tag{4.5.17}$$

is a direct-summand inclusion in $\mathsf{D}^b(\mathcal{A}^{\mathrm{ani}} \times \Sigma)$ for all j. It defines a split increasing filtration on $\boldsymbol{h}^{\mathrm{ani}}_{B,*}\mathbf{C}$, known as the **perverse filtration** and denoted $\mathbf{P}_{\leq *}$. It is convenient to shift the filtration numbering so that

$$\mathbf{P}_{\leq j}\boldsymbol{h}^{\mathrm{ani}}_{B,*}\mathbf{C} = {}^p\tau_{\leq j + \dim \mathcal{A} \times C}\boldsymbol{h}^{\mathrm{ani}}_{B,*}\mathbf{C}. \tag{4.5.18}$$

With this convention, we have

$$\mathrm{Gr}^{\mathbf{P}}_j\, \boldsymbol{h}^{\mathrm{ani}}_{B,*}\mathbf{C} = {}^p\mathcal{H}^{j + \dim \mathcal{A} \times C}(\boldsymbol{h}^{\mathrm{ani}}_{B,*}\mathbf{C})[-j - \dim \mathcal{A} \times \Sigma], \tag{4.5.19}$$

so $\mathrm{Gr}^{\mathbf{P}}_j \, \boldsymbol{h}^{\mathrm{ani}}_{B,*}\mathbf{C}$ is a shifted perverse sheaf in perverse degree $j + \dim \mathcal{A} \times \Sigma$ for all j.

The Stable Summand Let $\pi_0(\mathcal{P}/\mathcal{A}^\heartsuit)$ be the maximal étale quotient of the smooth group stack over $\mathcal{A}^\heartsuit$ formed by $\mathcal{P}^\heartsuit$, so that at the level of stalks, we have

$$\pi_0(\mathcal{P}/\mathcal{A}^\heartsuit)_a = \pi_0(\mathcal{P}_a) \tag{4.5.20}$$

for all $a \in \mathcal{A}^\heartsuit$. Then the $\mathcal{P}^\heartsuit|_{\mathcal{A}^\heartsuit \times \Sigma}$-action on $\mathcal{M}^\heartsuit_B$ descends to a $\pi_0(\mathcal{P}/\mathcal{A}^\heartsuit)|_{\mathcal{A}^\heartsuit \times \Sigma}$-action on $\boldsymbol{h}^\heartsuit_{B,!}\mathbf{C}$.

Let $\pi_0(\mathcal{P}/\mathcal{A}^{\mathrm{ani}}) = \pi_0(\mathcal{P}/\mathcal{A}^\heartsuit)|_{\mathcal{A}^{\mathrm{ani}}}$. By the definition of the anisotropic locus, this is a sheaf of finite abelian groups. We have an $\pi_0(\mathcal{P}/\mathcal{A}^{\mathrm{ani}})|_{\mathcal{A}^{\mathrm{ani}} \times \Sigma}$ action on $\boldsymbol{h}^{\mathrm{ani}}_{B,*}\mathbf{C}$. Again using the decomposition theorem [4], we can define the **stable summand**

$$(\boldsymbol{h}^{\mathrm{ani}}_{B,*}\mathbf{C})_{st} \subseteq \boldsymbol{h}^{\mathrm{ani}}_{B,*}\mathbf{C} \tag{4.5.21}$$

to be the maximal direct summand of $\boldsymbol{h}^{\mathrm{ani}}_{B,*}\mathbf{C}$ on which $\pi_0(\mathcal{P}/\mathcal{A}^{\mathrm{ani}})$ acts trivially. Then for all $(a, v) \in \mathcal{A}^{\mathrm{ani}} \times C$, the stalk of $(\boldsymbol{h}^{\mathrm{ani}}_{B,*}\mathbf{C})_{st}$ at (a, v) is the summand

$$\mathrm{H}^*(\mathcal{M}_{B,a,v}, \mathbf{C})_{st} \subseteq \mathrm{H}^*(\mathcal{M}_{B,a,v}, \mathbf{C}) \tag{4.5.22}$$

on which $\pi_0(\mathcal{P}_a)$ acts trivially.

As in §4.4.4, the action of W on $\boldsymbol{h}^{\mathrm{ani}}_{B,*}\mathbf{C}$ commutes with that of $\pi_0(\mathcal{P}/\mathcal{A}^{\mathrm{ani}})|_{\mathcal{A}^{\mathrm{ani}} \times \Sigma}$. Therefore, it restricts to an action on $(\boldsymbol{h}^{\mathrm{ani}}_{B,*}\mathbf{C})_{st}$, hence also $\mathrm{H}^*(\mathcal{M}_{B,a,v}, \mathbf{C})_{st}$.

We again write $\mathbf{P}_{\leq *}$ to denote the restriction of the perverse filtration to $(\boldsymbol{h}^{\mathrm{ani}}_{B,*}\mathbf{C})_{st}$. The W-action on $(\boldsymbol{h}^{\mathrm{ani}}_{B,*}\mathbf{C})_{st}$ is induced by a degree-preserving action on $\boldsymbol{s}^{\mathrm{ani}}_*\mathbf{C}$, so it descends to bidegree-preserving actions on $\mathrm{Gr}^{\mathbf{P}}_* \, (\boldsymbol{h}^{\mathrm{ani}}_{B,*}\mathbf{C})_{st}$ and $\mathrm{Gr}^{\mathbf{P}}_* \, \mathrm{H}^*(\mathcal{M}_{B,a,v}, \mathbf{C})_{st}$.

Remark 4.5.2. Even though the decomposition of $\boldsymbol{h}^{\mathrm{ani}}_{B,*}\mathbf{C}$ into shifted simple perverse sheaves need not be canonical, the stable summand is canonical.

One can consider the more general isotypic decomposition of $\boldsymbol{h}^{\mathrm{ani}}_{B,*}\mathbf{C}$ under $\pi_0(\mathcal{P}/\mathcal{A}^{\mathrm{ani}})$. For $a \in \mathcal{A}^{\mathrm{ani}}$, Ngô's work on the Fundamental Lemma reveals a close relationship between the induced isotypic decomposition of $\mathrm{H}^*(\mathcal{M}_a, \mathbf{C})$ under $\pi_0(\mathcal{P}_a)$ and certain endoscopic groups of G attached to a. Roughly, the summands that correspond to nontrivial characters of $\pi_0(\mathcal{P}_a)$ can be matched with the stable summands of the cohomology of Hitchin fibers for smaller endoscopic groups.

The Weight Filtration The weight filtration on $\mathrm{H}^*(\mathcal{M}_{B,a,0}, \mathbf{C})$ is stable under the action of the finite group $\pi_0(\mathcal{P}_a)$, so it is compatible with the formation of the stable summand. Moreover, it is strictly compatible with the perverse filtration $\mathbf{P}_{\leq *}$ [81, 28]. So at last, we can form:

$$\mathrm{E}^{\mathbf{P}}(-t, \boldsymbol{q} \mid \mathcal{M}_{B,a,0})_{st} = \sum_{i,j,k} (-1)^{i+k} t^k \boldsymbol{q}^j \, \mathrm{Gr}^{\mathbf{W}}_k \, \mathrm{Gr}^{\mathbf{P}}_j \, \mathrm{H}^i(\mathcal{M}_{B,a,0}, \mathbf{C})_{st}. \tag{4.5.23}$$

This is the element of $\mathrm{K}_0(W)[\boldsymbol{q}, t]$ we desired. Our notation is meant to emphasize that it is a *modification* of the virtual weight series of $\mathcal{M}_{B,a,0}$, *cf.* §4.1.6.

Remark 4.5.3. The appearance of the stable summand may seem unmotivated. However, as we indicate in Section 4.10, we need to restrict to it in order to equip a certain action of the degenerate Hecke algebra of W with the right grading.

While this action does not appear in our main conjectures, it appears in the work of Oblomkov–Yun in [90], which we use in Section 4.11, and we will use it to state stronger conjectures in future work. We emphasize that *the evidence we offer in this paper for our conjectures will not detect whether* $(-)_{st}$ *is strictly necessary or not.*

4.6 Synthesis

4.6.1. From $\mathcal{A}$ to B(raids) Let $Br_W/{\sim}$ be the set of conjugacy classes of Br_W. The relation between Hitchin fibers and algebraic Artin braids rests on the following lemma.

Lemma 4.6.1. *There is a map*

$$\begin{aligned} \mathcal{A}^{\heartsuit}(\mathbf{C}) &\to (Br_W/{\sim})^{\Sigma^{\mathrm{an}}} \\ a &\mapsto \{[\beta_{a,v}]\}_{v\in\Sigma^{\mathrm{an}}} \end{aligned} \tag{4.6.1}$$

in which the $[\beta_{a,v}]$ are defined by Lemma 4.2.7. In particular, each $[\beta_{a,v}]$ is algebraic and only depends on $a|_{D_v} : D_v = \operatorname{Spec}\hat{\mathcal{O}}_{\Sigma,v} \to \mathfrak{c}_L$.

Proof. Fix $a \in \mathcal{A}^{\heartsuit}(\mathbf{C})$ and $v \in \Sigma(\mathbf{C})$. We recall how $a : \Sigma \to \mathfrak{c}_L$ gives rise to $a_v : D \to \mathfrak{c}$, recapitulating §4.5.2 but keeping track of more detail.

Pick a Zariski open $\Sigma' \subseteq \Sigma$ containing v such that L is trivializable over Σ'. After we choose the trivialization of L, the section $a|_{\Sigma'} : \Sigma' \to \mathfrak{c}_L|_{\Sigma'}$ becomes a map $a' : \Sigma' \to \mathfrak{c}$. So after choosing a uniformization $D_v \simeq D$, we get a composition

$$a_v : D \xrightarrow{\sim} D_v \to \Sigma' \xrightarrow{a'} \mathfrak{c}. \tag{4.6.2}$$

Again, the condition $a \in \mathcal{A}^{\heartsuit}$ ensures that a_v sends η into $\mathfrak{c}^{\circ}$. Therefore, a_v gives rise to a $\mathfrak{c}$-algebraic étale loop $[\beta^{\text{ét}}_{a,v}] \subseteq \pi_1^{\text{ét}}(\mathfrak{c}^{\circ})$ and (Σ', v) forms a model of finite type for a_v. By Lemma 4.2.7, there is a distinguished loop

$$[\beta_{a,v}] \subseteq \pi_1((\mathfrak{c}^{\circ})^{\mathrm{an}}) \simeq Br_W \tag{4.6.3}$$

that maps into $[\beta^{\text{ét}}_{a,v}]$ and only depends on a_v.

We know that $[\beta_{a,v}]$ does not depend on the uniformization of D_v, because any two uniformizers differ by an automorphism of the disk D, which induces the identity morphism on $\pi_1^{\text{ét}}(D)$. It remains to check that $[\beta_{a,v}]$ does not depend on the trivialization of L.

Any two trivalizations differ by pointwise multiplication by a nowhere-zero function $g : \Sigma' \to \mathbf{G}_m$. If multiplication by g changes $[\beta_{a,v}]$, then for some small analytic disk $D^{\mathrm{an}} \subseteq (\Sigma')^{\mathrm{an}}$ centered at v, the map $g|_{D^{\mathrm{an}}\setminus v} : D^{\mathrm{an}} \setminus v \to \mathbf{C}^{\times}$ must have nontrivial winding number. But g first restricts to a map $D^{\mathrm{an}} \to \mathbf{C}^\times$ and we know that D^{an} is simply-connected, so this cannot happen. □

4.6.2. Delta Invariants As motivation for the conjectures to come, we show how the local and global δ-invariants defined by Ngô [87, §3.7, §4.9] can be expressed using braid-theoretic data via Lemmata 4.2.7 and 4.6.1. Recall that:

1. For all $f : (D, \eta) \to (\mathfrak{c}, \mathfrak{c}^\circ)$, the **local δ-invariant** of f is

$$\delta(f) = \dim \mathcal{F}l_{G,f} = \dim P_f. \tag{4.6.4}$$

2. For all $a \in \mathcal{A}^\heartsuit$, the **global δ-invariant** of a is

$$\delta(a) = \dim \mathcal{R}_a, \tag{4.6.5}$$

   where we write $\mathcal{R}_a$ for the maximal affine part of $\mathcal{P}_a$.

There are explicit formulæ for (1) and (2), respectively due to Bezrukavnikov [10] (see also [87, Prop. 3.7.5]) and Ngô [87, §4.8]. When $G = \mathrm{SL}_n$, these invariants recover the classical δ-invariants defined by Serre for algebraic curves and their germs, as we briefly discuss in Section 4.8.

We first discuss the local δ-invariant. Consider a map $f : (D, \eta) \to (\mathfrak{c}, \mathfrak{c}^\circ)$. Following Ngô, let the **cameral cover** $\tilde{D}_f \to D$ be the pullback of $\mathfrak{t} \to \mathfrak{c}$ along f. The monodromy of $\tilde{D}_f|_\eta \to \eta$ defines a morphism

$$\hat{\mathbf{Z}} \simeq \pi_1^{\text{ét}}(\eta, \bar{\eta}) \to W. \tag{4.6.6}$$

In what follows, recall that r is the rank of G and $r(w) = \dim (\mathbf{X}_{\mathbf{C}})^w$.

Theorem 4.6.2 (Bezrukavnikov)**.** *For all* $f : (D, \eta) \to (\mathfrak{c}, \mathfrak{c}^\circ)$*, we have*

$$2\delta(f) = \operatorname{val}_\varpi \mathfrak{D}_f - r + r(w_f), \tag{4.6.7}$$

where $\mathfrak{D}_f = f^{-1}(\mathfrak{D})$ *(see Section 4.3) and* $w_a \in W$ *denotes the image of* $1 \in \hat{\mathbf{Z}}$ *under* (4.6.6)*.*

Corollary 4.6.3. *If* $f : D \to \mathfrak{c}$ *is transverse to or does not intersect* $\mathfrak{D}$*, then* $\delta(f) = 0$*.*

Proof of Corollary 4.6.3. If f is transverse to $\mathfrak{D}$, then we have $\operatorname{val}_\varpi \mathfrak{D}_f = r - r(w_f) = 1$. If instead f does not intersect $\mathfrak{D}$, then we have $\operatorname{val}_\varpi \mathfrak{D}_f = r - r(w_f) = 0$. □

We note that (4.6.6) factors through the morphism $\pi_1^{\text{ét}}(\eta, \bar{\eta}) \to \pi_1^{\text{ét}}(\mathfrak{c}^\circ)$ induced by $f|_\eta$. We further note that if f admits a model of finite type, then $\operatorname{val}_\varpi \mathfrak{D}_f = |\beta_f|$ by Theorem 4.3.12. So in this case, Bezrukavnikov's formula can be expressed purely in terms of $[\beta_f]$:

Theorem 4.6.4. *If* $f : (D, \eta) \to (\mathfrak{c}, \mathfrak{c}^\circ)$ *admits a model of finite type, then we have*

$$2\delta(f) = |\beta_f| - r + r(w_f), \tag{4.6.8}$$

where $[w_f] \subseteq W$ *denotes the image of* $[\beta_f] \subseteq Br_W$*. In particular,* $\delta(f)$ *only depends on* $[\beta_f]$*.*

Remark 4.6.5. If $\gamma \in \mathfrak{g}(D) \cap \mathfrak{g}(\bar{\eta})$ lifts f, then the split rank of G_γ equals $r(w_f)$. In particular, γ is elliptic if and only if $r(w_f) = 0$.

We now turn to the global δ-invariant.

Proposition 4.6.6 (Ngô)**.** *For all* $a \in \mathcal{A}^\heartsuit(\mathbf{C})$*, we have*

$$\delta(a) = \sum_{v \in \Sigma(\mathbf{C})} \delta(a_v). \tag{4.6.9}$$

Note that by Corollary 4.6.3, the sum on the right can be restricted to $\mathfrak{D}_a = a^{-1}(\mathfrak{D}_L)$*.*

Corollary 4.6.7. *For all $a \in \mathcal{A}^{\heartsuit}(\mathbf{C})$, we have*

$$2\delta(a) = \sum_{v \in \mathfrak{D}_a} (|\beta_{a,v}| - r + r(w_{a,v})), \tag{4.6.10}$$

where $[w_{a,v}] \subseteq W$ denotes the image of $[\beta_{a,v}] \subseteq Br_W$. In particular, $\delta(a)$ only depends on the collection $\{[\beta_{a,v}]\}_{v \in \Sigma^{\mathrm{an}}}$.

The following formulæ of Ngô are Prop. 4.9.5 and Cor. 4.13.3 of [87].

Proposition 4.6.8 (Ngô)**.** *For all $a \in \mathcal{A}^{\heartsuit}(\mathbf{C})$, we have*

$$2\delta(a) = |\Phi| \deg L - \sum_{v \in \mathfrak{D}_a} (r - r(w_{a,v})), \tag{4.6.11}$$

$$2 \dim \mathcal{P}_a = |\Phi| \deg L - r\chi(\Sigma^{\mathrm{an}}), \tag{4.6.12}$$

where χ denotes Euler characteristic (so that $\chi(\Sigma^{\mathrm{an}}) = 2 - 2g$).

Note that if G is semisimple, then $|\Phi|$ equals the writhe of the full twist $\pi \in Br_W$ (see Section 4.3), via $|\Phi| = 2|w_0| = |\pi|$. Combining Corollary 4.6.7 with the first formula of Proposition 4.6.8, we deduce:

Corollary 4.6.9. *For all $a \in \mathcal{A}^{\heartsuit}(\mathbf{C})$, we have*

$$\sum_{v \in \mathfrak{D}_a} |\beta_{a,v}| = |\pi| \deg L. \tag{4.6.13}$$

In particular, the sum on the left only depends on W and L.

4.6.3. Interlude: Orbifold Curves Up to this point, Σ has been a smooth projective curve, a fortiori a scheme. However, for the purposes of Section 4.11, we adapt the ideas of the previous two subsections to a setting where Σ has stacky structure. The exposition that follows is based on §1.3.6 of Behrend's article [3].

An **orbifold curve** is a 1-dimensional, reduced, separated Deligne–Mumford $\mathbf{C}$-stack of finite type that is generically schematic. If Σ is an orbifold curve, then by hypothesis, the set of stacky points of Σ is finite. If $v \in \Sigma(\mathbf{C})$ is a nonsingular stacky point, then étale-locally around x, we can model Σ as the stack quotient of a smooth curve by a finite cyclic group. Ultimately, one can check that any *smooth* orbifold curve arises as follows: Given a pair $(\Sigma^\dagger, \Delta)$, where $\Sigma^\dagger$ is a smooth (non-orbifold) curve and Δ is a divisor on $\Sigma^\dagger$, there is an orbifold curve Σ with coarse space $\Sigma^\dagger$ such that for all $v \in \Sigma(\mathbf{C})$, the automorphism group at v is cyclic of order $1 + \mathrm{ord}_v(\Delta)$.

Each divisor on Σ corresponds to a $\mathbf{Q}$-linear divisor on $\Sigma^\dagger$, say $\sum_i a_i v_i$, that satisfies $a_i|\mathrm{Aut}(v_i)| \in \mathbf{Z}$ for all i. Conversely, each such $\mathbf{Q}$-linear divisor on $\Sigma^\dagger$ lifts to a $\mathbf{Z}$-linear divisor on Σ. Thus, there is a degree map of the form

$$\deg : \mathrm{Pic}(\Sigma) \to \tfrac{1}{n}\mathbf{Z}, \tag{4.6.14}$$

where we can take n to be any common multiple of the numbers $1 + \mathrm{ord}_v(\Delta)$ for $v \in \Delta$. For any line bundle $L \to \Sigma$, we have the following version of Riemann–Roch:

$$\chi(\Sigma, L) = \lfloor \deg L \rfloor + 1 - g, \tag{4.6.15}$$

where g denotes the genus of $\Sigma^\dagger$ [3, Cor. 1.189].

Below, we explain how Lemma 4.6.1, Theorem 4.3.12, and Proposition 4.2.12 generalize to orbifold curves. For brevity, we discuss everything in the global setting only.

Suppose that $v \in \Sigma(\mathbf{C})$ is a stacky point and that $n = |\mathrm{Aut}(v)|$. Slightly abusing notation, we set $D_v = \mathrm{Spec}\,\hat{\mathcal{O}}_{\Sigma,v}$ and $\eta_v = D_v \setminus 0$. Then

$$D_v \simeq D/\mu_n, \tag{4.6.16}$$

$$\eta_v \simeq \eta/\mu_n. \tag{4.6.17}$$

Since $\eta = \operatorname{Spec}\mathbf{C}(\!(\varpi)\!)$ is an n-fold étale cover of η_v, we can fix an identification $\eta_v \simeq \operatorname{Spec}\mathbf{C}(\!(\varpi^n)\!)$. Then the isomorphism $\mathbf{C}(\!(\varpi)\!) \simeq \mathbf{C}(\!(\varpi^n)\!)$ induces an isomorphism

$$\eta \simeq \eta_v. \tag{4.6.18}$$

This does *not* extend to an isomorphism between D and D_v. Nonetheless, for all $a \in \mathcal{A}^{\heartsuit}$, we can use (4.6.18) to obtain a map $a_v : \eta \to \mathfrak{c}^{\circ}$ as in the proof of Lemma 4.6.1. Since the construction of the algebraic braid $[\beta_{a,v}]$ in Lemma 4.2.7 only depends on a *punctured* analytic disk at ∞, it too remains well-defined.

To generalize Theorem 4.3.12, observe that $\operatorname{ord}_{\varpi^n}(-) = \frac{1}{n}\operatorname{ord}_{\varpi}(-)$. This implies

$$|\beta_{a,v}| = \tfrac{1}{n}|\beta_{a,v}^n| = \tfrac{1}{n}\operatorname{ord}_{\varpi}(\mathfrak{D}_{a_v}) = \operatorname{ord}_{\varpi^n}(\mathfrak{D}_{a_v}), \tag{4.6.19}$$

so the formula in the theorem extends to orbifold curves. We deduce that Theorem 4.6.4 and Corollaries 4.6.7 and 4.6.9 also extend with their formulæ unchanged.

Finally, we extend the notion of homogeneity to the stacky setting. Observe that with respect to the coordinate $\eta_v \simeq \operatorname{Spec}\mathbf{C}(\!(\varpi^n)\!)$, the $\mathbf{G}_m$-action on η_v is given by

$$s \cdot \varpi^n = s^n \varpi^n. \tag{4.6.20}$$

Accordingly, the $(\mathbf{G}_m \times \mathbf{G}_m)$-action on $a_v \in \mathfrak{c}^{\circ}(\eta_v)$ is given by

$$(s,t) \cdot f(\varpi^n) = t \cdot f(s^n \varpi^n). \tag{4.6.21}$$

Defining $\mathbf{G}_m(\nu)$ as the same subtorus of $\mathbf{G}_m \times \mathbf{G}_m$ as in §4.2.4, we get the following version of Proposition 4.2.12:

Proposition 4.6.10. *Let $v \in \Sigma(\mathbf{C})$ be a stacky point, and let $n = |\mathrm{Aut}(v)|$. If $a_v : \eta_v \to \mathfrak{c}^{\circ}$ is a fixed point of the $\mathbf{G}_m(\nu)$-action on $\mathfrak{c}^{\circ}(\eta_v)$, then $[\beta_{a,v}]$ is fractional of slope $\frac{\nu}{n}$.*

The second formula of Proposition 4.6.8 does not extend unchanged to orbifold curves, because its proof relies on Riemann–Roch. To state the correction, let $d_1, \ldots, d_r$ denote the weights of the $\mathbf{G}_m$-action on $\mathfrak{t}$.

Proposition 4.6.11. *For a general orbifold curve Σ and $a \in \mathcal{A}^{\heartsuit}(\mathbf{C})$, we have*

$$2 \dim \mathcal{P}_a = |\Phi| \deg L - 2 \left(-rg + \sum_i \varrho((d_i - 1) \deg L) \right), \tag{4.6.22}$$

where g is the genus of $\Sigma^\dagger$ and

$$\varrho(\alpha) = \begin{cases} 1 & \alpha \in \mathbf{Z} \\ \alpha - \lfloor \alpha \rfloor & \alpha \notin \mathbf{Z} \end{cases} \tag{4.6.23}$$

for all $\alpha \in \mathbf{R}$.

Proof. We have $\dim \mathcal{P}_a = -\chi(\Sigma, \mathrm{Lie}(J_a)) = -\chi(\Sigma, \mathfrak{c}_L^\vee \otimes L))$, so by (4.6.15),

$$\begin{aligned} \dim \mathcal{P}_a &= \sum_{1 \le i \le r} -\chi(\Sigma, \mathcal{O}(-(d_i - 1) \deg L)) \\ &= rg + \sum_{1 \le i \le r} (-\lfloor -(d_i - 1) \deg L \rfloor - 1). \end{aligned} \tag{4.6.24}$$

Let n be the common denominator of the numbers $(d_i - 1) \deg L \in \mathbf{Q}$. To finish, we invoke the following identities that hold for all $\alpha \in \frac{1}{n}\mathbf{Z}$:

$$-\lfloor -\alpha \rfloor - 1 = \lfloor \alpha - \tfrac{1}{n} \rfloor, \tag{4.6.25}$$

$$\alpha - \lfloor \alpha - \tfrac{1}{n} \rfloor = \varrho(\alpha). \tag{4.6.26}$$

Indeed, it is a formula of Kostant [87, 79] that $\sum_i (d_i - 1) = \frac{1}{2}|\Phi|$. □

Remark 4.6.12. This proof was inspired by the author's attempts to parse [90, Prop. 6.5.2].

If Σ is not stacky, then the sum in (4.6.22) simplifies to r, recovering the second formula of Proposition 4.6.8. In Section 4.11, we will encounter a situation where Σ is stacky, yet (4.6.22) still simplifies drastically. Until then, it is convenient to set

$$\varrho(G, \Sigma, L) = r - \sum_i \varrho((d_i - 1) \deg L) \geq 0, \tag{4.6.27}$$

so that $2 \dim \mathcal{P}_a = |\Phi| \deg L - r\chi(\Sigma^{\mathrm{an}}) + 2\varrho(G, \Sigma, L)$.

4.7 The Main Conjectures

4.7.1. Annular Braid Homology Let $\mathrm{K}_0^+(W) \subseteq \mathrm{K}_0(W)$ denote the semiring of actual, not virtual, characters. Let $(-,-)_W$ be the $\mathbf{Z}$-valued pairing on $\mathrm{K}_0(W)$ defined by

$$(\phi, \psi)_W = \frac{1}{|W|} \sum_{w \in W} \phi(w)\psi(w^{-1}). \tag{4.7.1}$$

For any ring R, we extend $(-,-)_W$ to $\mathrm{K}_0(W) \otimes R$ by linearity. We write 1 for the trivial character of W, so that $(1,-)_W$ is the functor of W-invariants.

We recall from [105] that for all $\beta \in Br_W$, the **annular braid homology** of β can be viewed as a bivariate series:

$$\mathrm{AH}(\beta) = \bigoplus_{j,k} \boldsymbol{q}^{\frac{j}{2}} t^k \mathrm{AH}_{j,k}(\beta) \in \mathrm{K}_0^+(W)[\![\boldsymbol{q}^{\frac{1}{2}}]\!][t]. \tag{4.7.2}$$

It only depends on the conjugacy class $[\beta] \in Br_W/{\sim}$. As in the introduction, we set

$$\mathfrak{A}\mathrm{H}(\beta) = \left(\frac{1 - \boldsymbol{q}}{1 + \boldsymbol{q}t}\right)^{r(w)} \mathrm{AH}(\beta), \tag{4.7.3}$$

where $w \in W$ is the image of β. Since $w \mapsto r(w)$ is a class function on W, we again find that $\mathfrak{A}\mathrm{H}(\beta)$ only depends on $[\beta]$. The **annular character** of β is

$$\textsc{ann}(\beta) = \mathrm{AH}(\beta)|_{t \to -1} = \mathfrak{A}\mathrm{H}(\beta)|_{t \to -1}. \tag{4.7.4}$$

We refer to [105] for the technical definition of AH, which uses the homological algebra of mixed perverse sheaves on $\mathcal{B} \times \mathcal{B}$ and G. In [106], we proved a simpler, more manifestly geometric formula for $\textsc{ann}$ in terms of virtual weight polynomials, which will be restated in Section 4.9. In that paper, we also conjectured a similar formula for AH. Combining results proved in [105, §4] and [106, §5], we have the following constraint on $\textsc{ann}$:

Theorem 4.7.1. *For all $\beta \in Br_W$, the annular character $\textsc{ann}(\beta)$ is a rational function of $\boldsymbol{q}$ of degree $|\beta| - r$ such that $(1, \textsc{ann}(\beta))|_{\boldsymbol{q}\to 0} = 1$. Moreover,*

$$\textsc{ann}(\beta) \in \frac{1}{(1-\boldsymbol{q})^{r(w)}} \mathrm{K}_0(W)[\boldsymbol{q}] \cap (\boldsymbol{q}^{\frac{1}{2}})^{|\beta|-r} \mathrm{K}_0(W)(\boldsymbol{q}^{\frac{1}{2}} - \boldsymbol{q}^{-\frac{1}{2}}), \tag{4.7.5}$$

where $w \in W$ is the image of β.

4.7.2. Statements Fix an orbifold curve Σ and a basepoint $0 \in \Sigma(\mathbf{C})$. To ease notation, let

$$\tilde{\boldsymbol{h}} : \tilde{\mathcal{M}} \to \mathcal{A} \tag{4.7.6}$$

be the pullback of $\boldsymbol{h} : \mathcal{M}_B \to \mathcal{A} \times \Sigma$ to 0. Thus, for all $a \in \mathcal{A}^{\mathrm{ani}}(\mathbf{C})$, we write

$$\tilde{\mathcal{M}}_a = \mathcal{M}_{B,a,0}, \tag{4.7.7}$$

$$\mathrm{E}^{\mathbf{P}}(-t, \boldsymbol{q} \mid \tilde{\mathcal{M}}_a)_{st} = \mathrm{E}^{\mathbf{P}}(-t, \boldsymbol{q} \mid \mathcal{M}_{B,a,0})_{st}, \tag{4.7.8}$$

where $\mathrm{E}^{\mathbf{P}}(-t, \boldsymbol{q} \mid \mathcal{M}_{B,a,0})_{st} \in \mathrm{K}_0(W)[\boldsymbol{q}, t]$ was defined in §4.5.3. We let

$$U_a = \Sigma \setminus (\mathfrak{D}_a \cup 0), \tag{4.7.9}$$

a dense open of Σ.

Conjecture A. *For all* $a \in \mathcal{A}^{\mathrm{ani}}(\mathbf{C})$*, we have*

$$\mathrm{E}^{\mathbf{P}}(-t, \boldsymbol{q} \mid \tilde{\mathcal{M}}_a)_{st} = (1+\boldsymbol{q}t)^{N_a} \cdot \mathfrak{A}\mathrm{H}(\beta_{a,0}) \cdot \prod_{v \in \mathfrak{D}_a \setminus 0} (1, \mathfrak{A}\mathrm{H}(\beta_{a,v}))_W \tag{4.7.10}$$

in $\mathrm{K}_0(W)[\boldsymbol{q}, t]$*, where* $N_a = 2\varrho(G, \Sigma, L) - r\chi(U_a^{\mathrm{an}})$*.*

In the introduction, we stated Conjecture A for non-stacky curves in an abstract form that did not involve $\mathfrak{D}_a$ or U_a explicitly. To present it, we need a multiplicative version of Lebesgue integration over $\Sigma \setminus 0$ with respect to Euler characteristic.

Definition 4.7.2. Let X be a $\mathbf{C}$-scheme of finite type and R a commutative ring. For any constructible function $f : X^{\mathrm{an}} \to R$, we define the **product integral** of f over X^{an} with respect to the Lebesgue measure $d\chi$ to be:

$$\prod_{x \in X^{\mathrm{an}}} f(x)^{d\chi(x)} := \prod_{\lambda \in R} \lambda^{\chi(f^{-1}(\lambda))} \tag{4.7.11}$$

The right-hand product is finite because f is constructible and X is of finite type.

Observe that for fixed $a \in \mathcal{A}^{\heartsuit}(\mathbf{C})$, the function

$$\begin{aligned} \Sigma^{\mathrm{an}} &\to \mathrm{K}_0(W)[\![\boldsymbol{q}^{\frac{1}{2}}, t]\!] \\ v &\mapsto \mathfrak{A}\mathrm{H}(\beta_{a,v}) \end{aligned} \tag{4.7.12}$$

is constructible because it only changes value at the points of $\mathfrak{D}_a$. As for the points of U_a:

Lemma 4.7.3. *If* $v \notin \mathfrak{D}_a$*, then* $\beta_{a,v} = 1$*, the identity of* Br_W*. Moreover,*

$$\mathfrak{A}\mathrm{H}(1) = \frac{1}{(1+\boldsymbol{q}t)^r}\, \mathbf{C}[W], \tag{4.7.13}$$

where $\mathbf{C}[W]$ *denotes the regular representation of* W*.*

Proof. If $v \notin \mathfrak{D}_a$, then the map a_v sends D, not just η, into $\mathfrak{c}^\circ$. This implies $\beta_{a,v} = 1$. The claim about $\mathfrak{A}\mathrm{H}(1)$ follows from the calculation of AH(1) in [105, §7]. □

Lemma 4.7.3 implies that in the non-stacky case, Conjecture A specializes to the following simple "integral formula."

Conjecture B. *If Σ is non-stacky, then for all $a \in \mathcal{A}^{\mathrm{ani}}(\mathbf{C})$, we have*

$$\mathrm{E}^{\mathbf{P}}(-t, \boldsymbol{q} \mid \tilde{\mathcal{M}}_a)_{st} = \mathfrak{A}\mathrm{H}(\beta_{a,0}) \cdot \prod_{v \in \Sigma_a^{\mathrm{an}} \setminus 0} (1, \mathfrak{A}\mathrm{H}(\beta_{a,v}))_W^{d\chi(v)} \tag{4.7.14}$$

in $\mathrm{K}_0(W)[\boldsymbol{q}, t]$.

Remark 4.7.4 (Dimension)*.* By relative perverse hard Lefschetz (*cf.* [81, §2.16]), the top perverse degree of $\mathrm{Gr}_*^{\mathbf{P}} \mathrm{H}^*(\tilde{\mathcal{M}}_a)_{st}$ coincides with the top cohomological degree, which is twice the dimension of $\tilde{\mathcal{M}}_a$. Combining this fact with Theorem 4.7.1 and Corollary 4.6.9, we see that Conjecture A implicitly asserts:

$$\begin{aligned} 2 \dim \tilde{\mathcal{M}}_a &= 2\varrho(G, \Sigma, L) - r\chi(U_a^{\mathrm{an}}) + \sum_{v \in \mathfrak{D}_a \cup 0} (|\beta_{a,v}| - r) \\ &= 2\varrho(G, \Sigma, L) - r\chi(\Sigma^{\mathrm{an}}) + |\Phi| \deg L. \end{aligned} \tag{4.7.15}$$

At the same time, by [113, Cor. 2.4.2],

$$\dim \tilde{\mathcal{M}}_a = \dim \mathcal{M}_a = \dim \mathcal{P}_a. \tag{4.7.16}$$

In this way, Conjecture A recovers Proposition 4.6.11. Similarly, Conjecture B recovers the second formula of Proposition 4.6.8.

Remark 4.7.5 (Symmetry)*.* The relative perverse hard Lefschetz theorem ensures that the underlying bigraded dimension of

$$\boldsymbol{q}^{-\dim \tilde{\mathcal{M}}_a} \mathrm{E}^{\mathbf{P}}(-t, \boldsymbol{q} \mid \tilde{\mathcal{M}}_a)_{st} \tag{4.7.17}$$

is invariant under the substitution $\boldsymbol{q} \mapsto \boldsymbol{q}^{-1} t^{-2}$. We speculate that this symmetry respects the $\mathrm{K}_0(W)$-structure as well.

In [106], we proposed a similar conjecture for annular braid homology, stating that if $w \in W$ is the image of $\beta \in Br_W$, then

$$(\boldsymbol{q}^{-\frac{1}{2}})^{|\beta|-r+r(w)}(1-\boldsymbol{q})^{r(w)}\mathrm{AH}(\beta) \tag{4.7.18}$$

is invariant under $\boldsymbol{q}^{\frac{1}{2}} \mapsto \boldsymbol{q}^{-\frac{1}{2}}t^{-1}$. Theorem 4.7.1 shows that the $t \to -1$ limit holds: That is,

$$(\boldsymbol{q}^{-\frac{1}{2}})^{|\beta|-r}\textsc{ann}(\beta) \tag{4.7.19}$$

is invariant under $\boldsymbol{q}^{\frac{1}{2}} \mapsto -\boldsymbol{q}^{-\frac{1}{2}}$.

Remark 4.7.6. Taking W-invariants on both sides of Conjecture B, we get a non-parabolic or "spherical" version with $\mathcal{M}_a$ in place of $\tilde{\mathcal{M}}_a$, stating that

$$\mathrm{E}^{\mathbf{P}}(-t, \boldsymbol{q} \mid \tilde{\mathcal{M}}_a)_{st} = \prod_{v \in \Sigma_a^{\mathrm{an}}} (1, \mathfrak{A}\mathrm{H}(\beta_{a,v}))_W^{d\chi(v)} \tag{4.7.20}$$

in $\mathbf{Z}[\![\boldsymbol{q}^{\frac{1}{2}}, t]\!]$. Here, no basepoint needs to be chosen on Σ.

Remark 4.7.7. When we form the product integral of a function $f : X^{\mathrm{an}} \to R$, we only use the multiplicative structure of R, not its additive structure. This suggests that we should only define the product integral when we know that f takes values in $R^\times$.

Accordingly, we expect to show in forthcoming work that $\mathfrak{A}\mathrm{H}(\beta) \in \mathrm{K}_0^+(W)[\![\boldsymbol{q}^{\frac{1}{2}}, t]\!]^\times$ for all $\beta \in Br_W$. In particular, we expect that $(1, \mathfrak{A}\mathrm{H}(\beta))_W|_{\boldsymbol{q} \to 0} = 1$, generalizing the analogous statement for $\textsc{ann}(\beta)$ in Theorem 4.7.1.

4.7.3. Special Cases We enunciate two situations where the collection of Artin braids $\{\beta_{a,v}\}_v$ is particularly simple.

Let $\mathcal{A}^{\diamondsuit_0} \subseteq \mathcal{A}^{\mathrm{ani}}$ be the (open) locus of points a such that for all $v \neq 0$, the map $a_v : D \to \mathfrak{c}$ is either transverse to, or does not intersect, $\mathfrak{D}$. By an argument similar to the proof of [87, Prop. 4.7.1], we can show that:

Lemma 4.7.8. *If* $\deg L > 2g$*, then* $\mathcal{A}^{\diamondsuit_0}$ *is nonempty, hence dense in* $\mathcal{A}^{\mathrm{ani}}$*.*

Remark 4.7.9. The locus $\mathcal{A}^{\diamondsuit_0}$ is different from the locus $\mathcal{A}^{\diamondsuit} \subseteq \mathcal{A}^{\mathrm{ani}}$ introduced by Ngô in [87, §4.7], only because the defining condition of $\mathcal{A}^{\diamondsuit_0}$ excludes the basepoint 0.

Lemma 4.3.11 shows that if $a \in \mathcal{A}^{\diamondsuit_0}(\mathbf{C})$, then for each $v \in \mathfrak{D}_a$, we have some $s_v \in S$ such that $[\beta_{a,v}] = [\beta_{s_v}]$. We can calculate $(1, \mathrm{AH}(\beta_s))_W$, and hence $(1, \mathfrak{A}\mathrm{H}(\beta_s))_W$, from the definition of AH in [105]: more precisely, from the explicit description of the sheaf-theoretic Rouquier complex of β_s and the comparison between annular braid homology and Khovanov–Rozansky homology. Below, we just state the formula.

Lemma 4.7.10. *We have*

$$(1, \mathfrak{A}\mathrm{H}(\beta_s))_W = \frac{1}{(1+\boldsymbol{q}t)^{r-1}} \tag{4.7.21}$$

for all $s \in S$*.*

Plugging Lemma 4.7.10 directly into Conjecture A leads us to:

Conjecture C. *If* $a \in \mathcal{A}^{\diamondsuit_0}(\mathbf{C})$*, then*

$$\mathrm{E}^{\mathbf{P}}(-t, \boldsymbol{q} \mid \tilde{\mathcal{M}}_a)_{st} = (1+\boldsymbol{q}t)^{N_a} \cdot \mathfrak{A}\mathrm{H}(\beta_{a,0}) \tag{4.7.22}$$

in $\mathrm{K}_0(W)[\boldsymbol{q}, t]$*, where* $N_a = 2\varrho(G, \Sigma, L) - r\chi(U_a^{\mathrm{an}}) - (r-1)|\mathfrak{D}_a \setminus 0|$*.*

Another useful situation occurs when we can cleanly compare Hitchin fibers to affine Springer fibers. For this, we want

$$\dim P_{a_0} = \dim \mathcal{R}_a = \dim \mathcal{P}_a. \tag{4.7.23}$$

By Propositions 4.6.6 and 4.6.11, the above is equivalent to requiring:

1. $\sum_{v \in \mathfrak{D}_a} (r - r(w_{a,v})) = r\chi(\Sigma^{\text{an}}) - 2\varrho(G, \Sigma, L)$.

2. $\delta(a_v) = 0$ for all $v \neq 0$.

The left-hand side of (1) is nonnegative, which forces Σ to have genus 0 or 1. The genus-0 case is the interesting case. Below, we will also assume that $r(w_{a,0}) = 0$.

Definition 4.7.11. Suppose that Σ is of genus 0, i.e., Σ is a projective line possibly with orbifold points. Then we say that $a \in \mathcal{A}^{\text{ani}}(\mathbf{C})$ is **tidy** iff:

1. $r(w_{a,0}) = 0$. In particular, $0 \in \mathfrak{D}_a$.

2. $\dim P_{a_0} = \dim \mathcal{P}_a$.

3. $\delta(a_v) = 0$ for all $v \neq 0$.

By Remark 4.6.5, condition (1) implies that $\mathcal{F}l_{B,a_0}$ is projective. By Theorem 4.6.4, condition (3) is equivalent to the condition that $|\beta_{a,v}| = r - r(w_{a,v})$ for all $v \neq 0$.

Lemma 4.7.12. *Suppose that $\beta \in Br_W$ maps to $w \in W$. If $|\beta| = r - r(w)$, then*

$$\textsc{ann}(\beta) \in \frac{1}{(1 - \boldsymbol{q})^{r(w)}} \,\mathrm{K}_0(W) \tag{4.7.24}$$

and $(1, \textsc{ann}(\beta))_W = (1 - \boldsymbol{q})^{-r(w)}$.

Proof. By Theorem 4.7.1, the hypothesis $|\beta| = r - r(w)$ implies that $(1 - \boldsymbol{q})^{r(w)}\textsc{ann}(\beta)$ is a polynomial in $\boldsymbol{q}$ of degree 0, hence an element of $\mathrm{K}_0(W)$. The claim about $(1, \textsc{ann}(\beta))_W$ also follows from Theorem 4.7.1. □

Corollary 4.7.13. *If Σ is of genus 0 and $a \in \mathcal{A}^{\text{ani}}(\mathbf{C})$ is tidy, then the $t \to -1$ limit of the right-hand side of Conjecture A/B simplifies to $\textsc{ann}(\beta_{a,0})$.*

Let us assume that $a \in \mathcal{A}^{\text{ani}}$ is tidy and explain how to derive a conjecture about affine Springer fibers.

The product formula of §4.5.2 is defined via a morphism:

$$P_{a_0} \times \prod_{v \in \mathfrak{D}_a \setminus 0} P_{a_v} \to \mathcal{P}_a. \tag{4.7.25}$$

Let $S_{a,0}$ be the kernel of the map $P_{a,0} \to \mathcal{P}_a$. Explicitly, this map sends a J_a-torsor $Q \to D_0$ to the J_a-torsor $Q' \to \Sigma$ formed by gluing Q to the trivial J_a-torsor over $\Sigma \setminus 0$. We deduce that $S_{a,0}$ is the automorphism group of Q'. Then, by [87, Cor. 4.11.3] (see also the proof of [90, Prop. 6.6.3]), we can embed $S_{a,0}$ into $T^{w_{a,0}}$, and by $r(w_{a,0}) = 0$, we know that the latter is a finite group. Therefore, $S_{a,0}$ is also finite.

Letting $\mathcal{Q}_{a,0}$ be the kernel of the map $P_{a,0} \to \mathcal{P}_a$, we have the exact sequence:

$$0 \to S_{a,0} \to P_{a_0} \to \mathcal{P}_a \to \mathcal{Q}_{a,0} \to 0. \tag{4.7.26}$$

Since $S_{a,0}$ is finite and $\dim P_{a_0} = \dim \mathcal{P}_a$, we know that $\mathcal{Q}_{a,0}$ is discrete. Letting $\mathcal{Q}_a$ be the kernel of the map (4.7.25), there is a surjection $\mathcal{Q}_{a,0} \twoheadrightarrow \mathcal{Q}_a$. At the same time, we have a surjection $\pi_0(\mathcal{P}_a) \twoheadrightarrow \pi_0(\mathcal{Q}_{a,0}) = \mathcal{Q}_{a,0}$, so the condition $a \in \mathcal{A}^{\mathrm{ani}}$ now implies that $\mathcal{Q}_a$ is finite.

Finally, we observe that $\delta(a_v) = 0$ implies P_{a_v} and $\mathcal{F}l_{G,a_v}$ are discrete. Since P_a has dense open orbit on $\mathcal{F}l_{G,a_v}$, this forces $\mathcal{F}l_{G,a_v} = P_{a_v}^{\mathrm{red}}$. By the product formula, we conclude:

Lemma 4.7.14. *If Σ is of genus 0 and $a \in \mathcal{A}^{\mathrm{ani}}(\mathbf{C})$ is tidy, then there is a fiber bundle $\tilde{\mathcal{M}}_a \to \mathcal{Q}_a$, where $\mathcal{Q}_a$ is finite and the fibers are homeomorphic to $\mathcal{F}l_{B,a_0}/S_{a,0}$.*

By the lemma, we have a composition of morphisms:

$$\mathrm{H}^*(\tilde{\mathcal{M}}_a) \xrightarrow{\sim} \mathrm{H}^*(\mathcal{Q}_a) \otimes \mathrm{H}^*(\mathcal{F}l_{B,a_0})^{S_{a,0}} \twoheadrightarrow \mathrm{H}^*(\mathcal{F}l_{B,a_0})^{S_{a,0}}. \tag{4.7.27}$$

The first arrow sends the stable summand $\mathrm{H}^*(\tilde{\mathcal{M}}_a)_{st}$ into the $\mathcal{Q}_a$-invariant summand of the middle term, so we get an inclusion of $\mathrm{H}^*(\tilde{\mathcal{M}}_a)_{st}$ into $\mathrm{H}^*(\mathcal{F}l_{B,a_0})^{S_{a,0}}$. Let $\mathrm{H}^*(\mathcal{F}l_{B,a_0})_{st}$ denote the image. We again write $\mathbf{P}_{\leq *}$ for the filtration on $\mathrm{H}^*(\mathcal{F}l_{B,a_0})_{st}$ formed by the image of the perverse filtration on $\mathrm{H}^*(\tilde{\mathcal{M}}_a)_{st}$. Then we obtain:

$$\mathrm{E}^{\mathbf{P}}(1, \boldsymbol{q} \mid \tilde{\mathcal{M}}_a)_{st} = \sum_{i,j} (-1)^i \boldsymbol{q}^j \, \mathrm{Gr}_j^{\mathbf{P}} \, \mathrm{H}^i(\mathcal{F}l_{B,a_0})_{st}. \tag{4.7.28}$$

Below, we instead denote this series by $\mathrm{E}^{\mathbf{P}}(1, \boldsymbol{q} \mid \mathcal{F}l_{B,a_0})_{st}$ to emphasize that we are really studying $\mathcal{F}l_{B,a_0}$ rather than $\tilde{\mathcal{M}}_a$.

Conjecture D. *If Σ is of genus 0 and $a \in \mathcal{A}^{\mathrm{ani}}(\mathbf{C})$ is tidy, then*

$$\mathrm{E}^{\mathbf{P}}(1, \boldsymbol{q} \mid \mathcal{F}l_{B,a_0})_{st} = \textsc{ann}(\beta_{a,0}) \tag{4.7.29}$$

in $\mathrm{K}_0(W)[\boldsymbol{q}]$.

Remark 4.7.15. Above, the W-action on $\mathrm{E}^{\mathbf{P}}(1, \boldsymbol{q} \mid \mathcal{F}l_{B,a_0})_{st}$ is compatible with the Springer action of W on $\mathrm{H}^*(\mathcal{F}l_{B,a_0})$ from §4.4.4, by way of the product formula of §4.5.2.

We believe but have not shown that the summand $\mathrm{H}^*(\mathcal{F}l_{B,a_0})_{st} \subseteq \mathrm{H}^*(\mathcal{F}l_{B,a_0})$, together with its filtration $\mathbf{P}_{\leq *}$, depends only on a_0, not on a.

4.8 Links of Plane Curves

In this section, we show that Conjecture C implies the Oblomkov–Rasmussen–Shende conjecture. Most of the "heavy lifting" we need has been carried out in our paper [105] and in Section 9 of [50]. At the risk of rewriting [50], we will introduce the story gently.

4.8.1. The ORS Conjecture In §4.2.3, we sketched the relationship between plane curve germs and topological links. The ORS conjecture [88] relates the Hilbert scheme of such a germ to the so-called HOMFLY homology of its link.

The **HOMFLY series** is an isotopy invariant of links in 3-space, introduced jointly by multiple authors in 1985 [41]. It takes the form:

$$\textsc{homfly} : \{\text{links in } \mathbf{R}^3\}/\text{isotopy} \to \mathbf{Z}[(\boldsymbol{q}^{\frac{1}{2}} - \boldsymbol{q}^{-\frac{1}{2}})^{\pm 1}, \boldsymbol{a}^{\pm 1}]. \tag{4.8.1}$$

It is a common refinement of the Alexander polynomial, discovered in the 1920s, and the Jones polynomial, discovered in 1984.

Answering a conjecture of Dunfeld–Gukov–Rasmussen [32], Khovanov–Rozansky showed [62] that HOMFLY can be categorified to an invariant valued in triply-graded vector spaces, now called **HOMFLY** or **Khovanov–Rozansky homology**. We will write

$$\underline{\textsc{homfly}} : \{\text{links in } \mathbf{R}^3\}/\text{isotopy} \to \mathbf{Z}(\!(\boldsymbol{q}^{\frac{1}{2}})\!)[\boldsymbol{a}^{\pm 1}, t] \tag{4.8.2}$$

for the Poincaré series of their invariant, where the grading is normalized so that HOMFLY is the $t = -1$ limit of $\underline{\textsc{homfly}}$.

The Local Conjecture Fix a curve C and a point $p \in C(\mathbf{C})$ where C is locally planar. The **Hilbert scheme** of the germ (C, p), denoted $\mathrm{Hilb}(C, p)$, is a scheme locally of finite type over $\mathbf{C}$ that classifies the finite-length subschemes C supported at p, or equivalently, the ideals $I \subseteq \hat{\mathcal{O}}_{C,p}$ such that $\hat{\mathcal{O}}_{C,p}/I$ is a finite-dimensional vector space. There are constructible $\mathbf{Z}_{\geq 0}$-valued functions ℓ and m on $\mathrm{Hilb}(C, p)^{\mathrm{an}}$, defined by

$$\ell(I) = \dim_{\mathbf{C}} \hat{\mathcal{O}}_{C,p}/I, \tag{4.8.3}$$

$$m(I) = \dim_{\mathbf{C}} I/\mathfrak{m}_{C,p}I, \tag{4.8.4}$$

where $\mathfrak{m}_{C,p}$ is the maximal ideal of $\hat{\mathcal{O}}_{C,p}$. The function ℓ is called the **colength** and its values index the connected components of $\mathrm{Hilb}(C, p)$. Each connected component is a projective union of varieties. Note that by Nakayama's lemma, $m(I) \geq 1$ for all I.

As shown in [23] (see also [54, Appendix]), the virtual weight series $\mathrm{E}(-t \mid -)$ (see §4.1.6) defines a constructible Lebesgue measure $d\mathrm{E}(-t)$ on $\mathbf{C}$-schemes of finite type. In what follows, we set

$$\begin{aligned} Z^{\mathrm{local}}(\boldsymbol{q}, \boldsymbol{a}, t \mid C, p) &= \int_{\mathrm{Hilb}(C,p)^{\mathrm{an}}} \boldsymbol{q}^{\ell} \prod_{1 \leq k \leq m} (1 + t^{2k-1}\boldsymbol{a}^2)\, d\mathrm{E}(-t) \\ &= \sum_{\substack{\ell \geq 0 \\ m \geq 1}} \boldsymbol{q}^{\ell} \mathrm{E}(-t \mid \mathrm{Hilb}^{\ell,m}(C,p)) \prod_{1 \leq k \leq m} (1 + t^{2k-1}\boldsymbol{a}^2), \end{aligned} \tag{4.8.5}$$

where $\mathrm{Hilb}^{\ell,m}(C,p)$ denotes the stratum of $\mathrm{Hilb}(C,p)$ of ideals $I \subseteq \hat{\mathcal{O}}_{C,p}$ such that $\ell(I) = \ell$ and $m(I) = m$, a quasiprojective variety.

Recall that if $\hat{\mathcal{O}}_{C,p} \simeq \mathbf{C}[\![x,y]\!]/\langle f \rangle$, then the **Milnor number** $\mu = \mu_{C,p}$ is given by

$$\mu_{C,p} = \dim_{\mathbf{C}} \frac{\mathbf{C}[\![x,y]\!]}{J(f)}, \tag{4.8.6}$$

where $J(f) = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \rangle$, the Jacobian ideal of f. Now, the main conjecture of [88] states:

Conjecture 4.8.1 (Oblomkov–Rasmussen–Shende)**.** *Let $\lambda = \lambda_{C,p}$ be the link of C at p. Then we have the identity*

$$Z^{\mathrm{local}}(\boldsymbol{q}, \boldsymbol{a}, t \mid C, p) = (\boldsymbol{q}^{\frac{1}{2}}\boldsymbol{a}^{-1})^{\mu} \cdot \frac{1 + \boldsymbol{a}^2 t}{1 - \boldsymbol{q}} \cdot \underline{\mathit{HOMFLY}}(\lambda) \tag{4.8.7}$$

in $\mathbf{Z}(\!(\boldsymbol{q}^{\frac{1}{2}})\!)[\boldsymbol{a}^{\pm 1}, t]$.

As mentioned in the introduction, the $t \to -1$ limit of the ORS conjecture was originally announced by Oblomkov–Shende in [89] and proved by Maulik in [80] using Donaldson–Thomas theory. We refer to Migliorini's Bourbaki survey [83] for a thorough exposition of Maulik's work.

Theorem 4.8.2 (Maulik)**.** *The $t \to -1$ limit of Conjecture 4.8.1 holds. That is,*

$$\int_{\mathrm{Hilb}(C,p)} \boldsymbol{q}^{\ell} (1 - \boldsymbol{a}^2)^{m-1}\, d\chi = \frac{(\boldsymbol{q}^{\frac{1}{2}}\boldsymbol{a}^{-1})^{\mu}}{1 - \boldsymbol{q}} \cdot \mathit{HOMFLY}(\lambda) \tag{4.8.8}$$

in $\mathbf{Z}(\!(\boldsymbol{q}^{\frac{1}{2}})\!)[\boldsymbol{a}^{\pm 1}]$*, where* $d\chi$ *is the Lebesgue measure with respect to Euler characteristic.*

The Global Conjecture We need a version of Conjecture 4.8.1 in the setting of an actual curve, rather than its germ at a single point. Henceforth, *we assume that* C *is an integral projective curve, smooth away from (at most) a planar singularity at* p.

The **Hilbert scheme** of C, denoted $\mathrm{Hilb}(C)$, is a $\mathbf{C}$-scheme that classifies finite-length subschemes $\Delta \subseteq C$, without restrictions on the support of Δ. Analogously to (4.8.3), (4.8.4), we have functions $\ell, m_p : \mathrm{Hilb}(C) \to \mathbf{Z}_{\geq 0}$ defined by

$$\ell(\Delta) = \dim_{\mathbf{C}} \Gamma(\Delta, \mathcal{O}_\Delta), \tag{4.8.9}$$

$$m_p(\Delta) = \dim_{\mathbf{C}} \mathcal{I}_{\Delta,p}/\mathfrak{m}_{C,p}\mathcal{I}_{\Delta,p}, \tag{4.8.10}$$

where $\mathcal{I}_\Delta$ denotes the ideal sheaf of Δ. Once again, the connected components of $\mathrm{Hilb}(C)$ are indexed by the values of ℓ. Thus the series

$$Z^{\mathrm{global}}(\boldsymbol{q}, \boldsymbol{a}, t \mid C, p) = \sum_{\substack{\ell \geq 0 \\ m \geq 1}} \boldsymbol{q}^\ell \mathrm{E}(-t \mid \mathrm{Hilb}^{\ell, m_p}(C)) \prod_{1 \leq k \leq m} (1 + t^{2k-1}\boldsymbol{a}^2) \tag{4.8.11}$$

is a global analogue of $Z^{\mathrm{local}}(\boldsymbol{q}, \boldsymbol{a}, t \mid C, p)$. We note that the $\boldsymbol{a} \to 0$ limit,

$$Z(\boldsymbol{q}, t \mid C) = \sum_{\ell \geq 0} \boldsymbol{q}^\ell \mathrm{E}(-t \mid \mathrm{Hilb}^\ell(C)), \tag{4.8.12}$$

does not depend on the basepoint p. If C is smooth at p, then $Z^{\mathrm{global}}(\boldsymbol{q}, \boldsymbol{a}, t) = Z(\boldsymbol{q}, t)$, and moreover, $\mathrm{Hilb}^\ell(C) = \mathrm{Sym}^\ell(C)$ for all ℓ.

Let $\tilde{C}$ be the normalization of C, and let $\tilde{g}$ be the genus of $\tilde{C}$, i.e., the geometric genus of C. In the Grothendieck ring of varieties, we can write

$$[C] = [p] + [\tilde{C} \setminus \{p_1, \ldots, p_b\}], \tag{4.8.13}$$

where $b = b_{C,p}$ is the number of branches of (C, p) and $p_1, \ldots, p_b$ are the preimages of p along $\tilde{C} \to C$. By [52], we have the corresponding identity of formal series:

$$Z^{\mathrm{global}}(\boldsymbol{q}, \boldsymbol{a}, t \mid C, p) = Z^{\mathrm{local}}(\boldsymbol{q}, \boldsymbol{a}, t \mid C, p) \cdot Z(\boldsymbol{q}, t \mid \tilde{C} \setminus \{p_1, \ldots, p_b\}), \tag{4.8.14}$$

where on the right-hand side,

$$\begin{aligned} Z(\boldsymbol{q}, t \mid \tilde{C} \setminus \{p_1, \ldots, p_b\}) &= (1 - \boldsymbol{q})^b \cdot Z(\boldsymbol{q}, t \mid \tilde{C}) \\ &= \frac{(1 + \boldsymbol{q}t)^{2\tilde{g}}(1 - \boldsymbol{q})^{b-1}}{(1 - \boldsymbol{q}t^2)}. \end{aligned} \tag{4.8.15}$$

Thus, Conjecture 4.8.1 is equivalent to the following version from [50, §9]:

Conjecture 4.8.3. *For any integral projective curve C, smooth away from a planar germ at p, we have*

$$Z^{\mathrm{global}}(\boldsymbol{q}, \boldsymbol{a}, t \mid C, p) = \frac{(\boldsymbol{q}^{\frac{1}{2}}\boldsymbol{a}^{-1})^{\mu}(1 + \boldsymbol{q}t)^{2\tilde{g}}(1 - \boldsymbol{q})^{b-1}}{(1 - \boldsymbol{q}t^2)} \cdot \frac{1 + \boldsymbol{a}^2 t}{1 - \boldsymbol{q}} \cdot \underline{\mathit{HOMFLY}}(\lambda) \tag{4.8.16}$$

in $\mathbf{Z}(\!(\boldsymbol{q}^{\frac{1}{2}})\!)[\boldsymbol{a}^{\pm 1}, t]$, where we write $\tilde{g}$ for the geometric genus of C.

In the course of [50, §9], Gorsky–Oblomkov–Rasmussen–Shende explain how this version can be transformed into a conjecture about Hitchin fibers for the group $G = \mathrm{GL}_n$. We explain this step in the next subsection.

4.8.2. From Hilbert to Hitchin First, we explain how the Hilbert scheme of C is related to its so-called compactified Jacobian, by way of a generalized Abel–Jacobi map. Then, we explain how these objects are related to Hitchin fibers for GL_n.

Picard Objects The **Picard stack** of C, which we will denote $\mathcal{P}(C)$, classifies line bundles on C. We write $\mathcal{P}^{\ell}(C) \subseteq \mathcal{P}(C)$ for the connected component that corresponds to degree ℓ, and write $\mathrm{Pic}^{\ell}(C)$ for the 0-truncation (i.e., destackification) of $\mathcal{P}^0(C)$. That is, $\mathrm{Pic}^{\ell}(C)$

classifies degree-ℓ line bundles on C *up to isomorphism*. The natural map $\mathcal{P}^\ell(C) \to \mathrm{Pic}^\ell(C)$ is a $\mathbf{G}_m$-gerbe.

Recall that $\mathrm{Pic}^0(C)$ forms an algebraic group, known as the **Jacobian** of C and denoted $\mathrm{Jac}(C)$. It acts simply transitively on $\mathrm{Pic}^\ell(C)$ for all ℓ. If C is smooth, then $\mathrm{Jac}(C)$ is an abelian variety, but otherwise, it is not even proper.

We write $\overline{\mathcal{P}}^\ell(C)$ for the $\mathbf{C}$-scheme that classifies torsion-free coherent sheaves over C of degree ℓ and *generically* of rank 1, and write $\overline{\mathrm{Pic}}^\ell(C)$ for its 0-truncation. Since every line bundle is torsion-free, we have inclusions $\mathcal{P}^\ell(C) \subseteq \overline{\mathcal{P}}^\ell(C)$ and $\mathrm{Pic}^\ell(C) \subseteq \overline{\mathrm{Pic}}^\ell(C)$. The **compactified Jacobian** of C is defined to be $\overline{\mathrm{Jac}}(C) = \overline{\mathrm{Pic}}^0(C)$.

Altman–Iarrobino–Kleinman proved [1] that, under the assumption that C is integral and projective with at worst planar singularities, $\overline{\mathrm{Pic}}^\ell(C)$ is an irreducible projective variety of which $\mathrm{Pic}^\ell(C)$ is a dense open. A similar statement can be made for $\overline{\mathcal{P}}^\ell(C)$.

For all ℓ, there is an **Abel–Jacobi map**

$$\begin{array}{ccc} \mathrm{Hilb}^\ell(C) & \xrightarrow{AJ} & \overline{\mathrm{Pic}}^\ell(C) \\ \Delta & \mapsto & \mathcal{I}_\Delta^\vee = \underline{\mathrm{Hom}}(\mathcal{I}_\Delta, \mathcal{O}_C) \end{array} \tag{4.8.17}$$

The fibers of AJ are projective spaces. Indeed, if $\mathcal{F}$ is a $\mathbf{C}$-point of $\overline{\mathrm{Pic}}^\ell(C)$, viewed as a torsion-free sheaf over C, then there is a bijection between the nonzero global sections of $\mathcal{F}$ and embeddings $\mathcal{F}^\vee \hookrightarrow \mathcal{O}_C$ because C is Gorenstein. This yields an isomorphism from $AJ^{-1}(\mathcal{F})$ to the projective space $\mathbf{P}\Gamma(C, \mathcal{F})$ [89, §4].

When C is smooth, the Abel–Jacobi map is a fiber bundle. Via the isomorphisms $\overline{\mathrm{Pic}}^\ell(C) \simeq \overline{\mathrm{Jac}}(C)$ for each ℓ, a classical formula of Macdonald relates the cohomology of $\mathrm{Hilb}(C) = \coprod_\ell \mathrm{Hilb}^\ell(C)$ to that of $\overline{\mathrm{Jac}}(C)$.

If instead C has planar singularities, but remains integral projective, then there is a generalization of Macdonald's work, independently due to Maulik–Yun [81] and Migliorini–Shende [84]. We need a parabolic refinement of their formula, stated in [50, §9]. Before we introduce it, we explain how to interpret $\overline{\mathrm{Pic}}^\ell(C)$ via Hitchin systems for GL_n.

Spectral Covers In what follows, *we set $G = \mathrm{GL}_n$ in the notation of Section 4.4*. Thus, $W = S_n$ and $\mathfrak{t} = \mathbf{A}^n$. Recall from Remark 4.3.3 that this implies

$$(\mathfrak{c}_1, \mathfrak{c}_1^\circ) \simeq (\mathrm{Sym}^n(\mathbf{A}^1), \mathrm{Conf}^n(\mathbf{A}^1)). \tag{4.8.18}$$

As in Section 4.5, we fix a *non-stacky* smooth projective curve Σ and a line bundle $L \to \Sigma$.

In Example 4.2.9, we described a correspondence between:

1. Maps of the form $f : (D, \eta) \to (\mathrm{Sym}^n(\mathbf{A}^1), \mathrm{Conf}^n(\mathbf{A}^1))$.
2. Branched n-fold covers $D_f \to D$ embedded in $D \times \mathbf{A}^1$ and unramified over η.

The analogous statement with Σ instead of D is a correspondence between:

1. Points $a \in \mathcal{A}^\heartsuit(\mathbf{C})$, meaning sections

   $$a : \Sigma \to \mathfrak{c}_L \simeq \mathrm{Sym}^n(\mathbf{A}^1) \otimes L \tag{4.8.19}$$

   that send the generic point of Σ into $\mathfrak{c}_L^\circ \simeq \mathrm{Conf}^n(\mathbf{A}^1) \otimes L$.
2. Branched, generically unramified n-fold covers

   $$\pi_a : \Sigma_a \to \Sigma \tag{4.8.20}$$

   embedded in L. Note that π_a being generically unramified means Σ_a is reduced, so Σ_a is a (possibly singular, reducible) curve.

We say that Σ_a is the **spectral curve** corresponding to a.

To describe Σ_a more explicitly: Recall from Example 4.2.9 that we have $\mathrm{Sym}^n(\mathbf{A}^1) \simeq \mathrm{Spec}\, \mathbf{C}[\sigma_1, \ldots, \sigma_n]$, where σ_i has weight i under the $\mathbf{G}_m$-action on $\mathrm{Sym}^n(\mathbf{A}^1)$. This choice of coordinates yields an isomorphism

$$\mathcal{A} \simeq \Gamma(\Sigma, L \oplus L^{\otimes 2} \oplus \cdots \oplus L^{\otimes n}). \tag{4.8.21}$$

Under this isomorphism, we can express $a \in \mathcal{A}^\heartsuit(\mathbf{C})$ as a tuple $(a_1, \ldots, a_n)$, where a_i is a section of $L^{\otimes i}$. In terms of a vertical coordinate z on the fibers of $L \to \Sigma$, we find that Σ_a is the closed subscheme of L cut out by

$$z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n = 0. \tag{4.8.22}$$

Compare to (4.2.14).

Recall that GL_n-torsors correspond bijectively to rank-n vector bundles, via the map that sends a GL_n-torsor E to the vector bundle

$$V_E = E \wedge^{\mathrm{GL}_n} \mathbf{A}^n. \tag{4.8.23}$$

We write $\mathcal{M}^d \subseteq \mathcal{M}$ for the connected component of the Hitchin moduli space for $(\mathrm{GL}_n, \Sigma, L)$ that classifies Higgs pairs (E, θ) where V_E is of degree d. The following result, proved using a relative version of the Cayley–Hamilton theorem over Σ, is implicit in Hitchin's paper [55] and appears as Prop. 3.6 in [2].

Proposition 4.8.4 (Hitchin, Beauville–Narasimhan–Ramanan)**.** *For all $a \in \mathcal{A}^\heartsuit$, there is an isomorphism-preserving bijection between:*

1. *Pairs (V, θ), where:*
   (a) *$V \to \Sigma$ is a vector bundle of rank n and degree $\ell - \binom{n}{2} \deg L$.*
   (b) *θ is a section of $\underline{\mathrm{End}}(V) \otimes L$ having (4.8.22) as its characteristic polynomial.*
2. *Torsion-free coherent sheaves $\mathcal{F}$ over Σ_a of degree ℓ and generically of rank 1.*

Equivalently, we have an isomorphism

$$\mathcal{M}^d_a \simeq \overline{\mathcal{P}}^\ell(\Sigma_a), \tag{4.8.24}$$

where $\ell = d + \binom{n}{2} \deg L$. (Under the bijection, $\pi_{a,*}\mathcal{F}$ is the underlying sheaf of V.)

Remark 4.8.5. There is a similar isomorphism $\mathcal{P}_a \simeq \mathcal{P}^0(\Sigma_a)$. Together, these isomorphisms identify the $\mathcal{P}_a$-action on $\mathcal{M}_a$ with the $\mathcal{P}^0(\Sigma_a)$-action on $\overline{\mathcal{P}}^\ell(\Sigma_a)$.

Let $\mathcal{H}^d$ denote the *scheme* over $\mathbf{C}$ that classifies triples (E, θ, σ), where $(E, \theta) \in \mathcal{M}^d$ and $\sigma \in \mathbf{P}\Gamma(\Sigma, V_E)$. (The addition of σ makes the data rigid.) Via Proposition 4.8.4 and the section-theoretic description of the fibers of the Abel–Jacobi map, we obtain:

Corollary 4.8.6 (GORS). *For all $a \in \mathcal{A}^\heartsuit(\mathbf{C})$, we have an isomorphism*

$$\mathcal{H}^d_a \simeq \mathrm{Hilb}^\ell(\Sigma_a), \tag{4.8.25}$$

where $\ell = d + \binom{n}{2} \deg L$.

Let $\tilde{\mathcal{M}}^d \subseteq \tilde{\mathcal{M}}$ be the pullback of $\mathcal{M}^d$, and let $\tilde{\mathcal{H}}^d$ be the pullback of $\tilde{\mathcal{M}}^d$ along the forgetful map $\mathcal{H}^d \to \mathcal{M}^d$. Then both squares in the diagram below are cartesian:

$$\begin{array}{ccccc} \tilde{\mathcal{H}}^d & \longrightarrow & \tilde{\mathcal{M}}^d & \longrightarrow & [\mathfrak{b}/B]_L \\ \downarrow & & \downarrow & & \downarrow \boldsymbol{s} \\ \mathcal{H}^d & \longrightarrow & \mathcal{M}^d & \longrightarrow & [\mathfrak{gl}_n/\mathrm{GL}_n]_L \end{array} \tag{4.8.26}$$

For all $k \geq 0$, we set $\Lambda^k = \Lambda^k(\mathbf{C}^{n-1})$, viewed as the irreducible representation of S_n attached to the hook partition $(n-k, 1, \ldots, 1)$. By means of the combinatorics of Kostka polynomials for S_n, which encode the S_n-isotypic structure of the cohomology of (non-affine) Springer fibers for GL_n, Gorsky–Oblomkov–Rasmussen–Shende show [50, Lem. 9.5]:

Lemma 4.8.7 (GORS). *For all $a \in \mathcal{A}^\heartsuit(\mathbf{C})$ such that Σ_a is integral, we have*

$$Z^{\mathrm{global}}(\boldsymbol{q}, \boldsymbol{a}, t \mid \Sigma_a, p) = (1 + \boldsymbol{a}^2 t) \sum_{d,k} \boldsymbol{q}^d (\boldsymbol{a}^2 t)^k (\Lambda^k, \mathrm{E}(-t \mid \tilde{\mathcal{H}}^d_a))_{S_n}, \tag{4.8.27}$$

in $\mathbf{Z}[\![\boldsymbol{q}, t]\!][\boldsymbol{a}]$, where $\mathrm{E}(t, \tilde{\mathcal{H}}^d_a) \in \mathrm{K}_0(S_n)[\boldsymbol{q}, t]$ via the pullback of the S_n-action on $\boldsymbol{s}_*\mathbf{C}$.

Moreover, they introduce the following parabolic Macdonald formula, based on the non-parabolic version in [81, 84].

Theorem 4.8.8 (GORS, *après* Maulik–Yun, Migliorini–Shende)**.** *Let $a \in \mathcal{A}^{\heartsuit}(\mathbf{C})$ such that Σ_a is integral, and let $\tilde{\mathcal{M}}_a^{0,\dagger}$ be the 0-truncation of $\tilde{\mathcal{M}}_a$. Then*

$$\sum_{d,k \geq 0} \boldsymbol{q}^d t^k \operatorname{Gr}_k^{\mathbf{W}} \mathrm{H}^*(\tilde{\mathcal{H}}_a^d, \mathbf{C}) = \frac{1}{(1-\boldsymbol{q})(1-\boldsymbol{q}t^2)} \sum_{j,k \geq 0} \boldsymbol{q}^j t^k \operatorname{Gr}_j^{\mathbf{P}} \operatorname{Gr}_k^{\mathbf{W}} \mathrm{H}^*(\tilde{\mathcal{M}}_a^{0,\dagger}, \mathbf{C}) \tag{4.8.28}$$

in $\mathbf{Z}[\![\boldsymbol{q}, t]\!][\boldsymbol{a}]$. This isomorphism is equivariant with respect to the S_n-actions induced by pulling back the S_n-action on $\boldsymbol{s}_*\mathbf{C}$.

4.8.3. Conclusion

Our goal is to show:

Theorem 4.8.9. *Conjecture C, in the cases where $G = \mathrm{SL}_n$ for some n and $\Sigma = \mathbf{P}^1$, implies Conjecture 4.8.3.*

We first make a series of reductions. In Conjecture 4.8.3, we have freedom to choose the structure of C away from a planar affine chart at p. So it suffices to assume that:

(I) $C = \Sigma_a$, where $\Sigma = \mathbf{P}^1$ and $\deg L > 0$ and $a \in \mathcal{A}^{\heartsuit}(\mathrm{GL}_n, \Sigma, L)(\mathbf{C})$ for some n.

(II) p is the unique point of the fiber $\pi_a^{-1}(0) \subseteq C$.

After a change of coordinates, we can further assume that:

($\mathrm{III}^{\mathrm{SL}_n}$) $a = (a_1, \ldots, a_n)$ satisfies $a_1 = 0$.

This means a belongs to

$$\Gamma(\Sigma, 0 \oplus L^{\otimes 2} \oplus \cdots \oplus L^{\otimes n}) \simeq \mathcal{A}(\mathrm{SL}_n, \Sigma, L) \subseteq \mathcal{A}(\mathrm{GL}_n, \Sigma, L), \tag{4.8.29}$$

once we identify the Cartan subalgebra of $\mathfrak{gl}_n$ with the $\mathbf{C}$-vector space of diagonal $n \times n$ matrices, and that of $\mathfrak{sl}_n$ with the subspace of diagonal matrices of trace zero. Since SL_n

is semisimple and $\deg L > 0$, we know that $\mathcal{A}^{\mathrm{ani}}(\mathrm{SL}_n, \Sigma, L)$ is dense in $\mathcal{A}^{\heartsuit}(\mathrm{SL}_n, \Sigma, L)$ by Lemma 4.7.8. So after perturbing a, we can take:

(III$^{\diamondsuit_0}$) $a \in \mathcal{A}^{\diamondsuit_0}(\mathrm{SL}_n, \Sigma, L)(\mathbf{C})$.

If $E \to \Sigma$ is a GL_n-torsor, then a reduction-of-structure-group on E from GL_n to SL_n is equivalent to a trivialization of $\det V_E$. When $\Sigma = \mathbf{P}^1$, the latter exists for every vector bundle of degree 0. Therefore, for all $a \in \mathcal{A}^{\heartsuit}(\mathrm{SL}_n, \mathbf{P}^1, L)$, we have the following isomorphisms of 0-truncations:

$$\mathcal{M}(\mathrm{GL}_n, \mathbf{P}^1, L)_a^{0,\dagger} \simeq \mathcal{M}(\mathrm{SL}_n, \mathbf{P}^1, L)_a^{\dagger}, \tag{4.8.30}$$

$$\mathcal{P}(\mathrm{GL}_n, \mathbf{P}^1, L)_a^{0,\dagger} \simeq \mathcal{P}(\mathrm{SL}_n, \mathbf{P}^1, L)_a^{\dagger}. \tag{4.8.31}$$

Since the Grothendieck–Springer resolution of $\mathfrak{sl}_n$ is the restriction of that of $\mathfrak{gl}_n$, we also have the parabolic statement:

$$\tilde{\mathcal{M}}(\mathrm{GL}_n, \mathbf{P}^1, L)_a^{0,\dagger} \simeq \tilde{\mathcal{M}}(\mathrm{SL}_n, \mathbf{P}^1, L)_a^{\dagger}. \tag{4.8.32}$$

Note that $\tilde{\mathcal{M}}(\mathrm{SL}_n, \mathbf{P}^1, L)_a$ differs from $\tilde{\mathcal{M}}(\mathrm{SL}_n, \mathbf{P}^1, L)_a^{\dagger}$ essentially by the classifying stack of the finite group μ_n, so these spaces have the same rational cohomology. Finally, note that $\mathcal{P}(\mathrm{SL}_n, \mathbf{P}^1, L)_a$ is connected, so the cohomology of $\tilde{\mathcal{M}}(\mathrm{SL}_n, \mathbf{P}^1, L)_a$ is equal to its stable summand. (What we are really using is the irreducibility of Σ_a.)

By Lemma 4.8.7 and Theorem 4.8.8, we conclude that Conjecture 4.8.3 is equivalent to the following version:

Conjecture 4.8.10. *For all integers $n \geq 1$, line bundles $L \to \Sigma$, and $a \in \mathcal{A}^{\diamondsuit_0}(\mathrm{SL}_n, \mathbf{P}^1, L)(\mathbf{C})$*

satisfying (I), (II), (III$^{\diamondsuit_0}$), we have

$$\sum_{0\le k\le n}(\boldsymbol{a}^2t)^k(\Lambda^k,\mathrm{E}^{\mathbf{P}}(-t,\boldsymbol{q}\mid\tilde{\mathcal{M}}(\mathrm{SL}_n,\mathbf{P}^1,L)_a))_{S_n} = (\boldsymbol{q}^{\frac{1}{2}}\boldsymbol{a}^{-1})^{\mu}(1+\boldsymbol{q}t)^{2\tilde{g}}(1-\boldsymbol{q})^{b-1}\underline{\mathit{HOMFLY}}(\lambda) \tag{4.8.33}$$

in $\mathbf{Z}(\!(\boldsymbol{q}^{\frac{1}{2}})\!)[\boldsymbol{a}^{\pm1},t]$.

Let $[\beta]=[\beta_{a,0}]\subseteq Br_n$, and let $[w]\subseteq W$ be the image of $[\beta]$ under $Br_W\to W$. By assumption (II), the planar closure of β is precisely the link λ. In particular,

$$r(w) = \text{number of connected components of } \lambda = b-1. \tag{4.8.34}$$

From [105], we have the following comparison between the annular braid homology of β and the Khovanov–Rozansky homology of λ:

$$(\boldsymbol{q}^{\frac{1}{2}}a^{-1})^{|\beta|-r}\underline{\mathrm{HOMFLY}}(\lambda) = \sum_{0\le k\le n}(\boldsymbol{a}^2t)^k(\Lambda^k,\mathrm{AH}(\beta))_{S_n}. \tag{4.8.35}$$

To finish the proof, it remains to check the identities

$$\mu = |\beta|-r, \tag{4.8.36}$$

$$2\tilde{g}(\Sigma_a) = N_a - r(w), \tag{4.8.37}$$

where $N_a=-r\chi(U_a)-(r-1)|\mathfrak{D}_a\setminus 0|$ in the notation of Conjecture C.

To prove the first one, we invoke a formula of Milnor [85, Thm. 10.5]. Let $\delta=\delta_{C,p}$ be the δ-invariant of the plane curve germ (C,p) as defined in *loc. cit.*

Theorem 4.8.11 (Milnor). *For any plane curve germ,* $2\delta=\mu+b-1$.

Recall that at the same time, $\delta_{C,p}$ is equal to the local δ-invariant $\delta(a_0)$ defined by Ngô (see §4.6.2). Combining Theorem 4.6.4 and Milnor's theorem, we get

$$|\beta| - r + r(w) = 2\delta(a_0) = 2\delta_{C,p} = \mu + b - 1. \tag{4.8.38}$$

By (4.8.34), we arrive at $\mu = |\beta| - r$, as desired.

To prove the second identity, we use a Riemann–Hurwitz argument. Since we are assuming $a \in \mathcal{A}^{\diamondsuit_0}(\mathbf{C})$, we know that the only ramification points of $\Sigma_a \to \Sigma = \mathbf{P}^1$ are ordinary double points, and exactly one such point occurs above each point of $\mathfrak{D}_a \setminus 0$. This fact, plus the identity $n = r + 1$, yields:

$$\begin{aligned} \chi(\Sigma_a) &= 1 + n\chi(U_a) + (n-1)|\mathfrak{D}_a \setminus 0| \\ &= \chi(\Sigma) - N_a \\ &= 2 - N_a. \end{aligned} \tag{4.8.39}$$

Since p is the only point where Σ_a can have multiple branches,

$$\chi(\Sigma_a) = 2 - 2\tilde{g}(\Sigma_a) + b - 1. \tag{4.8.40}$$

Therefore, $-2\tilde{g}(\Sigma_a) + b - 1 = -N_a$. Again by (4.8.34), we're done.

4.9 P = W Phenomena

In this section, we show that our conjectures would imply numerical identities suggestive of P = W phenomena, in the sense of nonabelian Hodge theory.

4.9.1. From Braids to Varieties We briefly summarize the definition of the varieties $O(\beta)$, $\widetilde{O}(\beta)$, $\mathit{St}(\beta)$ mentioned in the introduction.

Recall that by Bruhat, the diagonal (left) G-orbits of $\mathcal{B} \times \mathcal{B}$ are indexed by the elements of W. We write O_w for the G-orbit corresponding to $w \in W$. Deligne showed in [27] that there is a map from elements $\beta \in Br_W^+$ to G-varieties $O(\beta)$ over $\mathcal{B} \times \mathcal{B}$ such that:

1. $O(\beta_w) = O_w$ for all w.

2. If $\beta = \beta'\beta''$ in Br_W^+, then there is a fixed isomorphism $O(\beta) \xrightarrow{\sim} O(\beta') \times_{\mathcal{B}} O(\beta'')$. Here, the fiber product is formed with respect to the right projection $O(\beta') \to \mathcal{B}$ and the left projection $O(\beta'') \to \mathcal{B}$. Moreover, these isomorphisms are associative in the sense that the diagram

$$
\begin{array}{ccc}
O(\beta'\beta''\beta''') & \longrightarrow & O(\beta'\beta'') \times_{\mathcal{B}} O(\beta''') \\
\downarrow & & \downarrow \\
O(\beta') \times_{\mathcal{B}} O(\beta''\beta''') & \longrightarrow & O(\beta') \times_{\mathcal{B}} O(\beta'') \times_{\mathcal{B}} O(\beta''')
\end{array}
\tag{4.9.1}
$$

   is commutative.

Explicitly, if $\beta = \beta_{w_1} \cdots \beta_{w_k}$ for some sequence of elements $w_1, \ldots, w_k \in W$, then

$$
O(\beta) \simeq O_{w_1} \times_{\mathcal{B}} O_{w_2} \times_{\mathcal{B}} \cdots \times_{\mathcal{B}} O_{w_k}. \tag{4.9.2}
$$

Deligne attributes the varieties $O(\beta)$ to Broué–Michel (see also [17]). We view the map $\beta \mapsto O(\beta)$ as an "algebro-geometric representation" of the positive Artin monoid Br_W^+.

The **horocycle correspondence** is the diagram of stacks

$$
\begin{array}{ccccc}
G\backslash(\mathcal{B} \times \mathcal{B}) & \xleftarrow{act} & G\backslash(\mathcal{B} \times G) & \xrightarrow{pr} & G\backslash G \\
(xB, gxB) & \longleftarrow\!\!\dashv & (xB, g) & \longmapsto & g
\end{array}
\tag{4.9.3}
$$

where G acts on itself by the adjoint action. As explained in [105, 106], pullback-pushforward of objects through this diagram from the left to the right is very roughly an analogue of the passage from elements of Br_W to their conjugacy classes. To make this more precise for positive braids $\beta \in Br_W^+$, let

$$\widetilde{O}(\beta) \xrightarrow{pr_\beta} G = pr_* act^*(O(\beta) \to \mathcal{B} \times \mathcal{B}). \tag{4.9.4}$$

In words, $pr_\beta : \widetilde{O}(\beta) \to G$ is the G-equivariant map formed by pulling back $O(\beta) \to \mathcal{B} \times \mathcal{B}$ along act and extending along pr. Then one can check directly that

$$G\backslash\widetilde{O}(\beta'\beta'') \simeq G\backslash\widetilde{O}(\beta''\beta') \tag{4.9.5}$$

as stacks over $G\backslash G$, for all $\beta', \beta'' \in Br_W^+$. Note that the isomorphism does not hold at the level of varieties, i.e., without the quotients by G.

Remark 4.9.1. The transitive closure of the relation $\beta'\beta'' \sim \beta''\beta'$ defines an equivalence relation on Br_W^+ called **conjugacy by circular permutations** [17, Déf. 3.16]. It is weaker than the relation of being conjugate in Br_W: For example, if s, t are distinct simple reflections in $W = S_n$ for some $n \geq 3$, then $[\beta_s] = [\beta_t]$ in $Br_W = Br_n$, but $\beta_s \not\sim \beta_t$. It appears to be unknown whether $\widetilde{O}(\beta)$ only depends on the conjugacy class $[\beta] \subseteq Br_W$.

Let $\tilde{G} = \widetilde{O}(1)$. By definition, $\tilde{G} \subseteq \mathcal{B} \times G$ is the subvariety of pairs (xB, g) such that $g \in xBx^{-1}$, so the map $pr_1 : \tilde{G} \to G$ is the analogue of the Grothendieck–Springer resolution $\boldsymbol{s} : \tilde{\mathfrak{g}} \to \mathfrak{g}$ with G in place of $\mathfrak{g}$. In particular, there is a W-action on $pr_{1,*}\mathbf{C} \in \mathsf{D}^b(G)$. Like in the Lie-algebra setting, all of these structures admit stacky versions in which we take quotients by G.

Let $\mathcal{U}$ be the unipotent locus of G. In [106], for any $\beta \in Br_W^+$, we defined the **Steinberg-like variety** $St(\beta)$ by the cartesian square:

$$\begin{array}{ccc} St(\beta) & \longrightarrow & \tilde{G} \times_G \widetilde{O}(\beta) \\ \downarrow & & \downarrow \\ \mathcal{U} & \longrightarrow & G \end{array} \tag{4.9.6}$$

Strictly speaking, the paper [105] adopts the setting of a finite field $\mathbf{F}_q$ rather than $\mathbf{C}$. Via the comparison theorems for weight structures on the cohomology of varieties over $\mathbf{F}_q$ versus $\mathbf{C}$, we can ignore the difference in what follows.

4.9.2. Virtual Weight Series As explained in *ibid.*, the W-action on $pr_{1,*}\mathbf{C}$ induces an action on the compactly-supported cohomology of $\mathit{St}(\beta)$, hence that of the **Steinberg-like stack** $G\backslash \mathit{St}(\beta)$. Therefore the virtual weight series $\mathrm{E}(\boldsymbol{q}^{\frac{1}{2}}, G\backslash \mathit{St}(\beta))$ (see §4.1.6) forms an element of $\mathrm{K}_0(W)[\![\boldsymbol{q}^{\frac{1}{2}}]\!]$.

In [106, §3], we showed the identity

$$\textsc{ann}(\beta) = (-1)^{|\beta|-r}\mathrm{E}(\boldsymbol{q}^{\frac{1}{2}} \mid G\backslash \mathit{St}(\beta)) \tag{4.9.7}$$

in $\mathrm{K}_0(W)[\![\boldsymbol{q}^{\frac{1}{2}}]\!]$. Moreover, in terms of

$$\mathrm{E}^{\mathrm{H}}(\boldsymbol{q}^{\frac{1}{2}}, t \mid -) = \bigoplus_{j,k} \boldsymbol{q}^{\frac{j}{2}} t^k \,\mathrm{Gr}_j^{\mathbf{W}}\, \mathrm{H}_c^k(-, \mathbf{C}), \tag{4.9.8}$$

we *conjectured* the refined identity

$$\mathrm{AH}(\beta) = \left(\frac{\boldsymbol{q}t^2 - 1}{1 - \boldsymbol{q}}\right)^{r(w)} \mathrm{E}^{\mathrm{H}}(\boldsymbol{q}^{\frac{1}{2}}, t \mid G\backslash \mathit{St}(\beta)) \tag{4.9.9}$$

in $\mathrm{K}_0(W)[\![\boldsymbol{q}^{\frac{1}{2}}]\!][t]$, where as usual $w \in W$ is the image of β. Note that (4.9.7) and (4.9.9) are compatible by way of:

Lemma 4.9.2. *For all $\beta \in Br_W$, the integer $|\beta| - r + r(w)$ is even.*

Remark 4.9.3. If $[\beta] = [\beta_a]$ and $a : (D, \eta) \to (\mathfrak{c}, \mathfrak{c}^\circ)$ admits a model of finite type, then this fact can also be seen from Theorem 4.6.4.

Proof. If β changes by β_s, then $|\beta|$ changes by 1. Simultaneously, w changes by s. By induction on the Bruhat order of W, we can show that this changes $r(w)$ by 1. □

Comparing (4.9.7) and (4.9.9) against Conjectures C and D and accounting for the lemma, we obtain:

Proposition 4.9.4. *Suppose that for some $a \in \mathcal{A}^{\diamondsuit_0}(\mathbf{C})$, Conjecture C holds and $[\beta_{a,0}] \subseteq Br_W$ admits a positive representative. Let*

$$\lambda_a(\boldsymbol{q}, t) = (1 + \boldsymbol{q}t)^{N_a - r(w_{a,0})}(\boldsymbol{q}t^2 - 1)^{r(w_{a,0})}, \tag{4.9.10}$$

where $[w_{a,0}] \subseteq W$ is the image of $[\beta_{a,0}]$. Then:

1. *We have*

$$\mathrm{E}^{\mathbf{P}}(1, \boldsymbol{q} \mid \tilde{\mathcal{M}}_a)_{st} = \lambda_a(\boldsymbol{q}, -1) \cdot \mathrm{E}(\boldsymbol{q}^{\frac{1}{2}} \mid G \backslash St(\beta)) \tag{4.9.11}$$

   in $\mathrm{K}_0(W)[\boldsymbol{q}]$.

2. *If (4.9.9) also holds, then we have*

$$\mathrm{E}^{\mathbf{P}}(-t, \boldsymbol{q} \mid \tilde{\mathcal{M}}_a)_{st} = \lambda_a(\boldsymbol{q}, t) \cdot \mathrm{E}^{\mathrm{H}}(\boldsymbol{q}^{\frac{1}{2}}, t \mid G \backslash St(\beta)) \tag{4.9.12}$$

   in $\mathrm{K}_0(W)[\boldsymbol{q}, t]$.

Significantly, in both of these identities, $\boldsymbol{q}$ tracks a perverse grading *on the left and $\boldsymbol{q}^{\frac{1}{2}}$ tracks a* weight grading *on the right.*

Proposition 4.9.5. *Suppose that Σ is of genus 0 and $a \in \mathcal{A}^{\mathrm{ani}}(\mathbf{C})$ is tidy. If Conjecture D holds for a and $[\beta_{a,0}] \subseteq Br_W$ admits a positive representative, then we have*

$$\mathrm{E}^{\mathbf{P}}(1, \boldsymbol{q} \mid \mathcal{F}l_{B,a_0})_{st} = \mathrm{E}(\boldsymbol{q}^{\frac{1}{2}} \mid G \backslash St(\beta)) \tag{4.9.13}$$

in $\mathrm{K}_0(W)[\boldsymbol{q}]$.

Remark 4.9.6. For $\beta \in Br_W$ not necessarily positive, we showed in [106, §3] that we still have

$$(-1)^{|\beta|-r}\mathrm{ANN}(\beta) = \sum_{w \in W} m_w(\boldsymbol{q})\mathrm{E}(\boldsymbol{q}^{\frac{1}{2}} \mid G\backslash St(\beta_w)) \tag{4.9.14}$$

in $\mathrm{K}_0(W)[\![\boldsymbol{q}^{\frac{1}{2}}]\!]$, where the $m_w(\boldsymbol{q}) \in \mathbf{Z}[\boldsymbol{q}]$ are polynomials such that

$$\boldsymbol{q}^{\frac{|\beta|}{2}}\beta = \sum_{w \in W} m_w(\boldsymbol{q})\boldsymbol{q}^{\frac{|w|}{2}}\beta_w \tag{4.9.15}$$

in the Iwahori–Hecke algebra of W over $\mathbf{Z}[\boldsymbol{q}^{\pm\frac{1}{2}}]$ (see *ibid.* for the definition). In this way, we can extend Propositions 4.9.4 and 4.9.5 to the setting where we do not assume that $[\beta_{a,0}]$ intersects Br_W^+. Nonetheless, we expect every algebraic braid class to intersect Br_W^+.

4.9.3. Unipotent vs. Nilpotent Loci Let $\mathcal{N}$ be the nilpotent locus of $\mathfrak{g}$, and let $\widetilde{\mathcal{N}}$ be the pullback of $\mathcal{N}$ along the Grothendieck–Springer map $\boldsymbol{s} : \tilde{\mathfrak{g}} \to \mathfrak{g}$. According to [63, §IV.3], we can fix a G-equivariant (algebraic) isomorphism

$$\upsilon : \mathcal{N} \xrightarrow{\sim} \mathcal{U}. \tag{4.9.16}$$

(Note that υ is not unique in general.) For all $e \in \mathcal{N}$, it induces an isomorphism from the Springer fiber $\mathcal{B}_e \subseteq \widetilde{\mathcal{N}}$ to the underlying reduced scheme of the fiber $\tilde{G}_{\upsilon(e)} \subseteq \tilde{G}$. This isomorphism respects the weight structures and W-actions on their cohomology.

The cohomology of the Springer fibers $\mathcal{B}_e$ can be endowed with "uninteresting" perverse and weight filtrations in the following sense.

- By the smallness of the Grothendieck–Springer map $\boldsymbol{s}$, the perverse truncations of $\boldsymbol{s}_*\mathbf{C}$ coincide with its standard truncations, so the perverse filtration on the cohomology of $\mathcal{B}_e$ coincides with the (increasing) cohomological filtration.
- The cohomology of $\mathcal{B}_e$ is pure [104] and concentrated in even degrees [24], so the weight filtration coincides with the cohomological filtration.

In formulæ, we can write:

$$\mathrm{E}^{\mathbf{P}}(1, \boldsymbol{q}^{\frac{1}{2}} \mid \mathcal{B}_e) = \mathrm{E}(\boldsymbol{q}^{\frac{1}{2}} \mid \mathcal{B}_e) = \sum_{i \geq 0} \boldsymbol{q}^{\frac{i}{2}} \mathrm{H}^i(\mathcal{B}_e, \mathbf{C}). \tag{4.9.17}$$

From these observations, we will draw a surprising corollary to Proposition 4.9.5.

Recall from [60] that an element $\gamma \in \mathfrak{g}(D) \cap \mathfrak{g}(\bar{\eta})$ is said to be **topologically nilpotent** iff $\gamma(0) \in \mathcal{N}(\mathbf{C})$. If γ maps to $a \in \mathfrak{c}(D) \cap \mathfrak{c}^\circ(\eta)$, then this condition is equivalent to the condition $a(0) = 0$. For such γ, we have a cartesian square:

$$\begin{array}{ccc} \mathcal{F}l_{B,\gamma} & \xrightarrow{ev_B} & \widetilde{\mathcal{N}}/B \\ \downarrow & & \downarrow {\scriptstyle s|_{\tilde{\mathcal{N}}}} \\ \mathcal{F}l_{G,\gamma} & \xrightarrow{ev_G} & \mathcal{N}/G \end{array} \tag{4.9.18}$$

For all $e \in \mathcal{N}$, we let $\mathcal{F}l_{G,\gamma,[e]}$ denote the fiber of $\mathcal{F}l_{G,\gamma}$ above the G-orbit $[e] \subseteq \mathcal{N}$. If we further assume that γ is elliptic (see Section 4.4), so that $\mathcal{F}l_{G,\gamma}$ is of finite type, then in the Grothendieck ring of varieties over $\mathbf{C}$, we have a decomposition:

$$[\mathcal{F}l_{B,\gamma}] = \sum_{[e] \in \mathcal{N}/G} [\mathcal{F}l_{G,\gamma,[e]}][\mathcal{B}_e]. \tag{4.9.19}$$

At the same time, we have a similar cartesian square involving the varieties $\widetilde{O}(\beta)$ and $St(\beta)$. Letting $\widetilde{\mathcal{U}}$ be the pullback of $\mathcal{U}$ along the map $pr_1 : \tilde{G} \to G$, the square is:

$$\begin{array}{ccc} St(\beta) & \longrightarrow & \widetilde{\mathcal{U}} \\ \downarrow & & \downarrow {\scriptstyle pr_1|_{\tilde{\mathcal{U}}}} \\ \mathcal{U} \times_G \widetilde{O}(\beta) & \longrightarrow & \mathcal{U} \end{array} \tag{4.9.20}$$

For all $u \in \mathcal{U}$, we let $\widetilde{O}(\beta)_{[u]}$ denote the fiber of $\widetilde{O}(\beta)$ above the G-orbit $[u] \subseteq \mathcal{U}$. Then in the Grothendieck ring of varieties,

$$[St(\beta)] = \sum_{[u] \in G\backslash\mathcal{U}} [\widetilde{O}(\beta)_{[u]}][\tilde{G}_u]. \tag{4.9.21}$$

Given that the formation of the virtual weight series $\mathrm{E}(\boldsymbol{q}^{\frac{1}{2}} \mid -)$ factors through the Grothendieck ring, we further obtain the identity:

$$\mathrm{E}(\boldsymbol{q}^{\frac{1}{2}} \mid G\backslash St(\beta)) = \sum_{[u] \in G\backslash\mathcal{U}} \mathrm{E}(\boldsymbol{q}^{\frac{1}{2}} \mid G\backslash\widetilde{O}(\beta)_{[u]})\mathrm{E}(\boldsymbol{q}^{\frac{1}{2}} \mid \tilde{G}_u). \tag{4.9.22}$$

By way of the isomorphisms $\mathcal{B}_e \simeq \tilde{G}^{\mathrm{red}}_{\upsilon(e)}$ for all $e \in \mathcal{N}(\mathbf{C})$, we conclude:

Corollary 4.9.7. *Under the hypotheses of Proposition 4.9.5, we can decompose:*

$$\mathrm{E}^{\mathbf{P}}(1, \boldsymbol{q} \mid \mathcal{F}l_{B,a_0})_{st} = \sum_{[e] \in \mathcal{N}/G} \mathrm{E}(\boldsymbol{q}^{\frac{1}{2}} \mid G\backslash\widetilde{O}(\beta_{a,0})_{[\upsilon(e)]})\mathrm{E}^{\mathbf{P}}(1, \boldsymbol{q}^{\frac{1}{2}} \mid \mathcal{B}_e). \tag{4.9.23}$$

That is: By way of (4.9.19), the perverse grading on $\mathrm{Gr}^{\mathbf{P}}_* \mathrm{H}^*(\mathcal{F}l_{B,a_0})_{st}$ *factors through the* halved *perverse gradings on* $\mathrm{H}^*(\mathcal{B}_e)$ *for each* e.

In particular, up to the original choice of $a \in \mathcal{A}^{\mathrm{ani}}$, *there is a canonical* $\boldsymbol{q}$*-deformation of the Euler characteristic of* $\mathcal{F}l_{G,a_0,[e]}$*: namely,*

$$\chi(\mathcal{F}l_{G,a_0,[e]}) = \mathrm{E}(\boldsymbol{q}^{\frac{1}{2}} \mid G\backslash\widetilde{O}(\beta_{a,0})_{[\upsilon(e)]})|_{\boldsymbol{q}^{1/2} \to 1}. \tag{4.9.24}$$

In general, this $\boldsymbol{q}$*-deformation bears no relation to* $\mathrm{E}(\boldsymbol{q}^{\frac{1}{2}} \mid \mathcal{F}l_{G,a_0,[e]})$ *itself.*

We speculate that there is an intrinsic definition of this $\boldsymbol{q}$-deformation of $\chi(\mathcal{F}l_{G,a_0,[e]})$, depending only on a_0 and $[e]$ but not involving the braid $\beta_{a,0}$.

4.10 The Degenerate Affine Hecke Action

Throughout this section, we write (1) to denote the Tate twist of any weight structure. It shifts weights down by 2, so the shift-twist [2](1) preserves weights.

4.10.1. The AHA The notation $\langle -, - \rangle : \mathbf{X} \times \mathbf{X}^\vee \to \mathbf{Z}$ will denote the evaluation pairing, as well as its base change from $\mathbf{X}$ to $\mathbf{X}_\mathbf{C}$.

Recall from Section 4.3 that $\mathrm{Ref}(W)$ is the set of reflections in W. For all $t \in \mathrm{Ref}(W)$, let

(4.10.1)
$$(\alpha_t, \alpha_t^\vee) \in \Phi_+ \times \Phi_+^\vee$$

be the corresponding positive root-coroot pair, so that $\langle \alpha_t, \alpha_t^\vee \rangle = 2$.

Following Lusztig [72], the **degenerate affine Hecke algebra (AHA)** of G is

(4.10.2)
$$\mathbf{H} = \frac{\mathbf{C}[u](W \otimes \mathrm{Sym}^*(\mathbf{X}_\mathbf{C}))}{\langle s\xi - {}^s\xi s - u\langle \xi, \alpha_s^\vee \rangle : \xi \in \mathbf{X}_\mathbf{C},\, s \in S \rangle}.$$

Its "rational" degeneration is

(4.10.3)
$$\mathbf{A} = \mathbf{C}[u](W \ltimes \mathrm{Sym}^*(\mathbf{X}_\mathbf{C})),$$

where the symbol $\ltimes$ means $s\xi = {}^s\xi s$ for all $\xi \in \mathbf{X}_\mathbf{C}$ and $s \in S$. This algebra appeared in our papers [105, 107].

Consider the following bigrading:

1. $\deg u = (2, 0)$.

2. $\deg \xi = (2, 1)$ for all $\xi \in \mathbf{X}_\mathbf{C}$.

3. $\deg w = (0, 0)$ for all $w \in W$.

We equip $\mathbf{H}$ with the *first* component of this bigrading, and $\mathbf{A}$ with *both* components.

4.10.2. The AHA Action Theorem B of Yun's paper [113] specializes to the following statement: There is a morphism of graded algebras

$$\mathbf{H} \to \bigoplus_{i \geq 0} \mathrm{End}^{2i}(\boldsymbol{h}^{\mathrm{ani}}_{B,*}\mathbf{C})(i), \tag{4.10.4}$$

where $\mathrm{End}^{2i}(\boldsymbol{h}^{\mathrm{ani}}_{B,*}\mathbf{C}) = \mathrm{Hom}_{\mathsf{D}^b(\mathcal{A}^{\mathrm{ani}}\times\Sigma)}(\boldsymbol{h}^{\mathrm{ani}}_{B,*}\mathbf{C}, \boldsymbol{h}^{\mathrm{ani}}_{B,*}\mathbf{C}[2i])$. Above,

1. The action of u is induced by cup product with the Chern class $c_1(L|_{\mathcal{A}^{\mathrm{ani}}\times\Sigma})$, where L is our fixed line bundle over Σ.
2. The action of $\mathbf{X}$ is induced by cup product with the Chern classes $c_1(\mathcal{O}(\lambda)|_{\mathcal{M}^{\mathrm{ani}}_B})$ for $\lambda \in \mathbf{X}$, where $\mathcal{O}(\lambda)$ is the line bundle over $\cdot/T$ defined by λ and we pull back along the composition of maps

$$\mathcal{M}^{\mathrm{ani}}_B \to [\mathfrak{b}/B]_L \to [\mathfrak{t}/T]_L \to \cdot/T. \tag{4.10.5}$$

3. The action of W on $\boldsymbol{h}^{\mathrm{ani}}_{B,*}\mathbf{C}$ is the one discussed in §4.5.2.

The graded $\mathbf{H}$-action on $\boldsymbol{h}^{\mathrm{ani}}_{B,*}\mathbf{C}$ induces a graded action on the stable summand:

1. The action of $c_1(L|_{\mathcal{A}^{\mathrm{ani}}\times\Sigma})$ on $\boldsymbol{h}^{\mathrm{ani}}_{B,*}\mathbf{C}$ commutes with that of $\pi_0(\mathcal{P}/\mathcal{A}^{\mathrm{ani}})|_{\mathcal{A}^{\mathrm{ani}}\times\Sigma}$, so it restricts to an action on $(\boldsymbol{h}^{\mathrm{ani}}_{B,*}\mathbf{C})_{st}$.
2. The $c_1(\mathcal{O}(\lambda)|_{\mathcal{M}^{\mathrm{ani}}_B})$-action on $\boldsymbol{h}^{\mathrm{ani}}_{B,*}\mathbf{C}_{st}$ is defined by

$$(\boldsymbol{h}^{\mathrm{ani}}_{B,*}\mathbf{C})_{st} \hookrightarrow \boldsymbol{h}^{\mathrm{ani}}_{B,*}\mathbf{C} \xrightarrow{c_1(\lambda)} \boldsymbol{h}^{\mathrm{ani}}_{B,*}\mathbf{C}[2](1) \twoheadrightarrow (\boldsymbol{h}^{\mathrm{ani}}_{B,*}\mathbf{C})_{st}[2](1), \tag{4.10.6}$$

where the first and last arrows are the direct-summand inclusion and projection maps, respectively.

The $\mathbf{H}$-action interacts with the perverse filtration $\mathbf{P}_{\leq *}$ in the following way:

1. The action of $c_1(L|_{\mathcal{A}^{\mathrm{ani}}\times\Sigma})$ takes place over $\mathcal{A}^{\mathrm{ani}} \times \Sigma$, so it preserves the perverse filtration and descends to:

$$\mathbf{P}_{\leq j}(\boldsymbol{h}^{\mathrm{ani}}_{B,*}\mathbf{C})_{st} \to \mathbf{P}_{\leq j}(\boldsymbol{h}^{\mathrm{ani}}_{B,*}\mathbf{C})_{st}[2](1). \tag{4.10.7}$$

2. The action of $c_1(\mathcal{O}(\lambda)|_{\mathcal{M}^{\mathrm{ani}}_B})$ is induced by a morphism $\mathbf{C} \to \mathbf{C}[2](1)$ over $\mathcal{M}^{\mathrm{ani}}_B$, so it descends to a morphism

$$\mathbf{P}_{\leq j}(\boldsymbol{h}^{\mathrm{ani}}_{B,*}\mathbf{C})_{st} \to \mathbf{P}_{\leq j+2}(\boldsymbol{h}^{\mathrm{ani}}_{B,*}\mathbf{C})_{st}[2](1). \tag{4.10.8}$$

   However, by Lem. 3.2.3 of [114], the post-composition of the above map with the projection to $\mathrm{Gr}^{\mathbf{P}}_{\leq j+2}(\boldsymbol{h}^{\mathrm{ani}}_{B,*}\mathbf{C})_{st}[2](1)$ is zero. So we obtain a morphism

$$\mathbf{P}_{\leq j}(\boldsymbol{h}^{\mathrm{ani}}_{B,*}\mathbf{C})_{st} \to \mathbf{P}_{\leq j+1}(\boldsymbol{h}^{\mathrm{ani}}_{B,*}\mathbf{C})_{st}[2](1). \tag{4.10.9}$$

By [90, Prop. 4.3.1], these properties imply that the graded $\mathbf{H}$-action on $(\boldsymbol{h}^{\mathrm{ani}}_{B,*}\mathbf{C})_{st}$ descends to a bigraded $\mathbf{A}$-action on $\mathrm{Gr}^{\mathbf{P}}_{*}(\boldsymbol{h}^{\mathrm{ani}}_{B,*}\mathbf{C})_{st}$. That is, we have a bigraded morphism

$$\mathbf{A} \to \bigoplus_{i,j} \mathrm{End}^{2i}(\mathrm{Gr}^{\mathbf{P}}_{j}\, \boldsymbol{h}^{\mathrm{ani}}_{B,*}\mathbf{C})_{st}(i), \tag{4.10.10}$$

extending the action $W \to \mathrm{End}^0(\mathrm{Gr}^{\mathbf{P}}_{*}\, \boldsymbol{h}^{\mathrm{ani}}_{B,*}\mathbf{C})$ from §4.5.2, where

$$\deg \mathrm{Gr}^{\mathbf{P}}_{j}\, \mathrm{End}^{2i}(\boldsymbol{h}^{\mathrm{ani}}_{B,*}\mathbf{C})_{st}(i) = (2i, j) \tag{4.10.11}$$

for all i, j.

Let $\tilde{\boldsymbol{h}}^{\mathrm{ani}} : \tilde{\mathcal{M}}^{\mathrm{ani}} \to \mathcal{A}^{\mathrm{ani}}$ be the pullback of $\boldsymbol{h} : \mathcal{M}^{\mathrm{ani}}_B \to \mathcal{A}^{\mathrm{ani}} \times \Sigma$ to the basepoint 0. Base change yields an $\mathbf{A}$-action on $\tilde{\boldsymbol{h}}^{\mathrm{ani}}\mathbf{C}$. So for all $a \in \mathcal{A}^{\mathrm{ani}}(\mathbf{C})$, we have a bigraded $\mathbf{A}$-action on

$$\bigoplus_{i,j,k} \mathrm{Gr}_k^{\mathbf{W}} \mathrm{Gr}_j^{\mathbf{P}} \mathrm{H}^i(\tilde{\mathcal{M}}_a, \mathbf{C})_{st}. \tag{4.10.12}$$

that preserves the weight grading. Note that while the action of $\mathrm{Sym}^*(\mathbf{X}_{\mathbf{C}}) \subseteq \mathbf{A}$ respects the perverse grading, it only respects the increasing filtration by cohomological degree, not the cohomological grading itself.

4.10.3. A Conjecture for A-Modules? In Section 4.7, we presented annular braid homology as a class function on Br_W valued in $\mathrm{K}_0^+(W)[\![\boldsymbol{q}^{\frac{1}{2}}]\!][t]$. However, the construction in [105, §7] shows something stronger:

Let $\mathbf{A}\text{-}\mathsf{Mod}_{\mathsf{gr}}^{(n)}$ be the category of graded $\mathbf{A}$-modules on which elements of $\mathbf{X}_{\mathbf{C}}$ act by degree n. Let $\mathrm{K}_0(\mathbf{A}\text{-}\mathsf{Mod}_{\mathsf{gr}}^{(n)})$ be the split Grothendieck group of $\mathbf{A}\text{-}\mathsf{Mod}_{\mathsf{gr}}^{(n)}$, and let $\mathrm{K}_0^+ \subseteq \mathrm{K}_0$ be the semiring of actual, not virtual, modules. Then our construction in [105] actually produces a class function:

$$\mathrm{AH} = \bigoplus_i t^i \mathrm{AH}_i : Br_W \to \mathrm{K}_0^+(\mathbf{A}\text{-}\mathsf{Mod}_{\mathsf{gr}}^{(2)})[t]. \tag{4.10.13}$$

At the same time, the preceding discussion shows that we can write:

$$\bigoplus_{i,k} (-1)^{i+k} t^k \bigoplus_j \mathrm{Gr}_j^{\mathbf{P}} \mathrm{Gr}_k^{\mathbf{W}} \mathrm{H}^i(\tilde{\mathcal{M}}_a, \mathbf{C})_{st} \in \mathrm{K}_0(\mathbf{A}\text{-}\mathsf{Mod}_{\mathsf{gr}}^{(1)})[t]. \tag{4.10.14}$$

In this light, it is natural to ask whether Conjecture A/B can be upgraded to a conjecture that compares bigraded $\mathbf{A}$-modules, possibly up to a reindexing that sends $\mathbf{A}\text{-}\mathsf{Mod}_{\mathsf{gr}}^{(1)}$ into $\mathbf{A}\text{-}\mathsf{Mod}_{\mathsf{gr}}^{(2)}$.

However, this cannot work. In the next section, we will explain that a variant of (4.10.14), involving $\mathbf{G}_m$-equivariant cohomology, forms a module over a larger algebra called the rational Cherednik algebra. The action of $\mathbf{X}$ will preserve the equivariant parameter. For cases where we can compute the $\mathbf{X}$-action on the $t \to -1$ limit, the result fails to match the $\mathbf{X}$-action on the $t \to -1$ limit of (4.10.13) when $\beta = \beta_{a,0}$. In fact, we discussed an example of this mismatch in the introduction to our paper [107].

Recall from Section 4.9 that for $\beta \in Br_W^+$, we have conjectured an identity that relates the underlying W-representation of $\mathrm{AH}(\beta)$ and the compactly-supported cohomology of the Steinberg-like stack $G\backslash St(\beta)$. There is an action of $\mathrm{H}^*(\,\cdot\,/T) \simeq \mathrm{Sym}^*(\mathbf{X}_\mathbf{C})$ on $\mathrm{H}_c^*(G\backslash St(\beta))$ by pullback of Chern classes along

$$G\backslash St(\beta) \to G\backslash \tilde{G} \xrightarrow{\sim} B\backslash B \to T\backslash T \to T\backslash\,\cdot \tag{4.10.15}$$

By direct calculation, one checks that the W- and $\mathbf{X}_\mathbf{C}$-actions on $\mathrm{H}_c^*(G\backslash St(\beta))$ generate an $\mathbf{A}$-action. From this observation, and the $\mathrm{P} = \mathrm{W}$ philosophy that underpins the discussion in Section 4.9, we suspect that the $\mathbf{A}$-module formed by

$$\bigoplus_k t^k \bigoplus_j \mathrm{Gr}_j^{\mathbf{W}} \mathrm{H}_c^k(G\backslash St(\beta), \mathbf{C}) \tag{4.10.16}$$

is—up to normalization—the replacement for (4.10.13) that fixes the mismatch of $\mathbf{X}$-actions, at least when $\beta_{a,0}$ is a positive braid. We will explore this possibility in a sequel.

4.11 The Oblomkov–Yun Examples

In this section, we establish infinitely many cases of Conjecture D for any irreducible Dynkin diagram, by combining the work of Oblomkov–Yun with our work from [107]. The crucial link between the two is Proposition 4.3.9. As a consequence, we get examples "beyond type A" of the $\mathrm{P} = \mathrm{W}$ phenomena proposed in Section 4.9.

4.11.1. The DAHA According to Oblomkov–Yun [90], the **rational Cherednik algebra** or **rational DAHA** of G is

$$\mathbf{A}^{\mathrm{rat}} = \frac{\mathbf{C}[u,\omega](W \ltimes (\mathrm{Sym}^*(\mathbf{X}_{\mathbf{C}}) \otimes \mathrm{Sym}^*(\mathbf{X}_{\mathbf{C}}^\vee)))}{\left\langle \eta\xi - \xi\eta - \delta\langle \xi, \eta\rangle - u \displaystyle\sum_{t \in \mathrm{Ref}(W)} \langle \xi, \alpha_t^\vee\rangle\langle \alpha_t, \eta\rangle t : \xi \in V,\, \eta \in V^\vee \right\rangle}. \tag{4.11.1}$$

It is equipped with the following bigrading:

1. $\deg u = \deg \omega = (2, 0)$.
2. $\deg \xi = (2, 1)$ for all $\xi \in V$.
3. $\deg \eta = (0, -1)$ for all $\eta \in V^\vee$.
4. $\deg w = (0, 0)$ for all $w \in W$.

Thus, there is an inclusion of bigraded algebras $\mathbf{A} \subseteq \mathbf{A}^{\mathrm{rat}}$. In the paper [107], we considered $\mathbf{A}^{\mathrm{rat}}$ without the second component of the bigrading.

For any $\nu \in \mathbf{C}$, we have two versions of the **rational DAHA of central charge** ν:

$$\mathbf{A}^{\mathrm{rat}}_{\nu\omega} = \mathbf{A}^{\mathrm{rat}}/\langle u + \nu\omega\rangle, \tag{4.11.2}$$

$$\mathbf{A}^{\mathrm{rat}}_{\nu} = \mathbf{A}^{\mathrm{rat}}_{\nu}/\langle \omega - 1\rangle. \tag{4.11.3}$$

The grading on $\mathbf{A}^{\mathrm{rat}}$ descends to a grading on $\mathbf{A}^{\mathrm{rat}}_{\nu\omega}$, which in turn descends to an increasing filtration on $\mathbf{A}^{\mathrm{rat}}_{\nu}$. Note that $\mathbf{A}^{\mathrm{rat}}_{\nu\omega}$ will not appear again until Theorem 4.11.6 below.

We write O_ν for the Bernstein–Gelfand–Gelfand category of modules over $\mathbf{A}^{\mathrm{rat}}_{\nu}$. The Verma modules in O_ν are indexed by $\mathrm{Irr}(W)$, the set of irreducible characters of W. For all $\phi \in \mathrm{Irr}(W)$, we write $\Delta_\nu(\phi) \in \mathsf{O}_\nu$ for the associated Verma module and $L_\nu(\phi)$ for its simple quotient. By definition,

$$\Delta_\nu(\phi) = \mathbf{A}^{\mathrm{rat}}_{\nu} \otimes_{\mathbf{A}} \phi. \tag{4.11.4}$$

In the literature, $\Delta_\nu(1)$, resp. $L_\nu(1)$, is sometimes called the **polynomial module**, resp. the **simple spherical module**.

Let $\boldsymbol{h} \in \mathbf{A}_\nu^{\mathrm{rat}}$ be the element formed by the Euler vector field on $\mathbf{X}_{\mathbf{C}}^\vee$ (see [38, Ch. 11]). Every object $M \in \mathsf{O}_\nu$ admits a decomposition into finite-dimensional, W-stable eigenspaces under $\boldsymbol{h}$. In our work, the eigenvalues will always be half-integers. Thus we can define the **graded W-character** of M to be

$$[M]_{\boldsymbol{q}} = \sum_{j \in \mathbf{Z}} \boldsymbol{q}^{\frac{j}{2}} M_{j/2} \in \mathrm{K}_0(W)(\!(\boldsymbol{q}^{\frac{1}{2}})\!), \tag{4.11.5}$$

where $M_{j/2}$ denotes the $\boldsymbol{h}$-eigenspace of eigenvalue $j/2$.

For generic $\nu \in \mathbf{C}$, the category O_ν is semisimple and $L_\nu(\phi) = \Delta_\nu(\phi)$ for all ϕ. For $\nu \in \mathbf{Q}$, its behavior is highly sensitive to ν, as we now describe.

Regularity Let $\zeta = e^{2\pi i \nu}$. We say that ν is a **regular slope** iff there exist $\xi \in \mathfrak{t}^\circ$ and $w \in W$ such that ξ is an eigenvector of w with eigenvalue ζ. In this case, we say that w is **ζ-regular**.

Springer showed [101] that all ζ-regular elements of W are conjugate. It follows from work of Broué–Michel [17, Prop. 3.11, Th. 3.12] that ν is regular if and only if there exists a fractional twist $\beta \in Br_W$ of slope ν (see Section 4.3). In this case, we showed in [107, §6] that $q^{r-|\beta|}\mathrm{ANN}(\beta)$ is the graded W-character of a virtual $\mathbf{A}^{\mathrm{rat}}$-module in which $L_\nu(1)$ occurs with multiplicity one.

Ellipticity We say that ν is a **regular elliptic slope** iff it is a regular slope and, in the situation above, we can choose w so that $\mathfrak{t}^w = 0$.

Varagnolo–Vasserot showed [109] that ν is regular elliptic if and only if $L_\nu(1)$ is finite-dimensional.

Cuspidality As before, $d_1, \dots, d_r$ are the weights of the W-action on $\mathfrak{t}$. We say that ν is a **cuspidal slope** iff there is a unique index i such that the denominator of ν in lowest terms divides d_i, or equivalently, such that $\zeta \in \mu_{d_i}$.

In [107, §6], we showed that if ν is cuspidal and $\beta \in Br_W$ is a fractional twist of slope ν, then $\textsc{ann}(\beta)$ is the graded W-character of a sum of simple $\mathbf{A}^{\mathrm{rat}}$-modules that each occur with multiplicity one. If, in addition, W is irreducible, then

$$q^{r-|\beta|}\textsc{ann}(\beta) = \begin{cases} [L_\nu(1)]_{\boldsymbol{q}} + [L_\nu(\mathfrak{t})]_{\boldsymbol{q}} & W = W(E_8) \text{ and } \nu \in \frac{1}{15}\mathbf{Z} \\ [L_\nu(1)]_{\boldsymbol{q}} & \text{else} \end{cases} \tag{4.11.6}$$

Notably, if ν is a **Coxeter slope**, in the sense that its denominator in lowest terms is the Coxeter number of W (see Example 4.3.10), then ν is cuspidal and $\textsc{ann}(\beta) = [L_\nu(1)]_{\boldsymbol{q}}$.

4.11.2. Homogeneity Revisited We turn to the geometric setup used by Oblomkov–Yun [90, §6] to construct representations of $\mathbf{A}^{\mathrm{rat}}_\nu$.

Fix $n \in \mathbf{Z}$. We take Σ to be the orbifold curve:

$$\mathbf{P}^1(n, 1) = (\mathbf{A}^2 \setminus (0,0))/\mathbf{G}_m, \tag{4.11.7}$$

where $\mathbf{G}_m$ acts on $\mathbf{A}^2$ with weights $(n, 1)$. In the weighted homogeneous coordinates that $\mathbf{P}^1(n, 1)$ inherits from $\mathbf{A}^2$, we set $0 = [0 : 1]$ and $\infty = [1 : 0]$. Then 0 is an ordinary point, whereas ∞ has automorphism group μ_n (and ∞ is the only stacky point).

Note that the degree map yields an isomorphism $\deg : \mathrm{Pic}(\Sigma) \xrightarrow{\sim} \frac{1}{n}\mathbf{Z}$. For a line bundle L of nonnegative degree, we have

$$\Gamma(\Sigma, L) = \{\text{homogeneous polynomials in } x, y \text{ of weight } n \deg L\}, \tag{4.11.8}$$

where $\deg x = n$ and $\deg y = 1$.

In §4.2.4, we introduced the notion of homogeneous loops. For our present choice of Σ, there is a global analogue of homogeneity, described below:

Via the inclusion $\mathbf{G}_m \subseteq \mathbf{P}^1(n,1) = \Sigma$, we have a $\mathbf{G}_m$-action on Σ. In coordinates, it is

$$s \cdot [x : y] = [sx : y]. \tag{4.11.9}$$

For any line bundle $L \to \Sigma$, we can lift this action to a $\mathbf{G}_m$-equivariant structure on L that contracts onto the fiber L_0. The induced action on global sections is

$$s \cdot f(x : y) = f(sx : y) \tag{4.11.10}$$

for all $f \in \Gamma(\Sigma, L)$. In this way, the $\mathbf{G}_m$-action on Σ gives rise to actions on the Hitchin moduli spaces $\mathcal{M}_P(G, \Sigma, L)$ and the Hitchin base $\mathcal{A}(G, \Sigma, L)$.

Concurrently, we have $\mathbf{G}_m$-actions on $\mathcal{M}_P$ and $\mathcal{A}$ that respectively arise from the $\mathbf{G}_m$-actions on $\mathfrak{g}$ and $\mathfrak{c}$ by dilation. So we actually have $(\mathbf{G}_m \times \mathbf{G}_m)$-actions on $\mathcal{M}_P$ and $\mathcal{A}$. In particular, the action on $\mathcal{A}$ is defined by

$$(s,t) \cdot a(x : y) = t \cdot a(sx : y) \tag{4.11.11}$$

once $a \in \mathcal{A}$ is viewed as a section $a : \Sigma \to \mathfrak{c}_L$. It restricts to an action on $\mathcal{A}^\heartsuit$.

As in §4.2.4, any rational number $\nu \in \mathbf{Q}$ defines a subtorus $\mathbf{G}_m(\nu) \subseteq \mathbf{G}_m \times \mathbf{G}_m$ (see (4.2.21)). Following [90], we say that $a \in \mathcal{A}$ is **homogeneous of slope** ν iff it is a fixed point of the $\mathbf{G}_m(\nu)$-action on $\mathcal{A}$. To make this explicit, suppose that the weights of the $\mathbf{G}_m$-action on $\mathfrak{c}$ are $d_1 \leq \ldots \leq d_r$. We can fix an isomorphism

$$\mathcal{A} \simeq \bigoplus_{1 \leq i \leq r} \Gamma(\Sigma, \mathcal{O}(d_i \deg L)). \tag{4.11.12}$$

In these coordinates, Lem. 6.4.2 of *ibid.* states:

Lemma 4.11.1 (Oblomkov–Yun)**.** *A point $a \in \mathcal{A}(\mathbf{C})$ is homogeneous of slope $\nu \in \mathbf{Q}$ if and only if there exist $c_1, \ldots, c_r \in \mathbf{C}$ such that*

$$a_i = c_i x^{d_i \nu} y^{d_i(\deg L - \nu)n} \tag{4.11.13}$$

for all i. In particular, we must have $d_i \nu \in \mathbf{Z}$ for all i such that $c_i \neq 0$.

We write $\mathcal{A}_\nu \subseteq \mathcal{A}$ for the (closed) locus of points that are homogeneous of slope ν. The lemma shows that a necessary and sufficient condition for $\mathcal{A}_\nu$ to be nonzero is $0 \leq \nu \leq \deg L$. We also set $\mathcal{A}_\nu^\heartsuit = \mathcal{A}_\nu \cap \mathcal{A}^\heartsuit$ and $\mathcal{A}_\nu^{\mathrm{ani}} = \mathcal{A}_\nu \cap \mathcal{A}^{\mathrm{ani}}$.

At this point, we draw the relation between Oblomkov–Yun's setup and the notions about Artin braids that we have introduced in this paper. By our discussion of algebraic braids in the orbifold setting from §4.6.3, and the explicit description of $\mathcal{A}_\nu$ from Lemma 4.11.1, we arrive at this result:

Proposition 4.11.2. *If $\nu \in \mathbf{Q}$ and $a \in \mathcal{A}_\nu^\heartsuit$, then:*

1. *$\mathfrak{D}_a \subseteq \{0, \infty\}$.*

2. *a_0 is homogeneous of slope ν.*

3. *a_∞ is homogeneous of slope $\deg L - \nu$.*

In particular, $[\beta_{a,0}]$ and $[\beta_{a,\infty}]$ are fractional twists.

Remark 4.11.3. Above, we see that $|\beta_{a,0}| + |\beta_{a,\infty}| = |\pi| \deg L$, which verifies that Corollary 4.6.9 continues to hold in this stacky setting.

The following corollary is implicit in [90]: The first statement can be deduced from their Thm. 3.2.5, which uses a more direct argument inspired by Moy–Prasad theory, and the second is essentially their Prop. 6.5.2.

Corollary 4.11.4 (Oblomkov–Yun)**.** *If $\mathcal{A}_\nu^\heartsuit$ is nonempty, then $\nu \in \mathbf{Q}$ is a regular slope. In this case,* $2 \dim \mathcal{P}_a = |\Phi| \deg L - r - r(w_{a,0})$.

Proof. The first statement follows from combining Proposition 4.11.2, Proposition 4.3.9, and the results of Broué–Michel that we mentioned in §4.11.1. As for the second statement, we must show in the notation of §4.6.3 that

$$2\varrho(G, \Sigma, L) = r + r(w_{a,0}). \tag{4.11.14}$$

By [101, Thm. 6.4(v)], the (diagonalizable) action of $w_{a,0}$ on $\mathfrak{t}$ has eigenvalues $e^{2\pi i(d_i-1)\nu}$ for $1 \leq i \leq r$. Let n be the denominator of ν in lowest terms, and for all m, let $\mathfrak{t}_m$ be the $e^{2\pi i \frac{m}{n}}$-eigenspace of $\mathfrak{t}$ under $w_{a,0}$. Then

$$\sum_{i=1}^{r} \varrho((d_i - 1)\deg L) = \sum_{m=1}^{n} \frac{m}{n} \dim \mathfrak{t}_m. \tag{4.11.15}$$

For $1 \leq m < n$, the Killing form restricts to a perfect pairing $\mathfrak{t}_m \times \mathfrak{t}_{n-m} \to \mathbf{C}$. Thus,

$$\sum_{m=1}^{n} \frac{m}{n} \dim \mathfrak{t}_m = \frac{1}{2}(\dim \mathfrak{t} + \dim \mathfrak{t}_0), \tag{4.11.16}$$

as needed. □

Corollary 4.11.5 (Oblomkov–Yun)**.** *Suppose that* $\deg L = \nu$*. Then $\nu \in \mathbf{Q}$ is a regular elliptic slope if and only if every point of $\mathcal{A}_\nu^{\mathrm{ani}}$ is tidy (see §4.7.3). In this case, $\mathcal{A}_\nu^\heartsuit = \mathcal{A}_\nu^{\mathrm{ani}}$.*

Proof. We see that ν is a regular elliptic slope if and only if both $\mathfrak{t}^{w_{a,0}} = 0$ and $w_{a,0}$ is $e^{2\pi i\nu}$-regular element of W. The first condition says $r(w_{a,0}) = 0$, and by the previous corollary, the second condition is equivalent to $\dim P_{a_0} = \dim \mathcal{P}_a$. Finally, the condition $\deg L = \nu$ ensures that $|\beta_{a,\infty}| = r - r(w_{a,\infty}) = 0$, whence $\delta(a_\infty) = 0$. □

4.11.3. The DAHA Action Henceforth, *we assume that the (connected) reductive group G is almost-simple and simply-connected.* Every crystallographic, irreducible finite Coxeter group W occurs as the Weyl group of some such G. *Also, we now assume that n is the denominator of ν in lowest terms.*

We keep the setup in which $\Sigma = \mathbf{P}^1(n,1)$, etc. Let

$$\tilde{\boldsymbol{h}}_\nu^\heartsuit : \tilde{\mathcal{M}}_\nu^\heartsuit \to \mathcal{A}_\nu^\heartsuit \tag{4.11.17}$$

be the pullback of $\tilde{\boldsymbol{h}}$ to $\mathcal{A}_\nu^\heartsuit$, so that $\tilde{\mathcal{M}}_\nu^\heartsuit$ and $\tilde{\boldsymbol{h}}_\nu^\heartsuit$ are fixed by the $\mathbf{G}_m(\nu)$-action on $\mathcal{M}_B$. As we will work with $\mathbf{G}_m(\nu)$-equivariant cohomology, we fix an isomorphism

$$\mathrm{H}^*(\,\cdot\,/\mathbf{G}_m(\nu), \mathbf{C}) \simeq \mathbf{C}[\omega], \tag{4.11.18}$$

where $\deg \omega = 2$, and abbreviate

$$\mathrm{H}^*_{\omega=\omega_0}(-) = \mathrm{H}^*_{\mathbf{G}_m(\nu)}(-)|_{\omega\to\omega_0}. \tag{4.11.19}$$

That is, $\mathrm{H}^*_{\omega=\omega_0}$ denotes $\mathbf{G}_m(\nu)$-equivariant cohomology where the parameter ω is specialized to ω_0. What follows are the main results of [90] regarding $\mathbf{A}^{\mathrm{rat}}$-actions on the cohomology of Hitchin fibers.

Theorem 4.11.6 (Oblomkov–Yun)**.** *Suppose that $\deg L = \nu$ and $\nu \in \mathbf{Q}$ is a regular elliptic slope of denominator n in lowest terms. Then:*

1. *The $\mathbf{A}$-action on $\mathrm{Gr}_*^{\mathbf{P}}(\tilde{\boldsymbol{h}}_*^{\mathrm{ani}}\mathbf{C})_{st}$ from Section 4.10 extends to a (bigraded) $\mathbf{A}^{\mathrm{rat}}$-action. The element $\omega \in \mathbf{A}^{\mathrm{rat}}$ acts by the equivariant parameter $\omega \in \mathrm{H}^*(\,\cdot\,/\mathbf{G}_m(\nu))$.*

2. *The $\mathbf{A}^{\mathrm{rat}}$-action on $\mathrm{Gr}_*^{\mathbf{P}}\,\tilde{\boldsymbol{h}}_*^{\mathrm{ani}}\mathbf{C}$ from (1) descends to an $\mathbf{A}^{\mathrm{rat}}_{\nu\omega}$-action on $\mathrm{Gr}_*^{\mathbf{P}}\,\tilde{\boldsymbol{h}}_{\nu,*}^\heartsuit\mathbf{C}$. Moreover, $(\tilde{\boldsymbol{h}}_{\nu,*}^\heartsuit\mathbf{C})_{st} = \tilde{\boldsymbol{h}}_{\nu,*}^\heartsuit\mathbf{C}$.*

The analogous statement to (1), resp. (2), holds in the $(\mathbf{G}_m \times \mathbf{G}_m)$-equivariant derived category of $\mathcal{A}^{\mathrm{ani}}$, resp. the $\mathbf{G}_m(\nu)$-equivariant derived category of $\mathcal{A}_\nu^\heartsuit$. Thus, for all $a \in \mathcal{A}_\nu^\heartsuit$, *there is a bigraded* $\mathbf{A}^{\mathrm{rat}}_{\nu\omega}$*-action on*

$$\mathrm{Gr}^{\mathbf{P}}_{*}\,\mathrm{H}^{*}_{\mathbf{G}_m(\nu)}(\tilde{\mathcal{M}}_a)_{st} = \mathrm{Gr}^{\mathbf{P}}_{*}\,\mathrm{H}^{*}_{\mathbf{G}_m(\nu)}(\tilde{\mathcal{M}}_a), \tag{4.11.20}$$

where $\deg \mathrm{Gr}^{\mathbf{P}}_{j}\,\mathrm{H}^{2i}_{\mathbf{G}_m(\nu)}(\tilde{\mathcal{M}}_a) = (2i, j)$.

Assume that $\deg L = \nu$ and $\nu \in \mathbf{Q}$ is at least a regular slope. Then following §5.4.6 of Oblomkov–Yun, we define the **local braid group** at a to be

$$Br_a = \pi_1(\mathcal{A}^{\heartsuit}_{\nu}, a), \tag{4.11.21}$$

It is mentioned in *loc. cit.* that for any $e^{2\pi i\nu}$-regular element $w \in W$, this is isomorphic to the Artin group attached to the so-called little Weyl group $C_W(w)$ (which may be a complex reflection group, rather than a Coxeter group).

Below, parts (1) and (2) summarize [90, Thm. 8.2.3], whereas part (3) is [90, Ex. 8.2.6].

Theorem 4.11.7 (Oblomkov–Yun)**.** *Suppose that* $\deg L = \nu$ *and* $\nu \in \mathbf{Q}$ *is a regular elliptic slope of denominator* n *in lowest terms. Then:*

1. *The monodromy action of* Br_a *on* $\mathrm{H}^{*}_{\mathbf{G}_m(\nu)}(\tilde{\mathcal{M}}_a)$ *descends to* $\mathrm{Gr}^{\mathbf{P}}_{*}\,\mathrm{H}^{*}_{\mathbf{G}_m(\nu)}(\tilde{\mathcal{M}}_a)$*, where it commutes with the* $\mathbf{A}^{\mathrm{rat}}_{\nu\omega}$*-action.*

2. *After specializing* $\omega \to 1$*, we have an isomorphism of* $\mathbf{A}^{\mathrm{rat}}_{\nu}$*-modules*

$$L_\nu(1) \simeq \mathrm{Gr}^{\mathbf{P}}_{*}\,\mathrm{H}^{*}_{\omega=1}(\tilde{\mathcal{M}}_a)^{Br_a}. \tag{4.11.22}$$

3. *If* ν *is a Coxeter slope, then* Br_a *acts trivially on* $\mathrm{H}^{*}_{\omega=1}(\tilde{\mathcal{M}}_a)$*.*

Tying everything together, we claim:

Corollary 4.11.8. *Suppose that:*

1. *G is almost-simple and simply-connected.*

2. *ν is a cuspidal slope of W such that either $W \neq W(E_8)$ or $\nu \notin \frac{1}{15}\mathbf{Z}$.*

3. *$\Sigma = \mathbf{P}^1(n,1)$, where n is the denominator of ν in lowest terms.*

4. *$L \to \Sigma$ is the line bundle of degree ν.*

Then $\textsc{ann}(\beta_{a,0})$ is a virtual summand of $\mathrm{E}^{\mathbf{P}}(1, \boldsymbol{q} \mid \tilde{\mathcal{M}}_a)_{st}$. If, more strongly, ν is a Coxeter slope of W, then Conjecture D holds for (G, Σ, L) and $a \in \mathcal{A}_\nu^\heartsuit$.

Proof. By hypothesis (2), $\textsc{ann}(\beta_{a,0}) = [L_\nu(1)]_{\boldsymbol{q}}$. By Corollary 4.11.5 and the fact that cuspidal implies regular elliptic, $a \in \mathcal{A}_\nu^{\mathrm{ani}}$ is tidy. So by the other hypotheses and Theorem 4.11.7, it remains to check that

$$\sum_{i,j} (-1)^i q^j \operatorname{Gr}_j^{\mathbf{P}} \mathrm{H}^i_{\omega=1}(\tilde{\mathcal{M}}_a)^{Br_a} = \sum_{i,j} (-1)^i q^j \operatorname{Gr}_j^{\mathbf{P}} \mathrm{H}^i_{\omega=0}(\tilde{\mathcal{M}}_a)^{Br_a} \tag{4.11.23}$$

as elements of $\mathrm{K}_0(W)[q]$. The graded dimensions are the same because we can write

$$\mathrm{H}^i_{\mathbf{G}_m(\nu)} \simeq \bigoplus_k \omega^k \mathrm{H}^{i-2k}.$$

The W-actions match because W commutes with ω in the definition of $\mathbf{A}^{\mathrm{rat}}$. □

We expect that via explicit computations of the monodromy action of Br_a, we will be able to generalize this result to all regular elliptic slopes ν. In more detail, we intend to prove:

Conjecture 4.11.9. *Suppose that we retain the hypotheses of Corollary 4.11.8, except we now assume that ν is only a regular elliptic slope of W. Then*

$$\sum_{i,j} (-1)^i q^j \operatorname{Gr}_j^{\mathbf{P}} \mathrm{H}^i(\tilde{\mathcal{M}}_a) = \textsc{ann}(\beta_{a,0}) \tag{4.11.24}$$

in $\mathrm{K}_0(W)[q]$.

Moreover, there is a bijective correspondence between the (virtual) Br_a-isotypic summands on the left-hand side and the (virtual) $\mathbf{A}_\nu^{\mathrm{rat}}$-isotypic summands on the right-hand side.

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