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PAUL APISA

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To Dena and Misha; to my parents - Jay and Carolyn; to Victor.

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Abstract

The moduli space of holomorphic one-forms on Riemann surfaces admits a natural action by $\mathrm{GL}(2, \mathbb{R})$. This thesis is concerned with using the results of Eskin, Mirzakhani, and Mohammadi to study the orbit closures of points under this action. The first two chapters show that for hyperelliptic components of strata of Abelian differentials every orbit is closed, dense, or contained in a locus of branched covers. The final chapter studies orbits of translation surfaces with marked points and relates the results to rational billiards and the existence of holomorphic sections of the universal curve restricted to subvarieties of moduli space.

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Chapter 1

Higher Rank Orbit Closures in Hyperelliptic Components of Strata

This chapter contains the results of Apisa [**Apib**].

The object of this chapter is to study $\mathrm{GL}_2\mathbb{R}$ orbit closures in hyperelliptic components of strata of abelian differentials. The main result is that all higher rank affine invariant submanifolds in hyperelliptic components are branched covering constructions, i.e. every translation surface in the affine invariant submanifold covers a translation surface in a lower genus hyperelliptic component of a stratum of abelian differentials. This result implies finiteness of algebraically primitive Teichmuller curves in all hyperelliptic components for genus greater than two. A classification of all $\mathrm{GL}_2\mathbb{R}$ orbit closures in hyperelliptic components of strata (up to computing connected components and up to finitely many nonarithmetic rank one orbit closures) is provided. The main theorem resolves a pair of conjectures of Mirzakhani in the case of hyperelliptic components of moduli space.

1.1 Introduction

Let $\mathcal{M}_g$ be the moduli space of genus g Riemann surfaces and let $\mathcal{H}_g$ be the sublocus of hyperelliptic Riemann surfaces. The inclusion of $\mathcal{H}_g$ into $\mathcal{M}_g$ is totally geodesic with respect to the Kobayashi metrics. Teichmüller geodesic flow and complex scalar multiplication generate a $\mathrm{GL}_2(\mathbb{R})$ action on $\Omega\mathcal{M}_g$ - the moduli space of holomorphic one-forms on closed genus g Riemann surfaces. Since $\mathcal{H}_g$ is totally geodesically embedded in $\mathcal{M}_g$, the collection $\Omega\mathcal{H}_g$ of holomorphic one-forms on genus g hyperelliptic Riemann surfaces is invariant under the $\mathrm{GL}_2(\mathbb{R})$ action.

Every genus g hyperelliptic Riemann surface may be written as the normalization of the projective curve defined in affine coordinates by the equation $y^2 = f(x)$ where f is a polynomial of degree $2g+1$ or $2g+2$ with simple roots. The hyperelliptic Riemann surface X defined by the equation $y^2 = f(x)$ admits a holomorphic one-form $\omega = \frac{dx}{y}$. The zeros of this one-form occur precisely at the points of the curve which are at infinity. This set contains either a single point fixed by the hyperelliptic involution or two points exchanged by the involution. The question we undertake in the sequel is - "what is the $\mathrm{GL}_2\mathbb{R}$ orbit closure of this one-form?" Call this orbit closure $\mathcal{M}$. Every such orbit closure is a complex subvariety of $\Omega\mathcal{M}_g$ by work of McMullen [**McM07**] in genus two and work of Eskin-Mirzakhani [**EM**], Eskin-Mirzakhani-Mohammadi [**EMM15**], and Filip [**Fil16b**] in general. Surprisingly, we find

Main Theorem 1. *Let $\mathcal{M}$ be as in the preceding paragraph. Let $\pi : \Omega\mathcal{M}_g \longrightarrow \mathcal{M}_g$ be the forgetful map that sends a translation surface to its underlying Riemann surface. If $g > 2$, then there is a finite union $C_g \subseteq \Omega\mathcal{M}_g$ of subvarieties of complex dimension at most three depending only on g, so that at least one of the following three possibilities occurs:*

1. *$\mathcal{M}$ is contained in C_g*

2. *$\pi(\mathcal{M})$ coincides with the hyperelliptic locus*

3. *$\pi(\mathcal{M})$ is a locus of branched covers contained in the hyperelliptic locus.*

Though it may not seem so, complex dimension three is not arbitrary. The reason it forms a natural threshold is rank. The tangent bundle of an orbit closure $\mathcal{M}$ at a point (X, ω) is naturally identified with a complex linear subspace of $H^1(X, Z(\omega); \mathbb{C})$ where $Z(\omega)$ is the zero set of ω, see [**Wri15a**] for details. If p is the projection from relative to absolute cohomology, then Avila-Eskin-Möller [**AEM**] showed that $p\left(T_{(X,\omega)}\mathcal{M}\right)$ is a complex symplectic vector space. The rank of an affine invariant submanifold is defined to be half the complex dimension of the vector space $p\left(T_{(X,\omega)}\mathcal{M}\right)$. An orbit closure is said to be higher rank if its rank is larger than one. In the cases that we consider an orbit closure is low rank if and only if its complex dimension is three or less.

The notion of rank allows us to recast the main question into the language of Teichmüller dynamics. The space $\Omega\mathcal{M}_g$ admits a $\mathrm{GL}_2\mathbb{R}$-invariant stratification by prescribing the number and degree of vanishing of the zeros of the holomorphic one-forms. Given a partition κ of $2g-2$, $\Omega\mathcal{M}_g(\kappa)$ denotes the stratum of holomorphic one-forms with $|\kappa|$ zeros whose degrees of vanishing form the set κ. The stratification on $\Omega\mathcal{M}_g$ induces a $\mathrm{GL}_2\mathbb{R}$-invariant stratification on $\Omega\mathcal{H}_g$. In this work, since we are interested in the orbit closures of abelian differentials of the form $\frac{dx}{y}$ we are interested in components of strata with either one zero or two zeros exchanged by the hyperelliptic involution. These are denoted by $\mathcal{H}^{hyp}(2g-2)$ and $\mathcal{H}^{hyp}(g-1,g-1)$ respectively. The main question then becomes "what are the higher rank orbit closures in $\mathcal{H}^{hyp}(2g-2)$ and $\mathcal{H}^{hyp}(g-1,g-1)$?"

Remark 1. *The notation change from $\Omega\mathcal{H}_g$ to $\mathcal{H}^{hyp}$ is to distinguish specific components of $\Omega\mathcal{H}_g$. While $\Omega\mathcal{H}_g(2g-2)$ is connected and equal to $\mathcal{H}^{hyp}(2g-2)$, $\Omega\mathcal{H}_g(g-1,g-1)$ is disconnected outside of genus two. For $g>2$, $\Omega\mathcal{H}_g(g-1,g-1)$ contains two components - $\mathcal{H}^{hyp}(g-1,g-1)$ - which contains abelian differentials with two zeros exchanged by the hyperelliptic involution and a second component that contains abelian differentials whose zeros are both Weierstrass points. In this work, we restrict our attention to $\mathcal{H}^{hyp}(2g-2)=\Omega\mathcal{H}_g(2g-2)$ and to the component $\mathcal{H}^{hyp}(g-1,g-1)$ of $\Omega\mathcal{H}_g(g-1,g-1)$.*

Posed in this way, the question already has a conjectural answer. Mirzakhani conjectured that higher rank orbit closures have field of definition $\mathbb{Q}$ and that they are branched covering constructions. These conjectures were first published in [**Wri15a**]. Recent work of McMullen, Mukamel, and

Wright [**MMW16**] provides a counterexample to this conjecture in a non-hyperelliptic component of a stratum in genus three. It remains an open question to determine in which components of which strata these conjectures hold. An orbit closure $\mathcal{M}$ is said to be branched covering construction if for every abelian differential (X, ω) in $\mathcal{M}$ - where X is a Riemann surface and ω a holomorphic one-form - there is a quadratic differential (Y, q) on a Riemann surface Y and a holomorphic map $f : X \longrightarrow Y$ such that $\omega^2 = f^*q$. In the case of the hyperelliptic locus, which is automatically a stratum of branched covers of quadratic differentials on punctured spheres, the definition is modified to mean a locus of pullbacks of abelian differentials on lower genus Riemann surfaces.

The main result of this work, which implies Theorem 1, is that the Mirzakhani conjectures hold in hyperelliptic components of strata.

Main Theorem 2. *Let $\mathcal{M}$ be an orbit closure in $\mathcal{H}^{hyp}(2g-2)$ or $\mathcal{H}^{hyp}(g-1, g-1)$ of complex dimension at least four. If $\dim \mathcal{M} = 2r$ then $\mathcal{M}$ is a branched covering construction over $\mathcal{H}^{hyp}(2r-2)$ and if $\dim \mathcal{M} = 2r+1$ then $\mathcal{M}$ is a branched covering construction over $\mathcal{H}^{hyp}(r-1, r-1)$. The covers are branched over the zeros of the holomorphic one-forms and commute with the hyperelliptic involution.*

The letter r was chosen in the above theorem statement since it coincides with the rank of $\mathcal{M}$.

Remark 2. *It is important to note that this result is about hyperelliptic components of strata and not about hyperelliptic loci of abelian differentials. An open problem related to this work is to find a classification of the orbit closures in other strata of the hyperelliptic locus, i.e. to analyze the* $\mathrm{GL}_2\mathbb{R}$ *dynamics of* $\Omega\mathcal{H}_g(\kappa)$ *for* κ *beyond* $(2g-2)$ *and* $(g-1, g-1)$.

As a corollary of Theorem 2 we have finiteness of algebraically primitive Teichmüller curves in hyperelliptic components. In the case of $\mathcal{H}^{hyp}(g-1, g-1)$ this result was the main result of Möller [**Möl08**].

Main Theorem 3. *In* $\mathcal{H}^{hyp}(2g-2)$ *and* $\mathcal{H}^{hyp}(g-1, g-1)$ *there are finitely many algebraically primitive Teichmüller curves for* $g > 2$.

Proof. Suppose not to a contradiction. Let C_i be an infinite sequence of distinct algebraically primitive Teichmüller curves. By Eskin-Mirzakhani [**EM**] a subsequence equidistributes in a finite union of connected affine invariant submanifolds $\mathcal{M} = \bigcup_i \mathcal{M}_i$. By Matheus-Wright [**MW15**] algebraically primitive Teichmüller curves cannot equidistribute in the connected component of any stratum when $g > 2$. Main Theorem 1 implies that no $\mathcal{M}_i$ is higher rank since this would imply that C_i is not geometrically primitive (and hence not algebraically primitive). Finally, no $\mathcal{M}_i$ is rank one since these orbit closures only contain finitely many nonarithmetic Teichmüller curves by Lanneau-Nguyen-Wright [**LNW**]. Therefore, we have a contradiction. □

Work of Eskin, Filip, and Wright [**EFW17**] establishes the following.

Theorem 1.1.1 (Eskin, Filip, Wright [**EFW17**] Theorem 1.5)**.** *Any infinite collection of nonarithmetic rank one* $\mathrm{GL}_2\mathbb{R}$ *orbit closures admits a subsequence that equidistributes in a rank two affine invariant submanifold.*

It follows immediately from Main Theorem 1 that

Main Theorem 4. *For $g > 2$, all but finitely many orbit closures in $\mathcal{H}^{hyp}(2g-2)$ and $\mathcal{H}^{hyp}(g-1, g-1)$ are branched covering constructions, and each of them has dimension at most three.*

Proof. Suppose to a contradiction that there is an infinite sequence C_i of orbit closures in $\mathcal{H}^{hyp}(2g-2)$ or $\mathcal{H}^{hyp}(g-1, g-1)$ that are not branched covering constructions. By Main Theorem 1 these orbit closures are rank one. Since arithmetic rank one orbit closures are torus covers, each C_i is nonarithmetic rank one. Therefore, Eskin, Filip, and Wright [**EFW17**] (Theorem 1.5) implies that the sequence equidistributes in a union of rank two affine invariant submanifolds. By Main Theorem 1 these rank two orbit closures are branched covering constructions and therefore so are the C_i, which is a contradiction. □

This theorem implies that outside of a subvariety of dimension at most three, we understand the closure of every complex geodesic in $\mathcal{H}^{hyp}(2g-2)$ and $\mathcal{H}^{hyp}(g-1,g-1)$. However, determining this subvariety remains open in all genera greater than two.

The results in this chapter represent the first such classification of orbit closures that holds in all genera. The argument is based on a degeneration argument that takes advantage of recent work of Mirzakhani and Wright **[MW17]** on partial compactifications of affine invariant submanifolds. This is the first time that such an argument has been used to study orbit closures.

1.1.1 Classification of Orbit Closures

We now offer a coarse classification of orbit closures in hyperelliptic components of strata. By coarse classification we mean a classification of orbit closures up to finitely many nonarithmetic closed orbits and up to classifying connected components of orbit closures.

Main Theorem 5 (Classification of Orbit Closures in $\mathcal{H}^{hyp}(2g-2)$)**.** *The affine invariant submanifolds in* $\mathcal{H}^{hyp}(2g-2)$ *are:*

1. *Countably many Teichmüller curves which arise from torus covers branched over one point.*

2. *(If* $g \equiv 2 \mod 3$*) Countably many Teichmüller curves whose trace fields are degree two and which equidistribute in an affine invariant submanifold of* $\mathcal{H}^{hyp}(2g-2)$ *arising from a branched cover construction over* $\mathcal{H}(2)$*.*

3. *Finitely many Teichmüller curves beyond the previous two families.*

4. *Finitely many rank* $r > 1$ *affine invariant submanifolds for each* $2r-1 \mid 2g-1$*. These are branched covering constructions over* $\mathcal{H}^{hyp}(2r-2)$*.*

Corollary 1.1.2 (Characterization of Optimal Dynamics in $\mathcal{H}^{hyp}(2g-2)$)**.** *Every orbit in* $\mathcal{H}^{hyp}(2g-2)$ *is either closed or equidistributed if and only if* $2g-1$ *is prime.*

Proof. Higher rank proper orbit closures arise if and only if $2r-1$ divides $2g-1$ for some $r > 1$. □

In the case of $g = 2$ this characterization of optimal dynamics follows from McMullen's classification of orbit closures in genus two, see **[McM03]**,

[McM06], and **[McM07]**. In the case of $g = 3$ this theorem was the main theorem of Nguyen-Wright **[NW14]**.

Main Theorem 6 (Classification of Orbit Closures in $\mathcal{H}^{hyp}(g-1, g-1)$)**.** *The affine invariant submanifolds in $\mathcal{H}^{hyp}(g-1, g-1)$ are:*

1. *Countably many Teichmüller curves which arise from torus covers branched over one point.*

2. *(If $g \equiv 0 \mod 3$) Countably many Teichmüller curves whose trace fields are degree two and which equidistribute in affine invariant submanifolds that are branched covering constructions over $\mathcal{H}(2)$.*

3. *Finitely many Teichmüller curves beyond the previous two families.*

4. *Countably many orbit closures that are branched covering constructions over $\mathcal{H}(0, 0)$.*

5. *(If g is even) Countably many orbit closures that cover genus two eigenform loci; these equidistribute in affine invariant submanifolds that are branched covering constructions over $\mathcal{H}(1, 1)$.*

6. *Finitely many three dimensional orbit closures beyond the previous two families. These are necessarily rank one and nonarithmetic.*

7. *Finitely many rank $r > 1$ affine invariant submanifolds for each $r \mid g$. These are branched covering constructions of $\mathcal{H}^{hyp}(r-1, r-1)$.*

8. *Finitely many rank $r > 1$ affine invariant submanifolds for each $2r-1 \mid g$. These are branched covering constructions of $\mathcal{H}^{hyp}(2r-2)$.*

In the case of $g = 2$ this theorem follows from McMullen's classification of orbit closures in genus two. In the case of $g = 3$ this theorem was one of the main theorems of Aulicino-Nguyen [**AN16**].

Corollary 1.1.3. *There are no higher rank proper even dimensional orbit closures in $\mathcal{H}^{hyp}(g-1, g-1)$ if and only if $g = 2^n$ for some n. There are no higher rank proper odd dimensional orbit closures in $\mathcal{H}^{hyp}(g-1, g-1)$ if and only if g is prime.*

1.1.2 Relation to Previous Results

The origin of this work begins in the study of orbit closures in genus two. In the early 2000s, Calta [**Cal04**] and McMullen [**McM03**] discovered an infinite family of closed complex geodesics in $\mathcal{M}_2$ that projected to curves W_D on Hilbert modular surfaces X_D. In a subsequent series of papers [**McM05**], [**McM06**] and [**McM07**], McMullen showed that the holomorphic one-forms on genus two Riemann surfaces whose $\mathrm{GL}_2(\mathbb{R})$ are not dense are exactly the eigenforms of real multiplication. Bainbridge determined the Euler characteristic of each Weierstrass curve W_D and the Lyapunov exponents of the Kontsevich-Zorich cocycle restricted to W_D in [**Bai07**]. Mukamel [**Muk14**] determined the orbifold points and homeomorphism type of each W_D. These results represent, up to classifying loci of torus covers, a classification of orbit closures in $\mathcal{H}^{hyp}(2)$ and $\mathcal{H}^{hyp}(1,1)$.

After McMullen's classification in genus two, attention turned to periodic points in genus two. Möller [**Möl06**] established that the only periodic points on Veech surfaces in $\Omega\mathcal{M}_2$ are Weierstrass points. The result was then extended to generic surfaces in all components of strata in Apisa [**Apia**].

For more general results, see Eskin-Mirzakhani [**EM**], Eskin-Mirzakhani-Mohammadi [**EMM15**], and Filip [**Fil16b**]. We suspect that orbit closures in hyperelliptic loci could be productively studied by exploring the connection between the braid group and Hodge structure, see for instance McMullen [**McM13a**].

1.1.3 Organization of the Chapter and Remarks on the Proof

In Section 2.2 we provide background on results that will be fundamental for the proofs of our results. In Section 1.3 we will discuss a combinatorial model - the Lindsey half-tree - created by Kathryn Lindsey in [**Lin15**] to study horizontally periodic translation surfaces in hyperelliptic components of strata. This model explains much of why orbit closures are particularly well-behaved in hyperelliptic components of strata. In Section 1.4 we will define branched covering constructions rigorously and devise a criterion for when an affine invariant submanifold is a branched covering construction. In Section 1.5 we will discuss Alex Wright's cylinder deformation theorem [**Wri15a**] and related constructions. In Section 1.7 we discuss Maryam Mirzakhani and Alex Wright's translation surface degeneration theorem [**MW17**]. In Section 1.8 we specialize these results to the setting of hyperelliptic components of strata. These sections establish the tools needed to run the basic mechanism of the proof: find a horizontally periodic translation surface in an affine invariant submanifold $\mathcal{M}$, use the cylinder deformation theorem to degenerate it to the boundary of $\mathcal{M}$, use the results of Section 1.3 to show that the boundary translation surface is a disjoint union of translation surfaces in hyperelliptic components of strata, and then use induction and the degeneration theorem to study the original translation surface.

In Section 1.6 we kick off the induction argument by establishing a host of nice properties satisfied by odd dimensional orbit closures in $\mathcal{H}^{hyp}(g-1, g-1)$. This is leveraged in Section 1.9 where we show that four-dimensional affine invariant submanifolds are branched covering constructions of $\mathcal{H}(2)$. This proof is representative of the more general proof, but without the technical difficulties. In Section 1.10 we study the flat geometry of translation surfaces in higher rank even complex-dimensional affine invariant submanifolds; this result represents the technical core of the chapter. In Section 1.11 we use the results of the preceding section to implement the strategy developed in Section 1.9 to the more general setting. Finally, we prove the main theorem in Section 1.12.

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1.2 Background

The cotangent bundle of the moduli space $\mathcal{M}_g$ of smooth genus g curves is naturally identified with the space of quadratic differentials over $\mathcal{M}_g$. Each quadratic differential on a Riemann surface X associates a natural flat structure - i.e. a metric that is flat away from finitely many cone points - to X, see Zorich [**Zor06**]. This flat structure is called a half-translation surface structure since it endows X with an atlas of charts to $\mathbb{C}$ with transition functions given by $z \longrightarrow \pm z + c$ for some $c \in \mathbb{C}$. The $\mathrm{GL}_2\mathbb{R}$ action on $\mathbb{C}$ induces a $\mathrm{GL}_2\mathbb{R}$ action on half-translation surfaces and hence a $\mathrm{GL}_2\mathbb{R}$ action on the cotangent bundle of $\mathcal{M}_g$.

Teichmüller's theorem states that if X and Y are distinct Riemann surfaces in $\mathcal{M}_g$ that are distance d apart in the Teichmüller metric then the geodesic from X to Y is given by fixing a quadratic differential q on X, associating the natural flat structure to (X, q), and then applying the matrix $\begin{pmatrix} e^d & 0 \\ 0 & e^{-d} \end{pmatrix}$. The $g_t := \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$ action on the space of quadratic differentials is called the Teichmüller geodesic flow. The $\mathrm{GL}_2\mathbb{R}$ action on the bundle of quadratic differentials is the smallest group action generated by complex scalar multiplication and Teichmüller geodesic flow.

The cotangent bundle of moduli space contains a subbundle $\Omega\mathcal{M}_g$ of quadratic differentials that are squares of abelian differentials. $\Omega\mathcal{M}_g$ is stratified by specifying the number of zeros and their degree of vanishing on the underlying one-form. Let $\mathcal{H}$ be such a stratum and let $(X, \omega) \in \mathcal{H}$. Let S be a basis of relative homology $H_1(X; Z(\omega))$ where $Z(\omega)$ is the zero set of ω. Local coordinates around (X, ω) are given by the map $\Phi(Y, \eta) = \left(\int_s \eta\right)_{s \in S}$. These coordinates are called period coordinates.

Kontsevich and Zorich classified the connected components of strata of abelian differentials in [**KZ03**].

Theorem 1.2.1 (Kontsevich-Zorich)**.** *Each stratum has at most three components, which are distinguished by hyperellipticity and spin parity.*

The only two strata that admit hyperelliptic components are $\Omega\mathcal{M}_g(2g-2)$ and $\Omega\mathcal{M}_g(g-1, g-1)$. These components will be denoted by $\mathcal{H}^{hyp}(2g-2)$ and $\mathcal{H}^{hyp}(g-1, g-1)$ respectively. To be completely explicit, $\mathcal{H}^{hyp}(2g-2)$ coincides with $\Omega\mathcal{H}_g(2g-2)$ and $\mathcal{H}^{hyp}(g-1, g-1)$ is the component of $\Omega\mathcal{H}_g(g-1, g-1)$ where the two zeros are exchanged by the hyperelliptic involution. The motivation of this work is to understand the dynamics of the $\mathrm{GL}_2\mathbb{R}$ action on the two hyperelliptic components of strata by classifying $\mathrm{GL}_2\mathbb{R}$ orbit closures.

The two hyperelliptic components admit another pleasantly simple description. For every genus g hyperelliptic Riemann surface X there is a degree $2g+1$ or $2g+2$ polynomial with complex coefficients and simple roots so that X is the normalization of the projectivization of the affine curve $\{(x,y) \in \mathbb{C}^2 : y^2 = f(x)\}$. Given any such affine curve, there is an associated holomorphic one-form $\frac{dx}{y}$. The abelian differentials in $\mathcal{H}^{hyp}(2g-2)$ (resp. $\mathcal{H}^{hyp}(g-1, g-1)$) are all pairs of Riemann surfaces and one-forms constructed above where $\deg f = 2g+1$ (resp. $2g+2$). Another way of phrasing the goal of this chapter is to take a simple polynomial, associate to it a hyperelliptic Riemann surface and holomorphic one-form, and to study the complex geodesic that this generates.

Work of Eskin, Mirzakhani, and Mohammadi implies that $\mathrm{GL}_2\mathbb{R}$ orbit closures in strata of abelian differentials are orbifolds:

Theorem 1.2.2 (Eskin-Mirzakhani [**EM**]; Eskin-Mirzakhani-Mohammadi [**EMM15**])**.** $\mathrm{GL}_2\mathbb{R}$ *orbit closures in strata of abelian differentials are affine invariant submanifolds, i.e.* $\mathrm{GL}_2\mathbb{R}$-*invariant orbifolds (possibly with self-intersections) that are locally cut out by real homogeneous linear equations* *in period coordinates.*

For a survey of this theorem and its applications to the study and classification of affine invariant submanifolds see Wright [**Wri15b**].

The tangent bundle of an affine invariant submanifold $\mathcal{M}$ at a point (X, ω) is naturally identified with a complex linear subspace of $H^1(X, Z(\omega); \mathbb{C})$, see [**Wri15a**] for details. Let $p : H^1(X, Z(\omega); \mathbb{C}) \longrightarrow H^1(X, \mathbb{C})$ be the projection from relative to absolute cohomology.

Theorem 1.2.3 (Avila-Eskin-Möller [**AEM**])**.** *If $\mathcal{M}$ is an affine invariant submanifold, then $p\left(T_{(X,\omega)}\mathcal{M}\right)$ is a complex symplectic vector space.*

Remark 1.2.4. *In fact, by Filip* [**Fil16a**] *(Theorem 1.1) $p\left(T_{(X,\omega)}\mathcal{M}\right)$ respects the Hodge structure on $H^1(X, \mathbb{C})$*

Define the rank of $\mathcal{M}$ to be $\mathrm{rk}(\mathcal{M}) := \frac{1}{2} \dim_{\mathbb{C}} p\left(T_{(X,\omega)}\mathcal{M}\right)$ and define $\mathrm{rel}(\mathcal{M}) := \dim_{\mathbb{C}} \mathcal{M} - 2 \cdot \mathrm{rk}(\mathcal{M})$. An affine invariant submanifold is said to be higher rank if its rank is larger than one.

1.2.1 Self-intersecting orbit closures and tangent spaces

It is important to discuss an issue arising from the fact that affine invariant submanifolds may contain self-intersections. If $\mathcal{M}$ is an affine invariant submanifold, then the self-intersection locus is a proper affine invariant submanifold. Throughout the sequel, we will wish to refer to the tangent space of an affine invariant submanifold, but this notion may break down at the self-intersection locus or at orbifold points. To avoid orbifold issues we will assume throughout that a tacit level three structure has been fixed. The self-intersection issue is a little trickier.

One approach to this issue would be to only work with translation surfaces that are generic in $\mathcal{M}$ with respect to the $\mathrm{GL}(2,\mathbb{R})$ action. However, this can become cumbersome, and so we elect to follow the approach outlined in Section 2.1 of Lanneau-Nguyen-Wright [**LNW**]. Let $\mathcal{M}$ be an affine invariant submanifold and let $\mathcal{M}'$ be the set of translation surfaces (X,ω) together with a maximal subspace $V \subseteq H^1(X,\Sigma;\mathbb{C})$ that is tangent to $\mathcal{M}$ and where Σ is the collection of zeros of ω. Notice that $\mathcal{M}'$ has a $\mathrm{GL}_2(\mathbb{R})$ action that acts in the usual way on (X,ω) and by parallel transport using the Gauss-Manin connection on V. Let $f : \mathcal{M}' \longrightarrow \mathcal{M}$ be the map that forgets the subspace V. This map is generically one-to-one and $\mathrm{GL}_2(\mathbb{R})$-equivariant.

Given an element v sufficiently close to 0 in relative cohomology $H^1(X,\Sigma;\mathbb{C})$ let $(X,\omega)+v$ be the unique element of the stratum near (X,ω) where ω is replaced by $\omega+v$. The condition that v is sufficiently close to 0 is necessary to prevent the translation surface from degenerating. A neighborhood basis of $\mathcal{M}'$ around $(X, \omega; V)$ will be the following. For any open set $U \subseteq V$ containing 0 and where $\{(X, \omega) + u\}_{u \in U}$ is a contractible set contained in $\mathcal{M}$ set the corresponding neighborhood of $(X, \omega; V)$ to be $\{(X, \omega + u; V) : u \in U\}$. With this topology, f is continuous and $\mathcal{M}'$ has a linear structure that makes f locally linear. Finally, the tangent space at a point $(X, \omega; V)$ of $\mathcal{M}'$ is canonically identified with V and so $\mathcal{M}'$ is smooth.

Throughout the text we will tacitly work on $\mathcal{M}'$, but write $\mathcal{M}$. This allows us to refer to a well-defined tangent space of an affine invariant submanifold (despite the fact that $\mathcal{M}$ may actually have self-intersection in a stratum). For a discussion of how to avoid issues of self-intersection when using the Mirzakhani-Wright partial compactification see the statement of Theorem 2.9 in Mirzakhani-Wright [**MW17**] and the surrounding discussion.

1.3 The Lindsey Half-Tree of Horizontally Periodic Translation Surfaces in Hyperelliptic Components of Strata

In this section we will review a construction of Lindsey that associates a half-tree to a horizontally periodic translation surface in a hyperelliptic component of a stratum of abelian differentials. We will show that any surface constructed in this way is guaranteed to be in $\mathcal{H}^{hyp}(2g-2)$ or $\mathcal{H}^{hyp}(g-1, g-1)$ for some $g \geq 1$. In particular, translation surfaces constructed in this way will have marked points if and only if the genus is one.

We begin by making two combinatorial definitions. A half-graph Γ consists of a set of vertices, a set of edges (each of which connects two vertices), and a set of half-edges that begin at a vertex but do not end at a vertex. A half-tree is a half-graph whose vertex and edge set form a tree.

Let (X, ω) be a horizontally periodic translation surface in a hyperelliptic component of a stratum of abelian differentials. Lindsey in **[Lin15]** showed the following results:

Theorem 1.3.1 (Lindsey **[Lin15]** Lemma 2.2)**.** *Every horizontal cylinder* *on (X, ω) is fixed by the hyperelliptic involution. Therefore, if C and D are two horizontal cylinders and C shares a saddle connection with D on its top boundary, then it shares a saddle connection with D on its bottom boundary as well.*

Theorem 1.3.2 (Lindsey [**Lin15**] Lemma 2.4)**.** *To each cylinder C one may associate a translation surface in the following way. Consider C as a cylinder in (X, ω) with boundary. If the hyperelliptic involution exchanges two saddle connections on the top and bottom boundary of C, then glue them together by translation. The resulting translation surface is hyperelliptic.*

These two observations lead to the following construction of a half-tree associated to (X, ω). To each horizontal cylinder associate a vertex. Connect two distinct vertices by an edge if the corresponding cylinders share a saddle connection on their boundary. By Lindsey [**Lin15**] Lemma 2.4, the resulting graph is a tree. To make the graph into a half-tree and not just a tree to each cylinder that has a saddle connection joining its top and bottom boundary, add a half-edge to the corresponding vertex of Γ. The total number of half-edges is $2g + |\Sigma| - 2$ where Σ is the singular set and where we count each full edge as two half-edges. We will call Γ the Lindsey tree associated to (X, ω).

Lindsey's result are actually more general than the results we have stated here. It associates a tree to any translation surface in a hyperelliptic component; in particular the translation surface need not be horizontally periodic. In this more complicated construction, each node represents either a horizontal cylinder or a minimal component of the horizontal line flow and a new kind of half-edge is required corresponding to horizontal lines beginning at a singularity, but never terminating at a singularity.

In Figures 1.3.1 and 1.3.2 below are all possible half-trees with four or fewer half-edges - i.e. the ones arising from surfaces in $\mathcal{H}(2)$ and $\mathcal{H}(1,1)$ - and corresponding horizontally periodic surfaces.

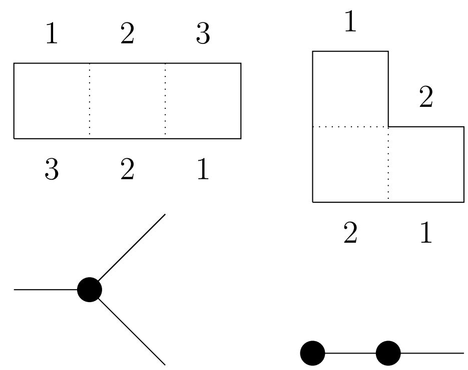


Figure 1.3.1: Lindsey trees of horizontally periodic translation surfaces in $\mathcal{H}(2)$

Define the combinatorial type of a horizontally periodic translation surface in $\mathcal{H}^{hyp}(2g-2)$ or $\mathcal{H}^{hyp}(g-1,g-1)$ to be the equivalence class of horizontally periodic translation surfaces that are related by some combina-

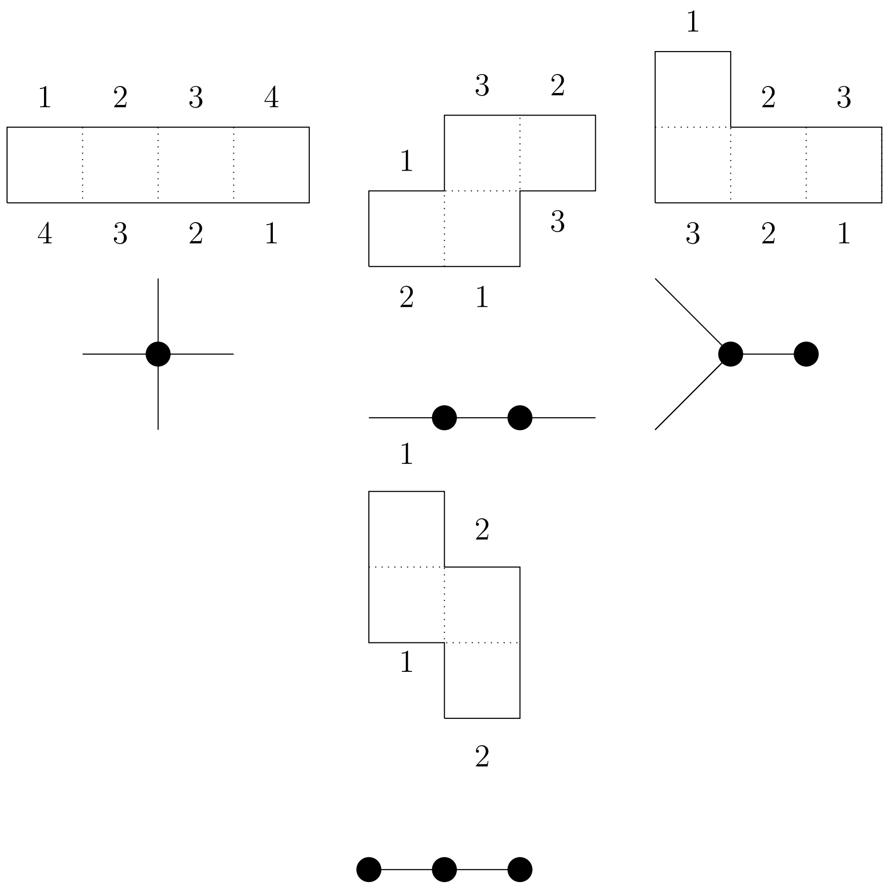


Figure 1.3.2: Lindsey trees of horizontally periodic translation surfaces in $\mathcal{H}(1,1)$

tion of the following three operations:

1. Horizontally shearing a horizontal cylinder.
2. Vertically dilating a horizontal cylinder.
3. Changing the lengths of a collection of horizontal saddle connections that is invariant under the hyperelliptic involution.

The notion of combinatorial type applies equally well to horizontal cylinders on translation surfaces.

Lemma 1.3.1. *Each node in the Lindsey half-tree corresponds to a horizontal cylinder of the same combinatorial type as the cylinder in Figure 1.3.3.*

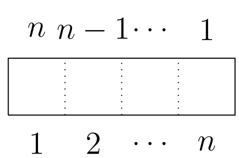


Figure 1.3.3: The combinatorial type of a horizontal cylinder in $\mathcal{H}^{hyp}(2g-2)$ or $\mathcal{H}^{hyp}(g-1,g-1)$

By identifying horizontal saddle connections that have the same label, we may choose to view this cylinder as a translation surface in a hyperelliptic component.

Proof. By Lindsey (**[Lin15]** Lemma 2.4) each node corresponds to a hyperelliptic surface that is contained in one horizontal cylinder and whose hyperelliptic involution is given by rotation by π. It follows that each node has the combinatorial type shown above. □

So that we may refer to it later the combinatorial type of the cylinder in Lemma 1.3.1 will be referred to as hyperelliptic combinatorial type.

Theorem 1.3.2. *The combinatorial types of horizontally periodic translation surfaces in $\mathcal{H}^{hyp}(2g-2)$ and $\mathcal{H}^{hyp}(g-1, g-1)$ are in bijective correspondence with planar embeddings of half-trees with $2g + |\Sigma| - 2$ half-edges up to precomposition with half-tree automorphisms (here we count full edges as two half-edges).*

Proof. For the forward direction of this correspondence, take the combinatorial type of a horizontally periodic translation surface in $\mathcal{H}^{hyp}(2g-2)$ or $\mathcal{H}^{hyp}(g-1, g-1)$ and associate to it the Lindsey half-tree. We form the planar embedding of the Lindsey half-tree (up to automorphism) by cyclically ordering the half-edges attached to a node according to the cyclic ordering of saddle connections on the upper boundary of the horizontal cylinder corresponding to the node.

For the reverse direction we must take a half-tree Γ and produce a translation surface $(X_\Gamma, \omega_\Gamma)$ in a hyperelliptic component of a stratum whose associated Lindsey half-tree is Γ. Associate to each node $v \in \Gamma$ the translation surface in Figure 1.3.3 taking n to be the number of edges and half-edges attached to v. Label the edges and half-edges attached to v clockwise $\{1, \ldots, n\}$. If an edge of the Lindsey tree connects the nodes v and w it will have two labels i and j coming from v and w respectively. To form $(X_\Gamma, \omega_\Gamma)$ slice open v along saddle connection i and w along saddle connection j and glue i to j. Do this for all edges in Γ.

The resulting surface has an involution given by $-I$ that fixes every horizontal cylinder. Since the surface in Figure 1.3.3 is hyperelliptic with hyperelliptic involution given by $-I$ the quotient of every node of the Lindsey tree is a copy of $\mathbb{P}^1$. The tree structure (ignoring half-edges) of the half-tree describes how the copies of $\mathbb{P}^1$ glue together. Since trees are contractible it follows that $(X_\Gamma, \omega_\Gamma)/-I$ is homeomorphic to $\mathbb{P}^1$. Consequently, $(X_\Gamma, \omega_\Gamma)$ is a hyperelliptic translation surface, i.e. X_Γ admits a hyperelliptic involution that takes ω_Γ to $-\omega_\Gamma$. To conclude that $(X_\Gamma, \omega_\Gamma)$ actually lies in $\mathcal{H}^{hyp}(2g-2)$ or $\mathcal{H}^{hyp}(g-1, g-1)$ and not in a hyperelliptic locus in another connected component it suffices to show that the surface has either one zero or two zeros that are exchanged by the hyperelliptic involution.

The translation surfaces in Figure 1.3.3 has a single Weierstrass point at the center of the rectangle and another $n+1$ Weierstrass points at the midpoints of each saddle connection on the boundary (whether or not the vertex is also a Weierstrass point will depend on the parity of n). Therefore, prior to identifying saddle connections, each node v of Γ contributes $d_v + 2$ Weierstrass points (not including potential Weierstrass points at vertices) where d_v is the number of full and half-edges attached to v. When two saddle connections on nodes v and w are identified the midpoints of both saddle connections are exchanged by the hyperelliptic involution and hence cease to be fixed points. Letting Z be the number of zeros fixed by the hyperelliptic involution, it follows that the number of Weierstrass points on $(X_\Gamma, \omega_\Gamma)$ is

$$2g + 2 = \left(\sum_{v \in V} d_v + 2 \right) - 2|E| + Z$$

where V (resp. E) are the vertices (resp. full edges) of Γ. Since Γ is a tree $2|V| - 2|E| = 2$ and since the total number of half-edges (counting each full edge as two half edges) is $2g + |\Sigma| - 2$ we also have that $\sum_{v \in V} d_v = 2g + |\Sigma| - 2$. Therefore,

$$2g + 2 = (2g + |\Sigma| - 2 + 2|V|) - 2|E| + Z$$

which shows that

$$|\Sigma| + Z = 2$$

It follows that $(X_\Gamma, \omega_\Gamma)$ has either one fixed zero or two zeros that are exchanged by the hyperelliptic involution; so $(X_\Gamma, \omega_\Gamma)$ lies in $\mathcal{H}^{hyp}(2g-2)$ or $\mathcal{H}^{hyp}(g-1, g-1)$. □

Corollary 1.3.3. *Suppose that (X, ω) is a horizontally periodic translation surface satisfying the following two conditions:*

1. *Every horizontal cylinder has hyperelliptic combinatorial type*

2. *The cylinder diagram of (X, ω) in the horizontal direction is a Lindsey tree*

then (X, ω) lies in a hyperelliptic component of a stratum of abelian differentials.

Proof. Suppose that (X, ω) was constructed as described. Let Γ be the cylinder diagram. Shearing individual cylinders preserves the stratum in which (X, ω) is contained. Moreover, given a horizontal cylinder and a saddle connection s on its top boundary there is a saddle connection s' on its lower boundary that has identical length by construction. Changing the length of

s and s' so that they remain of identical length preserves the stratum in which (X, ω) is contained. These two operations can be successively used to move (X, ω) to the surface $(X_\Gamma, \omega_\Gamma)$, which we proved belonged to a hyperelliptic component in Theorem 1.3.2. $\square$

1.4 Branched Covering Constructions

A translation covering $f : (Y, \eta) \longrightarrow (X, \omega)$ is a holomorphic map $f : Y \longrightarrow X$ such that $f^*\omega = \eta$. A simple translation covering is a translation covering that is branched over the zeros of ω and for which Y is connected. The goal of this section is to develop a criterion to recognize "branched covering constructions over $\mathcal{M}$" - i.e. affine invariant submanifolds all of whose elements are simple translation coverings of elements in a component $\mathcal{M}$ of a stratum of abelian differentials. We begin extending Mumford's compactness theorem to strata of abelian differentials.

Lemma 1.4.1 (Maskit-Mumford Compactness Lemma). *If $((X_n, \omega_n))_n$ is a sequence of translation surfaces in a fixed stratum that have area bounded from above and below and the length of their shortest saddle connection bounded below, then there is a convergent subsequence.*

Proof. By Maskit ([**Mas85**] Corollary 2) if the length of the shortest saddle connection is bounded away from zero then the length of the shortest hyperbolic curve on X_n is bounded away from zero. By Mumford's compactness theorem there is a convergent subsequence of X_n. Passing to this subsequence let X be the limit and let U be a precompact neighborhood of X on which the bundle of holomorphic one-forms is trivial. Since the area is bounded below and above (X_n, ω_n) eventually is contained in a bundle of compact annuli over the compact set $\overline{U}$. Therefore there is a convergent subsequence. Since no saddle connection becomes short the sequence remains in the same fixed stratum. □

Theorem 1.4.2. *Suppose that $f : (X, \omega) \longrightarrow (Y, \eta)$ is a simple translation covering. Let $\mathcal{M}$ be the $\mathrm{GL}_2\mathbb{R}$ orbit closure of (X, ω) and let $\mathcal{N}$ be the $\mathrm{GL}_2\mathbb{R}$ orbit closure of (Y, η). Every translation surface in $\mathcal{M}$ is a translation covering of a surface in $\mathcal{N}$.*

Proof. Let $\mathcal{M}$ be the orbit closure of (X, ω) and $\mathcal{N}$ the orbit closure of (Y, η). First notice that if $f : (X, \omega) \longrightarrow (Y, \eta)$ is a simple translation covering, then for any g in $\mathrm{GL}_2(\mathbb{R})$ we have that $gfg^{-1} : g(X, \omega) \longrightarrow g(Y, \eta)$ is a simple translation covering too. Therefore, to show that the orbit closure of (X, ω) only includes surfaces that are simple translation coverings of a surface in

$\mathcal{M}$ it suffices to show that if (g_i) is a sequence of elements of $\mathrm{GL}_2(\mathbb{R})$ and $g_i(X, \omega)$ converges to (X', ω') then (X', ω') is a simple translation cover of some translation surface in $\mathcal{N}$. Let $(X_i, \omega_i) = g_i(X, \omega)$ and $(Y_i, \eta_i) = g_i(Y, \eta)$. Let $f_i = g_i f g_i^{-1}$.

Notice that since each f_i has the same degree, say d, the systole of (Y_i, η_i) in the flat metric is bounded below by $\frac{\mathrm{sys}(X_i)}{d}$. Since (X_i, ω_i) converge to (X', ω') the length of the systole along the sequence (X_i, ω_i) is bounded below and hence the length of the systole along the sequence (Y_i, η_i) is also bounded below. Since a degree d map preserves area up to a factor of $\frac{1}{d}$, it follows from the Maskit-Mumford compactness lemma (Lemma 1.4.1) that (Y_i, η_i) has a convergent subsequence. After passing to this subsequence we may suppose that (Y_i, η_i) converges to a translation surface (Y', η') belonging to $\mathcal{N}$.

After deleting sufficiently many initial terms we may suppose that all (X_i, ω_i) and (Y_i, η_i) belong to a small neighborhood of (X', ω') and (Y', η') respectively where the zeros of the one-forms are labelled. Let $\phi_i : X' \longrightarrow X_i$ and $\psi_i : Y' \longrightarrow Y_i$ be quasiconformal maps of minimal dilatation that take labelled zeros to the corresponding labelled zeros. Suppose too, after perhaps again passing to a subsequence, that for all i the ramification type over a given labelled zero of (Y_i, η_i) is constant. Since (X_i, ω_i) and (Y_i, η_i) converge we have that the dilatation of these ϕ_i and ψ_i tends to 1 as i tends to infinity. Therefore, the dilatation of the map $F_i = \psi_i^{-1} \circ f_i \circ \phi_i : X' \longrightarrow Y'$ tends to 1 as i tends to infinity as well. Since a collection of quasiconformal maps of bounded dilatation is precompact, it follows that there is a subsequence of F_i that tends to a quasiconformal map $F : X' \longrightarrow Y'$ of dilatation 1, i.e. a holomorphic map. Moreover, the condition on labelled zeros implies that $\mathrm{div}(\omega') = \mathrm{div}(F^*\eta)$. Therefore, up to multiplication by scalars $F^*\eta = \omega'$. Therefore, (X', ω') is a simple translation covering of a surface in $\mathcal{N}$ as desired. □

Corollary 1.4.3. *An affine invariant submanifold $\mathcal{M}$ is a branched covering construction if there is an $\mathcal{M}$-generic point that is a simple translation covering of a lower genus translation surface.*

Proof. Suppose that (X, ω) is a translation surface that is generic in $\mathcal{M}$ with respect to the $\mathrm{GL}(2, \mathbb{R})$ action. Suppose that there is a map $f : (X, \omega) \longrightarrow (Y, \eta)$ that is a simple translation covering, where (Y, η) is a lower genus translation surface. Let $\mathcal{N}$ be the orbit closure of (Y, η). By Theorem 1.4.2 every point in $\mathcal{M}$ is a simple translation covering of a point in $\mathcal{N}$ and so $\mathcal{M}$ is a branched covering construction. □

1.5 Cylinder Deformations

Throughout this section $\mathcal{M}$ will be an affine invariant submanifold in a component $\mathcal{H}$ of a stratum. Suppose that (X, ω) is a translation surface in an affine invariant submanifold $\mathcal{M}$. Let C and C' be two cylinders on (X, ω) with core curves γ_C and $\gamma_{C'}$. If γ_C and $\gamma_{C'}$ are parallel in some neighborhood $U \subseteq \mathcal{M}$ of (X, ω), then C and C' are said to be $\mathcal{M}$-equivalent. When the affine invariant submanifold $\mathcal{M}$ is clear from context, $\mathcal{M}$-equivalent and $\mathcal{M}$-equivalence class will be shorted to "equivalent" and "equivalence class" respectively.

Theorem 1.5.1 (Cylinder Proportion Theorem, Proposition 3.2, Nguyen-Wright [**NW14**]). *If C and C' are two $\mathcal{M}$-equivalent cylinders and $\mathcal{V}$ is any equivalence class of cylinders, then*

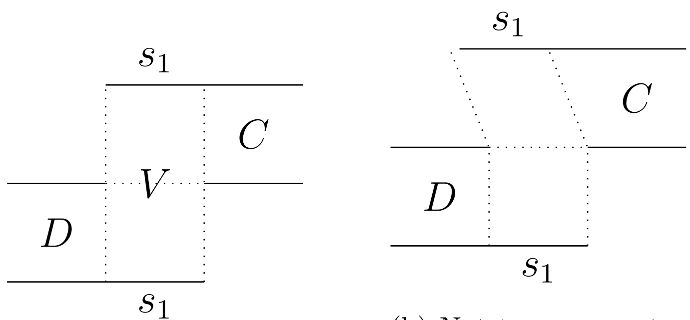
$$\frac{|C \cap \mathcal{V}|}{|C|} = \frac{|C' \cap \mathcal{V}|}{|C'|}$$

where $|\cdot|$ denotes area.

Applying the matrix $u_t := \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ to a horizontal $\mathcal{M}$-equivalence class of horizontal cylinders $\mathcal{C}$ will be called (horizontally) shearing $\mathcal{C}$. Applying the matrix $a_t := \begin{pmatrix} 1 & 0 \\ 0 & e^t \end{pmatrix}$ will be called (vertically) dilating $\mathcal{C}$.

Theorem 1.5.2 (Cylinder Deformation Theorem, Wright [**Wri15a**]). *Let $(X, \omega) \in \mathcal{M}$ be a translation surface and let $\mathcal{C}$ be an $\mathcal{M}$-equivalence class of horizontal cylinders on (X, ω). Horizontally shearing and vertically dilating $\mathcal{C}$ by t remains in $\mathcal{M}$ for all t.*

A special feature of the hyperelliptic component of a stratum is that if two horizontal cylinders share a horizontal saddle connection then they share exactly two - one on the top of each cylinder and one on the bottom of each cylinder. This feature holds because the graph of cylinder adjacencies is a tree and because if a cylinder C borders a cylinder D on its top boundary, then it borders D on its bottom boundary as well. The two saddle connections joining the cylinders are exchanged by the hyperelliptic involution. Given two adjacent cylinders C and D, which border each other along saddle connections s_1 and s_2, we say that the cylinders are in transverse standard position if there is a cylinder V that is contained in $C \cup D$, that contains s_1 and s_2, and that only intersects the core curves of cylinder C and D once. We say that C and D are in standard position if the core curve of V is perpendicular to the core curves of cylinders C and D. The definition is illustrated in Figure 1.5.1.



(a) Standard position

(b) Not transverse standard position

Figure 1.5.1: An illustration of the definition of transverse standard position

Lemma 1.5.3 (Standard Position)**.** *Suppose that* (X, ω) *is a translation surface in a hyperelliptic component of a stratum and suppose that* C *and* D *are adjacent horizontal cylinders that belong to distinct equivalence classes* $\mathcal{C}_1$ *and* $\mathcal{C}_2$ *respectively.*

1. *It is possible to shear* $\mathcal{C}_2$ *so that* C *and* D *are in transverse standard position.*

2. *It is possible to shear* $\mathcal{C}_1$ *and* $\mathcal{C}_2$ *so that* C *and* D *are in standard position.*

Proof. Let s_1 and s_2 be the horizontal saddle connections lying on the boundary of C and D. By Theorem 1.5.2 shear $\mathcal{C}_1$ so that s_1 lies directly over s_2; then shear $\mathcal{C}_2$ so that s_2 lies directly over s_1. Recall that s_1 and s_2 are exchanged by the hyperelliptic involution and hence have identical lengths. Choose V to be the vertical cylinder passing through s_1 and s_2. This proves part (2), the proof of part (1) is almost completely identical. $\square$

In the sequel, moving to the second configuration will be called putting C and D in standard position. Moving to the first will be called putting C and D in transverse standard position while fixing C.

If C is a cylinder call the distance h_C from one boundary of the cylinder to the other its height. Let γ_C^* be the cohomology class that is dual to the core curve of C under the intersection pairing. This cohomology class requires specifying an orientation on γ_C. Usually this orientation will not matter, but we will establish the conventions that when C is horizontal the orientation is left to right, when C is vertical it is top to bottom, and when $C_1, \ldots, C_n$ are all $\mathcal{M}$-equivalent cylinders we will assume that the holonomy vectors of the core curves point in the same directions.

Let $\mathcal{C}_1, \ldots, \mathcal{C}_n$ be an enumeration of the $\mathcal{M}$-equivalence classes of horizontal cylinders on (X, ω). For each equivalence class $\mathcal{C}_i$ there is an element of the tangent space called the standard shear which is defined to be $u_{\mathcal{C}_i} = \sum_{c \in \mathcal{C}_i} h_c \gamma_c^*$. A reformulation of the cylinder deformation theorem is that the standard shear is always in the tangent space of $\mathcal{M}$ at (X, ω).

Let $\mathcal{C}$ denote the collection of all horizontal cylinders on (X, ω). The twist space of $\mathcal{M}$ at (X, ω) is defined to be

$$\mathrm{Twist}_{(X,\omega)}\mathcal{M} := \mathrm{span}_{\mathbb{R}} \left(\gamma_c^*\right)_{c \in \mathcal{C}} \cap T^{\mathbb{R}}_{(X,\omega)}\mathcal{M}$$

where $T^{\mathbb{R}}_{(X,\omega)}\mathcal{M} = T_{(X,\omega)}\mathcal{M} \cap H^1(X, Z(\omega); \mathbb{R})$ where $T_{(X,\omega)}\mathcal{M}$ has been identified with a subspace of $H^1(X, Z(\omega); \mathbb{C})$. The standard shears are always in the twist space. Define the cylinder preserving space, denoted $\mathrm{Pres}_{(X,\omega)}\mathcal{M}$, to be all elements of $T^{\mathbb{R}}_{(X,\omega)}\mathcal{M}$ that pair trivially with every element of $(\gamma_c)_{c \in \mathcal{C}}$ under the intersection pairing. It is clear that $\mathrm{Twist}_{(X,\omega)}\mathcal{M} \subseteq \mathrm{Pres}_{(X,\omega)}\mathcal{M}$. The following theorem establishes that there is always a translation surface (X, ω) in an affine invariant submanifold $\mathcal{M}$ where $\mathrm{Twist}_{(X,\omega)}\mathcal{M} = \mathrm{Pres}_{(X,\omega)}\mathcal{M}$.

Theorem 1.5.4 (Lemma 8.6, Wright [**Wri15a**]). $\mathrm{Twist}_{(X,\omega)}\mathcal{M} = \mathrm{Pres}_{(X,\omega)}\mathcal{M}$ *whenever* (X, ω) *has the maximum number of horizontal cylinders in* $\mathcal{M}$.

The next theorem indicates why having $\mathrm{Twist}_{(X,\omega)}\mathcal{M} = \mathrm{Pres}_{(X,\omega)}\mathcal{M}$ is special. In particular, it says that whenever equality is achieved (X, ω) has at least $\mathrm{rk}(\mathcal{M})$ many $\mathcal{M}$-equivalence classes and the twist space projects to a Lagrangian in $p\left(T_{(X,\omega)}\mathcal{M}\right)$.

Theorem 1.5.5 (Lemma 8.12, Wright [**Wri15a**]). *If* (X, ω) *is a translation* *surface in* $\mathcal{M}$ *and* $\mathrm{Twist}_{(X,\omega)}\mathcal{M} = \mathrm{Pres}_{(X,\omega)}\mathcal{M}$ *then* $\mathrm{span}_{\mathbb{R}} p\left(u_{\mathcal{C}_i}\right)_{i=1}^n$ *is a Lagrangian subspace of* $p(T^{\mathbb{R}}_{(X,\omega)}\mathcal{M})$ *where* $\{\mathcal{C}_1, \ldots, \mathcal{C}_n\}$ *is an enumeration of the* $\mathcal{M}$*-equivalence classes of horizontal cylinders and* $u_{\mathcal{C}_i}$ *is the standard shear. In particular,* (X,ω) *contains at least* $\mathrm{rk}(\mathcal{M})$ *distinct* $\mathcal{M}$*-equivalence classes of horizontal cylinders.*

The combination of the previous two theorems is an engine that allows us to convert the rank of an affine invariant submanifold into geometric information that picks out a translation surface where a large dimensional subspace, the twist space, of the tangent space is geometrically meaningful. Recall that, given a translation surface (X,ω) belonging to a stratum $\mathcal{H}$ and with cone points Σ, the tangent space to $\mathcal{H}$ at (X,ω) can be identified with the relative cohomology group $H^1(X,\Sigma;\mathbb{C})$. Let p be the projection from $H^1(X,\Sigma;\mathbb{C})$ onto absolute cohomology.

Theorem 1.5.6 (Twist Space Decomposition Theorem, Theorem 4.7, Mirzakhani-Wright [**MW17**]). *Let* (X,ω) *be a translation surface in an affine invariant submanifold* $\mathcal{M}$*. Let* $\mathcal{C}_1, \ldots, \mathcal{C}_d$ *be an enumeration of the* $\mathcal{M}$*-equivalence classes of horizontal cylinders.*

1. *If* $v \in \mathrm{Twist}_{(X,\omega)}\mathcal{M}$ *then* $v = \sum_{i=1}^d v_i$ *where* $v_i \in \mathrm{Twist}_{(X,\omega)}\mathcal{M} \cap \mathrm{span}_{\mathbb{R}}(\gamma_c^*)_{c\in\mathcal{C}_i}$.

2. *If* $v_i \in \mathrm{Twist}_{(X,\omega)}\mathcal{M} \cap \mathrm{span}_{\mathbb{R}}(\gamma_c^*)_{c\in\mathcal{C}_i}$ *then* $v_i \in \mathbb{R}\cdot u_{\mathcal{C}_i} \oplus \ker p$ *where* $u_{\mathcal{C}_i}$ *is the standard shear.*

The last result regarding cylinder deformations that we need is the statement that given a collection of d $\mathcal{M}$-equivalence classes of cylinders, where d is no bigger than the rank of $\mathcal{M}$, it is possible to perturb the translation surface so that one $\mathcal{M}$-equivalence class becomes disjoint and vertical and all others remain horizontal. This result is crucial in establishing that all higher rank affine invariant submanifolds of complex dimension four in hyperelliptic components of strata are branched covering constructions over $\mathcal{H}(2)$.

Theorem 1.5.7 (Perturbation Theorem, Lemma 5.5, Mirzakhani-Wright [**MW17**])**.** *Suppose that* (X,ω) *is a translation surface in an affine invariant submanifold* $\mathcal{M}$*. Suppose that* $\mathcal{C}_1,\ldots,\mathcal{C}_d$ *are* $\mathcal{M}$*-equivalence classes of horizontal cylinders such that* $\{p(u_{\mathcal{C}_1}),\ldots,p(u_{\mathcal{C}_d})\}$ *spans a* d *dimensional subspace. Define* $\mathcal{C} := \mathcal{C}_1\cup\ldots\cup\mathcal{C}_{d-1}$*. There is a piecewise smooth path* $f:[0,1]\longrightarrow\mathcal{M}$ *such that* $f(0)=(X,\omega)$ *and along the path*

1. *All cylinders in* $\mathcal{C}$ *persist and are horizontal.*

2. *The cylinders in* $\mathcal{C}_d$ *persist, become nonhorizontal on* $f(t)$ *for* $t>0$*, and vertical on* $f(1)$*.*

3. At all points along the path any cylinder $\mathcal{M}$-equivalent to $\mathcal{C}_d$ is disjoint from any cylinder $\mathcal{M}$-equivalent to $\mathcal{C}_i$ for all $i \in \{1, \ldots, d-1\}$.

Proof. Let γ_i be the core curve of some cylinder in $\mathcal{C}_i$ for each $i \in \{1, \ldots, d\}$. Consider the linear functionals $(\gamma_i)_{i=1}^{d-1}$ on $T_{(X,\omega)}\mathcal{M}$. Since the linear functionals factor through $p : T_{(X,\omega)}\mathcal{M} \longrightarrow H^1(X; \mathbb{C})$ we see that the intersection of the kernel of these functionals on $p(T_{(X,\omega)}\mathcal{M})$ is at least dimension $(2r-d)+1$. Therefore there is a vector $v \in T_{(X,\omega)}\mathcal{M}$ that is not in the kernel of p, not in the cylinder preserving space, and such that $v(\gamma_i) = 0$ for $1 \leq i \leq d-1$.

Consider the path $(X, \omega) + tv$ for $t \geq 0$. This path is well-defined and remains in $\mathcal{M}$ for some range $t \in [0, T]$. Since $v(\gamma_i) = 0$ for each $1 \leq i \leq d-1$ it follows by definition of $\mathcal{M}$-equivalence that $\mathcal{C}_i$ persist (perhaps after decreasing T) and remain horizontal for $1 \leq i \leq d-1$. Since v is not in the cylinder preserving space we see that, perhaps after decreasing T, $\mathcal{C}_d$ also persists and becomes non-horizontal. By perhaps decreasing T again we may suppose by Mirzakhani-Wright Lemma 5.1 [**MW15**] that any cylinder $\mathcal{M}$-equivalent to $\mathcal{C}_d$ remains disjoint from any cylinder $\mathcal{M}$-equivalent to $\mathcal{C}_i$ at all points along the path.

Now horizontally shear $(X, \omega) + Tv$ until $\mathcal{C}_d$ becomes vertical. The equations on period coordinates cutting out $\mathcal{M}$ may be parallel translated along this path and so we see that along this path no new cylinders become $\mathcal{M}$-equivalent to cylinders in $\mathcal{C}_i$ for any i. $\square$

In the following proof we will say that two heights (of cylinders) h_1 and h_2 are a-close if $|h_1 - h_2| \leq a$.

Lemma 1.5.8. *Let $\mathcal{M}$ be an affine invariant submanifold. Suppose that for any horizontally periodic $(X, \omega) \in \mathcal{M}$ such that $\mathrm{Twist}_{(X,\omega)}\mathcal{M} = \mathrm{Pres}_{(X,\omega)}\mathcal{M}$, the heights of any two $\mathcal{M}$-equivalent horizontal cylinders are identical. Then the heights of any two equivalent cylinders on any translation surface in $\mathcal{M}$ are identical.*

Proof. The following proof is almost identical to the proof of Theorem 5.1 in Wright [**Wri15a**]. Let $\mathcal{C}$ be an equivalence class of horizontal cylinders on $(X_0, \omega_0) \in \mathcal{M}$. Let $\epsilon > 0$ be taken to be smaller than the heights of all cylinders in $\mathcal{C}$. Consider the following iterative process:

1. If (X_i, ω_i) is horizontally periodic and $\mathrm{Twist}_{(X_i,\omega_i)} = \mathrm{Pres}_{(X_i,\omega_i)}$ then end the process. Otherwise, Smillie-Weiss [**SW04**] (Corollary 6) implies that the horocycle flow accumulates on a horizontally periodic translation surface (Y_i, η_i). Since the horocycle flow of (X_i, ω_i) becomes arbitrarily close to (Y_i, η_i) we may assume that there is some

T so that there is a perturbation of $u_T(X_i, \omega_i)$ - through surfaces in $\mathcal{M}$ - to (Y_i, η_i) so that the cylinders on (X_i, ω_i) persist on (Y_i, η_i) and have heights that are $\frac{\epsilon}{2\cdot(g+|\Sigma|-1)}$-close to their height on (X_i, ω_i). By definition of $\mathcal{M}$-equivalence class, if $\mathcal{M}$-equivalent cylinders persist under a perturbation of a translation surface through translation surfaces in $\mathcal{M}$ then they remain $\mathcal{M}$-equivalent. Therefore, the cylinders in $\mathcal{C}$ persist on (Y_i, η_i), remain $\mathcal{M}$-equivalent there, and have heights that are $\frac{\epsilon}{2\cdot(g+|\Sigma|-1)}$ close to their original height.

2. If $\text{Twist}_{(Y_i,\eta_i)} = \text{Pres}_{(Y_i,\eta_i)}$ then end the process. Otherwise there is an element v in the cylinder preserving space that fails to be in the twist space. Flowing in the $\sqrt{-1}\cdot v$ direction for an arbitrarily small positive time leads to a surface (X_{i+1}, ω_{i+1}) on which the cylinders in $\mathcal{C}$ persist, are $\mathcal{M}$-equivalent and, have heights that are $\frac{\epsilon}{2\cdot(g+|\Sigma|-1)}$-close to their heights on (Y_i, η_i); but where the horizontal cylinders from (Y_i, η_i) although they persist, do not cover (X_{i+1}, ω_{i+1}). Now return to step 1.

Since the number of cylinders increases with each iteration and the largest possible number of horizontal cylinders is $g + |\Sigma| - 1$ the process terminates after at most $g + |\Sigma| - 1$ cycles. Each iteration alters the height of each cylinder in $\mathcal{C}$ by at most $\frac{\epsilon}{g+|\Sigma|-1}$. Since there are at most $g+|\Sigma|-1$ iterations, when the process terminates each cylinder has had its height altered by at most ϵ. Moreover, at the end of the process, $\mathcal{C}$ is a collection of $\mathcal{M}$-equivalent cylinders on a translation surface (X, ω) with heights ϵ-close to their original heights and where $\mathrm{Twist}_{(X,\omega)}\mathcal{M} = \mathrm{Pres}_{(X,\omega)}\mathcal{M}$. By hypothesis, these cylinders on (X, ω) have identical heights. Therefore, the cylinders in $\mathcal{C}$ on (X_0, ω_0) all have heights that are ϵ-close to one another for arbitrarily small ϵ. □

1.6 Odd Dimensional Orbit Closures

Throughout this section $\mathcal{M}$ will be an affine invariant submanifold in $\mathcal{H}^{hyp}(g-1, g-1)$ of odd complex dimension $2r+1$. The two main results of this section are the following:

Theorem 1.6.1. *If $\mathcal{M}$ is higher rank and (X, ω) is a horizontally periodic translation surface with the twist space and cylinder preserving space coinciding, then (X, ω) has $g + 1$ horizontal cylinders and equivalent horizontal cylinders are nonadjacent and have identical heights.*

This theorem will be the key to showing that if $\mathcal{M}$ is higher rank and odd dimensional then it is a branched covering of $\mathcal{H}^{hyp}(r-1, r-1)$. Before stating the second result we associate to a collection of horizontal cylinders $\mathcal{C}$ the deformation $\sigma_{\mathcal{C}} = \sum_{c \in \mathcal{C}} h_c \gamma_c^*$ where h_c is the height of the horizontal cylinder c, γ_c is the core curve of c (oriented from left to right), and γ_c^* is the dual cohomology class corresponding to γ_c. The second main result of this section is the following.

Theorem 1.6.2. *If $\mathcal{M}$ is rank one and (X, ω) is horizontally periodic and the twist space is spanned by $\sigma_{\mathcal{C}_1}$ and $\sigma_{\mathcal{C}_2}$ where $\mathcal{C}_1$ and $\mathcal{C}_2$ are disjoint collections of cylinders, then if two cylinders belong to the same $\mathcal{C}_i$, for $i = 1$ or 2, they have the same height and are not adjacent.*

The application of Theorem 1.6.2 to the study of higher rank affine invariant submanifolds is less obvious than the application of Theorem 1.6.1. After all, Theorem 1.6.1 is expressly about higher rank affine invariant submanifolds whereas Theorem 1.6.2 is expressly about rank one affine invariant submanifolds. The connection to our problem is that we will classify the rank two rel zero affine invariant submanifolds by degenerating to a rank one affine invariant submanifold whose twist space is as described in Theorem 1.6.2.

We now fix some notation that will be used in the sequel. Let $p : T_{(X,\omega)}\mathcal{M} \longrightarrow H^1(X, \mathbb{C})$ be the projection of the tangent space of $\mathcal{M}$ at (X, ω) onto absolute cohomology. Let η be a nonzero relative cohomology class contained in $\ker(p) \cap H^1(X, \Sigma; \mathbb{R})$ where Σ is the zero set of ω. In other words, η is a relative deformation on (X, ω) that preserves the horizontal cylinders.

Theorem 1.6.3. *Let (X, ω) be a translation surface in $\mathcal{M}$ with at least one horizontal cylinder. The following are equivalent:*

1. *(X, ω) has $g + 1$ horizontal cylinders.*
2. $\mathrm{Twist}_{(X,\omega)}\mathcal{M} = \mathrm{Pres}_{(X,\omega)}\mathcal{M}$
3. *The relative deformation η is contained in the twist space.*
4. *The Lindsey tree is a tree and not just a half-tree.*

When any of these equivalent conditions holds label the cylinders $\{c_0, \ldots, c_g\}$ and the core curves $\{\gamma_0, \ldots, \gamma_g\}$, it follows that $\eta = \sum_{i=0}^{g} (-1)^{d(c_0,c_i)} \gamma_i^*$ where $d(c_i, c_0)$ is the distance between c_i and c_0 in the Lindsey tree.

Proof. ($1 \Rightarrow 2$) By Theorem 1.5.4 if (X, ω) has $g + 1$ horizontal cylinders then $\mathrm{Twist}_{(X,\omega)}\mathcal{M} = \mathrm{Pres}_{(X,\omega)}\mathcal{M}$.

($2 \Rightarrow 3$) Since any relative deformation fixes the core curves of every cylinder it follows that if $\mathrm{Twist}_{(X,\omega)}\mathcal{M} = \mathrm{Pres}_{(X,\omega)}\mathcal{M}$ then η is contained in $\mathrm{Twist}_{(X,\omega)}\mathcal{M}$.

($3 \Rightarrow 1$) Now suppose that η is contained in the twist space, i.e $\eta = \sum_c a_c \gamma_c^*$ where the sum is taken over the collection of horizontal cylinders and $a_c \in \mathbb{R}$. Since (X, ω) is a translation surface in a hyperelliptic connected component whenever v and v' are adjacent cylinders there is an absolute period contained in $v \cup v'$ that intersects γ_v and $\gamma_{v'}$ exactly once with the same orientation. This follows since any two adjacent cylinders are adjacent on both boundaries. Since this period must be unchanged by the relative deformation η it follows that $a_v + a_{v'} = 0$ for any two adjacent cylinders v and v'. Since the vertices are arranged in a tree we have that up to scaling the purely relative deformation is $\sum_c (-1)^{d(c,c_0)} \gamma_c^*$ where c_0 is some fixed cylinder. Finally, the Lindsey tree of (X, ω) cannot have any half-edges since they yield nonzero elements of absolute homology that are supported in a single cylinder and hence will be altered by $\sum_c (-1)^{d(c,c_0)} \gamma_c^*$. Therefore, the Lindsey tree of (X, ω) is a tree (not just a half-tree) and (X, ω) has $g+1$ horizontal cylinders.

(1 if and only if 4) (X, ω) has $g+1$ horizontal cylinders if and only if there are g full edges in the Lindsey tree (equivalently $2g$ half-edges with each full edge counted as two half-edges). Since the total number of half edges for a Lindsey tree corresponding to a surface in $\mathcal{H}^{hyp}(g-1, g-1)$ is $2g$ there are

g full edges if and only if there are no half-edges.

$\square$

Theorem 1.6.4. *If $\mathcal{M}$ has rank $r > 1$ and if $(X, \omega) \in \mathcal{M}$ has $g + 1$ horizontal cylinders then (X, ω) has $r + 1$ equivalence classes. If $\mathcal{C}_0, \ldots, \mathcal{C}_r$ is an enumeration of the equivalence classes then*

$$\mathrm{Twist}_{(X,\omega)}\mathcal{M} = \mathrm{span}_{\mathbb{R}}\{u_{\mathcal{C}_0}, \ldots, u_{\mathcal{C}_r}\}$$

where $u_{\mathcal{C}_i}$ is the standard shear of $\mathcal{C}_i$. Any two cylinders in the same equivalence class have identical heights and are an even distance apart in the Lindsey tree. Moreover, $\mathcal{M}$ is defined over $\mathbb{Q}$.

Proof. If (X, ω) has $g + 1$ cylinders then $\mathrm{Twist}_{(X,\omega)}\mathcal{M} = \mathrm{Pres}_{(X,\omega)}\mathcal{M}$. Since $\mathcal{M}$ is higher rank it follows that there are at least two $\mathcal{M}$-equivalence classes of horizontal cylinders and so Theorem 1.6.3 implies that η is not supported on a single $\mathcal{M}$-equivalence class. By the twist space decomposition theorem (Theorem 1.5.6) it follows that the only element of the twist space supported on a single equivalence class is the standard shear and hence η is a real linear combination of standard shears, i.e.

$$\sum_{i=0}^{g}(-1)^{d(c_i,c_0)}\gamma_i^* = \sum_{i=1}^{n} a_i \sum_{c\in\mathcal{C}_i} h_c\gamma_c^*$$

Since γ_c^* are all linearly independent in $T_{(X,\omega)}\mathcal{H}$ where $\mathcal{H}$ is the component of the stratum containing (X,ω) it follows that any two equivalent cylinders have the same height and are an even distance apart in the Lindsey tree. Finally we see that $\mathrm{Twist}_{(X,\omega)}\mathcal{M}$ is spanned by standard shears and its projection to absolute cohomology has a one-dimensional kernel. It follows that (X,ω) has $r+1$ equivalence classes of cylinders.

By Theorem 7.1 [**Wri15a**] to show that $\mathcal{M}$ is defined over $\mathbb{Q}$ it suffices to show that the ratio of lengths of core curves of any two equivalent horizontal cylinders is always rational. Notice that the only element of the twist space supported on a single equivalence class is the standard shear. This implies that the ratio of moduli of any two equivalent cylinders is rational since otherwise there would be a deformation supported on the equivalence class and contained in the tangent space of $\mathcal{M}$ that was not the standard shear. Since the heights of any two equivalent cylinders are identical the result follows. □

Corollary 1.6.5. *Any two equivalent horizontal cylinders on any translation surface in $\mathcal{M}$ have identical heights when $\mathcal{M}$ is higher rank.*

Proof. This is immediate from Lemma 1.5.8 and Theorem 1.6.4. □

Corollary 1.6.6. *If $(X,\omega) \in \mathcal{M}$ is a translation surface with at least one horizontal cylinder and $\mathcal{M}$ is higher rank then the twist space is spanned by standard shears of the $\mathcal{M}$-equivalence classes of horizontal cylinders.*

Proof. By Theorem 1.6.3, Theorem 1.6.4 establishes this result when the twist space contains η. When the twist space does not contain η the result is immediate by the twist space decomposition theorem (Theorem 1.5.6). □

Theorem 1.6.7. *Suppose that $\mathcal{M}$ has rank one and (X,ω) has $g+1$ cylinders. Suppose furthermore that $\mathcal{C}_0, \mathcal{C}_1$ is a partition of the cylinders so that $T^{\mathbb{R}}_{(X,\omega)}\mathcal{M}$ contains $\displaystyle\sum_{c\in C_0} h_c\gamma_c^*$. Then any two cylinders in $\mathcal{C}_0$ (resp. $\mathcal{C}_1$) have identical heights and are an even distance apart in the Lindsey tree. Moreover, $\mathcal{M}$ is defined over $\mathbb{Q}$ and hence is a branched covering construction of $\mathcal{H}(0,0)$.*

Proof. Since $T^{\mathbb{R}}_{(X,\omega)}\mathcal{M}$ contains $u_0 := \displaystyle\sum_{c\in C_0} h_c\gamma_c^*$ and contains the standard shear $\displaystyle\sum_{c\in C_0\cup C_1} h_c\gamma_c^*$, it contains $u_1 := \displaystyle\sum_{c\in C_1} h_c\gamma_c^*$. Since $\mathcal{M}$ is one-dimensional and has one dimension of rel, it follows that the relative deformation η is a real linear combination of u_1 and u_2. The proof is now identical to the proof of Theorem 1.6.4. □

1.7 A Partial Compactification of Strata of Abelian Differentials

A natural partial compactification of a stratum of abelian differentials is the bundle of stable finite volume abelian differentials over the Deligne-Mumford compactification of moduli space. However, it is often more natural from the perspective of flat geometry to consider a quotient of this space that ignores components of the underlying curve on which the stable one-form vanishes. For the remainder of the chapter this quotient will be called the partial compactification of a stratum. It was introduced in McMullen [**McM13b**] and extensively studied in Mirzakhani-Wright [**MW17**].

One aspect of this partial compactification is that boundary translation surfaces may have marked points. In the following, if we specify a boundary translation surface (X, ω, Σ) then we understand that X may be a disjoint union of Riemann surfaces, ω is a holomorphic one form that does not have zero area on any component of X, and Σ is a collection of marked points that includes all the zeros of ω as well as potentially new marked points that arise. One of the difficulties that we will tackle in the next section is ensuring that marked points do not arise.

The following example of convergence to the boundary is Example 3.1 of [**MW17**]. Suppose that $\mathcal{M}$ is an affine invariant submanifold and let (X, ω) be a translation surface in $\mathcal{M}$ with an $\mathcal{M}$-equivalence class of horizontal cylinders $\mathcal{C}$. Suppose that $\mathcal{C}$ does not cover (X, ω) and that the union of cylinders in $\mathcal{C}$ contains a vertical saddle connection. Let (X_t, ω_t) be (X, ω) with $\mathcal{C}$ vertically shrunk by e^t. By the cylinder deformation theorem it follows that (X_t, ω_t) is a smooth path in $\mathcal{M}$. This sequence converges to a translation surface $(X_\infty, \omega_\infty)$ on the boundary of $\mathcal{M}$. To form $(X_\infty, \omega_\infty)$ use the following procedure:

1. Delete every cylinder in $\mathcal{C}$ from (X, ω) to form a translation surface with boundary. The boundary is a collection of saddle connections.

2. If there is a point p on the boundary of $(X, \omega) - \mathcal{C}$ that is joined to a zero or marked point of (X, ω) by a vertical line that is completely contained in $\mathcal{C}$ then mark p. On the boundary translation surface these points will be either marked points or zeros of the boundary holomorphic one-form. Adding in these marked points may divide saddle connections

into several smaller saddle connections.

3. If two saddle connections on the boundary of $(X,\omega)-\mathcal{C}$ were connected by a vertical line that was completely contained in $\mathcal{C}$ then glue the two saddle connections together. The resulting translation surface is $(X_\infty, \omega_\infty)$.

This construction is called a horizontal cylinder collapse. The analogous construction with an $\mathcal{M}$-equivalence class of vertical cylinders will be called a vertical cylinder collapse.

Theorem 1.7.1 (Mirzakhani-Wright [**MW17**]; Proposition 2.3)**.** *Given a sequence* $(X_n, \omega_n, \Sigma_n)$ *of translation surfaces converging to* (X, ω, Σ) *there are collapse maps* $f_n : X_n \longrightarrow X$ *such that*

1. *There is a neighborhood* U_n *of* Σ_n *so that* $f_n : X_n - U_n \longrightarrow X$ *is a diffeomorphism onto its image with inverse* g_n*.*

2. *The injectivity radius of* U_n *goes to zero uniformly in* n*.*

3. $g_n^*\omega_n$ *converges to* ω *in the compact open topology.*

Define the space of vanishing cycles to be

$$V_n = \ker\left(f_n : H_1(X_n, \Sigma_n; \mathbb{C}) \longrightarrow H_1(X, \Sigma; \mathbb{C})\right)$$

For large enough n this space is constant and will be called V.

Returning to the example of the horizontal cylinder collapse: the space of vanishing cycles will be the subspace spanned by the heights of the horizontal cylinders in $\mathcal{C}$ and by any homology classes that have a representative supported in a subsurface that collapses to a point in the limit. By “height” of a cylinder we mean any saddle connection joining a zero on one boundary of a cylinder to a zero on the other and that intersects the core curve of the cylinder exactly once.

Theorem 1.7.2 (Degeneration Theorem; Theorem 2.7, Mirzakhani-Wright [**MW17**]). *Let $\mathcal{M}$ be an affine invariant submanifold. Let $(X_n, \omega_n, \Sigma_n)$ be translation surfaces in $\mathcal{M}$ converging to $(X_\infty, \omega_\infty, \Sigma)$. Let (Y, η) be a component of $(X_\infty, \omega_\infty)$ and let $\iota : (Y, \eta) \hookrightarrow (X_\infty, \omega_\infty)$ be the inclusion map. Let V be the space of vanishing cycles. The $\mathrm{GL}_2\mathbb{R}$ orbit closure of (Y, η) is an affine invariant submanifold $\mathcal{M}'$ whose tangent space is*

$$T_{(Y,\eta)}\mathcal{M}' = \iota^*\left(T_{(X_n,\omega_n)} \cap \mathrm{Ann}(V_n)\right)$$

where $T_{(X_n,\omega_n)} \cap \mathrm{Ann}(V_n)$ has been identified with the tangent space at the boundary by parallel transport. As a consequence, $\dim_{\mathbb{C}} \mathcal{M}' < \dim_{\mathbb{C}} \mathcal{M}$ and

$\mathrm{rk}(\mathcal{N}) \leq \mathrm{rk}(\mathcal{M})$ *where the inequality is strict if* $\mathrm{rel}(\mathcal{M}) = 0$

1.8 Degenerating to the Boundary in Hyperelliptic Components of Strata

The strategy for classifying higher rank orbit closures in hyperelliptic components in this chapter is an inductive one. We will study affine invariant submanifolds $\mathcal{M}$ by studying their boundary in the Mirzakhani-Wright partial compactification. However, we immediately confront two potential obstacles to our approach. First, the boundary of $\mathcal{M}$ might contain marked points and second, it might not belong to a hyperelliptic component. The goal of this section is to devise degenerations that avoid these two potential problems.

Let (X, ω) be a horizontally periodic translation surface in a hyperelliptic component of a stratum of abelian differentials on genus $g > 1$ Riemann surfaces. Suppose that $\mathcal{V}$ is a collection of vertical cylinders that contains at least one horizontal saddle connection. Suppose too that $\mathcal{C}$ is a collection of horizontal cylinders that are not self-adjacent and that contains a horizontal saddle connection. Suppose that neither collection of cylinders covers the surface. We would like to collapse these cylinders to pass to a surface on the boundary of the stratum of abelian differentials. To be clear, collapsing a collection of vertical cylinders means collapsing them horizontally, i.e. applying the matrix $\begin{pmatrix} e^{-t} & 0 \\ 0 & 1 \end{pmatrix}$ to the vertical cylinders (while fixing the rest of the translation surface) and taking the limit as t goes to infinity. Similarly, collapsing a collection of horizontal cylinders means vertically collapsing them.

Every degeneration of a translation surface that we use will be collapsing a collection of vertical or horizontal cylinders. In this section, we will show the following

Theorem 1.8.1. *Collapsing either $\mathcal{V}$ or $\mathcal{C}$ degenerates to a disjoint union of translation surfaces in hyperelliptic components of strata of abelian differentials.*

Recall that in genus one we have defined the hyperelliptic components to be $\mathcal{H}(0)$ and $\mathcal{H}(0,0)$. Moreover, the boundary of $\mathcal{M}$ that is referred to in the theorem is the boundary in the sense of the Mirzakhani-Wright partial compactification. Finally we remark that a collection S of parallel cylinders is said to be self-adjacent if two cylinders in S are adjacent or if there is a single cylinder in S whose two boundaries are glued together along a saddle connection. To fix notation, let Γ be the Lindsey tree of (X, ω) and let J be the hyperelliptic involution on (X, ω).

We begin by establishing Theorem 1.8.1 in the case of vertical collapses of cylinders. The intuition for the result is the following. Corollary 1.3.3 tells us, roughly, that whenever we glue together cylinders of hyperelliptic combinatorial type (i.e. ones that look like the cylinders in Figure 1.3.3) along a tree that we must get a translation surface in a hyperelliptic component of a stratum. When we collapse a collection of vertical cylinders on a horizontally periodic translation surface, the horizontal cylinders persist on the boundary; they still have hyperelliptic combinatorial type; and they are still glued together in a disjoint union of trees. So the boundary translation surface must be a disjoint union of translation surfaces in hyperelliptic components of strata.

Lemma 1.8.2 (Vertical Cylinder Collapse Lemma)**.** *Collapsing $\mathcal{V}$ degenerates the translation surface to a disjoint union of translation surfaces in hyperelliptic components of strata of abelian differentials.*

Proof. By Corollary 1.3.3 a translation surface belongs to a hyperelliptic component if and only if it is constructed in the following way:

1. Fix a tree with a cyclic ordering around each vertex. For each degree n

vertex in the tree associate a horizontal cylinder of hyperelliptic combinatorial, i.e. the cylinder shown in Figure 1.3.3 up to changing the lengths of the saddle connections and horizontally shearing.

2. When two vertices are joined along an edge, open up the corresponding edges on the appropriate horizontal cylinders and glue the two cylinders together.

This provides both blueprints on how to build translation surfaces in hyperelliptic components and a certificate that a surface belongs to a hyperelliptic component.

Recall that any cylinder in (X, ω) is invariant under the hyperelliptic involution. Therefore, if the proportion of a horizontal saddle connection s contained in a cylinder V is p, then the proportion of the saddle connection $J(s)$ contained in V is also p. Collapsing a collection of vertical cylinders $\mathcal{V}$ passes to a boundary translation surface (Y, η) that can be constructed from (X, ω) in the following way.

1. Let Γ' be the tree that is formed from Γ when all the edges corresponding to saddle connections completely contained in $\mathcal{V}$ are deleted. If a node has no edges attached to it in Γ', then delete it.

2. The remaining nodes correspond to horizontal cylinders on (X, ω) that persist on the boundary translation surface (Y, η). To change a horizontal cylinder C on (X, ω) to the corresponding one on (Y, η) take each saddle connection s on the boundary C and change its length to the length of s not contained in $\mathcal{V}$. The new cylinder on (Y, η) still has hyperelliptic combinatorial type.

Since (Y, η) can be constructed by gluing together horizontal cylinders of hyperelliptic combinatorial type along a disjoint union of trees, Corollary 1.3.3 implies that (Y, η) is a disjoint union of translation surfaces in hyperelliptic components of strata of abelian differentials. □

Now we will complete the proof of Theorem 1.8.1 by analyzing degenerations that involve collapsing a collection of horizontal cylinders.

Lemma 1.8.3 (Horizontal Cylinder Collapse Lemma)**.** *Collapsing $\mathcal{C}$ degenerates to a disjoint union of translation surfaces in hyperelliptic components of strata.*

Proof. Recall that the boundary translation surface $(X_\infty, \omega_\infty)$ may be constructed from (X, ω) in the following way:

1. Delete every cylinder in $\mathcal{C}$ from (X, ω). The result is a translation surface with boundary where the boundary consists of saddle connections that formerly bordered cylinders in $\mathcal{C}$.

2. For each saddle connection on the boundary of $(X, \omega) - \mathcal{C}$ add a marked point to the saddle connection for each point p such that the vertical line contained in $\mathcal{C}$ passing through p terminates at a zero of ω. Since the newly added marked points are invariant under the hyperelliptic involution each cylinder on $(X, \omega) - \mathcal{C}$ continues to have hyperelliptic combinatorial type.

3. Glue together saddle connections on the boundary of $(X, \omega) - \mathcal{C}$ which were connected by a vertical line contained in $\mathcal{C}$. This saddle connection identification is again invariant under the hyperelliptic involution. Let Γ' be Γ with vertices in $\mathcal{C}$ deleted, edges connected to $\mathcal{C}$ deleted, and new edges added between two cylinders that are connected by a vertical line in $\mathcal{C}$.

By Corollary 1.3.3it remains to verify that the cylinder diagram Γ' is a tree. Notice that Γ' is constructed by deleting each vertex v in $\mathcal{C}$ and adding in edges between vertices that were adjacent to v. To show that Γ' is a tree it suffices to show that whenever a vertex v is deleted no cycle forms among the vertices that were formerly adjacent to v.

Rephrased, it suffices to show the following. Suppose that C is a single cylinder of hyperelliptic type. Let $s_1, \ldots, s_n$ be saddle connections on the boundary of C. Let G be a graph with n vertices labelled $\{1, \ldots, n\}$. Connect vertices i and j if s_i and $J(s_j)$ are connected by a vertical line. Then G is a disjoint union of trees. This follows immediately from the following lemma.

Sublemma 1.8.1. *Suppose that there is a graph G with vertices labelled $\{1, \ldots, n\}$, which we imagine as being cyclically ordered. Let C_i be the set of vertices connected to vertex i. Suppose that for all i there is an increasing subset of $\{1, \ldots, n\}$ (perhaps wrapping around 0) that we will denote $I_i = (k_i, k_i + 1, \ldots, \ell_i)$ such that*

1. *$C_i \subseteq I_i$ for all i.*

2. *$I_i \cap I_{i+1} = \{k_i\} = \{\ell_{i+1}\}$ if $n > 2$.*

Then G is a disjoint union of trees.

Proof. Proceed by induction on n. The $n = 2$ base case is trivial. Now suppose that $n > 2$. Suppose to a contradiction that G contains a cycle. Let γ be the shortest cycle in G. If the cycle fails to contain every vertex, then delete the vertices not contained in γ from G. The induction hypothesis implies that the resulting graph cannot contain a cycle, which is a contradiction. Suppose then without loss of generality that γ involves every vertex.

If the degree of a vertex i is greater than two then we may suppose that $C_i = \{k_i, k_{i+1}, \ldots, \ell_i\}$. By the hypotheses, vertices $k_i + 1, \ldots, \ell_i - 1$ only connect to vertex i. Since γ is the shortest cycle in G it does not pass through vertices $k_i + 1, \ldots, \ell_i - 1$ contrary to our assumption that γ passes through every vertex. It follows that every vertex in G has degree two.

Since every vertex in G has degree two and appears exactly once in γ it follows that $C_i = \{k_i, k_i + 1\}$ and $C_{i+1} = \{k_i - 1, k_i\}$ for all i. Therefore, the path γ is $(\gamma_1, \gamma_2, \ldots, \gamma_n) = (1, k_1, 2, k_1 - 1, 3, k_1 - 2, \ldots)$. Notice that the order of the odd vertices is $(\gamma_1, \gamma_3, \ldots) = (1, 2, \ldots, n)$. Since every vertex appears exactly once in γ this implies that $n = 2m + 1$ and the path γ is $(1, m + 2, 2, m + 3, \ldots, m, 2m + 1, m + 1)$. However the order of the even vertices must be $(\gamma_2, \gamma_4, \ldots) = (m + 2, m + 1, m, \ldots)$. Therefore the cyclic order $(1, 2, \ldots, n)$ and the cyclic order $(n, n - 1, \ldots, 1)$ must be the same order. This only occurs when $n = 2$. But we have supposed that $n > 2$, which is a contradiction. $\square$

$\square$

As mentioned earlier all of the degenerations that we will use in this chapter will be either a horizontal or a vertical cylinder collapse. So let's analyze this situation. Let $\mathcal{M}$ be a higher rank affine invariant submanifold in a hyperelliptic component of a stratum. Let (X, ω) be a horizontally periodic translation surface.

Lemma 1.8.4. *If (X, ω) is horizontally periodic then its twist space is spanned by standard shears*

Proof. By Theorem 1.5.6 if this is not the case then there is an $\mathcal{M}$-equivalence class of cylinders $\mathcal{C}$ and a twist space deformation supported on $\mathcal{C}$ that is rel. However, by Theorem 1.6.3 this is only possible if $\mathcal{C}$ contains $g+1$ horizontal cylinders. However, (X, ω) contains at most $g+1$ horizontal cylinders and if all of them belong to one equivalence class then (X, ω) belongs to a rank one orbit closure, which contradicts the hypothesis that $\mathcal{M}$ is higher rank. $\square$

Let $\mathcal{C}$ be an equivalence class of either vertical cylinders or horizontal cylinders that do not form a self-adjacent equivalence class on (X, ω). Let (Y, η) be the translation surface formed by collapsing $\mathcal{C}$ and let (Z, ζ) be a component of (Y, η). Let $\mathcal{N}$ be the affine invariant submanifold in the boundary of $\mathcal{M}$ that contains (Z, ζ).

Lemma 1.8.5 (Twist Space Degeneration Lemma)**.** *Let k be the number of pairwise $\mathcal{M}$-inequivalent horizontal cylinders that persist on (Z, ζ). If $k \geq 2$, the following hold:*

1. *The dimension of the twist space of (Z, ζ) in $\mathcal{N}$ is k.*

2. *If $\mathcal{N}$ is higher rank then two horizontal cylinders in (Z, ζ) are $\mathcal{N}$-equivalent if and only if their preimages on (X, ω) were $\mathcal{M}$-equivalent.*

3. *If $\mathcal{N}$ is rank one but (Z, ζ) contains two cylinders from distinct $\mathcal{M}$-equivalence classes $\mathcal{C}_1$ and $\mathcal{C}_2$ on (X, ω), then no two cylinders from $\mathcal{C}_i$ are adjacent on (Z, ζ) and any two such cylinders have identical height for $i = 1, 2$.*

4. *If the twist space and cylinder preserving space coincide on (X, ω) and a cylinder from every equivalence class persists on (Z, ζ), then $\mathcal{M}$ is even-dimensional, $\mathcal{N}$ is odd-dimensional, and any two equivalent cylinders that persist on (Z, ζ) are not adjacent and have identical heights.*

For the first claim, let $\mathcal{C}$ be a maximal collection of horizontal cylinders on (Z, ζ) that were equivalent on (X, ω). Let $u_{\mathcal{C}}$ be the standard shear of these cylinders on (Z, ζ). By the degeneration theorem of Mirzakhani-Wright **[MW17]** (Theorem 1.7.2 in this work, Theorem 2.7 in **[MW17]**) each $u_{\mathcal{C}}$ is a tangent vector on (Z, ζ). For any collection of cylinders C, the standard shear u_C belongs to the twist space. Since the equivalence classes are pairwise disjoint, the standard shears they induce on (Z, ζ) all belong to the twist space and so the twist space is at least k dimensional. We will complete the proof of the first claim after the proof of the third claim.

For the second claim, suppose that $\mathcal{N}$ is higher rank. Any two $\mathcal{M}$-equivalent cylinders that persist on the boundary remain $\mathcal{N}$-equivalent since the colinearity of their core curves is an algebraic equation that extends to the boundary. Suppose that $\mathcal{C}_1, \ldots, \mathcal{C}_n$ are the equivalence classes of cylinders that persist on $\mathcal{N}$. Any $\mathcal{N}$-equivalence class of horizontal cylinders must $\bigcup_S \mathcal{C}_i$ for some S a subset of $\{1, \ldots, n\}$. By Lemma 1.8.4 the twist space of $\mathcal{N}$ is spanned by standard shears of equivalence classes of horizontal cylinders. By the proof of the first claim the twist space contains $u_{\mathcal{C}_1}, \ldots, u_{\mathcal{C}_n}$. Therefore, $\mathcal{C}_1, \ldots, \mathcal{C}_n$ are exactly the $\mathcal{N}$-equivalence classes. In this case we see that the the dimension of the twist space is exactly k.

For the third claim, suppose that exactly two equivalence classes of cylinders $\mathcal{C}_1$ and $\mathcal{C}_2$ persist on $\mathcal{N}$ and suppose that $\mathcal{N}$ has rank one. By Theorem 1.6.2 the cylinders in $\mathcal{C}_i$ are not adjacent and all have identical heights for each $i = 1, 2$. In this case we also have that the dimension of the twist space is k. This completes the proof of the third claim.

For the fourth claim, suppose that the twist space and cylinder preserving space coincide on (X, ω) and that a cylinder from every equivalence class persists on (Z, ζ). By the first claim, the dimension of the twist space of (Z, ζ) in $\mathcal{N}$ and the dimension of the twist space of (X, ω) in $\mathcal{M}$ are identical. The assumption that the twist space and cylinder preserving space coincide on (X, ω) guarantees that the twist space on (X, ω) is maximal dimensional and hence has dimension $\text{rank}(\mathcal{M}) + \text{rel}(\mathcal{M})$. By the degeneration theorem of Mirzakhani-Wright **[MW17]** (Theorem 1.7.2 in this work, Theorem 2.7 in [**MW17**]),

$$\text{rank}(\mathcal{N}) \leq \text{rank}(\mathcal{M})$$

with strict inequality when $\text{rel}(\mathcal{M}) = 0$. Since $\mathcal{N}$ belongs to a hyperelliptic component of a stratum $\text{rel}(\mathcal{N}) \leq 1$. This observation implies that

$$\text{rank}(\mathcal{N}) + \text{rel}(\mathcal{N}) = \text{rank}(\mathcal{M}) + \text{rel}(\mathcal{M})$$

This expression implies that

$$\dim_{\mathbb{C}} \mathcal{N} = 2 \cdot \text{rank}(\mathcal{N}) + \text{rel}(\mathcal{N}) \leq 2 \cdot \text{rank}(\mathcal{M}) + \text{rel}(\mathcal{M}) = \dim_{\mathbb{C}} \mathcal{M}$$

The degeneration theorem of Mirzakhani-Wright states that the dimension of $\mathcal{N}$ is strictly less than the rank of $\mathcal{M}$ so we see that the rank of $\mathcal{N}$ is strictly less than the rank of $\mathcal{M}$. Since both (X, ω) and (Z, ζ) have the same dimensional twist space we also have that $\text{rel}(\mathcal{N}) = 1$. Since the twist space on (X, ω) is maximal dimensional we have that $\text{rel}(\mathcal{M}) = 0$. Finally since $\mathcal{N}$ is odd dimensional and since the twist space and cylinder preserving space coincide on (Z, ζ) - since the twist space is maximal dimensional - Theorem 1.6.1 and Theorem 1.6.2 implies that any two equivalent cylinders that persist on (Z, ζ) are not adjacent and have identical heights.

1.9 Rank Two Rel Zero Orbit Closures

Let $\mathcal{M}$ be a rank two rel zero affine invariant submanifold. The goal of this section will be to show that if $\mathcal{M}$ is contained in a hyperelliptic component of a stratum of abelian differentials then it is a branched covering construction of $\mathcal{H}(2)$. There are many reasons to single out this case. First, this case is the basis of our induction argument. Second, the proof that $\mathcal{M}$ is a branched covering is almost identical to the general case, but with fewer technical problems (so it makes the main ideas of the proof more transparent). Finally, the proof relies on a lemma, which we call the "Prototype Lemma", that has found application in several forthcoming results in flat geometry. Alex Wright suggested the formulation and proof of the Prototype Lemma.

Lemma 1.9.1. *Let* (X, ω) *be a translation surface in an rank two rel zero affine invariant submanifold* $\mathcal{M}$ *that belongs to any component of any stratum of abelian differentials. Suppose that* (X, ω) *contains two non-intersecting* $\mathcal{M}$*-equivalence classes of cylinders* $\mathcal{C}_1$ *and* $\mathcal{C}_2$*. If* $\mathcal{C}_1$ *contains a saddle connection parallel to the core curves of the cylinders in* $\mathcal{C}_2$*, then* (X, ω) *is periodic in that direction.*

Proof. Since $\mathcal{M}$ is rank two rel zero if there are two distinct equivalence classes of parallel cylinders on (X, ω) in the v-direction, then $\mathcal{M}$ is periodic in the v-direction. Therefore, since the statement is immediate when $\mathcal{C}_1$ and $\mathcal{C}_2$ are parallel, let's assume that they are not parallel. After rotating and shearing we may assume without loss of generality that $\mathcal{C}_1$ is a collection of horizontal cylinders and that $\mathcal{C}_2$ is a collection of vertical ones. The condition that $\mathcal{C}_1$ contains a saddle connection with period v_2 now becomes that $\mathcal{C}_1$ contains a vertical saddle connection. Let $(X_\infty, \omega_\infty)$ be the boundary translation surface formed by collapsing $\mathcal{C}_1$.

By the degeneration theorem of Mirzakhani-Wright [**MW17**] (Theorem

1.7.2 in this work, Theorem 2.7 in [**MW17**]), if $\mathcal{C}_1$ is collapsed and (Y, η) is any component of the boundary translation surface $(X_\infty, \omega_\infty)$, then the orbit closure of (Y, η) has complex dimension at most three. In particular, each component of the boundary translation surface is completely periodic. A completely periodic translation surface is characterized by the property that if there is one cylinder in a given direction, then that direction is periodic. Therefore, if we can show that some cylinder from $\mathcal{C}_2$ appears on each component of the boundary translation surface then we may conclude that every component of the boundary translation surface is vertically periodic. In particular, since $(X_\infty, \omega_\infty)$ was formed by vertically collapsing a collection of cylinders, each component of $(X_\infty, \omega_\infty)$ is vertically periodic if and only if (X, ω) is vertically periodic. To summarize, it suffices to show that a cylinder from $\mathcal{C}_2$ appears on each component of $(X_\infty, \omega_\infty)$.

Let C be a connected component of the translation surface with boundary formed from (X, ω) once $\mathcal{C}_1$ and $\mathcal{C}_2$ are removed. It suffices to show that each region C borders a cylinders in $\mathcal{C}_2$. Suppose to a contradiction that C is such a region that does not border a cylinder in $\mathcal{C}_2$. It follows that the region C borders a horizontal cylinder D in $\mathcal{C}_1$. Applying the cylinder deformation theorem, we may vertically shear the cylinders in $\mathcal{C}_2$ so that it contains a horizontal saddle connection while fixing the rest of the translation surface. Vertically shearing the cylinders in $\mathcal{C}_2$ does not alter the fact that the region C is not adjacent to a cylinder in $\mathcal{C}_2$. Let (Z, ζ) be the boundary translation surface formed by collapsing $\mathcal{C}_2$.

Since C does not border a vertical cylinder in $\mathcal{C}_2$ it persists isometrically on the boundary translation surface (Z, ζ). Since it is adjacent to the horizontal cylinder D, the region C and the cylinder D remain adjacent on (Z, ζ) and hence belong to the same component of the boundary translation surface. Since each component of the boundary translation surface is completely periodic and since the component containing the region C also contains the horizontal cylinder D, it follows that C is a union of horizontal cylinders.

Since C did not border a cylinder in $\mathcal{C}_2$, it was unaffected by the degeneration and hence the region C on (X, ω) is also covered by horizontal cylinders. By assumption, since C is in the complement of the cylinders in $\mathcal{C}_1$, the region C is covered by horizontal cylinders that are inequivalent to the horizontal cylinders in $\mathcal{C}_1$. Moreover, none of the horizontal cylinders in the region C intersect cylinders in $\mathcal{C}_2$. Therefore, (X, ω) contains two equivalence classes of horizontal cylinders, neither of which intersects cylinders in $\mathcal{C}_2$. However, since $\mathcal{M}$ is rank two, if (X, ω) contains two equivalence classes of horizontal cylinders then these two equivalence classes cover all of (X, ω) and so some horizontal cylinder must intersect a cylinder in $\mathcal{C}_2$, which is a contradiction. □

Lemma 1.9.2 (Prototype Lemma)**.** *Any rank two rel zero affine invariant submanifold $\mathcal{M}$ in any component of any stratum contains a translation surface that has exactly two horizontal and two vertical $\mathcal{M}$-equivalence classes of cylinders, so that one of the horizontal $\mathcal{M}$-equivalence classes does not intersect one of the vertical $\mathcal{M}$-equivalence classes.*

Proof. Let $\mathcal{M}$ be a rank two rel zero affine invariant submanifold. As described in the cylinder deformation section (Section 1.5 Theorem 1.5.4), by Wright Lemma 8.6 **[Wri15a]** there is a horizontally periodic translation surface (X, ω) in $\mathcal{M}$ on which the twist space and the cylinder preserving space coincide. Since we have assumed that $\mathcal{M}$ has rank two and no rel, this implies that (X, ω) has exactly two equivalence classes of horizontal cylinders. Call these equivalence classes $\mathcal{C}_1$ and $\mathcal{C}_2$ and let σ_1 and σ_2 be the two corresponding standard shears (defined immediately before the statement of Theorem 1.5.4). The absence of rel implies that the projections to absolute cohomology of σ_1 and σ_2 span a two dimensional subspace.

The perturbation theorem (Theorem 1.5.7 in this work; Lemma 5.5 in

Mirzakhani-Wright [**MW17**]) states that we can find a path in $\mathcal{M}$ that deforms (X, ω) to a new translation surface (Y, η) so that along the path (1) no cylinder in $\mathcal{C}_1$ or $\mathcal{C}_2$ vanishes, (2) if C_1 is a cylinder equivalent to a cylinder in $\mathcal{C}_1$ and C_2 is cylinder equivalent to a cylinder in $\mathcal{C}_2$ then C_1 and C_2 are disjoint, and (3) on (Y, η) the cylinders in $\mathcal{C}_1$ remain horizontal and the cylinders in $\mathcal{C}_2$ are vertical. After applying the cylinder deformation theorem we may ensure, perhaps after horizontally shearing the cylinders in $\mathcal{C}_1$ and vertically shearing the cylinders in $\mathcal{C}_2$, that $\mathcal{C}_1$ contains a vertical saddle connection and that $\mathcal{C}_1$ contains a horizontal saddle connection. By Lemma 1.9.1, the new translation surface is vertically and horizontally periodic and contains a equivalence class of vertical cylinders $\mathcal{C}_2$ that does not intersect a horizontal equivalence class of cylinders $\mathcal{C}_1$. Since the translation surface is horizontally and vertically periodic and contains a non-intersecting horizontal and vertical equivalence class of cylinders, it follows that there are at least two horizontal and two vertical equivalence classes. Since $\mathcal{M}$ is rank two rel zero, there are no more than two equivalence classes of cylinders in any given direction and so there are exactly two horizontal and two vertical equivalence classes of cylinders. □

Given a rank two rel zero affine invariant submanifold $\mathcal{M}$, we say that

(X, ω) is a prototype translation surface if (X, ω) belongs to $\mathcal{M}$, is vertically and horizontally periodic, has a non-intersecting horizontal and vertical equivalence class of cylinders. Given a horizontally periodic translation surface with Lindsey tree Γ in a hyperelliptic component of a stratum, we will say that two edges or half-edges of Γ are $\mathcal{M}$-equivalent if they connect the same two $\mathcal{M}$-equivalence classes of horizontal cylinders. Half-edges will be understood to connect an $\mathcal{M}$-equivalence class to itself.

From now on we assume that $\mathcal{M}$ is an affine invariant submanifold in a hyperelliptic component of a stratum of abelian differentials and that it is rank two rel zero. Let (X, ω) be a prototype surface on $\mathcal{M}$ with horizontal equivalence classes $\mathcal{C}_1$ and $\mathcal{C}_2$ and vertical equivalence classes $\mathcal{V}_1$ and $\mathcal{V}_2$. We assume without loss of generality that $\mathcal{C}_1$ and $\mathcal{V}_2$ do not intersect and that moreover they contain a vertical and horizontal saddle connection respectively. Finally, let $\mathcal{S}_1$ be the equivalence class of horizontal saddle connections that connect a cylinder in $\mathcal{C}_1$ to a cylinder in $\mathcal{C}_2$. Let $\mathcal{S}_1$ be the equivalence class of horizontal saddle connections that connect a cylinder in $\mathcal{C}_2$ to a cylinder in $\mathcal{C}_2$.

Lemma 1.9.3. *If (X, ω) is a prototype surface in $\mathcal{M}$ described above then*

1. *Equivalent horizontal cylinders have identical heights.*

2. *Equivalent horizontal saddle connections have identical lengths.*

3. *The saddle connections on the boundary of $\mathcal{C}_2$ alternate between $\mathcal{S}_1$ and $\mathcal{S}_2$.*

Proof. Given a prototype surface (X, ω) we can degenerate the surface by collapsing either $\mathcal{C}_1$ or $\mathcal{V}_2$. The vertical collapse lemma (Lemma 1.8.2) implies that the resulting boundary translation surface is a disjoint union of translation surfaces in hyperelliptic components of strata. Let $(X_\infty, \omega_\infty)$ be the boundary translation surface formed by collapsing $\mathcal{C}_1$.

Let (Y, η) be a component of $(X_\infty, \omega_\infty)$. By the degeneration theorem of Mirzakhani-Wright **[MW17]** (Theorem 1.7.2 in this work, Theorem 2.7 in **[MW17]**), the orbit closure $\mathcal{N}$ of (Y, η) is at most three complex-dimensional. Since cylinders from both $\mathcal{V}_1$ and $\mathcal{V}_2$ persist on (Y, η), the standard shears $\sigma_{\mathcal{V}_1}$ and $\sigma_{\mathcal{V}_2}$ are both tangent to $\mathcal{N}$ at (Y, η). Since $\mathcal{N}$ is rank one and since the two standard shears are not constant multiples of each other and pair to zero under the cup product pairing, it follows that $\mathcal{N}$ must have nonzero rel. Since $\mathcal{N}$ is at most three complex-dimensional, it follows that $\mathcal{N}$ is rank one rel one. In particular, some nonzero linear combination of $\sigma_{\mathcal{V}_1}$ and $\sigma_{\mathcal{V}_2}$ must be rel on (Y, η).

To summarize, any component of $(X_\infty, \omega_\infty)$ has rank one rel one orbit closure and contains $\sigma_{\mathcal{V}_1}$ and $\sigma_{\mathcal{V}_2}$ where $\mathcal{V}_1$ and $\mathcal{V}_2$ are disjoint collections of vertical cylinders, Since the twist space of a translation surface in a rank one rel one orbit closure is at most two dimensional, it follows that these two tangent vectors span the twist space. We are therefore exactly in the situation described in Theorem 1.6.2. By Theorem 1.6.2, no two cylinders in $\mathcal{V}_1$ and $\mathcal{V}_2$ are adjacent on $(X_\infty, \omega_\infty)$ and any two cylinders that both belonged to the same $\mathcal{V}_i$ for $i = 1$ or 2 have identical heights.

Since no two equivalent vertical cylinders can be adjacent in $\mathcal{C}_2$, claim 3 follows. Moreover, for any horizontal saddle connection on the boundary of a cylinder in $\mathcal{C}_2$ there is exactly one vertical cylinder that passes through it (otherwise two equivalent cylinders would be adjacent in $\mathcal{C}_2$). Therefore, the length of each saddle connection in $\mathcal{S}_i$ is also the height of a cylinder in $\mathcal{V}_i$ for $i = 1$ and 2. Claim 2 will follow from claim 1 by symmetry of hypotheses. It remains to establish claim 1, i.e. that equivalent horizontal cylinders have identical heights.

We begin by showing that any two cylinders in $\mathcal{C}_1$ have identical heights. We have already shown that if we degenerate $\mathcal{V}_2$ and two cylinders from $\mathcal{C}_1$ end up on the same component of the boundary translation surface, then those two cylinders have identical heights. A path γ in $X - Z(\omega)$ is called a staircase path if it is piecewise linear and each linear piece is either vertical or horizontal. Let C and D be horizontal cylinders in $\mathcal{C}_1$. For any staircase path γ between C and D it is possible to produce a collection of cylinders $\{C_i\}_{i=0}^n$ in $\mathcal{C}_1$ so that $C_0 = C$, $C_n = D$ and so that there are staircase paths γ_i from C_{i-1} to C_i that pass through at most one cylinder in $\mathcal{V}_2$.

The algorithm to produce this new collection of staircase paths is straightforward. Follow the staircase path γ until it has entered and exited a cylinder in $\mathcal{V}_2$. Since no two cylinders in $\mathcal{V}_2$ are adjacent, before entering and after exiting a cylinder in $\mathcal{C}_2$, it will be in a cylinder contained in $\mathcal{V}_1$. Modify γ with a vertical line so that once it exits $\mathcal{V}_2$ it travels vertically and enters a cylinder in $\mathcal{C}_1$. Now repeat the procedure.

Since any two cylinders may be joined by a staircase path it suffices to show that if C and D are cylinders in $\mathcal{C}_1$ that are joined by a staircase path γ that passes through exactly one cylinder V in $\mathcal{V}_2$ that C and D have identical heights. Let s_1 be the vertical saddle connection on the lefthand boundary of V that γ passes through and s_2 the one on the righthand boundary. By the cylinder deformation theorem, it is possible to vertically shear $\mathcal{V}_2$ while fixing the rest of the translation surface so that s_1 and s_2 are connected by a horizontal line contained in V and so that $\mathcal{V}_2$ contains some horizontal saddle connection. Collapsing $\mathcal{V}_2$ ensures that C and D land on the same component of the boundary translation surface and hence, as argued above, that they have identical heights. This establishes that all cylinders in $\mathcal{C}_1$ have identical heights as desired.

It remains to show that any two cylinders in $\mathcal{C}_2$ have identical heights. Let C be the cylinder in $\mathcal{C}_2$ that has smallest height. Let D be any cylinder in $\mathcal{C}_1$ adjacent to C. By the standard position lemma, it is possible to put C and D into standard position. If V is the resulting cylinder then every cylinder equivalent to V must spend the same percent of time in $\mathcal{C}_2$ as V does by the cylinder proportion theorem. Since every cylinder in $\mathcal{C}_1$ has identical height and no two are adjacent, it follows that if V' is a vertical cylinder equivalent to V then every cylinder in $\mathcal{C}_2$ that V' intersects has the same height as C. By the cylinder proportion theorem, every cylinder in $\mathcal{C}_2$ is intersected by a vertical cylinder equivalent to V and so every cylinder in $\mathcal{C}_2$ has the same height as cylinder C. □

We will now rephrase Lemma 1.9.3 in a form more obviously connected to branched covering constructions. First, we make a definition. Suppose that $\{C_1, \ldots, C_n\}$ is a collection of cylinders on a flat surface. We will take the cylinders to be marked at points on their boundary corresponding to cone points of the flat metric. We will say that the cylinders are mutually isogenous if there is a cylinder C with marked boundary and a local isometry $f_i : C_i \longrightarrow C$ for each $1 \leq i \leq n$ so that the preimage of the marked points on the boundary of C under f_i are exactly the marked points on the boundary of C_i. The upshot of Lemma 1.9.3 is the following

Corollary 1.9.4. *Let (X, ω) be the prototype surface in $\mathcal{M}$ as in Lemma 1.9.3. Let ℓ_i be the length of the saddle connections in $\mathcal{S}_i$ and let h_i be the heights of the cylinders in $\mathcal{C}_i$ for $i = 1, 2$. Each cylinder in $\mathcal{C}_1$ is isogenous to the cylinder in Figure 1.9.1.*

ℓ_1

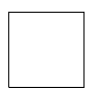
h_1 

Figure 1.9.1: The cylinder to which all cylinders in $\mathcal{C}_1$ are isogenous

and every cylinder in $\mathcal{C}_2$ is isogenous to the cylinder in Figure 1.9.2

ℓ_1 ℓ_2

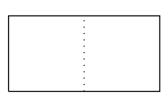
h_2 

Figure 1.9.2: The cylinder to which all cylinders in $\mathcal{C}_2$ are isogenous

where opposite sides are identified, the labels correspond to lengths, and all angles are right angles.

Proof. If C is a cylinder in $\mathcal{C}_1$ then it has height h_1 and every saddle connection on the boundary has length ℓ_1 by Lemma 1.9.3. Since every vertical cylinder that passes through C crosses through a saddle connection exactly once and must fully contain any saddle connection it passes through, it follows that every saddle connection on one boundary of C perfectly vertically aligns with a saddle connection on the other boundary. It follows that C is isogenous to the cylinder in Figure 1.9.1. The proof for cylinders in $\mathcal{C}_2$ is essentially identical and so we omit it. □

Theorem 1.9.5. *If $\mathcal{M}$ is a rank two rel zero affine invariant submanifold in a hyperelliptic component of a stratum then it is a branched covering construction of $\mathcal{H}(2)$.*

Proof. Let (X, ω) be a prototype surface in $\mathcal{M}$ as in Lemma 1.9.3. We will now construct the surface that (X, ω) is a translation covering of. Let (Y, η) be the translation surface in Figure 1.9.3.

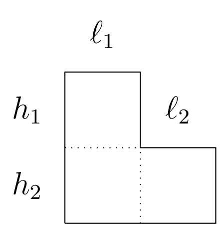


Figure 1.9.3: The translation surface that (X, ω) covers in $\mathcal{H}(2)$

In Figure 1.9.3 opposite sides are identified, the labels correspond to the lengths of the saddle connections, and all angles are right angles. For each horizontal cylinder on (X, ω) there is a local isometry that takes it to either the top cylinder, if the horizontal cylinder belonged to $\mathcal{C}_1$, or the bottom cylinder, if it belonged to $\mathcal{C}_2$. The maps on the horizontal cylinders of (X, ω) agree whenever two cylinders share a boundary and so the local isometries glue together to form a map $f : (X, \omega) \longrightarrow (Y, \eta)$ that is a translation covering.

By Corollary 1.4.3 to show that $\mathcal{M}$ is a branched covering construction of $\mathcal{H}(2)$ it suffices to show that we can find a generic prototype surface. By the cylinder deformation theorem we may suppose without loss of generality that $\ell_1 = \ell_2 = h_1 = 1$ and $h_2 = a$ where a is any transcendental number. Since the moduli of the cylinders in $\mathcal{C}_1$ and those in $\mathcal{C}_2$ are not rational multiples of each other, the orbit closure of (X, ω) contains the standard shears on $\mathcal{C}_1$ and $\mathcal{C}_2$. By Avila, Eskin, Möller [**AEM**], the projection to absolute cohomology of the tangent space of any orbit closure is complex symplectic. Since the tangent space of the orbit closure of (X, ω) contains a two dimensional complex isotropic subspace, it must have complex dimension at least four and hence coincide with the tangent space to $\mathcal{M}$ at (X, ω). Therefore,

(X, ω) is generic under the action of $\mathrm{GL}_2(\mathbb{R})$ and so $\mathcal{M}$ is a branched covering construction of $\mathcal{H}(2)$ as desired. □

Remark 1.9.1. *Remember the standing assumption that we are really working with*

1.10 The Flat Geometry of Translation Surfaces in Higher Rank Affine Invariant Submanifolds

In this section, we begin the inductive argument. Let $\mathcal{M}$ be a higher rank affine invariant submanifold in a hyperelliptic component of a stratum of abelian differentials. Assume throughout this section that any higher rank affine invariant submanifold in a hyperelliptic components that has dimension strictly smaller than $\mathcal{M}$ is a branched covering construction of a hyperelliptic component of a stratum. Under these hypotheses, the main theorem of the section is the following:

Theorem 1.10.1. *If (X, ω) is a horizontally periodic translation surface in $\mathcal{M}$ with twist space and cylinder preserving space coinciding then the follow-*

ing hold:

1. *Any two $\mathcal{M}$-equivalent horizontal cylinders in (X, ω) have identical heights.*

2. *If $\mathcal{M}$ is even complex-dimensional then there is exactly one self-adjacent equivalence class of horizontal cylinders.*

3. *If s is a saddle connection on the boundary of two equivalent horizontal cylinders, then there is a cylinder that contains s, intersects it exactly once, and is contained in the equivalence class of horizontal cylinders.*

By Theorem 1.6.1 this result holds for odd-dimensional $\mathcal{M}$. Therefore, throughout this section we will suppose that $\mathcal{M}$ is even-dimensional. Since we have already established the main theorem for rank two affine invariant submanifolds that are even-dimensional, we will suppose furthermore that the rank r of $\mathcal{M}$ is at least three. Suppose finally that (X, ω) is a horizontally periodic translation surface with twist space and cylinder preserving space coinciding.

We begin by finding an equivalence class of cylinders that we will use to degenerate (X, ω) to the boundary of $\mathcal{M}$.

Lemma 1.10.2 (Leaf Lemma)**.** *There is a horizontal cylinder that is only adjacent to one inequivalent cylinder and perhaps also itself.*

Proof. Let Γ be the Lindsey half-tree of (X, ω). Enumerate the equivalence classes of horizontal cylinders $\{1, \ldots, m\}$ and color the vertices of the tree by the corresponding equivalence class. Let Γ' be the quotient of Γ where each monochromatic connected subtree is collapsed to a single point (colored with the same color as the subtree). Let λ be a leaf of the quotient graph Γ', i.e. a vertex that connects to at most one other vertex in Γ'. Let T be the monochromatic connected subtree corresponding to λ. Since λ was a leaf in Γ' there is a single vertex w in T so that T is connected to the rest of Γ by an edge joining w to a vertex v of a different color.

Suppose to a contradiction that T contains more vertices than just w. Let C_w and C_v be the cylinders in (X, ω) corresponding to w and v respectively. Let C be the cylinder in T that is different from C_w. By the standard position lemma it is possible to shear the equivalence classes containing C_w and C_v so that there is a vertical cylinder V that is contained in $C_w \cup C_v$ and that contains the two saddle connections s and s' that connect C_w to C_v. By the cylinder proportion theorem, there is a cylinder V' that is equivalent to V and that passes through C. Since the monochromatic tree T is connected to the rest of Γ through w and since V contains the saddle connections connecting C_v to C_w it follows that V' must be contained in T (since it cannot escape into the rest of Γ through s and s'). By the cylinder proportion theorem, since V' is contained entirely in one equivalence class so is V. This contradicts the assumption that V intersects both C_w and C_v, which are inequivalent cylinders. Therefore, T contains a single vertex w and w is only adjacent to one inequivalent cylinder v and perhaps also itself. $\square$

We now fix notation that we will use for the remainder of the section. Let L be a horizontal leaf cylinder on (X, ω) and let $\mathcal{C}_0$ be the equivalence class of horizontal cylinders that it belongs to. Suppose that the only distinct cylinder that L is adjacent to is L', which belongs to the equivalence class $\mathcal{C}_1$. Suppose furthermore, using the standard position lemma, that L and L' are in standard position and that W is the resulting vertical cylinder that passes between them. Let $\mathcal{W}$ be the equivalence class of vertical cylinders that contains W.

By the vertical collapse lemma (Lemma 1.8.2), collapsing $\mathcal{W}$ results in a translation surface (Y, η) that is a disjoint union of translation surfaces in hyperelliptic components of strata. Since collapsing $\mathcal{W}$ is a path in $\mathcal{M}$, the surface (Y, η) belongs to the Mirzakhani-Wright partial compactification of

$\mathcal{M}$.

Lemma 1.10.3. *If (Z, ζ) is a component of (Y, η) that contains a cylinder from $\mathcal{C}_1$ and $\mathcal{N}$ is the affine invariant submanifold in the boundary of $\mathcal{M}$ that contains (Z, ζ) then the following hold:*

1. *$\mathcal{N}$ is either $2r-3$ or $2r-2$ complex-dimensional*

2. *The twist space and cylinder preserving space on (Z, ζ) coincide.*

Proof. Suppose that (Z, ζ) is a component of (Y, η) that contains a cylinder that belonged to $\mathcal{C}_1$. Suppose that $\mathcal{C}$ is any equivalence class of horizontal cylinders excluding $\mathcal{C}_0$. We will show that (Z, ζ) contains a cylinder belonging to $\mathcal{C}$ by inducting on the distance d from $\mathcal{C}$ to $\mathcal{C}_1$ in the Lindsey tree. The $d = 0$ base case, i.e. that (Z, ζ) contains a cylinder that belonged to $\mathcal{C}_1$ holds by assumption.

Now suppose that $\mathcal{C}$ is distance $d > 0$ away from $\mathcal{C}_1$ in the Lindsey tree and that a cylinder from any equivalence class that is distance less than d away from $\mathcal{C}$ appears on each component of (Y, η). Let C be a cylinder in $\mathcal{C}$ that is adjacent to a cylinder D that is distance $d-1$ away from $\mathcal{C}_1$ in the Lindsey tree. Let $\mathcal{C}'$ be the equivalence class of cylinders containing D. The standard position lemma implies that we may apply a standard shear to

$\mathcal{C}$ to put C and D in transverse standard position. In particular, there is a cylinder V that is contained in $C \cup D$ and intersects the saddle connection s' on their boundary exactly once. Let $\mathcal{V}$ be the equivalence class of cylinders containing V. By the cylinder proportion theorem, any cylinder in $\mathcal{V}$ must be contained exclusively in cylinders contained in $\mathcal{C}$ or $\mathcal{C}'$. In particular, the cylinders do not pass through saddle connections connecting $\mathcal{C}_0$ to $\mathcal{C}_1$, which are precisely the ones altered in the collapse. Therefore, all of the cylinders in $\mathcal{V}$ persist on (Y, η). By the cylinder proportion theorem, every cylinder in $\mathcal{C}$ and $\mathcal{C}'$ are intersected by a cylinder in $\mathcal{V}$. Since there is a cylinder in $\mathcal{C}'$ on (Z, ζ) by the induction hypothesis, it follows that there is a cylinder in $\mathcal{V}$ on (Z, ζ) as well. Therefore, there is a cylinder equivalent to $\mathcal{C}$ on (Z, ζ) as desired.

The degeneration theorem of Mirzakhani-Wright [**MW17**] (Theorem 1.7.2 in this work, Theorem 2.7 in [**MW17**]) implies that the rank of $\mathcal{N}$ is strictly less than r. Since a cylinder from all but one horizontal equivalence class persists on (Z, ζ) it follows from the twist space degeneration lemma (Lemma 1.8.5 part 1) that the dimension of the twist space of (Z, ζ) is $r-1$. The dimension of the twist space is at most $\text{rank}(\mathcal{N}) + \text{rel}(\mathcal{N})$. Since $\mathcal{N}$ is contained in a hyperelliptic component, $\text{rel}(\mathcal{N}) \leq 1$ and so $\mathcal{N}$ is either rank $r-1$ or rank

$r-2$ and rel 1. Since the complex-dimension of an affine invariant submanifold $\mathcal{N}$ is $2 \cdot \text{rank}(\mathcal{N}) + \text{rel}(\mathcal{N})$ we see that $\mathcal{N}$ has complex dimension at least $2r-3$.

We will now argue that $\mathcal{N}$ has dimension at most $2r-2$. Let C be a cylinder in $\mathcal{C}_1$ that belongs to (Z, ζ). Let (Y_1, η_1) be the surface formed by collapsing $\mathcal{C}_0$. While we cannot apply the horizontal collapse lemma to (Y_1, η_1) - since we have not verified that $\mathcal{C}_0$ is not self-adjacent - we can still apply the degeneration theorem. Let (Z_1, η_1) be the component of $\mathcal{C}_0$ that contains C and let $\mathcal{N}_1$ be the affine invariant submanifold in the boundary of $\mathcal{M}$ that contains (Z_1, ζ_1). Since the cylinders in $\mathcal{W}$ continue to persist on (Z_1, ζ_1) we may collapse them to pass to a translation surface (Y_2, η_2). However, (Y_2, η_2) could have been produced in a single degeneration by degenerating $\mathcal{W}$. In particular, (Y_2, η_2) is a union of components of (Y, η). Since the cylinder C persists on (Y_2, η_2) one of those components that is contained in (Y_2, η_2) is (Z, ζ).

This implies that $\mathcal{N}_1$ is in the boundary of $\mathcal{N}$ and that $\mathcal{N}$ is in the boundary of $\mathcal{N}_1$. By the degeneration theorem of Mirzakhani-Wright [**MW17**], the boundary of an affine invariant submanifold has dimension that is strictly smaller than that of the affine invariant submanifold. This implies that $\mathcal{N}$

has dimension at most $2r-2$. Since the dimension of the twist space is maximal dimensional on (Z,ζ) it follows that the twist space and cylinder preserving space coincide. ☐

The upper bound on the size of the dimension of $\mathcal{N}$ in the proof of Lemma 1.10.3 came from realizing the degeneration as two successive degenerations. In the following, lemma we will exploit this idea again. We will show that if $\mathcal{C}_0$ contains and is distinct from an equivalence class of cylinders $\mathcal{V}$, then we can realize the degeneration from (X,ω) to (Y,η) as three successive degenerations. This will force the dimension of any component of (Y,η) to be at most $2r-3$ complex-dimensional.

Lemma 1.10.4. *Suppose that there is an equivalence class of cylinders $\mathcal{V}$ that is contained in and distinct from $\mathcal{C}_0$, then if (Z,ζ) is any component of (Y,η) that contains a cylinder from $\mathcal{C}_1$ it has orbit closure of dimension $2r-3$. In particular, $\mathcal{C}_0$ is the only self-adjacent equivalence class of horizontal cylinders on (X,ω).*

Proof. Fix a component (Z,ζ) of (Y,η) that contains a cylinder C that was previously part of $\mathcal{C}_1$ on (X,ω). We arrived at (Z,ζ) from a single degeneration, namely degenerating $\mathcal{W}$, but now we will show that we can also arrive at (Z, ζ) through three successive degenerations.

Let (Y_1, η_1) be the translation surface that results from collapsing $\mathcal{V}$. By the vertical collapse lemma, (Y_1, η_1) belongs to a disjoint union of translation surfaces in hyperelliptic components of strata. Let (Z_1, ζ_1) be that component that contains the cylinder C that was previously in $\mathcal{C}_1$. Let $\mathcal{N}_1$ be the affine invariant submanifold in the boundary of $\mathcal{M}$ that contains (Z_1, ζ_1).

Since cylinders from $\mathcal{C}_0$ appear on (Z_1, ζ_1) we may collapse them. Let (Y_2, η_2) be the resulting translation surface. Let (Z_2, ζ_2) be the component of (Y_2, η_2) that contains C and let $\mathcal{N}_2$ be the affine invariant submanifold containing (Z_2, ζ_2) that is contained in the boundary of $\mathcal{N}_1$.

Finally, cylinders from $\mathcal{W}$ persist on (Z_2, ζ_2) and we may collapse them. Let (Y_3, η_3) be the resulting translator surface; let (Z_3, ζ_3) be a component of (Y_3, η_3) containing C and let $\mathcal{N}_3$ be the orbit closure of (Z_3, ζ_3).

Observe that the three successive degenerations that we constructed to produce (Z_3, ζ_3) produce the same translation surface as produced if $\mathcal{W}$ is collapsed. Since both (Z_3, ζ_3) and (Z, ζ) contain the image of the cylinder C, it follows that they are the same translation surface. By the degeneration theorem of Mirzakhani-Wright [**MW17**] (Theorem 1.7.2 in this work, Theorem 2.7 in [**MW17**]) the complex dimension of $\mathcal{N}_i$ is at most $2r - i$. Since the the orbit closure $\mathcal{N}$ of (Z, ζ) has complex-dimension at least three and since $\mathcal{N}$ corresponds to $\mathcal{N}_3$, which has complex dimension at most $2r-3$ we see that the $\mathcal{N}$ is exactly $2r-3$ complex-dimensional.

Every component of (Y, η) either contains only cylinders that belonged to $\mathcal{C}_0$ or it contains a cylinder that belonged to $\mathcal{C}_1$ and has orbit closure of dimension $2r-3$ with twist space and cylinder preserving space coinciding. In the latter case, Theorem 1.6.1 and Theorem 1.6.2 imply that no two equivalent cylinders on (X, ω) remain adjacent on a component of (Y, η) that contains a cylinder from $\mathcal{C}_1$. This implies that only cylinders in $\mathcal{C}_0$ were self-adjacent. □

Now we are almost in a position to show that (X, ω) satisfies the final two properties in Theorem 1.10.1. The following lemma almost proves two out of three of the properties we must show to prove the main theorem of this section. The "almost" is needed because we are about to show that (X, ω) has at most one self-adjacent equivalence class, whereas Theorem 1.10.1 states that it has exactly one. The final result of the section will be circling around and improving "at most one" to "exactly one".

Lemma 1.10.5. *There is at most one horizontal equivalence class on (X, ω) that is self-adjacent and given any saddle connection s connecting two equiv-*

alent horizontal cylinders there is a cylinder V contained in the equivalence class, containing s, and intersecting s exactly once.

Proof. Suppose first that $\mathcal{C}$ is an equivalence class that is not equal to $\mathcal{C}_0$ or $\mathcal{C}_1$ that is self-adjacent. Let s be any saddle connection on the boundary of two cylinders in $\mathcal{C}$. When the equivalence class $\mathcal{W}$ is collapsed to form (Y, η) the horizontal cylinders in $\mathcal{C}$ remain unaltered. Let (Z, ζ) be the component of (Y, η) on which s persists and let $\mathcal{N}$ be the affine invariant submanifold in the boundary of $\mathcal{M}$ that contains (Z, ζ). By Lemma 1.10.3, the twist space and cylinder preserving space of (Z, ζ) coincide. It follows from Theorem 1.6.1 and Theorem 1.6.2 that $\mathcal{N}$ cannot have odd dimension since it contains two cylinders from $\mathcal{C}$ that are adjacent. Therefore, $\mathcal{N}$ is higher rank. By the induction hypothesis, there is a cylinder V that is contained in $\mathcal{C}$, that contains s, and that intersects s exactly once.

Since the collapse of $\mathcal{W}$ fixes the cylinders in $\mathcal{C}$, it follows that on (X, ω) the cylinder V persists. Let $\mathcal{V}$ be the cylinders that are equivalent to V on (X, ω). By the cylinder proportion theorem, each cylinder in $\mathcal{V}$ is contained in the union of cylinders in $\mathcal{C}$ and each cylinder in $\mathcal{C}$ intersects a cylinder in $\mathcal{V}$. The components of (Y, η) either consist entirely of cylinders from $\mathcal{C}_0$ or they contain every equivalence class of horizontal cylinder except for $\mathcal{C}_0$. In the latter case, the component contains a cylinder from $\mathcal{C}$ and hence a cylinder from $\mathcal{V}$. This implies that on these components $\mathcal{C}$ remains self-adjacent. By the induction hypothesis, there is exactly one equivalence class of adjacent cylinders on any such component of (Y, η) and so we have shown that the only equivalence classes of cylinders on (X, ω) that were self-adjacent are $\mathcal{C}$ and possibly $\mathcal{C}_0$.

Suppose now to a contradiction, that both $\mathcal{C}$ and $\mathcal{C}_0$ are self-adjacent on (X, ω). Let (Y_1, η_1) be the translation surface that results from collapsing $\mathcal{V}$. By the vertical collapse lemma, (Y_1, η_1) is a disjoint union of translation surface in hyperelliptic components of strata. Let (Z_1, ζ_1) be any component of (Y_1, η_1) that contains two adjacent cylinders in $\mathcal{C}_0$. This component necessarily contains a cylinder from $\mathcal{C}_0$ and $\mathcal{C}_1$ and hence its twist space is at least two-dimensional. However, since two cylinders from $\mathcal{C}_0$ remain adjacent, Theorem 1.6.2 implies that the orbit closure cannot be three complex-dimensional. Therefore, the orbit closure of (Z_1, ζ_1) is higher rank. By the induction hypothesis, it follows that there is a cylinder V' that passes through the saddle connection connecting the two cylinders in $\mathcal{C}_0$. However, by Lemma 1.10.4 this contradicts the claim that $\mathcal{C}$ is self-adjacent and we have a contradiction.

We have shown that if $\mathcal{C}$ is an equivalence class of cylinders that does not coincide with $\mathcal{C}_0$ or $\mathcal{C}_1$ and that is self-adjacent, then the conclusion of the lemma holds. Therefore, it suffices to consider the case where the only two equivalence classes of horizontal cylinders that are possibly self-adjacent are $\mathcal{C}_0$ and $\mathcal{C}_1$. The arguments that we are about to make are very similar to the preceding arguments and can be skipped by readers who are only looking for the thrust of the argument.

Let $\mathcal{C}$ be any other equivalence class of cylinders. By assumption, $\mathcal{C}$ is not self-adjacent. Collapse it and let (Y_2, η_2) be the resulting translation surface. By the horizontal collapse lemma (Lemma 1.8.3), the translation surface (Y_2, η_2) is a disjoint union of translation surfaces in hyperelliptic components of strata of abelian differential.

Suppose first that $\mathcal{C}_0$ is self-adjacent on (X, ω) and let s be any saddle connection on the boundary of two cylinders in $\mathcal{C}_0$. Let (Z_2, ζ_2) be the component of (Y_2, η_2) that contains the saddle connection s. Let $\mathcal{N}_2$ be the affine invariant submanifold in the boundary of $\mathcal{M}$ that contains (Z_2, ζ_2). Since (Z_2, ζ_2) contains a horizontal cylinder from $\mathcal{C}_0$ and a horizontal cylinder from $\mathcal{C}_1$, the twist space is at least two dimensional. The dimension of $\mathcal{N}_2$ cannot be three since then Theorem 1.6.2 would imply that no two cylinders in $\mathcal{C}_0$ could be adjacent. Therefore, $\mathcal{N}_2$ is higher rank and so the induction hypothesis implies that there is a cylinder V that is contained in $\mathcal{C}_0$, contains s, and intersects s exactly once. Lemma 1.10.4 implies that $\mathcal{C}_1$ cannot be self-adjacent on (X, ω) and so $\mathcal{C}_0$ is the unique self-adjacent equivalence class.

The final case to consider is the case where only $\mathcal{C}_1$ is self-adjacent. Let s be a saddle connection on the boundary of two cylinders in $\mathcal{C}_1$. Shrink the cylinders in $\mathcal{W}$ so that no cylinder has length longer than the saddle connection s. Collapse $\mathcal{C}_0$ to form a translation surface (Y_3, η_3). Since we have supposed that $\mathcal{C}_0$ is not self-adjacent, the horizontal collapse lemma (Lemma 1.8.2) implies that (Y_3, η_3) is a disjoint union of translation surfaces in hyperelliptic components of strata. Let (Z_3, ζ_3) be the component of (Y_3, η_3) that contains s. By the induction hypothesis there is a cylinder V that is entirely contained in $\mathcal{C}_1$, that contains s, and that passes through s exactly once. Moreover, since all the horizontal saddle connections in $\mathcal{W}$ are smaller than s, it follows that V passes through no horizontal saddle connection contained in $\mathcal{W}$. In particular, V persists as a cylinder on (X, ω) as desired. $\square$

It remains for us to show that equivalent horizontal cylinders have identical heights on (X,ω). The strategy for establishing this claim is revealed in the following result.

Lemma 1.10.6. *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two adjacent equivalence classes of horizontal cylinders on (X,ω) and suppose that at least one of them is not self-adjacent. If all cylinders in $\mathcal{C}_1$ have identical heights, then all cylinders in $\mathcal{C}_2$ do as well.*

Proof. If $\mathcal{C}_1$ is self-adjacent then let C_2 be the tallest cylinder in $\mathcal{C}_2$ that is adjacent to a cylinder in $\mathcal{C}_1$. Otherwise, let C_2 be the smallest cylinder in $\mathcal{C}_2$ that is adjacent to a cylinder in $\mathcal{C}_1$. Let h_2 be the height of C_2. Since all the cylinders in $\mathcal{C}_1$ have identical height by assumption, call that height h_1. Let C_1 be a cylinder in $\mathcal{C}_1$ that borders C_2.

By the standard position lemma, assume that C_1 and C_2 are in standard position and that V is the vertical cylinder that passes through them. Let V' be any other vertical cylinder equivalent to V. Let n_i be the number of times that V' intersects a cylinder in $\mathcal{C}_i$. If $\mathcal{C}_1$ is self-adjacent then every time that V' enters a cylinder in $\mathcal{C}_2$ it must exit the cylinder into a cylinder in $\mathcal{C}_1$. This implies that $n_2 \leq n_1$. If $\mathcal{C}_1$ is not self-adjacent then the same reasoning implies that $n_1 \leq n_2$. Let P' be the proportion of the area of V' contained in $\mathcal{C}_2$.

If $\mathcal{C}_1$ is self-adjacent, then C_2 was assumed to be the tallest cylinder in $\mathcal{C}_2$ bordering a cylinder in $\mathcal{C}_1$ and so

$$P \leq \frac{n_1 h_2}{n_1 h_1 + n_1 h_2} = \frac{h_2}{h_1 + h_2}$$

Otherwise, C_2 was assumed to be the smallest cylinder in $\mathcal{C}_2$ bordering a cylinder in $\mathcal{C}_1$ and so

$$P \geq \frac{n_1 h_2}{n_1 h_1 + n_1 h_2} = \frac{h_2}{h_1 + h_2}$$

By the cylinder proportion theorem, the percent of time that V' spends in $\mathcal{C}_2$ is identical to the percent of time that V spends in $\mathcal{C}_2$, which is $\frac{h_2}{h_1+h_2}$. Therefore, $n_1 = n_2$ and the height of every cylinder in $\mathcal{C}_2$ that V' passes through is h_2. By the cylinder proportion theorem, for every cylinder in $\mathcal{C}_2$ there is a cylinder equivalent to V that passes through it and so all cylinders in $\mathcal{C}_2$ have height h_2 as desired. □

We will use the lemma predominantly in the following rephrased form:

Lemma 1.10.7 (Identical Heights Lemma)**.** *If (X, ω) has exactly one self-adjacent equivalence class of horizontal cylinders, then if any equivalence* *class of horizontal cylinders has the property that all cylinders in it have identical heights, then all equivalence classes of horizontal cylinders have this property.*

We will use the identical heights lemma to establish the first claim of the main theorem of the section (Theorem 1.10.1).

Lemma 1.10.8. *Any two equivalent horizontal cylinders on (X, ω) have identical heights.*

Proof. By the identical heights lemma (Lemma 1.10.7) it suffices to show that any two cylinders in $\mathcal{C}_1$ have identical heights. Let $\mathcal{C}$ be any equivalence class of cylinders apart from $\mathcal{C}_0$ and $\mathcal{C}_1$. If $\mathcal{C}$ is not self-adjacent then let (Y_1, η_1) be the translation surface formed by collapsing $\mathcal{C}$. If $\mathcal{C}$ is self-adjacent, then it contains an equivalence class $\mathcal{V}$ of transverse cylinders and let (Y_1, η_1) be the translation surface formed by collapsing $\mathcal{V}$.

In the case where $\mathcal{C}$ is self-adjacent, the vertical collapse lemma (Lemma 1.8.2) implies that (Y_1, η_1) is a disjoint union of translation surfaces in hyperelliptic components of strata. Any component (Z_1, η_1) of (Y_1, η_1) that contains a cylinder from $\mathcal{C}_1$ must also contain a cylinder from $\mathcal{C}_0$ and hence have twist space of dimension at least two. Let $\mathcal{N}_1$ be the orbit closure of (Z_1, ζ_1). If $\mathcal{N}_1$

is higher rank, then the induction hypothesis implies that any two cylinders in $\mathcal{C}_1$ that persist on (Z_1, ζ_1) have identical heights. Otherwise, $\mathcal{N}_1$ is three dimensional and Theorem 1.6.2 implies that any two cylinders in $\mathcal{C}_1$ that persist on (Z_1, ζ_1) have identical heights.

In the case where $\mathcal{C}$ is not self-adjacent, the horizontal collapse lemma (Lemma 1.8.3) implies that (Y_1, η_1) is a disjoint union of translation surfaces in hyperelliptic components of strata. Any component (Z_1, ζ_1) of (Y_1, η_1) that contains a cylinder from $\mathcal{C}_1$ contains a cylinder from $\mathcal{C}_0$. The preceding argument implies that any cylinder in $\mathcal{C}_1$ that persist on the same component of (Y_1, η_1) have identical heights.

Let us rephrase this observation in the language of Lindsey trees. Let Γ be the Lindsey tree of (X, ω). Let T be any connected subtree of Γ that only consists of vertices corresponding to cylinders in $\mathcal{C}_0$ or $\mathcal{C}_1$. We have shown that any two cylinders in $\mathcal{C}_1$ that belong to T have identical heights. Suppose that T_1 and T_2 are connected subtrees of Γ that only consist of vertices in $\mathcal{C}_0$ and $\mathcal{C}_1$. Suppose too that T_1 and T_2 are maximal in the sense that they cannot be enlarged by adding another vertex in $\mathcal{C}_0$ or $\mathcal{C}_1$. We will say that T_1 and T_2 are adjacent if there is a path between them that does not pass though a vertex in $\mathcal{C}_0$ or $\mathcal{C}_1$. It suffices to show that if T_1 and T_2 are adjacent then all cylinders in $\mathcal{C}_1$ in T_1 and T_2 have identical heights. For any two such trees, there are cylinders in $\mathcal{C}_1$ - say C_1 on T_1 and C_2 on T_2 - that belong to the same component of (Y, η) - the translation surface that we formed by collapsing the cylinders $\mathcal{W}$ at the beginning of the section.

Notice that we already have shown that if two cylinders from $\mathcal{C}_1$ belong to the same component of (Y, η) then they have identical heights. This follows because every component of (Y, η) that has a cylinder from $\mathcal{C}_1$ is a translation surface with twist space and cylinder preserving space coinciding and with an orbit closure of dimension at least $2r - 3$. If the orbit closure is exactly three dimensional, then Theorem 1.6.2 implies that any two cylinders from $\mathcal{C}_1$ appearing together on the component have identical heights. Otherwise, the orbit closure of the component is higher rank and the any two cylinders from $\mathcal{C}_1$ appearing on the component have identical heights by the induction hypothesis. $\square$

Now we have almost completed the proof of the main theorem of the section (Theorem 1.10.1), but as discussed earlier there is one missing ingredient. So far we have shown that on (X, ω) there is at most one self-adjacent equivalence class. We must show that there is exactly one.

Lemma 1.10.9. (X, ω) *contains a self-adjacent $\mathcal{M}$-equivalence class of horizontal cylinders.*

Proof. Suppose not to a contradiction. Since, a fortiori, no cylinder can be adjacent to itself the Lindsey tree Γ of (X, ω) has no half-edges and hence is a tree (and not just a half-tree). By Theorem 1.6.3, if $V(\Gamma)$ is the collection of horizontal cylinders on (X, ω) then

$$\eta = \sum_{c \in V(\Gamma)} (-1)^{d(c,c_0)} \gamma_c^*$$

is a relative deformation where c_0 is a fixed horizontal cylinder and $d(c, c_0)$ is the distance between cylinders c and c_0 in the Lindsey tree. Since $\mathcal{M}$ is an even dimensional affine invariant submanifold in $\mathcal{H}^{hyp}(g-1, g-1)$ its tangent space contains no relative deformation. Therefore, we can produce a contradiction if we can show that η belongs to the tangent space of $\mathcal{M}$ at (X, ω).

First we formulate the problem as a graph theory problem using trees. In the Lindsey tree the vertices correspond to horizontal cylinders. Since cylinders divide into equivalence classes, if there are m equivalence classes of horizontal cylinders we imagine that the Lindsey tree is colored using colors

$\{1, \ldots, m\}$ so that a vertex is colored by the equivalence class it belongs to. The assumption that no equivalent cylinders are adjacent means that no two vertices of the same color are adjacent. We will find one more constraint on the Lindsey tree using the cylinder proportion theorem. By the standard position lemma, given two adjacent cylinders C and D we can shear their equivalence classes to put them in standard position and in particular find a cylinder V that intersects C and D and that is contained in $C \cup D$. Let $\mathcal{C}$ be the equivalence class containing C and $\mathcal{D}$ the one containing D. By the cylinder proportion theorem, if V' is a cylinder equivalent to V then V' must intersect at least one cylinder in $\mathcal{C}$ and at least one cylinder in $\mathcal{D}$ and V' must be contained in the union of cylinders in $\mathcal{C}$ and $\mathcal{D}$. Since equivalent cylinders are not adjacent, if V' is equivalent to V it must alternate between passing through cylinders in $\mathcal{C}$ and cylinders in $\mathcal{D}$. The cylinder proportion theorem guarantees that for any cylinder in $\mathcal{C}$ or $\mathcal{D}$ there is a cylinder V' equivalent to V that intersects the cylinder. In particular, this means that every cylinder in $\mathcal{C}$ is adjacent to one in $\mathcal{D}$ and vice versa.

Sublemma 1.10.1. *Let Γ be a finite tree whose vertices are colored with colors $\{1, \ldots, m\}$. Suppose that*

1. *(No self-adjacency) No two vertices of the same color are adjacent.*

2. (Cylinder Proportion Theorem) If v and w are vertices of the same color and v borders a vertex of color c then w does as well.

Then vertices of the same color are an even distance apart in Γ.

Proof. Induct on the number of colors m. For $m = 2$ the result is clear. Suppose now that $m > 2$. Let a be a leaf of Γ connected to vertex b and suppose that a and b have colors 1 and 2 respectively. Since a is a leaf, the only vertex it borders is b. The cylinder proportion hypothesis then implies that any vertex of color 1 can only border a vertex of color 2.

Let v and w be two vertices of the same color and let $[v, w]$ be the geodesic between them in Γ. Let Γ' be the colored tree that results from collapsing each maximal connected subtree containing only vertices of color 1 or 2 to a point of color 0. We see that the two properties we assumed about Γ - no self-adjacency and the cylinder proportion theorem - descend to Γ'. Let $[v]$ and $[w]$ be the images of v and w in Γ'. By the inductive hypothesis, $d([v], [w])$ is even. It suffices to show that $d(v, w)$ is even.

First suppose that v and w are not color 1. Let Z be the collection of points of color 0 on $[[v], [w]]$ and for each $z \in Z$ let T_z be the maximal connected subtree of vertices of color 1 or 2 in Γ corresponding to z. As discussed above, if v and w are not color 1 then whenever the geodesic $[v, w]$

enters the tree T_z it does so through a vertex of color 2. Similarly, if the geodesic exits the tree T_z it does so through a vertex of color 2. It follows that the length ℓ_z the geodesic $[v, w]$ travels in the tree T_z is even, since it is a path in T_z between two vertices of color 2. Since $d(v, w) = d([v], [w]) + \sum_{z \in Z} \ell_z$ is the sum of even numbers it is even.

Now suppose v and w are of color 1. There are unique vertices of color 2, call them v' and w', such that $d(v, w) = 2 + d(v', w')$. Since the distance between any two points of color 2 is even, it follows that the distance between v and w is even as well. □

Now we will proceed with the proof of Lemma 1.10.9. Since equivalent horizontal cylinders have identical heights in (X, ω) by assumption it follows that if $\mathcal{C}$ is an equivalence class then, up to scaling, the standard shear is $\sigma_{\mathcal{C}} = \sum_{c \in \mathcal{C}} \gamma_c^*$. Fix a cylinder c_0. By the sublemma, the collection of cylinders that are an even distance away from c_0 is a union of equivalence classes $\mathcal{C}_0, \dots, \mathcal{C}_k$ and the collection of cylinders that are an odd distance away from c_0 is also a union of equivalence classes $\mathcal{C}_{k+1}, \dots, \mathcal{C}_m$. It follows that the relative deformation η can be written as follows:

$$\eta = \sum_{i=1}^{k} \sigma_{\mathcal{C}_i} - \sum_{i=k+1}^{m} \sigma_{\mathcal{C}_i}$$

Since each standard shear belongs to the tangent space of $\mathcal{M}$ at (X, ω) and since η is a linear combination of them, it follows that the tangent space also contains the relative deformation η. This is a contradiction. $\square$

1.11 The Isogenous Cylinder Lemma

In Section 1.9 we showed that if $\mathcal{M}$ is an affine invariant submanifold in a hyperelliptic component and $\mathcal{M}$ is four complex-dimensional then $\mathcal{M}$ is a branched covering construction of $\mathcal{H}(2)$. The proof revolved around showing that equivalent cylinders were isogenous. To show that equivalent cylinders were isogenous we needed three results about translation surfaces (X, ω) with twist space and cylinder preserving space coinciding.

1. Equivalent horizontal cylinders have identical heights.
2. Equivalent saddle connections have identical lengths.
3. Equivalent saddle connections alternate around the boundary of horizontal cylinders.

We will now prove these claims for a general affine invariant submanifold $\mathcal{M}$. Throughout this section we will suppose that $\mathcal{M}$ is a higher rank affine invariant submanifold in a hyperelliptic component of a stratum and that $\mathcal{M}$ has complex-dimension at least 5. Let (X, ω) be a translation surface in $\mathcal{M}$ that is horizontally periodic and for which the twist space and cylinder preserving space coincide. Suppose furthermore that any higher rank affine invariant submanifold in a hyperelliptic component that has dimension strictly smaller than $\mathcal{M}$ is a branched covering construction. The main result of this section will be that equivalent horizontal cylinders on (X, ω) are isogenous.

If S is a collection of equivalent saddle connections on (X, ω) then let $S_t \cdot (X, \omega)$ be the translation surface where all saddle connections in S have been dilated by t.

Lemma 1.11.1. *If $\mathcal{M}$ is even complex-dimensional then (X, ω) has exactly one equivalence class of horizontal cylinders, say $\mathcal{C}$, that is self-adjacent. If S is the collection of horizontal saddle connections connecting two cylinders in $\mathcal{C}$ then all saddle connections in S have identical lengths and $S_t \cdot (X, \omega)$ belongs to $\mathcal{M}$ for all $t > 0$.*

Proof. By Lemma 1.10.9 there is at least one self-adjacent $\mathcal{M}$-equivalence class $\mathcal{C}$ of horizontal cylinders on (X, ω). Let S be the collection of horizontal saddle connections that connect two cylinders in $\mathcal{C}$. Suppose that s is the longest saddle connection in S and suppose that it lies on the boundary of cylinders A and B in $\mathcal{C}$. By Theorem 1.10.1 there is a cylinder V that is contained in $\mathcal{C}$, passes through s exactly once, and contains s in its interior. Let $\mathcal{V}$ be the equivalence class of cylinders that contains V. Without loss of generality, after perhaps, shearing the surface we may suppose that the cylinders in $\mathcal{V}$ are vertical. Every cylinder in $\mathcal{V}$ has the same height as V and only passes through horizontal saddle connections that lie on the boundary of two cylinders in $\mathcal{C}$. Since s is the longest saddle connection in S, it follows that every cylinder in $\mathcal{V}$ passes exclusively through horizontal saddle connections of length s. Let $\mathcal{S}$ be the collection of horizontal saddle connections through which a cylinder in $\mathcal{V}$ passes. Notice that the standard shear on $\mathcal{V}$ is equivalent to $\mathcal{S}_t \cdot (X, \omega)$ for some real number t

By the vertical collapse lemma (Lemma 1.8.2), collapsing $\mathcal{V}$ results in a translation surface (Y, η) that is a disjoint union of translation surfaces in hyperelliptic components of strata. Since collapsing $\mathcal{V}$ is a path in $\mathcal{M}$, the surface (Y, η) belongs to the Mirzakhani-Wright partial compactification of $\mathcal{M}$.

Sublemma 1.11.1. *A cylinder from every $\mathcal{M}$-equivalence class of horizontal cylinders persists on each component of (Y, η).*

Proof. Let $\mathcal{C}'$ be an $\mathcal{M}$-equivalence class of horizontal cylinders on (X, ω). We wish to show that a cylinder from $\mathcal{C}'$ persists on every component of

(Y, η). Assume without loss of generality that we have applied $\mathcal{S}_t$ so that every saddle connection in $\mathcal{S}$ is shorter than every other horizontal saddle connection on (X, ω). Proceed by induction on the distance d from $\mathcal{C}$ to $\mathcal{C}'$ in the Lindsey tree. The base case, $d = 0$, is immediate since a cylinder from $\mathcal{C}$ necessarily appears on each component of (Y, η).

Now suppose that $\mathcal{C}'$ is distance $d > 0$ away from $\mathcal{C}$ in the Lindsey tree and that a cylinder from any equivalence class that is distance less than d away from $\mathcal{C}$ appears on each component of (Y, η). Let C be a cylinder in $\mathcal{C}'$ that is adjacent to a cylinder D that belongs to an equivalence class that is distance $d - 1$ away from $\mathcal{C}$ in the Lindsey tree. The standard position lemma implies that we may apply a standard shear to $\mathcal{C}'$ to put C and D in transverse standard position. In particular, there is a cylinder W that is contained in $C \cup D$ and intersects the saddle connection s' on their boundary exactly once. The horizontal distance across W is the length of s' and every cylinder equivalent to W is the same horizontal distance across. In particular, this means that W cannot intersect any horizontal saddle connection in $\mathcal{S}$ since all of these are strictly shorter than s'. Let $\mathcal{W}$ be the collection of cylinders equivalent to W. Since no cylinder in $\mathcal{W}$ passes through a saddle connection in $\mathcal{S}$, all of the cylinders in $\mathcal{W}$ persist on (Y, η). By the cylinder proportion theorem, every cylinder in $\mathcal{C}'$ and every cylinder equivalent to D is intersected by a cylinder in $\mathcal{W}$. Since there is a cylinder equivalent to D on every component of (Y, η) by the induction hypothesis, it follows that there is a cylinder equivalent to $\mathcal{W}$ on every component of (Y, η). Therefore, there is a cylinder equivalent to $\mathcal{C}'$ on every component of (Y, η) as desired. $\square$

Let (Z, ζ) be any component of (Y, η) and let $\mathcal{N}$ be the orbit closure of (Z, ζ). By assumption a cylinder from every equivalence class of horizontal cylinders persists on (Z, ζ) and the twist space of (Z, ζ) contains the image of the standard shears of the horizontal equivalence classes of cylinders on (X, ω) by the degeneration theorem of Mirzakhani-Wright [**MW17**] (Theorem 1.7.2 in this work, Theorem 2.7 in [**MW17**]). In particular, this implies that the twist space on (Z, ζ) is at least r dimensional. The dimension of the twist space of a horizontally periodic translation surface in an affine invariant submanifold $\mathcal{N}$ is at most dimension $\text{rank}(\mathcal{N}) + \text{rel}(\mathcal{N})$. Since $\mathcal{N}$ belongs to a hyperelliptic component, $\text{rel}(\mathcal{N}) \leq 1$. By the degeneration theorem of Mirzakhani-Wright, $\text{rank}(\mathcal{N}) < r$. These observations imply that the twist space is exactly r dimensional and that $\text{rank}(\mathcal{N}) = r - 1$ and $\text{rel}(\mathcal{N}) = 1$.

Since the twist space has maximal dimension, the twist space and cylinder preserving space on (Z, ζ) coincide. By the twist space degeneration lemma

(Lemma 1.8.5 part 3), if $\mathcal{N}$ is rank one then $\mathcal{M}$ equivalent horizontal cylinders are not adjacent on (Z, ζ). If $\mathcal{N}$ is higher rank, then two horizontal cylinders that persist on (Z, ζ) are $\mathcal{N}$-equivalent if and only if they were $\mathcal{M}$-equivalent on (X, ω). Theorem 1.6.1 then implies that when $\mathcal{N}$ is higher rank, $\mathcal{M}$-equivalent cylinders are not adjacent on (Z, ζ). Since these conclusions hold for all components of (Y, η) it follows that once $\mathcal{V}$ is collapsed that no $\mathcal{M}$-equivalent cylinders remain adjacent. □

One of the main ingredients in showing that rank two rel zero affine invariant submanifolds were branched covers of $\mathcal{H}(2)$ was to show that on horizontally periodic translation surfaces with twist space and cylinder preserving space coinciding, equivalent saddle connections had identical lengths. This was established in Lemma 1.9.3. We will now establish the same result in the present more general setting.

Lemma 1.11.2 (Saddle Connection Dilation Lemma)**.** *Let S be an equivalence class of horizontal saddle connections on (X, ω), then every element of S has the same length and $S_t \cdot (X, \omega) \in \mathcal{M}$ for all t.*

Proof. Let $\mathcal{C}_0$ and $\mathcal{C}_1$ be two distinct adjacent $\mathcal{M}$-equivalence classes of horizontal cylinders on (X, ω). Let S be the equivalence class of horizontal saddle connections connecting them. Let $s \in S$ have length ℓ and be the longest element of S. By Lemma 1.11.1 suppose without loss of generality that any saddle connection connecting two $\mathcal{M}$-equivalent cylinders has length strictly smaller than ℓ. Suppose that s lies on the boundary of $C_0 \in \mathcal{C}_0$ and $C_1 \in \mathcal{C}_1$. By the standard position lemma suppose without loss of generality that C_0 and C_1 are in standard position and let V be the resulting cylinder. Let $\mathcal{V}$ be the $\mathcal{M}$-equivalence class of vertical cylinders that contains V. Since every cylinder in $\mathcal{V}$ has width ℓ it follows that no cylinder in $\mathcal{V}$ passes through a horizontal saddle connection that connects two $\mathcal{M}$-equivalent cylinders.

Sublemma 1.11.2. *$\mathcal{V}$ contains every element of S.*

Proof. Suppose to a contradiction that C is a cylinder in $\mathcal{C}_0$ and D is an adjacent cylinder in $\mathcal{C}_1$ so that $\mathcal{V}$ does not contain the saddle connection connecting them. Collapse $\mathcal{V}$ and let (Y, η) be the resulting boundary translation surface. By the vertical collapse lemma (Y, η) is a disjoint union of connected translation surfaces in hyperelliptic components of strata. Let (Y', η') be the component of (Y, η) containing C and D. By assumption, (Y', η') contains a cylinder belonging to $\mathcal{C}_0$ and $\mathcal{C}_1$. Since collapsing $\mathcal{V}$ only alters saddle connections connecting $\mathcal{C}_0$ to $\mathcal{C}_1$ it follows from the cylinder proportion theorem that every equivalence class of horizontal cylinders persists on (Y', η'). By the twist space degeneration lemma (Lemma 1.8.5 part 4) (Y', η') cannot contain the image of two adjacent $\mathcal{M}$-equivalent cylinders and $\dim_{\mathbb{C}} \mathcal{M} = 2r$. By Theorem 1.10.1 (specifically, by the result shown in Lemma 1.10.9), (Y', η') must contain the image of two adjacent $\mathcal{M}$-equivalent cylinders, which is a contradiction.

□

By hypothesis, every cylinder in $\mathcal{V}$ has the same height. Since s was chosen to be the longest saddle connection in S and since its length is the height V, it follows that all saddle connections in S have identical lengths. Since every element of S is contained in $\mathcal{V}$ and since $\mathcal{V}$ only passes through saddle connections in S, the cylinder deformation theorem implies that dilating $\mathcal{V}$ horizontally by t for arbitrary t remains in $\mathcal{M}$. Now undoing the shears that put C_0 and C_1 in standard position remains in $\mathcal{M}$ by the cylinder deformation theorem and is the translation surface $S_t \cdot (X, \omega)$ as desired. □

To summarize our progress, we have shown that equivalent horizontal cylinders on (X, ω) have identical heights and that equivalent saddle connections have identical lengths. It remains for us to study the order in which saddle connections appear on the boundary of the horizontal cylinders (and ultimately to use that to show that equivalent cylinders are isogenous).

First we will need a combinatorial lemma. Before stating it consider the following - the saddle connections on the boundary of a horizontal cylinder are cyclically ordered - suppose that they are labelled $\{1, \dots, n\}$ from left to right. Each saddle connection belongs to some equivalence class of saddle connections and so we imagine it being colored by the equivalence class that it belongs to. Given two saddle connections i and j we will let (i, j) be the collection of saddle connections running left to right from i to j excluding saddle connections i and j. Let $C_{(i,j)}$ be the colors (i.e. equivalence classes) of saddle connections in (i, j) recorded with multiplicity.

Lemma 1.11.3. *Suppose that there is a horizontal cylinder C on (X, ω) with the following property: if $i \neq j$ and $j \neq k$ are saddle connections on the boundary of C so that*

1. *All three saddle connections belong to an equivalence class c*

2. *Neither $C_{(i,j)}$ nor $C_{(j,k)}$ contains the equivalence class c*

then $C_{(i,j)} = C_{(j,k)}$ as multi-sets. Then the cyclic order of equivalence classes on the boundary of C is periodic, i.e. after renaming the equivalence classes $\{1, \dots, m\}$ appearing on the boundary of C, the equivalence classes of saddle connections on the boundary of C appear in the cyclic order $(1, 2, \dots, m, 1, 2,$

$\ldots, m, \ldots)$.

Proof. Let $\{1, \ldots, m\}$ be the colors of saddle connections appearing on the boundary of C. If every color appears exactly once on the boundary of C then the result holds trivially. Suppose that this is not the case. Let i and j be any two distinct saddle connections of the same color (without loss of generality color 1) such that (i, j) has the fewest possible elements. Construct an order set of saddle connections of color 1 as follows. Let $s_1 = i$. Given s_a let s_{a+1} be the first saddle connection to the right of s_a that has color 1. Stop the process when $s_k = s_1$. It follows that every saddle connection either has color 1 or belongs to (s_a, s_{a+1}) for some $1 \leq a \leq k-1$. Since by assumption $C_{(s_a, s_{a+1})}$ is the same regardless of a, it follows that every color $\{2, \ldots, m\}$ appears in (s_a, s_{a+1}) for each a. Since (i, j) has the fewest number of elements conditional on i and j having the same color, it follows that every color must appear exactly once in $C_{(i,j)}$.

If (s_1, s_2) is empty, then $m = 1$ and the result trivially holds. Therefore, suppose that (s_1, s_2) is nonempty. Suppose after renaming the colors, that the order of colors in $[s_1, s_3]$ is $(1, \ldots, m, 1, c_2, \ldots, c_m, 1)$ where $(c_2, \ldots, c_m)$ is a permutation of $(2, \ldots, m)$. By assumption (s_1, s_2) has the fewest possible elements given that s_1 and s_2 have the same color. Therefore, $c_m = m$. This in turn implies that $c_{m-1} = m - 1$ and so we have that the order of the colors on $[s_1, s_3)$ is $(1, \ldots, m, 1, \ldots, m)$. Iterating this argument gives that the order of the color of the balls is periodic with period m. □

We now use this lemma to show that the cyclic order of equivalence classes of saddle connections appearing on the boundary of a horizontal cylinder C in (X, ω) is indeed periodic.

Lemma 1.11.4 (Periodic Ordering Lemma 1)**.** *Let C be a horizontal cylinder on (X, ω) and suppose that the equivalence classes of saddle connections on the boundary are labelled $\{1, \ldots, m\}$. The cyclic ordering of saddle connections on the boundary of C is (perhaps after relabelling) $(1, 2, \ldots, m, 1, 2, \ldots, m, \ldots)$.*

Proof. By Lemma 1.11.3 it suffices to show the following: suppose that i, j, k are saddle connections on the top boundary of C in equivalence class c. Suppose furthermore that (in the notation of Lemma 1.11.3) (i, j) and (j, k) do not contain saddle connections in equivalence class c. It suffices to show that $C_{(i,j)}$ and $C_{(k,\ell)}$ coincide as multi-sets.

Let a be any transcendental number. By the saddle connection dilation theorem we may suppose without loss of generality that each saddle connection in equivalence class k has length a^k. By using the cylinder deformation theorem we may shear C so that the saddle connection j lies above its image $J(j)$ under the hyperelliptic involution. Suppose moreover that j is contained in a vertical cylinder V that passes through it exactly once and only passes through saddle connections equivalent to j. Such a vertical cylinder exists by the standard position lemma in the case that j lies on the boundary of two inequivalent cylinders and by Theorem 1.10.1 otherwise.

Let $\mathcal{V}$ be the $\mathcal{M}$-equivalence class of cylinders containing V. By the saddle connection dilation lemma, every saddle connection equivalent to j is contained in a cylinder in $\mathcal{V}$ that intersects it exactly once. Moreover, all the cylinders in $\mathcal{V}$ have identical heights. Since the saddle connections of color c appear in the order (i, j, k) on the top boundary of C, their images under the hyperelliptic involution appear in the order $(J(k), J(j), J(i))$ on the bottom boundary, where J is the hyperelliptic involution. Since all saddle connections of color c are contained in cylinders in $\mathcal{V}$, it follows that i lies directly vertically above $J(k)$. In particular, this means that the sum of the lengths of the saddle connections in (i, j) coincides with the sum of the length of the saddle connections in $(J(k), J(i))$, which is equal to the sum of the lengths of the saddle connections in (j, k). Since we have assumed that $\mathcal{M}$-equivalence class k of horizontal saddle connections all have length a^k where a is transcendental, it follows that $C_{(i,j)} = C_{(j,k)}$. □

For the proof of the following lemma it will be useful to introduce the following definition. We will say that an equivalence class of cylinders $\mathcal{C}$ subsumes an equivalence class of saddle connections $\mathcal{S}$ on the boundary of two equivalence classes $\mathcal{C}_1$ and $\mathcal{C}_2$ if every saddle connection in $\mathcal{S}$ is contained in a cylinder in $\mathcal{C}$ that intersects it exactly once and if each cylinder in $\mathcal{C}$ is contained in collection of cylinders in $\mathcal{C}_1$ and $\mathcal{C}_2$. In the case that $\mathcal{C}_1$ and $\mathcal{C}_2$ are distinct, the standard position lemma implies that it is always possible to find an equivalence class of cylinders that subsumes $\mathcal{S}$. Otherwise, Theorem 1.10.1 implies that this is possible.

Lemma 1.11.5 (Periodic Ordering Lemma 2)**.** *If C and D are equivalent horizontal cylinders on (X, ω), then the boundaries of C and D contain the same $\mathcal{M}$-equivalence classes of horizontal saddle connections in the same cyclic order.*

Proof. Suppose that C belongs to the equivalence class $\mathcal{C}_0$. We begin by showing that the boundaries of C and D contain the same equivalence classes of saddle connections. If not, then the saddle connection dilation lemma implies that it is possible to alter the length of one cylinder in $\{C, D\}$ while fixing the other. However, by Wright **[Wri15a]** Lemma 4.7, the ratio of the lengths of core curves of equivalent cylinders is constant. This yields a contradiction and therefore, the same equivalence classes of cylinders appear on the boundaries of C and D.

Let $\{1, \ldots, m\}$ be the equivalence classes of cylinders appearing on the boundary of C and suppose that they occur in the cyclic order $(1, 2, \ldots, m, 1, 2, \ldots, m, \ldots)$. By the saddle connection dilation lemma we may suppose that each equivalence class of saddle connection appearing on the boundary of C has length 1. By the cylinder deformation theorem we may suppose that both C and D have height 1. We will suppose furthermore that if $\mathcal{C}_0$ is a self-adjacent equivalence class of cylinders, then the equivalence class of saddle connections joining $\mathcal{C}_0$ to itself is labelled 1. Let $\mathcal{T}_1$ be a vertical equivalence class of cylinders that subsumes all saddle connection in equivalence class 1. For every other equivalence class k of saddle connection on the boundary of C, there is a saddle connection that connects C to an inequivalent cylinder C_k in equivalence class $\mathcal{C}_k$. By the standard position lemma, it is possible to shear $\mathcal{C}_k$ to put $\mathcal{C}_0$ and $\mathcal{C}_k$ in transverse standard position. Let $\mathcal{T}_k$ be the resulting collection of cylinders, which pass through $\mathcal{C}_0$ and $\mathcal{C}_k$ and so every saddle connection in equivalence class k appears in a cylinder in $\mathcal{T}_k$ and is intersected exactly once by that cylinder. The slope of the core curve of cylinders in $\mathcal{T}_k$ is $\frac{1}{2k}$. By the cylinder proportion theorem D contains a cylinder in $\mathcal{T}_k$ for each k and that D is contained in the union of cylinders in some $\mathcal{T}_k$. The decreasing slopes force the periodic ordering on the boundary of D agrees with the periodic ordering on the boundary of C. □

Lemma 1.11.6 (Isogeny Lemma)**.** *All $\mathcal{M}$-equivalent horizontal cylinders on (X, ω) are isogeneous.*

Proof. Let $\mathcal{C}$ be an $\mathcal{M}$-equivalence class of horizontal cylinders. Let $C \in \mathcal{C}$ be a cylinder and let D be an adjacent $\mathcal{M}$-inequivalent cylinder. Put C and D into standard position by using the cylinder deformation theorem (suppose that $\mathcal{C}$ is sheared by $\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$) - and let V be the resulting vertical cylinder. Let $\mathcal{V}$ be the $\mathcal{M}$-equivalence class of cylinders containing V. Label the equivalence class containing the edge connecting C to D by 1. For any cylinder $v \in \mathcal{C}$ let $\{1, \ldots, m\}$ be the edge equivalence classes attached to v and suppose that they are ordered by $(1, \ldots, m, \ldots)$ around v. By assumption every cylinder in $\mathcal{C}$ has height h. By the saddle connection dilation lemma every saddle connection in equivalence class i has length ℓ_i and $\mathcal{V}$ contains every saddle connection in edge equivalence class 1. Therefore all cylinders in $\mathcal{C}$ are isogenous to $\begin{pmatrix} 1 & -t \\ 0 & 1 \end{pmatrix}$ applied to the cylinder depicted in Figure 1.11.1

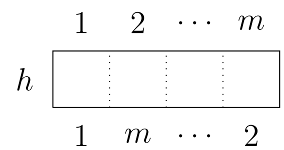


Figure 1.11.1: The cylinder to which all cylinders in $\mathcal{C}$ are isogenous

where labels describe how the top boundary glues to the bottom boundary and where a saddle connection labelled k has length ℓ_k. □

1.12 Higher Rank Affine Invariant Submanifolds are Branched Covering Constructions

The goal of this section is to show that all higher rank affine invariant submanifolds in hyperelliptic components of strata are branched covering constructions. We begin by proving an analogous statement at the level of trees. Suppose that Γ and Γ' are graphs. Define a degree d branched covering of finite graphs to be a simplicial map between graphs $\pi : \Gamma \longrightarrow \Gamma'$ such that

1. $|E_\Gamma| = d \cdot |E_{\Gamma'}|$ where E_Γ (resp. $E_{\Gamma'}$) is the edge set of Γ (resp. Γ').

2. For each vertex v in Γ the ramification index $e_v := \frac{\deg(v)}{\deg \pi(v)}$ is an integer.

3. For each vertex w in Γ', $\sum_{\pi(v)=w} e_v = d$.

Lemma 1.12.1. *If $f : \Gamma \longrightarrow \Gamma'$ is branched covering of finite graphs where Γ is a disjoint union of trees and Γ' is connected, then Γ' is a tree as well.*

Proof. Let χ_Γ denote the Euler characteristic of the graph Γ. We will first show a Riemann-Hurwitz type formula for branched covers of graphs.

$$\chi_\Gamma = |V_\Gamma| - |E_\Gamma| = d \cdot |V_{\Gamma'}| - \sum_{v \in V_\Gamma} (e_v - 1) - d|E_{\Gamma'}| = d \cdot \chi_{\Gamma'} - \sum_{v \in B} (e_v - 1)$$

By assumption χ_Γ is positive. The Riemann-Hurwitz type formula therefore implies that $\chi_{\Gamma'}$ is positive too and therefore must be equal to 1 since Γ' is connected. The connected graphs of Euler characteristic one are exactly trees. □

Theorem 1.12.2. *Suppose that $\mathcal{M}$ is a rank $r > 1$ affine invariant submanifold in a hyperelliptic component of a stratum of abelian differentials, then $\mathcal{M}$ is a branched covering construction of $\mathcal{H}^{hyp}(2r-2)$ if it is even complex-dimensional and of $\mathcal{H}^{hyp}(r-1, r-1)$ if it is odd complex dimensional. The* *branched covers are branched over zeros of the one-forms and commute with the hyperelliptic involution.*

Proof. We will proceed by induction on the dimension of $\mathcal{M}$. The base case has been established in Section 1.9. By Corollary 1.4.3 it suffices to produce an $\mathcal{M}$-generic horizontally periodic translation surface (X, ω) with $\text{Twist}_{(X,\omega)}\mathcal{M} = \text{Pres}_{(X,\omega)}\mathcal{M}$ and so that (X, ω) is a simple translation covering of a generic surface (Y, η) in $\mathcal{H}^{hyp}(2r-2)$ or $\mathcal{H}^{hyp}(r-1, r-1)$. Enumerate the $\mathcal{M}$-equivalence classes of horizontal cylinders $\{1, \ldots, m\}$ and the equivalence classes of edges $\{1, \ldots, n\}$. Choose two transcendental numbers a and b so that $\mathbb{Q}(a, b)$ is isomorphic as a field to $\mathbb{Q}(x, y)$ where x and y are indeterminates. Using the cylinder deformation theorem and the saddle connection dilation lemma, assume without loss of generality that the height of the cylinders in equivalence class k is a^k and the lengths of saddle connections in equivalence class k is b^k.

We will first build the Lindsey tree Γ_Y of (Y, η). For each $\mathcal{M}$-equivalence class in (X, ω) add a corresponding node in (Y, η). If two distinct equivalence classes are adjacent in (X, ω), then add an edge connecting the corresponding nodes in Γ_Y. If an equivalence class is self-adjacent in (X, ω) then add a half-edge to the corresponding node in Γ_Y. The cyclic order around each node is specified by the periodic ordering lemmas. This completely specifies Γ_Y.

We will now build (Y, η) and the simple translation covering $f : (X, \omega) \longrightarrow (Y, \eta)$. To each $\mathcal{M}$-equivalence class $\mathcal{C}_i$ of horizontal cylinders let C_i be the shortest cylinder isogenous to the cylinders in $\mathcal{C}_i$ (this will be the one constructed in Lemma 1.11.6). For each $c \in \mathcal{C}_i$ let $f_c : c \longrightarrow C_i$ be the local isometry constructed in Lemma 1.11.6. Gluing the cylinders C_i together according to Γ_Y now constructs a translation surface (Y, η). Let f be the map that sends a point $x \in c$ to $f_c(x)$. This map defines a holomorphic degree d covering map $f : X - Z(\omega) \longrightarrow Y - Z(\eta)$ so that f pulls η back to ω. By the Riemann extension theorem f extends to a holomorphic map $f : X \longrightarrow Y$ such that $f^*\eta = \omega$.

Let Γ' be Γ_Y with the half-edge deleted (should there be one). Let Γ be the Lindsey tree Γ_X of (X, ω) with edges connecting equivalent cylinders deleted. Since f is a holomorphic map between $X - Z(\omega)$ and $Y - Z(\eta)$ it induces a degree d branched covering of graphs between Γ and Γ'. Lemma 1.12.1 implies that Γ' is a tree and hence that Γ_Y is a half-tree.

By Corollary 1.3.3, (Y, η) is a translation surface in a hyperelliptic component of a stratum. Moreover, the translation covering $f : (X, \omega) \longrightarrow (Y, \eta)$ is a simple translation covering. Since Γ_Y is a half-tree on $\mathrm{rk}(\mathcal{M}) + \mathrm{rel}(\mathcal{M})$

vertices with an extra half-edge if and only if $\mathrm{rel}(\mathcal{M}) = 0$, it follows that (Y, η) belongs to $\mathcal{H}^{hyp}(2r-2)$ when $\mathrm{rel}(\mathcal{M}) = 0$ and to $\mathcal{H}^{hyp}(r-1, r-1)$ when $\mathrm{rel}(\mathcal{M}) = 1$. Moreover the map f is branched over zeros and commutes with the hyperelliptic involution by construction.

It remains to show that (X, ω) is generic. Let $\mathcal{N}$ be the orbit closure of (X, ω). Notice that by construction two horizontal cylinders on (X, ω) have a ratio of lengths of core curves that is algebraic if and only if they are $\mathcal{M}$-equivalent. By Wright [**Wri15a**] Lemma 4.7, if two cylinders are $\mathcal{N}$-equivalent then they have an algebraic ratio of lengths of their core curves. Therefore, there are $\mathrm{rank}(\mathcal{M}) + \mathrm{rel}(\mathcal{M})$ $\mathcal{N}$-equivalence classes of horizontal cylinders on (X, ω). Since $\mathcal{N}$ is contained in $\mathcal{M}$ its rank is bounded above by the rank of $\mathcal{M}$ and rel is bounded above by the rel of $\mathcal{M}$. Moreover, the dimension of the twist space of (X, ω) in $\mathcal{N}$ is bounded above by $\mathrm{rank}(\mathcal{N}) + \mathrm{rel}(\mathcal{N})$.Combining these inequalities yields

$$\mathrm{rank}(\mathcal{N}) + \mathrm{rel}(\mathcal{N}) \leq \mathrm{rank}(\mathcal{M}) + \mathrm{rel}(\mathcal{M}) \leq \mathrm{rank}(\mathcal{N}) + \mathrm{rel}(\mathcal{N})$$

Therefore $\mathrm{rank}(\mathcal{N}) = \mathrm{rank}(\mathcal{M})$ and $\mathrm{rel}(\mathcal{N}) = \mathrm{rel}(\mathcal{M})$ and hence $\mathcal{M}$ and $\mathcal{N}$ coincide as affine invariant submanifolds. The exact same reasoning implies that (Y, η) is generic in the component of the stratum of abelian differentials to which it belongs. $\square$

Chapter 2

Rank One Orbit Closures in Hyperelliptic Components of Strata

This material contains the results of [**Api17b**].

In this chapter, it will be established that all $\mathrm{GL}(2,\mathbb{R})$ orbits in hyperelliptic components of strata of abelian differentials in genus greater than two are closed, dense, or contained in a locus of branched covers.

2.1 Introduction

Teichmüller geodesic flow induces a $\mathrm{GL}(2,\mathbb{R})$ action on the moduli space of holomorphic one-forms $\Omega\mathcal{M}_g$. The space $\Omega\mathcal{M}_g$ admits a $\mathrm{GL}(2,\mathbb{R})$-invariant stratification by specifying the number and order of zeros of the holomorphic one-forms. The components of these strata were classified by Kontsevich and Zorich [**KZ03**] and shown to be distinguished by hyperellipticity and spin parity. The closure of a $\mathrm{GL}(2,\mathbb{R})$ orbit is a linear submanifold by work of Eskin-Mirzakhani [**EM**] and Eskin-Mirzakhani-Mohammadi [**EMM15**]. In Apisa [**Apib**], it was shown that in hyperelliptic components of strata of $\Omega\mathcal{M}_g$ orbit closures of dimension greater than three are either loci of branched covers or the entire component of the stratum. In the sequel, we strengthen this theorem to show the following:

Theorem 2.1.1. *In hyperelliptic components of strata of abelian differentials in genus greater than two every* $\mathrm{GL}(2,\mathbb{R})$ *orbit is closed, dense, or contained in a locus of branched covers.*

Theorem 2.1.1 and Eskin-Filip-Wright [**EFW17**, Theorem 1.5] imply that in these components there are only finitely many closed orbits that are not contained in a locus of branched covers. All invariant measures and orbit closures in hyperelliptic components are known up to computing this finite collection of Teichmüller curves.

The proof uses the explicit geometry of abelian differentials in $\mathcal{H}^{hyp}(g-1, g-1)$ (developed in Lindsey [**Lin15**] and extended in Apisa [**Apib**]) and cylinder deformation results of Wright [**Wri15a**] to reduce the problem to the classification of orbit closures in genus two, which was achieved by McMullen in [**McM03**], [**McM05**], [**McM06**], and [**McM07**].

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2.2 Background

Throughout this section we make the following assumption

Assumption 2.2.1. *Let (X, ω) be a horizontally periodic translation surface in an affine invariant submanifold $\mathcal{M}$. Suppose that $(C_i)_{i=1}^n$ is the collection of horizontal cylinders and that C_i has height h_i, modulus m_i, and core curve* *oriented form left to right γ_i of length c_i for $i = 1, \dots, n$. If a cylinder C is specified without an index, then these quantities will be denoted h_C, m_C, γ_C, and c_C respectively. Let $W \subseteq \mathbb{Q}^n$ be the collection of rational homogeneous linear relations satisfied by $m = (m_i)_{i=1}^n$, i.e. a vector w belongs to W if and only if $w \cdot m = 0$.*

Lemma 2.2.2 (Wright [**Wri15a**], Corollary 3.4)**.** *If $v \in \mathbb{R}^n$ belongs to $W^\perp$ then*

$$\sum_{i=1}^{n} c_i v_i \gamma_i^* \in T_{(X,\omega)}\mathcal{M}$$

where γ_i^* is the cohomology class dual to γ_i.

Definition 2.2.3. *The twist space of a horizontally periodic translation (X, ω) in an affine invariant submanifold $\mathcal{M}$ is the complex linear span of all tangent vectors of $T_{(X,\omega)}\mathcal{M}$ of the form $\sum_{i=1}^n a_i \gamma_i^*$. The cylinder preserving space is the subspace of $T_{(X,\omega)}\mathcal{M}$ consisting of cohomology classes that evaluate to zero on the core curves of all horizontal cylinders. An element of this space is called a cylinder preserving deformation.*

The tangent space to $\mathcal{M}$ at (X, ω) is a subspace of the relative cohomology group $H^1(X, \Sigma; \mathbb{C})$ where Σ is the collection of zeros of ω. Let $p : H^1(X, \Sigma; \mathbb{C}) \longrightarrow H^1(X; \mathbb{C})$ be the projection from relative to absolute cohomology. The image of $T_{(X,\omega)}\mathcal{M}$ under p is always complex-symplectic by Avila-Eskin-Möller [**AEM**].

Lemma 2.2.4 (Wright [**Wri15a**], proof of Theorem 1.10)**.** *For any affine invariant submanifold $\mathcal{M}$ there is a horizontally periodic translation surface $(X, \omega) \in \mathcal{M}$ whose twist space and cylinder preserving space coincide. On such a surface, the twist space contains all of $\ker p \cap T_{(X,\omega)}\mathcal{M}$ and projects to a Lagrangian subspace of $p\left(T_{(X,\omega)}\mathcal{M}\right)$.*

Definition 2.2.5. *The rank of an affine invariant submanifold is defined to be half the complex dimension of $p\left(T_{(X,\omega)}\mathcal{M}\right)$ for any $(X, \omega) \in \mathcal{M}$. This definition is independent of the choice of (X, ω). The rel is defined to be the complex dimension of the intersection of $\ker p$ and $T_{(X,\omega)}\mathcal{M}$. An element of this intersection is called a rel deformation.*

.

2.3 Sub-equivalence Classes in $\mathcal{H}^{hyp}(g-1, g-1)$

Throughout this section Assumption 2.2.1 will remain in effect. Moreover, we will make the following assumption.

Assumption 2.3.1. *$\mathcal{M}$ is a rank one rel one nonarithmetic orbit closure in $\mathcal{H}^{hyp}(g-1, g-1)$ for some $g > 2$.*

Definition 2.3.2. *By Apisa* [**Apib**, Theorem 6.3]*, if (X, ω) is a horizontally periodic translation surface whose twist space contains a rel deformation, then the rel deformation is given (up to scaling) by*

$$\eta := i \cdot \sum_{j=1}^{n} q_j \gamma_j^* \qquad \textit{where} \qquad q_j = (-1)^{d(C_j, C_1)}$$

where $d(C_j, C_1)$ is the distance between C_j and C_1 in the cylinder graph - the graph whose vertices are horizontal cylinders and edge relations correspond to cylinder adjacencies. Notice that η is characterized by the property that one of the coefficients (and hence all coefficients) of $\{\gamma_j^*\}_{j=1}^n$ belongs to $\{\pm i\}$. Hence η is defined up to multiplication by ± 1. Since η is a tangent vector on a linear submanifold, we may move in the η direction, and this alters the heights of the horizontal cylinders. When a cylinder reaches height zero, it is said to collapse.

Lemma 2.3.3. *There is a horizontally periodic translation surface (X, ω) in $\mathcal{M}$ that contains η in its twist space. By moving an arbitrarily small distance in the η direction it is possible to ensure that the modulus of any horizontal* *cylinder that expands under η is not a rational multiple of the modulus of any horizontal cylinder that contracts under η.*

Proof. By Lemma 2.2.4, there is a translation surface (X, ω) whose twist space contains the rel deformation η. Moving in the η direction causes some horizontal cylinders to expand and others to contract. The collection of times in which one contracting and one expanding horizontal cylinder have a rational ratio of moduli is countable. Therefore, moving in the η direction for an arbitrarily small amount of time that lies outside of this countable set produces the desired surface. □

Lemma 2.3.4. *If (X, ω) contains η in its twist space and not all its horizontal cylinders have a rational ratio of moduli, then*

$$W^{\perp} = \operatorname{span}_{\mathbb{C}} \{m, (q_i/c_i)_{i=1}^{n}\}$$

and the vector space of rational homogeneous linear relations satisfied by $(1/c_i)_{i=1}^{n}$ is exactly $W \cdot \operatorname{diag}(q_i)$.

Proof. Recall that W is the vector space of rational homogeneous linear relations satisfied by the moduli m of the horizontal cylinders. By Lemma 2.2.2, each element of $W^{\perp}$ corresponds to an element of the twist space, which is at most two-dimensional since $\mathcal{M}$ is rank one rel one. Therefore, $W^\perp$ is no more than two dimensional. Since not all horizontal cylinders have a rational ratio of moduli, W is at least, and hence exactly, codimension two. Since $W^\perp$ contains m by definition and since η belongs to the twist space by assumption we see that $W^\perp$ is spanned by m and $(q_i/c_i)_{i=1}^n$.

By Assumption 2.3.1, $\mathcal{M}$ is nonarithmetic and so, by Wright [**Wri15a**, Theorem 1.9], the $\mathbb{Q}$-span of $\{1/c_i\}_{i=1}^n$ in $\mathbb{R}$ has dimension at least two. Therefore, the subspace U of rational homogeneous linear relations satisfied by $(1/c_i)_{i=1}^n$ is at least codimension two. Since $(1/c_i)_{i=1}^n$ satisfies all the rational linear relations in the codimension two subspace $W \cdot \mathrm{diag}(q_i)$ we see that U coincides with this space. □

Remark 2.3.5. *The second statement of Lemma 2.3.4 is one of the key observations of this work. It shows that the rational homogeneous linear relations that the moduli of horizontal cylinders satisfy can be recovered from the relations satisfied by the reciprocals of the lengths of core curves (and vice versa).*

Lemma 2.3.6. *It is not the case that all contracting horizontal cylinders in both the $+\eta$ and $-\eta$ directions collapse simultaneously.*

Proof. Suppose not; then all contracting (resp. expanding) horizontal cylinders have identical heights, i.e. there are two disjoint groups of horizontal cylinders, $\mathcal{A}$ and $\mathcal{B}$, so that $\sum_{a\in\mathcal{A}}\gamma_a^*$ and $\sum_{b\in\mathcal{B}}\gamma_b^*$ span the tangent space. This implies that all cylinders in a given group have a constant, and hence rational, ratio of moduli. Since cylinders in a given group have identical heights, any two such cylinders have a rational ratio of lengths of core curves. The linear relation on homology classes of core curves given by η now implies that all core curves have a rational ratio of lengths and hence, by Wright [**Wri15a**, Theorem 1.9], that $\mathcal{M}$ is a rank one arithmetic orbit closure, which contradicts Assumption 2.3.1. □

Definition 2.3.7. *Two cylinders on a translation surface in $\mathcal{M}$ will be said to be sub-equivalent if they are equivalent and they have a generically rational ratio of moduli. A maximal collection of sub-equivalent cylinders will be called a sub-equivalence class.*

Proposition 2.3.8. *If (X, ω) contains a rel deformation in its twist space, then there are three sub-equivalence classes and sub-equivalent horizontal cylinders have identical heights.*

Remark 2.3.9. *If Theorem 2.1.1 holds, then there is a map from (X, ω) to a horizontally periodic translation surface in $\mathcal{H}(1,1)$ with three horizontal cylinders. A sub-equivalence class in (X, ω) is then just a collection of* *horizontal cylinders that map to a given cylinder on the surface in* $\mathcal{H}(1,1)$*. Proposition 2.3.8 must hold if Theorem 2.1.1 does.*

Proof. If necessary, by Lemma 2.3.3, we move an arbitrarily small distance in the η direction to guarantee that no horizontal cylinder that expands in the η direction has a modulus that is a rational multiple of one that contracts (this allows us to apply Lemma 2.3.4). By Lemma 2.3.6, we may assume (perhaps after multiplying η by -1) that in the η direction, not all contracting horizontal cylinders simultaneously collapse. Let $\mathcal{A}_1$ be those that collapse first and let $\mathcal{A}_2$ be the remaining horizontal cylinders that contract in the η direction. Let $\mathcal{A}_3$ be the horizontal cylinders that expand in the η direction. Finally, suppose that the cylinders in $\mathcal{A}_2 \cup \mathcal{A}_3$ are the ones labelled $(C_1, \ldots, C_k)$.

Notice that a cylinder in $\mathcal{A}_i$ and a cylinder in $\mathcal{A}_j$ are sub-equivalent only if $i \neq j$. We will show conversely that $\mathcal{A}_i$ is a sub-equivalence class for each $i = 1, 2, 3$.

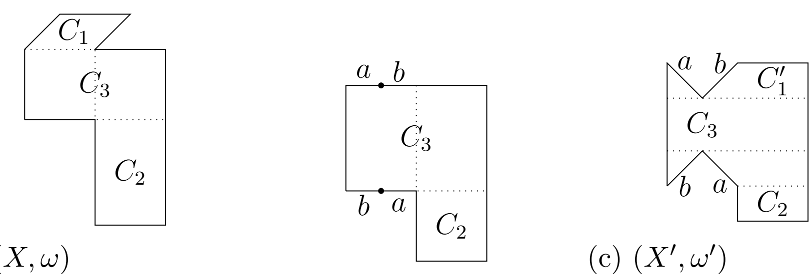
Suppose without loss of generality, perhaps after applying $\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ for arbitrarily small t, that when $\mathcal{A}_1$ collapses, no vertical saddle connection vanishes. Let (Y, ζ) be the translation surface formed when $\mathcal{A}_1$ collapses (by moving in the η direction). This surface is horizontally periodic and the horizontal cylinders are precisely the ones belonging to $\mathcal{A}_2 \cup \mathcal{A}_3$. Since $\mathcal{M}$ is nonarithmetic, Wright [**Wri15a**, Theorem 1.9] implies that not all elements of $(c_i)_{i=1}^k$ are rational multiples of each other. Let $U \subseteq \mathbb{Q}^k$ be the subset of rational homogeneous linear relations that $(1/c_i)_{i=1}^k$ satisfy. By Lemma 2.3.4 applied to (X, ω), the $\mathbb{Q}$-linear span of $\{1/c_i\}_{i=1}^n$ in $\mathbb{R}$ is two-dimensional, and so U is a codimension two subspace.

Notice that while η no longer lies in the twist space of (Y, ζ), it still corresponds to a tangent vector in $T_{(Y,\zeta)}\mathcal{M}$ and so we may continue to move in the η direction. Doing so creates a horizontally periodic translation surface (X', ω') that contains η in its twist space (notice that the surface is horizontally periodic because it contains a horizontal cylinder and any such surface in $\mathcal{M}$ is horizontally periodic by Wright [**Wri15a**, Theorem 1.5] since $\mathcal{M}$ is rank one).

On (X', ω') the η direction remains well-defined, although it becomes a different linear combination of the core curves of horizontal cylinders. Since the coefficients of η are purely imaginary on (X, ω), they remain purely imaginary on (X', ω') and so the η direction on (X', ω') agrees with the direction defined in Definition 2.3.2. As at the start of this proof we may apply

Lemma 2.3.3 to assume that no horizontal cylinder that expands in the η direction has a modulus that is a rational multiple of one that contracts (and so Lemma 2.3.4 holds on (X', ω')).

To visualize these definitions in the case of a surface in $\mathcal{H}(1,1)$ see Figure 2.3.1; in that figure, cylinder C_i belongs to $\mathcal{A}_i$ for $i = 1, 2, 3$, a and b are used to indicate side identifications (along with the convention that unlabelled opposite sides are identified). The η direction is written as a linear combination of core curves whenever possible with γ_1' denoting the core curve of the cylinder C_1'.



(a) (X, ω)
$-i\eta = \gamma_3^* - \gamma_1^* - \gamma_2^*$

(b) (Y, ζ)

(c) (X', ω')
$-i\eta = \gamma_1'^* - \gamma_3^* - \gamma_2^*$

Figure 2.3.1: An illustration of the preceding discussion

Recall that $(q_i)_{i=1}^k \in \{\pm 1\}_{i=1}^k$ is chosen so that $-i\eta$ restricted to the cylinder in $\mathcal{A}_2 \cup \mathcal{A}_3$ on (X, ω) is $\sum_{i=1}^k q_i \gamma_i^*$. Define $(q_i')_{i=1}^k$ analogously for (X', ω'). Let $V \subseteq \mathbb{Q}^k$ be the subspace of rational homogeneous linear equations that the moduli of cylinders in $\mathcal{A}_2 \cup \mathcal{A}_3$ satisfy on (X, ω) and let V' be the analogous subspace for (X', ω'). By Lemma 2.3.4,

$$V \cdot \operatorname{diag}(q_1, \ldots, q_k) = U = V' \cdot \operatorname{diag}(q_1', \ldots, q_k') \tag{2.3.1}$$

Define $D = \operatorname{diag}(q_1 q_1', \ldots, q_k q_k')$, which is a diagonal matrix with all diagonal entries belonging to $\{\pm 1\}$. Notice that a cylinder in $\mathcal{A}_2 \cup \mathcal{A}_3$ has D-eigenvalue $+1$ if it is contracted (or expanded) in the η direction on both (X, ω) and (X', ω'). If a cylinder is contracted in the η direction on one surface, but not the other then its D-eigenvalue is -1.

Step 1: The $+1$-eigenspace of D contains $\mathcal{A}_2$ and the -1-eigenspace of D is nonempty

By Definition 2.3.2, no two horizontal cylinders that contract in the η direction are adjacent. In particular, no cylinder in $\mathcal{A}_2$ is adjacent to a cylinder in $\mathcal{A}_1$. Since the cylinders in $\mathcal{A}_1$ are exactly those that collapse on (Y, ζ), it follows that if γ is a cross curve of a cylinder in $\mathcal{A}_2$ on (X, ω), then it remains one on (X', ω'). Since the period of γ is linear in the η direction, any cylinder in $\mathcal{A}_2$ belongs to the $+1$-eigenspace of D.

When a cylinder C_1 in $\mathcal{A}_1$ collapses, it contains a cross curve s - the imaginary part of whose period is decreasing in the η direction - that remains a saddle connection on (Y, ζ) on the boundary of a horizontal cylinder C_3. Since cylinders in $\mathcal{A}_1$ can only border cylinders in $\mathcal{A}_3$, C_3 belongs to $\mathcal{A}_3$. Since the period of s is linear in the η direction, moving in the η direction from (Y, ζ) causes the imaginary part of the period of s to continue to decrease and therefore to form a new cylinder C_1' adjacent to C_3 (see Figure 2.3.1). Since cylinders that expand in the η direction, like C_1', can only border cylinders that contract in the η direction, C_3 contracts in the η direction on (X', ω') and so it belongs to the -1-eigenspace of D.

Step 2: If two cylinders have the same D-eigenvalue, then they have a rational ratio of moduli

Suppose to a contradiction that two cylinders with D-eigenvalue $+1$ do not have a rational ratio of moduli and suppose without loss of generality that their moduli are m_1 and m_2. Since V is codimension two, all moduli of horizontal cylinders are a rational linear combination of m_1 and m_2. Suppose without loss of generality that C_3 belongs to the -1 eigenspace of D and that

$$m_3 = \alpha_1 m_1 + \alpha_2 m_2 \qquad \text{for } \alpha_1, \alpha_2 \in \mathbb{Q}$$

is an equation that holds in V. By Equation 2.3.1,

$$-m_3 = \alpha_1 m_1 + \alpha_2 m_2$$

is an equation that holds in V'. Since both equations must hold on (Y, ζ) it follows that $m_3 = 0$ on (Y, ζ), which contradicts the fact that C_3 is a positive-area cylinder there. The same argument holds for two cylinders of D-eigenvalue -1.

Step 3: Any two cylinders in $\mathcal{A}_i$, for $i = 1, 2, 3$, have identical heights and a rational ratio of moduli

Since cylinders in $\mathcal{A}_2$ belong to the $+1$-eigenspace of D and since any other such horizontal cylinder must have a modulus that is a rational multiple of those of cylinders in $\mathcal{A}_2$, it follows that the $+1$-eigenspace of D cannot contain any cylinder in $\mathcal{A}_3$. Therefore, the cylinders in $\mathcal{A}_i$, for $i = 2, 3$ have a rational ratio of moduli by the previous step. The same holds for the cylinders in $\mathcal{A}_1$ since they have identical heights and hence a generically constant, hence rational, ratio of moduli.

Since any two cylinders in $\mathcal{A}_1$ have identical heights and since any two cylinders in $\mathcal{A}_2$, or $\mathcal{A}_3$, have a rational ratio of moduli,

$$i\eta = \sum_{c\in\mathcal{A}_1\cup\mathcal{A}_2} \gamma_c^* - \sum_{c\in\mathcal{A}_3} \gamma_c^* = \sum_{a\in\mathcal{A}_1} \gamma_a^* + \alpha \sum_{b\in\mathcal{A}_2} h_b\gamma_b^* - \beta \sum_{c\in\mathcal{A}_3} h_c\gamma_c^*$$

where α and β are constants. Therefore, any two cylinders in $\mathcal{A}_i$ have identical heights for $i = 1, 2, 3$. □

Remark 2.3.10. *Proposition 2.3.8 immediately implies that the field of definition of $\mathcal{M}$ is quadratic. The proof is omitted since a stronger result will be established in the following section.*

2.4 Proof of Theorem 2.1.1

By Apisa [**Apib**], Theorem 2.1.1 holds for orbit closures of complex dimension greater than three. Since closed orbits are exactly those whose orbit closures are dimension two, it suffices to consider orbit closures of complex dimension three. Those that are arithmetic are loci of branched covers of tori. Therefore, throughout this section, Assumption 2.3.1 remains in effect. Using Proposition 2.3.8, we will make the following assumption.

Assumption 2.4.1. *Fix a horizontally periodic translation surface (X, ω) that belongs to $\mathcal{M}$ and that has the relative deformation η in its twist space.*

Let $\mathcal{A}_i$, for $i = 1, 2, 3$, denote the three sub-equivalence classes and suppose that a cylinder in $\mathcal{A}_i$ has height h_i. Suppose that the relative deformation is

$$\sum_{c \in \mathcal{A}_1 \cup \mathcal{A}_2} \gamma_c^* - \sum_{c \in \mathcal{A}_3} \gamma_c^*$$

We will say, abusing notation, that a horizontal saddle connection belongs to $\mathcal{A}_i$ for $i = 1, 2$ if it borders a cylinder in $\mathcal{A}_i$. Notice that every saddle connection belongs to exactly one of $\mathcal{A}_1$ or $\mathcal{A}_2$.

We will repeatedly use the following construction in the sequel.

Definition 2.4.2. *A horizontal saddle connection s on (X, ω) borders two cylinders, say C_1 and C_2. We will say that s may be put in standard position if a cylinder preserving deformation may be applied to (X, ω) so that there is a vertical cylinder V that is contained in $C_1 \cup C_2$, that only intersects the core curves of C_1 and C_2 once, and that contains s as a cross curve, i.e. s is contained in V and intersects the core curve of V exactly once.*

Lemma 2.4.3. *Any horizontal saddle connection on (X, ω) may be put in standard position.*

Proof. Suppose that C_1 and C_2 are the two horizontal cylinders bordering a horizontal saddle connection s and suppose furthermore that C_1 belongs to $\mathcal{A}_1 \cup \mathcal{A}_2$ and that C_2 belongs to $\mathcal{A}_3$. Shear all horizontal cylinders so that s and its image under the hyperelliptic involution lie directly above one another in C_2. Notice that the tangent space contains

$$\sum_{c \in \mathcal{A}_1 \cup \mathcal{A}_2} \gamma_c^* - \sum_{c \in \mathcal{A}_3} \gamma_c^* + \sum_{i=1}^{3} \sum_{c \in \mathcal{A}_i} \frac{h_i}{h_3} \gamma_c^* = \sum_{c \in \mathcal{A}_1 \cup \mathcal{A}_2} \left(1 + \frac{h_c}{h_3}\right) \gamma_c^*$$

This deformation fixes all cylinders in $\mathcal{A}_3$ while shearing those in $\mathcal{A}_1 \cup \mathcal{A}_2$. This allows s and its image under the hyperelliptic involution to be put directly above each other in C_1, putting s in standard position. □

Proposition 2.4.4. *For $i = 1, 2$, all saddle connections in $\mathcal{A}_i$ have identical lengths and if any one is put in standard position, all occur as cross curves of vertical cylinders that only pass through saddle connections in $\mathcal{A}_i$.*

Remark 2.4.5. *This proposition is the main result of this section. Theorem 2.1.1 follows directly from it via the arguments in Apisa* [**Apib**, Sections 11 and 12]. *Those arguments will be presented in the following theorem for completeness.*

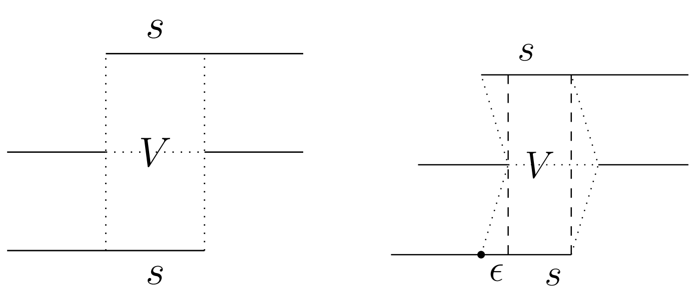
Proof. Since $\mathcal{M}$ is rank one it is completely periodic by Wright [**Wri15a**, Theorem 1.5] and so whenever a saddle connection is put in standard position, the translation surface is vertically periodic.

Step 1: When a horizontal saddle connection is put in standard position, there are two vertical sub-equivalence classes

Let s be a horizontal saddle connection and put it in standard position. Let V be the resulting vertical cylinder.

Suppose first to a contradiction that there is only one sub-equivalence class in the vertical direction. By Proposition 2.3.8, every vertical cylinder has identical height and so all horizontal core curves of cylinders have a rational ratio of length. By Wright [**Wri15a**, Theorem 1.9], $\mathcal{M}$ is arithmetic, contradicting Assumption 2.3.1.

Suppose now that there are at least two vertical sub-equivalence classes. Apply the rel deformation $\epsilon\left(\sum_{c\in\mathcal{A}_1\cup\mathcal{A}_2}\gamma_c^* - \sum_{c\in\mathcal{A}_3}\gamma_c^*\right)$. This creates a gap next to V that can only be entered by a cylinder of height at most ϵ (see Figure 2.4.1).



(a) Standard position (b) The gap created by η

Figure 2.4.1: The gap created using a rel deformation

For ϵ smaller than $h/2$ where h is the smallest height of a vertical cylinder on (X, ω) we see that a new equivalence class must pass through this gap and hence by Proposition 2.3.8 that on (X, ω) there were only two sub-equivalence classes.

Step 2: There are two numbers ℓ and x so that all horizontal saddle connections on (X, ω) have length ℓ or nx for some integer n

Let ℓ be the length of the longest horizontal saddle connection. Let t be a horizontal saddle connection of length ℓ on (X, ω). By Lemma 2.4.3, we may replace (X, ω) with a surface on which t is in standard position. Let V be the vertical cylinder containing t as a cross curve. Every cylinder sub-equivalent to V has height ℓ and therefore contains every saddle connection it passes through as a height.

Let x be the height of the sub-equivalence class of vertical cylinders not containing V. Since any saddle connection not contained as a cross-curve of a cylinder sub-equivalent to V is crossed finitely many times by cylinders of height x it follows that all other saddle connections have length nx for some integer n.

Since the ratios of lengths of the core curves generate a field that properly contains $\mathbb{Q}$, we see that ℓ and x are not rational multiples of each other.

Therefore, when one saddle connection of length ℓ is put into standard position, all saddle connections of length ℓ occur as cross curves of vertical cylinders.

Step 3: All horizontal saddle connections have length ℓ or x

Let u be the longest horizontal saddle connection of length Nx for some integer N. Put u into standard position and let W be the resulting vertical cylinder.

Suppose first that some cylinder sub-equivalent to W passes through a saddle connection of length ℓ. Since ℓ is not an integer multiple of x, it follows that cylinders sub-equivalent to W account for all but $\ell - mx$ of the horizontal saddle connection for some integer m. Therefore, the second vertical sub-equivalence class has height $\frac{\ell - mx}{n}$ for some integer n. None of these cylinders can pass through a saddle connection of length kx, where k is an integer, since the only solution to

$$kx = a\left(\frac{\ell - mx}{n}\right) + bmx$$

with a and b integers occurs when $a = 0$ (because the $\mathbb{Q}$-span of ℓ and x in $\mathbb{R}$ is two-dimensional). Therefore, all horizontal saddle connections of length kx are only intersected by cylinders equivalent to W. Since all these cylinders have identical heights - Nx - and since Nx is the length of the longest horizontal saddle connection that has length of the form nx for some integer n - it follows that all horizontal saddle connections that do not have length ℓ have length Nx. If necessary, set x to Nx so that all horizontal saddle connections have length ℓ or x.

We see too from the argument that whenever a saddle connection of length x is put into standard position, all other saddle connections of length x occur as cross curves of vertical cylinders.

Suppose now that cylinders sub-equivalent to W do not pass through saddle connections of length ℓ. If such a cylinder passes through a saddle connection, then it contains that saddle connection as a cross curve. The other sub-equivalence class of vertical cylinders therefore has cylinders of height $\frac{\ell}{m}$ for some integer m. Since no saddle connection of the length nx for some integer n can be contained in finitely many vertical cylinders of height $\frac{\ell}{m}$ we see that every saddle connection of length nx for some integer n is contained in a cylinder sub-equivalent to W and hence has length Nx and occurs in a vertical cylinder as a cross curve. We are now done as before.

Step 4: All saddle connections in $\mathcal{A}_i$, for $i = 1, 2$, have identical

lengths

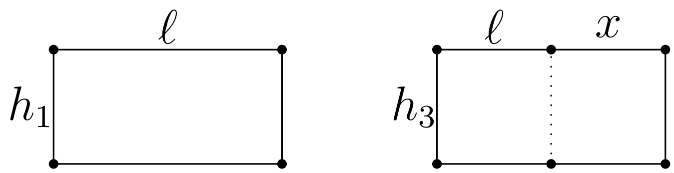
Suppose now, without loss of generality, that t borders a cylinder in $\mathcal{A}_1$ and $\mathcal{A}_3$. Therefore, the cylinder V, which is formed when t is put in standard position, has height ℓ and length $h_1 + h_3$. All sub-equivalent cylinders have a length that is a rational multiple of this length and which must be of the form $a\,(h_1 + h_3) + b\,(h_2 + h_3)$ where a and b are integers. Since the ratio of lengths of core curves in the vertical direction must span a field that properly contains $\mathbb{Q}$ we see that $a\,(h_1 + h_3) + b\,(h_2 + h_3)$ is a rational multiple of $h_1 + h_3$ if and only if $b = 0$. This shows that all cylinders sub-equivalent to V only pass through sub-equivalence classes $\mathcal{A}_1$ and $\mathcal{A}_3$. Therefore, all horizontal saddle connections of length ℓ border $\mathcal{A}_1$ and $\mathcal{A}_3$. The same argument shows that when u is put in standard position, all cylinders sub-equivalent to W pass exclusively through $\mathcal{A}_2$ and $\mathcal{A}_3$ and hence that all saddle connections of length x border $\mathcal{A}_2$ and $\mathcal{A}_3$. □

The following immediately implies Theorem 2.1.1.

Theorem 2.4.6. *$\mathcal{M}$ is a branched covering construction of an eigenform locus in $\mathcal{H}(1,1)$.*

Proof. Suppose to a contradiction that there is a cylinder C in $\mathcal{A}_3$ on whose boundary the saddle connections in $\mathcal{A}_1$ and $\mathcal{A}_2$ do not appear in alternating order. Suppose without loss of generality that the saddle connections s_1, t_1, t_2 appear on the top boundary, where s_1 belongs to $\mathcal{A}_1$ and t_i to $\mathcal{A}_2$ for $i = 1, 2$. On the bottom of the cylinder the saddle connections appear in the order t_2', t_1', s_1' where a prime denotes the image of a saddle connection under the hyperelliptic involution. Put t_1 into standard position and let V be the resulting vertical cylinder. By Proposition 2.4.4, every cylinder sub-equivalent to V must contain all horizontal saddle connections in $\mathcal{A}_2$ and must only pass through cylinders in $\mathcal{A}_2 \cup \mathcal{A}_3$. However, since a vertical line from some point in t_2' will intersect s_1 and hence pass into a cylinder in $\mathcal{A}_1$, this is not the case and we have a contradiction.

Suppose without loss of generality that the longest saddle connection (length ℓ) belongs to $\mathcal{A}_1$ and put it in standard position. By Proposition 2.4.4, all cylinders in $\mathcal{A}_i$ are horizontally tiled by C_i for $i = 1, 3$, where C_i for $i = 1, 2, 3$ will refer to the cylinders in Figure 2.4.2 with labels corresponding to lengths, all angles right angles, and x the length of the saddle connections in $\mathcal{A}_2$. These tilings induce local isometries from cylinders in $\mathcal{A}_i$ to C_i, for $i = 1, 3$, that are undefined at zeros and that take horizontal saddle connections to horizontal saddle connections.



(a) C_i: The cylinder by which all cylinders in $\mathcal{A}_i$ are tiled for $i = 1, 2$

(b) C_3: The cylinder by which all cylinders in $\mathcal{A}_3$ are tiled

Figure 2.4.2: The cylinders that tile those in $\mathcal{A}_i$ for $i = 1, 2, 3$

Apply an element of the cylinder preserving space to put a saddle connection of length x into standard position. As in Lemma 2.4.3 we may assume that this deformation acts as the identity on $\mathcal{A}_3$ and by the matrix $g = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}$ on $\mathcal{A}_1$ for some $\alpha \in \mathbb{R}$. By Proposition 2.4.4, all cylinders in $\mathcal{A}_2$ are tiled by C_2 and admit a local isometry to C_2 defined away from zeros (see Figure 2.4.2).

Combining these local isometries we have a map $f : (X, \omega) \longrightarrow (Y, \zeta)$ where (Y, ζ) is the translation surface that consists of $g \cdot C_1, C_2, C_3$ glued together as follows (numbered edges indicate gluing).

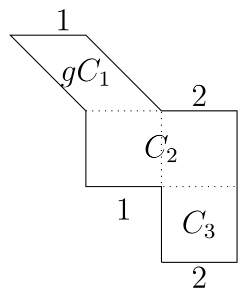


Figure 2.4.3: The translation surface (Y, ζ)

Since (X, ω) may be taken to have dense orbit in $\mathcal{M}$, by [**Apib**, Corollary 4.3], it follows that $\mathcal{M}$ is a locus of branched covers of translation surfaces in $\mathcal{H}(1,1)$. Since $\mathcal{M}$ is nonarithmetic and has complex dimension 3, it must be a locus of branched covers of nonarithmetic eigenforms in $\mathcal{H}(1,1)$, since these are the only such orbit closures of complex dimension 3 in $\mathcal{H}(1,1)$ by McMullen [**McM07**]. $\square$

Chapter 3

Marked Points in Strata of Abelian Differentials

This material contains results of Apisa [**Apia**].

In this chapter, it is shown that nontrivial $\mathrm{GL}(2, \mathbb{R})$ invariant point markings for translation surfaces in strata of Abelian differentials exist only when the translation surface belongs to a hyperelliptic component. As an application, we establish strong constraints on sections of the universal curve restricted to orbifold covers of subvarieties of the moduli space of Riemann surfaces that contain a Teichmüller disk. We also solve the finite blocking problem for the generic translation surface.

3.1 Introduction

Let $\mathcal{QM}_g$ be the moduli space of quadratic differentials on genus g Riemann surfaces. The space admits a $\mathrm{GL}(2,\mathbb{R})$ action generated by Teichmüller geodesic flow and complex scalar multiplication. The space $\mathcal{QM}_g$ also admits a $\mathrm{GL}(2,\mathbb{R})$-invariant stratification given by specifying the number of zeros and poles of the quadratic differentials and their orders of vanishing.

A point of $\mathcal{QM}_g$ will be denoted by (X,q) where X is a Riemann surface and q is a quadratic differential. A point (X,q) will be called generic if its $\mathrm{GL}(2,\mathbb{R})$ orbit is dense in the stratum containing it. We are interested in studying collections of points P on quadratic differentials (X,q). We will always assume that P does not contain any zeros or poles of q. Say that P is generic if the orbit closure of $(X,q;P)$ is as big as possible, i.e.

$$\dim_{\mathbb{C}} \overline{\mathrm{GL}(2,\mathbb{R})\cdot(X,q;P)} = |P| + \dim_{\mathbb{C}} \overline{\mathrm{GL}(2,\mathbb{R})\cdot(X,q)}$$

A non-generic point is called a periodic point. We will now characterize the non-generic collections of points on generic translation surfaces.

Theorem 3.1.1. *If (X,ω) is a generic translation surface of genus at least two and P is a non-generic collection of points then (X,ω) belongs to a* *hyperelliptic component of a stratum and P contains either a Weierstrass point or two points exchanged by the hyperelliptic involution.*

Remark 3.1.2. *In Apisa-Wright* [**AW17**]*, it is shown by separate methods that if (X, q) is a generic quadratic differential that does not belong to a stratum of Abelian differentials then the conclusion of Theorem 3.1.1 holds. Neither result implies the other.*

Holomorphic Sections of the Universal Curve.

In Hubbard [**Hub72**], it was shown that holomorphic sections of the universal curve over $\mathrm{Teich}_{g,n}$ - the Teichmüller space of genus g Riemann surfaces with n punctures - exist only when $(g, n) = (2, 0)$, in which case the sections mark fixed points of the hyperelliptic involution. Earle and Kra [**EK76**] generalized this result by allowing the sections to mark punctured points. They showed that in this more general setting, the only new sections that could arise were in genus two and are given by taking a punctured point and either marking it or its image under the hyperelliptic involution.

Inspired by these questions, we show the following. Let C be a subvariety of $\mathcal{M}_{g,n}$ and let Γ be a torsionfree finite index subgroup of the mapping class group $\mathrm{Mod}_{g,n}$. Let $C(\Gamma)$ be the preimage of C on $\mathrm{Teich}_{g,n}/\Gamma$.

Theorem 3.1.3. *If C contains the Teichmüller disk generated by the quadratic differential (X, q), then any holomorphic section of the universal curve over $C(\Gamma)$ marks a point p on X that is a pole, zero, or periodic point of (X, q).*

Remark 3.1.4. *By Eskin-Filip-Wright* [**EFW17**, Theorem 1.5]*, there are finitely many periodic points on (X, q) if and only if its holonomy double cover is not a torus cover. Therefore, for such quadratic differentials, Theorem 3.1.3 shows that a holomorphic section of the universal curve can only mark finitely many points and explicitly describes which points may be marked.*

Two Applications of Theorems 3.1.1 and 3.1.3.

Apart from $\mathcal{M}_{g,n}$ itself, other examples of algebro-geometrically interesting varieties that contain a Teichmüller disk are the theta-null divisor (see Müller [**Mül13**] and Grushevsky-Zakharov [**GZ14**]), the anti-ramification locus (see Farkas-Verra [**FV13**]), and the Weierstrass divisor (see Cukierman [**Cuk89**]). More examples are listed in Mullane [**Mul17**] and the Kodaira dimensions of many such loci are computed in Gendron [**Gen15**]. Each of these examples is the projection of a non-hyperelliptic component $\mathcal{H}$ of some stratum of Abelian differentials to $\mathcal{M}_{g,n}$. Therefore, the following is immediate from Theorems 3.1.1 and 3.1.3.

Corollary 3.1.5. *If $\mathcal{C}$ is any of the preceding loci and X is a Riemann surface in $\mathcal{C}$, then there is an Abelian differential ω so that (X, ω) belongs to $\mathcal{H}$ and the only points on X that may be marked by a holomorphic section of the universal curve defined on $\mathcal{C}$ are zeros of ω.*

Remark 3.1.6. *In particular, this corollary recovers a classical fact that for any finite-index torsionfree subgroup Γ of the mapping class group, there are no holomorphic sections of the universal curve defined over* $\text{Teich}_{g,n}/\Gamma$ *unless* $g = 2$.

Another application of Theorem 3.1.3 is to sections of the universal curve defined over Hilbert modular surfaces in genus two. In [**McM03**], [**McM05**], [**McM06**], and [**McM07**], McMullen showed that the closures of Teichmüller disks generated by an Abelian differential on a genus two Riemann surface is either $\mathcal{M}_2$ or is contained in a locus of torus covers or a Hilbert modular surface. The periodic points on a torus cover (X, ω) are exactly the preimages of torsion points on the torus. It was shown in Apisa [**Api17a**] that for any other Abelian differential (X, ω) in genus two - except those whose orbit closures project to E_5, i.e. the Hilbert modular surface parameterizing curves whose Jacobian admits real multiplication by the maximal order in $\mathbb{Q}[\sqrt{5}]$ - that the only periodic points are Weierstrass points.

Corollary 3.1.7. *The only holomorphic sections of the universal curve defined over genus two Hilbert modular surfaces (excluding E_5) mark Weierstrass points.*

Remark 3.1.8. *Up to the hyperelliptic involution there is one additional point that may be marked on the universal curve above E_5 and it is described in Kumar-Mukamel* [**KM16**].

Finite Blocking on Translation Surfaces. One motivation for studying marked points on translation surfaces is rational billiards. Given a rational billiard table and two points p and q one may ask whether there is a billiard trajectory from p to q. More ambitiously, one may seek to ascertain whether there is a finite collection of points that block all shots from p to q. To study rational billiards, one often applies the unfolding construction of Katok-Zemlyakov [**ZK75**] to turn the billiard table into a translation surface where billiard trajectories correspond to straight lines on the translation surface. The previously posed problem then becomes the finite blocking problem - given two points p and q on a translation surface is there a finite collection of points that intersects every straight line from p to q. We will continue to assume that neither p nor q coincide with a singularity of the flat metric.

Theorem 3.1.9. *A generic translation surface contains a pair of finitely blocked points if and only if it belongs to a hyperelliptic component, in which case the pair consists of a point and its image under the hyperelliptic involution.*

Proof of Theorem 3.1.9 given Theorem 3.1.1: By Apisa-Wright [**AW17**, Theorem 3.15] and Theorem 3.1.1, finitely blocked pairs of points only occur when the generic translation surfaces belongs to a hyperelliptic component; and in this case, the pair of points consists of either two Weierstrass points or two points exchanged by the hyperelliptic involution and in both cases the finite blocking set is the collection of Weierstrass points.

Since the collection of translation surfaces represented by strictly convex $2n$-gons with opposite sides identified is open, nonempty, and $\mathrm{GL}(2,\mathbb{R})$-invariant in hyperelliptic components of strata, its complement consists of non-generic translation surfaces. Therefore, the generic translation surface may be represented by a strictly convex $2n$-gon with opposite sides identified. The Weierstrass points are the midpoints of the polygon and its edges (and the vertices when n is even). By convexity a Weierstrass point is at most finitely blocked from itself and so the pairs of points finitely blocked from each other are exactly the ones containing a point and its image under the hyperelliptic involution. □

For more on the finite blocking problem see Lelièvre-Monteil-Weiss [**LMW16**]. For applications to billiards see Mirzakhani-Wright [**MW16**, Sections 6-7], Apisa-Wright [**AW17**, Section 3], and Apisa [**Api17a**].

Organization. The proof of Theorem 3.1.3 is independent of the rest of the chapter and appears in Section 3.3. The outline of the proof of Theorem 3.1.1 is given in Section 3.6 and reduced to two more technical results that are established in the subsequent two sections. The two main tools used in the proof of Theorem 3.1.1 are the construction of generic horizontally and vertically periodic translation surfaces in every component of every stratum of holomorphic one-forms in Section 3.4 and results in Section 3.5 that constrain the positions of periodic points using cylinders.

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3.2 Background

In this section, we will summarize the main tools used in the sequel. Let $\Omega\mathcal{M}_g$ be the moduli space of holomorphic one-forms (equivalently Abelian differentials or translation surfaces). As mentioned above this space admits a $\mathrm{GL}(2,\mathbb{R})$ action and a $\mathrm{GL}(2,\mathbb{R})$-invariant stratification. Each stratum of Abelian differentials admits a coordinate system - period coordinates - given by specifying the periods of the holomorphic one-forms in a basis of homology relative to the collection of zeros. The change of coordinates between two charts in period coordinates is a constant volume-preserving linear function, which endows strata of $\Omega\mathcal{M}_g$ with a linear structure and with a well-defined Lebesgue measure, see Zorich [**Zor06**] for details.

Lebesgue measure on strata of $\Omega\mathcal{M}_g$ induces a finite $\mathrm{SL}(2,\mathbb{R})$-invariant measure on $\mathcal{U}$ - the locus of unit area translation surfaces. An affine invariant submanifold is a closed subset of a stratum of $\Omega\mathcal{M}_g$ that is linear in period coordinates. Eskin and Mirzakhani [**EM**] showed that the collection of $\mathrm{SL}(2,\mathbb{R})$-invariant ergodic measures on $\mathcal{U}$ are precisely Lebesgue measure on affine invariant submanifolds restricted to $\mathcal{U}$. Eskin-Mirzakhani-Mohammadi [**EMM15**] showed that orbit closure of any holomorphic one-form in a stratum is exactly an affine invariant submanifold.

Definition 3.2.1. *Given an affine invariant submanifold $\mathcal{M}$, let $\mu_{\mathcal{M}}$ be Lebesgue measure on $\mathcal{M} \cap \mathcal{U}$. Given a $\mathrm{GL}(2,\mathbb{R})$-equivariant measurable map $f : \mathcal{M} \longrightarrow \mathcal{N}$ between two affine invariant submanifolds, define the pushforward of $\mathcal{M}$, denoted $f_*\mathcal{M}$, to be the affine invariant submanifold represented by the ergodic $\mathrm{SL}(2,\mathbb{R})$-invariant measure given by $f_*\mu_{\mathcal{M}}$. Notice that $f(\mathcal{M})$ coincides with $f_*\mathcal{M}$ up to sets of measure zero.*

A fundamental tool in the sequel will be cylinder deformations.

Definition 3.2.2. *Suppose that (X,ω) is a translation surface in an affine invariant submanifold $\mathcal{M}$. If C_1 and C_2 are two cylinders in $\mathcal{M}$ then C_1 and C_2 will be said to be $\mathcal{M}$-equivalent if C_1 and C_2 are parallel at all surfaces in an open neighborhood of (X,ω) in $\mathcal{M}$. A maximal collection of equivalent cylinders on (X,ω) will be called an $\mathcal{M}$-equivalence class. Given a collection $\mathcal{C}$ of cylinders, the standard shear, $\sigma_{\mathcal{C}}$, and the standard dilation, $a_{\mathcal{C}}$, are the* *following cohomology classes*

$$\sigma_{\mathcal{C}} := \sum_{c \in \mathcal{C}} h_c \gamma_c^* \qquad a_{\mathcal{C}} := i \sum_{c \in \mathcal{C}} h_c \gamma_c^*$$

where h_c is the height of cylinder c and γ_c is the core curve of the cylinder c oriented from left to right.

Theorem 3.2.3 (Wright [**Wri15a**], Corollary 3.4)**.** *Suppose that (X, ω) is a horizontally periodic translation surface whose $\mathrm{GL}(2, \mathbb{R})$ orbit closure is $\mathcal{M}$. Let $(C_1, \ldots, C_n)$ be an enumeration of the horizontal cylinders and suppose that cylinder C_i has modulus m_i and core curve (oriented from left to right) γ_i of length c_i for $i = 1, \ldots, n$. Let $W \subseteq \mathbb{Q}^n$ be the subset of rational homogeneous linear relations that the moduli $(m_i)_{i=1}^n$ satisfy, i.e. $w \in W$ if and only if $w \cdot m = 0$. If $(v_i)_{i=1}^n \in \mathbb{C}^n$ belongs to $W^\perp$, then*

$$\sum_{i=1}^n c_i v_i \gamma_i^* \in T_{(X,\omega)} \mathcal{M}$$

Theorem 3.2.4 (Wright [**Wri15a**], Lemma 4.11)**.** *If $\mathcal{C}$ is an equivalence class of horizontal cylinders on a translation surface (X, ω) contained in affine invariant submanifold $\mathcal{M}$ then the standard shear and the standard dilation are both contained in $T_{(X,\omega)} \mathcal{M}$.*

Finally, the fundamental object of study in the sequel will be the following.

Definition 3.2.5. *Given an affine invariant submanifold $\mathcal{M}$, let $\mathcal{M}(0^n)$ be the collection of quadratic differentials $(X, q) \in \mathcal{M}$ together with n distinct marked points that do not coincide with zeros or poles of q (where n is a positive integer). Let $\overline{\mathcal{M}(0^n)}$ be the partial compactification of $\mathcal{M}(0^n)$ where marked points are allowed to coincide with each other and with zeros and poles of q. Both $\mathcal{M}(0^n)$ and $\overline{\mathcal{M}(0^n)}$ admit a* $\mathrm{GL}(2, \mathbb{R})$*-action and all results stated in this section continue apply to both of these spaces.*

Notice that $\overline{\mathcal{M}(0^n)}$ is contained in the Mirzakhani-Wright partial compactification of $\mathcal{M}(0^n)$ (see [**MW17**])

Theorem 3.2.6 (Mirzakhani-Wright [**MW17**], Corollary 1.2)**.** *Suppose that $\mathcal{M}$ is an affine invariant submanifold and (X_n, ω_n) is a sequence of points in $\mathcal{M}$ that converges to a, possibly disconnected, translation surface (Y, η) in the boundary of $\mathcal{M}$. The orbit closure of any component of (Y, η) has smaller dimension than* $\dim \mathcal{M}$*.*

Lemma 3.2.7. *Let $\mathcal{M}$ be an affine invariant submanifold and let $\mathcal{N}$ be an affine invariant submanifold in $\mathcal{M}(0)$. Let $\pi : \mathcal{M}(0) \longrightarrow \mathcal{M}$ be the map that* *forgets the marked point. If $\pi_* \mathcal{N} = \mathcal{M}$ then $\pi(\mathcal{N})$ is an open dense subset of $\mathcal{M}$ and the fiber over a generic point in $\mathcal{M}$ is nonempty. Moreover, π is open.*

Proof. Since all fibers of the forgetful map $\pi : \overline{\mathcal{M}(0)} \longrightarrow \mathcal{M}$ are compact, π is proper and its image is closed. The image of π is full measure in $\pi_* \mathcal{N}$, which is $\mathcal{M}$ by assumption. If $\mathcal{N} = \mathcal{M}(0)$ then the result is immediate, so suppose instead that $\mathcal{N}$ and $\mathcal{M}$ have the same dimension. By Mirzakhani-Wright [**MW17**, Corollary 1.2], the components of the boundary of $\overline{\mathcal{N}} \cap \overline{\mathcal{M}(0)}$ have strictly smaller dimension than $\mathcal{M}$ and hence their pushfoward $\mathcal{C}$ cannot be $\mathcal{M}$ by Sard's theorem. Therefore, $\pi(\mathcal{N})$ contains the complement of $\mathcal{C}$, which is an open dense set in $\mathcal{M}$. Since $\pi : \mathcal{N} \longrightarrow \mathcal{M}$ is a finite holomorphic map between equidimensional varieties, it is open.

□

Definition 3.2.8. *If $\mathcal{M}$ is an affine invariant submanifold then let $\mathcal{M}^{ord}(0^n)$ be the finite cover of $\mathcal{M}(0^n)$ where the marked points are labelled. Let $\pi_k : \mathcal{M}(0^n) \longrightarrow \mathcal{M}(0^{n-1})$ be the map that forgets the kth-marked point where $k \in \{1, \ldots, n\}$.*

Lemma 3.2.9. *Let $\mathcal{N}$ be an affine invariant submanifold in $\mathcal{M}(0^n)$ that pushes forward to $\mathcal{M}$ under the map π that forgets all marked points. If*

$(X, \omega; P)$ is generic in $\mathcal{M}(0^n)$, then there is an open set U of $(X, \omega; P)$ in $\mathcal{N}$ so that $\pi(U)$ is an open set around (X, ω) in $\mathcal{M}$.

Proof. Without loss of generality, we will work on $\mathcal{M}^{ord}(0^n)$. Since each π_k restricted to $\mathcal{N}$ is open by Lemma 3.2.7 and since $\pi = \pi_1 \circ \ldots \circ \pi_n$, π restricted to $\mathcal{N}$ is open as well. □

3.3 Holomorphic Sections over Varieties containing a Teichmüller Disk - Proof of Theorem 3.1.3

Throughout this section we will make the following assumption.

Assumption 3.3.1. *Let Γ be a torsionfree finite index subgroup of the mapping class group. Let C be a complex analytic subvariety of $\mathrm{Teich}_{g,n}/\Gamma$ where $\mathrm{Teich}_{g,n}$ is the Teichmüller space of a closed genus g surface with n punctures and so that $3g-3+n > 0$. Let $\pi : \mathcal{C}_{g,n} \longrightarrow \mathrm{Teich}_{g,n}/\Gamma$ be the universal curve. Suppose additionally that C contains a Teichmüller disk generated by the quadratic differential (X, q) where X is a Riemann surface and q a quadratic differential on X. Suppose that $\mathcal{Q}$ is the stratum of quadratic dif-*

ferentials to which (X, q) belongs and let $\mathcal{M}$ be the orbit closure of (X, q) in $\mathcal{Q}$.

Lemma 3.3.2. *Every holomorphic section of π defined over C induces a* $\mathrm{GL}(2, \mathbb{R})$*-equivariant section of the forgetful map from $\overline{\mathcal{M}(0)}$ to $\mathcal{M}$.*

Proof. Let $s : C \longrightarrow \mathcal{C}_{g,n}$ be a holomorphic section of π. Let $\iota : \mathbb{D} \longrightarrow C$ be the inclusion of the Teichmüller disk into C. The inclusion is an isometry with respect to the underlying Kobayashi hyperbolic metrics. Since $\iota = \pi \circ s$ and π and s are contractions in the Kobayashi hyperbolic metrics, it follows that s restricted to the embedded Teichmüller disk in C is an isometry in the Kobayashi metrics. Therefore, $s(\iota(\mathbb{D}))$ is a Teichmüller disk in $\mathcal{C}_{g,n}$.

Sufficiently close Riemann surfaces X_1 and X_2 contained in $\iota(\mathbb{D})$ are joined by a dilatation minimizing homeomorphism given by a geodesic in $\iota(\mathbb{D})$. By Teichmüller's theorem, this homeomorphism is unique up to pre- and post-composition with a conformal automorphism. However, since Γ is torsionfree there are no such automorphisms that fix X_1 or X_2. If γ is the geodesic from X_1 to X_2 in $\iota(\mathbb{D})$, then $s(\gamma)$ is a geodesic of the same length in $\mathcal{C}_{g,n}$ and hence corresponds to a homeomorphism with the same dilatation. By uniqueness this path must correspond to the same homeomorphism and so if Teichmüller geodesic flow along (X_1, q_1) produces the geodesic γ, Teichmüller geodesic flow along $(s(X), q)$ produces $s(\gamma)$.

Let $\widetilde{s}: \mathcal{M} \longrightarrow \overline{\mathcal{M}(0)}$ be the section of the forgetful map from $\overline{\mathcal{M}(0)}$ to $\mathcal{M}$ given by sending a quadratic differential (X, q) to $(s(X), q)$. The argument above shows that this map is $\mathrm{GL}(2, \mathbb{R})$-equivariant on Teichmüller disks since it is equivariant under complex scalar multiplication and Teichmüller geodesic flow. Since $\mathcal{M}$ is foliated by $\mathrm{GL}(2, \mathbb{R})$ invariant Teichmüller disks the claim follows. □

Remark 3.3.3. *The proof of Lemma 3.3.2 only uses the hypothesis that* Γ *is torsionfree, not that it is finite index.*

Proof of Theorem 3.1.3: Let s be a holomorphic section of π defined over C. Since it is a continuous section, it follows that $s(C)$ and hence $\widetilde{s}(\mathcal{M})$ is closed. By Lemma 3.3.2, $\widetilde{s}(\mathcal{M})$ is closed and $\mathrm{GL}(2, \mathbb{R})$-invariant and therefore it is an affine-invariant submanifold by Eskin-Mirzakhani-Mohammadi [**EMM15**]. Notice that this application of Eskin-Mirzakhani-Mohammadi uses the fact that Γ is finite-index since this implies that the Lebesgue measure of the collection of unit area half-translation surfaces in $\mathcal{M}$ is finite. Since $\widetilde{s}(\mathcal{M})$ is an affine invariant-submanifold that does not coincide with $\mathcal{M}(0)$, it follows that the point that s marks above X is a periodic point, zero, or pole of (X, q). □

Remark 3.3.4. *The same proof shows that measurable equivariant sections of the forgetful map from* $\overline{\mathcal{M}}(0)$ *to* $\mathcal{M}$ *only mark periodic points, zeros, or poles. In the measurable setting, the section is used to pushforward Lebesgue measure to a measure on* $\overline{\mathcal{M}(0)}$*, which must be Lebesgue measure on an affine invariant submanifold by Eskin-Mirzakhani* [**EM**]*. The details are omitted.*

3.4 Explicit Translation Surfaces in Every Component of Every Stratum

In this section we will construct explicit generic translation surfaces in each connected component of every stratum of Abelian differentials. The connected components were classified by Kontsevich and Zorich [**KZ03**].

Theorem 3.4.1 (Kontsevich-Zorich [**KZ03**])**.** *All strata are connected except for the following:*

- *For* $g > 3$, $\mathcal{H}(2g-2)$ *has three connected components characterized by odd spin, even spin, and hyperellipticity.*
- *For odd* $g > 3$, $\mathcal{H}(g-1, g-1)$ *has three connected components characterized by odd spin, even spin, and hyperellipticity.*

- *For even $g > 3$, $\mathcal{H}(g-1, g-1)$ has two connected components characterized by hyperellipticity and nonhyperellipticity.*

- *For $g > 3$, $\mathcal{H}(2k_1, \ldots, 2k_n)$ has two connected components characterized by odd and even spin (excluding the case $\mathcal{H}(g-1, g-1)$ for odd $g > 3$, which, as mentioned above, has three components).*

- *$\mathcal{H}(4)$ and $\mathcal{H}(2, 2)$ have two connected components - a hyperelliptic and an odd one.*

To distinguish which connected component of a stratum a specific translation surface belongs to, we will use the following criterion.

Theorem 3.4.2 (Kontsevich-Zorich [**KZ03**], Corollary 2)**.** *Let $\mathcal{H}$ be a stratum of Abelian differentials. For each connected component C of the minimal stratum there is a unique component of $\mathcal{H}$ that contains C in its closure.*

We are now in a position to create horizontally and vertically periodic translation surfaces in each component of each stratum. First, we establish a convention:

Convention for Figures: We will often use polygons, all of whose edges will be vertical or horizontal, to represent translation surfaces using the following two conventions. The edge of a polygon will mean a line segment in the boundary of the polygon that connects two vertices and has no vertex in its interior.

1. The intersection of a dotted line and an edge is a vertex of the polygon.

2. If a pair of unmarked vertical (resp. horizontal) edges contain interior points that can be connected by a horizontal (resp. vertical) line that lies in the interior of the polygon then they are identified.

Under this convention the two translation surfaces in Figure 3.4.1 are identical:

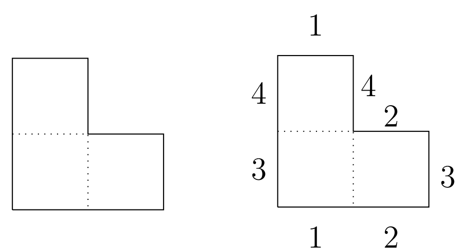
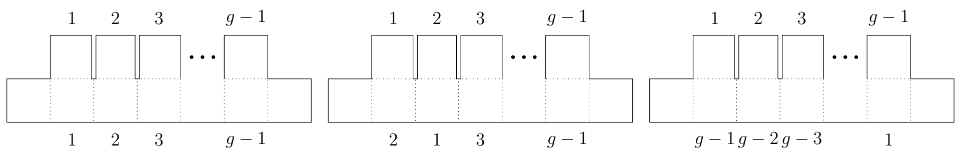


Figure 3.4.1: Equivalent representations of the same translation surface

Using separatrix diagrams, Kontsevich and Zorich produce surfaces that belong to each component of the minimal stratum, see [**KZ03**, Figure 4]. The surfaces are represented as translation surfaces in Figure 3.4.2.



(a) $\mathcal{H}^{odd}(2g-2)$ (b) $\mathcal{H}^{even}(2g-2)$ (c) $\mathcal{H}^{hyp}(2g-2)$

Figure 3.4.2: Surfaces in each component of the minimal stratum

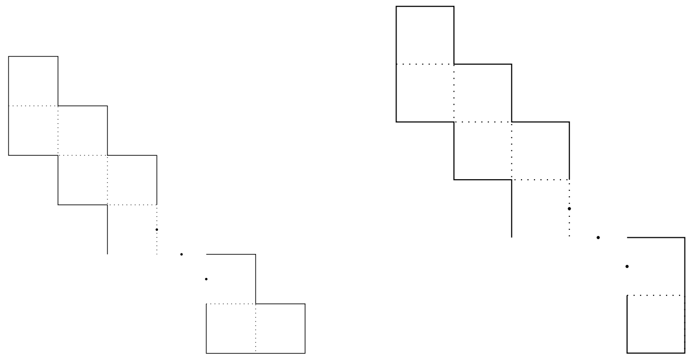
Proposition 3.4.3 (Genericity Criterion)**.** *Suppose that (X, ω) is a translation surface in a component $\mathcal{H}$ of a stratum of Abelian differentials of genus g surfaces with n zeros and no marked points. If (X, ω) has $g + n - 1$ horizontal cylinders whose moduli satisfy no rational linear relation, then (X, ω) is generic.*

Proof. Let $\mathcal{M}$ be the orbit closure of (X, ω). It suffices to show that $\mathcal{M}$ coincides with a component of $\mathcal{H}$. By Wright [**Wri15a**, Corollary 3.4] (Theorem 3.2.3), since the moduli satisfy no rational linear relation the tangent space of $\mathcal{M}$ at (X, ω) includes $\{\gamma_1^*, \ldots, \gamma_{g+n-1}^*\}$ where γ_i are core curves oriented from left to right of the horizontal cylinders and γ_i^* denotes the dual cohomology class under the intersection pairing. For details on the identification of $T_{(X,\omega)}\mathcal{M}$ with a subspace of $H^1(X, \Sigma; \mathbb{C})$, for Σ the zero set of ω, see Avila, Eskin, Möller [**AEM**]. The dual cohomology classes span a complex vector space of dimension $g + n - 1$.

Let $p : T_{(X,\omega)}\mathcal{M} \longrightarrow H^1(X,\mathbb{C})$ be the projection from relative to absolute cohomology. By Avila, Eskin, Möller [**AEM**] the image of the projection is a complex symplectic vector space. The kernel of the projection has (complex) dimension at most $n-1$. Since the projection of $\{\gamma_1^*, \ldots, \gamma_{g+n-1}^*\}$ spans an isotropic subspace, which has dimension at most g, it follows that the kernel of p has dimension exactly $n-1$ and that the projection of $\{\gamma_1^*, \ldots, \gamma_{g+n-1}^*\}$ spans a Lagrangian subspace. Since $p\left(T_{(X,\omega)}\mathcal{M}\right)$ is complex symplectic it follows that p is a surjection with maximal dimensional kernel. It follows that $T_{(X,\omega)}\mathcal{M}$ is isomorphic to $H^1(X,\Sigma;\mathbb{C})$ and hence that $\mathcal{M}$ has full dimension. Since $\mathcal{M}$ is open and closed it must coincide with a component of $\mathcal{H}$. □

Generic Surfaces in $\mathcal{H}^{hyp}(2g-2)$ and $\mathcal{H}^{hyp}(g-1, g-1)$

It is straightforward to verify that the translation surfaces in Figure 3.4.3 are in the indicated components; see for example [**Apib**, Section 2]. The genericity criterion (Proposition 3.4.3) implies that the translation surfaces are generic provided that all moduli of horizontal cylinders satisfy no rational linear relation.



(a) $\mathcal{H}^{hyp}(2g-2)$ (b) $\mathcal{H}^{hyp}(g-1, g-1)$

Figure 3.4.3: Hyperelliptic translation surfaces

Generic Surfaces in Even Components of Strata

To find a surface in $\mathcal{H}^{even}(2k_1, \ldots, 2k_n)$ start with the surface in Figure 3.4.4, set $g = 1+\sum_i k_i$, and then collapse every saddle connected labelled a_i except those in $S := \{a_0, a_{k_1}, a_{k_1+k_2}, \ldots, a_{\sum_{i=1}^{n-1} k_i}\}$. This surface is in the even component since collapsing $S - \{a_0\}$ is a path in the stratum whose endpoint is a surface in the even minimal component (specifically the surface in Figure 3.4.2b). By the genericity criterion (Proposition 3.4.3) whenever the vertical cylinders have rationally unrelated moduli this surface is generic.

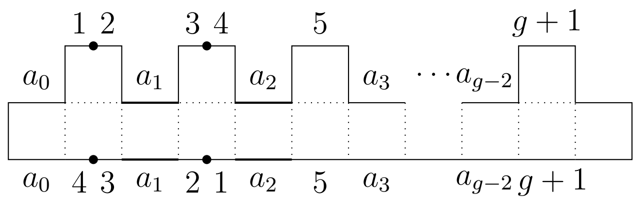


Figure 3.4.4: $\mathcal{H}^{even}(2,\ldots,2)$

Lemma 3.4.4. *Let (X,ω) be the surface just constructed in an even component $\mathcal{H}$ of a stratum of Abelian differentials. Let C be the unique horizontal cylinder that intersects every vertical cylinder. If $g>2$ and $\mathcal{H}$ is not hyperelliptic, then there are two equivalence classes of vertical cylinder $\mathcal{D}_1$ and $\mathcal{D}_2$, so that $C\cap\mathcal{D}_i$ is connected for $i=1,2$. Notice that $C-(\mathcal{D}_1\cup\mathcal{D}_2)$ consists of two disjoint rectangles. For generic choices of the lengths of the horizontal saddle connections, these two rectangles have different horizontal lengths.*

Proof. If $g>4$ then we may set $\mathcal{D}_1$ and $\mathcal{D}_2$ equal to the vertical cylinder that passes through the horizontal saddle connection labelled 5 and 6 in Figure 3.4.4 respectively. Suppose now that $g\in\{3,4\}$.

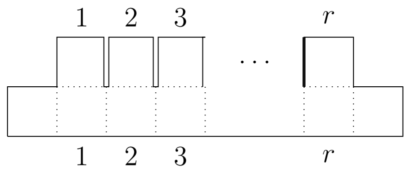
Suppose first that the horizontal saddle connection labelled a_1 is uncollapsed. When $g=4$, set $\mathcal{D}_1$ and $\mathcal{D}_2$ to be the vertical cylinders passing through the horizontal saddle connections labelled a_1 and 5 respectively. When $g=3$ we take them to be the vertical cylinders passing through a_0

and a_1.

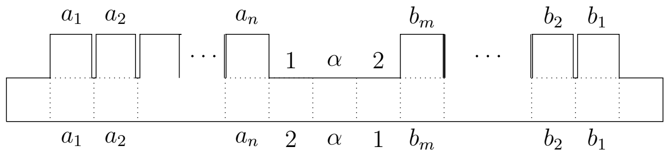
Suppose now that a_1 is collapsed. Since $\mathcal{H}$ is not hyperelliptic, $g = 4$. Set $\mathcal{D}_1$ to be the equivalence class that contains the two cylinders that intersect the horizontal saddle connections labelled $\{1, 2, 3, 4\}$. Set $\mathcal{D}_2$ to be the equivalence class of vertical cylinders intersecting the horizontal saddle connection labelled 5. $\square$

Generic Surfaces in Remaining Components

To find generic surfaces in all other connected components of the remaining strata we glue together copies of the surfaces in Figure 3.4.5 along the horizontal cylinders that intersects all vertical cylinders. By the genericity criterion (Proposition 3.4.3) whenever the vertical cylinders have rationally unrelated moduli this surface is generic.



(a) $\mathcal{H}(\ldots, 2r, \ldots)$



(b) $\mathcal{H}(\ldots, 2n+1, 2m+1, \ldots)$

Figure 3.4.5: Surfaces in $\mathcal{H}^{odd}$, $\mathcal{H}^{nonhyp}$, and connected strata

Lemma 3.4.5. *Lemma 3.4.4 holds for the surfaces just constructed in genus greater than two.*

Proof. Suppose first that the surface has two zeros of even order. Then the surface contains two copies of the surface in Figure 3.4.5a. It is sufficient to take the two vertical cylinders that pass through the horizontal saddle connection labelled 1. Similarly, if there are four zeros of odd order we have two copies of the surface in Figure 3.4.5b and we take the two vertical cylinders that pass through the horizontal saddle connection labelled α.

If there is one zero of even order and two zeros of odd order, then we have a surface like the one in Figure 3.4.5a and one like the one in Figure 3.4.5b and we take the vertical cylinder passing through the saddle connection labelled

1 on the first and the one passing through the horizontal saddle connection labeled α on the second.

If there is only one zero, then it is of even order and we take the two vertical cylinders that pass through 1 and 2. If there are only two zeros and both are of odd order then we take the vertical cylinder passing through α and the vertical cylinder passing through a_1 or b_1 (whichever exists). $\square$

Definition 3.4.6. *Each of the nonhyperelliptic surfaces constructed in this section contain a horizontal cylinder that intersects every vertical cylinder. This cylinder will be called the central horizontal cylinder.*

3.5 Marked Points in Cylinders

In this section we will prove results about marked points and cylinders that form the technical core of the chapter. We will make the following standing assumption:

Assumption 3.5.1. $\mathcal{M}$ *is an affine invariant submanifold and* (X, ω) *is a translation surface whose* $\mathrm{GL}(2, \mathbb{R})$ *orbit closure is* $\mathcal{M}$

Lemma 3.5.2. *Let P be a collection of distinct points on (X, ω) and suppose that $\mathcal{M}'$ is the orbit closure of $(X, \omega; P)$. If $\mathcal{C}$ is an $\mathcal{M}$-equivalence class of cylinders on (X, ω), then $\mathcal{C}'$ is an $\mathcal{M}'$-equivalence class where $\mathcal{C}'$ contains the cylinders in $\mathcal{C}$ divided into subcylinders by the points in P.*

Proof. First we will show that any two cylinders in $\mathcal{C}'$ are $\mathcal{M}'$ equivalent. Let C_i be two cylinders in $\mathcal{C}'$ and let γ_i be their core curves for $i = 1, 2$. By assumption, there is a neighborhood U of (X, ω) in $\mathcal{M}$ on which γ_1 and γ_2 are collinear. Let U' be a preimage of the U in $\mathcal{M}'$ on which C_1 and C_2 persist as cylinders. In this neighborhood, γ_1 and γ_2 must remain collinear and hence $\mathcal{M}'$-equivalent.

It remains to show that if C_1 and C_2 are $\mathcal{M}'$-equivalent cylinders with core curves γ_1 and γ_2, then γ_1 and γ_2 must be collinear on a neighborhood of (X, ω) in $\mathcal{M}$. By Lemma 3.2.9 there is a neighborhood of $(X, \omega; P)$ in $\mathcal{M}'$ that projects to an open neighborhood of (X, ω) in $\mathcal{M}$. Let U' be such a neighborhood of $(X, \omega; P)$ in $\mathcal{M}'$ where γ_1 and γ_2 are collinear and let U be its image under the forgetful map. Since γ_1 and γ_2 are collinear on U' they are collinear on U and hence are homotopic to core curves of $\mathcal{M}$-equivalent cylinders. $\square$

Definition 3.5.3. *Given a cylinder C in a translation surface we say that* *the height of the cylinder is the distance between the two boundaries in the flat metric. Suppose that p is a marked point contained in a cylinder C. Let h_C be the height of the cylinder and let h_p be the distance from the point to one of the two boundary curves of C. We say that p lies at rational height in C if the ratio $\frac{h_p}{h_C}$ is rational.*

Lemma 3.5.4 (Rational Height Lemma)**.** *Let $\mathcal{C}$ be an equivalence class of cylinders so that any two have a rational ratio of moduli. If a periodic point belongs to the interior of a cylinder in $\mathcal{C}$ then it lies at rational height.*

Proof. Let p be a periodic point contained in the interior of a cylinder in $\mathcal{C}$. Let $\mathcal{M}'$ be the orbit closure of $(X, \omega; p)$. Let $\mathcal{C}'$ be the collection of subcylinders on $(X, \omega; p)$ into which $\mathcal{C}$ is divided. By Lemma 3.5.2, $\mathcal{C}'$ is an $\mathcal{M}'$ equivalence class. Let $\sigma_{\mathcal{C}'}$ be the standard shear on $\mathcal{C}'$. Since the cylinders in $\mathcal{C}$ have a rational ratio of moduli, the flow along $\sigma_{\mathcal{C}}$ is periodic. Suppose to a contradiction that p does not have rational height. In this case, the flow along $\sigma_{\mathcal{C}'}$ is not periodic and so the orbit closure of $(X, \omega; p)$ contains $(X, \omega; q)$ where q is any point in C along the core curve of C that intersects p.

Let γ_1 and γ_2 be the two core curves of the cylinders into which p divides C. Since p may be moved along the core curve of C while fixing all cylinders in (X, ω), it follows that the tangent space of $\mathcal{M}'$ at $(X, \omega; p)$ contains the deformation $\gamma_1^* - \gamma_2^*$. Let U' be a neighborhood as in Lemma 3.2.9 of $(X, \omega; p)$ in $\mathcal{M}'$ on which the cylinders in $\mathcal{C}$ persist. This neighborhood projects to a neighborhood U of (X, ω) in $\mathcal{M}$. Since the tangent space contains the deformation $\gamma_1^* - \gamma_2^*$ the fiber of the projection from $\mathcal{M}'$ to $\mathcal{M}$ that forgets marked points has real dimension at least one. Therefore, the dimension of $\mathcal{M}'$ is strictly larger than the dimension of $\mathcal{M}$, which contradicts the assumption that p is a periodic point. □

Lemma 3.5.5. *Let (X, ω) be a generic translation surface in an affine invariant submanifold $\mathcal{M}$. Let C be a horizontal cylinder, and let $\mathcal{D}_1, \mathcal{D}_2$ be two vertical distinct $\mathcal{M}$-equivalence classes of cylinders such that*

1. *The intersection of $\mathcal{D}_i$ with the interior of C is connected and nonempty for $i = 1, 2$.*

2. *Any cylinder equivalent to C has a modulus that is an integer multiple of the modulus of C.*

If p is an $\mathcal{M}$-periodic point in the interior of C, then up to relabelling $\mathcal{D}_1$ and $\mathcal{D}_2$, the point p is at the center of the rectangle given by the intersection of $\mathcal{D}_1$ and C. Furthermore, removing $\mathcal{D}_1$ and $\mathcal{D}_2$ divides C into two rectangles *of equal size.*

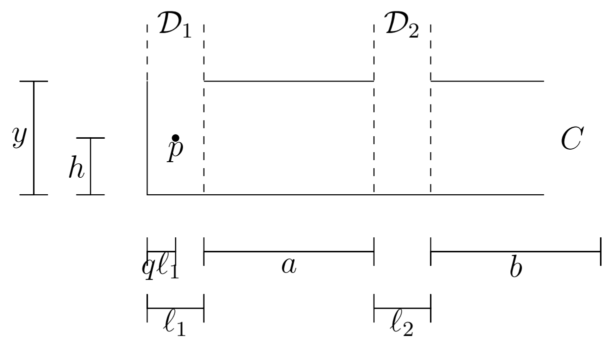


Figure 3.5.1: The lemma shows that $a = b$ and that, after scaling so C has unit height, $q = h = \frac{1}{2}$

Proof. Let $\mathcal{M}'$ be the orbit closure of $(X, \omega; p)$. Let $\mathcal{C}$ be the collection of horizontal cylinders equivalent to C on (X, ω).

For simplicity we begin by applying a element of $\mathrm{GL}_2(\mathbb{R})$ so that C has unit height. By the rational height lemma (Lemma 3.5.4) the marked point p lies at rational height h in C. Since we have normalized the height of C to be one this means that h is rational, in particular suppose that $h = \frac{n}{m}$ where n and m are coprime positive integers.

Part 1: We may assume that p belongs to $\mathcal{D}_1$

Suppose not. By Lemma 3.5.2, since p is not contained in $\mathcal{D}_1$, the standard shear $\sigma_{\mathcal{D}_1}$ is tangent to $\mathcal{M}'$. Traveling in the $\sigma_{\mathcal{D}_1}$ direction in $\mathcal{M}'$ from $(X, \omega; p)$ widens $\mathcal{D}_1$ while fixing the part of the translation surface (and marked point) in the complement $\mathcal{D}_1$. Travel in this direction until the intersection of $\mathcal{D}_1$ and C accounts for at least $\frac{m-1}{m}$ proportion of the area of C. Let (Y, η) be the new translation surface.

Part 1a: (Y, η) may be taken to be generic in $\mathcal{M}$

We formed (Y, η) by traveling in the $\sigma_{\mathcal{D}_1}$ direction from (X, ω) in $\mathcal{M}$. Let ℓ be the segment joining (X, ω) to (Y, η) in $\mathcal{M}$. Each proper affine invariant submanifold (of which there are only countably many by Eskin-Mirzakhani-Mohammadi [**EMM15**]) contained in $\mathcal{M}$ intersects ℓ in a closed set. If a neighborhood U of (Y, η) had the property that every point in $U \cap \ell$ was contained in some proper affine invariant submanifold, then by the Baire category theorem there would be a proper affine invariant submanifold $\mathcal{N}$ so that $\mathcal{N} \cap \ell$ had interior in $U \cap \ell$. Since affine invariant submanifolds are linear this would imply that all of ℓ was contained in $\mathcal{N}$, which contradicts the fact that ℓ contains a translation surface (X, ω) that is generic in $\mathcal{M}$. Therefore, we may assume that the point (Y, η) was chosen to be generic in $\mathcal{M}$.

Part 1b: The hypotheses of the lemma continue to hold on (Y, η) and the fiber of $\mathcal{M}'$ over (Y, η) contains $(Y, \eta; p)$ where p belongs to $\mathcal{D}_1$

Traveling in the $\sigma_{\mathcal{D}_1}$ direction keeps p fixed in the complement of $\mathcal{D}_1 \cup \mathcal{D}_2$

and keeps the heights of the cylinders in $\mathcal{C}$ constant. By Wright [**Wri15a**, Theorem 1.9], the ratio of lengths of core curves of cylinders in $\mathcal{C}$ are constant, and so the condition on moduli of cylinders in $\mathcal{C}$ continues to hold.

Letting $\mathcal{C}'$ be the collection of cylinders on $(Y, \eta; p)$ that project to cylinders in $\mathcal{C}$ we have that $\mathcal{C}'$ is an $\mathcal{M}'$-equivalence class by Lemma 3.5.2. Travel from $(Y, \eta; p)$ in the $\sigma_{\mathcal{C}'}$ direction until one complete Dehn twist has been performed in C. The resulting unmarked translation surface is (Y, η) since all cylinders in $\mathcal{C}$ have moduli that are integer multiples of the modulus of C. Since $\mathcal{D}_1 \cap C$ is connected and accounts for at least $\frac{m-1}{m}$ of the area of C, it follows that the marked point p now belongs to $\mathcal{D}_1$ and we have shown that $(Y, \eta; p)$ belongs to $\mathcal{M}'$.

Part 1c: If the conclusion of the lemma holds on (Y, η), it does so on (X, ω) as well

To pass from (X, ω) to (Y, η), we first traveled along $\sigma_{\mathcal{D}_1}$ and then traveled along $\sigma_{\mathcal{C}'}$ to perform one Dehn twist in C. If the conclusion of the lemma holds on (Y, η), then traveling in the opposite direction along $\sigma_{\mathcal{C}'}$ to perform the opposite Dehn twist in C moves p to the midpoint of the rectangle $\mathcal{D}_2 \cap C$. Let $\mathcal{D}_1'$ be the collection of subcylinders into which the cylinders in $\mathcal{D}_1$ are divided by p. Traveling along $\sigma_{\mathcal{D}_1'}$ back to (X, ω) keeps the complement of

$\mathcal{D}_1$ fixed and so the conclusion of the lemma held on $(X, \omega; p)$ as desired. Therefore, we may suppose without loss of generality (by replacing (X, ω) with (Y, η)) that p belongs to $\mathcal{D}_1$.

Part 2: Determining the position of p

Suppose now that p belongs to $\mathcal{D}_1$. As before, $\sigma_{\mathcal{D}_2}$ is tangent to $\mathcal{M}'$. Travel from $(X, \omega; p)$ in the $\sigma_{\mathcal{D}_2}$ direction in $\mathcal{M}'$ until the intersection of $\mathcal{D}_2$ and C accounts for at least $\frac{m-1}{m}$ proportion of the area of C. Without loss of generality, we may replace $(X, \omega; p)$ with the resulting marked translation surface.

Let ℓ_i be the horizontal length of the rectangle $C \cap \mathcal{D}_i$ for $i = 1, 2$. The complement of $\mathcal{D}_1 \cup \mathcal{D}_2$ in C is two disjoint rectangles. Let a (resp. b) be horizontal length of the rectangle to the right (resp. left) of $\mathcal{D}_1 \cap C$, see Figure 3.5.1. Let ℓ be the length of the core curve of C. Let $q \in [0, 1]$ be chosen so that p is a distance of $q\ell_1$ from the left boundary of $\mathcal{D}_1 \cap C$. Let $\mathcal{D}_1'$ be the collection of subcylinders that p divides $\mathcal{D}_1$ into on $(X, \omega; p)$.

Travel in the $\sigma_{\mathcal{D}_1'}$ direction from $(X, \omega; p)$ so that the length of the core curve of C increases by s and then travel in the $\sigma_{C'}$ direction to perform exactly one Dehn twist in C. The distance of the marked point from the lefthand boundary of $\mathcal{D}_2 \cap C$ is the following,

$$h\,(\ell + s) - (1-q)(\ell_1 + s) - a$$

Traveling back along the $\sigma_{\mathcal{D}_1'}$ direction returns to the unmarked surface (X, ω) while leaving the position of the marked point fixed in the complement of $\mathcal{D}_1$. Since p is a periodic point, the fiber of the forgetful map from $\mathcal{M}'$ to $\mathcal{M}$ over (X, ω) is finite.

Therefore, $h\,(\ell + s) - (1-q)(\ell_1 + s) - a$ is constant as a function of s. In other words, $h = (1-q)$. If we sheared C in the other direction we would have by symmetry that $h = q$ and so $q = h = \frac{1}{2}$. By symmetry, after shearing the marked point into $\mathcal{D}_2$ the distance from the lefthand boundary of $\mathcal{D}_2 \cap C$ is $\ell_2/2$, i.e.

$$\frac{1}{2}\,(\ell - \ell_1) - a = \frac{\ell_2}{2}$$

Since $\ell = \ell_1 + a + \ell_2 + b$ we see that $a = b$ as desired. □

3.6 Proof of Theorem 3.1.1

Throughout this section, we make the following assumption.

Assumption 3.6.1. *Let* $\mathcal{M}$ *be an affine invariant submanifold in* $\mathcal{H}(0^n)$ *where* $\mathcal{H}$ *is an unmarked stratum of Abelian differentials. Suppose that* $\mathcal{M}$ *contains marked points on a translation surface* (X, ω) *that is generic in* $\mathcal{H}$ *and is one of the surfaces constructed in Section 3.4. Finally, suppose after passing to a finite cover that the marked points are labelled as* $\{p_1, \ldots, p_n\}$*. Let* $\pi_k : \mathcal{H}^{ord}(0^n) \longrightarrow \mathcal{H}^{ord}(0^{n-1})$ *be the map that forgets the* k*th marked point for* $k \in \{1, \ldots, n\}$*.*

Theorem 3.6.2. *Periodic points exist on* (X, ω) *if and only if* $\mathcal{H}$ *is hyperelliptic, in which case they are Weierstrass points.*

Theorem 3.6.3. *If* $n \geq 2$ *and* $(\pi_k)_* \mathcal{M} = \mathcal{H}^{ord}(0^{n-1})$ *for all* $k \in \{1, \ldots, n\}$ *then* $\mathcal{H}$ *is hyperelliptic,* $n = 2$*, and the fiber of* $\mathcal{M}$ *over* (X, ω) *contains all pairs of distinct points exchanged by the hyperelliptic involution.*

If $\mathcal{H}$ is hyperelliptic, then given a pair of integers $\{i, j\}$ integers in $\{1, \ldots, n\}$, let $\mathcal{H}_{ij}$ denote the subset of $\mathcal{H}(0^n)$ where p_i and p_j are exchanged by the hyperelliptic involution. If $i = j$, this will means that p_i is a fixed point of the hyperelliptic involution. We will prove the following strengthening of Theorem 3.1.1.

Theorem 3.6.4. *The stratum* $\mathcal{H}$ *is hyperelliptic and there is a subset* S *of* *pairs of integers in* $\{1,\dots,n\}$ *so that* $\mathcal{M} = \bigcap_{\{i,j\}\in S} \mathcal{H}_{ij}$.

Proof of Theorem 3.6.4 given Theorem 3.6.2 and Theorem 3.6.3: Proceed by induction on n. The $n = 1$ case is Theorem 3.6.2. Suppose now that $n > 1$.

Suppose first that for some $k \in \{1,\dots,n\}$, $(\pi_k)_*\mathcal{M}$ has dimension $\dim \mathcal{M} - 1$. Suppose without loss of generality after relabelling that $k = 1$. By the induction hypothesis, $\mathcal{H}$ is hyperelliptic and $(\pi_1)_*\mathcal{M} = \bigcap_{\{i,j\}\in S} \mathcal{H}_{ij}(0^{n-1})$ where S is some subset of pairs of integers in $\{2,\dots,n\}$. It follows that $\mathcal{M}$ is contained in $\bigcap_{\{i,j\}\in S} \mathcal{H}_{ij}(0^n)$ and therefore coincides with it since both manifolds are connected, closed, and of the same dimension.

Suppose now that for all $k \in \{1,\dots,n\}$, $(\pi_k)_* \mathcal{M}$ has the same dimension as $\mathcal{M}$ and that it does not coincide with $\mathcal{H}(0^{n-1})$. By the induction hypothesis, for each k, there is a subset S_k of pairs of integers in $\{1,\dots,n\} - \{k\}$, so that $(\pi_k)_* \mathcal{M} = \bigcap_{\{i,j\}\in S_k} \mathcal{H}_{ij}$. The number of elements of S_k is exactly the codimension of $(\pi_k)_* \mathcal{M}$ in $\mathcal{H}(0^{n-1})$. Suppose without loss of generality that $\{1, \ell_1\}$ is contained in S_{ℓ_2}. Then this pair cannot be contained in S_1 and so $S_1 \cup S_{\ell_2}$ contains at least as many elements as the codimension of $\mathcal{M}$ in $\mathcal{H}(0^n)$. Hence, $\mathcal{M} = \bigcap_{\{i,j\}\in S_1\cup S_{\ell_2}} \mathcal{H}_{ij}(0^n)$ since both manifolds are connected, closed, and of the same dimension.

The only case that remains is when $(\pi_k)_*\mathcal{M} = \mathcal{H}^{ord}(0^{n-1})$ for all $k \in$

$\{1, \ldots, n\}$, which follows from Theorem 3.6.3. ☐

3.7 Proof of Theorem 3.6.3

Assumption 3.6.1 will remain in effect for this section. Assume too that $\mathcal{M}$ is a proper affine invariant submanifold in $\mathcal{H}^{ord}(0^n)$ where $n \geq 2$ and suppose that $(\pi_k)_* \mathcal{M} = \mathcal{H}^{ord}(0^{n-1})$ for all $k \in \{1, \ldots, n\}$. If $\mathcal{H}$ is non-hyperelliptic then let C be the central horizontal cylinder and if $\mathcal{H}$ is hyperelliptic let C be any horizontal cylinder that intersects two vertical cylinders. By Lemma 3.4.4 and 3.4.5, there are two equivalence classes of vertical cylinders $\mathcal{D}_1$ and $\mathcal{D}_2$ whose intersection with C is connected and nonempty.

Lemma 3.7.1. *There are two marked points. If one marked point lies in a cylinder in $\mathcal{D}_1$ or $\mathcal{D}_2$, the other one lies in that cylinder as well.*

Proof. Let $P = \{p_1, \ldots, p_{n-1}\}$ be a collection of $n-1$ points where p_1 lies in $\mathcal{D}_1$ and divides the cylinders in $\mathcal{D}_1$ into subcylinders whose moduli admit no rational homogeneous linear relation. Suppose too that $\{p_2, \ldots, p_{n-1}\}$ belong to $\mathcal{D}_2$ and divide it into vertical sub-cylinders whose moduli also admit no rational homogeneous linear relation. The genericity criterion (Proposition 3.4.3) implies that $(X, \omega; P)$ is generic in $\mathcal{H}(0^{n-1})$. By Lemma 3.2.7, there is a point p_n so that $(X, \omega; P \cup \{p_n\})$ belongs to $\mathcal{M}$.

Let $\mathcal{D}$ be an equivalence class in $\{\mathcal{D}_1, \mathcal{D}_2\}$ that does not contain p_n. Since $\mathcal{D}$ is its own equivalence class in $\mathcal{H}$ and since it is divided into sub-cylinders whose moduli admit no rational homogeneous linear relation, it follows that each subcylinder in $\mathcal{D}$ may be sheared while fixing the rest of the surface. Phrased differently, in the fiber of $\mathcal{M}$ over (X, ω) any marked point in $\mathcal{D}$ may be moved freely while fixing all other points in P. However, if p_k is a point (for $k \in \{1, \dots, n-1\}$) that belongs to $\mathcal{D}$, then the fiber of $\pi_k : \mathcal{M} \longrightarrow \mathcal{H}^{ord}(0^{n-1})$ is one-dimensional and, by assumption $(\pi_k)_* \mathcal{M} = \mathcal{H}^{ord}(0^{n-1})$. This implies that $\mathcal{M} = \mathcal{H}^{ord}(0^n)$, which is a contradiction. Therefore, there are no points belonging to $\mathcal{D}$ and so $n = 2$.

For the final statement, suppose to a contradiction that $\{p_1, p_2\}$ is a fiber of $\mathcal{M}$ over (X, ω) under the map that forgets marked points and suppose too that p_1 belongs to $\mathcal{D}$ for $\mathcal{D} \in \{\mathcal{D}_1, \mathcal{D}_2\}$, but that p_2 does not. Since the map π_2 that forgets the second point is open by Lemma 3.2.9, there is a nearby surface $(X, \omega; p_1', p_2')$ in $\mathcal{M}$ where p_1' divides the cylinders in $\mathcal{D}$ into subcylinders whose moduli admit no rational homogeneous linear relation and so that p_2' remains outside of $\mathcal{D}$. This contradicts the previous paragraph. □

Since $\pi_2 : \mathcal{M} \longrightarrow \mathcal{H}(0)$ is a finite holomorphic map, we see that given a point $(X, \omega; p_1, p_2) \in \mathcal{M}$ we move p_1 to a new point p_1' (at least locally) and there will be a unique nearby point $(X, \omega; p_1', p_2') \in \mathcal{M}$. Since the equations that define the affine invariant submanifold $\mathcal{M}$ have real coefficients, if p_1 moves horizontally (resp. vertically), so does p_2.

Now suppose without loss of generality that p_1 lies at irrational height in $\mathcal{D}_1$ and at irrational height in C. Let p_2 be a point so that $(X, \omega; p_1, p_2) \in \mathcal{M}$. By Lemma 3.7.1, p_2 must also lie in the interior $\mathcal{D}_1 \cap C$, which is a rectangle. Moving p_1 to the left we see that p_1 reaches the left boundary of $\mathcal{D}_1 \cap C$ at the same moment that p_2 reaches the vertical boundary. Now reversing direction and moving p_1 to the right we see again that p_1 and p_2 arrive at the vertical boundary of $\mathcal{D}_1$ at the same moment. This implies either that one point lies above the other and both move at the same speed in the same direction (horizontally) or that both points at some point were on opposite boundaries of $\mathcal{D}_1$ and move in opposite directions (horizontally) at the same speed. The first case cannot occur, since if it does we may simply shear the central horizontal cylinder and find two marked points that do not lie above each other and that still move in the same direction at the same speed. The same argument applied to the vertical direction shows that when one point moves at unit speed in the v direction, the other point moves at unit speed in the $-v$ direction. Coupled with the fact that the points arrive at the $\mathcal{D}_1 \cap C$ boundary at the same times we have that when $\mathcal{H}$ is hyperelliptic the two points are exchanged by the hyperelliptic involution.

We will now show that $\mathcal{H}$ must be hyperelliptic. The present situation is pictured in Figure 3.7.1. If $\mathcal{H}$ is not hyperelliptic then we may ensure that $a < b$ (by Lemmas 3.4.4 and 3.4.5). If p_2 moves to the right at unit speed, then p_1 moves to the left at unit speed and hence p_2 arrives in the interior of $\mathcal{D}_2$ before p_1, contradicting Lemma 3.7.1.

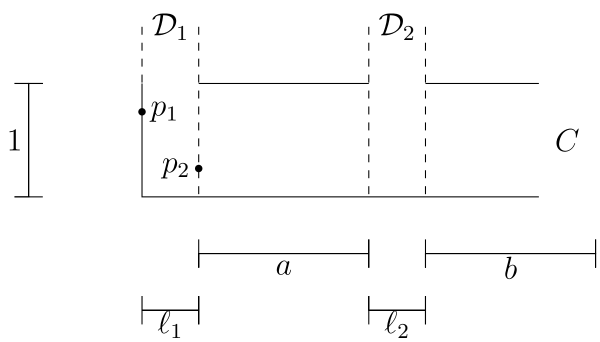


Figure 3.7.1: Two marked points in the central horizontal cylinder

3.8 Proof of Theorem 3.6.2

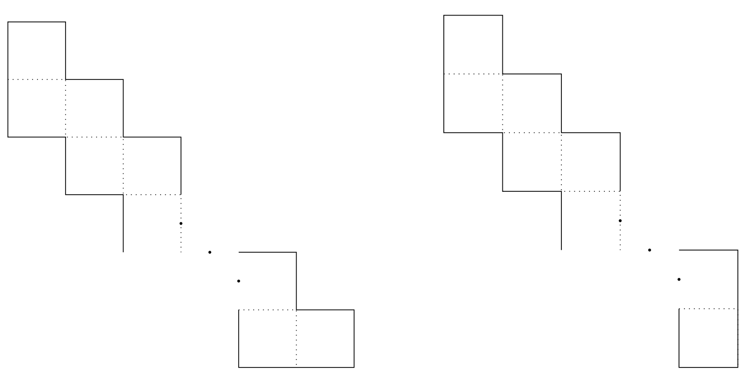
Throughout this section we will make the following assumption.

Assumption 3.8.1. *Let $\mathcal{H}$ be a stratum of Abelian differentials and let (X, ω) be a generic translation surface in $\mathcal{H}$ constructed in Section 3.4. Let $\mathcal{M}$ be an affine invariant submanifold properly contained in $\mathcal{H}(0)$. By Lemma 3.2.7, the fiber in $\mathcal{M}$ over (X, ω) is nonempty and any point p in the fiber is a periodic point.*

Definition 3.8.2. *A cylinder C is called $\mathcal{H}$-free if $\{C\}$ is an $\mathcal{H}$-equivalence class. This is equivalent to the condition that there is no other parallel cylinder C with a core curve that is homologous to the core curve of C.*

Proposition 3.8.3. *The periodic points on $\mathcal{H}^{hyp}(2g-2)$ and $\mathcal{H}^{hyp}(g-1, g-1)$ are exactly the Weierstrass points.*

Proof. Let $\mathcal{H}$ be either $\mathcal{H}^{hyp}(2g-2)$ or $\mathcal{H}^{hyp}(g-1, g-1)$. The surface (X, ω) is pictured again in Figure 3.8.1



(a) $\mathcal{H}^{hyp}(2g-2)$ (b) $\mathcal{H}^{hyp}(g-1, g-1)$

Figure 3.8.1: Hyperelliptic translation surfaces

Each horizontal and vertical cylinder is $\mathcal{H}$-free. By Lemma 3.5.5 if p is a periodic point in (X, ω) that lies in the interior of a horizontal cylinder that intersects two vertical cylinders, it is automatically a Weierstrass point. The same holds if it lies in the interior of a vertical cylinder that intersects two horizontal ones. We may therefore assume (up to exchanging each instance of the word "horizontal" for the word "vertical" and vice versa) that p lies in the interior of a horizontal cylinder that intersects only one vertical cylinder and on the boundary of a vertical cylinder. By Lemma 3.5.2, we may shear this cylinder and remain in $\mathcal{M}$. Shearing the horizontal cylinder so as to perform one complete Dehn twist moves the periodic point into the interior of a vertical cylinder that intersects two horizontal cylinders and so we have that p is a Weierstrass point by Lemma 3.5.5. □

Assumption 3.8.4. *Assume now that $\mathcal{H}$ is nonhyperelliptic and let C be the central horizontal cylinder in (X, ω).*

Proposition 3.8.5. *A periodic point on (X, ω) must lie on the boundary of C and in the interior of a vertical cylinder that is not $\mathcal{H}$-free.*

Proof. Let p be a periodic point in (X, ω). By Lemma 3.4.4, Lemma 3.4.5, and Lemma 3.5.5 the periodic point cannot lie in the interior of C. We will proceed now by cases based on the containment of p in vertical cylinders.

Case 1: p is contained in a vertical cylinder V that is $\mathcal{H}$-free, is contained in C, and only intersects the core curve of C once

In this case, Lemma 3.5.2 implies that we may shear V so as to perform one complete Dehn twist and remain in $\mathcal{M}$. This moves p to a periodic point in the interior of C, which is a contradiction.

Case 2: p is contained in a vertical cylinder V that is $\mathcal{H}$-free and is contained in C

By the previous case, V must intersect the core curve of C at least twice. By construction of the surfaces in Section 3.4 the situation must be as depicted in Figure 3.8.2. The marked point is then contained in a $\mathcal{H}$-free cylinder (drawn in dashed lines). Using Lemma 3.5.2 to shear this cylinder to perform one complete Dehn twist we see that p may be moved to a periodic point in the interior of C, which is a contradiction.

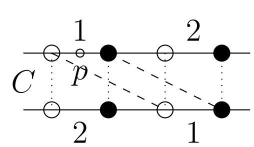


Figure 3.8.2: The translation surface in Case 2

Case 3: p is contained in a $\mathcal{H}$-free vertical cylinder V that is not contained in C

By construction of the surfaces in Section 3.4, V only intersects two horizontal cylinders - C and H - and core curves intersect exactly once, see Figure 3.8.3. Applying Lemma 3.5.5, with $\mathcal{D}_1 = \{C\}$ and $\mathcal{D}_2 = \{H\}$ we see that if p lies in the interior of V, then there must be a periodic point in C, which is a contradiction. If p does not lie in the interior of V, then by Lemma 3.5.2 we may shear H to perform one complete Dehn twist while fixing the remainder of the translation surface and remaining in $\mathcal{M}$. This shear moves p to the interior of V and so we are done.

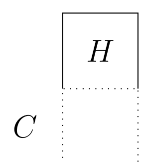


Figure 3.8.3: The vertical cylinder V

Case 4: p is contained in a vertical cylinder that is not contained in C and that is not $\mathcal{H}$-free

By construction of the surfaces in Section 3.4, $\mathcal{H}$ is an even component of a stratum of Abelian differentials. Moreover, either p is contained on the boundary of a vertical cylinder, as in Figure 3.8.4 or is contained in one of the cylinders passing through the saddle connections labelled $\{1, \ldots, 4\}$ in Figure 3.8.4.

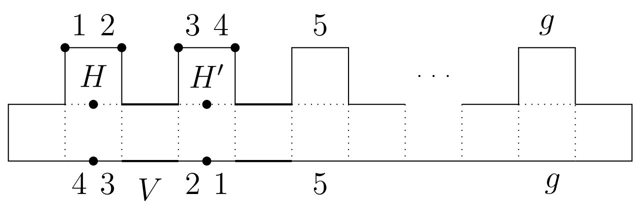


Figure 3.8.4: Points on the boundary of C and on the boundary of a vertical cylinder

Let H and H' be the indicated horizontal cylinder in Figure 3.8.4, which are $\mathcal{H}$-free. Suppose without loss of generality that p is contained in the horizontal cylinder H or its boundary. By Lemma 3.5.2 we may shear the cylinders H and H' to arrive at the surface in Figure 3.8.5. Let D be the diagonal cylinder with dashed boundary that passes through the horizontal saddle connection labelled 1. By Lemma 3.5.2 we may shear H if necessary to perform one complete Dehn twist and move p into the interior of D while remaining in $\mathcal{M}$. By Lemma 3.5.5 - where D is intersected by the equivalence classes $\{H\}$ and $\{C\}$ - it follows that there is a periodic point contained in the interior of C. By Lemma 3.5.2, we may shear H and H' while fixing the remainder of the translation surface to return to (X, ω) with a periodic point p in the interior of C, a contradiction.

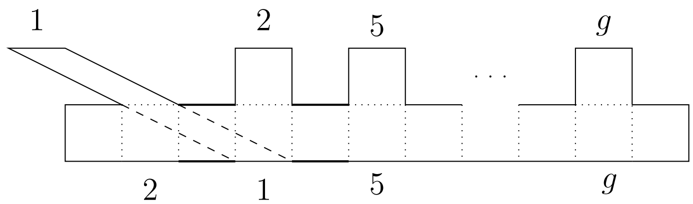


Figure 3.8.5: Moving potentially periodic points into the interior of C

Therefore, p lies on the boundary of C and is contained in the interior of a vertical cylinder which is not $\mathcal{H}$-free. $\square$

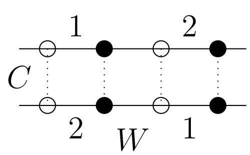
Corollary 3.8.6. *The only component of* $\mathcal{H}(2g-2)$ *with periodic points is* $\mathcal{H}^{hyp}(2g-2)$*.*

Proof. Let $\mathcal{H}$ be a minimal nonhyperelliptic stratum of Abelian differentials. If $\mathcal{M}$ is an affine invariant submanifold properly contained in $\mathcal{H}(0)$ and that pushes forward to $\mathcal{H}$ under the forgetful map, then its fiber over (X, ω) is nonempty by Lemma 3.2.7. If $(X, \omega; p)$ is an element of the fiber then p is contained in a vertical cylinder that is contained in C and that is not $\mathcal{H}$-free. By construction of the surfaces in Section 3.4 there are no such cylinders and so we are done. $\square$

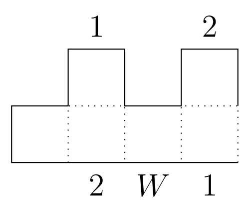
Proof of Theorem 3.6.2: Let $\mathcal{H}$ be a component of a stratum of Abelian differentials with at least two zeros. Proceed by induction on $\dim_{\mathbb{C}} \mathcal{H}$. The result has already been established for hyperelliptic components (Proposition 3.8.3), which establishes the base case, and allows us to assume without loss of generality that $\mathcal{H}$ is not a hyperelliptic component. Assume that p is a periodic point on (X, ω). By Proposition 3.8.5, p is contained on the boundary of C and in a vertical cylinder V that is contained in C and that is not $\mathcal{H}$-free.

Case 1: C contains an $\mathcal{H}$-free vertical cylinder W

By construction of the surfaces in Section 3.4, there are only two types of $\mathcal{H}$-free vertical cylinders contained in C, they are pictured in Figure 3.8.6.



(a) A subsurface of (X, ω) in components with two zeros of odd order



(b) A subsurface of (X, ω) in even components

Figure 3.8.6: The two types of $\mathcal{H}$-free cylinders W in C

By Lemma 3.5.2, if (X, ω) contains an $\mathcal{H}$-free cylinder W we may travel in $\mathcal{M}$ in the direction of the standard shear σ_W to shrink the horizontal cross curve of W until it vanishes. This degenerates $\mathcal{M}$ to the boundary of $\mathcal{H}(0)$. Notice that in both cases this degeneration causes two distinct zeros to collide on the boundary, but, by considering Euler characteristic, no genus is lost. By Mirzakhani-Wright [**MW17**, Corollary 1.2], the resulting translation surface has an orbit closure of strictly smaller dimension than $\mathcal{M}$. By the genericity criterion (Proposition 3.4.3), the boundary translation surface remains generic in the component of the stratum to which it belongs, which necessarily has complex dimension one less than $\mathcal{M}$. Therefore, p remains a periodic point.

By the induction hypothesis, p is a Weierstrass point and the boundary translation surface belongs to a hyperelliptic component. In particular, p must lie halfway across V on the boundary of C, dividing V into two subcylinders of equal modulus, call them V_1 and V_2. By construction, on $(X, \omega; p)$ the only rational linear homogeneous equation that holds on moduli of cylinders equivalent to V is that the moduli of V_1 and V_2 are equal. By Wright [**Wri15a**] - in particular Theorems 3.2.3 and 3.2.4 - and Lemma 3.5.2, V may be sheared on $(X, \omega; p)$ while remaining in $\mathcal{M}$ and fixing the remainder of the translation surface. Shearing so as to perform one complete Dehn twist moves p into the interior of C on (X, ω), which contradicts Proposition 3.8.5.

Notice that as a corollary of this step, $\mathcal{H}$ does not contain two zeros of odd order.

Case 2: The surface belongs to an even component

Let H and H' be the horizontal cylinders labelled in Figure 3.8.4. By

Lemma 3.5.2, shearing them while fixing the remainder of the surface remains in $\mathcal{M}$. Therefore, we shear them to find the surface in $\mathcal{M}$ depicted in Figure 3.8.7, which contains a vertical cylinder that contains H and H' and that passes through them exactly once.

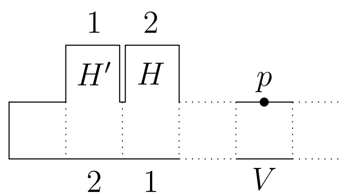


Figure 3.8.7: A translation surface in the even component

By Lemma 3.5.2, applying the standard dilation $a_{\{H\}}$ to H causes its vertical cross curve to vanish and passes to a surface on the boundary of $\mathcal{M}$. When the cross curve vanishes, the boundary translation surface (shown in Figure 3.8.8) has the zero of order $2k+2$ (where k is a positive integer) on the boundary of H split into two zeros - one of order 1 and one of order $2k-1$.

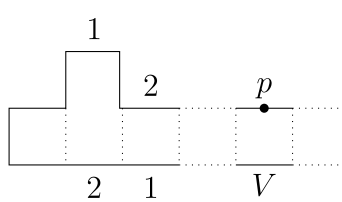
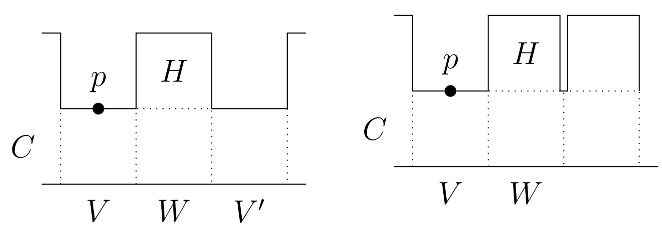


Figure 3.8.8: The boundary translation surface (Y, η)

Let $\mathcal{M}'$ be the orbit closure of $(Y, \eta; p)$. By the genericity criterion (Proposition 3.4.3), (Y, η) remains generic. By Mirzakhani-Wright [**MW17**, Corollary 1.2], the dimension of $\mathcal{M}'$ must be strictly smaller than the dimension of $\mathcal{M}$. It follows that p is a periodic point on (Y, η). However, no such points exist on translation surfaces of genus greater than two with a zero of odd order. Therefore, (X, ω) cannot belong to an even component.

Case 3: The surface does not belong to an even component

The marked point is contained in one of the two configurations shown in Figure 3.8.9, where V is adjacent on the right to a vertical cylinder W that contains a horizontal cylinder H (both shown in the figure).

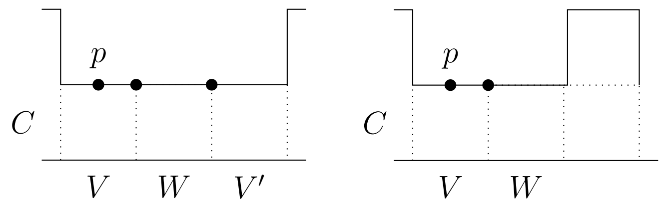


(a) The periodic point borders an order 2 zero

(b) The periodic point borders a zero of order 4 or more

Figure 3.8.9: Two possible configurations

By Lemma 3.5.2, applying the standard dilation $a_{\{H\}}$ to H causes its vertical cross curve to vanish and passes to a surface on the boundary of $\mathcal{M}$. The underlying translation surface moves from $\mathcal{H}(2k, \ldots)$ to $\mathcal{H}(0, 2k-2, \ldots)$ (see Figure 3.8.10).



(a) The surfaces collapses to $\mathcal{H}(0,0,\ldots)$

(b) The surfaces collapses to $\mathcal{H}(0,2k-2,\ldots)$

Figure 3.8.10: The result of collapsing H

Let $(Y,\eta;\{p\}\cup Q)$ be the boundary translation surface where Q are the marked points that arise in the degeneration. Let $\mathcal{M}'$ be the orbit closure of $(Y,\eta;\{p\}\cup Q)$. The genericity criterion (Proposition 3.4.3) implies that $(Y,\eta;Q)$ is generic in the stratum $\mathcal{H}'$ that contains it. By Mirzakhani-Wright [**MW17**, Corollary 1.2], $\mathcal{M}'$ is an affine invariant submanifold that is properly contained in $\mathcal{H}'(0)$. The induction hypothesis implies that (Y,η) belongs to a hyperelliptic component and that either p is a Weierstrass point or is exchanged with a point in Q under the hyperelliptic involution.

Consider first the configuration in Figure 3.8.10b. Since $V\cup W$ must be fixed by the hyperelliptic involution and since W may be made arbitrarily long horizontally we see that p is neither a Weierstrass point nor a point exchanged under the hyperelliptic involution with a point in Q.

Consider now the configuration in Figure 3.8.10a and let V_a and V_b be the left and right sub-cylinders that p splits V into. We see that p must be exchanged under the hyperelliptic involution with the rightmost point in Q and so V_a and V' have identical moduli. Repeating the argument with the vertical cylinder W' that V borders on the left shows that (X, ω) must contain the subsurface shown in Figure 3.8.11 and satisfy the property that the modulus of V' is the same as the modulus of V_a and the modulus of V'' the same as the modulus of V_b.

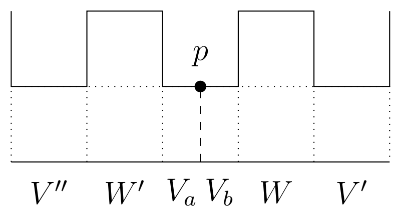


Figure 3.8.11: The surface $(X, \omega; p)$

Letting $\text{Mod}(D)$ denote the modulus of a cylinder D we see that on this surface

$$\text{Mod}(V) = \text{Mod}(V_a) + \text{Mod}(V_b) = \text{Mod}(V') + \text{Mod}(V'')$$

which is a rational linear homogeneous relation on moduli satisfied by vertical cylinders on (X, ω), which contradicts the fact that (X, ω) is constructed so as to prevent the moduli of vertical cylinders from satisfying such a relation. $\square$

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